Problem Set 3 Artur Suchii

$\frac{6.25}{7}$

1. (a) Let A be bounded below

$B = \{b \in \mathbb{R}: b \text{ is a lower bound for } A\}$

$$\sup(B) = \inf(A)$$

Prf:

$\forall b \in B$ b is a lower bound for A,

so $\forall a \in A$ $b \leq a$ by the defn of lower bound.

Thus, $\forall a \in A$, a is an upper bound of B by the defn of the upper bound

$\sup(B) \leq a$ $\forall a \in A$ by the def of the least upper bound

✓ $\sup(B)$ is ~~…~~ a lower bound of A. By the defn of lower bound

B is ~~…~~ a set of all lower bounds of A

So as $\sup(B) \geq b$ $\forall b \in B$ (by the defn of supreme) $\sup(B) = \inf(A)$ by the defn of the greatest lower bound

(b) (a) shows that for any set A bounded below, there is a set B of all lower bounds, least upper bound of which ~~…~~ equals to the greatest lower bound of A, which make $\inf A$ exist.

ok

2. A and B are nonempty and bounded above

$B \subseteq A$ $\sup(B) \leq \sup(A)$

Prf: $\forall b \in B$ $b \in A$ by the definition of a subset.

✓ $\forall a \in A$ $a \leq \sup(A)$ by the defn of upper bound.

Thus, $\forall b \in B$ $b \leq \sup(A)$, as $\forall b \in B$ $b \in A$

As $b \leq \sup(A)$ $\forall b \in B$ $\sup(A)$ is an upper bound of B

$\sup(B) \leq \sup(A)$ by the definition of the least upper bound

3. if $a \in A$ and a is an upper Bound, $a = \sup(A)$

Prf:

Assume not:

assume $\exists\, b = \sup(A)$ and $b \neq a$

a is an upper Bound and b is a least upper Bound

Thus, $a \geq b$ by the defn of the least upper b

$\forall \alpha \in A \quad b \geq \alpha$ by the defn of the upper Bound

$a \in A$, so $b \geq a$, which is a contradiction

So....

I'm giving you this as a technicality. You never quite finish your proofs. Let's talk.

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4. $\forall x \in \mathbb{R}$, there exist $n \in \mathbb{N}$ with $x < n$ (1)

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$\forall y \in \mathbb{R}^+$, there exist $n \in \mathbb{N}$ with $\frac{1}{n} < y$

This is backwards.

Prf: $\frac{1}{n} < y \implies n > \frac{1}{y}$

$y \in \mathbb{R}^+ \implies \frac{1}{y} \in \mathbb{R}^+$

Therefore, from (1) $\frac{1}{y} < n \implies \frac{1}{n} < y$

$(\forall \frac{1}{y} \in \mathbb{R},\ \exists\, n \in \mathbb{N}$ with $\frac{1}{y} < n)$

5. Given two real numbers $a + \sqrt{2}$ and $b + \sqrt{2}$,

there is a rational number $r \in \mathbb{Q}$

such that $a + \sqrt{2} < r < b + \sqrt{2}$ by the Density of $\mathbb{Q}$ in $\mathbb{R}$

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$a < r - \sqrt{2} < b$

because...

$\sqrt{2}$ is irrational, so $r - \sqrt{2}$ is irrational as well

6. $\bigcap_{n=1}^{\infty} (0, \frac{1}{n}) = \emptyset$ Assume not: $\exists\, x \in \mathbb{R}$ with $x \in \bigcap_{n=1}^{\infty} (0, \frac{1}{n})$

$x \in (0, \frac{1}{n})$ to every $n \in \mathbb{N}$

✓ However by the Archimedean Property N2: given $x \in \mathbb{R}$, $\exists\, n \in \mathbb{N}$ with $\frac{1}{n} < x$

which contradicts $x \in (0, \frac{1}{n})$

$\bigcap (0, \frac{1}{n}) = \emptyset$