Algorithm for solving structural problems

2D bar elements

1 Geometry

The following terms are defined

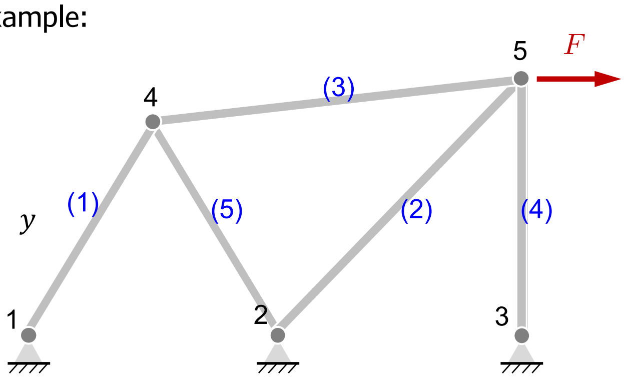
n_d : Problem dimension, i.e. $n_\mathrm{d} = 2$ (2D).
n_el : Total number of bars.
n_nod : Total number of nodes (joints).
n_ne : Number of nodes in a bar, i.e. $n_\mathrm{ne} = 2$.
n_i : Degrees of freedom per node, i.e. $n_\mathrm{i} = 2$.
n_dof : Total number of degrees of freedom, i.e. $n_\mathrm{dof} = n_\mathrm{nod} \times n_\mathrm{i}$.

$\mathbf{x}$: Nodal coordinates array ($n_\mathrm{nod} \times n_\mathrm{d}$):

$$\mathbf{x} = \begin{bmatrix} x_1 & y_1 \\ & \vdots & \\ x_n & y_n \end{bmatrix}.$$

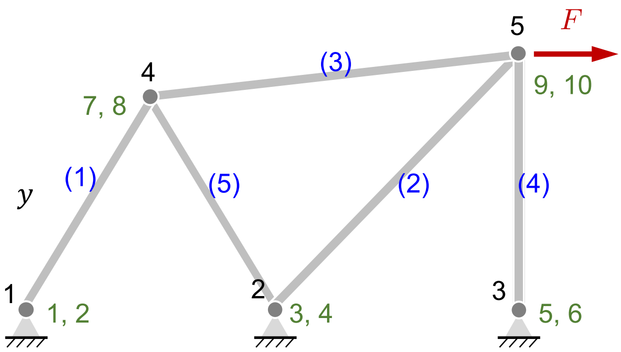
$\mathbf{T}_\mathrm{n}$: <u>Nodal</u> connectivity table ($n_\mathrm{el} \times n_\mathrm{ne}$). Example:

$$\mathbf{T}_\mathrm{n} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 4 & 5 \\ 3 & 5 \\ 2 & 4 \end{bmatrix}.$$



$\mathbf{T}_\mathrm{d}$: <u>Degrees of freedom</u> connectivity table ($n_\mathrm{el} \times (n_\mathrm{ne} \times n_\mathrm{i})$). Example:

$$\mathbf{T}_\mathrm{d} = \begin{bmatrix} 1 & 2 & 7 & 8 \\ 3 & 4 & 9 & 10 \\ 7 & 8 & 9 & 10 \\ 5 & 6 & 9 & 10 \\ 3 & 4 & 7 & 8 \end{bmatrix}.$$



2 Computation of the element stiffness matrices

For each bar $e = 1 \dots n_{\mathrm{el}}$

a) Compute element stiffness matrix

$x_1^e = \mathbf{x}(\mathbf{T}_{\mathrm{n}}(e,1),1)$

$y_1^e = \mathbf{x}(\mathbf{T}_{\mathrm{n}}(e,1),2)$

$x_2^e = \mathbf{x}(\mathbf{T}_{\mathrm{n}}(e,2),1)$

$y_2^e = \mathbf{x}(\mathbf{T}_{\mathrm{n}}(e,2),2)$

$l^e = \sqrt{(x_2^e - x_1^e)^2 + (y_2^e - y_1^e)^2}$

$s^e = \frac{(y_2^e - y_1^e)}{l^e}$

$c^e = \frac{x_2^e - x_1^e}{l^e}$

$$\mathbf{K}^e = \frac{A^e E^e}{l^e}\begin{bmatrix} (c^e)^2 & c^e s^e & -(c^e)^2 & -c^e s^e \\ c^e s^e & (s^e)^2 & -c^e s^e & -(s^e)^2 \\ -(c^e)^2 & -c^e s^e & (c^e)^2 & c^e s^e \\ -c^e s^e & -(s^e)^2 & c^e s^e & (s^e)^2 \end{bmatrix}$$

b) Store element matrix

For each $r = 1 \dots n_{\mathrm{ne}} \times n_{\mathrm{i}}$

For each $s = 1 \dots n_{\mathrm{ne}} \times n_{\mathrm{i}}$

$\mathbf{K}_{\mathrm{el}}(r,s,e) = \mathbf{K}^e(r,s)$

Next s

Next r

Next bar e

3 Global stiffness matrix assembly

Assembly operator:

$$\mathbf{K}_{\mathrm{G}} = \mathop{\mathbf{A}}_{e=1}^{n_{\mathrm{el}}} \mathbf{K}^e$$

Initialization:

$$\mathbf{K}_{\mathrm{G}} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix},\quad \text{dimensions: } n_{\mathrm{dof}} \times n_{\mathrm{dof}}$$

For each bar $e = 1 \dots n_{\text{el}}$

For each local degree of freedom $i = 1 \dots n_{\text{ne}} \times n_{\text{i}}$ (rows)

$I = \mathbf{T}_{\text{d}}(e, i)$ (corresponding global degree of freedom)

For each local degree of freedom $j = 1 \dots n_{\text{ne}} \times n_{\text{i}}$ (columns)

$J = \mathbf{T}_{\text{d}}(e, j)$ (corresponding global degree of freedom)

$\mathbf{K}_{\text{G}}(I, J) = \mathbf{K}_{\text{G}}(I, J) + \mathbf{K}_{\text{el}}(i, j, e)$

Next j

Next i

Next bar e

4 Global system of equations

Global system:

$$\mathbf{K}_{\text{G}} \hat{\mathbf{u}} = \widehat{\mathbf{F}}^{\text{ext}} + \hat{\mathbf{R}}$$

Compute external forces vector:

$\widehat{\mathbf{F}}^{\text{ext}} = [\dots]$: Array with the applied external forces.

Apply conditions:

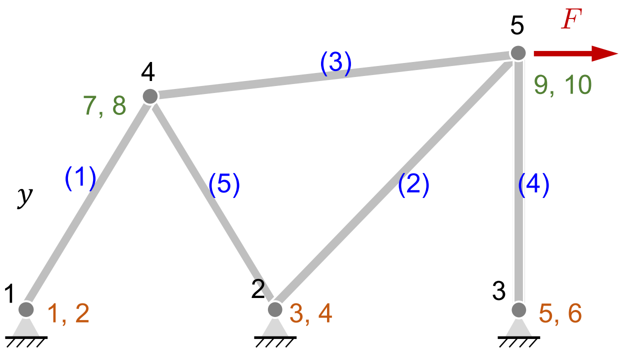
$\hat{\mathbf{u}}_{\text{R}} = [\dots]$: Array with the displacements on the imposed degrees of freedom.

$\nu_{\text{R}} = [\dots]$: Array with the <u>imposed</u> degrees of freedom.

$\nu_{\text{L}} = [\dots]$: Array with the <u>free</u> degrees of freedom.

* In the previous example:

$$\widehat{\mathbf{F}}^{\text{ext}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ F \\ 0 \end{bmatrix},$$



$$\hat{\mathbf{u}}_{\text{R}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \nu_{\text{R}} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}, \quad \nu_{\text{L}} = \begin{bmatrix} 7 \\ 8 \\ 9 \\ 10 \end{bmatrix}.$$

Partitioned system of equations:

$$\begin{aligned}
\mathbf{K}_{\mathrm{LL}} &= \mathbf{K}_{\mathrm{G}}(\nu_{\mathrm{L}}, \nu_{\mathrm{L}}) \\
\mathbf{K}_{\mathrm{LR}} &= \mathbf{K}_{\mathrm{G}}(\nu_{\mathrm{L}}, \nu_{\mathrm{R}}) \\
\mathbf{K}_{\mathrm{RL}} &= \mathbf{K}_{\mathrm{G}}(\nu_{\mathrm{R}}, \nu_{\mathrm{L}}) \\
\mathbf{K}_{\mathrm{RR}} &= \mathbf{K}_{\mathrm{G}}(\nu_{\mathrm{R}}, \nu_{\mathrm{R}}) \\
\widehat{\mathbf{F}}_{\mathrm{L}}^{\mathrm{ext}} &= \widehat{\mathbf{F}}^{\mathrm{ext}}(\nu_{\mathrm{L}}, 1) \\
\widehat{\mathbf{F}}_{\mathrm{R}}^{\mathrm{ext}} &= \widehat{\mathbf{F}}^{\mathrm{ext}}(\nu_{\mathrm{R}}, 1)
\end{aligned}$$

$$\begin{bmatrix} \mathbf{K}_{\mathrm{RR}} & \mathbf{K}_{\mathrm{RL}} \\ \mathbf{K}_{\mathrm{LR}} & \mathbf{K}_{\mathrm{LL}} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{u}}_{\mathrm{R}} \\ \hat{\mathbf{u}}_{\mathrm{L}} \end{bmatrix} = \begin{bmatrix} \widehat{\mathbf{F}}_{\mathrm{R}}^{\mathrm{ext}} \\ \widehat{\mathbf{F}}_{\mathrm{L}}^{\mathrm{ext}} \end{bmatrix} + \begin{bmatrix} \hat{\mathbf{R}}_{\mathrm{R}} \\ \mathbf{0} \end{bmatrix}$$

Data:

$\hat{\mathbf{u}}_{\mathrm{R}}$: Imposed displacement vector
$\widehat{\mathbf{F}}^{\mathrm{ext}}$: External force vector

Unknowns:

$\hat{\mathbf{u}}_{\mathrm{L}}$: Free displacement vector
$\hat{\mathbf{R}}_{\mathrm{R}}$: Reactions vector

System resolution:

$$\mathbf{K}_{\mathrm{LL}} \hat{\mathbf{u}}_{\mathrm{L}} = \widehat{\mathbf{F}}_{\mathrm{L}}^{\mathrm{ext}} - \mathbf{K}_{\mathrm{LR}} \hat{\mathbf{u}}_{\mathrm{R}} \quad \rightarrow \quad \hat{\mathbf{u}}_{\mathrm{L}} = \mathbf{K}_{\mathrm{LL}}^{-1} \left(\widehat{\mathbf{F}}_{\mathrm{L}}^{\mathrm{ext}} - \mathbf{K}_{\mathrm{LR}} \hat{\mathbf{u}}_{\mathrm{R}} \right)$$

$$\hat{\mathbf{R}}_{\mathrm{R}} = \mathbf{K}_{\mathrm{RR}} \hat{\mathbf{u}}_{\mathrm{R}} + \mathbf{K}_{\mathrm{RL}} \hat{\mathbf{u}}_{\mathrm{L}} - \widehat{\mathbf{F}}_{\mathrm{R}}^{\mathrm{ext}}$$

Obtain generalized displacement vector:

$$\hat{\mathbf{u}}(\nu_{\mathrm{L}}, 1) = \hat{\mathbf{u}}_{\mathrm{L}}$$

$$\hat{\mathbf{u}}(\nu_{\mathrm{R}}, 1) = \hat{\mathbf{u}}_{\mathrm{R}}$$

5 Strains and stresses

Strains and stresses for each bar can be computed by means of the following procedure:

For each bar $e = 1 \dots n_{el}$

a) Compute the rotation matrix

$$x_1^e = \mathbf{x}(\mathbf{T}_{\mathrm{n}}(e,1), 1)$$

$$y_1^e = \mathbf{x}(\mathbf{T}_{\mathrm{n}}(e,1), 2)$$

$$x_2^e = \mathbf{x}(\mathbf{T}_{\mathrm{n}}(e,2), 1)$$

$$y_2^e = \mathbf{x}(\mathbf{T}_{\mathrm{n}}(e,2), 2)$$

$l^e = \sqrt{(x_2^e - x_1^e)^2 + (y_2^e - y_1^e)^2}$

$s^e = \frac{(y_2^e - y_1^e)}{l^e}$

$c^e = \frac{x_2^e - x_1^e}{l^e}$

$$\mathbf{R}^e = \begin{bmatrix} c^e & s^e & 0 & 0 \\ -s^e & c^e & 0 & 0 \\ 0 & 0 & c^e & s^e \\ 0 & 0 & -s^e & c^e \end{bmatrix}$$

b) Obtain element displacement in <u>global</u> coordinates

For each local degree of freedom $i = 1 \dots n_{ne} \times n_i$

$I = \mathbf{T}_d(e, i)$

$\hat{\mathbf{u}}^e(i, 1) = \hat{\mathbf{u}}(I)$

Next i

c) Compute element displacement in <u>local</u> coordinates

$\hat{\mathbf{u}}^{e\prime} = \mathbf{R}^e \hat{\mathbf{u}}^e$

d) Compute element strain

$\hat{\varepsilon}^e = \frac{1}{l^e}[-1 \quad 0 \quad 1 \quad 0]\hat{\mathbf{u}}^{e\prime}$

e) Compute element stress

$\hat{\sigma}^e = E^e \hat{\varepsilon}^e$

Next bar e