Algorithm for solving structural problems
3D bar elements

1 Geometry

The following terms are defined

n_d : Problem dimension, i.e. $n_\mathrm{d} = 3$ (3D).
n_el : Total number of bars.
n_nod : Total number of nodes (joints).
n_ne : Number of nodes in a bar, i.e. $n_\mathrm{ne} = 2$.
n_i : Degrees of freedom per node, i.e. $n_\mathrm{i} = 3$.
n_dof : Total number of degrees of freedom, i.e. $n_\mathrm{dof} = n_\mathrm{nod} \times n_\mathrm{i}$.

$\mathbf{x}$: Nodal coordinates array ($n_\mathrm{nod} \times n_\mathrm{d}$):

$$\mathbf{x} = \begin{bmatrix} x_1 & y_1 & z_1 \\ & \vdots & \\ x_n & y_n & z_n \end{bmatrix}.$$

$\mathbf{T}_\mathrm{n}$: Nodal connectivity table ($n_\mathrm{el} \times n_\mathrm{ne}$).

$\mathbf{T}_\mathrm{d}$: Degrees of freedom connectivity table ($n_\mathrm{el} \times\ \ n_\mathrm{ne} \times n_\mathrm{i}$).

2 Computation of the element stiffness matrices

For each bar $e = 1 \dots n_\mathrm{el}$

a) Compute element stiffness matrix

$$x_1^e = \mathbf{x}\ \ \mathbf{T}_n\ \ e,1\ \ ,1$$

$$x_2^e = \mathbf{x}\ \ \mathbf{T}_n\ \ e,2\ \ ,1$$

$$y_1^e = \mathbf{x}\ \ \mathbf{T}_n\ \ e,1\ \ ,2$$

$$y_2^e = \mathbf{x}\ \ \mathbf{T}_n\ \ e,2\ \ ,2$$

$$z_1^e = \mathbf{x}\ \ \mathbf{T}_n\ \ e,1\ \ ,3$$

$$z_2^e = \mathbf{x}\ \ \mathbf{T}_n\ \ e,2\ \ ,3$$

$$l^e = \sqrt{\ x_2^e - x_1^e\ \ ^2 + \ y_2^e - y_1^e\ \ ^2 + \ z_2^e - z_1^e\ \ ^2}$$

$$\mathbf{R}^e = \frac{1}{l^e}\begin{bmatrix} x_2^e - x_1^e & y_2^e - y_1^e & z_2^e - z_1^e & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2^e - x_1^e & y_2^e - y_1^e & z_2^e - z_1^e \end{bmatrix}$$

$$\mathbf{K}^{e\prime} = \frac{A^e E^e}{l^e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\mathbf{K}^e = \mathbf{R}^{e\mathrm{T}}\mathbf{K}^{e\prime}\mathbf{R}^e$$

b) Store element matrix

For each $r = 1 \dots n_{\mathrm{ne}} \times n_{\mathrm{i}}$

For each $s = 1 \dots n_{\mathrm{ne}} \times n_{\mathrm{i}}$

$$\mathbf{K}_{\mathrm{el}}\ \ r,s,e\ \ = \mathbf{K}^{e}(r,s)$$

Next s

Next r

Next bar e

3 Global stiffness matrix assembly

Assembly operator:

$$\mathbf{K}_{\mathrm{G}} = \mathop{\mathbf{A}}_{e=1}^{n_{\mathrm{el}}} \mathbf{K}^{e}$$

Initialization:

$$\mathbf{K}_{\mathrm{G}} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}, \quad \text{dimensions: } n_{\mathrm{dof}} \times n_{\mathrm{dof}}$$

For each bar $e = 1 \dots n_{\mathrm{el}}$

For each local degree of freedom $i = 1 \dots n_{\mathrm{ne}} \times n_{\mathrm{i}}$ (rows)

$I = \mathbf{T}_{\mathrm{d}}\ \ e,i$ (corresponding global degree of freedom)

For each local degree of freedom $j = 1 \dots n_{\mathrm{ne}} \times n_{\mathrm{i}}$ (columns)

$J = \mathbf{T}_{\mathrm{d}}\ \ e,j$ (corresponding global degree of freedom)

$$\mathbf{K}_{\mathrm{G}}\ \ I,J\ \ = \mathbf{K}_{\mathrm{G}}\ \ I,J\ \ + \mathbf{K}_{\mathrm{el}}\ \ i,j,e$$

Next j

Next i

Next bar e

4 Global system of equations

Global system:

$$\mathbf{K}_{\mathrm{G}}\mathbf{u} = \widehat{\mathbf{F}}^{\mathrm{ext}} + \mathbf{R}$$

Compute external forces vector:

$\widehat{\mathbf{F}}^{\mathrm{ext}} = [\dots]$: Array with the applied external forces.

Apply conditions:

$\mathbf{u}_{\mathrm{R}} = [\ldots]$: Array with the displacements on the imposed degrees of freedom.

$\nu_{\mathrm{R}} = [\ldots]$: Array with the <u>imposed</u> degrees of freedom.

$\nu_{\mathrm{L}} = [\ldots]$: Array with the <u>free</u> degrees of freedom.

Partitioned system of equations:

$$
\begin{aligned}
&\mathbf{K}_{\mathrm{LL}} = \mathbf{K}_{\mathrm{G}}\ \nu_{\mathrm{L}},\nu_{\mathrm{L}} \\
&\mathbf{K}_{\mathrm{LR}} = \mathbf{K}_{\mathrm{G}}\ \nu_{\mathrm{L}},\nu_{\mathrm{R}} \\
&\mathbf{K}_{\mathrm{RL}} = \mathbf{K}_{\mathrm{G}}\ \nu_{\mathrm{R}},\nu_{\mathrm{L}} \\
&\mathbf{K}_{\mathrm{RR}} = \mathbf{K}_{\mathrm{G}}\ \nu_{\mathrm{R}},\nu_{\mathrm{R}} \\
&\widehat{\mathbf{F}}^{\mathrm{ext}}_{\mathrm{L}} = \widehat{\mathbf{F}}^{\mathrm{ext}}\ \nu_{\mathrm{L}},1 \\
&\widehat{\mathbf{F}}^{\mathrm{ext}}_{\mathrm{R}} = \widehat{\mathbf{F}}^{\mathrm{ext}}\ \nu_{\mathrm{R}},1
\end{aligned}
$$

$$
\begin{bmatrix} \mathbf{K}_{\mathrm{RR}} & \mathbf{K}_{\mathrm{RL}} \\ \mathbf{K}_{\mathrm{LR}} & \mathbf{K}_{\mathrm{LL}} \end{bmatrix}
\begin{bmatrix} \mathbf{u}_{\mathrm{R}} \\ \mathbf{u}_{\mathrm{L}} \end{bmatrix}
=
\begin{bmatrix} \widehat{\mathbf{F}}^{\mathrm{ext}}_{\mathrm{R}} \\ \widehat{\mathbf{F}}^{\mathrm{ext}}_{\mathrm{L}} \end{bmatrix}
+
\begin{bmatrix} \mathbf{R}_{\mathrm{R}} \\ \mathbf{0} \end{bmatrix}
$$

Data:

$\mathbf{u}_{\mathrm{R}}$	:	Imposed displacement vector
$\widehat{\mathbf{F}}^{\mathrm{ext}}$	:	External force vector

Unknowns:

$\mathbf{u}_{\mathrm{L}}$	:	Free displacement vector
$\mathbf{R}_{\mathrm{R}}$	:	Reactions vector

System resolution:

$$
\mathbf{K}_{\mathrm{LL}}\mathbf{u}_{\mathrm{L}} = \widehat{\mathbf{F}}^{\mathrm{ext}}_{\mathrm{L}} - \mathbf{K}_{\mathrm{LR}}\mathbf{u}_{\mathrm{R}} \quad \rightarrow \qquad \mathbf{u}_{\mathrm{L}} = \mathbf{K}_{\mathrm{LL}}^{-1}\left(\widehat{\mathbf{F}}^{\mathrm{ext}}_{\mathrm{L}} - \mathbf{K}_{\mathrm{LR}}\mathbf{u}_{\mathrm{R}}\right)
$$

$$
\mathbf{R}_{\mathrm{R}} = \mathbf{K}_{\mathrm{RR}}\mathbf{u}_{\mathrm{R}} + \mathbf{K}_{\mathrm{RL}}\mathbf{u}_{\mathrm{L}} - \widehat{\mathbf{F}}^{\mathrm{ext}}_{\mathrm{R}}
$$

Obtain generalized displacement vector:

$$\mathbf{u}\ \nu_{\mathrm{L}},1 \ = \mathbf{u}_{\mathrm{L}}$$

$$\mathbf{u}\ \nu_{\mathrm{R}},1 \ = \mathbf{u}_{\mathrm{R}}$$

5 Strains and stresses

Strains and stresses for each bar can be computed by means of the following procedure:

For each bar $e = 1 \ldots n_{el}$

a) Compute the rotation matrix

$$x_1^e = \mathbf{x}\ \mathbf{T}_n\ e,1\ ,1$$

$$x_2^e = \mathbf{x}\ \mathbf{T}_n\ e,2\ ,1$$

$y_1^e = \mathbf{x}\ \ \mathbf{T}_n\ \ e,1\ \ ,2$

$y_2^e = \mathbf{x}\ \ \mathbf{T}_n\ \ e,2\ \ ,2$

$z_1^e = \mathbf{x}\ \ \mathbf{T}_n\ \ e,1\ \ ,3$

$z_2^e = \mathbf{x}\ \ \mathbf{T}_n\ \ e,2\ \ ,3$

$l^e = \sqrt{\ x_2^e - x_1^e\ \ ^2 + \ y_2^e - y_1^e\ \ ^2 + \ z_2^e - z_1^e\ \ ^2}$

$\mathbf{R}^e = \frac{1}{l^e}\begin{bmatrix} x_2^e - x_1^e & y_2^e - y_1^e & z_2^e - z_1^e & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2^e - x_1^e & y_2^e - y_1^e & z_2^e - z_1^e \end{bmatrix}$

b) Obtain element displacement in global coordinates

For each local degree of freedom $i = 1 \dots n_{ne} \times n_i$

$I = \mathbf{T}_\mathrm{d}\ \ e,i$

$\mathbf{u}^e\ \ i,1\ \ = \mathbf{u}\ \ I$

Next i

c) Compute element displacement in local coordinates

$\mathbf{u}^{e\prime} = \mathbf{R}^e \mathbf{u}^e$

d) Compute element strain

$\varepsilon^e = \frac{1}{l^e}[-1 \quad 1]\mathbf{u}^{e\prime}$

e) Compute element stress

$\sigma^e = E^e \varepsilon^e$

Next bar e