Problem Set 2 $\frac{7.5}{9}$ Artur Savchii

1.

a) $f(x)=x^2$, $A=[0,4]$ $B=[-1,1]$

$f^{-1}(A)$ — ? $f^{-1}(A)=\{x\in\mathbb{R}: f(x)\in A\}$

$\forall x\in\mathbb{R}$ $x^2\in[0,4] \Rightarrow f^{-1}(A)=[-2,2]$

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$\forall x\in\mathbb{R}$ $x^2\in[-1,1] \Rightarrow f^{-1}(B)=[-1,1]$

b)

$A\cap B=[0,1]$

$f^{-1}(A\cap B)=[-1,1]$

$f^{-1}(A)\cap f^{-1}(B)=[-2,2]\cap[-1,1]=[-1,1]$

Thus, $f^{-1}(A\cap B)=f^{-1}(A)\cap f^{-1}(B)$

$A\cup B=[-1,2]$

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$f^{-1}(A\cup B)=[-2,2]$

$f^{-1}(A)\cup f^{-1}(B)=[-2,2]\cup[-1,1]=[-2,2]$

Thus $f^{-1}(A\cup B)=f^{-1}(A)\cup f^{-1}(B)$.

c)

1) for all $A,B\in\mathbb{R}$ and $f:\mathbb{R}\to\mathbb{R}$

$f^{-1}(A\cap B)=f^{-1}(A)\cap f^{-1}(B)$

Prf:

($\rightarrow$) Assume: $x\in f^{-1}(A\cap B)$

$f^{-1}(A)=\{x\in\mathbb{R}: f(x)\in A\}$ by definition of preimage

Therefore $f(x)\in A\cap B \Rightarrow \begin{cases} f(x)\in A \\ f(x)\in B \end{cases}$ by defn of intersection

$\begin{cases} x\in f^{-1}(A) \\ x\in f^{-1}(B) \end{cases}$ by the defn of preimage

$\Downarrow$

$x\in f^{-1}(A)\cap f^{-1}(B)$ by the defn of intersection

Thus, $f^{-1}(A \cap B) \subseteq f^{-1}(A) \cap f^{-1}(B)$ (1)

($\leftarrow$) Assume $x \in f^{-1}(A) \cap f^{-1}(B)$

$\begin{cases} x \in f^{-1}(A) \\ x \in f^{-1}(B) \end{cases}$ by the defn of intersection

$\begin{cases} f(x) \in A \\ f(x) \in B \end{cases}$ by the defn of preimage

$f(x) \in A \cap B$ by the defn of intersection

$x \in f^{-1}(A \cap B)$ by the defn of preimage

Thus, $f^{-1}(A) \cap f^{-1}(B) \subset f^{-1}(A \cap B)$ (2)

Because of (1) and (2)

$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

2) $\forall A, B \in \mathbb{R}$ and $f: \mathbb{R} \to \mathbb{R}$

$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

Prf:

($\rightarrow$) Assume: $x \in f^{-1}(A \cup B)$

It mean $f(x) \in A \cup B$ by the defn. of preimage

$f(x) \in A$ or $f(x) \in B$ by the defn of union

$x \in f^{-1}(A)$ or $x \in f^{-1}(B)$ by the defn of preimage

$x \in f^{-1}(A) \cup f^{-1}(B)$ by the defn of union

Thus, $f^{-1}(A \cup B) \subseteq f^{-1}(A) \cup f^{-1}(B)$ (1)

($\leftarrow$) Assume: $x \in f^{-1}(A) \cup f^{-1}(B)$

This means $x \in f^{-1}(A)$ or $x \in f^{-1}(B)$ By the defn of union

$f(x) \in A$ or $f(x) \in B$ by the defn of preimage

$f(x) \in A \cup B$ by the defn of union

$x \in f^{-1}(A \cup B)$ by the defn of preimage

Thus, $f^{-1}(A) \cup f^{-1}(B) \subseteq f^{-1}(A \cup B)$ (2)

Because of (1) and (2):

$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$$

2. $X_1 = 1 \qquad X_{n+1} = \frac{1}{2} X_n + 1$

$X_n \leq 2$ for any $n \in \mathbb{N}$

Prf by Induction:

(Basis) $n=1$: $X_1 = 1 \leq 2$

$X_1 \leq 2$

Assume: $X_n \leq 2$

Let's show that $X_{n+1} \leq 2$

$X_n \leq 2$

$\frac{1}{2} X_n \leq 1$

$\frac{1}{2} X_n + 1 \leq 2$

$X_{n+1} \leq 2$

✓

Thus, $X_n \leq 2$ for any $n \in \mathbb{N}$

3.

$y_1 = 1 \qquad y_{n+1} = \frac{3y_n + 4}{4}$

a) $y_n \leq 4$ for all $n \in \mathbb{N}$

Prf by Induction:

(Basis) $n=1$: $y_1 = 1 \quad 1 \leq 4$

$y_1 \leq 4$

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Assume: $y_n \leq 4$

Let's show that $y_{n+1} \leq 4$

$y_n \leq 4 \quad / \cdot 3$

$3y_n \leq 12 \quad / +4$

$3y_n + 4 \leq 16 \quad / : 4$

$\frac{3y_n + 4}{4} \leq 4$

$y_{n+1} \leq 4$

Thus, $y_n \leq 4$ for all $n \in \mathbb{N}$

These are both a little sparse.

b) $y_1, y_2, \ldots$ is increasing

Prf by Induction:

(Basis) $y_1 \leq y_2$? $y_1 = 1$

$y_2 = \frac{3y_1+4}{4} = \frac{7}{4} = 1.75$

This is confusing

$1.75 > 1$

$y_2 > y_1$ ✓

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Assume: $y_{n-1} \leq y_n$

Let's show that $y_n \leq y_{n+1}$

$y_{n-1} \leq y_n \quad / \cdot 3$

$3y_{n-1} \leq 3y_n \quad / +4$

$3y_{n-1} + 4 \leq 3y_n + 4 \quad / : 4$

$\frac{3y_{n-1}+4}{4} \leq \frac{3y_n+4}{4}$

$y_n \leq y_{n+1}$

Thus, $y_1, y_2, \ldots$ is increasing

4.

$\mathbb{Z}_5 = \{0,1,2,3,4\}$

a) $\forall z \in \mathbb{Z}_5 \ \exists \ y \in \mathbb{Z}_5$ such that $(z + y) \bmod 5 = 0$

Prf: for $z=0$ $y=0$ $0+0=0$ $0 \bmod 5 = 0$

for $z=1$ $y=4$ $1+4=5$ $5 \bmod 5 = 0$

for $z=2$ $y=3$ $2+3=5$ $5 \bmod 5 = 0$

for $z=3$ $y=2$ $3+2=5$ $5 \bmod 5 = 0$

for $z=4$ $y=1$ $4+1=5$ $5 \bmod 5 = 0$

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b) $\forall z \in \mathbb{Z}_5, z \neq 0 \ \exists \ y \in \mathbb{Z}_5$ such that $z \cdot y \bmod 5 = 1$

Prf: for $z=1$ $y=1$ $1 \cdot 1 = 1$ $1 \bmod 5 = 1$

for $z=2$ $y=3$ $2 \cdot 3 = 6$ $6 \bmod 5 = 1$

for $z=3$ $y=2$ $3 \cdot 2 = 6$ $6 \bmod 5 = 1$

for $z=4$ $y=4$ $4 \cdot 4 = 16$ $16 \bmod 5 = 1$

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c) $\mathbb{Z}_4 = \{0, 1, 2, 3\}$

Additive inverse:

$\forall z \in \mathbb{Z}_4 \; \exists y \in \mathbb{Z}_4$ such that $(z+y) \bmod 4 = 0$?

for $z=0$ $y=0$ $0+0=0$ $0 \bmod 4 = 0$
for $z=1$ $y=3$ $1+3=4$ $4 \bmod 4 = 0$
for $z=2$ $y=2$ $2+2=4$ $4 \bmod 4 = 0$
for $z=3$ $y=1$ $3+1=4$ $4 \bmod 4 = 0$

Yes, $\mathbb{Z}_4$ has additive inverse

Multiplicative inverse:

$\forall z \in \mathbb{Z}_4, z \neq 0 \; \exists y \in \mathbb{Z}_4$ such that $z \cdot y \bmod 4 = 1$?

for $z=1$ $y=1$ $1 \cdot 1 = 1$ $1 \bmod 4 = 1$
for $z=2$ there is no y such that $z \cdot y \bmod 4 = 1$

ok.

Thus, $\mathbb{Z}_4$ doesn't have multiplicative inverse

Therefore $\mathbb{Z}_4$ is not a field