$z^a = e^{a \operatorname{Ln} z}$, $a \in \mathbb{C}$

Частный случай ($a = \frac{1}{n}$) $z^{\frac{1}{n}} = \sqrt[n]{z} = e^{\frac{1}{n}(\ln|z| + i(\arg z + 2\pi k))} =$
$= \sqrt[n]{|z|} \cdot e^{i\frac{\arg z + 2\pi k}{n}}$, $k = 0, 1, \dots$

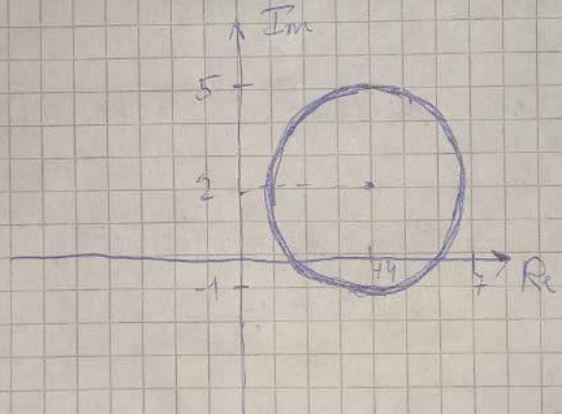
(7) Общая показательная функция

$a^z = e^{z \operatorname{Ln} a}$, $a \in \mathbb{C}$

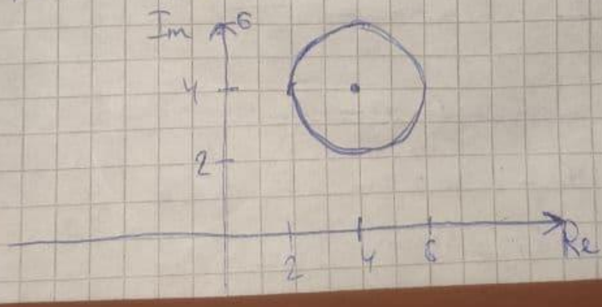
Занятие 13. ФКП

(I) Постройте линии, заданные уравнениями и области D, заданные системой условий. Проверьте, принадлежит ли точка $z_0$ области D

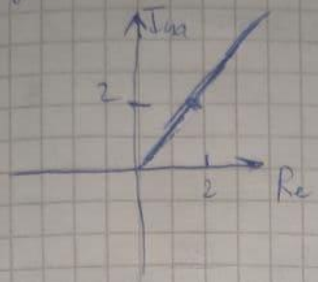
(1) $|z - 2i - 4| = 3$




(2) $|z - 4 - 4i| = 2$

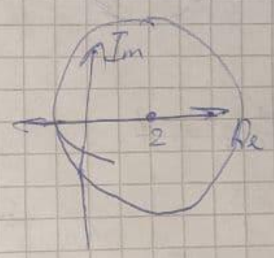


(3) $\arg z = \frac{\pi}{4}$



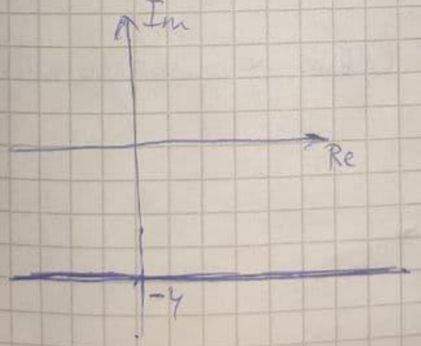

(4) $\operatorname{Re} z = 1$


(5) $|z-2| = 3$




(6)

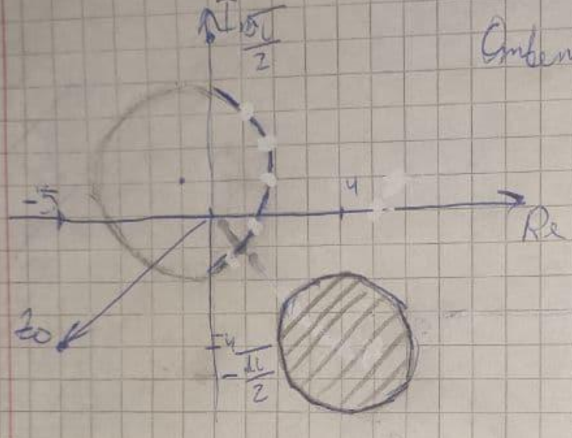
$\operatorname{Im} z = -4$



(7)

$$D: \begin{cases} |z - i + 1| \geq 3 \\ |z - 4 + 4i| \leq 2 \\ -\frac{\pi}{2} \leq \arg z \leq \frac{\pi}{2} \end{cases}$$

$z_0 = 5 + 4i$



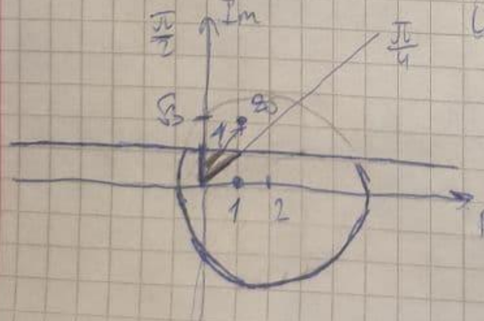

Ответ: не принадлежит

(8)

~~$|z| - \mathrm{Im}\, z < 1$~~

$$D: \begin{cases} |z - 2| \leq 3 \\ \mathrm{Re}\, z \leq 1 \\ \frac{\pi}{4} \leq \arg z < \frac{\pi}{2} \end{cases}$$

$z = 1 + i\sqrt{3}$




Ответ: не принадлежит

(9)

II.

f

От

w =

z =

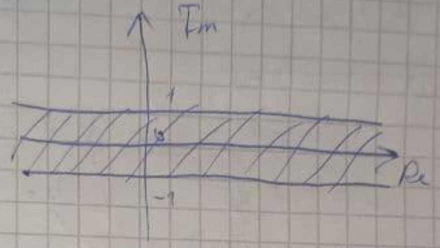
f(z)

− x

= (

(9) $|z| - Im\, z < 1$

$|z| < 1 + Im\, z$


II. (1) $w = z^2$

$w = f(z) = f(x+iy) = u(x;y) + i\,v(x;y)$

$f(z) = (x+iy)^2 = x^2 + i^2y^2 + 2xiy = (x^2 - y^2) + i(2xy)$

Ответ: $(x^2 - y^2) = u(xy)$, $(2xy) = v(xy)$

(2) $f(z) = iz^2 - \bar{z}$ $\quad u(xy)$ - ? $\quad v(xy)$ - ?

$w = f(z) = f(x+iy) = \underbrace{u(x;y)}_{действ} + i\,\underbrace{v(x;y)}_{мним}$

$z = x + iy$

$f(z) = f(x+iy) = i(x+iy)^2 - (x-iy) = i(x^2 + 2ixy + (iy)^2) -$

$- x + iy = (x^2 - y^2)i + 2i^2xy - x + iy = -(2xy + x) + i(x^2 - y^2 + y) =$

$= \underbrace{(-2xy - x)}_{u(xy)} + i\underbrace{(x^2 - y^2 + y)}_{v(xy)}$

(III) Определите функцию $w = f(z)$ по известной действительной ($u(x;y)$) и мнимой части ($v(x;y)$).

$u(x;y) = x+y, \quad v(x;y) = x-y.$

$\cancel{(x+y)+i(x-y)}$

$$\begin{cases} z = x+iy \\ \bar{z} = x-iy \end{cases} \Rightarrow \begin{cases} x = \frac{1}{2}(z+\bar{z}) \\ y = \frac{1(-i)}{2i(-i)}(z-\bar{z}) \end{cases} \quad y = \frac{-i}{2}(z-\bar{z})$$

$$f(z) = x(1+i) + y(1-i) = \frac{1}{2}(z+\bar{z})(1+i) - \frac{i}{2}(z-\bar{z})$$

$$= x+y+i(x-y)$$

Ответ: $f(z) = \frac{1}{2}(z+\bar{z})(1+i) - \frac{i}{2}(z-\bar{z})$.

(IV) Найти все значения ф-ии $w$ в заданной т-ке $z_0$.

(1) $w = \frac{\sqrt{2}}{2} - \sqrt{z}, \quad z_0 = i$

$$\sqrt[n]{z} = \sqrt[n]{|z|}\cdot\left(\cos\frac{\varphi+2\pi k}{n} + i\sin\frac{\varphi+2\pi k}{n}\right)$$

$$w(i) = \begin{cases} \frac{\sqrt{2}}{2} - z_0, & k=0 \\ \frac{\sqrt{2}}{2} - z_1, & k=1 \end{cases}$$

$$\sqrt{z} = \begin{cases} 1\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = z_0, & k=0 \\ 1\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = z_1, & k=1 \end{cases}$$

$$w(i) = \begin{cases} \frac{\sqrt{2}}{2} - \left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = -i\frac{\sqrt{2}}{2}, & k=0 \\ \frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = \sqrt{2} + i\frac{\sqrt{2}}{2} \end{cases}$$

отн.фул $(x;y)$.

$(z-\bar{z})$

$(z\bar{z}-\bar{z})$

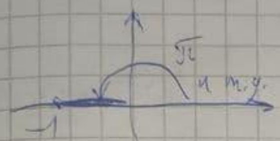
ном т.ке

② $w = z + \sqrt[4]{z}$, $z_0 = 1$

$w(-1) - ?$

$\sqrt[4]{-1} = \begin{cases} z_0 \\ z_1 \\ z_2 \\ z_3 \end{cases}$

$w(-1) = -1 + \sqrt[4]{-1}$




$\sqrt[n]{z} = \sqrt[n]{|z|(\cos\varphi + i\sin\varphi)}$

$\sqrt[4]{-1} = \sqrt[4]{1} \cdot \left(\cos\frac{\pi+2\pi k}{4} + i\sin\frac{\pi+2\pi k}{4}\right)$

$k = 0, 1, 2, 3.$

V вычислить

① $\sin i$

$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$ $\qquad \cos z = \frac{1}{2}(e^{iz} + e^{-iz})$

$z = i$

$\sin i = \frac{1(-i)}{2i(-i)}(e^{i^2} - e^{-i^2}) = \frac{-i}{2}(e^{-1} - e) = \frac{i}{2}\left(e - \frac{1}{e}\right) + 0 =$

$= \frac{e^2 - 1}{2e} \cdot i$

② $\operatorname{ch}(2-3i)$

$\operatorname{ch} z = \frac{1}{2}(e^{z} + e^{-z})$ $\qquad \operatorname{sh} z = \frac{1}{2}(e^{z} - e^{-z})$

$\operatorname{ch}(2-3i) = \frac{1}{2}(e^{2-3i} + e^{-2+3i}) = \frac{1}{2}e^{2}\left(\frac{1}{e^{3i}} \cdot e^{-3i} + \frac{e^{3i}}{e^{2}}\right) =$

$= \frac{e^{2} \cdot e^{3i}}{2}\left(\frac{1}{e^{6i}} + \frac{1}{e^{2}}\right)$

③ $\cos 2i$

$\cos z = \frac{1}{2}(e^{i \cdot 2i} + e^{-i \cdot 2i}) = \frac{e^{-2} + e^{2}}{2} = \frac{\frac{1}{e^2} + e^2}{2} =$

$= \frac{e^4 + 1}{2e^2}$

(4) $sh(3-i)$

$sh(3+i) = \frac{1}{2}(e^{z} - e^{-z})$

$sh(3-i) = \frac{1}{2}(e^{3-i} - e^{i-3}) = \frac{1}{2} e^{-i} \cdot \left(e^{3} - \frac{1}{e^{3}} \cdot e^{2i}\right) =$

$= \frac{1}{2} e^{-i}\left(e^{3} + \frac{1}{e^{3}}\right) = \frac{e^{6}+e^{3}}{2e^{3}} \cdot e^{i} =$ $\qquad e^{ix} = \cos x + i \sin x$

$= \frac{e^{6}+e^{3}}{2e^{3}} \cdot (\cos 1 + i \sin 1)$

VI 1) $Arccos\, z$ $\qquad Arccos\, 2 - ?$

$Arccos\, z = w \iff \cos w = z$

$w \in [-1, 1]$, $\theta \in [0, \pi]$

$z = \frac{1}{2}(e^{iw} + e^{-iw})$ Решим ур-ние отн-но $w$

$e^{iw} = t$

$2z = t + \frac{1}{t} \quad | \cdot t$

$t_{1,2} = z + \sqrt{z^2 - 1}$

$e^{iw} = z + \sqrt{z^2 - 1}$

$iw = \ln\left(z + \sqrt{z^2 - 1}\right) \quad (\cdot -i)$ $\qquad \boxed{Arccos\, z = -i \cdot \ln\left(z + \sqrt{z^2 - 1}\right)}$

- $Arccos\, 2 = -i \ln(2 + \sqrt{3}) = -i\left(\ln(2+\sqrt{3}) + i(0 + 2\pi k)\right) =$

$= 2\pi k - i \ln(2 + \sqrt{3})$, $k \in \mathbb{Z}$

$\boxed{Ln\, z = \ln|z| + i\varphi}$