Problem

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{1}{m}u$$

$$y = x$$

Let us put that in a standard form

$$x_1 \coloneqq x$$

$$x_2 \coloneqq \frac{d}{dt}x_1 = \frac{d}{dt}x = \dot{x}$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \ddot{x}_1 = \ddot{x}$$

Using those new terms,

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 + \frac{c}{m}x_2 + \frac{k}{m}x_1 = \frac{1}{m}u$$

$$y = x_1$$

Arranging them,

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$[y] = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

There is a parametric uncertainty in the given dynamical system,

$$m = m_0(1 + \eta_m \delta_m)$$

$$c = c_0(1 + \eta_c \delta_c)$$

$$k = k_0(1 + \eta_k \delta_k)$$

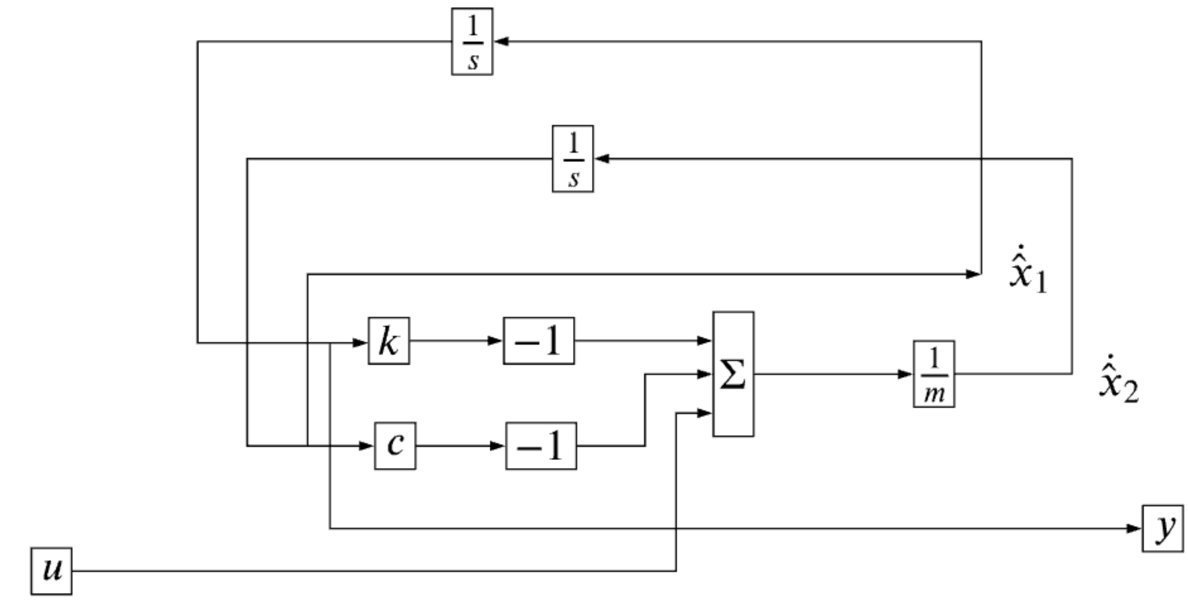
η_m term signifies the percentage, let us say it is 0.1, then it means we expect that there is 10% parametric uncertainty in that parameter.

This problem is a typical robust controller design problem. The first step in solving this problem is to formulate it systematically, otherwise there is a strong chance of making a mistake somewhere. In this tutorial, this process of formulating mu-synthesis problem [by taking advantage of matlab] is discussed in detail.

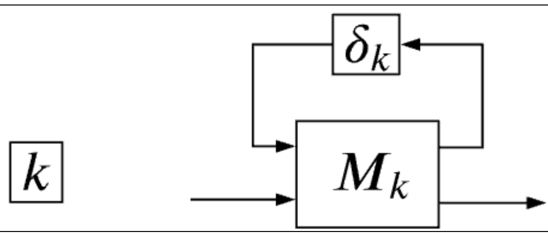
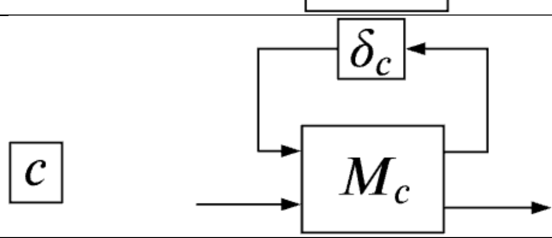
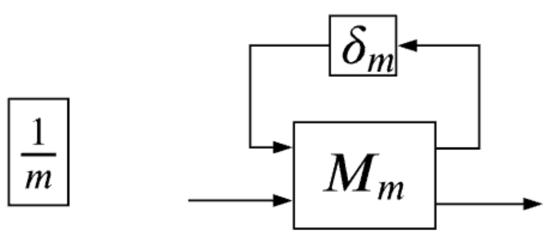
Step 1: by using the given state-space construct the block diagram given below.

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

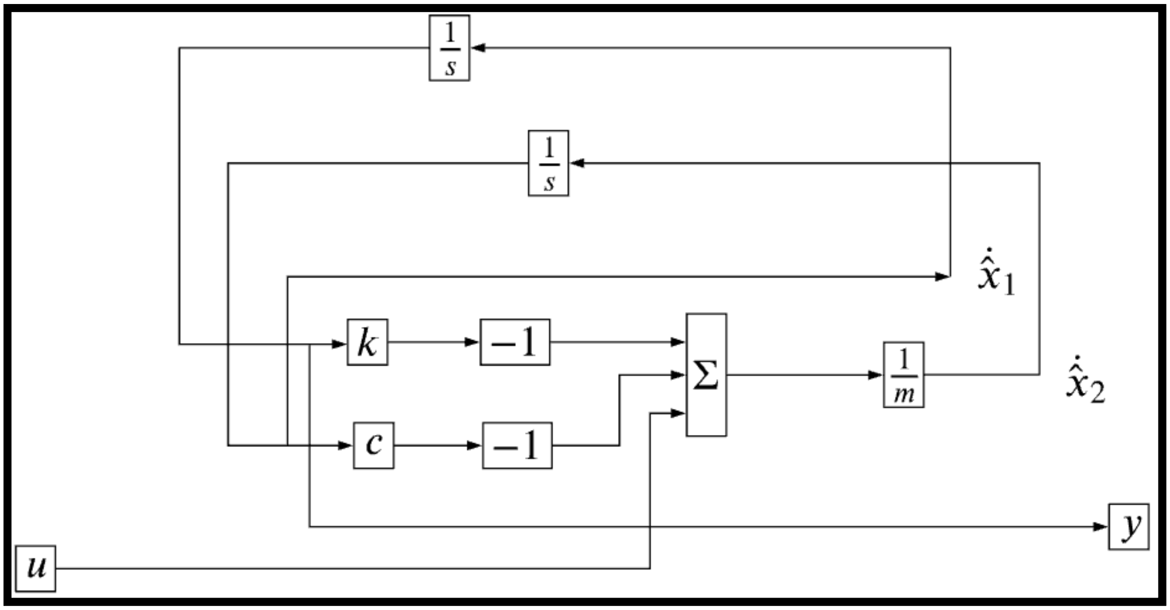
$$[y] = [1 \quad 0]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



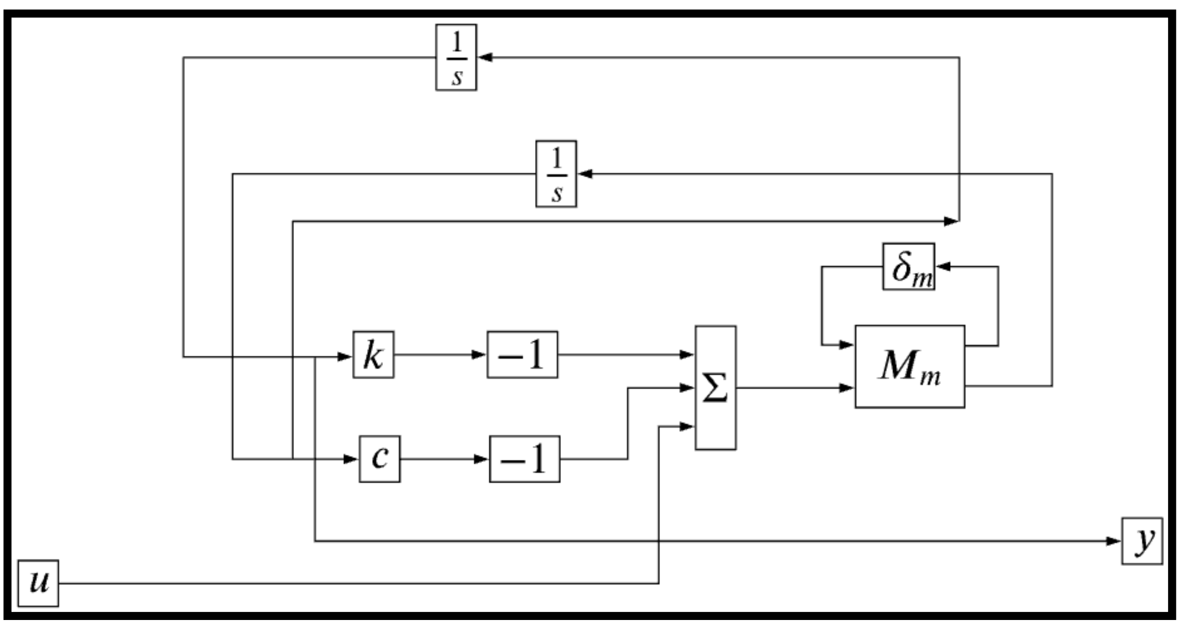
Step 2: Define the uncertainty-subblocks for each uncertain parameter

	$k = k_0(1+\eta_k\delta_k)$ $k_0 = nominal\ value$ $\eta_k = unc - percentage$ $\lvert\delta_k\rvert \le 1$ $M_k = \begin{bmatrix} 0 & k_0 \\ \eta_k & k_0 \end{bmatrix}$
	$c = c_0(1+\eta_c\delta_c)$ $c_0 = nominal\ value$ $\eta_c = unc - percentage$ $\lvert\delta_c\rvert \le 1$ $M_c = \begin{bmatrix} 0 & c_0 \\ \eta_c & c_0 \end{bmatrix}$
	$m = m_0(1+\eta_m\delta_m)$ $m_0 = nominal\ value$ $\eta_m = unc - percentage$ $\lvert\delta_m\rvert \le 1$ $M_m = \begin{bmatrix} -\eta_m & \frac{1}{m_0} \\ -\eta_m & \frac{1}{m_0} \end{bmatrix}$

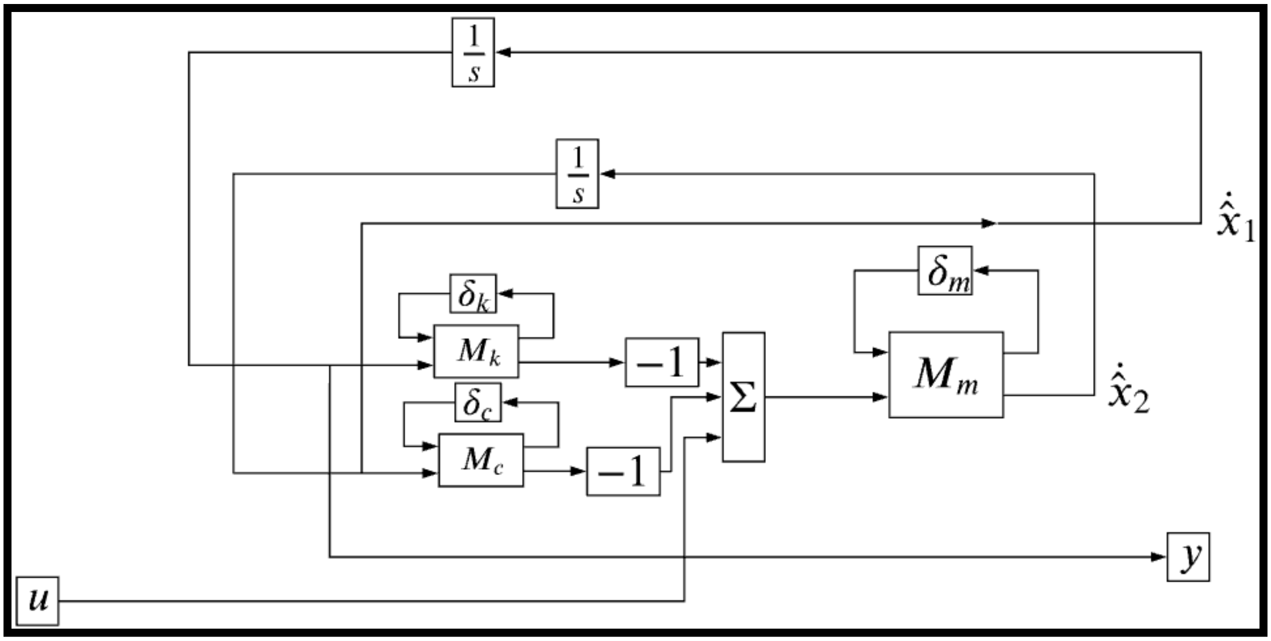
Step 3: substitute uncertainty-subblocks for parameters in the block diagram



First substitute the uncertainty-subblocks for m-parameter

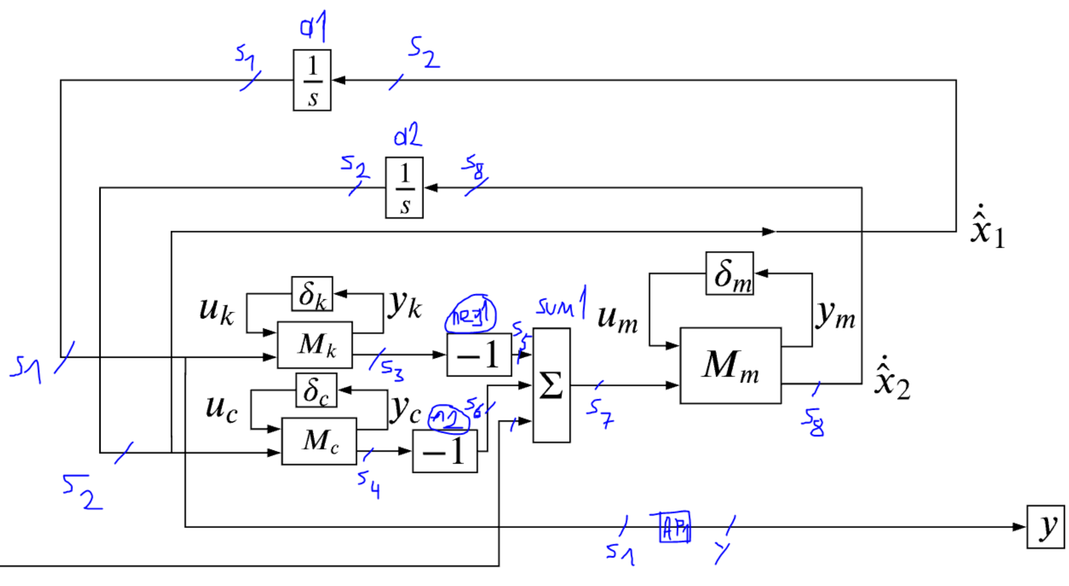


Then substitute the uncertainty-subblocks for parameter-k and parameter-c

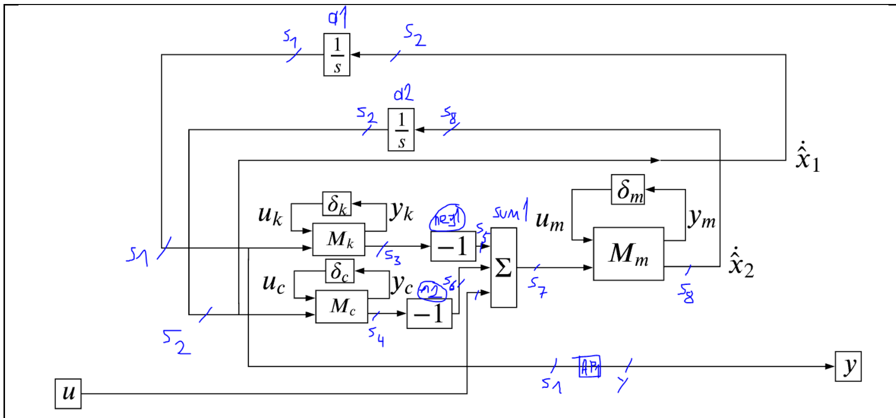


Step 4: name each signal that enters a block in the given block-diagram

Since there will be usually many signals, do not use alphabetical naming convention, use number naming convention such as s1,s2,s3, etc.



Step 5: define each block in the block-diagram



Systems:

aa_1
aa_2
$Mk-sys$
$Mc-sys$
$Mm-sys$
$neg-1$
$neg-2$
$AP-1$
$summer-1$

block	i/p	o/p	block	i/p	o/p
aa_1	s_2	s_1	$neg-1$	s_3	s_5
aa_2	s_8	s_2	$neg-2$	s_4	s_6
$Mk-sys$	$\begin{bmatrix}u_k\\s_1\end{bmatrix} = \{'uk','s1'\}$	$\begin{bmatrix}y_k\\s_3\end{bmatrix} = \{'yk','s3'\}$	$AP-1$	s_1	y
$Mc-sys$	$\begin{bmatrix}u_c\\s_2\end{bmatrix} = \{'uc','s2'\}$	$\begin{bmatrix}y_c\\s_4\end{bmatrix} = \{'yc','s4'\}$	$summer-1$	The relation $s_7 = s_5 + s_6 + u$	
$Mm-sys$	$\begin{bmatrix}u_m\\s_7\end{bmatrix} = \{'um','s7'\}$	$\begin{bmatrix}y_m\\s_8\end{bmatrix} = \{'ym','s8'\}$			

Step 6: Code the problem and use connect-function of matlab

1

```
clear all;close all;clc;
%%
m0=1
eta_m=0.1;
k0=1
eta_k=0.1;
c0=1
eta_c=0.1;
%%
aa1=tf([1],[1 0])
aa1.u='s2'
aa1.y='s1'
%
aa2=tf([1],[1 0])
aa2.u='s8'
aa2.y='s2'
```

2

```
Mm=[-eta_m,1/m0;-eta_m,1/m0];
Mm_sys=ss(Mm)
Mm_sys.u={'um','s7'};
Mm_sys.y={'ym','s8'};
Mk=[0,k0;eta_k,k0];
Mk_sys=ss(Mk)
Mk_sys.u={'uk','s1'};
Mk_sys.y={'yk','s3'};
Mc=[0,c0;eta_c,c0];
Mc_sys=ss(Mc)
Mc_sys.u={'uc','s2'};
Mc_sys.y={'yc','s4'};
```

3

```
neg1=ss(-1);
neg1.u='s3';
neg1.y='s5';
neg2=ss(-1);
neg2.u='s4';
neg2.y='s6';
summer1=sumblk('s7=s5+s6+u');
AP1=ss(1);
AP1.u='s1';
AP1.y='y';
```

The final line

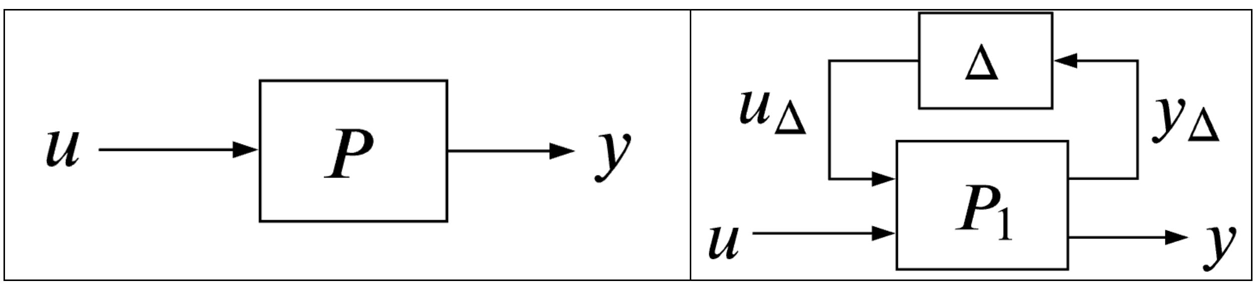
```
P1=connect(aa1,aa2,Mm_sys,Mk_sys,Mc_sys,neg1,neg2,summer1,AP1,…
         {'um','uk','uc','u'},…
         {'ym','yk','yc','y'})
```

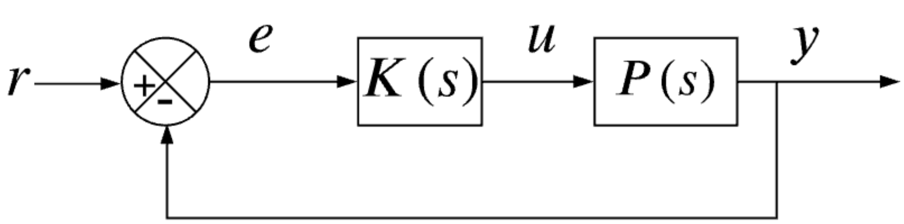
Code output

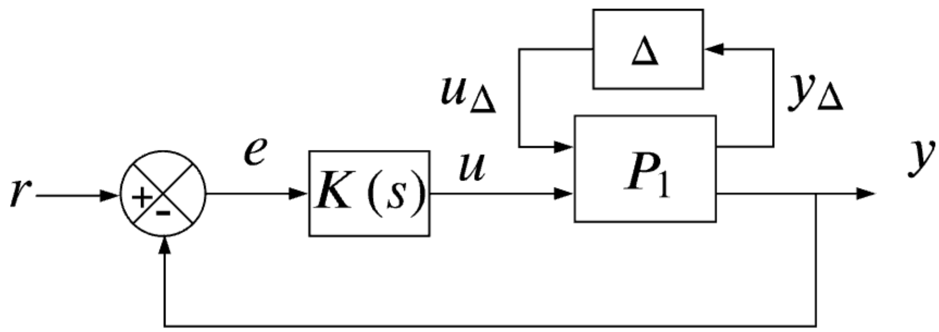
```
P1 =

  A =
        x1   x2
   x1    0    1
   x2   -1   -1

  B =
          um     uk     uc      u
   x1      0      0      0      0
   x2   -0.1   -0.1   -0.1      1
```

```
  C =
       x1   x2
   ym  -1   -1
   yk   1    0
   yc   0    1
   y    1    0

  D =
         um     uk     uc      u
   ym  -0.1   -0.1   -0.1      1
   yk     0      0      0      0
   yc     0      0      0      0
   y      0      0      0      0

Continuous-time state-space model.
```

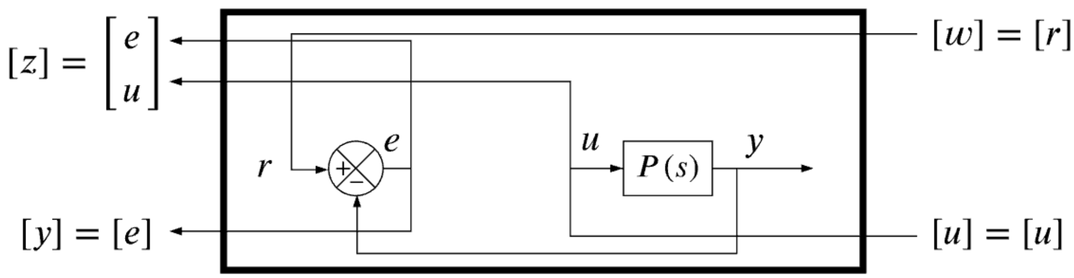
Step 7: to obtain the 9-block rep. do the following



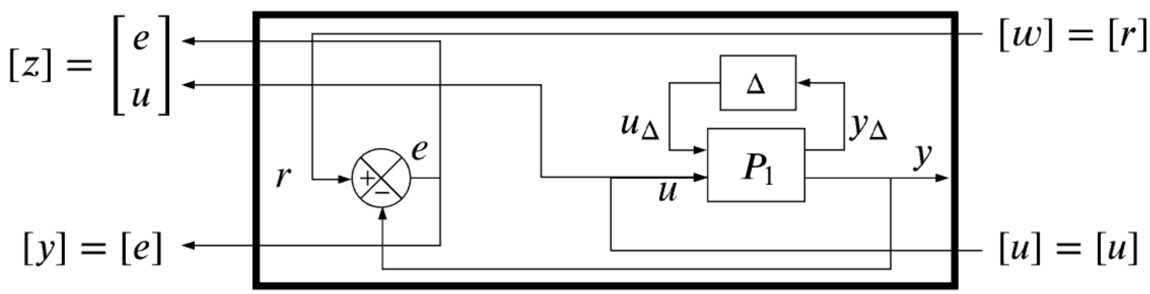




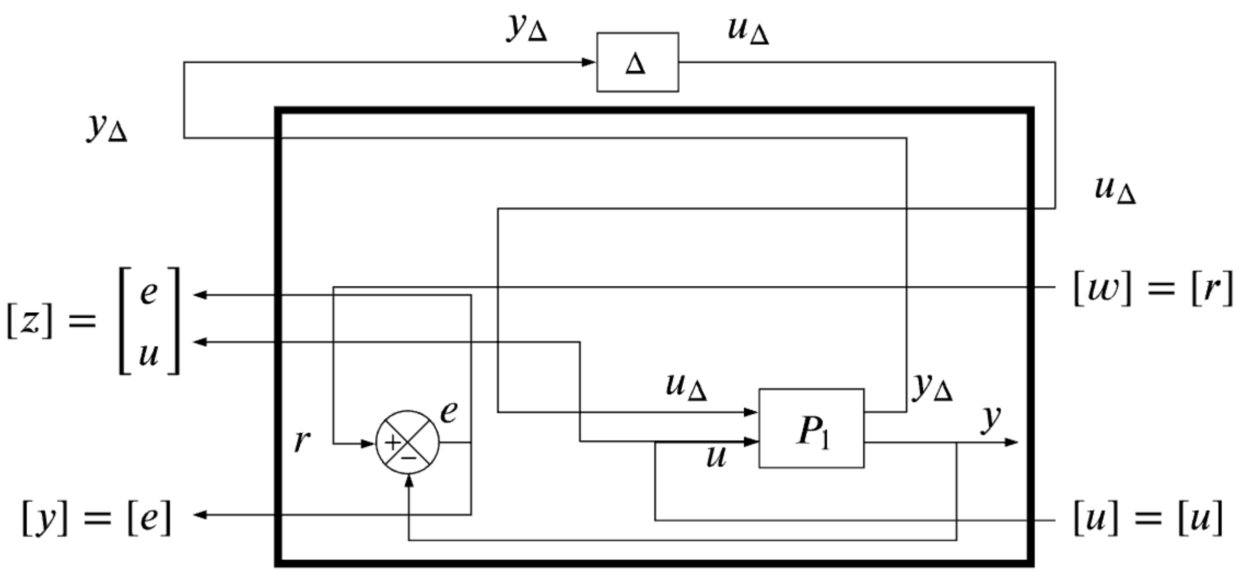
Δ
u_Δ
y_Δ
e
u
y
r
+
-
$K(s)$
P_1



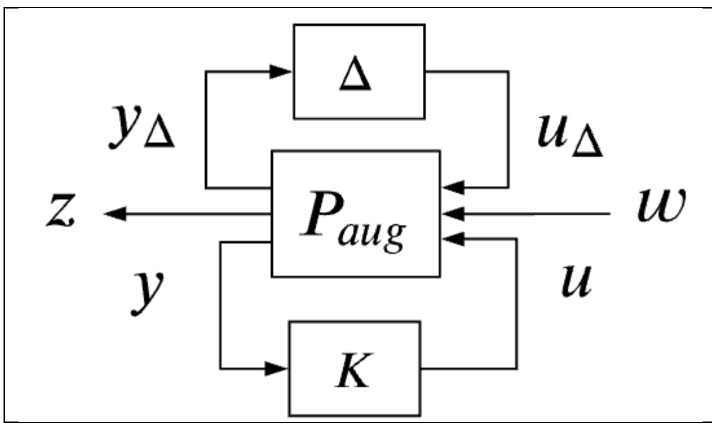
$[z] = \begin{bmatrix} e \\ u \end{bmatrix}$
$[w] = [r]$
r
e
u
y
$P(s)$
$[y] = [e]$
$[u] = [u]$



$[z] = \begin{bmatrix} e \\ u \end{bmatrix}$
$[w] = [r]$
Δ
u_Δ
y_Δ
r
e
P_1
y
u
$[y] = [e]$
$[u] = [u]$




```matlab
%% P1 to Paug(9 block rep.)

summer1=sumblk('e=r-y');
P2=connect(P1,summer1,{'um','uk','uc','r','u'},{'ym','yk','yc','e','u','e'});
```



$$u_\Delta = \begin{bmatrix} u_m \\ u_k \\ u_c \end{bmatrix}$$
$$w = [r]$$
$$u = [u]$$

$$y_\Delta = \begin{bmatrix} y_m \\ y_k \\ y_c \end{bmatrix}$$
$$z = \begin{bmatrix} e \\ u \end{bmatrix}$$
$$y = [e]$$

Matlab code output

```
P2 =

  A =
       x1   x2
   x1   0    1
   x2  -1   -1

  B =
          um     uk     uc      r      u
   x1      0      0      0      0      0
   x2   -0.1   -0.1   -0.1      0      1

  C =
       x1   x2
   ym  -1   -1
   yk   1    0
   yc   0    1
   e   -1    0
   u    0    0
   e   -1    0

  D =
          um     uk     uc      r      u
   ym   -0.1   -0.1   -0.1      0      1
   yk      0      0      0      0      0
   yc      0      0      0      0      0
   e       0      0      0      1      0
   u       0      0      0      0      1
   e       0      0      0      1      0

Continuous-time state-space model.
```

$\mathcal{H}_\infty$ *paradigm*	$\mu - synthesis$ *paradigm*