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Algorithm for solving structural problems

1D beam elements

1 Geometry

The following terms are defined

n_d : Problem dimension, i.e. $n_\mathrm{d} = 1$ (3D).
n_el : Total number of beams.
n_nod : Total number of nodes (joints).
n_ne : Number of nodes in a beam, i.e. $n_\mathrm{ne} = 2$.
n_i : Degrees of freedom per node, i.e. $n_\mathrm{i} = 2$ (deflection and section rotation)
n_dof : Total number of degrees of freedom, i.e. $n_\mathrm{dof} = n_\mathrm{nod} \times n_\mathrm{i}$.

$\mathbf{x}$: Nodal coordinates array ($n_\mathrm{nod} \times n_\mathrm{d}$):

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_{n_\mathrm{nod}} \end{bmatrix}.$$

$\mathbf{T}_\mathrm{n}$: Nodal connectivity table ($n_\mathrm{el} \times n_\mathrm{ne}$).
$\mathbf{T}_\mathrm{d}$: Degrees of freedom connectivity table ($n_\mathrm{el} \times (n_\mathrm{ne} \times n_\mathrm{i})$).

2 Computation of the element stiffness matrices

For each beam $e = 1 \dots n_\mathrm{el}$

a) Compute element stiffness matrix

$$x_1^e = \mathbf{x}(\mathbf{T}_n(e,1),1)$$

$$x_2^e = \mathbf{x}(\mathbf{T}_n(e,2),1)$$

$$l^e = |x_2^e - x_1^e|$$

$$\mathbf{K}^e = \frac{I_z^e E^e}{l^{e3}} \begin{bmatrix} 12 & 6l^e & -12 & 6l^e \\ 6l^e & 4l^{e2} & -6l^e & 2l^{e2} \\ -12 & -6l^e & 12 & -6l^e \\ 6l^e & 2l^{e2} & -6l^e & 4l^{e2} \end{bmatrix}$$

b) Store element matrix

For each $r = 1 \dots n_\mathrm{ne} \times n_\mathrm{i}$

For each $s = 1 \dots n_\mathrm{ne} \times n_\mathrm{i}$

$$\mathbf{K}_\mathrm{el}(r,s,e) = \mathbf{K}^e(r,s)$$

Next s

Next r

Next beam e

3 Computation of the element force vector

For each beam $e = 1 \dots n_{\mathrm{el}}$

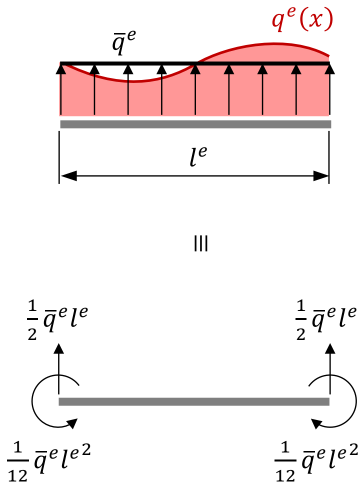
a) Compute element force vector

$x_1^e = \mathbf{x}(\mathbf{T}_n(e,1),1)$

$x_2^e = \mathbf{x}(\mathbf{T}_n(e,2),1)$

$l^e = |x_2^e - x_1^e|$

$$\mathbf{F}^e = \frac{\bar{q}^e l^e}{2}\begin{bmatrix} 1 \\ l^e/6 \\ 1 \\ -l^e/6 \end{bmatrix}$$



a) Store element force vector

For each $r = 1 \dots n_{\mathrm{ne}} \times n_{\mathrm{i}}$

$\mathbf{F}_{\mathrm{el}}(r,e) = \mathbf{F}^e(r)$

Next r

Next beam e

4 Global stiffness matrix and force vector assembly

Initialization:

$\widehat{\mathbf{F}}^{\mathrm{ext}} = [\dots]$: Array with the applied external **point forces** (dimensions: $n_{\mathrm{dof}} \times 1$).

$$\mathbf{K}_{\mathrm{G}} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}, \quad \text{dimensions: } n_{\mathrm{dof}} \times n_{\mathrm{dof}}$$

For each beam $e = 1 \dots n_{\mathrm{el}}$

For each local degree of freedom $i = 1 \dots n_{\mathrm{ne}} \times n_{\mathrm{i}}$ (rows)

$I = \mathbf{T}_{\mathrm{d}}(e,i)$ (corresponding global degree of freedom)

$\widehat{\mathbf{F}}^{\mathrm{ext}}(I) = \widehat{\mathbf{F}}^{\mathrm{ext}}(I) + \mathbf{F}_{\mathrm{el}}(i,e)$

For each local degree of freedom $j = 1 \dots n_{\mathrm{ne}} \times n_{\mathrm{i}}$ (columns)

$J = \mathbf{T}_{\mathrm{d}}(e,j)$ (corresponding global degree of freedom)

$\mathbf{K}_{\mathrm{G}}(I,J) = \mathbf{K}_{\mathrm{G}}(I,J) + \mathbf{K}_{\mathrm{el}}(i,j,e)$

Next j

Next i

Next beam e

5 Global system of equations

Global system:

$$\mathbf{K}_{\mathrm{G}}\hat{\mathbf{u}} = \widehat{\mathbf{F}}^{\mathrm{ext}} + \hat{\mathbf{R}}$$

Apply conditions:

$\hat{\mathbf{u}}_{\mathrm{R}} = [\dots]$: Array with the displacements on the imposed degrees of freedom.

$\nu_{\mathrm{R}} = [\dots]$: Array with the <u>imposed</u> degrees of freedom.

$\nu_{\mathrm{L}} = [\dots]$: Array with the <u>free</u> degrees of freedom.

Partitioned system of equations:

$$\begin{aligned}
\mathbf{K}_{\mathrm{LL}} &= \mathbf{K}_{\mathrm{G}}(\nu_{\mathrm{L}}, \nu_{\mathrm{L}}) \\
\mathbf{K}_{\mathrm{LR}} &= \mathbf{K}_{\mathrm{G}}(\nu_{\mathrm{L}}, \nu_{\mathrm{R}}) \\
\mathbf{K}_{\mathrm{RL}} &= \mathbf{K}_{\mathrm{G}}(\nu_{\mathrm{R}}, \nu_{\mathrm{L}}) \\
\mathbf{K}_{\mathrm{RR}} &= \mathbf{K}_{\mathrm{G}}(\nu_{\mathrm{R}}, \nu_{\mathrm{R}}) \\
\widehat{\mathbf{F}}_{\mathrm{L}}^{\mathrm{ext}} &= \widehat{\mathbf{F}}^{\mathrm{ext}}(\nu_{\mathrm{L}}, 1) \\
\widehat{\mathbf{F}}_{\mathrm{R}}^{\mathrm{ext}} &= \widehat{\mathbf{F}}^{\mathrm{ext}}(\nu_{\mathrm{R}}, 1)
\end{aligned}$$

$$\begin{bmatrix} \mathbf{K}_{\mathrm{RR}} & \mathbf{K}_{\mathrm{RL}} \\ \mathbf{K}_{\mathrm{LR}} & \mathbf{K}_{\mathrm{LL}} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{u}}_{\mathrm{R}} \\ \hat{\mathbf{u}}_{\mathrm{L}} \end{bmatrix} = \begin{bmatrix} \widehat{\mathbf{F}}_{\mathrm{R}}^{\mathrm{ext}} \\ \widehat{\mathbf{F}}_{\mathrm{L}}^{\mathrm{ext}} \end{bmatrix} + \begin{bmatrix} \hat{\mathbf{R}}_{\mathrm{R}} \\ \mathbf{0} \end{bmatrix}$$

Data:

$\hat{\mathbf{u}}_{\mathrm{R}}$: Imposed displacement vector
$\widehat{\mathbf{F}}^{\mathrm{ext}}$: External force vector

Unknowns:

$\hat{\mathbf{u}}_{\mathrm{L}}$: Free displacement vector
$\hat{\mathbf{R}}_{\mathrm{R}}$: Reactions vector

System resolution:

$$\mathbf{K}_{\mathrm{LL}}\hat{\mathbf{u}}_{\mathrm{L}} = \widehat{\mathbf{F}}_{\mathrm{L}}^{\mathrm{ext}} - \mathbf{K}_{\mathrm{LR}}\hat{\mathbf{u}}_{\mathrm{R}} \quad \rightarrow \quad \hat{\mathbf{u}}_{\mathrm{L}} = \mathbf{K}_{\mathrm{LL}}^{-1}\left(\widehat{\mathbf{F}}_{\mathrm{L}}^{\mathrm{ext}} - \mathbf{K}_{\mathrm{LR}}\hat{\mathbf{u}}_{\mathrm{R}}\right)$$

$$\hat{\mathbf{R}}_{\mathrm{R}} = \mathbf{K}_{\mathrm{RR}}\hat{\mathbf{u}}_{\mathrm{R}} + \mathbf{K}_{\mathrm{RL}}\hat{\mathbf{u}}_{\mathrm{L}} - \widehat{\mathbf{F}}_{\mathrm{R}}^{\mathrm{ext}}$$

Obtain generalized displacement vector:

$$\hat{\mathbf{u}}(\nu_{\mathrm{L}}, 1) = \hat{\mathbf{u}}_{\mathrm{L}}$$

$$\hat{\mathbf{u}}(\nu_{\mathrm{R}}, 1) = \hat{\mathbf{u}}_{\mathrm{R}}$$

6 Compute internal distributions

For each beam $e = 1 \dots n_{\text{el}}$

$$x_1^e = \mathbf{x}(\mathbf{T}_n(e,1),1)$$

$$x_2^e = \mathbf{x}(\mathbf{T}_n(e,2),1)$$

$$l^e = |x_2^e - x_1^e|$$

a) Element nodes displacement

For each local degree of freedom $i = 1 \dots n_{\text{ne}} \times n_{\text{i}}$

$$I = \mathbf{T}_{\text{d}}(e,i)$$

$$\hat{\mathbf{u}}^e(i,1) = \hat{\mathbf{u}}(I)$$

Next i

b) Internal forces at element nodes

$$\widehat{\mathbf{F}}_{\text{int}}^e = \mathbf{K}_{\text{el}}(:,:,e)\hat{\mathbf{u}}^e$$

c) Shear force and bending moment at element nodes

$$\left.\begin{aligned} \hat{F}_{y'}^e(e,1) &= -\widehat{\mathbf{F}}_{\text{int}}^e(1) \\ \hat{F}_{y'}^e(e,2) &= \widehat{\mathbf{F}}_{\text{int}}^e(3) \end{aligned}\right\} \text{Shear force}$$

$$\left.\begin{aligned} \widehat{M}_{z'}^e(e,1) &= -\widehat{\mathbf{F}}_{\text{int}}^e(2) \\ \widehat{M}_{z'}^e(e,2) &= \widehat{\mathbf{F}}_{\text{int}}^e(4) \end{aligned}\right\} \text{Bending moment}$$

d) Third-order polynomial coefficients for element's deflection and section rotation

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \frac{1}{l^{e3}} \begin{bmatrix} 2 & l^e & -2 & l^e \\ -3l^e & -2l^{e2} & 3l^e & -l^{e2} \\ 0 & l^{e3} & 0 & 0 \\ l^{e3} & 0 & 0 & 0 \end{bmatrix} \hat{\mathbf{u}}^e$$

$$\hat{p}_{u_y^e}(e,[1,2,3,4]) = [a,b,c,d] \qquad \left(u_{y'}^e(x') = a(x')^3 + b(x')^2 + cx' + d,\ \ \forall x \in [0,l^e]\right)$$

$$\hat{p}_{\theta_z^e}(e,[1,2,3]) = [3a,2b,c] \qquad \left(\theta_{z'}^e(x') = 3a(x')^2 + 2bx' + c,\ \ \forall x \in [0,l^e]\right)$$

Next beam e