The Existence of Convex Lyapunov Functions

Definition 2.1 (Lyapunov Function)

Consider a system of the form $\dot{x}(t) = f_{i(\cdot)}(x(t))$ for which $x = 0$ is an equilibrium point. A continuous function $\Psi : \mathbb{R}^n \to \mathbb{R}$ is said to be a Lyapunov function if

- $\kappa_1(\|x\|) \leq \Psi(x) \leq \kappa_2(\|x\|)$, where $\kappa_1, \kappa_2 \in \mathcal{K}$,
- $\Psi(x(t)) \leq \alpha(\Psi(x(0)), t), \forall t \geq 0$,
- $\alpha(\psi, t)$ is a continuous function defined for $\psi, t \geq 0$
- for fixed t, strictly increasing w.r.t. ψ
- for fixed ψ, strictly decreasing w.r.t. t
- $\alpha(\psi, 0) = \psi$

Remark (Exponential Stability)

In case of a Lyapunov fcn assuring exponential stability we just have:

$$\alpha(\Psi(x(0)), t) = \Psi(x(0))e^{-\sigma t} \text{ for some } \sigma > 0$$

Definition 2.2 (Confinement Set)

$$\mathcal{C}(T; \mathcal{R}) = \{x(t) \in \mathbb{R}^n \mid \dot{x} = f(x), x(0) \in \mathcal{R}, t \geq T\}$$

Remark (Confinement Set for Stable Systems)

For stable systems: $\boxed{\lim_{T\to\infty} \mathcal{C}(T; \mathcal{R}) \to \{\mathbf{0}\}}$

Lemma 2.1 (Fundamental Lemma of the paper)

Consider $\dot{x} = f_{i(\cdot)}(x(t))$ and let $\mathcal{R}$ be a compact set.
IF the system admits a convex Lyapunov function,
THEN $\forall T > 0, \mathcal{R} \nsubseteq \text{conv}\{\mathcal{C}(T; \mathcal{R})\}$.

Lemma 2.1 PROOF

Consider the convex Lyap fcn $\Psi(x)$ and let $\boxed{\bar{\mu} = max_{x\in\mathcal{R}}\Psi(x)}$:

$$\Psi(x) \le \alpha(\bar{\mu}, T) < \bar{\mu}, \forall x \in \mathcal{C}(T;\mathcal{R})$$

FOR THE FIRST PART:

$$\Psi(x(t)) \le \alpha(\Psi(x(0)), t), \forall t$$

$$\Psi(x(T)) \le \alpha(\Psi(x(0)), T), \forall x(0) \in \mathcal{R}, x(T) \in \mathcal{C}(T;\mathcal{R})$$

choose: $x(0) = argmax_{x\in\mathcal{R}}\Psi(x)$

$$\Psi(x(T)) \le \alpha(\boxed{max_{x\in\mathcal{R}}\Psi(x)}, T), \forall x(T) \in \mathcal{C}(T;\mathcal{R})$$

$$\Psi(x(T)) \le \alpha(\bar{\mu}, T), \forall x(T) \in \mathcal{C}(T;\mathcal{R})$$

$$\Psi(x) \le \alpha(\bar{\mu}, T), \forall x \in \mathcal{C}(T;\mathcal{R})$$

FOR THE SECOND PART:

$$\boxed{\begin{matrix}\alpha(\bar{\mu},0)=\bar{\mu}\\ \alpha(\bar{\mu},0)<\alpha(\bar{\mu},T)\end{matrix}} \implies \boxed{\alpha(\bar{\mu}, T) \le \bar{\mu}}$$

$$\boxed{\begin{matrix}\Psi(x)\le\alpha(\bar{\mu},T)<\bar{\mu}\\ \forall x\in\mathcal{C}(T;\mathcal{R})\end{matrix}} \implies \boxed{\begin{matrix}\Psi(x)\le\alpha(\bar{\mu},T)<\bar{\mu}\\ \forall x \text{conv}\{\mathcal{C}(T;\mathcal{R})\}\end{matrix}}$$

consider the inequality : $\Psi(x) < \bar{\mu}, \forall x \in \mathcal{C}(T;\mathcal{R})$

Since $\Psi(x)$ is convex, the set, $\{x \in \mathbb{R}^n : \Psi(x) < \bar{\mu}\}$ is also convex (Sublevel set of a cvx-fcn is a cvx-set)

$$\underbrace{\mathcal{C}(T;\mathcal{R})}_{\text{confinement set}} \subseteq \underbrace{\{x \in \mathbb{R}^n : \Psi(x) < \bar{\mu}\}}_{\text{subset of a cvx-fcn}}$$

$$\text{conv}\{\mathcal{C}(T;\mathcal{R})\} \subseteq \{x \in \mathbb{R}^n : \Psi(x) < \bar{\mu}\}$$

$$\boxed{\begin{matrix}\Psi(x)\le\alpha(\bar{\mu},T)<\bar{\mu}\\ \forall x\in\text{conv}\{\mathcal{C}(T;\mathcal{R})\}\end{matrix}} \implies \boxed{\sup_{\text{conv}\{\mathcal{C}(T;\mathcal{R})\}} \Psi(x)} \le \alpha(\bar{\mu}, T) < \bar{\mu}$$

If we assume $\boxed{\mathcal{R} \subseteq \text{conv}\{\mathcal{C}(T;\mathcal{R})\}}$:

$$\boxed{\boxed{\max_{x\in\mathcal{R}}\Psi(x)} \le \boxed{\sup_{\text{conv}\{\mathcal{C}(T;\mathcal{R})\}} \Psi(x)} \le \alpha(\bar{\mu}, T) < \bar{\mu}} \implies \boxed{\boxed{\max_{x\in\mathcal{R}}\Psi(x)} < \bar{\mu}}$$

CONTRADICTION!

$$\therefore \boxed{\mathcal{R} \nsubseteq \text{conv}\{\mathcal{C}(T;\mathcal{R})\}} \qquad \mathcal{Q.E.D.}$$

Source: *F. Blanchini and C. Savorgnan, "Stabilizability of switched linear systems does not imply the existence of convex Lyapunov functions," Automatica, vol. 44, no. 4, pp. 1166–1170, Apr. 2008.*