Problem Set 4

1. Let $\mathbb{I}$ be the irrational numbers.
   a) Show that for $a, b \in \mathbb{Q}$, $a \cdot b$ and $a+b \in \mathbb{Q}$.

   b) Show that for $a \in \mathbb{Q}$, $t \in \mathbb{I}$, then if $a \neq 0$, $a+t \in \mathbb{I}$ and $a \cdot t \in \mathbb{I}$.

   c) Part (a) shows that $\mathbb{Q}$ is closed under addition and multiplication. Is $\mathbb{I}$ under addition and multiplication? For $s, t \in \mathbb{I}$, what can we say about $s \cdot t$ and $s+t$?

2. We say that for two sets A and B, $A \sim B$ means that A and B have the same cardinality.

   a) Show that $A \sim B$ is equivalent to $B \sim A$.

   b) Show that if $A \sim B$ and $B \sim C$, then $A \sim C$. This means that "$\sim$" is an <u>equivalence relation</u>.

3. a) Show that the set of all finite subsets of $\mathbb{N}$ is countable.

   b) Explain why the set of all subsets of $\mathbb{N}$ is uncountable.

4. a) Is the set of all functions from $\{0,1\} \to \mathbb{N}$ countable or uncountable?

   b) Is the set of all functions from $\mathbb{N} \to \{0,1\}$ countable or uncountable?

   c) Given a Set B, $A \subseteq P(B)$ is called an <u>antichain</u> if no element of A is a subset of any other element of A. Does $P(\mathbb{N})$ contain an uncountable antichain?