Guidelines for Assignment 3

Enginyeria Aeroespacial Computacional

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November 30, 2020

1 Part 1 (basic)

1. Consider the cantilever box beam depicted in Figure 1. The length of the beam is $L_X = 2\ m$, its width and height $h_y = h_z = 0.25$ m, and its thickness $e = 0.05$ m. The material is isotropic, with $E = 70 \cdot 10^6\ kN/m^2$, and $\nu = 0.3$. The beam is subjected to a uniformly distributed load of value $t^{(2)} = -500\ kN/m^2$ on its top surface ($y = y_{max}$).

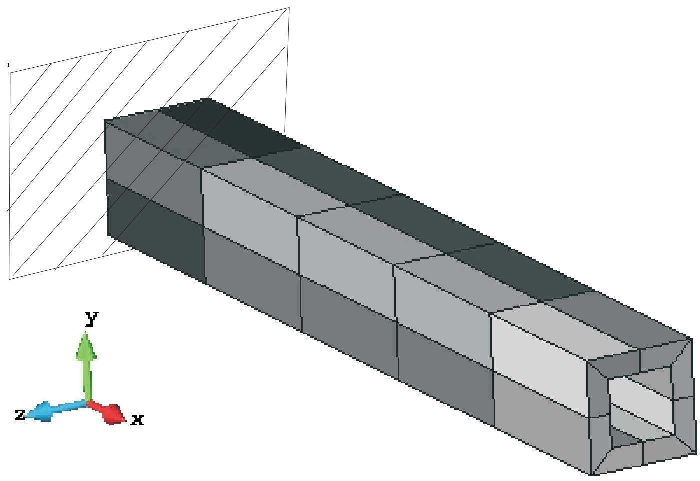

Figure 1

Assess the **convergence upon mesh refinement**, by launching **4 different analyses** with increasing number of finite elements (hexahedral). In particular, use semi-structured meshes with

- MESH 1: $n_{ex} = 2$ divisions in the $x-$direction, and typical size in the $y-z$ plane of 0.1 m.
- MESH 2: $n_{ex} = 10$ divisions in the $x-$direction, and typical size in the $y-z$ plane of 0.05 m.
- MESH 3: $n_{ex} = 15$ divisions in the $x-$direction, and typical size in the $y-z$ plane of 0.05 m.
- MESH 4: $n_{ex} = 20$ divisions in the $x-$direction, and typical size in the $y-z$ plane of 0.025 m.

For these four cases, plot the distribution of vertical displacements (in the y-direction) along one of the edges on the top surface (parallel to the x-axis). Check also whether the maximum displacement at the tip of the beam converges to the analytical predictions provided by the theory of Strength of Materials.

2. Once the performance of the code has been properly assessed, study, for MESH 4, the finite element solution corresponding to the load state in which a torque T = 100 kN m is applied at the free

end (see figure below). HINT: replace this point torque by any statically equivalent system of point/distributed forces.

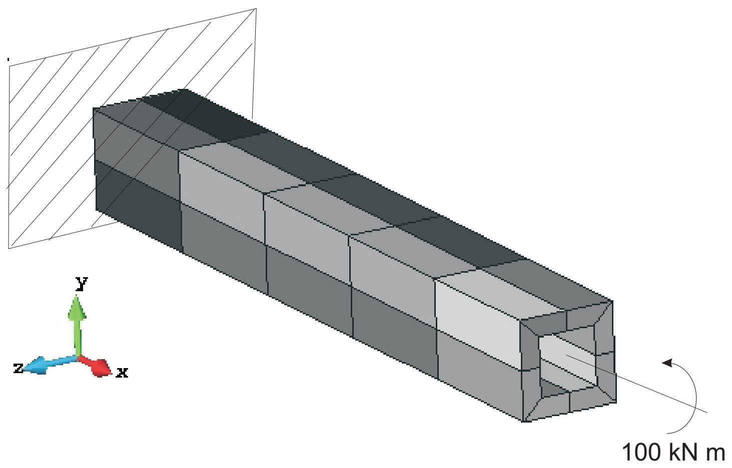

Figure 2

3. Code a matlab routine able to automatically compute the resultant of the reaction forces at the fixed end (R_x, R_y, R_z as well as M_x, M_y and M_z). Check that such reactions are in equilibrium with the prescribed forces.

1.1 Guidelines

- The matlab program for carrying out the calculations can be invoked by running the script:

  mainELASTOSTATIC.m

  There are some parts which are missing in the code, and you have to complete them in order to generate the results. More specifically, you have to implement

  – The assembly of the stiffness matrix (function ComputeK.m), the shape function routines of trilinear hexahedral elements (inside function ComputeElementShapeFun.m), and the solution of the final system of equations (see video Video1_Kshape.avi).

- The way boundary conditions are introduced are similar to that explained in assignment 2.

- The guidelines are provided for the case of a square cross-section in the following videos.

  1. How to adapt the input data file: see video Video01_InputData.avi
  2. How to create the 3D geometry in GID (by extrusion): see video Video02_Extrusion.avi
  3. Labeling surfaces and assigning material: see video Video03_Label.avi
  4. How to mesh the geometry. Differences between unstructured, structured and semi-structured mesh. Semi-structured mesh is a special type of mesh for extruded geometries, in which one meshes a “master surface” (in this case the cross-section of the beam in the plane y-z) and then repeat the elements in the perpendicular direction (in this case the x-axis): see video Video04_mesh.avi
  5. How to post-process results: see video Video05_post.avi

2 Part 2 (advanced)

2.1 Theoretical introduction

2.1.1 Thermoelasticity (see video Video06_thermoINTRO.avi)

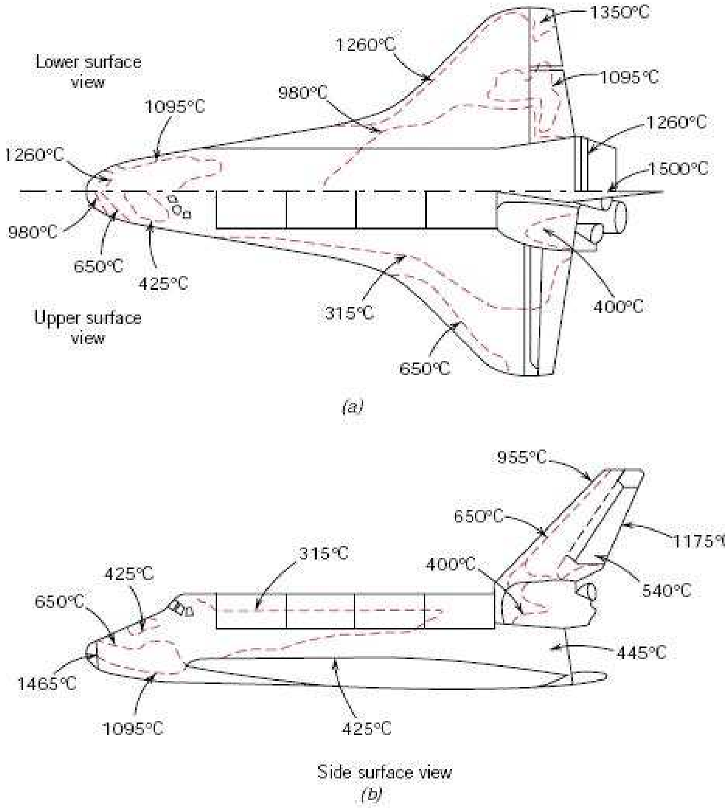

Figure 3 *Approximate maximum outer surface temperature profiles for the Space Shuttle Orbiter during reentry: (a) upper and lower views; (b) side view. Reprinted from [1]*

It is well-known that changes of temperature in elastic bodies produce strains. In the linear regime, such strains are calculated by

$$\varepsilon_{ij}^{thermal} = \alpha_{ij}\Delta T \tag{2.1}$$

where α_{ij} denotes the tensor of *thermal expansion coefficients* ($i, j = 1, 2, 3$ for 3D problems), while $\Delta T = T - T_0$ represents the variation of temperature with respect to the reference temperature $T_0 = T_0(\boldsymbol{x})$. In isotropic materials, α_{ij} can be expressed by

$$\boldsymbol{\alpha} = \alpha \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.2}$$

In Voigt notation, the preceding expressions adopt the form

$$\boldsymbol{\varepsilon}^{thermal} = \boldsymbol{\alpha}\Delta T \tag{2.3}$$

where

$$\boldsymbol{\alpha} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad (for\ 3D) \qquad \boldsymbol{\alpha} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad (for\ 2D) \tag{2.4}$$

2.1.2 Thermo-mechanical constitutive equation

The weak statement of the boundary value equilibrium problem reads (see page 30 of the theory document)

$$\text{W.F.} \begin{cases} \text{Given } \boldsymbol{f} : \Omega \rightarrow \mathbb{R}^{n_{sd}},\ \bar{\boldsymbol{t}} : \Gamma_\sigma \rightarrow \mathbb{R}^{n_{sd}}, \ \text{ find } \boldsymbol{u} \in \mathcal{S} \ \text{ such that} \\ \displaystyle\int_\Omega \boldsymbol{\nabla}^s \boldsymbol{v}^T \boldsymbol{\sigma} \ d\Omega = \int_\Omega \boldsymbol{v}^T \boldsymbol{f} \ d\Omega + \int_{\Gamma_\sigma} \boldsymbol{v}^T \bar{\boldsymbol{t}} \ d\Gamma \qquad \forall \boldsymbol{v} \in \mathcal{V} \end{cases} \tag{2.5}$$

In the isothermal case, we know that $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\nabla}^s\boldsymbol{u}$, that is, the stresses depend exclusively on the symmetric gradient of the displacements. To account for thermal effects, it is necessary to modify this constitutive equation and introduce the thermal strains defined in Eq.(2.3). This is done as follows (see e.g. Reference [2]):

$$\boldsymbol{\nabla}^s\boldsymbol{u} = \boldsymbol{C}^{-1}\boldsymbol{\sigma} + \boldsymbol{\alpha}\Delta T \tag{2.6}$$

By solving this equation for $\boldsymbol{\sigma}$, we obtain the desired *constitutive* equation

$$\boldsymbol{\sigma} = \boldsymbol{C}(\boldsymbol{\nabla}^s\boldsymbol{u} - \boldsymbol{\alpha}\Delta T) = \boldsymbol{C}\boldsymbol{\nabla}^s\boldsymbol{u} - \boldsymbol{\beta}\Delta T \tag{2.7}$$

where

$$\boldsymbol{\beta} := \boldsymbol{C}\boldsymbol{\alpha} \tag{2.8}$$

For an isotropic material, we have that

$$\boldsymbol{\beta} = \boldsymbol{C}\boldsymbol{\alpha} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} \alpha \\ \alpha \\ \alpha \\ 0 \\ 0 \\ 0 \end{bmatrix} = \kappa\alpha \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.9}$$

where κ stands for the *bulk* modulus of the material.

$$\kappa := 3\lambda + 2\mu = 3\frac{\nu E}{(1+\nu)(1-2\nu)} + 2\frac{E}{2(1+\nu)} = \frac{E}{1-2\nu} \tag{2.10}$$

2.2 Requested tasks

2.2.1 Derivation of thermo-mechanical equilibrium equations

Let $\Omega = \cup_{e=1}^{n_{el}} \Omega^e$ be a given finite element discretization. Suppose we solve the heat conduction problem explained in chapter 2 under given boundary conditions, and obtains the vector of temperatures, $\boldsymbol{\theta}$, at the nodes of the discretization (with respect to the reference temperature T_0). With this vector at our disposal, we can interpolate the relative temperature at any point within an element Ω^e by the corresponding shape functions:

$$\Delta T(\boldsymbol{x}) = \boldsymbol{\mathcal{N}}^e(\boldsymbol{x})\boldsymbol{\theta}^e, \qquad \boldsymbol{x} \in \Omega^e \tag{2.11}$$

where $\boldsymbol{\theta}^e$ denotes the vector of nodal temperatures of element Ω^e. With this consideration in mind, **derive the discrete system of equilibrium equations arising from the variational principle** (2.5) when the **stresses depend on the temperature through equation** (2.7).

- **HINT**: The contribution of the temperature to the final system of equation is a vector of "thermal forces": $\boldsymbol{F}_{th}$, similar to that of, for instance, the body forces, in the sense that it can be obtained by assembly of element contributions $\boldsymbol{F}_{th}^{e}$. The student is requested to show how to obtain the element vector $\boldsymbol{F}_{th}^{e}$ as a function of $\boldsymbol{\theta}^{e}$ using the variational formulation (2.5).

References

[1] William D Callister et al. *Fundamentals of materials science and engineering*. Wiley London, 2000.

[2] Junuthula Narasimha Reddy. *Energy principles and variational methods in applied mechanics*. John Wiley & Sons, 2017.