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Analytical solution for a steady flow of enclosed rotating disks

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1. Introduction

In a microgravitational environment, fluids may be handled adequately by inducing a small centrifugal force. Experimental results [1] have shown the decisive importance of the fluid dynamical behavior of the multi-disk stirrer for low-Reynolds number flow. In such a disk-cylinder system, referred to as *Quirl-tank*, several circular disks rotate in a container filled with a viscous fluid. Further, the experiments have shown that the flow pattern occurred in one compartment does not interfere with that of the neighbouring one. Thus representatively, the flow between two confined rotating disks is investigated here.

The first attempt of a solution for an infinite rotating disk in a viscous fluid was made by von Kármán [2] and has occupied a prime position in the literature of rotating disks. The bulk of the investigations in this area deal with infinite rotating disks in an unbounded fluid whereas those dealing with confined finite disk flows are limited.

The flow due to a rotating disk in a cylindrical container has been first investigated by Schultz Grunow [3]. He has presented an analytical solution of the flow without satisfying the boundary conditions on the finite cylindrical container. The agreement of his results with the experimental results is very good, provided the aspect ratio $h/l(\ll 1)$ is very small, h and l being the height and radius of the cylindrical container. The low-Reynolds number flow in a particular disk-cylinder system with one disk rotating, the other along with the cylindrical container at rest, was given by Pao [4].

From the practical point of view, however, the most important problems involve conditions on the outer boundaries and different rotation rates. It is found experimentally [1] that the case of counter-rotating disks and also the case in which disks and container rotate in opposite directions are of more practical interest as the total moment of inertia vanishes.

Following this, here, an analytical solution for the general case of boundary conditions and aspect ratios in confined disk-cylinder systems is given. The solution is presented in terms of Fourier series and Bessel functions. The significant result is that the solution for the flows in confined rotating disks is unique, unlike the similarity solutions for the unbounded case. An expression for the torque acting on the disks is also obtained. The results bringing out the structure of the flow development are shown in figures for different cases of physical interest.

2. Formulation of the problem

Consider the steady, laminar and axisymmetric flow of an incompressible viscous fluid between two rotating disks confined by a cylinder. In cylindrical polar coordinates (r, θ, z) the disks are chosen to be situated at $z = 0$, h and the cylindrical container is given by $r = l$. When the average radial and axial velocity components u, w are very small compared with the tangential velocity component v ($u, w \ll v$, which corresponds to the case of small Reynolds number, $\mathrm{Re} \leq 10$), the Navier-Stokes equations reduce to a single equation for the tangential velocity. Using the nondimensional variables defined by

$$\bar{v} = \frac{v}{l\omega}, \qquad \bar{r} = \frac{r}{l}, \qquad \bar{z} = \frac{z}{h}, \tag{1}$$

the governing equation for tangential velocity is (bars dropped)

$$\delta^2 \left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{v}{r^2} \right) + \frac{\partial^2 v}{\partial z^2} = 0, \tag{2}$$

with $\delta = h/l$ being the aspect ratio.

The corresponding boundary conditions for the case of lower and upper disk rotating with ω_1 and ω_2, respectively and the cylindrical container at rest are

$$\left.\begin{array}{ll} v = r & \text{on } z = 0 \\ v = sr & \text{on } z = 1 \end{array}\right\} \quad 0 \leq r < 1, \tag{3}$$

$$\left.\begin{array}{ll} v = 0 & \text{on } r = 1 \\ v = 0 & \text{on } r = 0 \end{array}\right\} \quad 0 < z < 1, \tag{4}$$

where $s = \omega_2/\omega_1$.

3. Analytical solution

Upon substitution of $v_1(r, z) = f(z) \cdot r$ in eq. (2), one obtains

$$v_1(r, z) = [(s-1)z + 1]r, \tag{5}$$

which satisfies the boundary conditions at the axis of symmetry and the disks, whereas it produces the non-vanishing function $g(z) = v_1(r = 1, z) = (s-1) \cdot z + 1$ at the cylindrical container.

One now has to seek for a solution $v_2(r, z)$ of (2), which satisfies the boundary conditions

$$v_2(r, 0) = 0, \qquad v_2(r, 1) = 0, \qquad v_2(0, z) = 0, \tag{6}$$

$$v_2(1, z) = -g(z). \tag{7}$$

Because of the linearity of the problem, if $v_1(r, z)$ and $v_2(r, z)$ each separately satisfies equation (2), then so does $v_1(r, z) + v_2(r, z)$.

By taking $v_2(r, z) = f_1(z) \cdot f_2(r)$ and introducing the separation constant λ, eq. (2) provides two ordinary differential equations for f_1 and f_2 given by

$$-\frac{1}{f_1}\frac{d^2 f_1}{dz^2} = \lambda^2 \tag{8}$$

$$\frac{\delta^2}{f_2}\left(\frac{d^2 f_2}{dr^2} + \frac{1}{r}\frac{df_2}{dr} - \frac{f_2}{r^2}\right) = \lambda^2 \quad \text{(const)}. \tag{9}$$

The general solution of (8) satisfying the boundary conditions at the disks is

$$f_1(z) = A_n \sin n\pi z, \tag{10}$$

whereas the solution of (9) is given by

$$f_2(r) = c I_1(\lambda_n r), \tag{11}$$

in which c is an arbitrary constant and I_1 is the modified Bessel function of the first kind and first order with $\lambda_n = n\pi/\delta$.

Thus $v_2(r, z)$ is given by

$$v_2(r, z) = \sum_{n=1}^{\infty} A_n I_1(\lambda_n r) \sin n\pi z. \tag{12}$$

The Fourier coefficients A_n in (12) are determined using (7) and are given by

$$A_n = \frac{2}{n\pi I_1(\lambda_n)}[(-1)^n s - 1]. \tag{13}$$

Finally, the solution of (2) satisfying (3) and (4) is

$$v(r, z) = [(s-1)z + 1]r + \sum_{n=1}^{\infty} A_n I_1(\lambda_n r) \sin n\pi z. \tag{14}$$

To obtain the solution for the general case in which each of the disks and the cylindrical container rotate with different, but constant angular velocities ($\omega_1, \omega_2, \omega_3$, respectively), the parameters s_i ($s_i = \omega_i/\omega, i = 1, 2, 3$) are introduced with ω being some characteristic rotation speed. A similar approach as described above will lead to

$$v_1(r, z) = [(s_2 - s_1)z + s_1]r, \tag{15}$$

and $v_2(r, z)$ remains the same with the Fourier coefficients

$$A_n = \frac{2}{n\pi I_1(\lambda_n)}[(s_2 - s_3)(-1)^n + (s_3 - s_1)]. \tag{16}$$

Finally, one obtains

$$v(r, z) = [(s_2 - s_1)z + s_1]r + \sum_{n=1}^{\infty} A_n I_1(\lambda_n r) \sin n\pi z \tag{17}$$

as the solution of (2) satisfying the general boundary conditions mentioned above with A_n given in (16).

It is easily seen from (16) and (17) that the case $s_1 = s_2 = s_3$ corresponds to that of rigid body rotation. If the angular velocity of the lower disk is used as the characteristic velocity and the cylindrical container is at rest, then $s_1 = 1$, $s_2 = \omega_2/\omega_1$ and $s_3 = 0$. By taking $s_1 = 1$, $s_2 = s$ and $s_3 = 0$ in (17), the solution discussed by Khalili [5] is given. The solution given by Pao [4] corresponds to $s_2 = s_3 = 1$ and $s_1 = 0$ in (17), but he presents the solution in terms of Bessel series. Both Khalili and Pao have presented their solutions in terms of circulation $\Omega(r, z)$ which is $r \cdot v(r, z)$. The angular or azimuthal velocity of the flow $\omega(r, z)$ is $v(r, z)/r$.

Another physical quantity of interest is the torque acting on the disks at $z = 0$ or h, which is given by

$$M = -2\pi\mu \int_0^l r^2 \left.\frac{\partial v}{\partial z}\right|_{z=0 \text{ or } h} \cdot dr. \tag{18}$$

By substituting (17) in (18), the nondimensional torque on the disk $z = 0$ is

$$\bar{M} = -2\pi \left[\frac{s_2 - s_1}{4} + \sum_{n=1}^{n} \frac{A_n I_2(\lambda_n)}{\lambda_n I_1(\lambda_n)}\right], \tag{19}$$

where I_2 is the modified Bessel function of order 2. The above results (eqs. (17) and (19)) are valid only when the Reynolds number is small. For higher values of the Reynolds number, the radial and axial velocities are not negligible and the nonlinear terms become very important. These effects are considered both analytically and numerically in a subsequent paper.

4. Convergence of the series and discussion of the results

The circulation $\Omega(r, z)$, the tangential velocity $v(r, z)$ and the angular velocity $\omega(r, z)$ are computed using (17) for different rotation rates s_1, s_2 and s_3 of the disks and the container, and also for different aspect ratio δ. It is observed that the series in (17) converges at the truncation order $n = 120$ for the values of δ considered. Below, a few typical pictures giving the flow structure are presented. Investigations made for the case $s_1 = 1$, $s_2 = 1$, $s_3 = 0$ and $\delta = 1$ have shown that the lines of constant circulation, tangential and angular velocity have the similar type of behavior. Thus, the lines of constant tangential velocity only are depicted in the discussion. The lines of constant angular velocity for this case given in Fig. 1 agree very well with those of Duck [6] which were given by using a numerical method when Reynolds number $\mathrm{Re} = 0$. It is observed that these lines are symmetric with respect to the middle plane $z = 1/2$.

Lines of constant tangential velocity when $s_1 = s_2 = 1$, $s_3 = 0$ for different values of δ are shown in Fig. 2. It is easily seen from Fig. 1 for $\delta = 1$, Fig. 2(a) for $\delta = 4$ and Fig. 2(b) for $\delta = 2$, that with an increase in the aspect ratio the flow moves towards the disks leaving the core at rest.

Whereas from Fig. 1 for $\delta = 1$, Fig. 2(c) for $\delta = 0.5$, Fig. 2(d) for $\delta = 0.25$ it is observed that with a decrease in aspect ratio, the flow near the disk moves towards the cylinder at rest. Further, the region of rigid rotation near the axis increases with a decrease in aspect ratio. From Fig. 2(d) is clear that more than 50% of the region near the axis is in state of rigid rotation and in the other region the tangential velocity decreases and becomes zero on the cylindrical container.

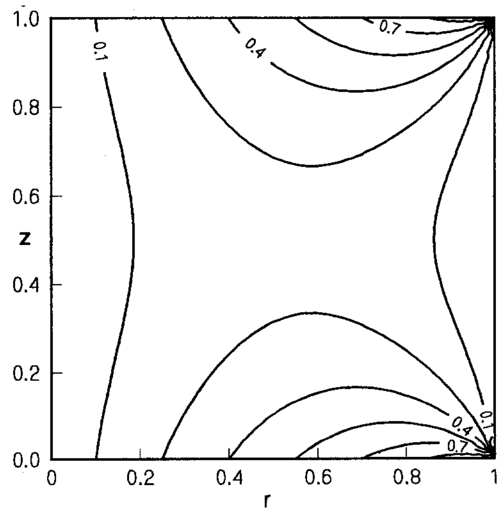


Figure 1
Lines of constant tangential velocity for $s_1 = 1$, $s_2 = 1$, $s_3 = 0$, $\delta = 1$.

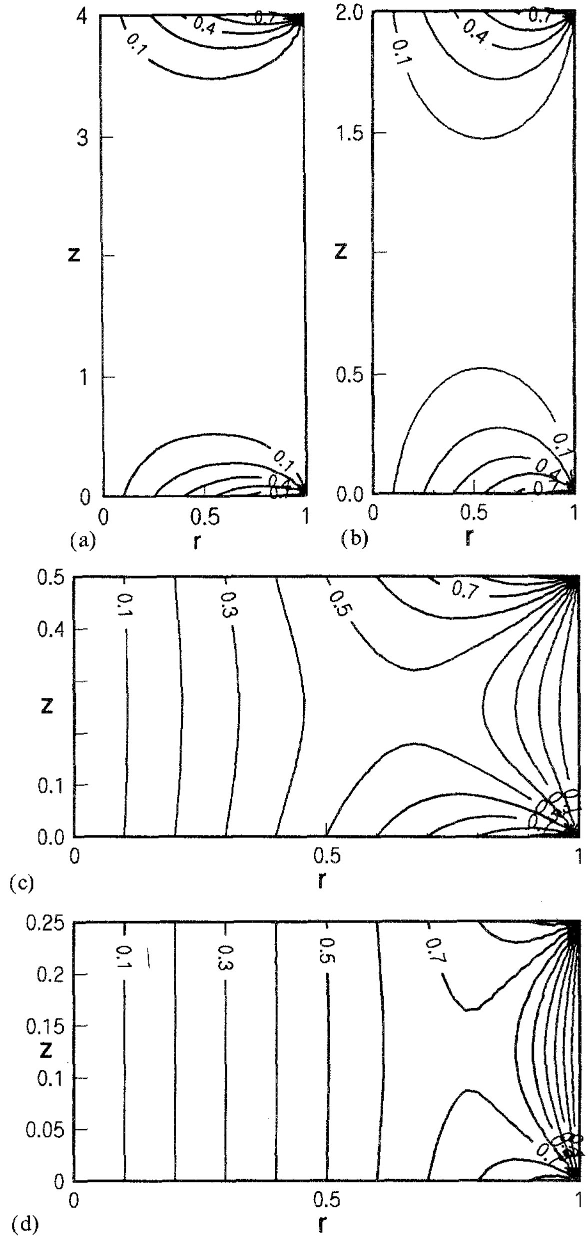


Figure 2
Lines of constant tangential velocity for $s_1 = 1$, $s_2 = 1$, $s_3 = 0$ and (a) $\delta = 4$ (b) $\delta = 2$ (c) $\delta = 0.5$ (d) $\delta = 0.25$.

In the following, the effect of aspect ratio δ on the tangential velocity is discussed. We depict in Fig. 3 the lines of constant tangential velocity distributions for fixed $\delta = 0.5$, $s_1 = 1$, $s_3 = 0$ and for different s_2. The flow is not symmetric about $z = 0.5$ as demonstrated in Fig. 3(a), (b) and (c). Fig.

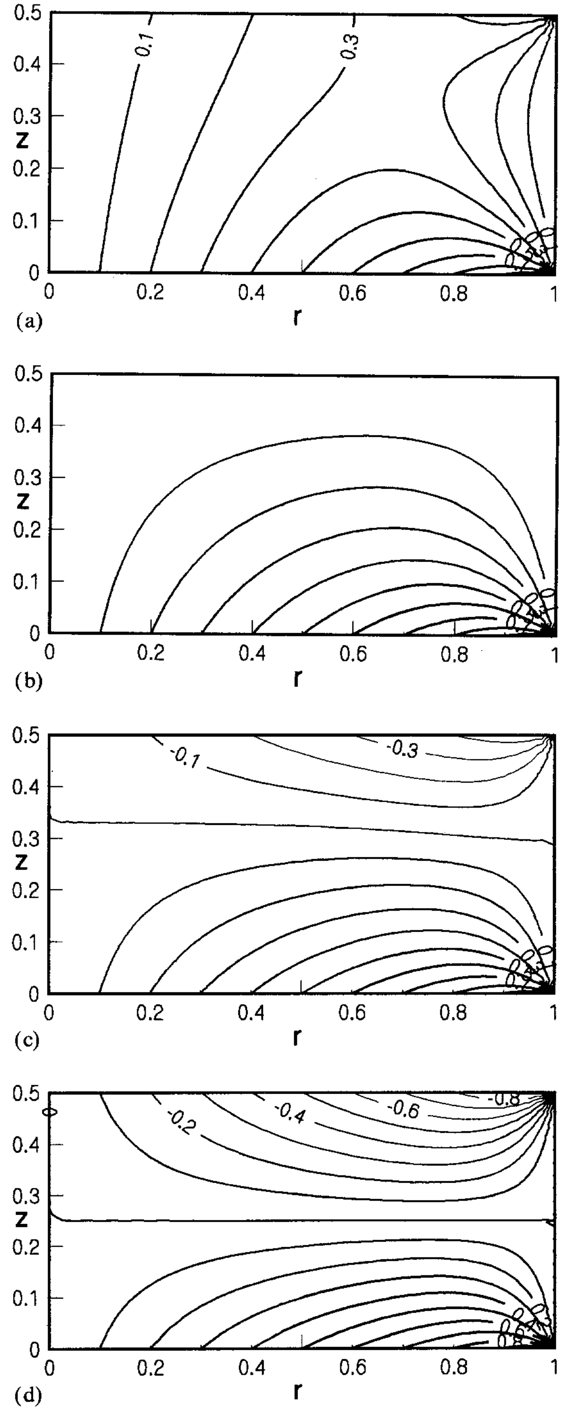


Figure 3
Lines of constant tangential velocity for $s_1 = 1$, $s_3 = 0$, $\delta = 0.5$ and (a) $s_2 = 0.5$ (b) $s_2 = 0$ (c) $s_2 = -0.5$ (d) $s_2 = -1$.

3(a) and 3(b) show the changes in flow pattern as s_2 decreases from 0.5 to 0. For negative values of s_2, a line of zero tangential velocity separates the flow into two counter-rotating regions as seen from Fig. 3(c) for $s_2 = -0.5$ and this shifts to $z = 0.5$ for $s_2 = -1$, Fig. 3(d).

In connection with Fig. 3(b) we recall the situation, which is related to the original von Kármán problem, where the fluid at infinity is in solid body rotation while the disk (at $z = 0$) is at rest. This problem was first considered by Bödewadt [7] and later by Mellor *et al.* [8] who generalized the Bödewadt problem and studied the steady axisymmetric flow of a viscous incompressible fluid between two infinite coaxial disks, one rotating and one stationary. As shown in [8], the solution of the Bödewadt problem is non-unique and solutions of both the Batchelor [9] and Stewartson [10] type may occur for different Reynolds number. For the problem discussed in the present paper, however, a unique solution for the well-defined differential equation (2) exists, for the boundary conditions (3) and (4). Another significant difference between an infinite rotating disk and a confined one can be seen from the tangential velocity distribution depicted in Figs. 4(a) and 4(b). In terms of tangential velocity, the von Kármán similarity hypothesis was written by Bödewadt [7] in the form

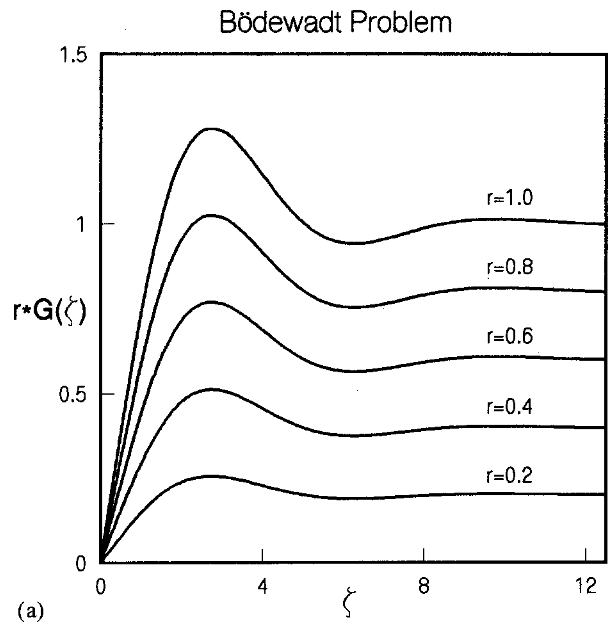
$$v = r \cdot g(z) \quad \text{with} \quad g(z) = \omega \cdot G(\zeta). \tag{20}$$

With ζ denoting the non-dimensional length scale $\zeta = \sqrt{\omega/\nu}$ (ν represents the coefficient of viscosity), the tangential velocity was given by

$$v = \omega \cdot r \cdot G(\zeta). \tag{21}$$

In Fig. 4(a), $r \cdot G(\zeta)$ has been plotted as a function of ζ for different values of r. Whereas, Fig. 4(b) exhibits the tangential velocity v for the case of a confined disk as a function of r for different values of z. In contrast to the infinite disk case, the tangential velocity for the confined disk does not exhibit oscillation and behaves monotonically. Furthermore, as the fluid particles approach the cylindrical shroud, the linear dependency of the tangential velocity on r also vanishes and v reduces to 0 at the shroud (see Fig. 4(b)) unlike the unbounded case presented in Fig. 4(a).

The case presented in Fig. 3(b), where the fluid is driven by the constant rotation of the bottom endwall, also gives rise to the known phenomenon of swirling flows in confined geometries [11], susceptible to the occurrence of a phenomenon known as vortex breakdown, which is associated with an abrupt change in the character of the vortex near the axial region and is followed by a region of recirculation. In the situation described in this paper, however, vortex breakdown can not occur as the flow is unidirectional (in the azimuthal direction) and the radial and axial components of the velocity vector and consequently, the azimuthal component of vorticity vector, do not exist.



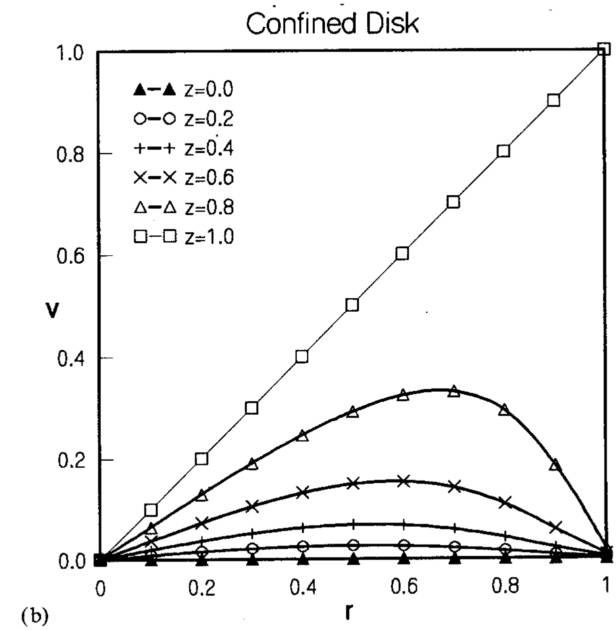


Figure 4
Tangential velocity distribution $s_1 = 1$, $s_2 = 0$, $s_3 = 0$, $\delta = 1.0$ (a) (b).

The flow pattern is also investigated for different values of s_3. The significant result is that, when the container rotates in opposite direction, a counter rotating region is developed near the container and is intensified for larger negative values of s_3.

It is interesting to note that at high rotation rates, the above mentioned effects of the aspect ratio δ on the flow structure is not discussed amply in the literature. This aspect needs further careful examination and is the aim of a future study.

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Abstract

Low-Reynolds-number flow plays an important role in the centrifugal separation of fluid particles under microgravity conditions and also in micromechanics due to the miniaturization of fluid mechanical parts. In this situation, the governing equations may be simplified. Here an analytical solution is presented for the steady flow of an incompressible viscous fluid between two finite disks enclosed by a cylindrical container for small Reynolds number ($\mathrm{Re} \leq 10$). The general solution is valid for all choices of the aspect ratio (δ) and different cases of disk to cylinder rotation rates (s). An expression for the torque acting on the disk is obtained. The tangential velocity distribution is calculated and presented graphically for different values of δ and s. Known results in the literature for a single rotating disk and similar problems follow as a particular case of the general solution presented.

Zusammenfassung

Zahlreiche hydrodynamische Vorgänge unter der Bedingung verminderter Schwerkraft aber auch Vorgänge in der Mikromechanik finden im Bereich kleiner Reynoldszahlen statt. In solchen Situationen können die Bewegungsgleichungen vereinfacht und eventuell analytische Lösungen gefunden werden. In dieser Arbeit wird die stationäre Strömung einer viskosen, inkompressiblen Flüssigkeit für kleine Reynolds- und unterschiedliche Aspektzahlen untersucht. Die Flüssigkeit ist zwischen zwei rotierenden Scheiben und einem zylindrischen Behälter eingeschlossen. Eine analytische Lösung für die Tangentialkomponente des Geschwindigkeitsvektors ist für den allgemeinen Fall, dass die Scheiben und der Behälter unterschiedliche Winkelgeschwindigkeiten besitzen können, dargestellt. Des weiteren wurde eine Beziehung für das Widerstandsmoment der rotierenden Scheibe angegeben. Der Verlauf der Tangentialgeschwindigkeiten für verschiedene Rotations- und Aspektverhältnisse wird graphisch dargestellt und diskutiert. Bereits angegebene Lösungen in der Literatur bezüglich dieser Geometrie können als Sonderfall der hier dargestellten Lösung entwickelt werden.

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