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Growth of Disturbances in Both Space and Time

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Betchov and Criminale found certain singularities in the eigenvalue relations for the combined stability problem in space and time. It is shown why these singularities must occur, and how they influence the perturbation generated by a pulse.

I. INTRODUCTION

The behavior of all possible perturbations of a given mean flow determine whether that flow is stable or not. In the case of parallel flows, or flows such as boundary layers or wakes which are approximately parallel, the most general two-dimensional perturbation takes the form of a traveling wave

$$\psi(y; x, t) = \Phi(y)e^{i(\alpha x-\beta t)}; \qquad (1)$$

ψ is a stream function, α is the wavenumber, and β the frequency of the motion. The mean flow, which is in the x direction, is a function of y only.

In the classical theory of small disturbances the linearized equations of motion for the perturbation Φ reduce to the Orr–Sommerfeld equation. Solutions of this equation which satisfy the usual homogeneous hydrodynamic boundary conditions yield a characteristic equation relating the parameters α and β. In all the early studies of flow instability α was taken to be a real quantity so that the wave motion defined by (1) grew or decayed exponentially in time depending on the sign of the imaginary part of β. If modes existed with $\beta_i < 0$, the system was considered to be stable. Experimental investigations of flow instabilities[1] made on a forced wave system generated by a periodic perturbation of the flow, produced a motion which had a wave amplitude that increased or decreased with x depending on the sign of α_i, the imaginary part of α. When the rates of growth were small ($|\alpha_i| \ll 1$), this spatial instability could be related to temporal instability through the group velocity.[2] But, in very unstable flows, such as jets or wakes, where no simple relationship existed between the two types of mode, the behavior of spatially growing disturbances could only be found by solving the Orr–Sommerfeld equation for complex wavenumbers. At moderately large Reynolds numbers these very unstable velocity profiles are almost unaffected by viscosity and the inviscid part of the Orr–Sommerfeld equation adequately describes any wavy perturbation. The direct numerical solution of this equation and the evaluation of the eigenvalue relations for spatially growing waves in unstable unbounded flows have been given by Davey[3] for the case of a wake, and Michalke[4] for a shear flow profile.

Temporally growing waves, which have $\alpha_i = 0$, and spatially growing waves, which have $\beta_i = 0$, represent only two particular types of mode. More general waves having complex values of both α and β have been investigated by Betchov and Criminale[5] for jets and wakes in inviscid flow. Solutions of the characteristic equation, $F(\alpha, \beta) = 0$, were shown in the form of contours of constant α_r and α_i plotted on both the c plane and the β plane; the wave speed c is equal to β/α. Although it was not obvious what physical significance these solutions had, some interesting features were revealed. Lines of constant α_r and α_i were shown to be orthogonal except at isolated points. This implied that α was a regular function of c or β except for certain singularities. Betchov and Criminale stated that these singularities were completely unexpected. They suggested that these points had some special significance regarding likely modes of instability, although they were unable to explain in what way the flow was influenced by singularities in the eigenvalue relationships. The present paper, indicates why these singularities arise and explains their physical importance.

The following analysis applies to the full Orr–Sommerfeld equation. Singularities arising in the eigenmode relations are not associated with any singular behavior of the equations describing the eigenfunction. Solutions of the inviscid equation, such as those of Betchov and Criminale, and Davey, can be expected to yield results which are valid for high Reynolds numbers except when the eigenfunction has a singularity. Singularities arise at the critical layer when $c_i = 0$ and c_r lies in the range of u.

[1] G. B. Schubauer and H. K. Skramstad, NACA Technical Report No. 909 (1947).
[2] M. Gaster, J. Fluid Mech. 14, 222 (1962).
[3] M. Gaster, Progr. Aeron. Sci. 6, 251 (1965).
[4] A. Michalke, J. Fluid Mech. 23, 521 (1965).
[5] R. Betchov and W. O. Criminale, Phys. Fluids 9, 2 (1966).

Therefore, the analysis can be applied to the numerical data provided we exclude eigenvalues with $c_i = 0$.

II. CHARACTERISTIC FUNCTION DESCRIBING THE EIGENMODES

Betchov and Criminale found different types of singularity in the eigenvalue relations for the jet and the wake. In the case of the jet profile they treated α as a function of c and plotted contours of α_r and α_i on the complex c plane. At two points in the region of the c plane covered by their calculations the orthogonal relationship between the α_r and α_i contours broke down. At points where α was not a regular function of c there were singularities of the square root type arising from zeros in the derivative $dc/d\alpha$. The wake profile did not show this behavior and α was found to be a regular function of c within the domain investigated. However, when α was treated as a function of β, a singularity appeared where $d\beta/d\alpha$ was zero.

The characteristic equation relating the parameters α and β can be written in the form

$$F(\alpha, \beta) = 0. \tag{2}$$

Since $F(\alpha, \beta)$ is an entire function of the two variables[6] α and β a Taylor expansion can be used about (α_0, β_0),

$$\begin{aligned} F(\alpha, \beta) &= F(\alpha_0, \beta_0) + (\alpha - \alpha_0)\frac{\partial F}{\partial \alpha}(\alpha_0, \beta_0) \\ &+ \tfrac{1}{2}(\alpha - \alpha_0)^2 \frac{\partial^2 F}{\partial \alpha^2}(\alpha_0, \beta_0) + O(\alpha - \alpha_0)^3 \\ &+ (\beta - \beta_0)\frac{\partial F}{\partial \beta}(\alpha_0, \beta_0) + \tfrac{1}{2}(\beta - \beta_0)^2 \frac{\partial^2 F}{\partial \beta^2}(\alpha_0, \beta_0) \\ &+ \cdots O(\beta - \beta_0)^3 \cdots \\ &+ (\alpha - \alpha_0)(\beta - \beta_0)\frac{\partial^2 F}{\partial \alpha\, \partial \beta}(\alpha_0, \beta_0) + \cdots \\ &\cdot O[(\alpha - \alpha_0)(\beta - \beta_0)^2 + (\alpha - \alpha_0)^2(\beta - \beta_0)]. \end{aligned} \tag{3}$$

If (α_0, β_0) is chosen so that $F(\alpha_0, \beta_0)$ is zero we can find the relationship between α and β in the neighborhood of (α_0, β_0) by equating (3) to zero. Then

$$\begin{aligned} (\beta - \beta_0) &= -(\alpha - \alpha_0)\frac{\partial F}{\partial \alpha}(\alpha_0, \beta_0) \Big/ \frac{\partial F}{\partial \beta}(\alpha_0, \beta_0) + \cdots O(\alpha - \alpha_0)^2 \\ \text{or} \\ (\alpha - \alpha_0) &= -(\beta - \beta_0)\frac{\partial F}{\partial \beta}(\alpha_0, \beta_0) \Big/ \frac{\partial F}{\partial \alpha}(\alpha_0, \beta_0) + \cdots O(\beta - \beta_0)^2, \end{aligned} \tag{4}$$

provided $\partial F(\alpha_0, \beta_0)/\partial\alpha$ and $\partial F(\alpha_0, \beta_0)/\partial\beta$ are not zero. Clearly β is a regular function of α in the neighborhood of α_0, and α is a regular function of β close to β_0.

When $F(\alpha, \beta)$ is equated to zero, $\partial F/\partial\alpha$ can be considered solely as a function of α. This derivative will, therefore, also be an entire function of α so that it must have at least one zero on the α plane or have a constant value. At such a point the above analysis is invalid and extra terms of the series expansion are needed to represent the function adequately.

If $\partial F(\alpha_0, \beta_0)/\partial\alpha$ is zero we obtain

$$(\beta - \beta_0) = -\tfrac{1}{2}(\alpha - \alpha_0)^2 \frac{\partial^2 F}{\partial \alpha^2}(\alpha_0, \beta_0) \Big/ \frac{\partial F}{\partial \beta}(\alpha_0, \beta_0) + \cdots O(\alpha - \alpha_0)^3. \tag{5}$$

α is no longer a regular function at β_0 where there is a square-root singularity. Note that α *is* a regular function of c. This is the type of singularity that appeared in the wake calculations of Betchov and Criminale.

Similarly when $\partial F(\alpha_0, \beta_0)/\partial\beta$ is zero we have,

$$(\alpha - \alpha_0) = -\tfrac{1}{2}(\beta - \beta_0)^2 \frac{\partial^2 F}{\partial \beta^2}(\alpha_0, \beta_0) \Big/ \frac{\partial F}{\partial \alpha}(\alpha_0, \beta_0) + \cdots O(\beta - \beta_0)^3. \tag{6}$$

The group velocity $d\beta/d\alpha$, which is usually a complex quantity, is equal to $-(\partial F/\partial\alpha)/(\partial F/\partial\beta)$. The two special cases leading to singularities in (5) and (6) arise when the group velocity is either zero or infinite.

If F is treated as a function of α and c, we can apply similar arguments to show that there must be singularities on the c plane where $dc/d\alpha$ is zero:

$$c = \frac{\beta}{\alpha},$$

therefore,

$$\frac{dc}{d\alpha} = \frac{1}{\alpha}\left[\frac{d\beta}{d\alpha} - \frac{\beta}{\alpha}\right]. \tag{7}$$

Singularities thus arise when $d\beta/d\alpha$ is equal to β/α, or when the wave speed is equal to the group velocity. Betchov and Criminale found two such singular points for the jet profile.

III. MOTION GENERATED BY AN IMPULSE

Any physical significance of the singularities discussed in Sec. II can be revealed by considering a simple example of forced motion. Here we will consider the motion resulting from a pulse input. Since such a disturbance contains modes of all frequencies and wavenumbers, we can expect any significant

[6] C. C. Lin, Quart. Appl. Math. 3, 117 (1945).

irregularity in the eigenvalue relationship to be reflected in the motion. The solution will be obtained in the form of an integral of traveling-wave modes evaluated over all wavenumbers. The asymptotic expansion of this integral is used to explain the significance of various types of singularity.

It has been shown[7] for a boundary layer that a perturbation of one of the boundary conditions generates a motion given by the integral

$$v(y;x,t) = \frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} v(0;x,t) \cdot e^{-i(\alpha x-\beta t)}\,dx\,dt\right]\frac{\Phi(y;\alpha,\beta)}{\Phi(0;\alpha,\beta)}\, e^{i(\alpha x-\beta t)}\,d\alpha\,d\beta, \qquad (8)$$

where v is the disturbance velocity in the y direction and Φ is the perturbation stream function which satisfies the Orr–Sommerfeld equation with the remaining homogeneous boundary conditions.

In the present case where unbounded flows are being discussed forcing through one of the outer boundary conditions leads to a similar integral. In particular, the motion arising from a point impulse, $v(0;x,t) = \delta(x)\,\delta(t)$, takes the form

$$v \sim \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{e^{i(\alpha x-\beta t)}}{F(\alpha,\beta)}\,d\alpha\,d\beta. \qquad (9)$$

$F(\alpha,\beta) = 0$ is the characteristic equation defining all the possible modes of the system. Since $F(\alpha,\beta)$ is an entire function of the parameters α and β, the only singularities appearing in (9) are poles arising from the zeros of $F(\alpha,\beta)$. The perturbation is, therefore, composed of a number of discrete modes associated with these zeros. In general only one zero, that describing the most highly amplified mode, need be considered.

A. Evaluation of the Integral

We integrate (9) with respect to β along a contour passing above the pole (the integral then vanishes for $t < 0$ thus satisfying the causality condition). For $t > 0$ we obtain

$$v \sim \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{e^{i[\alpha x-\beta(\alpha)t]}}{\partial F[\alpha,\beta(\alpha)]/\partial\beta}\,d\alpha, \qquad (10)$$

where $F(\alpha,\beta) = 0$ defines the behavior of β in terms of α. The asymptotic expansion of (10) for large t can be found in the usual way by deforming the contour to pass through the point where the real part of the exponent is at a maximum in such a way that the imaginary part is stationary.

[7] M. Gaster, J. Fluid Mech. 22, 433 (1965).

Let the saddle point lie at α^*, then

$$\frac{d\beta}{d\alpha}(\alpha^*) = \frac{x}{t}.$$

Splitting this into real and imaginary parts gives

$$\frac{\partial\beta_r}{\partial\alpha_r}(\alpha^*) = \frac{x}{t}$$

and

$$\frac{\partial\beta_i}{\partial\alpha_i}(\alpha^*) = 0. \qquad (11)$$

These two equations define the point α^* for any x/t. As x/t is varied the saddle-point traces out a path on the α plane—each point on the physical plane can be related to a point on this line through (11). Neglecting a phase factor we thus obtain the leading term of the expansion

$$v \sim \frac{e^{i[\alpha^* x-\beta(\alpha^*)t]}}{\dfrac{\partial F}{\partial\beta}[\alpha^*,\beta(\alpha^*)]\left[\dfrac{d^2\beta}{d\alpha^2}(\alpha^*)t\right]^{\frac{1}{2}}} + \cdots O(t^{-1}). \qquad (12)$$

For large t the motion is dominated by the exponential term. Along rays in the physical $x \sim t$ plane the disturbance has the form of a traveling wave having a constant value of the complex wavenumber and frequency. The wave motion appears to grow in both space and time with an amplification rate given by

$$-[\alpha_i^* x - \beta_i(\alpha^*)t].$$

This may be written in the form of a temporal growth

$$-\left[\alpha_i^*\frac{\partial\beta_r}{\partial\alpha_r}(\alpha^*) - \beta_i(\alpha^*)\right]t.$$

The most highly amplified waves arise when the above expressions are at a maximum. Keeping t constant (viewing the wave packet as a whole at one instant of time), we differentiate the temporal growth with respect to α_i and equate to zero:

$$\left[\frac{\partial\beta_r}{\partial\alpha_r}(\alpha^*) + \alpha_i^*\frac{\partial^2\beta_r}{\partial\alpha_r\,\partial\alpha_i}(\alpha^*) - \frac{\partial\beta_i}{\partial\alpha_i}(\alpha^*)\right]t = 0.$$

But from the Cauchy–Riemann relations we have $\partial\beta_r/\partial\alpha_r = \partial\beta_i/\partial\alpha_i$ and maximum (or minimum) amplification with respect to time occurs when $\alpha_i^* = 0$. Putting α_i^* to zero we find that this maximum amplification of the wave packet is $\beta_i(\alpha^*)t$. Since α^* is defined by the condition $\partial\beta_i/\partial\alpha_r(\alpha^*) = 0$, this maximum growth rate is identical to that of the most highly amplified temporally growing wave mode. Similarly, the most rapid spatial growth, $-\alpha_i^* x$, occurs when the saddle point lies on the $\beta_i = 0$ contour.

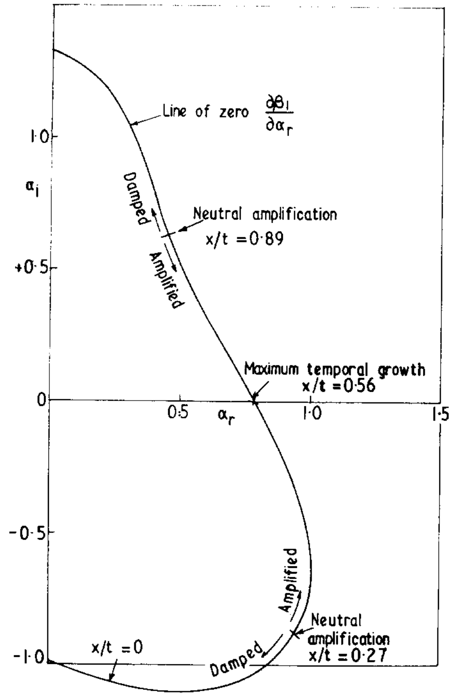

FIG. 1. Path of the saddle point on the α plane.

The boundary of the wave packet can be defined by the rays along which the amplification is zero. To obtain these neutral rays containing the amplified region it is necessary to know how the parameters

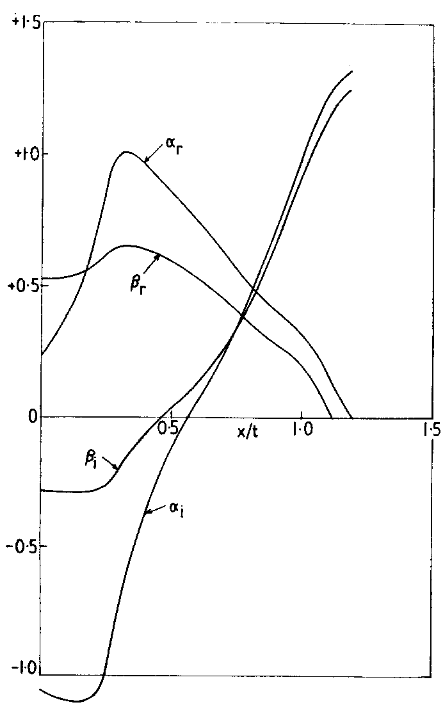

FIG. 2. Behavior of the real and imaginary parts of α and β as a function of x/t.

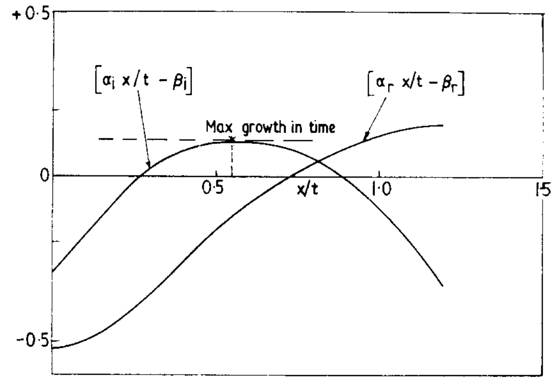

FIG. 3. Real and imaginary parts of the exponential term describing the wave packet.

α, β, and $\partial\beta_r/\partial\alpha_r$ vary along the saddle-point trajectory. Solutions of the eigenvalue relations are needed for complex values of both α and β. Betchov and Criminale have computed some complex solutions for a wake and a jet, but these data are not presented in a form suitable for the present discussion. Davey carried out similar computations on the wake profile

$$u/U_\infty = 1 - 0.692e^{-0.693y^2}.$$

(This profile was used by Sato and Kuriki[8] for calculating temporally growing modes). He obtained the path of the saddle point $[\partial\beta_i(\alpha^*)/\partial\alpha_r = 0]$ for a range of $\partial\beta_r/\partial\alpha_r$ values. The α plane on Fig. 1 shows the trajectory of the saddle point. Figure 2 shows the variation of both the real and imaginary parts of α and β with x/t. These numerical values have been used to obtain the real and imaginary parts of the exponent of (12) which describe the motion for a range of x/t [see Fig. 3]. The two neutral rays propagate with speeds of 0.89 and 0.27 of the free stream velocity.

IV. EFFECT OF SINGULARITIES ON THE DISTURBED MOTION

The general treatment of pulse excitation has been discussed in Sec. III when the eigenrelations are regular. We now consider the influence of various singularities on the nature of the solution.

(a) $\partial F/\partial\alpha$ has a zero at (α_0, β_0).

Expanding $F(\alpha, \beta)$ about this point in a Taylor series we obtain

$$\beta^* = \beta_0 - \tfrac{1}{2}(\alpha - \alpha_0)^2 \frac{\partial^2 F}{\partial\alpha^2}(\alpha_0, \beta_0) \Big/ \frac{\partial F}{\partial\beta}(\alpha_0, \beta_0) + \cdots + O(\alpha - \alpha_0)^3, \quad (13)$$

[8] H. Sato and K. Kuriki, J. Fluid Mech. 11, 321 (1961).

where β^* is the pole on the β plane. v becomes

$$\int_{-\infty}^{+\infty} \exp\left\{i\left[\alpha_0 x - \beta_0 t - \tfrac{1}{2}(\alpha-\alpha_0)^2\left[\frac{\partial^2 F}{\partial\alpha^2}\Big/\frac{\partial F}{\partial\beta}\right]t\right] + \cdots O(\alpha-\alpha_0)^3 t\right\}\left[\frac{\partial F}{\partial\alpha}(\alpha_0,\beta_0)\right]^{-1} d\alpha.$$

For large t/x this leads to the asymptotic form

$$\exp\left\{i\left[(\alpha_0 x - \beta_0 t) + \frac{x^2}{t}\frac{\partial F}{\partial\beta}\Big/\frac{\partial^2 F}{\partial\alpha^2}\right]\right\}\left[\frac{\partial F}{\partial\beta}\frac{\partial^2 F}{\partial\alpha^2}t\right]^{-\frac{1}{2}}$$

or

$$e^{i(\alpha_0 x-\beta_0 t)}e^{ikx^2/t}/t^{\frac{1}{2}}. \tag{14}$$

The above expression is only valid for small values of x/t. At these values the saddle point lies close to α_0 and F is expressed adequately by (13). For small x the dominant term is $[e^{-i\beta_0 t}]/t^{\frac{1}{2}}$. Thus some time after the disturbance has been initiated, the motion at the origin consists of periodic oscillation which is either damped ($\beta_{0i} < 0$) or amplified ($\beta_{0i} > 0$) with time. For slightly larger values of x the motion consists of a traveling-wave modified by a diffusion term, the disturbance existing both upstream and downstream of the source. In most flows a pulse disturbance generates a wave packet which is convected downstream. Even when such a flow is unstable, in the sense that waves forming the packet increase in amplitude with time (or distance), the motion close to the source decays. In the present example where modes with zero group velocity exist, the energy is not convected away from the source and there is the possibility of a true time-growing instability. However, in the example of such a flow given by Betchov and Criminale, β_{0i} is positive and the motion decays near the source.

(b) $\partial F/\partial\beta$ has zero at (α_0, β_0).

Again expanding about (α_0, β_0) we obtain

$$(\beta^* - \beta_0)^2 = -2\left[\frac{\partial F}{\partial\alpha}\Big/\frac{\partial^2 F}{\partial\beta^2}\right](\alpha-\alpha_0) + \cdots$$

or

$$\beta^* = \beta_0 \pm \left[-2\frac{\partial F}{\partial\alpha}\Big/\frac{\partial^2 F}{\partial\beta^2}\right]^{\frac{1}{2}}(\alpha-\alpha_0)^{\frac{1}{2}}.$$

Thus, there are two similar poles close together on the β plane. As β^* approaches β_0 these poles run together and form a second-order pole. The contributions from the two poles are of equal magnitude but of opposite sign, and to first order the integral vanishes. No disturbances with infinite group velocity are, therefore, generated by a pulse input.

(c) $dc/d\alpha$ is zero at some point α_0 on the α plane.

From (7) the above situation arises when $d\beta/d\alpha = \beta/\alpha$. All the expansions used to derive (10) and (12) are still valid when α_0 is a saddle point. The exponent in (12) will vanish along either the leading or trailing edge of the wave packet, but no interesting physical phenomena arise when the group and phase velocity are equal.

V. DISCUSSION

It has been shown that the relationship between the eigenvalues *must* contain various singular points. Such singularities were found by Betchov and Criminale in their numerical work. The wave packet generated by a pulse contains waves of all wavenumbers, and thus the meaning of these singularities might be expected to show up in some way by considering this type of forced motion. A general solution for such a wave packet was obtained and the results were applied to the wake profile as an example. The influence of various types of singularity in the general solution was investigated. The only condition leading to a physically significant result arose when the group velocity became zero. In the example of a wake profile the leading and trailing edges of the wave packet moved with velocities of 0.89 and 0.27 of the free stream. The disturbance is thus convected downstream and the mode with zero group velocity decays in time. The case of a jet discussed by Betchov and Criminale also had a negative value of β_i at the point where the group velocity was zero. However, there may be flows which do exhibit a true time-growing instability, where an applied disturbance amplifies with time without being convected from the region of initial excitation. The divergent channel flows treated by Eagles[9] can have this property. In such cases the flow is unstable in the true sense—any infinitesimal disturbances will grow exponentially in time at a point. One can only conclude that the steady laminar flow cannot exist in this case.

ACKNOWLEDGMENTS

The author wishes to thank Dr. A. Davey for many useful discussions concerning this work and for carrying out the computations. The author also thanks Professor J. T. Stuart for a helpful discussion.

[9] P. M. Eagles, J. Fluid Mech. **24**, 191 (1966).