Problem Set 5 Artur Savchii

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(1) a) $\lim \frac{1}{(6n^2+1)} = 0$

Prf:

Let $\varepsilon > 0$ be arbitrary.

Chose $N \in \mathbb{N}$ with $N > \sqrt{\frac{1-\varepsilon}{6\varepsilon}}$ ✓

$\forall n \in \mathbb{N}$ with $n \geq N$ $n > \sqrt{\frac{1-\varepsilon}{6\varepsilon}}$, $\Longrightarrow \frac{1}{6n^2+1} < \varepsilon$,

which implies that $\left|\frac{1}{6n^2+1} - 0\right| < \varepsilon$ $\forall n \geq N$ as desired.

b) $\lim \left(\frac{3n+1}{2n+5}\right) = \frac{3}{2}$

Prf:

Let $\varepsilon > 0$ be arbitrary

Chose $N \in \mathbb{N}$ with $N > \frac{13-10\varepsilon}{4\varepsilon}$

$\forall n \in \mathbb{N}$ with $n \geq N$ $n > \frac{13-10\varepsilon}{4\varepsilon} \Longrightarrow \frac{13}{4n+10} < \varepsilon$

which implies that $\left|\frac{3n+1}{2n+5} - \frac{3}{2}\right| < \varepsilon$ $\forall n \geq N$

as $\left|\frac{3n+1}{2n+5} - \frac{3}{2}\right| = \frac{13}{4n+10}$. $\Longrightarrow$ Thus, $\lim \left(\frac{3n+1}{2n+5}\right) = \frac{3}{2}$

✓ by the definition of convergence

c) $\lim \left(\frac{2}{\sqrt{n+3}}\right) = 0$

Prf: Let $\varepsilon > 0$ be arbitrary

Chose $N \in \mathbb{N}$ with $N > \frac{4}{\varepsilon^2} - 3$

$\forall n \in \mathbb{N}$ with $n \geq N$ $n > \frac{4}{\varepsilon^2} - 3 \Longrightarrow \frac{2}{\sqrt{n+3}} < \varepsilon$

which implies that $\left|\frac{2}{\sqrt{n+3}} - 0\right| < \varepsilon$ $\forall n \geq N$

Thus, $\lim \left(\frac{2}{\sqrt{n+3}}\right) = 0$ by the defn of convergence

✓

(2)

a) $|a_n - a| < \varepsilon \ \forall n \in A$

where $A = \{a \in \mathbb{N} : a > N\}$

for a larger N, N_ℓ

assume $|a_n - a| < \varepsilon \ \forall n \in B$

where $B = \{b \in \mathbb{N} : b > N_\ell\}$

$N_\ell > N$, so $B \subseteq A$

✓ Therefore, a larger N works for the same ε

Similarly, for a smaller N, set of all ~~possible~~ $n \geq N_s$ will contain additional elements, which can be out of neighborhood of ε.

Therefore, a smaller N doesn't work

b) $\forall n \geq N$. a_n is in neighborhood of a, i.e. $|a_n - a| < \varepsilon$.

Taking a larger ε, the neighborhood becomes wider, and therefore still contains all elements a_n $\forall n \geq N$. Thus, a larger ε works for the same N.

✓ Taking a smaller ε, the neighborhood becomes narrower, and therefore can lack some values of a_n for $n \geq N$. Thus a smaller ε doesn't work

(3) A sequence (a_n) converges to ∞ if for any arbitrary chosen $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N, n \in \mathbb{N}$ $a_n > \varepsilon$

a) Let $\varepsilon > 0$ be arbitrary

✓ Chose $N \in \mathbb{N}$ with $N > \varepsilon^2$

$\forall n \in \mathbb{N}$ with $n \geq N$, $n > \varepsilon^2 \implies \sqrt{n} > \varepsilon$

Thus $\sqrt{n}$ converges to ∞ by the defn.

b) All elements on odd positions are negative.
And as $|a_n| > |a_{n-1}|$, $a_{n+2} < a_n$ ~~for~~ $\forall n \in \mathbb{N}, n$ is odd
Therefore, we cannot chose such $N \in \mathbb{N}$ that $\forall n \geq N$ $a_n > \varepsilon$ because odd elements form decreasing sequence.
✓ Thus, $(n(-1)^n)$ does not converge to ∞

c) Every second element of the sequence is 0
For a sequence to converge to ∞, $a_n > \varepsilon$ $\forall n \geq N$, where $N \in \mathbb{N}$. for any $\varepsilon > 0$.
$\begin{matrix} a_n > \varepsilon \\ \varepsilon > 0 \end{matrix} \Rightarrow a_n > 0$, which is not possible for any N, as every second elemet is 0.
✓ Thus, $(1,0,2,0,3,0,\dots)$ does not converge to ∞

(4) a) Assume (a_n) is eventually in A
i.e. $\exists N \in \mathbb{N}$ such that $\forall n \geq N$ $a_n \in A$
(1) For any $N_B > N$ it is still true that $\forall n \geq N_B$ $a_n \in A$, as a set of all possible n for N_B ~~[illegible]~~ is a subset of a set of all possible n for N.
(2) For any $N_S < N$ we can still find such $n \geq N$, which implies $n \geq N_S$, that $a_n \in A$.
(1) and (2) implies that $\forall N \in \mathbb{N}$ $\exists n \geq N$ such that $a_n \in A$.
Thus, (a_n) is frequently in A ✓
In the contrary, if (a_n) is frequently in A, i.e. $\forall N \in \mathbb{N}$, $\exists n \geq N$ with $a_n \in A$, it doesn't necessary imply that ~~[illegible]~~ there exist $N \in \mathbb{N}$ such that for all $n \geq N$ $a_n \in A$

Thus, the definition of (a_n) being eventually in A is stronger

b) (X_n) is not eventually in $(1.9, 2.1)$.

Let's take a sequence $(X_n) = (2, 0, 2, 0, 2, 0 \ldots)$

It has infinite number of terms that are equal to 2.

However, if we select any $N \in \mathbb{N}$, there will exist such $n \geq N$ that $X_n = 0$, and $0 \notin (1.9, 2.1)$

(X_n) is frequently in $(1.9, 2.1)$

no matter what large N we choose, there still exists $n \geq N$ such that $a_n = 2 \Rightarrow a_n \in (1.9, 2.1)$,
ok
as there ~~are~~ an infinite number of 2 in a sequence.