Problem Set 4 $\quad \frac{8.75}{8}$ $\quad$ Artur Savchii

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a) $a, b \in \mathbb{Q}$ $\qquad \mathbb{Q} = \{\frac{a}{b} \mid a, b \in \mathbb{Z}\}$

As $a, b \in \mathbb{Q}$ let $a = \frac{p_a}{q_a}$, $b = \frac{p_b}{q_b}$ with $p_a, p_b, q_a, q_b \in \mathbb{Z}$

$$a \cdot b = \frac{p_a}{q_a} \cdot \frac{p_b}{q_b} = \frac{p_a p_b}{q_a q_b},$$

and $p_a p_b \in \mathbb{Z}$, $q_a q_b \in \mathbb{Z}$ as $\mathbb{Z}$ is closed under multiplication

$$a \cdot b = \frac{p_a p_b}{q_a q_b} \text{ with } p_a p_b,\ q_a q_b \in \mathbb{Z}$$

Therefore, $\underline{ab \in \mathbb{Q}}$ by the definition of rational numbers

$$a + b = \frac{p_a}{q_a} + \frac{p_b}{q_b} = \frac{p_a q_b + p_b q_a}{q_a q_b}$$

$p_a q_b, p_b q_a, q_a q_b \in \mathbb{Z}$, as $\mathbb{Z}$ is closed under multiplication

$p_a q_b + p_b q_a \in \mathbb{Z}$, as $\mathbb{Z}$ is closed under addition

$$a + b = \frac{p_a q_b + p_b q_a}{q_a q_b} = \frac{m}{k}, \text{ where } m, k \in \mathbb{Z}$$

Therefore, $\underline{a + b \in \mathbb{Q}}$ by the definition of rational numbers

b) $a \in \mathbb{Q}$, $t \in \mathbb{I}$, $a \neq 0$

$(a + t) + (-a) = t$, $t \in \mathbb{I}$

As $\mathbb{Q}$ is closed under addition and $(a+t) + (-a) \in \mathbb{I}$, with $-a \in \mathbb{Q}$

$$\underline{a + t \in \mathbb{I}}$$

$at \cdot \frac{1}{a} = t$, $t \in \mathbb{I}$

As $\mathbb{Q}$ is closed under multiplication and $at \cdot \frac{1}{a} \in \mathbb{I}$ with $\frac{1}{a} \in \mathbb{Q}$, $\underline{at \in \mathbb{I}}$

c) $s, t \in \mathbb{I}$

Let $s = a + \sqrt{2}$, $t = a - \sqrt{2}$ with $a \in \mathbb{Q}$

$s, t \in \mathbb{I}$ by (B)

$s + t = a + \sqrt{2} + a - \sqrt{2} = 2a$

$2a \in \mathbb{Q}$ as $\mathbb{Q}$ is closed under multiplica

$s + t = 2a$ with $2a \notin \mathbb{I}$

Therefore, $\mathbb{I}$ is not closed under addition.

$s \cdot t = (a + \sqrt{2})(a - \sqrt{2}) = a^2 - 2$

$a^2 \in \mathbb{Q}$, as $\mathbb{Q}$ is closed under multiplicat

$a^2 - 2 \in \mathbb{Q}$, as $\mathbb{Q}$ is closed under addition

/ $s \cdot t = a^2 - 2$ with $a^2 - 2 \notin \mathbb{I}$

Therefore, $\mathbb{I}$ is not closed under multiplication

2.

a) $A \sim B \Rightarrow \exists f : A \to B$

f is one-to-one and onto

then if $g = f^{-1}$

$g : B \to A$ ~~is one-to-one and onto~~

by the definition of preimage

/ $f : A \to B$ is one-to-one if $f(x) = f(y)$ $x = y$

$f^{-1}(f(x)) = g(f(x)) = x$

$f^{-1}(f(y)) = g(f(y)) = y$ by the defn of pre.

so $g : B \to A$ is one-to-one.

$f : A \to B$ is onto if $\forall b \in B \ \exists a \in A \ f(a) = b$

Thus $\forall a \in A \ \exists b \in B \ f^{-1}(b) = a$

so $g : B \to A$ is onto

$g : B \to A$ is one-to-one and onto

Therefore $B \sim A$ by the defn of cardinality

b) $A \sim B \implies f_1 : A \to B$ is one-to-one and onto

$B \sim C \implies f_2 : B \to C$ is one-to-one and onto

Let $g = f_2(f_1)$

Range of f_1 equals to domain of f_2

so ~~range~~ domain of $f_2(f_1)$ = ~~range~~ domain of f_1

and range of $f_2(f_1)$ = range of f_2

Therefore $g : A \to C$

f_1 and f_2 are one-to-one functions

if $f_1(x) = f_1(y)$, $x = y$ as f_1 is one-to-one

if $f_2(f_1(x)) = f_2(f_1(y))$, $f_1(x) = f_1(y)$ as f_2 is one-to-one

Thus $g : A \to C$ is one-to-one.

$\forall c \in C \; \exists \; b \in B$ such that $f_2(b) = c$ as f_2 is onto

$\forall b \in B \; \exists \; a \in A$ such that $f_1(a) = b$ as f_1 is onto

Thus $\forall c \in C \; \exists \; a \in A$ such that $f_2(f_1(a)) = c$

Therefore $g : A \to C$ is onto

$g : A \to C$ is one-to-one and onto

Thus, $A \sim C$ by the defn of cardinality

3

a) Let A_n be a set containing all subsets of $\mathbb{N}$ with size n.

$A_0 = \{\emptyset\}$, $A_1 = \{\{1\}, \{2\}, \{3\} \ldots\}$ ...

(3,14) A_n is always countable

for A_n we can take B_K be a set of all elements (sets) of A which sum of element equals to K.

Therefore $A_n \sim \mathbb{N}$

Thus, set of all finite subsets of $\mathbb{N}$ is equal to $\bigcup_{n=1}^{\infty} A_n$

$\bigcup_{n=1}^{\infty} A_n$ is countable, as A_n is countable for all n (by Theorem 1.4.13)

b) There does not exist a function $f : \mathbb{N} \to P(\mathbb{N})$ that is onto by the Cantor's theorem. Thus $\mathbb{N} \nsim P(\mathbb{N})$ by the defn of cardinality. Therefore $P(\mathbb{N})$ is uncountable

4

a) $f: \{0,1\} \to \mathbb{N}$

$f_i(0) = a_i, \; a_i \in \mathbb{N}$

$f_i(1) = b_i, \; b_i \in \mathbb{N}$

We can represent set of fuctions f_i as a set of pairs (a_i, b_i)

Let $A_n = \{(a,b) \mid a+b=n, \; a, b \in \mathbb{N}\}$

$A_2 = \{(1,1)\}$

$A_3 = \{(1,2), (2,1)\}$

$\vdots$

We can correspond all sets A_n to $\mathbb{N}$ and A_n will include all possible pairs (a,b).

ok

Therefore, the set is countable, as it has the same cardinality as $\mathbb{N}$.

b) $f: \mathbb{N} \to \{0,1\}$

$\mathbb{N}$ is domain, which is mapped by function f. $\mathbb{N}$ is infinite, so all outputs of the fuction f is infinite sequence of 1 and 0.

$f_1 = \{0, 0, 0, 0, 0 \ldots\}$

$f_2 = \{1, 0, 1, 0, 1 \ldots\}$

$f_3 = \{0, 0, 1, 1, 0, \ldots\}$

$\vdots \qquad f_{new} = \{\cancel{1}, 1, 0 \ldots\}$

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Similarly to the Cantor's proof ~~of~~ that $\mathbb{R}$ is uncountable ~~We can take~~ we always can take a function, with n-th output be different from n-th output of f_n, and therefore this function will be new and unique.

c) ?

Thus, set of all functions $f: \mathbb{N} \to \{0,1\}$ is uncountable.