COMPARISON FUNCTIONS

Class-$\mathcal{K}$

Class-$\mathcal{K}_\infty$

Class-$\mathcal{L}$

Class-$\mathcal{KL}$

Class-$\mathcal{K}$ Function

Definition

A continuous function $\alpha : [0, a) \to [0, \infty)$, where $a > 0$ (possibly $a = \infty$), is said to be of class-$\mathcal{K}$ if:

- $\alpha(0) = 0$,
- $\alpha(r) > 0$ for all $r \in (0, a)$,
- α is strictly increasing.

Class-$\mathcal{K}$ Function

Definition

A continuous function $\alpha : [0, a) \to [0, \infty)$, where $a > 0$ (possibly $a = \infty$), is said to be of class-$\mathcal{K}$ if:

- $\alpha(0) = 0$,
- $\alpha(r) > 0$ for all $r \in (0, a)$,
- α is strictly increasing.

Examples

- $\alpha(r) = r$
- $\alpha(r) = r^p$, for any $p > 0$
- $\alpha(r) = \tan^{-1}(r)$

Class-$\mathcal{K}_\infty$ Function

Definition

A function $\alpha : [0, \infty) \to [0, \infty)$ is of class-$\mathcal{K}_\infty$ if:

- $\alpha \in \mathcal{K}$,
- $\lim_{r \to \infty} \alpha(r) = \infty$.

Class-$\mathcal{K}_\infty$ Function

Definition

A function $\alpha : [0, \infty) \to [0, \infty)$ is of class-$\mathcal{K}_\infty$ if:

- $\alpha \in \mathcal{K}$,
- $\lim_{r\to\infty} \alpha(r) = \infty$.

Examples

- $\alpha(r) = r$
- $\alpha(r) = r^p$ for $p > 0$
- $\alpha(r) = \ln(1 + r^2)$

Class-$\mathcal{L}$ Function

Definition

A function $\beta : [0, \infty) \to [0, \infty)$ is of class-$\mathcal{L}$ if:

- β is continuous,
- β is strictly decreasing,
- $\lim_{t \to \infty} \beta(t) = 0$.

Class-$\mathcal{L}$ Function

Definition

A function $\beta : [0, \infty) \to [0, \infty)$ is of class-$\mathcal{L}$ if:

- β is continuous,
- β is strictly decreasing,
- $\lim_{t \to \infty} \beta(t) = 0$.

Examples

- $\beta(t) = e^{-\lambda t}$ for $\lambda > 0$
- $\beta(t) = \frac{1}{1+t}$
- $\beta(t) = \frac{1}{\sqrt{1+t}}$

Class-$\mathcal{KL}$ Function

Definition

A function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is of class-$\mathcal{KL}$ if:

- For each fixed $t \geq 0$, $\beta(\cdot, t) \in \mathcal{K}$,
- For each fixed $r \in [0, a)$, $\beta(r, \cdot) \in \mathcal{L}$.

Class-$\mathcal{KL}$ Function

Definition

A function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is of class-$\mathcal{KL}$ if:

- For each fixed $t \geq 0$, $\beta(\cdot, t) \in \mathcal{K}$,
- For each fixed $r \in [0, a)$, $\beta(r, \cdot) \in \mathcal{L}$.

Definition

A function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is of class-$\mathcal{KL}$ if:

- $\beta(0, t) = 0, \forall t$,
- For each fixed t, strictly increasing w.r.t. r,
- For each fixed r, strictly decreasing w.r.t. t,
- For each fixed r, $\lim_{t\to\infty} \beta(r, t) = 0$.

Class-$\mathcal{KL}$ Function

Definition

A function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is of class-$\mathcal{KL}$ if:

- For each fixed $t \geq 0$, $\beta(\cdot, t) \in \mathcal{K}$,
- For each fixed $r \in [0, a)$, $\beta(r, \cdot) \in \mathcal{L}$.

Definition

A function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is of class-$\mathcal{KL}$ if:

- $\beta(0, t) = 0, \forall t$,
- For each fixed t, strictly increasing w.r.t. r,
- For each fixed r, strictly decreasing w.r.t. t,
- For each fixed r, $\lim_{t \to \infty} \beta(r, t) = 0$.

Examples

- $\beta(r, t) = re^{-\lambda t}$ for $\lambda > 0$
- $\beta(r, t) = \frac{r}{1+t}$
- $\beta(r, t) = \frac{r}{\sqrt{1+t}}$

Examples-1

Lyapunov Fcn

Consider a system of the form $\dot{x}(t) = f_{i(\cdot)}(x(t))$ for which $x = 0$ is an equilibrium point.
A continuous function $\Psi : \mathbb{R}^n \to \mathbb{R}$ is said to be a Lyapunov function if

- $\kappa_1(\|x\|) \leq \Psi(x) \leq \kappa_2(\|x\|)$, where $\kappa_1, \kappa_2 \in \mathcal{K}$,
- $\Psi(x(t)) \leq \alpha(\Psi(x(0)), t), \forall t \geq 0$,
- $\alpha(\psi, t)$ is a continuous function defined for $\psi, t \geq 0$
- for fixed t, strictly increasing w.r.t. ψ
- for fixed ψ, strictly decreasing w.r.t. t
- $\alpha(\psi, 0) = \psi$

Source: *F. Blanchini and C. Savorgnan, "Stabilizability of switched linear systems does not imply the existence of convex Lyapunov functions," Automatica, vol. 44, no. 4, pp. 1166–1170, Apr. 2008.*

Examples-2

GAS

The origin is globally asymptotically stable for $\dot{x} = f(x)$ if there exists a function $\beta \in \mathcal{KL}$ such that, for all $x_0 \in \mathbb{R}^n$,

$$\|x(t)\| \leq \beta(\|x_0\|, t), \forall t \in \mathbb{R}_{\geq 0}$$

Source: *Kellett, C.M. A compendium of comparison function results. Math. Control Signals Syst. 26, 339–374 (2014)*

Examples-3

Lyapunov Fcn-1

A Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ for $\dot{x} = f(x)$ is a continuously differentiable function such that there exists $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, a continuous positive definite function $\rho : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \forall x \in \mathbb{R}^n$$

$$\dot{V}(x) \leq -\rho(x), \forall x \in \mathbb{R}^n$$

Examples-3

Lyapunov Fcn-1

A Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ for $\dot{x} = f(x)$ is a continuously differentiable function such that there exists $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, a continuous positive definite function $\rho : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \forall x \in \mathbb{R}^n$$

$$\dot{V}(x) \leq -\rho(x), \forall x \in \mathbb{R}^n$$

Lyapunov Fcn-2

A Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ for $\dot{x} = f(x)$ is a continuously differentiable function such that there exists $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\beta \in \mathcal{KL}$ and

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \forall x \in \mathbb{R}^n$$

$$V(x(t)) \leq \beta(V(x_0), t), \forall t \in \mathbb{R}_{\geq 0}$$

Source: *Kellett, C.M. A compendium of comparison function results. Math. Control Signals Syst. 26, 339–374 (2014)*

Examples-4

ISS

Consider a time-invariant system of ordinary differential equations of the form

$$\dot{x} = f(x, u), x(t) \in \mathbb{R}^n,$$

where $u : \mathbb{R}_+ \to \mathbb{R}^m$ is a Lebesgue measurable essentially bounded external input and f is a Lipschitz continuous function w.r.t. the first argument uniformly w.r.t. the second one. This ensures that there exists a unique absolutely continuous solution of the system.
System is called input-to-state stable (ISS) if $\exists \gamma \in \mathcal{K}, \exists \beta \in \mathcal{KL} \ni \forall x_0, \forall u \in \mathcal{U}_{adm}, \forall t \geq 0$:

$$\|x(t)\| \leq \beta(\|x_0\|, t) + \gamma(\|u\|_\infty)$$

The function γ in the above inequality is called the **gain**.

Source: *wikipedia.org/wiki/Input-to-state_stability*