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Course: Advanced Topics: Linear Algebra
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Problem Set 6

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Problem 1
Let S and T be subspaces of V. Prove that:
a) $S + T = L(S \cup T)$
b) $S \cap (S + T) = S$
c) $S + T = T + S$
d) If $S \subseteq T$, then $S + T = T$

Proof. a) $S + T = L(S \cup T)$

Let $A \in S + T \implies A = \vec{s} + \vec{t}$ where $\vec{s} \in S, \vec{t} \in T \implies \vec{s}, \vec{t} \in S \cup T$. $\vec{s} + \vec{t}$ is a linear combination of vectors in $S \cup T$, and therefore, $\vec{s} + \vec{t} = A \in L(S \cup T)$ by the defn. of linear span. Thus, $S + T \subseteq L(S \cup T)$

Let $A \in L(S \cup T)$. This implies that A can be represented as linear combination of vectors in $S \cup T$: $A = r_1\vec{a_1} + r_2\vec{a_2} + ... + r_k\vec{a_k}$, where $\vec{a_i} \in S \cup T$. $\vec{a_i} \in S$ or $\vec{a_i} \in T$ or both, by the defn. of union. Hence, denote those $\vec{a_i} \in S$ as $\vec{s_i}$, $\vec{a_i} \in T$ as $\vec{t_i}$ and $\vec{a_i} \in S, T$ as $\vec{t_i}$ as well. $A = r_1\vec{a_1} + r_2\vec{a_2} + ...r_k\vec{a_k} = r_{s1}\vec{s_1} + r_{t1}\vec{t_1} + ... + r_{sn}\vec{s_n} + r_{tm}\vec{t_m}$. Because S is a subspace of V, and, therefore, a vector space, and, therefore, closed under vector addition and scalar multiplication, $(r_{s1}\vec{s_1} + ... + r_{sn}\vec{s_n}) \in S$. Similarly, for T, $(r_{t1}\vec{t_1} + ... + r_{tn}\vec{t_n}) \in T$. Thus, $A = (r_{s1}\vec{s_1} + ... + r_{sn}\vec{s_n}) + (r_{t1}\vec{t_1} + ... + r_{tn}\vec{t_n}) = (\vec{s} + \vec{t}) \in S + T$ by the definition of the sum of subspaces. Thus, $L(S \cup T) \subseteq S + T$.

$S + T \subseteq L(S \cup T)$ and $L(S \cup T) \subseteq S + T$. This implies $S + T = L(S \cup T)$

b) $S \cap (S + T) = S$

Let $A \in S \cap (S + T) \implies A \in S$ by the defn. of intersection. Thus, $S \cap (S + T) \subseteq S$.

Let $A \in S$. T is a subspace of V, and, therefore, $\vec{0} \in T$. Thus, $A + \vec{0} \in S + T$ by the definition of the sum of the subspaces. $A + \vec{0} = A$ by Axiom 3. Therefore, $A \in S + T$. $A \in S + T$ and $A \in S$ $\implies A \in S \cap (S + T)$ by the definition of the intersection. Thus, $S \subseteq S \cap (S + T)$

$S \cap (S + T) \subseteq S$ and $S \subseteq S \cap (S + T)$. This implies $S \cap (S + T) = S$.

c) $S + T = T + S$

S+T is a set of $\vec{s} + \vec{t}$ with $\vec{s} \in S$ and $\vec{t} \in T$ by defn.

T+S is a set of $\vec{t} + \vec{s}$ with $\vec{t} \in T$ and $\vec{s} \in S$ by defn.

T and S are subspaces, and vector addition is commutative by Axiom 1.

Therefore, T+S is a set of $\vec{t} + \vec{s} = \vec{s} + \vec{t}$ with $\vec{t} \in T$ and $\vec{s} \in S$ by defn.

This matches our definition of S+T. Thus, S+T = T+S.

d) Let $A \in S + T$. Then $A = \vec{s} + \vec{t}$ with $\vec{s} \in S$ and $\vec{t} \in T$ by defn. $S \subseteq T$. Therefore, if $\vec{s} \in S$, $\vec{s} \in T$ by defn. of a subset. $\vec{s} \in T$ and $\vec{t} \in T$. Thus, $\vec{s} + \vec{t} = A \in T$, as T is a subspace of V, and, therefore, a vector space, and, therefore, closed under vector addition. Thus, $S + T \subseteq T$

Let $A \in T$. S is a subspace of V, and, therefore, $\vec{0} \in S$. Thus, $\vec{0} + A \in S + T$ by the definition of the sum of the subspaces. $A + \vec{0} = A$ by Axiom 3. Therefore, $A \in S + T$. Thus, $T \subseteq S + T$.

$S + T \subseteq T$ and $T \subseteq S + T$. This implies S+T = T.

□

Problem 2
Which of the following sets of vectors in $\mathbb{R}^3$ are linearly dependent? Which are linearly independent? $E = \{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\}$, $F = \{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}\}$, $G = \{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\}$, $H = \{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}\}$, $K = \{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\}$

Solution:

E) $\begin{bmatrix}1\\1\\1\end{bmatrix} - \begin{bmatrix}0\\1\\0\end{bmatrix} - \begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} = \vec{0}$. Therefore, by the definition, **E is linearly dependent.**

F) $r_1 \begin{bmatrix}1\\1\\1\end{bmatrix} + r_2 \begin{bmatrix}1\\1\\0\end{bmatrix} + r_3 \begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} = \vec{0}$.

For the third term we have $r_1 * 1 + r_2 * 0 + r_3 * 0 = 0 \implies r_1 = 0$. For the second term we have $r_1 * 1 + r_2 * 1 + r_3 * 0 = 0 * 1 + r_2 * 1 + r_3 * 0 = 0 \implies r_2 = 0$. For the first term we have $r_1 * 1 + r_2 * 1 + r_3 * 1 = 0 * 1 + 0 * 1 + r_3 * 1 = 0 \implies r_3 = 0$. However, by the definition, it has to be non-trivial linear combination. Therefore, **F is linearly independent.**

G) $r_1 \begin{bmatrix}1\\1\\1\end{bmatrix} + r_2 \begin{bmatrix}1\\1\\0\end{bmatrix} + r_3 \begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} = \vec{0}$

For the third term we have $r_1 * 1 + r_2 * 0 + r_3 * 1 = 0 \implies r_1 + r_3 = 0$. For the second term we have $r_1 * 1 + r_2 * 1 + r_3 * 0 = 0 \implies r_1 + r_2 = 0$. For the first term we have $r_1 * 1 + r_2 * 1 + r_3 * 1 = 0 \implies r_1 + r_2 + r_3 = 0$. We know from the second term that $r_1 + r_2 = 0$. Thus, we have $0 + r_3 = 0 \implies r_3 = 0$. We know from the third term that $r_1 + r_3 = 0 = r_1 + 0 \implies r_1 = 0$. Thus, we get $r_1 = 0, r_2 = 0, r_3 = 0$. However, by the definition, it has to be non-trivial linear combination. Therefore, **G is linearly independent.**

H) $r_1 \begin{bmatrix}1\\0\\0\end{bmatrix} + r_2 \begin{bmatrix}0\\1\\0\end{bmatrix} + r_3 \begin{bmatrix}1\\2\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} = \vec{0}$

For the third term we have $r_1 * 0 + r_2 * 0 + r_3 * 1 = 0 \implies r_3 = 0$. For the second term we have $r_1 * 0 + r_2 * 1 + r_3 * 2 = 0 \implies r_2 + 2r_3 = 0 \implies r_2 = 0$. For the first term we have $r_1 * 1 + r_2 * 0 + r_3 * 1 = 0 \implies r_1 + r_3 = 0 \implies r_1 = 0$. However, by the definition, it has to be non-trivial linear combination. Therefore, **H is linearly independent.**

K) $r_1 \begin{bmatrix}1\\1\\1\end{bmatrix} + r_2 \begin{bmatrix}0\\1\\0\end{bmatrix} + r_3 \begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} = \vec{0}$

For the first term we have $r_1 * 1 + r_2 * 0 + r_3 * 0 = 0 \implies r_1 = 0$. For the second term we have $r_1 * 1 + r_2 * 1 + r_3 * 0 = 0 \implies r_1 + r_2 = 0 \implies r_2 = 0$. For the third term we have $r_1 * 1 + r_2 * 0 + r_3 * 1 = 0 \implies r_1 + r_3 = 0 \implies r_3 = 0$. Thus, we get $r_1 = 0, r_2 = 0, r_3 = 0$. However, by the definition, it has to be non-trivial linear combination. Therefore, **K is linearly independent.**

Problem 3
Let $E = \{(1,1,0,0,1),(1,1,0,1,1),(0,1,1,1,1),(2,1,-1,0,1)\} \subseteq \mathbb{R}^5$, and let $L(E) = U$. Find $F \subseteq E$ such that $L(F) = U$ and F is linearly independent.

Solution: (2, 1, -1, 0, 1) = (1, 1, 0, 0, 1) + (1, 1, 0, 1, 1) - (0, 1, 1, 1, 1). This means (2, 1, -1, 0, 1) is linearly dependent on a set $\{(1,1,0,0,1),(1,1,0,1,1),(0,1,1,1,1)\}$. Therefore, let $F = E \setminus \{(2,1,-1,0,1)\}$. Then we have $L(F) = L(E) = U$ by Theorem 2.
Now let's prove that F is linearly independent. Assume not. Then we have:
$r_1(1,1,0,0,1) + r_2(1,1,0,1,1) + r_3(0,1,1,1,1) = \vec{0} = (0,0,0,0,0)$.
$((r_1+r_2),(r_1+r_2+r_3),r_3,(r_2+r_3),(r_1+r_2+r_3)) = (0,0,0,0,0)$ We get $r_3 = 0$. $r_2+r_3 = 0 \implies r_2 = 0$. $r_1 + r_2 = 0 \implies r_1 = 0$. However, by the definition, it has to be non-trivial linear combination. Therefore, F is linearly independent.

Problem 4
Suppose that $E \subseteq F$ are sets of vectors in V, a vector space. Prove that if F is linearly independent, so is E.

Proof. Assume not. Assume F is linearly independent, but E is linearly dependent. By the defn, $\exists A_1, ..., A_k \in E$ and $a_1, ..., a_k \in \mathbb{R}$ (not all zeros), such that $a_1A_1 + ... + a_kA_k = \vec{0}$.

$E \subseteq F \implies A_1, ..., A_k \in F$. Denote those vectors that are in F but not in E as $B_1, ..., B_k$. Then we have $a_1A_1 + ... + a_kA_k + 0B_1 + ... + 0B_k = (a_1A_1 + ... + a_kA_k) + 0(B_1 + ... + B_k) = \vec{0} + \vec{0} = \vec{0}$. This implies that F is linearly dependent (by the definition), which contradicts our initial assumption. Therefore, if F is linearly independent, so is E.
□

Problem 5
Suppose that $E \subseteq F$ are sets of vectors in V, a vector space. Prove that if E is linearly dependent, so is F.

Proof. By the defn, $\exists A_1, ..., A_k \in E$ and $a_1, ..., a_k \in \mathbb{R}$ (not all zeros), such that $a_1A_1+...+a_kA_k = \vec{0}$. Denote those vectors that are in F but not in E as $B_1, ..., B_k$. Then we have $a_1A_1 + ... + a_kA_k + 0B_1 + ... + 0B_k = (a_1A_1 + ... + a_kA_k) + 0(B_1 + ... + B_k) = \vec{0} + \vec{0} = \vec{0}$. This implies that F is linearly dependent by the definition. □