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Course: Advanced Topics: Linear Algebra

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Problem 1
Let $U \in P(\mathbb{R})$ be $U = L(E)$ for $E = \{x^3, x^3 - x^2, x^3 + x^2, x^3 - 1\}$
Find F $\subseteq$ E such that L(F) = L(E) = U and F in linearly independent

Solution: $x^3 = 0.5 * (x^3 - x^2) + 0.5 * (x^3 + x^2) + 0 * (x^3 - 1)$. Therefore, x^3 is linearly dependent on a set $E \setminus \{x^3\}$. Thus, $L(E \setminus \{x^3\}) = L(E) = U$ by Theorem 2.

Assume $E \setminus \{x^3\}$ is linearly dependent. Therefore, $r_1(x^3 - x^2) + r_2(x^3 + x^2) + r_3(x^3 - 1) = \vec{0} = 0 \implies (r_1 + r_2 + r_3)x^3 + (r_2 - r_1)x^2 - r_3 = 0 \implies r_3 = 0, r_1 + r_2 = 0, r_2 - r_1 = 0$. From the later two we have $2r_2 = 0 \implies r_2 = 0 \implies r_1 = 0$. However, for a set to be linearly dependent, the equation must hold true for a non-trivial linear combination. Therefore, $E \setminus \{x^3\}$ is linearly independent by the definition.

F = $E \setminus \{x^3\}$, L(F) = U and F is linearly independent.

Problem 2
Let $S = \{x_1, x_2, x_3\}$ and let $f, g, h \in Fun(S)$.
If $f(x_1) = 0$, $f(x_2) = 1$, $f(x_3) = 1$
$g(x_1) = 1$, $g(x_2) = 0$, $g(x_3) = 1$
$h(x_1) = 1$, $h(x_2) = 1$, $h(x_3) = 0$
Does $\{f, g, h\}$ for a linearly independent set?

Solution:
By the definition, $\{f, g, h\}$ is linearly dependent if there exists non-trivial linear combination $r_1 f + r_2 g + r_3 h = \vec{0} \implies (r_1 f + r_2 g + r_3 h)(x_1) = 0, (r_1 f + r_2 g + r_3 h)(x_2) = 0, (r_1 f + r_2 g + r_3 h)(x_3) = 0$
$(r_1 f + r_2 g + r_3 h)(x_1) = r_1 f(x_1) + r_2 g(x_1) + r_3 h(x_1) = r_2 + r_3 = 0$
$(r_1 f + r_2 g + r_3 h)(x_2) = r_1 f(x_2) + r_2 g(x_2) + r_3 h(x_2) = r_1 + r_3 = 0$
$(r_1 f + r_2 g + r_3 h)(x_3) = r_1 f(x_3) + r_2 g(x_3) + r_3 h(x_3) = r_1 + r_2 = 0$
Solving the system of equations we get $r_3 - r_2 = 0, r_2 + r_3 = 0 \implies r_3 = 0 \implies r_2 = 0 \implies r_1 = 0$
However, the linear combination had to be non-trivial. Therefore, $\{f, g, h\}$ is linearly independent.

Problem 3
Let A and B be linearly independent sets of vectors in V.
Show that:
1) $A \cap B$ is linearly independent.
2) $A \cup B$ is linearly independent iff $L(A) \cap L(B) = \{\vec{0}\}$

Proof. 1) Assume not. Assume A and B are linearly independent sets, but $A \cap B$ is linearly dependent. By the defn, for $\vec{A_1}, ..., \vec{A_k} \in A \cap B$ and $a_1, ..., a_k \in \mathbb{R}$ (not all zeros), such that $a_1\vec{A_1} + ... + a_k\vec{A_k} = \vec{0}$. $\vec{A_1}, ..., \vec{A_k} \in A \cap B \implies \vec{A_1}, ..., \vec{A_k} \in A$ and $\vec{A_1}, ..., \vec{A_k} \in B$ by the definition of intersection. Denote those $\vec{B_1}, ..., B_m \in A$ such that $\vec{B_1}, ..., B_m \notin B$. Then linear combination of all vectors in A is $(a_1\vec{A_1} + ... + a_k\vec{A_k}) + (0 * \vec{B_1} + ... + 0 * \vec{B_m}) = (a_1\vec{A_1} + ... + a_k\vec{A_k}) + 0 * (\vec{B_1} + ... + \vec{B_m}) = (a_1\vec{A_1} + ... + a_k\vec{A_k}) + \vec{0} = \vec{0} + \vec{0} = \vec{0}$. Thus, A is linearly dependent by the definition. Similarly, B is linearly dependent by the definition. However, this contradicts our initial assumption that A and B are linearly independent sets. Thus, $A \cap B$ is linearly independent.

2) ($\rightarrow$) $A \cup B$ is linearly independent.
Assume $L(A) \cap L(B) \neq \{\vec{0}\}$. Thus, $\exists \vec{v} \in L(A) \cap L(B), \vec{v} \neq \vec{0} \implies \vec{v} \in L(A)$ and $\vec{v} \in L(B)$
By the definition of span, $\vec{v} = a_1\vec{A_1} + ... + a_n\vec{A_n}$ with $a_1, ..., a_n \in \mathbb{R}$ and $\vec{A_1}, ..., \vec{A_n} \in A$ and also $\vec{v} = b_1\vec{B_1} + ... + b_m\vec{B_m}$ with $b_1, ..., b_m \in \mathbb{R}$ and $\vec{B_1}, ..., \vec{B_m} \in B$. By the definition of union, $\vec{B_1}, ..., \vec{B_m} \in A \cup B$ and $\vec{A_1}, ..., \vec{A_n} \in A \cup B$. $a_1\vec{A_1} + ... + a_n\vec{A_n} - b_1\vec{B_1} - .... - b_m\vec{B_m} = (a_1\vec{A_1} + ... + a_n\vec{A_n}) - (b_1\vec{B_1} + ... + b_m\vec{B_m}) = \vec{v} - \vec{v} = \vec{0}$ by Axiom 4. This mean that there is non-trivial linear combinations in A and B, making this sets linearly dependent by definition. However, this contradicts our initial assumption. Therefore, $\vec{v} \in L(A) \cap L(B), \vec{v} \neq \vec{0}$ does not exist. Thus, $L(A) \cap L(B) = \{\vec{0}\}$

($\leftarrow$) $L(A) \cap L(B) = \{\vec{0}\}$
$A \subseteq L(A)$ as just one vector alone is a linear combination with all of the other vectors having zero coefficients. Similarly, $B \subseteq L(B)$. Therefore, if $\exists \vec{v} \in A \cap B, \vec{v} \in L(A), L(B) \implies \vec{v} \in L(A) \cap L(B)$, which contradicts our initial assumption that $L(A) \cap L(B) = \{\vec{0}\}$ ($\vec{0} \notin A, B$, as the sets wouldn't be linearly independent then). Thus, such vector $\vec{v} \in A \cap B$ does not exist, and, therefore, $A \cap B = \emptyset$
This implies that every vector in $A \cup B$ can be denoted as either $\vec{A} \in A$ or $\vec{B} \in B$, and there is no vector that is both in A and B. Assume $A \cup B$ is linearly dependent. Thus, there has to exist non-trivial linear combination $a_1\vec{A_1} + ... + a_n\vec{A_n} + b_1\vec{B_1} + ... + b_m\vec{B_m} = \vec{0}$
$(a_1\vec{A_1} + ... + a_n\vec{A_n}) + (b_1\vec{B_1} + ... + b_m\vec{B_m}) = \vec{0}$ This mean that one of the following must be true:

1) both $(a_1\vec{A_1} + ... + a_n\vec{A_n}) = \vec{0}$ and $(b_1\vec{B_1} + ... + b_m\vec{B_m}) = \vec{0}$. However, this makes A and B linearly dependent by definition, which contradicts our initial condition.

2) $(a_1\vec{A_1} + ... + a_n\vec{A_n}) = -(b_1\vec{B_1} + ... + b_m\vec{B_m})$. $(a_1\vec{A_1} + ... + a_n\vec{A_n}) \in L(A)$ by definition. $(b_1\vec{B_1} + ... + b_m\vec{B_m}) \in L(B)$ by definition. $(b_1\vec{B_1} + ... + b_m\vec{B_m}) = -(a_1\vec{A_1} + ... + a_n\vec{A_n}) = -a_1\vec{A_1} - ... - a_n\vec{A_n} \in L(A)$ be definition. Thus, $(b_1\vec{B_1} + ... + b_m\vec{B_m}) \in L(A) \cap L(B)$ However, this contradicts our initial assumption that $L(A) \cap L(B) = \{\vec{0}\}$.

Hence, there does not exist a non-trivial linear combination such that $a_1\vec{A_1} + ... + a_n\vec{A_n} + b_1\vec{B_1} + ... + b_m\vec{B_m} = \vec{0}$, and, therefore, $A \cup B$ is linearly independent by the definition.

□

Problem 4
Let $S, T \subseteq V$ be subspaces, and $S \cap T = \{\vec{0}\}$.
For non-zero $\vec{A} \in S$ and $\vec{B} \in T$, show that $\{\vec{A}, \vec{B}\}$ is a linearly independent set.

Proof. Assume not. Assume $\{\vec{A}, \vec{B}\}$ is linearly dependent. Therefore, there mus exist non-trivial linear combination $a\vec{A} + b\vec{B} = \vec{0}$ with non-zero $a, b \in \mathbb{R}$. This mean that one of the following must be true:

1) $a\vec{A} = \vec{0}$ and $b\vec{B} = \vec{0}$. However, $\vec{A}, \vec{B} \neq \vec{0}$ by the condition of the problem, and a and b cannot be both equal zero by definition of a non-trivial linear combination.

2) $a\vec{A} = -b\vec{B}$. In this case, sum of two inverse vectors is zero vector by Axiom 4. $\vec{A} \in S \implies a\vec{A} \in S$, as S is a subspace, and, therefore, a vector space, and, therefore, closed under scalar multiplication. $a\vec{A} \in S, b\vec{B} = -a\vec{A} \implies b\vec{B} \in S$, as a unique inverse of a vector in a vector space by Axiom 4. $b\vec{B} \in S, b\vec{B} \in T \implies b\vec{B} \in S \cap T$ by the definition of the intersection. However, this contradicts our initial condition that $S \cap T = \{\vec{0}\}$.

Hence, there does not exist a non-trivial linear combination such that $a\vec{A} + b\vec{B} = \vec{0}$, and, therefore, $\{\vec{A}, \vec{B}\}$ is linearly independent by definition. □

Problem 5
Show that any 4 vectors in $\mathbb{R}^3$ must be linearly dependent.

Proof. Let's prove that any linearly independent set of n vector spans $\mathbb{R}^n$

Proof by induction.

Base Case: n = 1. Any non-zero real number spans $\mathbb{R}$ (See Problem Set 5, 2).
Hypothesis: Assume for any $k \in \mathbb{R}$, linearly independent set of k vector spans $\mathbb{R}^k$
Inductive: We want to show that a linearly independent set of k+1 vector spans $\mathbb{R}^{k+1}$. We know that k of those vectors span $\mathbb{R}^k$. Because the set is still linearly independent, the (k+1)-th vector is not in the span of other k vectors. Therefore, it basically expands this k-span/surface to the k+1 dimension, spanning $\mathbb{R}^{k+1}$.

Now, let's assume we have set of four vectors in $\mathbb{R}^3$: $\{v_1, v_2, v_3, v_4\}$.
If $\{v_1, v_2, v_3\}$ are linearly dependent, there has to exist a non-trivial linear combination $r_1v_1 + r_2v_2 + r_3v_3 = \vec{0}$. Then, $r_1v_1 + r_2v_2 + r_3v_3 + 0r_4 = \vec{0} + \vec{0} = \vec{0}$, and therefore, $\{v_1, v_2, v_3, v_4\}$ is linearly dependent by definition.
If $\{v_1, v_2, v_3\}$ are linearly independent, then $L(v_1, v_2, v_3) = \mathbb{R}^3$, as was proven above. $v_4 \in \mathbb{R}^3 \implies v_4 \in L(v_1, v_2, v_3) \implies v_4$ is linearly dependent on set $\{v_1, v_2, v_3\}$. Thus, $\{v_1, v_2, v_3, v_4\}$ is linearly dependent. $\square$