4.5.2 Delay-Dependent Condition

In the last section, we have considered the delay-independent condition for the stability of the time delay system (4.47). In this section, we further present a delay-dependent condition. For this purpose, we first rewrite the system as

$$\begin{aligned}\dot{x}(t) &= Ax(t) + A_d x(t-d) \\ &= (A + A_d)x(t) - A_d\,(x(t) - x(t-d)) \\ &= (a + A_d)x(t) - A_d \int_{-d}^{0} \dot{x}(t+s)\,\mathrm{d}s \\ &= (A + A_d)x(t) - A_d \int_{-d}^{0} [Ax(t+s) + A_d x(t-d+s)]\,\mathrm{d}s. \end{aligned} \tag{4.51}$$

The main result in this section is given as follows.

$$\dot{x}(t) = Ax(t) + A_d x(t-d)$$

$$= (A + A_d)x(t) - A_d\big(x(t) - x(t-d)\big)$$

$$= (A + A_d)x(t) - A_d \int_{-d}^{0} \dot{x}(t+s)\,ds$$

$$= (A + A_d)x(t) - A_d \int_{-d}^{0} [Ax(t+s) + A_d x(t-d+s)]\,ds$$

Theorem 4.9 The time-delay system (4.47) is uniformly asymptotically stable if there exist a symmetric positive definite matrix X and a scalar $0 < \beta < 1$, such that

$$\begin{bmatrix} \Phi(X) & \bar{d}XA^{\mathrm{T}} & \bar{d}XA_d^{\mathrm{T}} \\ \bar{d}AX & -\bar{d}\beta I & 0 \\ \bar{d}A_dX & 0 & -\bar{d}(1-\beta)I \end{bmatrix} < 0, \tag{4.52}$$

$$\begin{bmatrix} \Phi(X) & d_mXA^T & d_mXA_d^T \\ d_mAX & -d_m\beta I & 0 \\ d_mA_dX & 0 & -d_m(1-\beta)I \end{bmatrix} < 0$$

$$\Phi(X) = X(A+A_d)^T + (A+A_d)X + d_mA_dA_d^T$$

PROOF:

$$V(x_t) = x^TPx + \frac{1}{\beta}\int_{-d}^{0}\int_{t+s}^{t} x^T(\theta)A^TAx(\theta)d\theta\, ds + \frac{1}{1-\beta}\int_{-d}^{0}\int_{t-d+s}^{t} x^T(\theta)A_d^TA_dx(\theta)d\theta\, ds$$

$$V(x_t) = T_1 + T_2 + T_3$$

$$V(x_t) = \underbrace{x^TPx}_{T_1} + \underbrace{\frac{1}{\beta}\int_{-d}^{0}\int_{t+s}^{t} x^T(\theta)A^TAx(\theta)d\theta\, ds}_{T_2} + \underbrace{\frac{1}{1-\beta}\int_{-d}^{0}\int_{t-d+s}^{t} x^T(\theta)A_d^TA_dx(\theta)d\theta\, ds}_{T_3}$$

$$\frac{d}{dt}V(x_t) = \frac{d}{dt}T_1 + \frac{d}{dt}T_2 + \frac{d}{dt}T_3$$

$$\frac{d}{dt}T_1 = \frac{d}{dt}x^TPx = \dot{x}^TPx + x^TP\dot{x}$$

$$= \left[(A+A_d)x(t) - A_d\int_{-d}^{0}[Ax(t+s) + A_dx(t-d+s)]\,ds\right]^T Px + x^TP\left[(A+A_d)x(t) - A_d\int_{-d}^{0}[Ax(t+s) + A_dx(t-d+s)]\,ds\right]$$

$$= \left[(A+A_d)x(t) - A_d\int_{-d}^{0}[Ax(t+s)]\,ds - A_d\int_{-d}^{0}[A_dx(t-d+s)]\,ds\right]^T Px + x^TP\left[(A+A_d)x(t) - A_d\int_{-d}^{0}[Ax(t+s)]\,ds - A_d\int_{-d}^{0}[A_dx(t-d+s)]\,ds\right]$$

$$= \left[(A+A_d)x(t) - A_d\int_{-d}^{0}[Ax(t+s)]\,ds - A_d\int_{-d}^{0}[A_dx(t-d+s)]\,ds\right]^T Px + x^TP\left[(A+A_d)x(t) - A_d\int_{-d}^{0}[Ax(t+s)]\,ds - A_d\int_{-d}^{0}[A_dx(t-d+s)]\,ds\right]$$

$$\dot{T}_{1r} = [x^T(A+A_d)^TPx + x^TP(A+A_d)x]$$

Now focus on the green terms

$$\dot{T}_{1g} = \left[-A_d\int_{-d}^{0}[Ax(t+s)]\,ds\right]^T Px + x^TP\left[-A_d\int_{-d}^{0}[Ax(t+s)]\,ds\right]$$

Since each of these two terms is scalar, each can be transposed. Let's transpose the first term.

$$\dot{T}_{1g} = x^T P\left[-A_d \int_{-d}^{0} [Ax(t+s)]\, ds\right] + x^T P\left[-A_d \int_{-d}^{0} [Ax(t+s)]\, ds\right]$$

$$\dot{T}_{1g} = -2x^T P A_d \left[\int_{-d}^{0} [Ax(t+s)]\, ds\right]$$

Now focus on the blue terms,

$$\dot{T}_{1b} = \left[-A_d \int_{-d}^{0} [A_d x(t-d+s)]\, ds\right]^T Px + x^T P\left[-A_d \int_{-d}^{0} [A_d x(t-d+s)]\, ds\right]$$

Since each of these two terms is scalar, each can be transposed. Let's transpose the first term.

$$\dot{T}_{1b} = x^T P\left[-A_d \int_{-d}^{0} [A_d x(t-d+s)]\, ds\right] + x^T P\left[-A_d \int_{-d}^{0} [A_d x(t-d+s)]\, ds\right]$$

$$\dot{T}_{1b} = -2x^T P A_d \left[\int_{-d}^{0} [A_d x(t-d+s)]\, ds\right]$$

Now let us focus on the derivative of T2 wrt t

$$\dot{T}_2 = \frac{d}{dt}\left[\frac{1}{\beta}\int_{-d}^{0}\int_{t+s}^{t} x^T(\theta)A^T Ax(\theta)d\theta\, ds\right]$$

$$\dot{T}_2 = \frac{1}{\beta}\int_{-d}^{0} \frac{d}{dt}\left(\int_{t+s}^{t} x^T(\theta)A^T Ax(\theta)d\theta\right) ds$$

Using Leibnitz integral formulas,

$$\frac{d}{dt}\int_{a}^{b} f(t,x)dx = \int_{a}^{b}\left(\frac{\partial}{\partial t} f(t,x)\right) dx$$

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(t,x)dx = \int_{a(t)}^{b(t)}\left(\frac{\partial}{\partial t} f(t,x)\right) dx + f\big(t,b(t)\big)\frac{db(t)}{dt} - f\big(t,a(t)\big)\frac{da(t)}{dt}$$

$$\frac{d}{dt}\left(\int_{t+s}^{t} x^T(\theta)A^T Ax(\theta)d\theta\right) = \int_{t+s}^{t}\left(\frac{d}{dt}x^T(\theta)A^T Ax(\theta)\right)d\theta + \left(x^T(\theta)A^T Ax(\theta)\right)\Big|_{\theta\leftarrow t}(1) - \left(x^T(\theta)A^T Ax(\theta)\right)\Big|_{\theta\leftarrow t+s}(1)$$

$$\frac{d}{dt}\left(\int_{t+s}^{t} x^T(\theta)A^T Ax(\theta)d\theta\right) = \left(x^T(t)A^T Ax(t)\right) - \left(x^T(t+s)A^T Ax(t+s)\right)$$

$$\dot{T}_2 = \frac{1}{\beta}\int_{-d}^{0}\frac{d}{dt}\left(\int_{t+s}^{t} x^T(\theta)A^T Ax(\theta)d\theta\right)ds$$

$$\dot{T}_2 = \frac{1}{\beta}\int_{-d}^{0}\left[\left(x^T(t)A^T Ax(t)\right) - \left(x^T(t+s)A^T Ax(t+s)\right)\right]ds$$

$$\dot{T}_2 = \frac{1}{\beta}\int_{-d}^{0}\left[\left(x^T(t)A^T Ax(t)\right)\right]ds - \frac{1}{\beta}\int_{-d}^{0}\left[\left(x^T(t+s)A^T Ax(t+s)\right)\right]ds$$

$$\dot{T}_2 = \frac{d}{\beta}\left(x^T(t)A^T Ax(t)\right) - \frac{1}{\beta}\int_{-d}^{0}\left[\left(x^T(t+s)A^T Ax(t+s)\right)\right]ds$$

Now let us focus on the derivative of T3 wrt t

$$\dot{T}_3 = \frac{d}{dt}\left[\frac{1}{1-\beta}\int_{-d}^{0}\int_{t-d+s}^{t} x^T(\theta)A_d^T A_d x(\theta)d\theta\, ds\right]$$

$$\dot{T}_3 = \frac{1}{1-\beta}\int_{-d}^{0}\frac{d}{dt}\left(\int_{t-d+s}^{t} x^T(\theta)A_d^T A_d x(\theta)d\theta\right)ds$$

By using Leibnitz's integral formulae,

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(t,x)dx = \int_{a(t)}^{b(t)}\left(\frac{\partial}{\partial t}f(t,x)\right)dx + f\big(t,b(t)\big)\frac{db(t)}{dt} - f\big(t,a(t)\big)\frac{da(t)}{dt}$$

$$\frac{d}{dt}\left(\int_{t-d+s}^{t} x^T(\theta)A_d^T A_d x(\theta)d\theta\right) = \int_{t-d+s}^{t}\left(\frac{d}{dt}x^T(\theta)A_d^T A_d x(\theta)\right)d\theta + \left(x^T(\theta)A_d^T A_d x(\theta)\right)\Big|_{\theta\leftarrow t}(1) - \left(x^T(\theta)A_d^T A_d x(\theta)\right)\Big|_{\theta\leftarrow t-d+s}(1)$$

$$\frac{d}{dt}\left(\int_{t+s}^{t} x^T(\theta)A^T Ax(\theta)d\theta\right) = \left(x^T(t)A_d^T A_d x(t)\right) - \left(x^T(t-d+s)A_d^T A_d x(t-d+s)\right)$$

$$\dot{T}_3 = \frac{1}{1-\beta}\int_{-d}^{0}\frac{d}{dt}\left(\int_{t-d+s}^{t} x^T(\theta)A_d^T A_d x(\theta)d\theta\right)ds$$

$$\dot{T}_3 = \frac{1}{1-\beta}\int_{-d}^{0}\left[\left(x^T(t)A_d^T A_d x(t)\right) - \left(x^T(t-d+s)A_d^T A_d x(t-d+s)\right)\right]ds$$

$$\dot{T}_3 = \frac{1}{1-\beta}\int_{-d}^{0}\left[\left(x^T(t)A_d^T A_d x(t)\right)\right]ds - \frac{1}{1-\beta}\int_{-d}^{0}\left[\left(x^T(t-d+s)A_d^T A_d x(t-d+s)\right)\right]ds$$

$$\dot{T}_3 = \frac{d}{1-\beta}\left(x^T(t)A_d^T A_d x(t)\right) - \frac{1}{1-\beta}\int_{-d}^{0}\left[\left(x^T(t-d+s)A_d^T A_d x(t-d+s)\right)\right]ds$$

As a result of this,

$$V(x_t) = \underbrace{x^T P x}_{T_1} + \underbrace{\frac{1}{\beta}\int_{-d}^{0}\int_{t+s}^{t} x^T(\theta)A^T A x(\theta)d\theta\, ds}_{T_2} + \underbrace{\frac{1}{1-\beta}\int_{-d}^{0}\int_{t-d+s}^{t} x^T(\theta)A_d^T A_d x(\theta)d\theta\, ds}_{T_3}$$

$$\frac{d}{dt}V(x_t) = \dot{T}_1 + \dot{T}_2 + \dot{T}_3$$

$$\begin{aligned}\frac{d}{dt}V(x_t) = {} & [x^T(A+A_d)^T P x + x^T P(A+A_d)x] - 2x^T P A_d\left[\int_{-d}^{0}[Ax(t+s)]\, ds\right] - 2x^T P A_d\left[\int_{-d}^{0}[A_d x(t-d+s)]\, ds\right] \\ & + \left[\frac{d}{\beta}\left(x^T(t)A^T A x(t)\right) - \frac{1}{\beta}\int_{-d}^{0}\left[\left(x^T(t+s)A^T A x(t+s)\right)\right]ds\right] \\ & + \left[\frac{d}{1-\beta}\left(x^T(t)A_d^T A_d x(t)\right) - \frac{1}{1-\beta}\int_{-d}^{0}\left[\left(x^T(t-d+s)A_d^T A_d x(t-d+s)\right)\right]ds\right]\end{aligned}$$

And

$$\begin{aligned}\frac{d}{dt}V(x_t) = {} & [x^T(A+A_d)^T P x + x^T P(A+A_d)x] - 2x^T P A_d\left[\int_{-d}^{0}[Ax(t+s)]\, ds\right] - 2x^T P A_d\left[\int_{-d}^{0}[A_d x(t-d+s)]\, ds\right] \\ & + \left[\frac{d}{\beta}\left(x^T(t)A^T A x(t)\right) - \frac{1}{\beta}\int_{-d}^{0}\left[\left(x^T(t+s)A^T A x(t+s)\right)\right]ds\right] \\ & + \left[\frac{d}{1-\beta}\left(x^T(t)A_d^T A_d x(t)\right) - \frac{1}{1-\beta}\int_{-d}^{0}\left[\left(x^T(t-d+s)A_d^T A_d x(t-d+s)\right)\right]ds\right]\end{aligned}$$

Now let us focus on the red-terms,

$-2x^T P A_d\left[\int_{-d}^{0}[Ax(t+s)]\, ds\right]$	**Lemma 2.1** Let $X, Y \in \mathbb{R}^{m\times n}$, $F \in \mathbb{S}^m$, $F > 0$, and $\delta > 0$ be a scalar, then $X^T FY + Y^T FX \leq \delta X^T FX + \delta^{-1} Y^T FY.$ (2.1) Particularly, when $X = x$ and $Y = y$ are vectors, the aforementioned inequality reduces to $2x^T Fy \leq \delta x^T Fx + \delta^{-1} y^T Fy.$ (2.2) **29**

$$2\underbrace{\left(-x^T P A_d\right)}_{\mathbb{X}^T}\underbrace{(d)}_{F}\underbrace{\left[\frac{1}{d}\int_{-d}^{0}[Ax(t+s)]\, ds\right]}_{\mathbb{Y}} \leq \underbrace{\boxed{\beta}}_{\delta}\underbrace{\left(-x^T P A_d\right)}_{\mathbb{X}^T}\underbrace{(d)}_{F}\underbrace{\left(-x^T P A_d\right)^T}_{\mathbb{X}} + \underbrace{\boxed{\frac{1}{\beta}}}_{\delta^{-1}}\underbrace{\left[\frac{1}{d}\int_{-d}^{0}[Ax(t+s)]\, ds\right]^T}_{\mathbb{Y}^T}\underbrace{(d)}_{F}\underbrace{\left[\frac{1}{d}\int_{-d}^{0}[Ax(t+s)]\, ds\right]}_{\mathbb{Y}}$$

$$2\underbrace{\left(-x^T P A_d\right)}_{\mathbb{X}^T}\underbrace{(d)}_{F}\underbrace{\left[\frac{1}{d}\int_{-d}^{0}[Ax(t+s)]\, ds\right]}_{\mathbb{Y}} \leq \beta d x^T P A_d A_d^T P x + \frac{d}{\beta}\underbrace{\left[\int_{-d}^{0}\frac{1}{d}[Ax(t+s)]\, ds\right]^T}_{\mathbb{Y}^T}\underbrace{\left[\int_{-d}^{0}\frac{1}{d}[Ax(t+s)]\, ds\right]}_{\mathbb{Y}}$$

Now by using Cauchy-Schwarz $\left(\int_a^b f(x)\, dx\right)^2 \leq (b-a)\int_a^b [f(x)]^2\, dx$	$\left(\frac{d}{\beta}\right)\underbrace{\left[\int_{-d}^{0}\frac{1}{d}[Ax(t+s)]\, ds\right]^T}_{\mathbb{Y}^T}\underbrace{\left[\int_{-d}^{0}\frac{1}{d}[Ax(t+s)]\, ds\right]}_{\mathbb{Y}}$

	$\leq \left(\frac{d}{\beta}\right) d \int_{-d}^{0} \left(\frac{1}{d}\right)^2 [x^T(t+s)A^TAx(t+s)]\, ds$ $\leq \left(\frac{1}{\beta}\right) \int_{-d}^{0} [x^T(t+s)A^TAx(t+s)]\, ds$

Therefore,

$$2\underbrace{(-x^TPA_d)}_{\mathbb{X}^T}\underbrace{(d)}_{F}\underbrace{\left[\frac{1}{d}\int_{-d}^{0}[Ax(t+s)]\,ds\right]}_{\mathbb{Y}} \leq \beta d x^T P A_d A_d^T P x + \left(\frac{1}{\beta}\right)\int_{-d}^{0}[x^T(t+s)A^TAx(t+s)]\,ds$$

$$2(-x^TPA_d)(d)\left[\frac{1}{d}\int_{-d}^{0}[Ax(t+s)]\,ds\right] - \left(\frac{1}{\beta}\right)\int_{-d}^{0}[x^T(t+s)A^TAx(t+s)]\,ds \leq \beta d x^T P A_d A_d^T P x$$

And the red terms are eliminated,

$$\begin{aligned}\frac{d}{dt}V(x_t) = [x^T(A+A_d)^TPx + x^TP(A+A_d)x] &- 2x^TPA_d\left[\int_{-d}^{0}[Ax(t+s)]\,ds\right] - 2x^TPA_d\left[\int_{-d}^{0}[A_dx(t-d+s)]\,ds\right] \\ &+ \left[\frac{d}{\beta}\left(x^T(t)A^TAx(t)\right) - \frac{1}{\beta}\int_{-d}^{0}\left[\left(x^T(t+s)A^TAx(t+s)\right)\right]ds\right] \\ &+ \left[\frac{d}{1-\beta}\left(x^T(t)A_d^TA_dx(t)\right) - \frac{1}{1-\beta}\int_{-d}^{0}\left[\left(x^T(t-d+s)A_d^TA_dx(t-d+s)\right)\right]ds\right]\end{aligned}$$

Now let us focus on the blue-terms,

$$-2x^TPA_d\left[\int_{-d}^{0}[A_dx(t-d+s)]\,ds\right] \leq d(1-\beta)x^TPA_dA_d^TPx + \frac{1}{1-\beta}\int_{-d}^{0}\left[\left(x^T(t-d+s)A_d^TA_dx(t-d+s)\right)\right]ds$$

This inequality is obtained by the same technique applied to the red-terms.

Therefore,

$$\frac{d}{dt}V(x_t) \leq [x^T(A+A_d)^TPx + x^TP(A+A_d)x] + \beta d x^TPA_dA_d^TPx + \left[\frac{d}{\beta}\left(x^T(t)A^TAx(t)\right)\right] + d(1-\beta)x^TPA_dA_d^TPx + \left[\frac{d}{1-\beta}\left(x^T(t)A_d^TA_dx(t)\right)\right]$$

$$\frac{d}{dt}V(x_t) \leq x^T[P(A+A_d) + (A+A_d)^TP]x + x^T(\beta dPA_dA_d^TP)x + x^T\left[\frac{d}{\beta}A^TA\right]x + x^T[d(1-\beta)PA_dA_d^TP]x + x^T\left[\frac{d}{1-\beta}A_d^TA_d\right]x$$

$$\frac{d}{dt}V(x_t) \leq x^T[P(A+A_d) + (A+A_d)^TP]x + x^T\left[\frac{d}{\beta}A^TA\right]x + x^T[dPA_dA_d^TP]x + x^T\left[\frac{d}{1-\beta}A_d^TA_d\right]x$$

$$\frac{d}{dt}V(x_t) \leq x^T[P(A+A_d) + (A+A_d)^TP]x + x^T\left[\frac{d}{\beta}A^TA + dPA_dA_d^TP + \frac{d}{1-\beta}A_d^TA_d\right]x$$

And since $0 < d \leq d_m$ $\boxed{d_m := d_{max}}$

$$\frac{d}{dt}V(x_t) \leq x^T[P(A+A_d) + (A+A_d)^TP]x + x^T\left[\frac{d_m}{\beta}A^TA + d_mPA_dA_d^TP + \frac{d_m}{1-\beta}A_d^TA_d\right]x$$

$$\frac{d}{dt}V(x_t) \leq x^T\left[P(A+A_d) + (A+A_d)^TP + \frac{d_m}{\beta}A^TA + d_mPA_dA_d^TP + \frac{d_m}{1-\beta}A_d^TA_d\right]x$$

For this term to be negative-definite, the RHS-matrix must be negative-definite

$$\left[P(A+A_d) + (A+A_d)^TP + \frac{d_m}{\beta}A^TA + d_mPA_dA_d^TP + \frac{d_m}{1-\beta}A_d^TA_d\right] < 0$$

By pre- and post-multiplying the LHS, and denoting $\boxed{X := P^{-1}}$

$$\left[P(A+A_d)+(A+A_d)^T P+d_m P A_d A_d^T P+\frac{d_m}{\beta}A^T A+\frac{d_m}{1-\beta}A_d^T A_d\right]<0$$

$$\left[XP(A+A_d)X+X(A+A_d)^T PX+Xd_m P A_d A_d^T PX+X\frac{d_m}{\beta}A^T AX+X\frac{d_m}{1-\beta}A_d^T A_d X\right]<0$$

$$\left[(A+A_d)X+X(A+A_d)^T+d_m A_d A_d^T+\frac{d_m}{\beta}XA^T AX+\frac{d_m}{1-\beta}XA_d^T A_d X\right]<0$$

Now define, $\boxed{\Phi(X)\coloneqq(A+A_d)X+X(A+A_d)^T+d_m A_d A_d^T}$

$$\left[\Phi(X)+\frac{d_m}{\beta}XA^T AX+\frac{d_m}{1-\beta}XA_d^T A_d X\right]<0$$

$$\left[\Phi(X)+d_m XA^T\left(\frac{1}{d_m\beta}\right)AXd_m+d_m XA_d^T\left(\frac{1}{d_m(1-\beta)}\right)A_d Xd_m\right]<0$$

$$d_m XA^T\left(\frac{1}{d_m\beta}\right)AXd_m+d_m XA_d^T\left(\frac{1}{d_m(1-\beta)}\right)A_d Xd_m=\begin{bmatrix}d_m XA^T & d_m XA_d^T\end{bmatrix}\begin{bmatrix}\left(\frac{1}{d_m\beta}\right) & 0\\ 0 & \left(\frac{1}{d_m(1-\beta)}\right)\end{bmatrix}\begin{bmatrix}d_m AX\\ d_m A_d X\end{bmatrix}$$

$$\Phi(X)+\begin{bmatrix}d_m XA^T & d_m XA_d^T\end{bmatrix}\begin{bmatrix}\left(\frac{1}{d_m\beta}\right) & 0\\ 0 & \left(\frac{1}{d_m(1-\beta)}\right)\end{bmatrix}\begin{bmatrix}d_m AX\\ d_m A_d X\end{bmatrix}<0$$

$$\Phi(X)-\begin{bmatrix}d_m XA^T & d_m XA_d^T\end{bmatrix}\begin{bmatrix}\left(\frac{1}{-d_m\beta}\right) & 0\\ 0 & \left(\frac{1}{-d_m(1-\beta)}\right)\end{bmatrix}\begin{bmatrix}d_m AX\\ d_m A_d X\end{bmatrix}<0$$

$$\Phi(X)-\begin{bmatrix}d_m XA^T & d_m XA_d^T\end{bmatrix}\begin{bmatrix}-d_m\beta & 0\\ 0 & -d_m(1-\beta)\end{bmatrix}^{-1}\begin{bmatrix}d_m AX\\ d_m A_d X\end{bmatrix}<0$$

$$\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}<0\leftrightarrow\begin{bmatrix}A_{11} & 0\\ 0 & A_{22}-A_{21}[A_{11}]^{-1}A_{12}\end{bmatrix}<0\leftrightarrow\begin{bmatrix}A_{11}-A_{12}[A_{22}]^{-1}A_{21} & 0\\ 0 & A_{22}\end{bmatrix}<0$$

$\begin{bmatrix}A_{11}-A_{12}[A_{22}]^{-1}A_{21} & 0\\ 0 & A_{22}\end{bmatrix}<0$ $A_{11}\coloneqq\Phi(X)$ $A_{12}\coloneqq\begin{bmatrix}d_m XA^T & d_m XA_d^T\end{bmatrix}$ $A_{21}\coloneqq\begin{bmatrix}d_m AX\\ d_m A_d X\end{bmatrix}$ $A_{22}\coloneqq\begin{bmatrix}-d_m\beta & 0\\ 0 & -d_m(1-\beta)\end{bmatrix}$	$\begin{bmatrix}A_{11}-A_{12}[A_{22}]^{-1}A_{21} & 0\\ 0 & A_{22}\end{bmatrix}<0\leftrightarrow\begin{bmatrix}\Phi(X) & \begin{bmatrix}d_m XA^T & d_m XA_d^T\end{bmatrix}\\ \begin{bmatrix}d_m AX\\ d_m A_d X\end{bmatrix} & \begin{bmatrix}-d_m\beta I & 0\\ 0 & -d_m(1-\beta)I\end{bmatrix}\end{bmatrix}<0$

Finally, the delay-dependent LMI condition is given by

$$\begin{bmatrix}\Phi(X) & \begin{bmatrix}d_m XA^T & d_m XA_d^T\end{bmatrix}\\ \begin{bmatrix}d_m AX\\ d_m A_d X\end{bmatrix} & \begin{bmatrix}-d_m\beta I & 0\\ 0 & -d_m(1-\beta)I\end{bmatrix}\end{bmatrix}<0$$

Therefore, the LMI conditions are given as,

$$For\ a\ given\ \boxed{d_m>0},Find\ \boxed{\begin{matrix}X>0\\ 0<\beta<1\\ \begin{bmatrix}\Phi(X) & d_m XA^T & d_m XA_d^T\\ d_m AX & -d_m\beta I & 0\\ d_m A_d X & 0 & -d_m(1-\beta)I\end{bmatrix}<0\end{matrix}}$$

Delay Independent LMI Condition

4.5 Time-Delay Systems

The problem of stability analysis for a time-delay system can be stated as follows.

Problem 4.3 Given matrices $A, A_d \in \mathbb{R}^{n\times n}$, check the stability of the following linear time-delay system

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d x(t-d) \\ x(t) = \phi(t), \ t \in [-d, 0], \ 0 < d \le \bar{d}, \end{cases} \quad (4.47)$$

where
$\phi(t)$ is the initial condition
d represents the time-delay
$\bar{d}$ is a known upper bound of d

4.5.1 Delay-Independent Condition

The following theorem gives a sufficient condition for the stability problem in terms of an LMI.

Theorem 4.8 The system (4.47) is asymptotically stable if there exist two symmetric matrices $P, S \in \mathbb{S}^n$, such that

$$\begin{cases} P > 0 \\ \begin{bmatrix} A^{\mathrm{T}}P + PA + S & PA_d \\ A_d^{\mathrm{T}}P & -S \end{bmatrix} < 0. \end{cases} \quad (4.48)$$

PROOF:

$$V(x_t) = x^T(t)Px(t) + \int_{t-d}^{t} x^T(\tau)Sx(\tau)d\tau \,, \boxed{x_t \coloneqq x(t+\theta), \theta \in [-d, 0]}$$

Let us obtain time-derivative of Lyap-fcn wrt the time

$$\frac{d}{dt}V(x_t) = \dot{x}^T P x + x^T P \dot{x} + \frac{d}{dt}\left[\int_{t-d}^{t} x^T(\tau)Sx(\tau)d\tau\right]$$

$$= [Ax + A_d x(t-d)]^T Px + x^T P[Ax + A_d x(t-d)] + \frac{d}{dt}\left[\int_{t-d}^{t} x^T(\tau)Sx(\tau)d\tau\right]$$

$$= [Ax + A_d x(t-d)]^T Px + x^T P[Ax + A_d x(t-d)] + \frac{d}{dt}\left[\int_{t-d}^{t} x^T(\tau)Sx(\tau)d\tau\right]$$

By using Leibnitz's integral formulae,

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(t,x)dx = \int_{a(t)}^{b(t)} \left(\frac{\partial}{\partial t} f(t,x)\right) dx + f\big(t, b(t)\big)\frac{db(t)}{dt} - f\big(t, a(t)\big)\frac{da(t)}{dt}$$

$$\frac{d}{dt}\left[\int_{t-d}^{t} x^T(\tau)Sx(\tau)d\tau\right] = \left[\int_{t-d}^{t} \left(\frac{d}{dt} x^T(\tau)Sx(\tau)\right) d\tau\right] + \big(x^T(\tau)Sx(\tau)\big)\Big|_{\tau\leftarrow t} (1) - \big(x^T(\tau)Sx(\tau)\big)\Big|_{\tau\leftarrow t-d} (1)$$

$$= \big(x^T(t)Sx(t)\big) - \big(x^T(t-d)Sx(t-d)\big)$$

and

$$\frac{d}{dt}V(x_t) = [Ax + A_d x(t-d)]^T Px + x^T P[Ax + A_d x(t-d)] + \left[\big(x^T(t)Sx(t)\big) - \big(x^T(t-d)Sx(t-d)\big)\right]$$

$$\frac{d}{dt}V(x_t) = x^T[A^TP + PA + S]x + x^TPA_dx(t-d) + x^T(t-d)A_d^TPx - x^T(t-d)Sx(t-d)$$

$$\frac{d}{dt}V(x_t) = \begin{bmatrix} x(t) \\ x(t-d) \end{bmatrix}^T \begin{bmatrix} A^TP + PA + S & PA_d \\ A_d^TP & -S \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-d) \end{bmatrix}$$

And since $\boxed{\frac{d}{dt}V(x_t) < 0}$ is required for the stability, RHS matrix must be negative-definite,

$$\begin{bmatrix} A^TP + PA + S & PA_d \\ A_d^TP & -S \end{bmatrix} < 0$$

Therefore, the conditions are given as

$$\boxed{\begin{gathered} P > 0 \\ \begin{bmatrix} A^TP + PA + S & PA_d \\ A_d^TP & -S \end{bmatrix} < 0 \end{gathered}}$$