Reachable Set
&
Confinement Set

Reachable Set Definition

System Description

Consider a control system described by the differential equation:

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0,$$

where:

- $x(t) \in \mathbb{R}^n$ is the state,
- $u(t) \in \mathcal{U} \subseteq \mathbb{R}^m$ is the control input,
- $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ defines the system dynamics.

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Reachable Set at Time T

The **reachable set** at time $T > 0$, from an initial state $x_0 \in \mathbb{R}^n$, is defined as:

$$\mathcal{R}(T, x_0) \triangleq \left\{ x(T) \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists\, u(\cdot) : [0, T] \to \mathcal{U} \text{ measurable such that} \\ x(0) = x_0, \\ \dot{x}(t) = f(x(t), u(t)) \text{ for all } t \in [0, T] \end{array} \right\}.$$

Reachable Set types

Reachable Set Over Time Interval

$$\mathcal{R}([0, T], x_0) \triangleq \bigcup_{t \in [0, T]} \mathcal{R}(t, x_0).$$

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Reachable Set from a Set of Initial Conditions

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Reachable Set Over Time Interval from a Set of Initial Conditions

$$\mathcal{R}(T, \mathcal{X}_0) \triangleq \bigcup_{t \in [0, T]} \bigcup_{x_0 \in \mathcal{X}_0} \mathcal{R}(t, x_0).$$

CONFINEMENT SET

Reachable Set Over Time Interval

Given a system: $\dot{x} = f(x), x \in \mathbb{R}^n$, a set of initial conditions $\mathcal{R}_0 \subseteq \mathbb{R}^n$, and a time $T > 0$, let $\mathcal{C}(T; \mathcal{R}_0)$ be the set of reachable points of solutions of system, after time T, starting in $\mathcal{R}_0$, that is,

$$\mathcal{C}(T; \mathcal{R}_0) \triangleq \{x(t) \in \mathbb{R}^n \mid \dot{x} = f(x), x(0) \in \mathcal{R}_0, t \geq T\}$$

Source: *F. Blanchini and C. Savorgnan, "Stabilizability of switched linear systems does not imply the existence of convex Lyapunov functions," Automatica, vol. 44, no. 4, pp. 1166–1170, Apr. 2008.*

Source: *M. Della Rossa, A. Tanwani and L. Zaccarian, "Max-Min Lyapunov Functions for Switching Differential Inclusions," 2018 IEEE Conference on Decision and Control (CDC), Miami, FL, USA, 2018, pp. 5664-5669.*