Polytopic LDI Example

$$\dot{x} = A(t)x$$
$$A(t) \in \mathrm{Co}\{A_1, A_2\}$$

$$A_1 = \begin{bmatrix} -100 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 8 & 9 \\ 120 & -18 \end{bmatrix}$$

polytopic LDI

$$V(x) = \max\left\{\|x\|_{P_1}^2, \|x\|_{P_2}^2\right\}$$
$$P_1 = \begin{bmatrix} 14 & -1 \\ -1 & 1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Robust Stability

Definition [Robust Stability]

$$\dot{x} = (A_0 + \Delta)x, \quad \forall \Delta \in \boldsymbol{\Delta}$$

is **RS** over $\boldsymbol{\Delta}$ if

$$A_0 + \Delta \quad \text{is Hurwitz} \quad \forall \Delta \in \boldsymbol{\Delta}$$

Definition [Robust Stability (LMI definition)]

$$\dot{x} = (A_0 + \Delta)x, \quad \forall \Delta \in \boldsymbol{\Delta}$$

is **RS** over $\boldsymbol{\Delta}$ if

$$\forall \Delta \in \boldsymbol{\Delta}, \exists P \succ \mathbf{0} : \langle P(A_0 + \Delta) \rangle_s \prec 0$$

Quadratic Stability

Definition [Robust Stability (LMI definition)]

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Quadratic Stability

Definition [Quadratic Stability (LMI definition)]

$$\dot{x} = A(t)x, \quad \forall A(t) \in Co\{A_1, \ldots, A_N\}, \quad \mathbb{I} \triangleq \{1, \ldots, N\}$$

is **QS** if

$$\exists P \succ \mathbf{0}, \forall i \in \mathbb{I} : \langle PA_i \rangle_s \prec 0$$

Theorem [Quadratic Stability (LMI definition)]

$$\dot{x} = A(t)x, \quad \forall A(t) \in Co\{A_1, \ldots, A_N\}, \quad \mathbb{I} \triangleq \{1, \ldots, N\}$$

if $\exists R_i, i \in \mathbb{I}$ s.t.

$$\sum_{i \in \mathbb{I}} \langle R_i A_i \rangle_s \succ \mathbf{0}$$

Then the system does not admit $P \succ \mathbf{0}$ s.t.

$$\langle PA_i \rangle_s \prec \mathbf{0}, \quad \forall i \in \mathbb{I}$$

S-Lemma

Lemma [S-Lemma]

$$Q(x; F, g, h) \triangleq x^\top F x + 2g^\top x + h$$

Assume $\boxed{\exists x_0 : Q(x_0; F_1, g_1, h_1) < 0}$, Then the implication

$$\boxed{Q(x; F_1, g_1, h_1) \leq 0} \implies \boxed{Q(x; F_2, g_2, h_2) \leq 0}$$

holds iff $\exists \lambda \geq 0$ s.t.

$$\lambda \begin{bmatrix} F_1 & g_1 \\ g_1^\top & h_1 \end{bmatrix} - \begin{bmatrix} F_2 & g_2 \\ g_2^\top & h_2 \end{bmatrix} \succeq \mathbf{0}$$

[S-Lemma Special Case]

$$\boxed{\|x\|_{P_1}^2 \leq \mathbf{0} \implies \|x\|_{P_2}^2 \leq \mathbf{0}} \iff \boxed{\exists \lambda \geq 0 : \lambda P_1 - P_2 \succeq \mathbf{0}}$$

Polytopic LDI Example

$$\dot{x} = A(t)x, \quad A(t) \in \mathrm{Co}\{A_1, A_2\}$$

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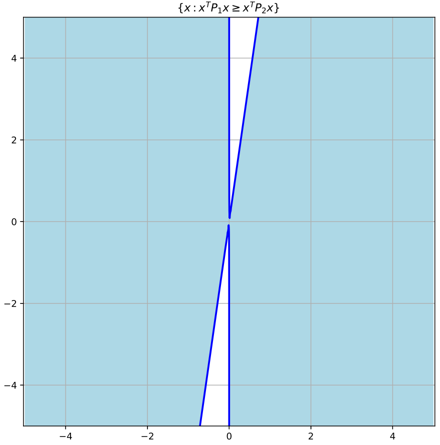
polytopic LDI

$$V(x) = \max\left\{\|x\|_{P_1}^2, \|x\|_{P_2}^2\right\}$$

$$P_1 = \begin{bmatrix} 14 & -1 \\ -1 & 1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Python Code and Figure Side-by-Side

```
import numpy as np
import matplotlib.pyplot as plt
# Define matrices
P1 = np.array([[14, -1],[-1, 1]])
P2 = np.array([[0, 0],[0, 1]])
# Create a grid of points
x_vals = np.linspace(-5, 5, 400)
y_vals = np.linspace(-5, 5, 400)
X, Y = np.meshgrid(x_vals, y_vals)
# Evaluate the inequality at each point
XY = np.stack([X, Y], axis=-1) # shape
      (400,400,2)
# Compute x’P1x and x’P2x
xP1x = (P1[0,0]*X**2 + (P1[0,1]+P1[1,0])*X*Y +
       P1[1,1]*Y**2)
xP2x = (P2[0,0]*X**2 + (P2[0,1]+P2[1,0])*X*Y +
       P2[1,1]*Y**2)
region = (xP1x >= xP2x)# Find the region
plt.figure(figsize=(8,8))# Plot
plt.contourf(X, Y, region, levels=[0.5, 1],
      alpha=0.5, colors=[’lightblue’])
plt.contour(X, Y, xP1x - xP2x, levels=[0],
      colors=’blue’, linewidths=2)
plt.xlabel(’$x_1$’);plt.ylabel(’$x_2$’);
plt.title("$\{x:x^{T} P_1 x \geq x^{T} P_2 x\}
      $")
plt.grid(True)
plt.axis(’equal’)
plt.show()
# Save the figure as EPS
plt.savefig(’reg_01.eps’, format=’eps’)
```



$\|x\|^2_{P_1} \geq \|x\|^2_{P_2}$

$$\begin{aligned}
\|x\|^2_{P_1} \geq \|x\|^2_{P_2} \quad &\Longrightarrow \quad \|x\|^2_{\langle P_1, \bar{A}\rangle_s} \leq 0 \\
\|x\|^2_{P_2} - \|x\|^2_{P_1} \leq 0 \quad &\Longrightarrow \quad \|x\|^2_{\langle P_1, \bar{A}\rangle_s} \leq 0 \\
\|x\|^2_{(P_2 - P_1)} \leq 0 \quad &\Longrightarrow \quad \|x\|^2_{\langle P_1, \bar{A}\rangle_s} \leq 0
\end{aligned}$$

$$\Updownarrow$$

$$\exists \lambda \geq 0 \quad : \quad \lambda(P_2 - P_1) - \langle P_1 \bar{A}\rangle_s \succeq \mathbf{0}, \quad \forall \bar{A} \in Co\{A_1, A_2\}$$

$$\Updownarrow$$

$$\begin{aligned}
\exists \lambda_1 \geq 0 : \lambda_1(P_2 - P_1) - \langle P_1 A_1\rangle_s \succeq \mathbf{0} \\
\exists \lambda_2 \geq 0 : \lambda_2(P_2 - P_1) - \langle P_1 A_2\rangle_s \succeq \mathbf{0}
\end{aligned}$$

$\|x\|_{P_2}^2 \geq \|x\|_{P_1}^2$

$$
\begin{aligned}
\|x\|_{P_2}^2 \geq \|x\|_{P_1}^2 \quad &\Longrightarrow \quad \|x\|^2_{\langle P_2, \bar{A}\rangle_s} \leq 0 \\
\|x\|_{P_1}^2 - \|x\|_{P_2}^2 \leq 0 \quad &\Longrightarrow \quad \|x\|^2_{\langle P_2, \bar{A}\rangle_s} \leq 0 \\
\|x\|^2_{(P_1 - P_2)} \leq 0 \quad &\Longrightarrow \quad \|x\|^2_{\langle P_2, \bar{A}\rangle_s} \leq 0
\end{aligned}
$$

$$\Updownarrow$$

$$\exists \lambda \geq 0 \quad : \quad \lambda(P_1 - P_2) - \langle P_2 \bar{A}\rangle_s \succeq \mathbf{0}, \quad \forall \bar{A} \in Co\{A_1, A_2\}$$

$$\Updownarrow$$

$$
\begin{aligned}
\exists \lambda_3 \geq 0 : \lambda_3(P_1 - P_2) - \langle P_2 A_1\rangle_s \succeq \mathbf{0} \\
\exists \lambda_4 \geq 0 : \lambda_4(P_1 - P_2) - \langle P_2 A_2\rangle_s \succeq \mathbf{0}
\end{aligned}
$$

Putting the conditions together

$$\exists \lambda_1 \geq 0 : \lambda_1(P_2 - P_1) - \langle P_1 A_1 \rangle_s \succeq \mathbf{0}$$
$$\exists \lambda_2 \geq 0 : \lambda_2(P_2 - P_1) - \langle P_1 A_2 \rangle_s \succeq \mathbf{0}$$
$$\exists \lambda_3 \geq 0 : \lambda_3(P_1 - P_2) - \langle P_2 A_1 \rangle_s \succeq \mathbf{0}$$
$$\exists \lambda_4 \geq 0 : \lambda_4(P_1 - P_2) - \langle P_2 A_2 \rangle_s \succeq \mathbf{0}$$

Feasible for $\boxed{\lambda_1 = 50, \lambda_2 = 0, \lambda_3 = 1, \lambda_4 = 100}$

SOURCE: LMIs in system and control theory
Stephen Boyd, Laurent El Ghaoui
Eric Feron, Venkataramanan Balakrishnan