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Course: Advanced Topics: Linear Algebra

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Problem 1

Show that the set of vectors

$E = \{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \} \in \mathbb{R}^3$

is linearly dependent, but the any set of 3 of them is linearly independent.

Proof. $1 * \begin{bmatrix}1\\0\\0\end{bmatrix} + 1 * \begin{bmatrix}0\\1\\0\end{bmatrix} + 1 * \begin{bmatrix}0\\0\\1\end{bmatrix} - 1 * \begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} = \vec{0}$

We found non-trivial linear combination of E equals to zero vector. Therefore, E is linearly dependent by definition.

Removing $\begin{bmatrix}1\\1\\1\end{bmatrix}$ vector we are left with $\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \}$, which is a basis for $\mathbb{R}^3$, and, therefore, linearly independent by definition.

Removing any other vector we are left with $\begin{bmatrix}1\\1\\1\end{bmatrix}$ vector and two vectors that both have common zero component. As $\begin{bmatrix}1\\1\\1\end{bmatrix}$ does not have any zero components, it would be impossible to create it by the linear combination. Therefore, $\begin{bmatrix}1\\1\\1\end{bmatrix}$ is linearly independent on the other two vector. Thus, the set is linearly independent. □

Problem 2

Which of the following sets of vectors are bases for $\mathbb{R}^3$?

$E = \{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix} \}$ $F = \{ \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \}$ $G = \{ \begin{bmatrix}-1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\-1\end{bmatrix} \}$

$H = \{ \begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix} \}$

Solution:

By Basis Extension Theorem for linearly independent $\vec{A_1}, ..., \vec{A_m} \in V$ $\exists n = dim(V) - m$ $\vec{B_1}, ..., \vec{B_n}$ such that $\{\vec{A_1}, ..., \vec{A_m}, \vec{B_1}, ..., \vec{B_n}\}$ is a basis. In our case we have for linearly independent $\vec{A_1}, ..., \vec{A_3} \in \mathbb{R}^3$ $\exists n = dim(V) - m = 3 - 3 = 0$ $\vec{B_1}, ..., \vec{B_n}$ such that $\{\vec{A_1}, ..., \vec{A_m}, \vec{B_1}, ..., \vec{B_n}\}$ is a basis. Thus, $\{\vec{A_1}, ..., \vec{A_3}\}$ is a basis of $\mathbb{R}^3$, if $\{\vec{A_1}, ..., \vec{A_3}\}$ is linearly independent.

$E = \{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix} \}$ E is linearly dependent if $r_1 \begin{bmatrix}1\\1\\1\end{bmatrix} + r_2 \begin{bmatrix}1\\1\\0\end{bmatrix} + r_3 \begin{bmatrix}1\\0\\0\end{bmatrix} = \vec{0} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

We get $\begin{cases} r_1 + r_2 + r_3 = 0, \\ r_1 + r_2 = 0, \\ r_1 = 0 \end{cases}$ Subtracting second from first and third from second we get $r_1 = 0, r_2 = 0, r_3 = 0$. However, this is not a non-trivial linear combination. Therefore, E is linearly independent by definition. Thus, E is a **basis** of $\mathbb{R}^3$.

$F = \{ \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \}$

$\begin{bmatrix}1\\0\\1\end{bmatrix} + \begin{bmatrix}0\\1\\0\end{bmatrix} - \begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} = \vec{0}$. We found a non-trivial linear combination that gives a zero vector. Therefore, F is linearly dependent by definition. Thus, F is **not a basis** of $\mathbb{R}^3$.

$G = \{ \begin{bmatrix}-1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\-1\end{bmatrix} \}$ G is linearly dependent if $r_1 \begin{bmatrix}-1\\1\\1\end{bmatrix} + r_2 \begin{bmatrix}1\\-1\\1\end{bmatrix} + r_3 \begin{bmatrix}1\\1\\-1\end{bmatrix} = \vec{0} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

We get $\begin{cases} -r_1 + r_2 + r_3 = 0, \\ r_1 - r_2 + r_3 = 0, \\ r_1 + r_2 - r_3 = 0 \end{cases}$ Adding each two of them will give us $r_1 = 0, r_2 = 0, r_3 = 0$. However, this is not a non-trivial linear combination. Therefore, G is linearly independent by definition. Thus, G is a **basis** of $\mathbb{R}^3$.

$H = \{ \begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix} \}$

$0 * \begin{bmatrix}1\\-1\\0\end{bmatrix} + \begin{bmatrix}1\\0\\-1\end{bmatrix} + \begin{bmatrix}-1\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} = \vec{0}$. We found a non-trivial linear combination that gives a zero vector. Therefore, F is linearly dependent by definition. Thus, F is **not a basis** of $\mathbb{R}^3$.

Problem 3
Find a basis for each of the following subspaces of $\mathbb{R}^3$

Solution:

$U = \{(x, y, z) | x - z = 0\}$

Possible subspaces of $\mathbb{R}^3$ are $\mathbb{R}^3$ space (if a subspace is $\mathbb{R}^3$ itself), plane, line, and point. $U \neq \mathbb{R}^3$. However, all three coordinates are changing. Therefore, U is a plane. By proof in Problem 2, two independent vectors span the plane.
Thus, $\{(1, 0, 1), (0, 1, 0)\}$ is a basis. (We cannot create one vector out of another one, as we cannot make 1 out of 0 term. Thus, they are linearly independent.)

$S = \{(x, y, z) | x + y + z = 0\}$

All three coordinates are changing. Therefore, S is a plane. Therefore two independent vectors in S span S.
Thus, $\{(3, -1, -2), (1, 3, -4)\}$ is a basis. $((3, -1, -2) = r(1, 3, -4) \implies r = 3, r = -1/3 \implies$

they are linearly independent, as r should be a constant)

$T = \{(x, y, z)|x + y - z = 0\}$
All three coordinates are changing. Therefore, T is a plane. Therefore two independent vectors in T span T.
Thus, $\{(3, 2, 5), (-1, 3, 2)\}$ is a basis ($(3, 2, 5) = r(-1, 3, 2) \implies r = 2/3, r = 5/2 \implies$ they are linearly independent, as r should be a constant).

$W = \{(x, y, z)|x = 0 \text{ and } y + z = 0\}$
X-coordinate is always zero, and two other coordinates are not free to choose (i.e. W is not a yz-plane) Therefore, W is a line. Therefore one independent vector in W span W.
Thus, $\{(0, -1, 1)\}$ is a basis. (any one vector is linearly independent)

Problem 4
Suppose that V is a finite dimensional vector space and S is a linear subspace of V. Show that there exists a linear subspace $T \subseteq V$ such that $S \cap T = \{\vec{0}\}$ and S + T = V.

Proof. Suppose dim(V) = n, and, therefore, $V = \{(v_1, v_2, ..., v_n)|v_i \in \mathbb{R}\}$.
Let $S = \{(0, v_2, ..., v_n)|v_i \in \mathbb{R}\}$ and $T = \{(v_1, 0, ..., 0)|v_i \in \mathbb{R}\}$.
$S \cap T = \{(0, 0, ..., 0)\} = \{\vec{0}\}$. $S + T = \{\vec{s} + \vec{t}|\vec{s} \in S, \vec{t} \in T\} = \{(0, v_2, ..., v_n) + (v_1, 0, ..., 0)\} = \{(v_1, v_2, ..., v_n)|v_i \in \mathbb{R}\} = V$

Example: Let $V = \mathbb{R}^2$ and let $S = \{(0, y)|y \in \mathbb{R}\}$ and $T = \{(x, 0)|x \in \mathbb{R}\}$.
$S \cap T = \{(0, 0)\} = \{\vec{0}\}$. $S + T = \{(0, y) + (x, 0)\} = \{(x, y)|x, y \in \mathbb{R}\} = \mathbb{R}^2$ □