# Problem Set 2.

1. Given $f: D \to \mathbb{R}$ and $B \subseteq \mathbb{R}$, let the <u>preimage</u> of $B$ - written $f^{-1}(B)$ be the set of all points in $D$ that get mapped to points in $B$, or $f^{-1}(B) = \{x \in D : f(x) \in B\}$

   a) Let $f(x) = x^2$, let $A = [0,4]$ and let $B = [-1,1]$.

   Find $f^{-1}(A)$ and $f^{-1}(B)$.

   b) Does $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ ?

   Does $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ ?

   c) Show that for all $A, B \subseteq \mathbb{R}$ and $f: \mathbb{R} \to \mathbb{R}$, the above statements hold.

2. Show that the sequence $x_1 = 1$, $x_{n+1} = \frac{1}{2} x_n + 1$ is ~~[illegible]~~ bounded above by 2. (I.e.: that $x_n \leq 2$ for all $n \in \mathbb{N}$)

3 a) Let $y_1 = 1$ and for each $n \in \mathbb{N}$ define $y_{n+1} = (3y_n + 4)/4$.

   Show that $y_n \leq 4$ for all $n \in \mathbb{N}$.

   b) Show that the sequence $y_1, y_2, \ldots$ is increasing.

4. Let $\mathbb{Z}_5 = \{0,1,2,3,4\}$ and define addition and multiplication modulo 5 (i.e. define them as we normally do, but subtract multiples of 5 until the sum/product $-5n \in \mathbb{Z}_5$.)

   a) Show that $\forall z \in \mathbb{Z}_5$, there exists an element $y$ for which $z + y = 0$. (This is called an additive inverse.)

   b) Show that for any $z \in \mathbb{Z}_5$, $z \neq 0$, there exists an element $y$ for which $z \cdot y = 1$. (This is called a multiplicative inverse)

   c) A set that has well defined additive and multiplicative inverses for all non-zero elements is called a field.
   Is $\mathbb{Z}_4 = \{0,1,2,3\}$ a field?