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Course: Advanced Topics: Linear Algebra

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Problem 1

Suppose $T : V \to W$ is a linear transformation with $ker(T) = \{\vec{0}\}$. Prove that for all $\vec{B} \in Im(T)$ there is exactly one vector $\vec{A} \in V$ such that $T(\vec{A}) = \vec{B}$.

Proof. Let $\vec{A_1}, \vec{A_2} \in V$ such that $T(\vec{A_1}) = \vec{B}$ and $T(\vec{A_2}) = \vec{B}$.
$T(\vec{A_1} - \vec{A_2}) = T(\vec{A_1}) - T(\vec{A_2})$ by the definition of a linear transformation.
$T(\vec{A_1} - \vec{A_2}) = T(\vec{A_1}) - T(\vec{A_2}) = B - B = \vec{0}$
$\vec{A_1} - \vec{A_2} \in V$ by the definition of a linear combination.
We have $\vec{A_1} - \vec{A_2} \in V$ and $T(\vec{A_1} - \vec{A_2}) = \vec{0}$
Knowing that $ker(T) = \{\vec{0}\}$, we have $\vec{A_1} - \vec{A_2} = \vec{0}$ by definition of kernel.
Therefore, $\vec{A_1} = \vec{A_2}$ by Axiom 4. Thus, there exists a unique vector $\vec{A} \in V$ such that $T(\vec{A}) = \vec{B}$.
□

Problem 2

Let $T : \mathbb{R}^3 \to \mathbb{R}$ by $T(x, y, z) = x - 3y + 2z$.
a) Show that T is a linear transformation
b) Find a basis for $ker(T)$
c) What are dim(ker(T)) and dim(Im(T))

Solution:

a) T is a linear transformation if 1) $T(\vec{A} + \vec{B}) = T(\vec{A}) + T(\vec{B}) \forall \vec{A}, \vec{B} \in V$ and 2) $T(a\vec{A}) = aT(\vec{A}) \forall a \in \mathbb{R}, \vec{A} \in V$ by definition.
$T((x_1, y_1, z_1) + (x_2, y_2, z_2)) = T(x_1 + x_2, y_1 + y_2, z_1 + z_2) = (x_1 + x_2) - 3 * (y_1 + y_2) + 2 * (z_1 + z_2)$
$T(x_1, y_1, z_1) + T(x_2, y_2, z_1) = (x_1 - 3y_1 + 2z_1) + (x_2 - 3y_2 + 2z_2) = (x_1 + x_2) - 3 * (y_1 + y_2) + 2 * (z_1 + z_2)$
$T((x_1, y_1, z_1) + (x_2, y_2, z_2)) = T(x_1, y_1, z_1) + T(x_2, y_2, z_2)$
$T(a(x, y, z)) = T(ax, ay, az) = (ax - 3ay + 2az) = a(x - 3y + 2z) = aT(x, y)$
Both 1) and 2) hold. Thus T is a linear transformation by definition

b) By definition, $ker(T) = \{(x, y, z) \in \mathbb{R}^3 | x - 3y + 2z = 0\}$.
x-3y+2z=0 is an equation of a plane. Therefore, dim(ker(T)) = 2. Thus, two linearly independent vectors on the plane span the plane and form a basis by Theorem 9. Let B = {(1, 1, 1), (3, 1, 0)}. $(1, 1, 1), (3, 1, 0) \in ker(T)$. If B is linearly dependent, then there exists a non-trivial linear combination $r_1(1, 1, 1) + r_2(3, 1, 0) = \vec{0} \implies r_1 + 3r_2 = 0, r_1 + r_2 = 0, r_1 = 0$. Plugging in the last one into first two we get $r_1 = 0, r_2 = 0$, which make it not a non-trivial linear combination. Therefore, B is linearly independent by definition.
B = {(1, 1, 1), (3, 1, 0)} is a basis of ker(T).

c) dim(ker(T)) = 2 (see b). Im(T) is a subspace of $\mathbb{R}$ by Prop 14. Thus, $dim(Im(T)) \leq dim(\mathbb{R}) \implies dim(Im(T)) \leq 1$ by Theorem 8. Because, dimension is a size of a set of coordinates, it has to be nonnegative integer by definition. dim(Im(T)) = 0 if all of the vectors are sent to zero vector. However, (1, 0, 0) is, for example, sent to 1. Hence, dim(Im(T)) = 1.

Problem 3
$T : V \to W$ is called injective if for all $\vec{A}, \vec{B} \in V$ with $\vec{A} \neq \vec{B}$, we have $T(\vec{A}) \neq T(\vec{B})$.
Show that T is injective iff $ker(T) = \{\vec{0}\}$

Proof. $(\to)$
T is injective $\implies \forall \vec{A}, \vec{B} \in V$ with $\vec{A} \neq \vec{B}, T(\vec{A}) \neq T(\vec{B})$.
Let $\vec{C} \in ker(T)$. Then $T(\vec{C}) = \vec{0}$ by definition. We also know that $T(\vec{0}) = \vec{0}$ because $\vec{0} \in ker(T)$.
For $\vec{A} \neq \vec{B}, T(\vec{A}) \neq T(\vec{B}) \implies$ If $T(\vec{A}) = T(\vec{B})$ then $\vec{A} = \vec{B}$.
Thus, $T(\vec{0}) = T(\vec{C}) \implies \vec{C} = \vec{0}$. Therefore, the only vector that is in ker(T) is zero vector. $ker(T) = \{\vec{0}\}$

$(\leftarrow)$
ker(T) = $\{\vec{0}\}$. Let $\vec{A_1}, \vec{A_2} \in V$ such that $\vec{A_1} \neq \vec{A_2}$ but $T(\vec{A_1}) = T(\vec{A_2}) = \vec{B}, \vec{B} \in Im(T)$ We know that there exists exactly one $\vec{A} \in V$, such that $T(\vec{A}) = \vec{B}$ (See Problem 1). However, then we get $\vec{A_1} = \vec{A_2}$, which contradicts our initial assumption. Thus, $T(\vec{A_1}) \neq T(\vec{A_2})$, and therefore, T is injective by definition.
□

Problem 4
Suppose V is a finite dimensional vector space and $T : V \to V$ is a linear transformation. Show that the following are equivalent:
a) T is an isomorphism
b) $ker(T) = \{\vec{0}\}$ (T is injective)
c) Im(T) = V (T is surjective)

Proof. 1) We will prove that T is an isomorphism iff $ker(T) = \{\vec{0}\}$ (T is injective).
$(\to)$
if T is an isomorphism $ker(T) = \{\vec{0}\}$ by Prop 16.
$(\leftarrow)$
If $ker(T) = \{\vec{0}\}$ T is injective, and for all $\vec{A}, \vec{B} \in V$ with $\vec{A} \neq \vec{B}$, we have $T(\vec{A}) \neq T(\vec{B})$ by definition (See Problem 3). Im(T) is a subspace of V by Prop 14. Therefore, $Im(T) \subseteq V$ Because elements from V are sent to Im(V), if there exists $\vec{A} \in V$, such that $\vec{A} \notin Im(T)$, then at least two elements of V were sent to the same element of Im(T). However, this contradicts our initial condition that T is injective and $T(\vec{A}) \neq T(\vec{B})$. Thus, there does not exist $\vec{A} \in V$, such that $\vec{A} \notin Im(T)$, and, therefore, Im(T) = V. Hence, T is an isomorphism by Prop 16

2) We will prove that T is an isomorphism iff Im(T) = V.
$(\to)$
if T is an isomorphism Im(T) = V by Prop 16.
$(\leftarrow)$
Let Im(T) = V. $T(\vec{0} + A) = T(\vec{0}) + T(A)$ by definition of linear transformation. Then we have $T(A) = T(\vec{0}) + T(A)$. Because $T(A) \in V$ a subspace, $T(\vec{0}) = \vec{0}$ by Axiom 3. Thus, $\vec{0} \in ker(T)$
Let $\vec{C} \in ker(T) \implies T(C) = \vec{0}$.
By definition of Image, there has to exist linear transformation $S : V \to V$, such that S(T(A)) = A. We have $S(T(\vec{0})) = \vec{0}$ and $S(T(\vec{C})) = \vec{C}$. But that means we have $S(\vec{0}) = \vec{0}$ and $S(\vec{0}) = C$, which cannot be by definition of a linear transformation. Thus, $\vec{C} = \vec{0}$. Hence, $ker(T) = \{\vec{0}\}$, and, therefore, T is an isomorphism by Prop 16.

1 is true iff 2 is true and 1 is true iff 3 is true. Thus, 1,2, and 3 are equivalent. □