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CS174A : Introduction to Computer Graphics

Royce 190
TT 4-6pm

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Curves

- Parametric – cubic blending functions
  - There is a slightly different way of looking at this interpolation process.
    - It allows us to see what exactly is going on.
  - Lets substitute those interpolating coefficients into the polynomial itself – generalize things a bit.

$$\mathbf{p}(u) = \mathbf{u}^T\mathbf{c} = \mathbf{u}^T\mathbf{M}_I\mathbf{p}, \quad \text{or} \quad \mathbf{p}(\mathrm{u}) = \mathbf{b}(\mathrm{u})^T\mathbf{p}, \quad \text{where} \quad \mathbf{b}(\mathrm{u}) = \mathbf{M}_I^T\mathbf{u}$$

$$\mathbf{b}(u) = \begin{bmatrix} b_0(u) \\ b_1(u) \\ b_2(u) \\ b_3(u) \end{bmatrix}$$

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Curves

- Parametric – cubic blending functions
  - If we express $\mathbf{p}(u)$ in terms of these cubic blending polynomials we get

$$\mathbf{p}(u) = b_0(u)\mathbf{p}_0 + b_1(u)\mathbf{p}_1 + b_2(u)\mathbf{p}_2 + b_3(u)\mathbf{p}_3$$

$$b_0(u) = -\frac{9}{2}\left(u-\frac{1}{3}\right)\left(u-\frac{2}{3}\right)(u-1),$$

$$b_1(u) = -\frac{27}{2}u\left(u-\frac{2}{3}\right)(u-1),$$

$$b_2(u) = -\frac{27}{2}u\left(u-\frac{1}{3}\right)(u-1),$$

$$b_3(u) = \frac{9}{2}u\left(u-\frac{1}{3}\right)\left(u-\frac{2}{3}\right).$$

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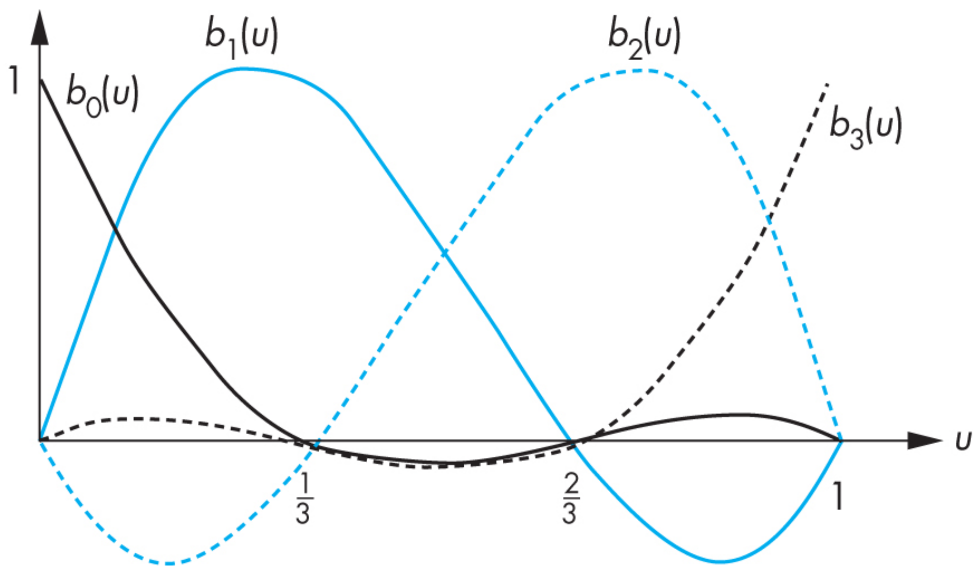
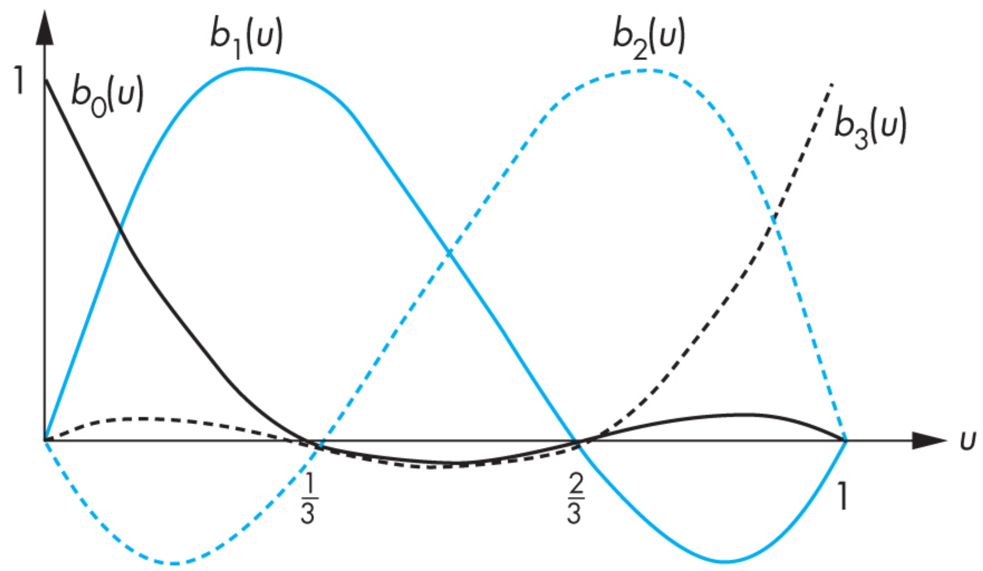
Curves

- Parametric – cubic blending functions
  - Which lets you see how each of the blending equations factors into the interpolation.
    - What we want to see



Curves

- Parametric – cubic blending functions
  - You can see how each is not particularly smooth
    - This is because we are insisting the interpolation pass through each of the control points.



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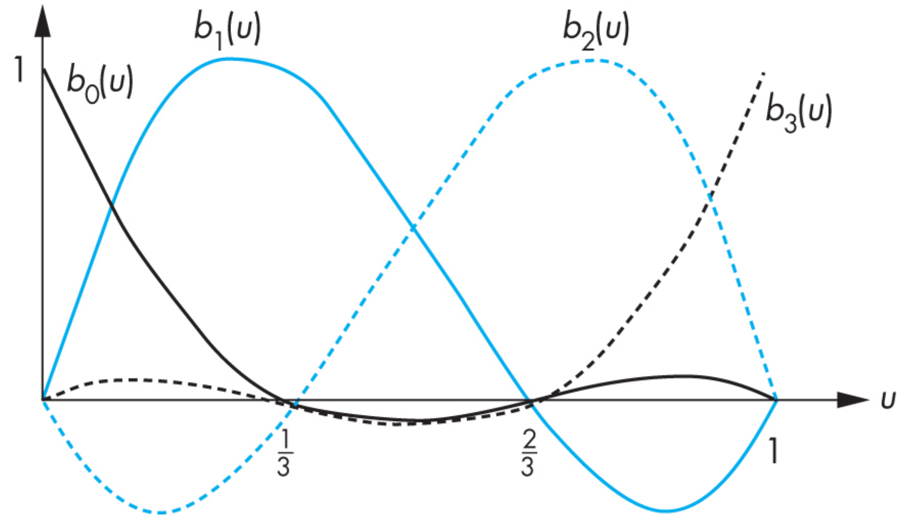
Curves

- Parametric – cubic blending functions
  - Higher degree polynomials would have even more pronounced swings.
  - Recall, there is also no way to enforce derivatives at the endpoints makes this form of limited use.



Curves

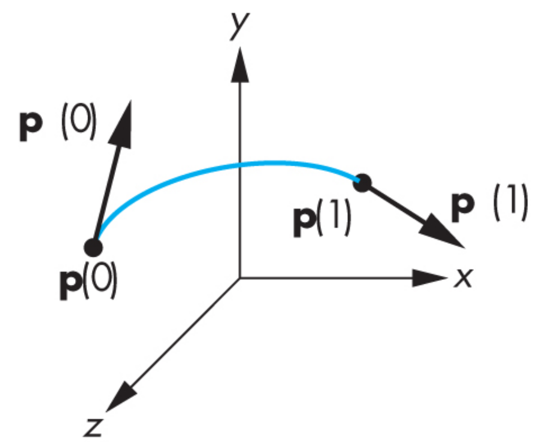
- Parametric – Hermite form
  - While able to form a curve (or surface) using the cubic interpolating polynomial, it has some issues.
  - Let's look at the Hermite form which allows some additional control over the derivatives at the ends of the curve.
  - Here we only consider the control points $\mathbf{p}_0$ and $\mathbf{p}_3$, which from our previous example we have the first two conditions

$$\mathbf{p}_0 = \mathbf{p}(0) = \mathbf{c}_0,$$
$$\mathbf{p}_3 = \mathbf{p}(1) = \mathbf{c}_0 + \mathbf{c}_1 + \mathbf{c}_2 + \mathbf{c}_3.$$

Curves

- Parametric – Hermite form
  - We can get the other two conditions if we assume the derivatives at u=0 and u=*1* are known.

$$\mathbf{p}'(u) = \mathbf{c}_1 + 2u\mathbf{c}_2 + 3u^2\mathbf{c}_3.$$
$$\mathbf{p}'_0 = \mathbf{p}'(0) = \mathbf{c}_1,$$
$$\mathbf{p}'_3 = \mathbf{p}'(1) = \mathbf{c}_1 + 2\mathbf{c}_2 + 3\mathbf{c}_3.$$



  - In matrix form

$$\mathbf{q} = \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_3 \\ \mathbf{p}'_0 \\ \mathbf{p}'_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix} \mathbf{c}.$$

Curves

- Parametric – Hermite form
  - Once again, solving for **c** we get

$$\mathbf{c} = \mathbf{M}_H \mathbf{q}, \quad \text{where}$$

$$\mathbf{M}_H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 3 & -2 & -1 \\ 2 & -2 & 1 & 1 \end{bmatrix}.$$

  - M_{H} is the Hermite geometry matrix

Curves

- Parametric – Hermite form
  - Once again we get the resulting polynomials

$$\mathbf{p}(u) = \mathbf{u}^T \mathbf{c} = \mathbf{u}^T \mathbf{M}_H q, \quad \text{or} \quad \mathbf{p}(\mathrm{u}) = \mathbf{b}(\mathrm{u})^T q, \quad \text{where} \quad \mathbf{b}(\mathrm{u}) = \mathbf{M}_H^T \mathbf{u}$$

$$\mathbf{b}(u) = \begin{bmatrix} 2u^3 - 3u^2 + 1 \\ -2u^3 + 3u^2 \\ u^3 - 2u^2 + u \\ u^3 - u^2 \end{bmatrix}$$

  - The blending functions can be used in the same way a before for the cubic interpolating polynomial.

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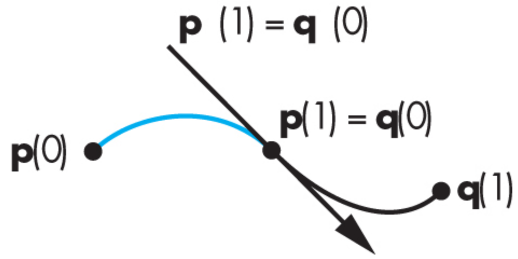
Curves

- Parametric – Hermite form
  - Here, if we hold the derivative to be the same across curve segments, at the join, we get continuity.



– Curves connecting at their endpoints have C^0 parametric continuity.

– If the derivatives also match we call that C^1 parametric continuity – like above figure.

– If the derivatives are only proportional to each other we call this G^1 geometric continuity.

Curves

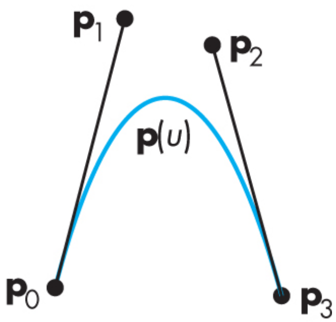
- Parametric – Bézier form
  - Cannot really compare cubic interpolating polynomial to Hermite.
    - Both are cubic in degree
    - But, they do not use the same data (control points)
  - We can use all **four** control points of the cubic interpolating polynomial to approximate the Hermite curve – this is called the Bézier form.
    - Probably heard of this one, named after Pierre Bézier who worked for Renault in France in the 1960's

Curves

- Parametric – Bézier form
  - We, again, use endpoints $\mathbf{p}_0$ and $\mathbf{p}_3$ and insist that our interpolation pass through these values.

$$\mathbf{p}_0 = \mathbf{p}(0) = \mathbf{c}_0,$$
$$\mathbf{p}_3 = \mathbf{p}(1) = \mathbf{c}_0 + \mathbf{c}_1 + \mathbf{c}_2 + \mathbf{c}_3.$$

  - Bézier, instead of using $\mathbf{p}_1$ and $\mathbf{p}_2$ for interpolation instead used them to approximate the tangents at $u=0$ and $u=1$.



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Curves

- Parametric – Bézier form
  - Approximating the tangent results in the following conditions.

$$\mathbf{p}'(0) \approx \frac{\mathbf{p}_1 - \mathbf{p}_0}{\frac{1}{3}} = 3(\mathbf{p}_1 - \mathbf{p}_0),$$

$$\mathbf{p}'(1) \approx \frac{\mathbf{p}_3 - \mathbf{p}_2}{\frac{1}{3}} = 3(\mathbf{p}_3 - \mathbf{p}_2).$$

gives

$$\mathbf{p}_0 = \mathbf{c}_0,$$

$$3\mathbf{p}_1 - 3\mathbf{p}_0 = \mathbf{c}_1,$$

$$3\mathbf{p}_3 - 3\mathbf{p}_2 = \mathbf{c}_1 + 2\mathbf{c}_2 + 3\mathbf{c}_3,$$

$$\mathbf{p}_3 = \mathbf{c}_0 + \mathbf{c}_1 + \mathbf{c}_2 + \mathbf{c}_3.$$

Curves

- Parametric – Bézier form
  - Once again, we have four equations and four unknowns.
  - Solving for **c**

$$\mathbf{c} = \mathbf{M}_B\mathbf{p}$$

where

$$\mathbf{M}_B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & -3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

  - $\mathbf{M}_B$ is the Bézier geometry matrix.

Curves

- Parametric – Bézier form
  - The cubic polynomial is then

$$\mathbf{p}(u) = \mathbf{u}^T \mathbf{M}_B \mathbf{p}.$$

  - Which is used in exactly the same manner as the cubic interpolating polynomial we saw earlier.
  - If the control points are overlapped you should see that we still have C^0 continuity at the join.
  - We do not have C^1 continuity like with Hermite because there are different approximations for the tangent.

Curves

- Parametric – Bézier form
  - However, there are some useful benefits to this form.
  - Let's look at the blending functions

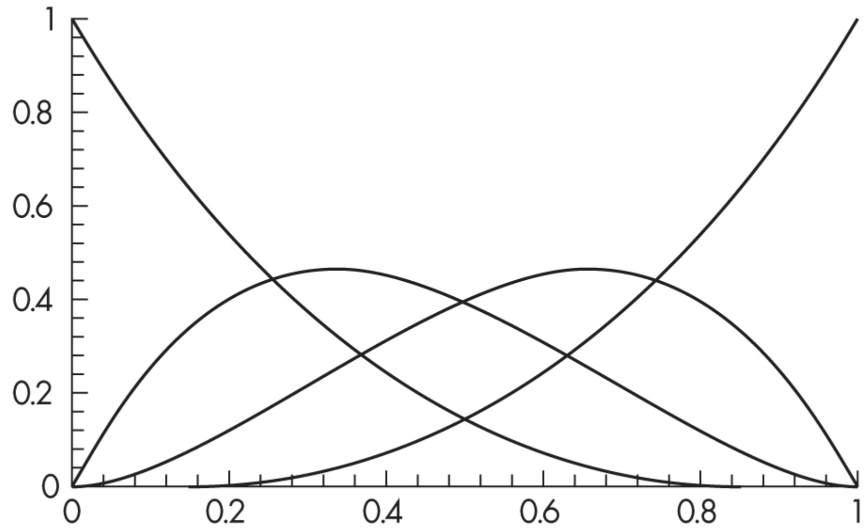
$$\mathbf{p}(u) = \mathbf{b}(u)^T \mathbf{p},$$

where

$$\mathbf{b}(\mathrm{u}) = \mathbf{M}_B^T \mathbf{u} = \begin{bmatrix} (1-u)^3 \\ 3u(1-u)^2 \\ 3u^2(1-u) \\ u^3 \end{bmatrix}.$$

Curves

- Parametric – Bézier form



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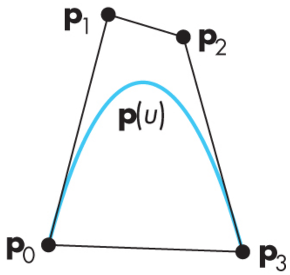
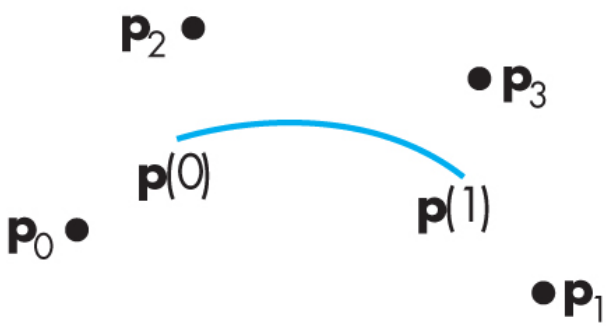
Curves

- Parametric – Bézier form
  - Zero values are only at the ends of the interval
    - Ensures a smooth interpolation over the interval [0,1]
  - We also see that while 0<u<1,
    - the blending functions are also $b(u) < 1$
    - This condition is called a *convex sum.*
    - Which implies that the curve will be contained within the *convex hull* of the control points.
    - Useful in interactive design of curves and surfaces.



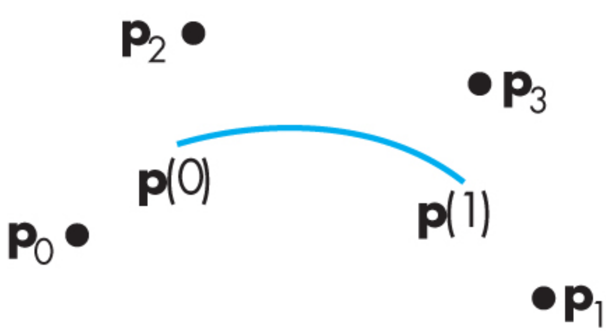
Curves

- Parametric – Cubic B-Splines
  - Bezier curves – only achieve C^0 continuity, mathematically speaking.
  - Can achieve C^1 by matching the tangents.
  - If we relax the condition that interpolation must pass though the control points we can achieve C^2 continuity using a cubic B-spline.



Curves

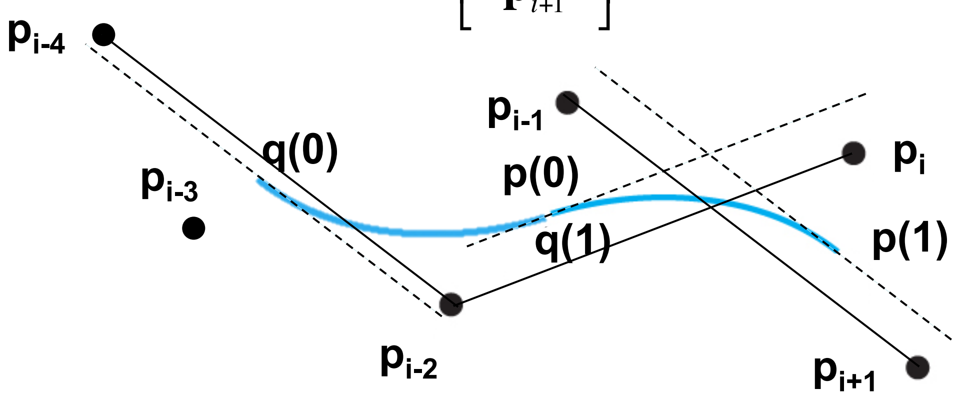
- Parametric – Cubic B-Splines
  - Previously, as we varied u from 0 to 1 the curve spanned over all four control points.
  - Consider, instead, spanning over only the middle two control points.



Curves

- Parametric – Cubic B-Splines
  - By matching conditions at $\mathbf{p}(0)$ with $\mathbf{q}(1)$ we can achieve C^2 continuity by solving for $\mathbf{M}$.

$$\mathbf{p}(u) = \mathbf{u}^T\mathbf{M}\mathbf{p}, \text{ where } \mathbf{p} = \begin{bmatrix} \mathbf{p}_{i-2} \\ \mathbf{p}_{i-1} \\ \mathbf{p}_i \\ \mathbf{p}_{i+1} \end{bmatrix}, \text{ and } \mathbf{q}(u) = \mathbf{u}^T\mathbf{M}\mathbf{q}, \text{ where } \mathbf{q} = \begin{bmatrix} \mathbf{p}_{i-3} \\ \mathbf{p}_{i-2} \\ \mathbf{p}_{i-1} \\ \mathbf{p}_i \end{bmatrix}.$$

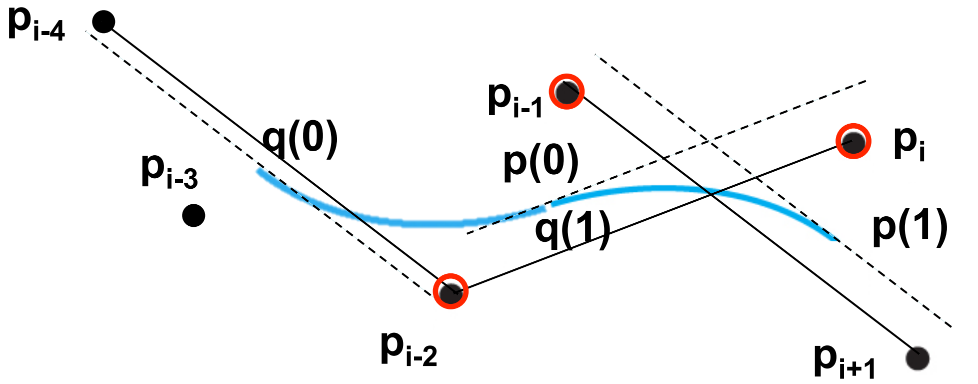


Curves

- Parametric – Cubic B-Splines
  - We again use symmetric approximations for the tangent at the joint point to get

$$\mathbf{p}(0) = \mathbf{q}(1) = \frac{1}{6}(\mathbf{p}_{i-2} + 4\mathbf{p}_{i-1} + \mathbf{p}_i) = \mathbf{c}_0,$$

$$\mathbf{p}'(0) = \mathbf{q}'(1) = \frac{1}{2}(\mathbf{p}_i - \mathbf{p}_{i-2}) = \mathbf{c}_1.$$



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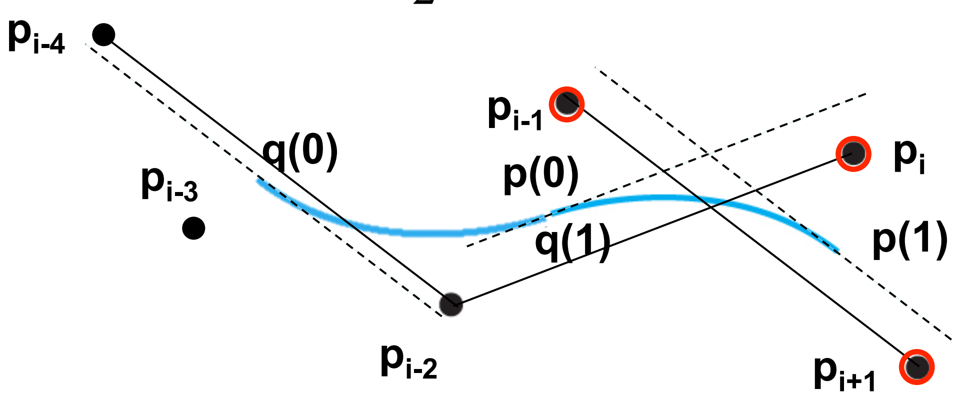
Curves

- Parametric – Cubic B-Splines
  - Applying the same conditions at $\mathbf{p}(1)$
    - We basically just slide down to the next set of control points.

$$\mathbf{p}(1) = \frac{1}{6}(\mathbf{p}_{i-1} + 4\mathbf{p}_i + \mathbf{p}_{i+1}) = \mathbf{c}_0 + \mathbf{c}_1 + \mathbf{c}_2 + \mathbf{c}_3,$$

$$\mathbf{p}'(1) = \frac{1}{2}(\mathbf{p}_{i+1} - \mathbf{p}_{i-1}) = \mathbf{c}_1 + 2\mathbf{c}_2 + 3\mathbf{c}_3.$$



Curves

- Parametric – Cubic B-Splines
  - We now have four equations for the coefficients of **c**.
  - Allowing us to solve for **M**
  - Really, $\mathbf{M}_S$, the B-spline geometry matrix, which is

$$\mathbf{M}_S = \frac{1}{6}\begin{bmatrix} 1 & 4 & 1 & 0 \\ -3 & 0 & 3 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

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Curves

- Parametric – Cubic B-Splines
  - We now have C^2 continuity at the joins but at a cost of three times as much work.
    - Have to interpolate between each set of control points.
  - C^2 continuity not only connects the segments
    - It matches the tangents
    - It also matches the curvature
    - Very useful properties when modeling real world materials
    - Difficult to use as the *curve does not pass though any of the control points*
      - » Not intuitive