Problem 1

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix} \; \& \; B = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

a. Find the eigenvalues and eigenvectors of the given matrices.
b. Find the transformation matrices to obtain JCF of the given matrices.
c. Find the JCF of the given matrices.
d. Find the characteristic polynomials of the given matrices.
e. Find the minimal polynomials of the given matrices.

A matrix Solution

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$det(A - \lambda I) = 0$$

$$det\left(\begin{bmatrix} 2 & 1 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right) = 0$$

$$det\left(\begin{bmatrix} 2-\lambda & 1 & 3 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{bmatrix} \right) = 0$$

$$(\lambda - 2)^2(\lambda - 5) = 0 \rightarrow \lambda(A) = \{2,2,5\}$$

There are two different eigenvalues 2 and 3.

If there were 3 distinct eigenvalues, we would say directly that the matrix A can be diagonalized.

However, for this situation, if we can find two linearly independent eigenvectors that correspond to $\boxed{\lambda = 2}$, we would say that the matrix A can be diagonalized. In other words, if

$$\dim\{\mathcal{N}(A - (2)I)\} = 2 \rightarrow A\ can\ be\ diagonalized.$$

If we see that that is not the case, we will conclude that “A cannot be diagonalized, but we can find its Jordan Canonical Form”.

Therefore, let us compute $\boxed{\dim\{\mathcal{N}(A - (2)I)\}}$.

$$\dim\{\mathcal{N}(A - (2)I)\} = 1$$

[For that we can simply use Matlab]

Ok, we can at this point conclude that A cannot be diagonalized and we can not only find its Jordan Canonical Form.

To obtain the JCF, let us check the eigenvectors that correspond to $\boxed{\lambda = 2}$ and $\boxed{\lambda = 5}$

<table>
<tr><th>$\lambda = 2$</th><th>$\lambda = 5$</th></tr>
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$$\mathcal{N}(A-(2)I) = span\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}\right\}$$

where, let's call the vector $\begin{bmatrix}1\\0\\0\end{bmatrix}$, v_1.
Now, we need to find the generalized eigenvector associated with v_1.
(which corresponds to $\boxed{\lambda = 2}$)
We only need one vector.
Let us call the generalized vector v_2.
v_2 can be computed using the natural basis vectors, that are,
$\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$.
If we will check all three of them,
Recall that the relation between v_1 and v_2 is given as,

$$[A-(2)I]v_2 = v_1$$

Let us check the three basis vectors.

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$$\mathcal{N}(A-(5)I) = span\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$$

Let's call this vector w_1.

$w_1$is the eigenvector that corresponds to $\boxed{\lambda = 5}$.

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$$[A-(2)I]\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \quad \#1$$

$$[A-(2)I]\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix} \quad \#2$$

$$[A-(2)I]\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}3\\0\\3\end{bmatrix} \quad \#3$$

Now the question is, which one of those satisfies the constraint [the relation b/w v_1 and v_2]

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$$[A-(2)I]\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \rightarrow v_2 \neq \begin{bmatrix}1\\0\\0\end{bmatrix}$$

Because v_1 is not linearly dependent on $\begin{bmatrix}0\\0\\0\end{bmatrix}$.

Let's check #2,

$$[A-(2)I]\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix} \rightarrow v_2\ can\ be\ \begin{bmatrix}0\\1\\0\end{bmatrix}.$$

Because v_1 is linearly dependent on, $\begin{bmatrix}1\\0\\0\end{bmatrix}$,
In fact,

$$v_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

Finally, [just for the sake of completeness]

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Let's check #3,

$$[A-(2)I]\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}3\\0\\3\end{bmatrix} \rightarrow v_2 \neq \begin{bmatrix}0\\0\\1\end{bmatrix}$$

Because v_1 is not linearly dependent on $\begin{bmatrix}3\\0\\3\end{bmatrix}$.

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Therefore, T [the transformation matrix], can be written as

$$T = [v_1\ v_2\ w_1] = \begin{bmatrix} \begin{bmatrix}1\\0\\0\end{bmatrix} & \begin{bmatrix}0\\1\\0\end{bmatrix} & \begin{bmatrix}1\\0\\1\end{bmatrix} \end{bmatrix} = \begin{bmatrix}1 & 0 & 1\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

And JCF is

$$J = T^{-1}AT$$

$$J = \begin{bmatrix}1 & 0 & 1\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}^{-1} \begin{bmatrix}2 & 1 & 3\\0 & 2 & 0\\0 & 0 & 5\end{bmatrix} \begin{bmatrix}1 & 0 & 1\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$$J = \begin{bmatrix}1 & 0 & -1\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} \begin{bmatrix}2 & 1 & 3\\0 & 2 & 0\\0 & 0 & 5\end{bmatrix} \begin{bmatrix}1 & 0 & 1\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$$J = \begin{bmatrix}1 & 0 & -1\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} \begin{bmatrix}2 & 1 & 3\\0 & 2 & 0\\0 & 0 & 5\end{bmatrix} \begin{bmatrix}1 & 0 & 1\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

Finally, JCF and the transformation matrix to obtain JCF are given as

$$J = \begin{bmatrix}2 & 1 & 0\\0 & 2 & 0\\0 & 0 & 5\end{bmatrix}, T = \begin{bmatrix}1 & 0 & 1\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

For this matrix, the characteristic polynomial is obtained as,

$$\chi_A(\lambda) = (\lambda - 2)^{\boxed{2}}(\lambda - 5)^{\boxed{1}}$$

For this matrix, the minimal polynomial is obtained as,

$$m_A(\lambda) = (\lambda - 2)^{\boxed{1}}(\lambda - 5)^{\boxed{1}}$$

[Algebraic multiplicity and geometric multiplicity]

For $\boxed{\lambda = 2}$, the algebraic multiplicity is $\boxed{2}$ and the geometric multiplicity is $\boxed{1}$.

For $\boxed{\lambda = 5}$, the algebraic multiplicity is $\boxed{1}$ and the geometric multiplicity is $\boxed{1}$.

B matrix Solution

$$B = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$det(B - \lambda I) = 0$$

$$det\left(\begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right) = 0$$

$$det\left(\begin{bmatrix} 2-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{bmatrix}\right) = 0$$

$$(\lambda - 2)^3 = 0 \rightarrow \lambda(A) = \{2,2,2\}$$

There is only one distinct eigenvalue.

If there were 3 distinct eigenvalues, we would know that "B can be diagonalized" but we cannot conclude that right away.

If we can compute 3 linearly independent eigenvectors that correspond to $\boxed{\lambda = 2}$, we can conclude that "B can be diagonalized".

In other words, if

$$\dim\{\mathcal{N}(A - (2)I)\} = 3 \rightarrow B \; can \; be \; diagonalized$$

If we see that $\boxed{\dim\{\mathcal{N}(A - (2)I)\} \neq 3}$ we will conclude that "B cannot be diagonalized only JCF can be computed"

Therefore, let's compute $\boxed{\dim\{\mathcal{N}(A - (2)I)\}}$,

$$\dim\{\mathcal{N}(A - (2)I)\} = 1$$

[For that we can simply use Matlab]

Ok, we can at this point conclude that B cannot be diagonalized and we can not only find its Jordan Canonical Form.

So, let's start computing the eigenvector that correspond to $\boxed{\lambda = 2}$,

$$\lambda = 2$$

$$\mathcal{N}(A-(2)I) = span\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}\right\}$$

And therefore,

$$Rank\{\mathcal{N}(A-(2)I)\} = 1$$

There is "1" Jordan block that corresponds to $\boxed{\lambda = 2}$.

So, the JCF can be written directly as

$$J = \begin{bmatrix}\boxed{\begin{matrix}2 & 1 & 0\\0 & 2 & 1\\0 & 0 & 2\end{matrix}}\end{bmatrix}$$

We need to find the generalized eigenvectors, $\boxed{v_1, v_2, v_3}$

The relation between those vectors can be written as

$$[A-(2)I]v_1 = 0$$
$$[A-(2)I]v_2 = v_1$$
$$[A-(2)I]v_3 = v_2$$

These constraints must be satisfied.

Starting from v_1, we write $\boxed{v_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}}$.

Then, we need to compute $\boxed{v_2, v_3}$ vectors.

Or we can start from v_3 and compute v_2 and then [by using v_2] compute v_1. [we have already found v_1]

First, we can see that,

$$Rank\{\mathcal{N}([A-(2)I]^1)\} = 1$$
$$Rank\{\mathcal{N}([A-(2)I]^2)\} = 2$$
$$Rank\{\mathcal{N}([A-(2)I]^3)\} = 3$$

v_3 can be one of the three standard basis vectors namely $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$.

Let's try three of them simultaneously.

$$[A-(2)I]v_3 = v_2$$

$$[A-(2)I]\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \rightarrow v_3 \neq \begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$[A-(2)I]\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix} \rightarrow v_3\ can\ be\ \begin{bmatrix}0\\1\\0\end{bmatrix}$$

$$[A-(2)I]\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}1\\2\\0\end{bmatrix} \rightarrow v_3\ can\ be\ \begin{bmatrix}0\\0\\1\end{bmatrix}$$

Thus, the pair (v_3, v_2), can be one of the pairs in the set $\left\{\left(\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}\mathbf{1}\\\mathbf{0}\\\mathbf{0}\end{bmatrix}\right), \left(\begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}\mathbf{1}\\\mathbf{2}\\\mathbf{0}\end{bmatrix}\right)\right\}$.

Then, by using v_2 [or candidate v_2 vectors] we will compute v_1.

$$[A-(2)I]v_2 = v_1$$

$$[A-(2)I]\begin{bmatrix}\mathbf{1}\\\mathbf{0}\\\mathbf{0}\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \rightarrow v_2 \neq \begin{bmatrix}\mathbf{1}\\\mathbf{0}\\\mathbf{0}\end{bmatrix}$$

$$[A-(2)I]\begin{bmatrix}\mathbf{1}\\\mathbf{2}\\\mathbf{0}\end{bmatrix} = \begin{bmatrix}2\\0\\0\end{bmatrix} \rightarrow v_2\ can\ be\ \begin{bmatrix}\mathbf{1}\\\mathbf{2}\\\mathbf{0}\end{bmatrix}$$

And we can conclude that, $\boxed{v_1, v_2, v_3}$ vectors can be given as $\left\{\begin{bmatrix}2\\0\\0\end{bmatrix}, \begin{bmatrix}\mathbf{1}\\\mathbf{2}\\\mathbf{0}\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$.

Therefore, the transformation matrix T can be obtained as

$$T = [v_1\ v_2\ v_3] = \begin{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{0} \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And JCF,

$$J = T^{-1}BT$$

$$J = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$J = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$J = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}, T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The characteristic polynomial is

$$\chi_A(\lambda) = (\lambda - 2)^{\boxed{3}}$$

And the minimal polynomial is

$$m_A(\lambda) = (\lambda - 2)^{\boxed{1}}$$

[Algebraic multiplicity and geometric multiplicity]

For $\boxed{\lambda = 2}$, the algebraic multiplicity is $\boxed{3}$ and the geometric multiplicity is $\boxed{1}$.

Problem 2

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

a. Find a set of basis vectors that span $\mathcal{R}(A)$.
b. Find a set of basis vectors that span $\mathcal{N}(A)$.

Part a

Let's do ref-procedure

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 5 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 0 & 5 \\ 0 & \boxed{1} & -2 \\ 0 & 0 & 0 \end{bmatrix} = ref(A)$$

It can be seen that the 1st and 2nd columns are pivot columns and therefore 1st and 2nd columns of the "original matrix" span $\mathcal{R}(A)$.

$$\mathcal{R}(A) = span\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}\right\}$$

Part b

$$\mathcal{N}(A) = \left\{x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 | Ax = 0\right\}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By using the ref expression,

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{matrix} x_1 + 5x_3 = 0 \\ x_2 - 2x_3 = 0 \end{matrix} \rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5x_3 \\ 2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -5 \\ 2 \\ 1 \end{bmatrix}$$

Therefore,

$$\mathcal{N}(A) = span\left\{\begin{bmatrix} -5 \\ 2 \\ 1 \end{bmatrix}\right\}$$

Problem 3

a. $\boxed{v_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}}$ is defined on the standard basis, i.e., $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$.

Compute the vector that corresponds to v_1 in $\left\{\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix}\right\}$ basis.

b. Defined on the basis $\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right\}$, the vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, is written on the basis $\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}\right\}$ as $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Therefore, Express the relation between $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

Part a

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 2 & 0 & 5 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 2 & 0 & 5 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \left(\frac{1}{17}\begin{bmatrix} 5 & -10 & 6 \\ 6 & 5 & -3 \\ -2 & 4 & 1 \end{bmatrix}\right)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \left(\frac{1}{17}\begin{bmatrix} 5 & -10 & 6 \\ 6 & 5 & -3 \\ -2 & 4 & 1 \end{bmatrix}\right)\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{1}{17}\begin{bmatrix} 1 \\ 25 \\ 3 \end{bmatrix}$$

Part b

$$\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right\}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}\right\}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 1 & 0 & 3 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \left(\frac{1}{2}\begin{bmatrix} 3 & 0 & -1 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3 & 5 & -3 \\ 2 & 1 & -3 \\ -1 & -1 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Problem 4

a. The basis $\left\{\begin{bmatrix} r \\ 2 \\ s \end{bmatrix}, \begin{bmatrix} r+1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ s \\ 1 \end{bmatrix}\right\}$ is known to span the vector space $(\mathbb{R}^3, \mathbb{R})$. What is the relation between $\boxed{r}$ and $\boxed{s}$?

b. The basis $\left\{\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} r \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}\right\}$ is known to not span the vector space $(\mathbb{R}^3, \mathbb{R})$. Find $\boxed{r}$.

Part a

$$span\left\{\begin{bmatrix} r \\ 2 \\ s \end{bmatrix}, \begin{bmatrix} r+1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ s \\ 1 \end{bmatrix}\right\} = \mathbb{R}^3 \to \det\left(\begin{bmatrix} r & r+1 & 3 \\ 2 & 2 & s \\ s & 1 & 1 \end{bmatrix}\right) \neq 0$$

$$\det\left(\begin{bmatrix} r & r+1 & 3 \\ 2 & 2 & s \\ s & 1 & 1 \end{bmatrix}\right) = r\begin{vmatrix} 2 & s \\ 1 & 1 \end{vmatrix} - (r+1)\begin{vmatrix} 2 & s \\ s & 1 \end{vmatrix} + 3\begin{vmatrix} 2 & 2 \\ s & 1 \end{vmatrix}$$

$$= r[2-s] - (r+1)[2-s^2] + 3[2-2s]$$

$$= [2r - sr] - [2(r+1) - s^2(r+1)] + [6-6s]$$

$$= [2r - sr] - [(2r+2) + (-rs^2 - s^2)] + [6-6s]$$

$$= [2r - sr] + [(-2r-2) + (\quad)] + [6-6s]$$

$$= [2r - sr - 2r - 2 + rs^2 + s^2 + 6 - 6s]$$

$$= [2r - sr - 2r + rs^2 + s^2 + 4 - 6s]$$

$$\det\left(\begin{bmatrix} r & r+1 & 3 \\ 2 & 2 & s \\ s & 1 & 1 \end{bmatrix}\right) = [-sr + rs^2 + s^2 + 4 - 6s]$$

Therefore the relation between $\boxed{r}$ and $\boxed{s}$ is given as

$$\boxed{-sr + rs^2 + s^2 + 4 - 6s \neq 0}$$

Part b

$$span\left\{\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} r \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}\right\} \subset \mathbb{R}^3 \to \det\left(\begin{bmatrix} 4 & r & 4 \\ 5 & 5 & 3 \\ 6 & 1 & 2 \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} 4 & r & 4 \\ 5 & 5 & 3 \\ 6 & 1 & 2 \end{bmatrix}\right) = 4\begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} - (r)\begin{vmatrix} 5 & 3 \\ 6 & 2 \end{vmatrix} + 4\begin{vmatrix} 5 & 5 \\ 6 & 1 \end{vmatrix}$$

$$= 4[7] - (r)[-8] + 4[-25]$$

$$= 28 + 8r - 100 \to 8r - 72 \to 8r = 72 \to \boxed{r = 9}$$

Problem 5

The set $\mathcal{U}$ is consist of the continuous real functions defined on $[0,1]$.

a. Show that $(\mathcal{U}, \mathbb{R})$ is a linear vector space. State the dimension of the $(\mathcal{U}, \mathbb{R})$.

b. $y = \mathcal{L}(x)$ is given where the operator is defined as $\boxed{\mathcal{L}: (\mathcal{U}, \mathbb{R}) \rightarrow (\mathcal{U}, \mathbb{R})}$. The operator is defined as $\boxed{y(t) = \int_0^1 g(t-\tau)x(\tau)d\tau}$ where $\boxed{g}$ is a continuous real function defined on $[-1,1]$. Show that the operator $\mathcal{L}(\cdot)$ is a linear operator.

PART A

$\mathbb{V}$ is a linear vector space defined on the field $\mathbb{F}$ (the field can be $\mathbb{R}$ or $\mathbb{C}$).

$\mathbb{V}$ must satisfy the constraints that are listed below,

Property 1	$\boldsymbol{u}, \boldsymbol{v} \in \mathbb{V} \rightarrow \boldsymbol{u} + \boldsymbol{v} \in \mathbb{V}$	Closed under vector addition
Property 2	$\boldsymbol{u} \in \mathbb{V}, c \in \mathbb{F} \rightarrow c \cdot \boldsymbol{u} \in \mathbb{V}$	Closed under scalar multiplication
Property 3	$\exists \boldsymbol{0} \in \mathbb{V}, \quad \boldsymbol{u} + \boldsymbol{0} = \boldsymbol{0} + \boldsymbol{u} = \boldsymbol{u}, \forall \boldsymbol{u} \in \mathbb{V}$	Identity element of vector addition
Property 4	$\boldsymbol{u}, \boldsymbol{v} \in \mathbb{V}, \boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u} \in \mathbb{V}$	Commutativity of vector addition
Property 5	$\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathbb{V}, (\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) \in \mathbb{V}$	Associativity of vector addition
Property 6	$\forall \boldsymbol{u} \in \mathbb{V}, \exists \boxed{-\boldsymbol{u}} \in \mathbb{V}, \quad \boldsymbol{u} + \boxed{-\boldsymbol{u}} = \boldsymbol{u} + \boxed{-\boldsymbol{u}} = \boldsymbol{0}$	Inverse elements of vector addition
Property 7	$\boldsymbol{u}, \boldsymbol{v} \in \mathbb{V}, c \in \mathbb{F} \rightarrow c(\boldsymbol{u} + \boldsymbol{v}) = c\boldsymbol{u} + c\boldsymbol{v}$	Distributivity of scalar multiplication with respect to vector addition
Property 8	$c, d \in \mathbb{F}, \boldsymbol{u} \in \mathbb{V} \rightarrow (c + d)\boldsymbol{u} = c\boldsymbol{u} + d\boldsymbol{u}$	Distributivity of scalar multiplication with respect to field addition
Property 9	$c, d \in \mathbb{F}, \boldsymbol{u} \in \mathbb{V} \rightarrow c(d\boldsymbol{u}) = (cd)\boldsymbol{u}$	Compatibility of scalar multiplication with field multiplication
Property 10	$\exists 1 \in \mathbb{F}\ s.t.\ 1\boldsymbol{u} = \boldsymbol{u}, \forall \boldsymbol{u} \in \mathbb{V}$	Identity element of scalar multiplication
Property 11	$\exists 0 \in \mathbb{F}\ s.t.\ 0\boldsymbol{u} = \boldsymbol{0}, \forall \boldsymbol{u} \in \mathbb{V}$	Identity element of field-addition

We need to check each one of them one by one,

Property 1	$\boldsymbol{u}, \boldsymbol{v} \in \mathbb{V} \rightarrow \boldsymbol{u} + \boldsymbol{v} \in \mathbb{V}$
	The sum of the given two continuous real functions that are defined on [0,1] are again a continuous real function that are defined on the same range.
Property 2	$\boldsymbol{u} \in \mathbb{V}, c \in \mathbb{F} \rightarrow c \cdot \boldsymbol{u} \in \mathbb{V}$
	The multiplication of a real scalar and a continuous real function defined on [0,1] is again a continuous real function defined on the same range.
Property 3	$\exists \boldsymbol{0} \in \mathbb{V}, \quad \boldsymbol{u} + \boldsymbol{0} = \boldsymbol{0} + \boldsymbol{u} = \boldsymbol{u}, \forall \boldsymbol{u} \in \mathbb{V}$
	The existence of an additive-identity [aka zero vector]. And additive-identity is a function that identically zero on [0,1]
Property 4	$\boldsymbol{u}, \boldsymbol{v} \in \mathbb{V}, \boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u} \in \mathbb{V}$
	The addition operation is commutative. $f(t) + g(t) = g(t) + f(t), \forall t \in [0,1]$
Property 5	$\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathbb{V}, (\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) \in \mathbb{V}$
	The associativity of the addition operation. $\big(f(t) + g(t)\big) + h(t) = f(t) + \big(g(t) + h(t)\big), \forall t \in [0,1]$
Property 6	$\forall \boldsymbol{u} \in \mathbb{V}, \exists \boxed{-\boldsymbol{u}} \in \mathbb{V}, \quad \boldsymbol{u} + \boxed{-\boldsymbol{u}} = \boldsymbol{u} + \boxed{-\boldsymbol{u}} = \boldsymbol{0}$

	The existence of an inverse for each element in the set. $$f(t)+\boxed{-f(t)}=0,\forall t\in[0,1]$$
Property 7	$\boldsymbol{u},\boldsymbol{v}\in\mathbb{V},c\in\mathbb{F}\rightarrow c(\boldsymbol{u}+\boldsymbol{v})=c\boldsymbol{u}+c\boldsymbol{v}$
	The Distributivity property. $$\alpha\big(f(t)+g(t)\big)=\alpha f(t)+\alpha g(t),\forall t\in[0,1]$$
Property 8	$c,d\in\mathbb{F},\boldsymbol{u}\in\mathbb{V}\rightarrow(c+d)\boldsymbol{u}=c\boldsymbol{u}+d\boldsymbol{u}$
	It is evident that holds, by using the equality $$(\alpha+\beta)f(t)=\alpha f(t)+\beta f(t),\forall t\in[0,1]$$
Property 9	$c,d\in\mathbb{F},\boldsymbol{u}\in\mathbb{V}\rightarrow c(d\boldsymbol{u})=(cd)\boldsymbol{u}$
	It is evident that holds, by using the equality $$\alpha\big(\beta f(t)\big)=(\alpha\beta)f(t),\forall t\in[0,1]$$
Property 10	$\exists 1\in\mathbb{F}\ s.t.\ 1\boldsymbol{u}=\boldsymbol{u},\forall\boldsymbol{u}\in\mathbb{V}$
	It is evident that holds, by using the equality $$1f(t)=f(t),\forall t\in[0,1]$$
Property 11	$\exists 0\in\mathbb{F}\ s.t.\ 0\boldsymbol{u}=\boldsymbol{0},\forall\boldsymbol{u}\in\mathbb{V}$
	It is evident that holds, by using the equality $$0f(t)=\boldsymbol{0}\equiv 0,\forall t\in[0,1]$$

The dimension of this space is infinite, the space can be spanned by the set of vectors given as,

$$\{1,x,x^2,x^3,\dots,x^\infty\},\forall x\in\mathbb{R}[0,1]$$

Problem 5

The set $\mathcal{U}$ is consist of the continuous real functions defined on $[0,1]$.

a. Show that $(\mathcal{U}, \mathbb{R})$ is a linear vector space. State the dimension of the $(\mathcal{U}, \mathbb{R})$.

b. $y = \mathcal{L}(x)$ is given where the operator is defined as $\boxed{\mathcal{L}: (\mathcal{U}, \mathbb{R}) \rightarrow (\mathcal{U}, \mathbb{R})}$. The operator is defined as $\boxed{y(t) = \int_0^1 g(t-\tau)x(\tau)d\tau}$ where $\boxed{g}$ is a continuous real function defined on $[-1,1]$. Show that the operator $\mathcal{L}(\cdot)$ is a linear operator.

PART B

To show that the operator $\mathcal{L}(\cdot)$ is a linear one, superposition property must be checked.

$$y_1 = \mathcal{L}(x_1) \,\&\, y_2 = \mathcal{L}(x_2) \rightarrow \alpha y_1 + \beta y_2 = \mathcal{L}(\alpha x_1 + \beta x_2)$$

If that is satisfied by the operator then it can be concluded that the operator is linear on the given vector space,

$$y_1(t) = \int_0^1 g(t-\tau)x_1(\tau)d\tau$$

$$y_2(t) = \int_0^1 g(t-\tau)x_2(\tau)d\tau$$

$$\alpha y_1(t) + \beta y_2(t) \overset{?}{=} \int_0^1 g(t-\tau)[\alpha x_1(\tau) + \beta x_2(\tau)]d\tau$$

$$= \int_0^1 g(t-\tau)[\alpha x_1(\tau)]d\tau + \int_0^1 g(t-\tau)[\beta x_2(\tau)]d\tau$$

$$= \alpha \int_0^1 g(t-\tau)[x_1(\tau)]d\tau + \beta \int_0^1 g(t-\tau)[x_2(\tau)]d\tau$$

$$= \alpha \underbrace{\boxed{\int_0^1 g(t-\tau)[x_1(\tau)]d\tau}}_{y_1(t)} + \beta \underbrace{\boxed{\int_0^1 g(t-\tau)[x_2(\tau)]d\tau}}_{y_2(t)}$$

It can be seen that the operator satisfies the superposition principle therefore, the operator is linear on the given vector space.

Problem 6

$T: \mathbb{R}^2 \to \mathbb{R}^2$ & $S: \mathbb{R}^2 \to \mathbb{R}^2$ are two functions, given as

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x + y \\ 0 \end{bmatrix}, S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + y \\ xy \end{bmatrix}$$

a. Determine if the function T is linear? If it is linear, explicitly obtain the matrix that corresponds to the given function.
b. Determine if the function T is linear? If it is linear, explicitly obtain the matrix that corresponds to the given function.
c. Determine if the function $S\left(T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)\right)$ is linear? If it is linear, explicitly obtain the matrix that corresponds to the given function.

Part a

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x + y \\ 0 \end{bmatrix} \to T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \to T \text{ is a linear operator on } \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$$

Part b

$$S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + y \\ xy \end{bmatrix} \to S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) \neq M\begin{bmatrix} x \\ y \end{bmatrix} \to S \text{ is NOT a linear operator on } \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$$

Part c

$$S\left(T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)\right) = S\left(\begin{bmatrix} 2x + y \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2x + y \\ 0 \end{bmatrix} \to S\left(T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)\right) = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$
$$\to S \circ T \text{ is a linear operator on } \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$$

Problem 7

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}, B = \begin{bmatrix} 5 & 1 & -1 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

a. Compute e^{At}.
b. Compute e^{Bt}.

A matrix solution

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

$$det(A - \lambda I) = 0$$

$$det\left(\begin{bmatrix} 3 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right) = 0$$

$$det\left(\begin{bmatrix} 3-\lambda & 1 & 1 \\ 0 & 3-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{bmatrix}\right) = 0$$

$$(\lambda - 3)^3 = 0 \rightarrow \lambda(A) = \{3,3,3\}$$

If there can be found 3 linearly independent eigenvectors that correspond to $\boxed{\lambda = 3}$ it can be concluded that "A can be diagonalized"

If,

$$\dim\{\mathcal{N}(A - (3)I)\} = 3$$

If that is the case, "A can be diagonalized", if not, only JCF can be found.

$$\dim\{\mathcal{N}(A - (3)I)\} = 1$$

Therefore, JCF of A must be computed.

$$\lambda = 3$$

$$\mathcal{N}(A-(3)I) = span\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}\right\}$$

And,

$$Rank\{\mathcal{N}(A-(3)I)\} = \boxed{1}$$

Which signifies that "there is $\boxed{1}$ Jordan block".

And JCF can be written directly as

$$J = \begin{bmatrix}\boxed{\begin{matrix}3 & 1 & 0\\0 & 3 & 1\\0 & 0 & 3\end{matrix}}\end{bmatrix}$$

But we need to determine the transformation matrix. We need 3 generalized eigenvectors that correspond to the one eigenvalue.

Let's call them $\boxed{v_1, v_2, v_3}$,

These vectors must satisfy the equalities,

$$[A-(3)I]v_1 = 0$$
$$[A-(3)I]v_2 = v_1$$
$$[A-(3)I]v_3 = v_2$$

Let us check,

$$Rank\{\mathcal{N}([A-(3)I]^1)\} = 1$$
$$Rank\{\mathcal{N}([A-(3)I]^2)\} = 2$$
$$Rank\{\mathcal{N}([A-(3)I]^3)\} = 3$$

We will start with v_3 and then we will find v_2 [by using v_3]. Then we will find v_1 [by using v_2].

First,

v_3 vector can be picked from the standard basis vectors, i.e., $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$.

But it is not clear at this point which one satisfies the constraints given above.

So, we will test all three of the basis vectors simultaneously.

The first constraint is $\boxed{[A-(3)I]v_3 = v_2}$, let's check that

$$[A-(3)I]\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}\mathbf{0}\\\mathbf{0}\\\mathbf{0}\end{bmatrix} \rightarrow \boldsymbol{it\ cannot\ be} \rightarrow v_3 \neq \begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$[A-(3)I]\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}\mathbf{1}\\\mathbf{0}\\\mathbf{0}\end{bmatrix} \rightarrow \boldsymbol{it\ can\ be}$$

$$[A-(3)I]\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}\mathbf{1}\\\mathbf{1}\\\mathbf{0}\end{bmatrix} \rightarrow \boldsymbol{it\ can\ be}$$

From this, (v_3, v_2) pair can be one of the given pairs that are $\left\{\left(\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}\mathbf{1}\\\mathbf{0}\\\mathbf{0}\end{bmatrix}\right), \left(\begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}\mathbf{1}\\\mathbf{1}\\\mathbf{0}\end{bmatrix}\right)\right\}$.

Now, we need to test the relation between v_2 and v_1. Recall that the relation is $\boxed{[A-(3)I]v_2 = v_1}$.

$$[A-(2)I]\begin{bmatrix}\mathbf{1}\\\mathbf{0}\\\mathbf{0}\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \rightarrow it\ cannot\ be \rightarrow (v_3, v_2, v_1) \neq \left\{\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}\mathbf{1}\\\mathbf{0}\\\mathbf{0}\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$$

Therefore, the transformation matrix can be obtained as,

$$T = [v_1\ v_2\ v_3] = \begin{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{0} \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And the JCF,

$$J = T^{-1}AT$$

$$J = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$J = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$J = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}, T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Characteristic polynomial of matrix A,

$$\chi_A(\lambda) = (\lambda - 3)^{\boxed{3}}$$

Minimal polynomial of matrix A,

$$m_A(\lambda) = (\lambda - 3)^{\boxed{1}}$$

To compute the term e^{At}, JCF of A is used as

$$A = TJT^{-1}$$

$$At = T(Jt)T^{-1}$$

$$\begin{bmatrix} 3 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} t = T\left(\begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} t \right) T^{-1}$$

$$\begin{bmatrix} 3t & t & t \\ 0 & 3t & t \\ 0 & 0 & 3t \end{bmatrix} = T\left(\begin{bmatrix} 3t & t & 0 \\ 0 & 3t & t \\ 0 & 0 & 3t \end{bmatrix} \right) T^{-1}$$

$$e^{At} = T(e^{Jt})T^{-1}$$

$$e^{\begin{bmatrix} 3t & t & t \\ 0 & 3t & t \\ 0 & 0 & 3t \end{bmatrix}} = T\left(e^{\begin{bmatrix} 3t & t & 0 \\ 0 & 3t & t \\ 0 & 0 & 3t \end{bmatrix}} \right) T^{-1}$$

$$e^{\begin{bmatrix} 3t & t & t \\ 0 & 3t & t \\ 0 & 0 & 3t \end{bmatrix}} = T\left(\begin{bmatrix} e^{(3t)} & te^{(3t)} & \frac{t^2}{2!}e^{(3t)} \\ 0 & e^{(3t)} & te^{(3t)} \\ 0 & 0 & e^{(3t)} \end{bmatrix} \right) T^{-1}$$

Recall that,

$$e^{\begin{bmatrix}\lambda & 1 & 0\\ 0 & \lambda & 1\\ 0 & 0 & \lambda\end{bmatrix}t} = e^{\begin{bmatrix}\lambda t & t & 0\\ 0 & \lambda t & t\\ 0 & 0 & \lambda t\end{bmatrix}} = \begin{bmatrix} e^{\lambda t} & \frac{t^1}{1!}e^{\lambda t} & \frac{t^2}{2!}e^{\lambda t} \\ 0 & e^{\lambda t} & \frac{t^1}{1!}e^{\lambda t} \\ 0 & 0 & e^{\lambda t}\end{bmatrix}$$

Which comes from

$$f(J_k(\lambda t)) = \begin{bmatrix} f(\lambda t) & tf'(\lambda t) & \frac{t^2}{2!}f''(\lambda t) & \cdots & \frac{t^{k-1}}{(k-1)!}f^{(k-1)}(\lambda t) \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & \frac{t^2}{2!}f''(\lambda t) \\ & & & \ddots & tf'(\lambda t) \\ 0 & & & & f(\lambda t) \end{bmatrix}$$

Continuing, we obtain

$$e^{\begin{bmatrix}3t & t & t\\ 0 & 3t & t\\ 0 & 0 & 3t\end{bmatrix}} = T\left(\begin{bmatrix} e^{(3t)} & te^{(3t)} & \frac{t^2}{2!}e^{(3t)} \\ 0 & e^{(3t)} & te^{(3t)} \\ 0 & 0 & e^{(3t)}\end{bmatrix}\right)T^{-1}$$

$$e^{At} = \begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\left(\begin{bmatrix} e^{(3t)} & te^{(3t)} & \frac{t^2}{2!}e^{(3t)} \\ 0 & e^{(3t)} & te^{(3t)} \\ 0 & 0 & e^{(3t)}\end{bmatrix}\right)\begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}^{-1}$$

$$e^{At} = \begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix} e^{(3t)} & te^{(3t)} & \frac{t^2}{2!}e^{(3t)} \\ 0 & e^{(3t)} & te^{(3t)} \\ 0 & 0 & e^{(3t)}\end{bmatrix}\begin{bmatrix}1 & -1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$e^{At} = \begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix} e^{(3t)} & te^{(3t)} - e^{(3t)} & \frac{t^2}{2!}e^{(3t)} \\ 0 & e^{(3t)} & te^{(3t)} \\ 0 & 0 & e^{(3t)}\end{bmatrix}$$

$$e^{At} = \begin{bmatrix} e^{(3t)} & te^{(3t)} - e^{(3t)} + e^{(3t)} & \frac{t^2}{2!}e^{(3t)} + te^{(3t)} \\ 0 & e^{(3t)} & te^{(3t)} \\ 0 & 0 & e^{(3t)}\end{bmatrix}$$

$$e^{At} = \begin{bmatrix} e^{(3t)} & te^{(3t)} & \frac{t^2}{2!}e^{(3t)} + te^{(3t)} \\ 0 & e^{(3t)} & te^{(3t)} \\ 0 & 0 & e^{(3t)}\end{bmatrix}$$

B matrix solution

$$B = \begin{bmatrix} 5 & 1 & -1 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$det(B - \lambda I) = 0$$

$$det\left(\begin{bmatrix} 5 & 1 & -1 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right) = 0$$

$$det\left(\begin{bmatrix} 5-\lambda & 1 & -1 \\ 0 & 5-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{bmatrix}\right) = 0$$

$$(\lambda - 5)^3 = 0 \rightarrow \lambda(B) = \{5,5,5\}$$

There is 1 distinct eigenvalue. If there were 3 distinct eigenvalues, we would certainly know that B matrix can be diagonalizable. In this case, for $\lambda = 5$, there must be 3 linearly independent eigenvector must be found for B to be diagonalizable.

If,

$$\dim\{\mathcal{N}(B - (5)I)\} = 3$$

It can be determined that B can be diagonalizable. If not, it cannot be diagonalizable. But, since,

$$\dim\{\mathcal{N}(B - (5)I)\} = 2$$

It can be determined that B cannot be diagonalizable. We can only transform it to a Jordan Normal Form.

$\lambda = 5$

$$\mathcal{N}(B-(5)I) = span\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$$

And therefore,

$$Rank\{\mathcal{N}(B-(5)I)\} = \mathbf{2}$$

That means, there are **2** Jordan block corresponds to $\lambda = 5$, i.e.

$$J = \begin{bmatrix}\boxed{\begin{matrix}5 & 1\\0 & 5\end{matrix}} & \begin{matrix}0\\0\end{matrix}\\ \begin{matrix}0 & 0\end{matrix} & \boxed{5}\end{bmatrix}$$

That means, we need to find v_1, v_2, w_1 vectors, such that,

$$[B-(5)I]v_1 = 0$$
$$[B-(5)I]w_1 = 0$$

$$[B-(5)I]v_2 = v_1$$

Such that,

$$\{v_1, v_2, w_1\}\ is\ linearly\ independent$$

At this stage, we do not know which one of these vectors $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$ must be chosen as v_1 and w_1.

Since we can determine,

$$Rank\{\mathcal{N}([B-(5)I]^1)\} = 2$$
$$Rank\{\mathcal{N}([B-(5)I]^2)\} = 3$$

We can start selecting v_2 among the standard basis vectors, i.e., $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$.

$$[B-(5)I]\begin{bmatrix}\mathbf{1}\\\mathbf{0}\\\mathbf{0}\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$[B-(5)I]\begin{bmatrix}\mathbf{0}\\\mathbf{1}\\\mathbf{0}\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$[B-(5)I]\begin{bmatrix}\mathbf{0}\\\mathbf{0}\\\mathbf{1}\end{bmatrix} = \begin{bmatrix}-1\\0\\0\end{bmatrix}$$

So, $(v_1, \boldsymbol{v_2})$ can be either, $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}\mathbf{0}\\\mathbf{1}\\\mathbf{0}\end{bmatrix}\right\}$ or $\left\{\begin{bmatrix}-1\\0\\0\end{bmatrix}, \begin{bmatrix}\mathbf{0}\\\mathbf{0}\\\mathbf{1}\end{bmatrix}\right\}$

And we can select w_1 as $\begin{bmatrix}0\\1\\1\end{bmatrix}$.

Therefore, we can select $\{v_1, v_2, w_1\}$ as, either,

$$\{v_1, v_2, w_1\} = \left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}\mathbf{0}\\\mathbf{1}\\\mathbf{0}\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$$

or

$$\{v_1, v_2, w_1\} = \left\{\begin{bmatrix}-1\\0\\0\end{bmatrix}, \begin{bmatrix}\mathbf{0}\\\mathbf{0}\\\mathbf{1}\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$$

Let use choose, $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}\mathbf{0}\\\mathbf{1}\\\mathbf{0}\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$.

So, we can construct T, transformation matrix, as,

$$T = [v_1\ v_2\ w_1] = \begin{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix} & \begin{bmatrix}\mathbf{0}\\\mathbf{1}\\\mathbf{0}\end{bmatrix} & \begin{bmatrix}0\\1\\1\end{bmatrix}\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix}$$

To compute a Jordan Normal Form of B,

$$J = T^{-1}BT$$

$$J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 5 & 1 & -1 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 1 & -1 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$J = \begin{bmatrix} \boxed{\begin{matrix} 5 & 1 \\ 0 & 5 \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \boxed{5} \end{bmatrix}, T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

The characteristic polynomial of B is given as,

$$\chi_A(\lambda) = (\lambda - 5)^{\boxed{3}}$$

Because B has $\lambda = 5$ with **algebraic multiplicity** $\boxed{3}$.

The minimal polynomial of B is given as,

$$m_A(\lambda) = (\lambda - 5)^{\boxed{2}}$$

Because B has $\lambda = 5$ with **geometric multiplicity** $\boxed{2}$.

To determine e^{Bt}, we need to use Jordan Normal Form of B matrix.

$$B = TJT^{-1}$$

$$Bt = T(Jt)T^{-1}$$

$$\begin{bmatrix} 5 & 1 & -1 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} t = T\left(\begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} t\right)T^{-1}$$

$$\begin{bmatrix} 5t & t & -t \\ 0 & 5t & 0 \\ 0 & 0 & 5t \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 5t & t & 0 \\ 0 & 5t & 0 \\ 0 & 0 & 5t \end{bmatrix}\right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

$$e^{Bt} = T(e^{Jt})T^{-1}$$

$$e^{\begin{bmatrix} 5t & t & -t \\ 0 & 5t & 0 \\ 0 & 0 & 5t \end{bmatrix}} = T\left(e^{\begin{bmatrix} 5t & t & 0 \\ 0 & 5t & 0 \\ 0 & 0 & 5t \end{bmatrix}}\right)T^{-1}$$

Let us first focus on the term, $e^{\begin{bmatrix} 5t & t & 0 \\ 0 & 5t & 0 \\ 0 & 0 & 5t \end{bmatrix}}$,

$$e^{\begin{bmatrix} 5t & t & 0 \\ 0 & 5t & 0 \\ 0 & 0 & 5t \end{bmatrix}} = e^{\begin{bmatrix} \boxed{\begin{matrix} 5t & t \\ 0 & 5t \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \boxed{5t} \end{bmatrix}} = \begin{bmatrix} e^{\boxed{\begin{matrix} 5t & t \\ 0 & 5t \end{matrix}}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & e^{\boxed{5t}} \end{bmatrix}$$

And,

$$e^{\boxed{\begin{matrix} 5t & t \\ 0 & 5t \end{matrix}}} = \begin{bmatrix} e^{5t} & te^{5t} \\ 0 & e^{5t} \end{bmatrix}$$

Therefore,

$$e^{\begin{bmatrix} 5t & t & 0 \\ 0 & 5t & 0 \\ 0 & 0 & 5t \end{bmatrix}} = \begin{bmatrix} e^{\boxed{\begin{matrix} 5t & t \\ 0 & 5t \end{matrix}}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & e^{\boxed{5t}} \end{bmatrix} = \begin{bmatrix} \boxed{\begin{matrix} e^{5t} & te^{5t} \\ 0 & e^{5t} \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \boxed{e^{5t}} \end{bmatrix}$$

As a result of this,

$$e^{\begin{bmatrix} 5t & t & -t \\ 0 & 5t & 0 \\ 0 & 0 & 5t \end{bmatrix}} = T\left(\begin{bmatrix} e^{(5t)} & te^{(5t)} & 0 \\ 0 & e^{(5t)} & 0 \\ 0 & 0 & e^{(5t)} \end{bmatrix}\right)T^{-1}$$

$$e^{Bt} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\left(\begin{bmatrix} e^{(5t)} & te^{(5t)} & 0 \\ 0 & e^{(5t)} & 0 \\ 0 & 0 & e^{(5t)} \end{bmatrix}\right)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

$$e^{Bt} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} e^{(5t)} & te^{(5t)} & 0 \\ 0 & e^{(5t)} & 0 \\ 0 & 0 & e^{(5t)} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$e^{Bt} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} e^{(5t)} & te^{(5t)} & -te^{(5t)} \\ 0 & e^{(5t)} & -e^{(5t)} \\ 0 & 0 & e^{(5t)} \end{bmatrix}$$

$$e^{Bt} = \begin{bmatrix} e^{(5t)} & te^{(5t)} & -te^{(5t)} \\ 0 & e^{(5t)} & e^{(5t)} - e^{(5t)} \\ 0 & 0 & e^{(5t)} \end{bmatrix}$$

$$e^{Bt} = \begin{bmatrix} e^{(5t)} & te^{(5t)} & -te^{(5t)} \\ 0 & e^{(5t)} & 0 \\ 0 & 0 & e^{(5t)} \end{bmatrix}$$