SCHEMES FOR THE MHD EQUATIONS WITH VARIABLE MAGNETIC PERMEABILITY...

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ABSTRACT. ....

1. Constructing an Analytical Solution for Maxwell Equations for Variable Permeability

$$\begin{cases} \partial_t(\mu\mathbf{H}) = -\nabla\times\mathbf{E} \qquad \text{in } \Omega, \\ \nabla\times\mathbf{H} = \sigma(\mathbf{E}+\mathbf{u}\times\mu\mathbf{H})+\mathbf{j} \qquad \text{in } \Omega_c, \\ \nabla\times\mathbf{H} = 0 \qquad \text{in } \Omega_v, \\ \operatorname{div}(\mu\mathbf{H}) = 0 \qquad \text{in } \Omega, \\ \nabla\cdot\mathbf{E} = 0 \qquad \text{in } \Omega_v. \end{cases} \tag{1.1}$$

We assume that Ω_v is connected, so as explained in [1], exists a scalar potential ϕ, up to a constant, such that $\mathbf{H}|_{\Omega_v} = \nabla\phi$. So we can define

$$\mathbf{H} = \begin{cases} \mathbf{H}^c & \text{in} \quad \Omega_c \\ \nabla\phi & \text{in} \quad \Omega_v \end{cases} \qquad \text{and} \quad \mu = \begin{cases} \mu^c & \text{in} \quad \Omega_c \\ 1 & \text{in} \quad \Omega_v \end{cases}$$

1.1. **Variable Permeability μ^c only in (r,z).** In the following we set Ω^c as a cylinder located at the origin with radius 1 and height 2.
Now, let

$$\mathbf{H} = \frac{1}{\mu^c}\nabla\psi, \tag{1.2}$$

where $\psi = \psi(r,z)$ and satisfies the Laplace equation in cylindrical coordinates,

$$\partial_{rr}\psi + \frac{1}{r}\partial_r\psi + \partial_{zz}\psi = 0. \tag{1.3}$$

If we also set $\mathbf{j} = \nabla\times\mathbf{H}$, $\mathbf{u} = 0$, and $\mathbf{E} = \mathbf{0}$. Then $\mathbf{H}$, defined as in (1.2), satisfies Maxwell equations (1.1).

Now, let

$$\mu^c = \mu^c(r,z) = \frac{1}{f(r,z)+1}, \tag{1.4}$$

where

$$f(r,z) = b\cdot r^3\cdot(1-r)^3\cdot(z^2-1)^3,$$

and $b \geq 0$ is a parameter which determines the variation of μ^c. Observe that

$$\partial_r f(r,z) = 3b\big(r(1-r)\big)^2(1-2r)(z^2-1)^3, \quad \partial_z f(r,z) = 6bz\big(r(1-r)\big)^3(z^2-1)^2.$$

Moreover, $f(r,z) \leq 0$ for $(r,\theta,z) \in \Omega^c$ and,

$$\sup_{\Omega^c} f(r,z) = f_{\max} = 0, \quad \inf_{\Omega^c} f(r,z) = f_{\min} = -\frac{b}{2^6},$$

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then,

$$\mu^c_{\min} = \frac{1}{1+f_{\max}}, \quad \mu^c_{\max} = \frac{1}{1+f_{\min}}, \qquad r_\mu = \frac{\mu_{\max}}{\mu_{\min}} = \frac{\frac{1}{1-\frac{b}{2^6}}}{1}, \quad \text{and} \quad b = 2^6\left(1-\frac{1}{r_\mu}\right).$$

To get an explicit solution in (1.2), equation (1.3) is solved using separation of variables, this is, letting $\psi(r,z) = R(r)Z(z)$ we solve the following system of ODEs,

$$\begin{aligned} Z'' - \lambda Z &= 0 \\ R'' + \frac{R'}{r} + \lambda R &= 0, \end{aligned}$$

where λ is any real number. Here we choose $\lambda = 1$, so

$$\psi(r,z) = J_0(r)\cosh(z). \tag{1.5}$$

Now, using $J_0'(r) = -J_1(r)$ and $\cosh'(z) = \sinh(z)$ we get,

$$\nabla\psi = \begin{bmatrix} -J_1(r)\cosh(z) \\ 0 \\ J_0(r)\sinh(z) \end{bmatrix} \tag{1.6}$$

then by (1.2),

$$\mathbf{H}^c = (f(r,z)+1)\begin{bmatrix} -J_1(r)\cosh(z) \\ 0 \\ J_0(r)\sinh(z) \end{bmatrix}, \tag{1.7}$$

To get $\nabla\times\mathbf{H}$, we use the identity

$$\nabla\times\left(\frac{1}{\mu^c}\nabla\psi\right) = \nabla\left(\frac{1}{\mu^c}\right)\times\nabla\psi + \frac{1}{\mu^c}\nabla\times\nabla\psi,$$

but $\nabla\times\nabla\psi = 0$. Then using equation (1.2),

$$\nabla\times\mathbf{H}^c = \nabla\left(\frac{1}{\mu^c}\right)\times\nabla\psi,$$

and

$$\nabla\frac{1}{\mu^c} = \begin{bmatrix} \partial_r f(r,z) \\ 0 \\ \partial_z f(r,z) \end{bmatrix}; \tag{1.8}$$

we obtain,

$$\nabla\times\mathbf{H}^c = \begin{bmatrix} 0 \\ -\partial_r f(r,z)J_0(r)\sinh(z) - \partial_z f(r,z)J_1(r)\cosh(z) \\ 0 \end{bmatrix}, \tag{1.9}$$

Complete Scheme:

$$
\begin{aligned}
&\mathbf{B}^c|_{t=0} = \mathbf{B}^c_0, \quad \phi|_{t=0} = \phi_0,\\
&\int_{\Omega_c} \frac{D\mathbf{B}^{c,n+1}}{\Delta t}\cdot\mathbf{b} + \int_{\Omega_v} \mu^v \frac{\nabla D\phi^{n+1}}{\Delta t}\cdot\nabla\varphi + \int_{\Omega_c} \left(\frac{R_m}{\sigma}\left(\nabla\times\frac{\boldsymbol{B}^{c,n+1}}{\mu^c} - \mathbf{j}^s\right) - \tilde{\boldsymbol{u}}\times\mathbf{B}^*\right)\cdot\nabla\times\mathbf{b}\\
&+ \int_{\Sigma_\mu} \left\{\frac{R_m}{\sigma}\left(\nabla\times\frac{\boldsymbol{B}^{c,n+1}}{\mu^c} - \mathbf{j}^s\right) - \tilde{\boldsymbol{u}}\times\mathbf{B}^*\right\}\cdot(\mathbf{b}_1\times\mathbf{n}^c_1 + \mathbf{b}_2\times\mathbf{n}^c_2)\\
&+ \beta_3 \sum_{F\in\Sigma_\mu} h_F^{-1}\int_F \left(\frac{\mathbf{B}_1}{\mu^c_1}\times\mathbf{n}^c_1 + \frac{\mathbf{B}_2}{\mu^c_2}\times\mathbf{n}^c_2\right)\cdot(\mathbf{b}_1\times\mathbf{n}^c_1 + \mathbf{b}_2\times\mathbf{n}^c_2)\\
&+ \beta_1 \sum_{F\in\Sigma_\mu} h_F^{-1}\int_F (\mathbf{B}_1\cdot\mathbf{n}^c_1 + \mathbf{B}_2\cdot\mathbf{n}^c_2)\cdot(\mu^c_1\mathbf{b}_1\cdot\mathbf{n}^c_1 + \mu^c_2\mathbf{b}_2\cdot\mathbf{n}^c_2)\\
&+ \int_{\Sigma} \left(\frac{R_m}{\sigma}\left(\nabla\times\frac{\boldsymbol{B}^{c}}{\mu^c} - \mathbf{j}^s\right) - \tilde{\boldsymbol{u}}\times\mathbf{B}^*\right)\cdot(\mathbf{b}\times\mathbf{n}^c + \nabla\varphi\times\mathbf{n}^v)\\
&+ \beta_2 \sum_{F\in\Sigma} h_F^{-1}\int_F \left(\frac{\mathbf{B}}{\mu^c}\times\mathbf{n}^c_1 + \nabla\phi\times\mathbf{n}^c_2\right)\cdot(\mathbf{b}\times\mathbf{n}^c + \nabla\varphi\times\mathbf{n}^v)\\
&+ \beta_1 \sum_{F\in\Sigma} h_F^{-1}\int_F (\mathbf{B}\cdot\mathbf{n}^c_1 + \nabla\phi\cdot\mathbf{n}^c_2)\cdot(\mu^c\mathbf{b}\cdot\mathbf{n}^c + \nabla\varphi\cdot\mathbf{n}^v)\\
&+ \beta_1 \left(\int_{\Omega_c} \mu^c\nabla p\cdot\mathbf{b} - \int_{\Omega_c}\mathbf{B}\cdot\nabla q + \sum_{K\in\mathcal{F}^c_h}\int_{K^{3D}} h_K^{2(1-\alpha)}\nabla p\cdot\nabla q + \sum_{K\in\mathcal{F}^c_h}\int_{K^{3D}} h_K^{2\alpha}\nabla\cdot\mathbf{B}\nabla\cdot(\mu^c\mathbf{b})\right)\\
&+ \int_{\Omega_v}\mu^v\nabla\phi^{n+1}\cdot\nabla\varphi - \int_{\partial\Omega_v}\mu^v\varphi\mathbf{n}\cdot\nabla\phi^{n+1}\\
&+ \int_{\Gamma^c_1} \left(\frac{R_m}{\sigma}\left(\nabla\times\frac{\boldsymbol{B}^{c,n+1}}{\mu^c} - \mathbf{j}^s\right) - \tilde{\boldsymbol{u}}\times\mathbf{B}^*\right)\cdot(\mathbf{b}\times\mathbf{n}^c) + \beta_3\left(\sum_{F\in\Gamma^c_1} h_F^{-1}\int_F \left(\frac{\mathbf{B}}{\mu^c}\times\mathbf{n}^c\right)\cdot(\mathbf{b}\times\mathbf{n}^c)\right)\\
&= \int_{\Gamma^c_2}(\mathbf{a}\times\mathbf{n})\cdot(\mathbf{b}\times\mathbf{n}) + \int_{\Gamma_v}(\mathbf{a}\times\mathbf{n})\cdot(\nabla\varphi\times\mathbf{n}) + \beta_3\left(\sum_{F\in\Gamma^c_1} h_F^{-1}\int_F \left(\mathbf{H}^{\text{given}}\times\mathbf{n}^c\right)\cdot(\mathbf{b}\times\mathbf{n}^c)\right).
\end{aligned}
$$

2. Variable Permeability μ^c only in Space

As before, we set Ω^c as a cylinder located at the origin with radius 1 and height 2. We also let,

$$\mathbf{H} = \frac{1}{\mu^c}\nabla\psi, \tag{2.1}$$

where $\psi = \psi(r,z)$ and satisfies the Laplace equation in cylindrical coordinates,

$$\partial_{rr}\psi + \frac{1}{r}\partial_r\psi + \partial_{zz}\psi = 0. \tag{2.2}$$

Again, we also set $\mathbf{j} = \nabla \times \mathbf{H}$, $\mathbf{u} = 0$, and $\mathbf{E} = \mathbf{0}$. Then $\mathbf{H}$, defined as in (2.1), satisfies Maxwell equations (1.1).

Now, let

$$\mu^c = \mu^c(r,\theta,z) = \frac{1}{f(r,\theta,z)\cos(m\theta)+1}, \tag{2.3}$$

where

$$f(r,z) = b\cdot r^3\cdot(1-r)^3\cdot(z^2-1)^3,$$

and $b \geq 0$ is a parameter which determines the variation of μ^c. Observe that

$$\partial_r f(r,z) = 3b\big(r(1-r)\big)^2(1-2r)(z^2-1)^3, \quad \partial_z f(r,z) = 6bz\big(r(1-r)\big)^3(z^2-1)^2.$$

Moreover, $f(r,z) \leq 0$ for $(r,\theta,z) \in \Omega^c$ and,

$$\sup_{\Omega^c}|f(r,z)| = |f|_{\max} = \frac{b}{2^6}, \quad \inf_{\Omega^c}|f(r,z)| = |f|_{\min} = 0.$$

Then if $m \neq 1$,

$$\mu^c_{\min} = \frac{1}{1+|f|_{\max}}, \quad \mu^c_{\max} = \frac{1}{1-|f|_{\max}}, \qquad r_\mu = \frac{\mu_{\max}}{\mu_{\min}} = \frac{1+|f|_{\max}}{1-|f|_{\max}}, \quad \text{then} \quad b = 2^6\left(\frac{r_\mu - 1}{r_\mu + 1}\right).$$

To get an explicit solution in (2.1), equation (2.2) is solved using separation of variables, this is, letting $\psi(r,z) = R(r)Z(z)$ we solve the following system of ODEs,

$$\begin{aligned} Z'' - \lambda Z &= 0 \\ R'' + \frac{R'}{r} + \lambda R &= 0, \end{aligned}$$

where λ is any real number. Here we choose $\lambda = 1$, so

$$\psi(r,z) = J_0(r)\cosh(z). \tag{2.4}$$

Now, using $J_0'(r) = -J_1(r)$ and $\cosh'(z) = \sinh(z)$ we get,

$$\nabla\psi = \begin{bmatrix} -J_1(r)\cosh(z) \\ 0 \\ J_0(r)\sinh(z) \end{bmatrix} \tag{2.5}$$

then by (2.1),

$$\mathbf{H}^c = (f(r,z)+1)\begin{bmatrix} -J_1(r)\cosh(z) \\ 0 \\ J_0(r)\sinh(z) \end{bmatrix}, \tag{2.6}$$

To get $\nabla \times \mathbf{H}$, we use the identity

$$\nabla \times \left(\frac{1}{\mu^c}\nabla\psi\right) = \nabla\left(\frac{1}{\mu^c}\right) \times \nabla\psi + \frac{1}{\mu^c}\nabla \times \nabla\psi,$$

but $\nabla \times \nabla\psi = 0$. Then using equation (2.1),

$$\nabla \times \mathbf{H}^c = \nabla \left(\frac{1}{\mu^c} \right) \times \nabla \psi,$$

and

$$\nabla \frac{1}{\mu^c} = \begin{bmatrix} (\partial_r f(r,z)) \cos(m\theta) \\ -\frac{m}{r} f(r,z) \sin(m\theta) \\ (\partial_z f(r,z)) \cos(m\theta) \end{bmatrix}; \tag{2.7}$$

we obtain,

$$\nabla \times \mathbf{H}^c = \begin{bmatrix} -\frac{m}{r} f(r,z) J_0(r) \sin(m\theta) \sinh(z) \\ -\partial_r f(r,z) J_0(r) \sinh(z) - \partial_z f(r,z) J_1(r) \cosh(z) \\ -\frac{m}{r} f(r,z) J_1(r) \sin(m\theta) \cosh(z) \end{bmatrix}, \tag{2.8}$$

We finally compute $\nabla \mu^c$ with,

$$\nabla \mu^c = \begin{bmatrix} \partial_r \mu^c \\ \frac{1}{r} \partial_\theta \mu^c \\ \partial_z \mu^c \end{bmatrix},$$

where

$$\partial_r \mu^c = -\frac{(\partial_r f(r,z)) \cos(m\theta)}{[f(r,z)\cos(m\theta) + 1]^2},$$

$$\partial_\theta \mu^c = \frac{m f(r,z) \sin(m\theta)}{[f(r,z)\cos(m\theta) + 1]^2},$$

and

$$\partial_z \mu^c = -\frac{(\partial_z f(r,z)) \cos(m\theta)}{[f(r,z)\cos(m\theta) + 1]^2}.$$

References

[1] J.-L. Guermond, J. Léorat, F. Luddens, C. Nore and A. Ribeiro, *Effects of Discontinuous Magnetic Permeability on Magnetohydronamic Problems* , Journal of Computational Physics, **230** (2011), 6299-6319.