**Artur Savchii** **Problem Set 9**

Course: Advanced Topics: Linear Algebra

Instructor: Mr. Blauss 21 February, 2024

---

**Problem 1**

Let S be the subspace of $\mathbb{R}^3$ given by: $S = \{(x, y, z)|y - z = 0\}$

Find a subspace $T \subseteq \mathbb{R}^3$ such that $S \cap T = \{\vec{0}\}$ and $S + T = \mathbb{R}^3$

*Solution:* $S = \{(x, y, z)|y - z = 0\} \implies S = \{(x, y, z)|y = z\}$. Thus, for $S \cap T = \{\vec{0}\}$, $y \neq z$ in T. $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is a basis for $\mathbb{R}^3$. Therefore, for $S + T = \mathbb{R}^3$, these three vectors have to be in either of the subspaces from the definition of the sum of subspaces.

$(1, 0, 0) \in S$ since y=z (0=0). Let $T = \{(x, y, z)|y \neq z\}$. Then $(0, 1, 0), (0, 0, 1) \in T$

Thus, if $T = \{(x, y, z)|y \neq z\}$, $S \cap T = \{\vec{0}\}$ and $S + T = \mathbb{R}^3$.

**Problem 2**

Let $\vec{P} = (a, b) \in \mathbb{R}^2$.

(a, b) are the coordinates of P relative to the basis $\vec{E_1} = (1, 0), \vec{E_2} = (0, 1)$

a) Find the coordinates of $\vec{P}$ relative to the basis $\vec{E_1} = (1, 0), \vec{E_2} = (0, 2)$

a) Find the coordinates of $\vec{P}$ relative to the basis $\vec{E_1} = (1, 1), \vec{E_2} = (-1, 2)$

*Solution:*

$\mathbb{R}^2$ is a finite dimensional vector space. By Theorem 6 and the definition of the coordinates, for ordered basis $\{\vec{E_1}, \vec{E_2}\}$, $\vec{P}$ can be written uniquely as $a_1\vec{A_1} + ... + a_n\vec{A_n} = \vec{P}$ and the sequence $(a_1, ..., a_n)$ is the coordinates. $a(1, 0) + b(0, 1) = (a, b)$

a) The vector $\vec{P}$ remains the same, but the coordinates and the basis change: $x_1(1, 0) + x_2(0, 2) = (a, b) \implies (x_1, 2x_2) = (a, b) \implies x_1 = a, x_2 = \frac{b}{2}$. The coordinates of $\vec{P}$ relative to the basis $\vec{E_1} = (1, 0), \vec{E_2} = (0, 2)$ are $(a, \frac{b}{2})$.

b) $x_1(1, 1) + x_2(-1, 2) = (a, b) \implies (x_1 - x_2, x_1 + 2x_2) = (a, b) \implies \begin{cases} x_1 - x_2 = a \\ x_1 + 2x_2 = b \end{cases}$.

Subtracting first from second we get $3x_2 = b - a \implies x_2 = \frac{b-a}{3}, x_1 = a + \frac{b-a}{3} = \frac{2a+b}{3}$.

the coordinates of $\vec{P}$ relative to the basis $\vec{E_1} = (1, 1), \vec{E_2} = (-1, 2)$ are $(\frac{2a+b}{3}, \frac{b-a}{3})$

**Problem 3**

Determine which of the followong are linear transformations?

*Solution:* T is a linear transformation if 1) $T(\vec{A} + \vec{B}) = T(\vec{A}) + T(\vec{B}) \forall \vec{A}, \vec{B} \in V$ and 2) $T(a\vec{A}) = aT(\vec{A}) \forall a \in \mathbb{R}, \vec{A} \in V$ by definition.

(1) $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that T(x, y) = (2x-y, x)

$T((x_1, y_1) + (x_2, y_2)) = T(x_1 + x_2, y_1 + y_2) = ((2(x_1 + x_2) - (y_1 + y_2)), (x_1 + x_2))$

$T(x_1, y_1) + T(x_2, y_2) = (2x_1 - y_1, x_1) + (2x_2 - y_2, x_2) = ((2(x_1 + x_2) - (y_1 + y_2)), (x_1 + x_2))$

$T((x_1, y_1) + (x_2, y_2)) = T(x_1, y_1) + T(x_2, y_2)$

$T(a(x, y)) = T(ax, ay) = (2ax - ay, ax) = a(2x - y, x) = aT(x, y)$

Both 1) and 2) hold. Thus T **is a linear transformation** by definition

2) $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(x, y) = (x^2, y^3)$

$T(a(x, y)) = T(ax, ay) = ((ax)^2, (ay)^3) = (a^2x^2, a^3y^3)$

$aT(x,y) = a(x^2, y^3) = (ax^2, ay^3)$
$T(a(x,y)) \neq aT(x,y)$. Thus, T **is not a linear transformation** by definition

3) $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that $T(x,y) = (xy, y, x)$
$T((x_1,y_1)+(x_2,y_2)) = T(x_1+x_2, y_1+y_2) = ((x_1+x_2)(y_1+y_2), y_1+y_2, x_1+x_2) = (x_1y_1 + x_1y_2 + x_1y_1 + x_2y_2, y_1+y_2, x_1+x_2)$
$T(x_1,y_1)+T(x_2,y_2) = (x_1y_1, y_1, x_1) + (x_2y_2, y_2, x_2) = (x_1y_1 + x_2y_2, y_1+y_2, x_1+x_2)$
$T((x_1,y_1)+(x_2,y_2)) = T(x_1,y_1)+T(x_2,y_2)$. Thus, T **is not a linear transformation** by definition

4) $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that $T(x,y) = (x+y, y, x)$
$T((x_1,y_1)+(x_2,y_2)) = T(x_1+x_2, y_1+y_2) = (x_1+x_2+y_1+y_2), y_1+y_2, x_1+x_2)$
$T(x_1,y_1)+T(x_2,y_2) = (x_1+y_1, y_1, x_1) + (x_2+y_2, y_2, x_2) = (x_1+y_1+x_2+y_2, y_1+y_2, x_1+x_2)$
$T((x_1,y_1)+(x_2,y_2)) = T(x_1,y_1)+T(x_2,y_2)$
$T(a(x,y)) = T(ax, ay) = (ax+ay, ay, ax) = a(x+y, y, x) = aT(x,y)$
Both 1) and 2) hold. Thus T **is a linear transformation** by definition

**Problem 4**
Let $T : \mathbb{R} \to \mathbb{R}$ be a linear transformation. Show that there exists $t \in \mathbb{R}$ depending only on T, such that T(x) = tx for all $x \in \mathbb{R}$

*Proof.* Let t = T(1). Because $T : \mathbb{R} \to \mathbb{R}$, $t \in \mathbb{R}$.
We know that $x \in \mathbb{R}$. Therefore, $xT(\vec{A}) = T(x\vec{A}) \forall \vec{A} \in \mathbb{R}$ by the definition of a linear transformation.
$xT(\vec{A}) = T(x\vec{A}) \implies xT(1) = T(x*1) = T(x) \implies xt = T(x)$ for any $x \in \mathbb{R}$ $\square$