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Course: Advanced Topics: Linear Algebra
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Problem Set 4
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Problem 1
Which of the following collections of vectors in $\mathbb{R}^3$ are subspaces?
a) $U = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 = 0\}$
b) $U = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_2 = 0\}$
c) $U = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 + x_2 = 0\}$
d) $U = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 + x_2 = 1\}$
e) $U = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 + x_2 \geq 0\}$

Solution:

a) $U = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 = 0\}$

Vector Addition: Let $A, B \in U$

$$A + B = (0, x_{A2}, x_{A3}) + (0, x_{B2}, x_{B3}) = (0, x_{A2} + x_{B2}, x_{A3} + x_{B3}) \in U$$

Because $A + B \in U$, U is closed under vector addition

Scalar Multiplication: Let $A \in U$ and $r \in \mathbb{R}$

$$rA = r * (0, x_2, x_3) = (r * 0, rx_2, rx_3) = (0, x_2, x_3) \in U$$

Because $rA \in U$, U is closed under scalar multiplication

U is closed under vector addition and scalar multiplication. Thus, **U is a subspace of** $\mathbb{R}^3$

b) $U = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_2 = 0\}$

Vector Addition: Let $A, B \in U$

$$A + B = (x_{A1}, 0, x_{A3}) + (x_{B1}, 0, x_{B3}) = (x_{A1} + x_{B1}, 0, x_{A3} + x_{B3}) \in U$$

Because $A + B \in U$, U is closed under vector addition

Scalar Multiplication: Let $A \in U$ and $r \in \mathbb{R}$

$$rA = r * (x_1, 0, x_3) = (rx_1, 0, rx_3) \in U$$

Because $rA \in U$, U is closed under scalar multiplication

U is closed under vector addition and scalar multiplication. Thus, **U is a subspace of** $\mathbb{R}^3$

c) $U = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 + x_2 = 0\}$

Vector Addition: Let $A, B \in U$

$$A + B = (x_{A1}, x_{A2}, x_{A3}) + (x_{B1}, x_{B2}, x_{B3}) = (x_{A1} + x_{B1}, x_{A2} + x_{B2}, x_{A3} + x_{B3})$$

$$x_1 + x_2 = (x_{A1} + x_{B1}) + (x_{A2} + x_{B2}) = (x_{A1} + x_{A2}) + (x_{B1} + x_{B2}) = 0 + 0 = 0$$

Thus, $A + B \in U \implies$ U is closed under vector addition

Scalar Multiplication: Let $A \in U$ and $r \in \mathbb{R}$

$$rA = r * (x_1, x_2, x_3) = (rx_1, rx_2, rx_3)$$

$$x_1' + x_2' = rx_1 + rx_2 = r(x_1 + x_2) = r * 0 = 0$$

Thus, $A + B \in U \implies$ U is closed under vector addition

U is closed under vector addition and scalar multiplication. Therefore, **U is a subspace of** $\mathbb{R}^3$

d) $U = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 + x_2 = 1\}$

Vector Addition: Let $A, B \in U$

$$A + B = (x_{A1}, x_{A2}, x_{A3}) + (x_{B1}, x_{B2}, x_{B3}) = (x_{A1} + x_{B1}, x_{A2} + x_{B2}, x_{A3} + x_{B3})$$

$$x_1 + x_2 = (x_{A1} + x_{B1}) + (x_{A2} + x_{B2}) = (x_{A1} + x_{A2}) + (x_{B1} + x_{B2}) = 1 + 1 = 2$$

Thus, $A + B \notin U \implies$ U is not closed under vector addition

U is not closed under vector addition. Therefore, **U is not a subspace of** $\mathbb{R}^3$

e) $U = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 + x_2 \geq 0\}$

Scalar Multiplication: Let $A \in U$ and $r \in \mathbb{R}$

$$rA = r * (x_1, x_2, x_3) = (rx_1, rx_2, rx_3)$$

$x_1^{'} + x_2^{'} = rx_1 + rx_2 = r(x_1 + x_2)$
For $r < 0$, $r(x_1 + x_2) \leq 0$
Thus, $A + B \in U \implies$ U is not closed under vector addition
U is not closed under scalar multiplication. Therefore, **U is not a subspace of** $\mathbb{R}^3$

Problem 2
Determine the subspace of $\mathbb{R}^3$ that is the linear span of the three vectors (1, 0, 1), (0, 1, 0), (0, 1, 1)

Solution: Let's check these vectors for linear dependence:
Try $(1, 0, 1) = r_1 * (0, 1, 0) + r_2 * (0, 1, 1) = (r_1 * 0 + r_2 * 0, r_1 * 1 + r_2 * 1, r_1 * 0 + r_2 * 1) = (0, (r_1 + r_2), r_2)$.
We have $1 = 0$. Thus, (1, 0, 1) is linearly independent of the span of (0, 1, 0) and (0, 1, 1)
Try $(0, 1, 0) = r_1 * (1, 0, 1) + r_2 * (0, 1, 1) = (r_1 * 1 + r_2 * 0, r_1 * 0 + r_2 * 1, r_1 * 1 + r_2 * 1) = (r_1, r_2, (r_1 + r_2))$.
We have $r_1 = 0, r_2 = 1$, and $r_1 + r_2 = 0$. Thus, (0, 1, 0) is linearly independent of a span of (1, 0, 1) and (0, 1, 1)
Try $(0, 1, 1) = r_1 * (1, 0, 1) + r_2 * (0, 1, 0) = (r_1 * 1 + r_2 * 0, r_1 * 0 + r_2 * 1, r_1 * 1 + r_2 * 0) = (r_1, r_2, r_1)$. We have $r_1 = 0$ and $r_1 = 1$. Thus, (0, 1, 1) is linearly independent of a span of (0, 1, 0) and (1, 0, 1)
Three vectors are linearly independent in $\mathbb{R}^3$. Therefore, their span is **the whole** $\mathbb{R}^3$ **space**.

Problem 3
Let $U = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 + x_2 + x_3 = a\}$ for some $a \in \mathbb{R}$
Show that U is a vector subspace if and only if a = 0.

Proof. $(\leftarrow)$ a = 0
Vector addition: Let $A, B \in U$
$A + B = (x_{A1}, x_{A2}, x_{A3}) + (x_{B1}, x_{B2}, x_{B3}) = (x_{A1} + x_{B1}, x_{A2} + x_{B2}, x_{A3} + x_{B3})$
$x_1 + x_2 + x_3 = (x_{A1} + x_{B1}) + (x_{A2} + x_{B2}) + (x_{A3} + x_{B3}) =$
$= (x_{A1} + x_{A2} + x_{A3}) + (x_{B1} + x_{B2} + x_{B3}) = 0 + 0 = 0$
Thus, $A + B \in U \implies$ U is closed under vector addition
Scalar multiplication: Let $A \in U$ and $r \in \mathbb{R}$
$r * A = (rx_{A1}, rx_{A2}, rx_{A3})$
$x_1 + x_2 + x_3 = rx_{A1} + rx_{A2} + rx_{A3} = r * (x_{A1} + x_{A2} + x_{A3}) = r * 0 = 0$
Thus, $rA \in U \implies$ U is closed under scalar multiplication
U is closed under vector addition and scalar multiplication. Thus, **U is a vector subspace**

$(\rightarrow)$ U is a vector subspace
Assume $a \neq 0$. Then we have for $A, B \in U$:
$A + B = (x_{A1}, x_{A2}, x_{A3}) + (x_{B1}, x_{B2}, x_{B3}) = (x_{A1} + x_{B1}, x_{A2} + x_{B2}, x_{A3} + x_{B3})$
$x_1 + x_2 + x_3 = (x_{A1} + x_{B1}) + (x_{A2} + x_{B2}) + (x_{A3} + x_{B3}) = (x_{A1} + x_{A2} + x_{A3}) + (x_{B1} + x_{B2} + x_{B3}) =$
$= a + a = 2a \neq 0$
We get $A + B \notin U$, which contradicts the property of U as a vector subspace.
Therefore, it is possible only for $\mathbf{a = 0}$.
$\square$

Problem 4
What is the span of $\{(1 + x), (1 - x)\}$ in $P(\mathbb{R})$?

Solution: Let's check these polynomials for linear dependence:
$1 - x = r * (1 + x) = r + rx$. But here we have r = 1 and r = -1. Thus, 1-x and 1+x are linearly independent.
The span of two linearly independent polynomials of the first order is all of the polynomials of the first order.
$\mathbf{L((1 + x), (1 - x)) = \{ax + b | a, b \in \mathbb{R}\}}$

Problem 5
Find a vector that spans the subspace $2x - 3y = 0$ of $\mathbb{R}$

Solution: The subspace $2x - 3y = 0$ is a line. Any vector on a line spans it, as we get all the vectors on a line by scalar multiplication of a single vector.
$y = \frac{2}{3}x$. For x = 1, $y = \frac{2}{3} * 1 = \frac{2}{3}$.
Thus, a vector $(1, \frac{2}{3})$ spans the subspace $2x - 3y = 0$

Problem 6
Is the plane $z = x + y + 1$ a vector subspace of of $\mathbb{R}^3$?

Solution: For a plane to be a subspace of $\mathbb{R}^3$, it has to contain a zero vector.
The zero vector in $\mathbb{R}^3$ is (0, 0, 0)
For z = x + y + 1, we have $0 = 0 + 0 + 1$, getting 0=1. Thus, the plane does not contain the zero vector.
Therefore, it is not a vector subspace of $\mathbb{R}^3$