Artur Savchii **Problem Set 5**
Course: Advanced Topics: Linear Algebra
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Problem 1
Let S and T be subspaces of a vector space V. When is $S \cup T$ a subspace of V?

Solution:

Let $s \in S$, $s \notin T \implies s \in S \cup T$ and $t \in T$, $t \notin S \implies s \in S \cup T$
For $S \cup T$ to be a subspace, it has to be closed under vector addition: $s + t \in S \cup T$
$s + t \in S \cup T \implies 1) s + t \in S$ or/and $2) s + t \in T$
1) $\exists$ -s $\in S$ by Axiom 4. S is a subspace, and therefore closed under vector addition:
$s + t + (-s) \in S \implies t \in S$
However, this contradicts our initial assumption that $t \notin S$
2) Similarly, for -t $\in T$, $s + t + (-t) \in T \implies s \in T$, which contradicts our assumption.
Thus, $S \cup T$ is not a subspace of V when there are vectors in S but not in T **and** vice versa.
This proves that for $S \cup T$ to be a subspace of V, $S \subseteq T$ **or** $T \subseteq S$

Problem 2
Show that any non-zero vector spans $\mathbb{R}$

Proof. For $A \in \mathbb{R}$ to span $\mathbb{R}$, the set $\{A\}$ has to be linearly independent.
By the definition of linearly independence, $rA = \vec{0}$ has to have no solutions for $r \neq 0$
$\vec{0}$ in $\mathbb{R}$ is 0. We have rA = 0. The only solutions are r = 0 or A = 0. However, $r \neq 0$ and A is a non-zero vector. The equation, therefore, has no solutions.
Thus, the set $\{A\}$ is linearly independent, and, therefore, spans $\mathbb{R}$.
□

Problem 3
Show that $A = \{(1, 1, 0), (0, 0, 1)\}$ and $B = \{(1, 1, 1), (-1, -1, 1)\}$ span the same subspace of $\mathbb{R}^3$

Proof. Denote $A = \{(1, 1, 0), (0, 0, 1)\} = \{a_1, a_2\}$ and $B = \{(1, 1, 1), (-1, -1, 1)\} = \{b_1, b_2\}$
Linear combination $a_1 + a_2 = (1, 1, 0) + (0, 0, 1) = (1, 1, 1) = b_1$
Linear combination $a_1 + a_2 = -(1, 1, 0) + (0, 0, 1) = (-1, -1, 1) = b_2$
$(a_1 + a_2)$, $(-a_1 + a_2) \in span(A)$, by the defn.
Thus, we have $b_1, b_2 \in span(A)$
span(A) is a subspace by Prop 4. Therefore, it is closed under vector addition and scalar multiplication.
Thus, all of the linear combinations of b_1 and b_2 are in span(A).
By the definition of a span, we have $span(B) \subseteq span(A)$

Linear combination $0.5b_1 - 0.5b_2 = (0.5, 0.5, 0.5) - (-0.5, -0.5, 0.5) = (1, 1, 0) = a_1$
Linear combination $0.5b_1 + 0.5b_2 = (0.5, 0.5, 0.5) + (-0.5, -0.5, 0.5) = (0, 0, 1) = a_2$
$(0.5b_1 - 0.5b_2)$, $(0.5b_1 + 0.5b_2) \in span(B)$, by the defn.
Thus, we have $a_1, a_2 \in span(B)$
span(B) is a subspace by Prop 4. Therefore, it is closed under vector addition and scalar multiplication.
Thus, all of the linear combinations of a_1 and a_2 are in span(A).
By the definition of a span, we have $span(A) \subseteq span(B)$

$span(B) \subseteq span(A)$ and $span(A) \subseteq span(B) \implies span(A) = span(B)$ □

Problem 4
Suppose S, T are subspaces of V and $S \cap T = \vec{0}$. Show that every vector $\vec{C} \in S + T$ can be written uniquely in the form $\vec{A} + \vec{B} = \vec{C}$ with $\vec{A} \in S$ and $\vec{B} \in T$. Construct an example to show that this is false if $S \cap T \neq \vec{0}$.

Proof. Assume not: Let $A_1, A_2 \in S$ and $B_1, B_2 \in T$. Then we have $A_1 + B_1 = C$ and $A_2 + B_2 = C$.

This implies $A_1 + B_1 = A_2 + B_2 \implies (A_1 - A_2) + (B_1 - B_2) = \vec{0}$

This is true for: 1) $(A_1 - A_2)$ and $(B_1 - B_2)$ are inverses; 2) $(A_1 - A_2)$ and $(B_1 - B_2)$ are zero vectors.

1) S is a subspace and, therefore, a vector space and, therefore closed under vector addition.

Thus $(A_1 - A_2) \in S$.

If $(B_1 - B_2)$ is an inverse of $(A_1 - A_2)$, then $(B_1 - B_2) \in S$ by Axiom 4.

T is a subspace and, therefore, a vector space and, therefore closed under vector addition.

Thus $(B_1 - B_2) \in T$.

If $(A_1 - A_2)$ is an inverse of $(B_1 - B_2)$, then $(B_1 - B_2) \in S$ by Axiom 4.

Thus, we have $(A_1 - A_2), (B_1 - B_2) \in S$ and $(A_1 - A_2), (B_1 - B_2) \in T \implies (A_1 - A_2), (B_1 - B_2) \in T \cap S$, which contradicts our condition that $T \cap S = \{\vec{0}\}$.

Therefore, as $(A_1 - A_2), (B_1 - B_2)$ cannot be inverses, for $(A_1 - A_2) + (B_1 - B_2) = \vec{0}$ to be true, $(A_1 - A_2)$ and $(B_1 - B_2)$ are zero vectors.

$A_1 - A_2 = \vec{0} \implies A_1 = A_2 = A$ by Axiom 4.

$B_1 - B_2 = \vec{0} \implies B_1 = B_2 = B$ by Axiom 4.

This shows that $\vec{C} \in S + T$ can be written uniquely in the form $\vec{A} + \vec{B} = \vec{C}$

Example: Let $S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_2 = 0\}$ and $T = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_3 = 0\}$

Then we have $S \cap T = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_2 = 0, x_3 = 0\} \neq \{\vec{0}\}$

$(2, 0, 0) \in S$ and $(3, 0, 0) \in T$. Then $(2, 0, 0) + (3, 0, 0) = (5, 0, 0) \in S + T$

$(1, 0, 0) \in S$ and $(4, 0, 0) \in T$. Then $(1, 0, 0) + (4, 0, 0) = (5, 0, 0) \in S + T$

Hence, $\vec{C} = (5, 0, 0) \in S + T$ can be written in **at least two ways** in the form $\vec{A} + \vec{B} = \vec{C}$ with $\vec{A} \in S$ and $\vec{B} \in T$.

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