Scaling of cluster heterogeneity in the two-dimensional Potts model

Jian-Ping Lv,[1] Xianqing Yang,[1] and Youjin Deng[2]
[1]*Department of Physics, China University of Mining and Technology, Xuzhou 221116, P. R. China*
[2]*Hefei National Laboratory for Physical Sciences at Microscale and Department of Modern Physics, University of Science and Technology of China, Hefei 230027, P. R. China*
(Received 15 June 2012; revised manuscript received 5 August 2012; published 27 August 2012)

Cluster heterogeneity, the number of clusters of mutually distinct sizes, has been recently studied for explosive percolation and standard percolation [H. K. Lee *et al.*, Phys. Rev. E **84**, 020101(R) (2011); J. D. Noh *et al.*, *ibid.* **84**, 010101(R) (2011)]. In this work we study the scaling of various quantities related with cluster heterogeneity in a broader context of two-dimensional q-state Potts model. We predict, via an analytic approach, the critical exponents for most of the measured quantities, and confirm these predications for various q values using extensive Monte Carlo simulations.

DOI: 10.1103/PhysRevE.86.022105 PACS number(s): 64.60.De, 68.35.Rh, 05.50.+q, 05.20.−y

The q-state Potts model plays an important role in the field of statistical mechanics and continues to attract intensive research attention [1,2]. Potts model can be defined on a finite graph $G = (V, E)$ with V (E) the vertex (edge) set, by placing on each vertex $i \in V$ a Potts spin $\sigma_i = 0, 1, \ldots, q-1$. The partition function of the Potts model reads

$$\mathcal{Z}_{\text{spin}} = \sum_{\{\sigma\}} \prod_{e_{ij}} e^{J\delta(\sigma_i,\sigma_j)}, \tag{1}$$

where e_{ij} is the edge connecting vertices i and j, δ is the Kronecker delta function, J is the coupling strength, and the sum is over all spin configurations. By the well-known Fortuin-Kasteleyn (FK) transformation [3], the Potts model can be mapped onto the random-cluster (RC) model [4] with partition sum

$$\mathcal{Z}_{\text{RC}} = \sum_{A \subseteq G} q^{k(A)} v^{|A|}, \tag{2}$$

where the sum is over all subgraphs of G, $|A|$ is the number of occupied edges, and $k(A)$ is the number of connected components (also called FK clusters). In this graphical representation, an occupied edge and a FK cluster have statistical weight $v = \exp(J) - 1$ and q, respectively, and thus q can take any real number with $q \geqslant 0$. In two dimensions, the RC model has a continuous and a first-order phase transition for $0 \leqslant q \leqslant 4$ and for $q > 4$, respectively.

The thermodynamic phase transition of the Potts model in the spin representation coincides with the geometric percolation threshold in the RC model, which for $q \to 1$, reduces to the standard bond percolation. Physical quantities in the Potts spins can be expressed in the form of geometric observables. Let s_i be the size of the ith cluster, and the nth moment of cluster-size distribution be

$$\mathcal{S}_n = \left\langle \sum_{i=1}^{k} s_i^n \right\rangle; \tag{3}$$

it can be shown that $\mathcal{S}_2/|V|$ is just the magnetic susceptibility, with $|V|$ the number of vertices. Energy-like quantities can be written as the occupied-bond number $|A|$, the cluster number $k(A)$, and the associated fluctuations. Therefore, the critical singularities of thermodynamic quantities are closely related to those of geometric observables, and they are governed by the same set of critical exponents. In the finite-size scaling (FSS) theory, one has at criticality [5]

$$\begin{aligned} &\mathcal{S}_n \propto L^{n y_h} \quad (n > 1), \\ &\rho_b \equiv \langle |A| \rangle / |E| \propto b_1 + a_1 L^{y_t - d}, \\ &\rho_k \equiv \langle k \rangle / |V| \propto b_2 + a_2 L^{y_t - d}, \\ &C_k \equiv (\langle k^2 \rangle - \langle k \rangle^2)/|V| \propto b_3 + a_3 L^{2y_t - d}, \end{aligned} \tag{4}$$

where d is the lattice dimension, L is the linear system size, $|E|$ is the number of edges, and y_t and y_h are the thermal and magnetic renormalization exponents, respectively. Parameters a_n and b_n ($n = 1$, 2, and 3) are nonuniversal constants, of which b_n accounts for background contribution. For the bond-percolation model ($q \to 1$), since bonds on different edges are uncorrelated, the nonuniversal parameters a_1, a_2, and a_3 are equal to 0, and the associated y_t-dependent terms vanish. In two dimensions, the exact values of y_t and y_h can be obtained by conformal field theory and Coloumb gas theory, and they read [6,7]

$$y_t = \frac{3g - 6}{g} \quad \text{and} \quad y_h = \frac{(g+2)(g+6)}{8g}, \tag{5}$$

where g is the Coloumb gas coupling strength and relates to q as $q = 4\cos^2(\pi g/4)$ with $2 \leqslant g \leqslant 4$.

Recently Lee and coworkers used a geometric quantity $\mathcal{K}_{\text{het}}$, the so-called cluster heterogeneity, to study explosive percolation [8] and standard percolation [9], and found that $\mathcal{K}_{\text{het}}$ is a good observable for locating the transition. The cluster heterogeneity $\mathcal{K}_{\text{het}}$ is defined as the number of clusters which have *mutually distinct* sizes. In other words, the clusters of the same size are counted only once. Besides $\mathcal{K}_{\text{het}}$, the scaling of other related quantities remains to be explored. Further, the bond percolation is a special case of the Potts model, where as mentioned above, the nonuniversal parameters a_1, a_2, and a_3 in Eq. (4) are zero. It is thus desirable to study a variety of observables associated with cluster heterogeneity in the general context of the q-state Potts model.

Theoretical predications. As in Ref. [9], we start with the critical scaling behavior of $\mathcal{K}(s, L) \equiv \langle k(s, L) \rangle$, with $k(s, L)$ the number of FK clusters of size s. It is known that at criticality, for a given cluster size s, $\mathcal{K}(s, L)$ behaves as [10]

$$\mathcal{K}(s, L) \sim L^d s^{-\tau} \, f(s/L^{y_h}), \tag{6}$$

where $\tau = 1 + d/y_h$ and $f(x) \approx \exp(-cx)$ is a universal function (c is a constant). Factor L^d comes from the fact that the total number of clusters is in order of volume L^d. Equation (6) suggests that $\mathcal{K}(s,L)$ decays algebraically for small s till some characteristic cluster size $s_h \sim L^{y_h}$. For small size $s \sim O(1)$, the cluster number $\mathcal{K}(s,L)$ is of order L^d, which, however, will be counted only once in defining the cluster heterogeneity $\mathcal{K}_{\rm het}$. For $s > s_h$ the cluster number $\mathcal{K}$ drops exponentially due to the limitation of finite size. This means that in a given configuration A and for a given size $s > s_h$, the probability is very *small* to find a cluster of size s, not to mention more than one cluster of size s. In fact, one expects that the counting-once effect would be already negligible for $s > s_{\rm het}$, with $s_{\rm het} < s_h$ another characteristic size. To find $s_{\rm het}$ for a given configuration A, we rank the FK clusters according to their size such that $s_i \geqslant s_{i+1}$ for $i \in [1, k(A) - 1]$. Then the average size $\langle s_i \rangle$ relates to i as [11]

$$i \sim \int_{\langle s_i \rangle}^{L^d} ds\ L^d s^{-\tau} f(s/L^{y_h}). \tag{7}$$

Straightforward calculation can yield the average-size difference of the $(i+1)$th and the ith cluster $\Delta_s \equiv \langle s_i \rangle - \langle s_{i+1} \rangle$. Since

$$i + 1 \sim \int_{\langle s_{i+1} \rangle}^{L^d} ds\ L^d s^{-\tau} f(s/L^{y_h}), \tag{8}$$

the subtraction of Eq. (7) from Eq. (8) gives

$$1 \sim \Delta_s L^d s_i^{-\tau} f(s_i/L^{y_h}). \tag{9}$$

Therefore, one has

$$\Delta_s \sim L^{-d} s_i^{\tau} / f(s_i/L^{y_h}). \tag{10}$$

The definition of $s_{\rm het}$ suggests that near $s_{\rm het}$, the size difference is $\Delta_s \sim O(1)$, and thus one has

$$s_{\rm het} \sim L^{d/\tau}. \tag{11}$$

Since $d/\tau = y_h/(1 + y_h/d) < y_h$, our expectation $s_{\rm het} < s_h$ holds for large L. The rank $i_{\rm het}$ for cluster of size $s_{\rm het}$ can be derived by combing Eqs. (7) and (11), which reads

$$i_{\rm het} \sim \int_{s_{\rm het}}^{L^d} ds\ L^d s^{-\tau}\ f(s/L^{y_h}) \tag{12}$$

$$\sim L^d \int_{s_{\rm het}}^{\infty} ds\ s^{-\tau} \sim L^{d/\tau}, \tag{13}$$

where we have neglected the cluster number for $s > s_h$. It is interesting to observe that the characteristic rank and size, $i_{\rm het}$ and $s_{\rm het}$, have the same scaling behavior.

In the cluster heterogeneity, the average number of clusters of size s is given by

$$\mathcal{K}_{\rm het}(s,L) = \langle k_{\rm het}(s,L) \rangle. \tag{14}$$

Here $k_{\rm het}(s,L)$ is defined as follows: for a given configuration, if there exists at least one cluster of size s, $k_{\rm het}(s,L) = 1$; otherwise, $k_{\rm het}(s,L) = 0$. Accordingly, one has $\mathcal{K}_{\rm het}(s,L) \sim 1$ for $s \leqslant s_{\rm het}$, while for $s > s_{\rm het}$, $\mathcal{K}_{\rm het}(s,L)$ is still described by Eq. (6). On this basis, the FSS of various quantities can be obtained. First, the cluster heterogeneity is just

$$\mathcal{K}_{\rm het}(L) = s_{\rm het}(L) + i_{\rm het}(L) \sim L^{d/\tau}. \tag{15}$$

Let the nth moment of cluster size in the cluster heterogeneity be

$$\mathcal{S}_{n,\rm het} = \sum_s \mathcal{K}_{\rm het}(s,L) s^n, \tag{16}$$

we have

$$\mathcal{S}_{n,\rm het} = \sum_{s=1}^{s_{\rm het}} s^n + a \sum_{s=s_{\rm het}}^{L^d} L^d s^{n-\tau} f(s/L^{y_h})$$

$$\sim L^{(n+1)d/\tau} + b L^{n y_h}, \tag{17}$$

where a and b are nonuniversal constants. Note that Eq. (17) holds for all integers $n \geqslant 1$. The leading scaling behavior of $\mathcal{S}_{n,\rm het}(L)$ will depend on the relative amplitude of $(n+1)d/\tau$ and $n y_h$. It follows that

$$\mathcal{S}_{n,\rm het} \sim \begin{cases} L^{(n+1)d/\tau} & [(n+1)d/\tau \geqslant n y_h], \\ L^{n y_h} & [(n+1)d/\tau < n y_h]. \end{cases} \tag{18}$$

Above theoretical predictions are valid for the continuous phase transitions of Potts model in arbitrary dimensions.

In following, we restrict ourselves to the two dimensions. According to Eq. (5), we have $1 + \sqrt{3}/2 \leqslant y_h \leqslant 2$. Thus, the relative amplitude in Eq. (18) can be evaluated as follows: for $n = 1$, since $y_h \leqslant 2$, one has $2d/\tau > y_h$; while for $n \geqslant 2$, since $y_h \geqslant 1 + \sqrt{3}/2$, one has $(n+1)d/\tau < n y_h$. Therefore, the FSS behavior of $\mathcal{S}_{n,\rm het}$ is described by

$$\mathcal{S}_{n,\rm het} \sim \begin{cases} L^{2d/\tau} & (n = 1), \\ L^{n y_h} & (n \geqslant 2). \end{cases} \tag{19}$$

Monte Carlo simulation. We examine the theoretical predictions by Monte Carlo simulating for the q-state Potts model on the square lattice for $q = 1$, 2, 3, 3.5, and 4. Periodic boundary conditions are implemented, and the simulations are carried out right at the critical point $v = \sqrt{q}$ [1] with system size up to $L = 1024$. To suppress the critical slowing down, we apply the Chayes-Machta-Swendsen-Wang cluster algorithm [12,13].

We plot in Fig. 1 $\mathcal{K}_{\rm het}(s,L)$ versus $s/L^{d/\tau}$ for $q = 3$. As predicted, there are two characteristic sizes $s_{\rm het} \sim L^{d/\tau}$ and $s_h \sim L^{y_h}$. For $s < s_{\rm het}$ one has $\mathcal{K}_{\rm het}(s,L) = 1$, and for $s > s_h$ $\mathcal{K}_{\rm het}(s,L)$ drops exponentially as s increases. In the intermediate region, $\mathcal{K}_{\rm het}(s,L)$ decays algebraically with exponent $-\tau$.

We measure $\mathcal{S}_{n,\rm het}$ with $n = 1$, 2, 3, and 4, as well as $\mathcal{K}_{\rm het}$. According to the least-squares criterion, the Monte Carlo data of these quantities are fitted by

$$O = c_1 L^X + c_2, \tag{20}$$

where c_1 and c_2 are unknown parameters, and X is the associated critical exponent. In the fitting procedure, the data for small system size $L < L_{\rm min}$ are gradually excluded to check how the residual χ^2 changes with respect to $L_{\rm min}$. The fitting results are considered reasonable if the χ^2 value is approximately equal to the degrees of freedom and does not drop dramatically as the data for one more system size are excluded.

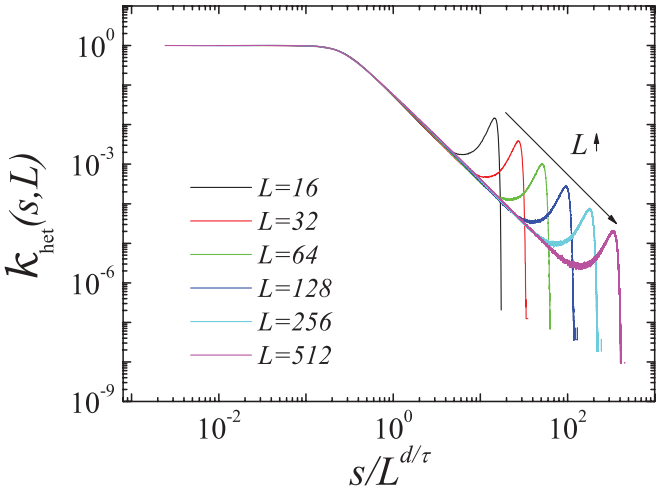

FIG. 1. (Color online) Cluster number $\mathcal{K}_{\rm het}(s,L)$ versus $s/L^{d/\tau}$ for $q=3$ at criticality.

The results are shown in Table I. For all the measured quantities and all the q values, the agreement with the theoretical prediction is good. It is worth mentioning that, due to the presence of logarithmic corrections [14], most of the numerical results for $q=4$ are less precise than those for $q=1$, 2, 3, and 3.5.

TABLE I. Numerical estimates of critical exponents. The theoretical values are rounded to four decimal places.

q	O	Theory	Simulation	$L_{\rm min}$
1	$\mathcal{K}_{\rm het}$	0.9733	0.973(1)	128
1	$\mathcal{S}_{1,\rm het}$	1.9465	1.947(1)	128
1	$\mathcal{S}_{2,\rm het}$	3.7917	3.792(1)	128
1	$\mathcal{S}_{3,\rm het}$	5.6875	5.688(1)	128
1	$\mathcal{S}_{4,\rm het}$	7.5833	7.584(1)	128
2	$\mathcal{K}_{\rm het}$	0.9677	0.968(1)	128
2	$\mathcal{S}_{1,\rm het}$	1.9355	1.934(2)	128
2	$\mathcal{S}_{2,\rm het}$	3.7500	3.751(2)	128
2	$\mathcal{S}_{3,\rm het}$	5.6250	5.625(1)	128
2	$\mathcal{S}_{4,\rm het}$	7.5000	7.501(1)	128
3	$\mathcal{K}_{\rm het}$	0.9655	0.966(1)	256
3	$\mathcal{S}_{1,\rm het}$	1.9310	1.930(1)	256
3	$\mathcal{S}_{2,\rm het}$	3.7333	3.733(1)	256
3	$\mathcal{S}_{3,\rm het}$	5.6000	5.601(1)	256
3	$\mathcal{S}_{4,\rm het}$	7.4667	7.467(2)	256
3.5	$\mathcal{K}_{\rm het}$	0.9654	0.964(1)	128
3.5	$\mathcal{S}_{1,\rm het}$	1.9308	1.927(2)	128
3.5	$\mathcal{S}_{2,\rm het}$	3.7325	3.732(1)	128
3.5	$\mathcal{S}_{3,\rm het}$	5.5987	5.599(1)	128
3.5	$\mathcal{S}_{4,\rm het}$	7.4649	7.465(1)	128
4	$\mathcal{K}_{\rm het}$	0.9677	0.97(1)	128
4	$\mathcal{S}_{1,\rm het}$	1.9355	1.93(1)	128
4	$\mathcal{S}_{2,\rm het}$	3.7500	3.75(1)	128
4	$\mathcal{S}_{3,\rm het}$	5.6250	5.62(1)	128
4	$\mathcal{S}_{4,\rm het}$	7.5000	7.49(1)	128

We also measured a specific-heat-like quantity that accounts for the fluctuation of $\mathcal{K}_{\rm het}$

$$C_{k,\rm het} = L^{-d}\left(\langle \mathcal{K}_{\rm het}^2\rangle - \langle \mathcal{K}_{\rm het}\rangle^2\right). \tag{21}$$

For this quantity, unfortunately, the critical FSS has not been theoretically derived.

We find for all $q=1$, 2, 3, 3.5, and 4, the $C_{k,\rm het}$ value converges to a finite constant as L increases. This behavior is different from that of C_k defined in Eq. (4), which diverges as $C_k \sim \ln L$ for $q=2$, $\sim L^{5/2}$ for $q=3$, $\sim L^{0.610\ldots}$ for $q=3.5$, and $\sim L^1$ for $q=4$. The Monte Carlo data are also fitted by Eq. (20), and we have the correction exponent $X=-1.00(1)$ for $q=1$, $-0.96(1)$ for $q=2$, $-0.76(1)$ for $q=3$, $-0.66(1)$ for $q=3.5$, and $-0.5(1)$ for $q=4$. We note that, the result for $q=4$ is less precise than that for other cases. This is also caused by the logarithmic corrections [14]. Assuming that in the cluster heterogeneity, the thermal exponent y_t is renormalized as $y_t^* = y_t/(1+y_h/d)$, one has $C_{k,\rm het} \sim L^{2y_t^*-d}$. This would yield $X=-1.229\ldots$ for $q=1$, $-0.967\ldots$ for $q=2$, $-0.758\ldots$ for $q=3$, $-0.649\ldots$ for $q=3.5$, and $-0.451\ldots$ for $q=4$, in good agreement with the numerical results for $q=2$, 3, 3.5, and 4. For $q=1$, since $-1.229\ldots$ is smaller than -1, it is conceivable that the leading correction exponent is -1 instead of $2y_t^*-d$.

As an illustration, we show in Fig. 2 the size-dependent behavior of measured quantities for $q=3$.

Discussion. In this work we study the critical finite-size scaling of various quantities related to cluster heterogeneity for the two-dimensional Potts model. The Monte Carlo results are compared to the theoretical predictions, and good agreement is found. We mention that unlike the standard geometric observables, quantities related to cluster heterogeneity cannot be directly mapped onto thermodynamic quantities. Thus, the physical significance is not clear yet.

Note that the cluster heterogeneity $\mathcal{K}_{\rm het}$ is found to be useful to the study of explosive percolation (EP) [8]. EP is a recently proposed theoretical model which has attracted a

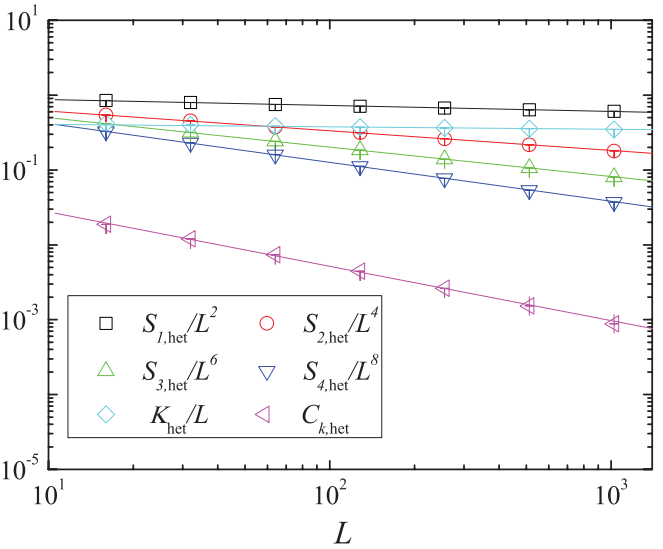

FIG. 2. (Color online) Various quantities versus L for $q=3$ at criticality. The solid lines are drawn according to the critical exponents obtained from the theoretical predications.

lot of research attention [15–20]. It is now generally accepted that the phase transition in EP is continuous [18–20]. Thus, it is desirable to study the scaling behavior of the measured quantities for EP, which may hopefully bring more information on its phase transition.

Acknowledgments. The work of J.P.L. was supported by the Fundamental Research Funds for the Central Universities (Grant No. 2012QNA43) and NSFC (Grant No. 11147013). The work of Y.D. was supported by NSFC (Grant No. 10975127) and CAS.

[1] F. Y. Wu, Rev. Mod. Phys. **54**, 235 (1982), and references therein.
[2] R. B. Potts, Proc. Cambridge Philos. Soc. **48**, 106 (1952).
[3] C. Fortuin and P. Kasteleyn, Physica **57**, 536 (1972).
[4] G. Grimmett, *The Random Cluster Model* (Springer-Verlag, New York, 2006).
[5] M. N. Barber, in *Finite-Size Scaling* in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J. L. Lebowitz, Vol. 8 (Academic Press, London, 1983) p. 167.
[6] B. Nienhuis, J. Stat. Phys. **34**, 731 (1984).
[7] M. P. M. den Nijs, Phys. Rev. B **27**, 1674 (1983).
[8] H. K. Lee, B. J. Kim, and H. Park, Phys. Rev. E **84**, 020101(R) (2011).
[9] J. D. Noh, H. K. Lee, and H. Park, Phys. Rev. E **84**, 010101(R) (2011).
[10] K. Christensen and N. R. Moloney, *Complexity and Criticality* (Imperial College Press, London, 2005), p. 39.
[11] N. Jan, D. Stauffer, and A. Aharony, J. Stat. Phys. **92**, 325 (1998).
[12] R. H. Swendsen and J. S. Wang, Phys. Rev. Lett. **58**, 86 (1987).
[13] L. Chayes and J. Machta, Physica A **254**, 477 (1998).
[14] J. Salas and A. D. Sokal, J. Stat. Phys. **88**, 567 (1997).
[15] D. Achlioptas, R. M. D' Souza, and J. Spencer, Science **323**, 1453 (2009).
[16] R. M. Ziff, Phys. Rev. Lett. **103**, 045701 (2009); Phys. Rev. E **82**, 051105 (2010).
[17] N. A. M. Araújo, J. S. Andrade, R. M. Ziff, and H. J. Herrmann, Phys. Rev. Lett. **106**, 095703 (2011).
[18] O. Riordan and L. Warnke, Science **333**, 322 (2011).
[19] P. Grassberger, C. Christensen, G. Bizhani, S. W. Son, and M. Paczuski, Phys. Rev. Lett. **106**, 225701 (2011).
[20] L. Tian and D. N. Shi, Phys. Lett. A **391**, 1234 (2012).