Quasiconcave programming

Notes by János Mayer, 16.7.2007

Contents

1 Introduction

In this note we give a short introduction to some classes of generalized concave functions and their applications in nonlinear programming. The main emphasis is on quasi–concave functions. For literature concerning generalized concavity see [2], [3].

Notation: Let

- $f : Y \mapsto \mathbb{R}$ be a real–valued function, defined on the open set $Y \subset \mathbb{R}^n$ and
- $X \subset Y$ be a convex set.

Keeping an eye on the applications in economics, we will assume throughout that f is continuous or even differentiable on Y.

The restriction that f is only defined on some subset Y, instead of the whole space $\mathbb{R}^n$, seems to be at the first glance an unnecessary complication of the presentation. In fact, frequently we have $Y = \mathbb{R}^n$ but in general, including some important cases, this cannot be assumed. Examples:

- ⊛ $u(x_1, x_2) = x_1^a x_2^b$, $a, b > 0$, $a+b = 1$, the Cobb–Douglas utility function. The domain of definition of this function is $\mathbb{R}^n_+ := \{x \mid x_i \geq 0, i = 1, \ldots, n\}$ with $n = 2$. For instance, choosing $a = b = 0.5$ $u(-1, -1) = \sqrt{-1} \cdot \sqrt{-1}$ is undefined for real numbers. Although the function is defined on $\mathbb{R}^n_+$, the derivatives do not exist on the boundary. Therefore, $Y = \mathbb{R}^n_{++} := \{x \mid x_i > 0, i = 1, \ldots, n\}$ with $n = 2$ will be chosen for this function.

- ⊛ $u(x_1, x_2) = \log x_1 + \log x_2$, an additively separable logarithmic utility function. The domain of definition is clearly $Y = \mathbb{R}^n_{++}$ with $n = 2$.

2 Quasi–concave functions

2.1 Mathematical definitions and properties

Recall that f is a concave function over X, if for any $x, y \in X$ and for any λ with $0 \leq \lambda \leq 1$ we have the inequality

$$f(\,\lambda x + (1-\lambda)y\,) \geq \lambda f(x) + (1-\lambda) f(y).$$

From this inequality we readily obtain

$$f(\,\lambda x + (1-\lambda)y\,) \geq \lambda f(x) + (1-\lambda) f(y) \;\geq\; \min\{f(x), f(y)\}$$

which suggests the following generalization of concavity:

Definition 1 *(quasi–concavity)*
*$f(x)$ is called **quasi–concave** on X, if $\forall x, y \in X$, $\forall \lambda$ with $0 \leq \lambda \leq 1$:*

$$\min\{f(x), f(y)\} \leq f(\,\lambda x + (1-\lambda)y\,),$$

or, equivalently,

$$f(x) \leq f(y) \implies f(x) \leq f(\,\lambda x + (1-\lambda)y\,).$$

Analogously, if f is a convex function over X then for any $x, y \in X$ and for any λ with $0 \leq \lambda \leq 1$ we have the inequality

$$f(\,\lambda x + (1-\lambda)y\,) \leq \lambda f(x) + (1-\lambda) f(y) \;\leq\; \max\{f(x), f(y)\}$$

which leads to the following generalization of convexity:

Definition 2 *(quasi–convexity)*
*$f(x)$ is called **quasi–convex** on X, if $\forall x, y \in X$, $\forall \lambda$ with $0 \leq \lambda \leq 1$:*

$$\max\{f(x), f(y)\} \geq f(\,\lambda x + (1-\lambda)y\,)$$

or, equivalently,

$$f(x) \geq f(y) \implies f(x) \geq f(\,\lambda x + (1-\lambda)y\,).$$

Finally, in analogy to linear functions, we introduce:

Definition 3 *(quasi–linearity)*
*f is called **quasi–linear** if it is both quasi–concave and quasi–convex.*

From the above discussion and noting that linear functions are both convex and concave, we clearly have the following implications:

- ▷ f is concave over $X \Rightarrow f$ is quasi–concave over X;
- ▷ f is convex over $X \Rightarrow f$ is quasi–convex over X;
- ▷ f is linear over $X \Rightarrow f$ is quasi–linear over X.

Relations between the two concepts:

- ▷ If f is quasi–concave then $-f$ is quasi–convex, and conversely,
- ▷ if f is quasi–convex then $-f$ is quasi–concave.

This can be readily seen by multiplying the defining inequality by -1 and noticing that

$$\begin{aligned} -\min\{f(x), f(y)\} &= \max\{-f(x), -f(y)\}, \\ -\max\{f(x), f(y)\} &= \min\{-f(x), -f(y)\}. \end{aligned}$$

In the sequel we will discuss properties of quasi–concave functions, the analogous results for quasi–convex functions can be derived in a straightforward way, due to the above–discussed relationship between the two concepts.

The following characterization is useful for checking whether a given function is quasi–concave. In fact, it is frequently chosen as a definition of quasi–concave functions. Recall that, by convention, the empty set is considered as convex.

Proposition 1 *(equivalent characterization)*

$$f(x) \text{ is quasi–concave } \iff \mathcal{L}_\alpha^U := X \cap \{x \mid f(x) \geq \alpha\} \text{ is a convex set}, \quad \forall \alpha.$$

Proof:

"$\Longrightarrow$" Let $x, y \in \mathcal{L}_\alpha^U$ and $\lambda \in [0,1]$. Then we have $x \in X$ and both $f(x) \geq \alpha$ and $f(y) \geq \alpha$. Utilizing the quasi–concavity of f, we get

$$f(\,\lambda x + (1-\lambda) y\,) \geq \min\{f(x), f(y)\} \geq \alpha.$$

Consequently, $\lambda x + (1-\lambda)y \in \mathcal{L}_\alpha^U$ holds implying that $\mathcal{L}_\alpha^U$ is a convex set.

"$\Longleftarrow$" Let $\alpha = \min\{f(x), f(y)\}$ then both $x \in \mathcal{L}_\alpha^U$ and $y \in \mathcal{L}_\alpha^U$ hold. Since $\mathcal{L}_\alpha^U$ is a convex set, we have $\lambda x + (1-\lambda)y \in \mathcal{L}_\alpha^U$. This implies $f(\,\lambda x + (1-\lambda)y\,) \geq \alpha = \min\{f(x), f(y)\}$, proving quasi–concavity of f. ⊠

That is, a function f is quasi–concave, if and only if for any α the upper level set of f is a convex set.

Proposition 2 *Let f be a quasi–concave function and Ψ a strictly increasing function, defined on the range of f. Then $F(x) := \Psi(\,f(x)\,)$ is also quasi–concave.*

Proof: This follows immediately from the fact that strictly monotone functions have strictly monotone inverse functions. Thus we have for any α:

$$\{x \mid F(x) \geq \alpha\} = \{x \mid \Psi(f(x)) \geq \alpha\} = \{x \mid f(x) \geq \Psi^{-1}(\alpha\})$$

which is a convex set, due to the quasi–concavity of f. ⊠

2.2 Examples and intuition

For checking whether a given function is quasi–concave, the easiest way is checking the property stated in Proposition 1: a function f is quasi–concave, if and only if all of its upper level sets are convex sets. That is, for any α, the set of points x where $f(x) \geq \alpha$ holds is convex.

We will explore the properties of quasi–concave functions in a step–by–step fashion.

2.2.1 Quasi–concave functions of a single variable

We begin with functions of a single variable. Let us observe that monotone functions in $\mathbb{R}^1$ are both quasi–concave and quasi–convex, because both the upper–and the lower level sets are convex (they are intervals). Consequently, monotone functions are quasi–linear. Examples:

⊛ The most simple example is $f_1(x) := x$, a strictly increasing linear function, see Figure 1.

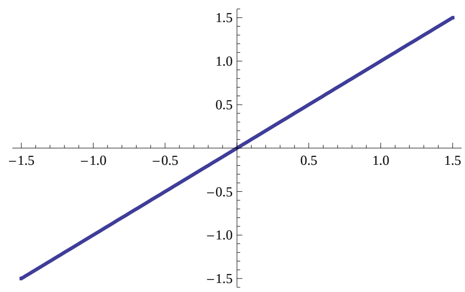

Figure 1: The linear function $f_1(x) = x$

⊛ Next let us consider the function

$$f_2(x) = \begin{cases} x & \text{if } x \leq 0 \\ 0 & \text{if } 0 \leq x \leq 1 \\ x-1 & \text{if } x \geq 1, \end{cases}$$

see Figure 2. This function is no more linear. It is monotonically increasing, therefore it is quasi–concave (and quasi–convex). Notice the horizontal piece on the graph of the function, corresponding to $x \in [0, 1]$.

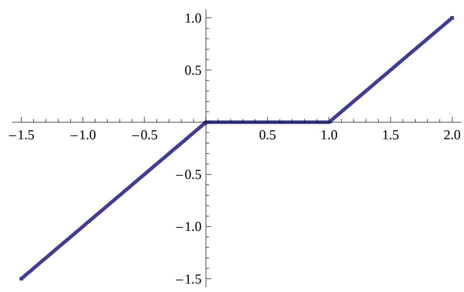

Figure 2: The function $f_2(x)$ with a horizontal piece on its graph

⊛ As a final example, let $f_3(x) := x^3$ which is a strictly increasing function, see Figure 3.

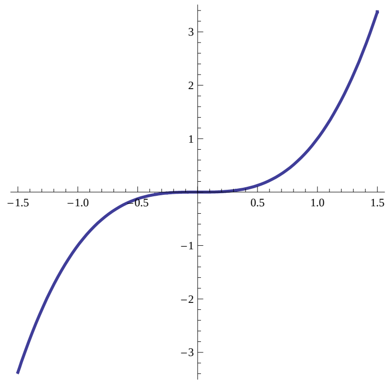

Figure 3: The function $f_3(x) = x^3$

⊛ All of the previous examples involved monotone functions. As a next example let us consider the function

$$f_4(x) = \begin{cases} -x^2 & \text{if } \ \leq 0 \\ 0 & \text{if } 0 \leq x \leq 1 \\ -(x-2)^2 + 1 & \text{if } x \geq 1, \end{cases}$$

shown in Figure 4. This is neither increasing nor decreasing and has a horizontal piece on its graph.

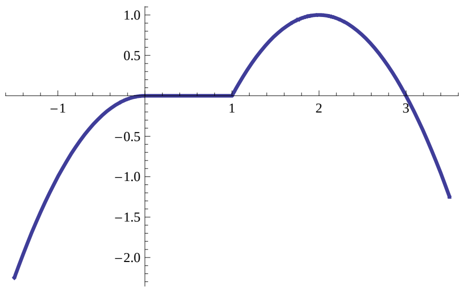

Figure 4: The function $f_4(x)$

f_4 is clearly quasi–concave because all of its upper level sets are convex sets (intervals). It is not quasi–convex because the lower level sets are not convex.

It can be shown that Figure 4 displays the general shape of quasi–concave functions of a single variable. In fact, each continuous quasi–concave function f of a single variable belongs to one of the following classes of functions (see, for instance, [4]):

- Either f is monotonically increasing on X, or
- it is monotonically decreasing on X, or
- $\exists a, b \in X$, $a \leq b$, such that f is monotonically increasing for $x < a$, it is constant for $x \in [a, b]$, and it is monotonically decreasing for $x > b$.

2.2.2 Strictly monotone transformations

Proposition 2 states an important property of quasi–concave functions: transforming them via strictly monotonically increasing functions leads to quasi–concave functions. This property makes quasi–concave functions especially well-suited for ordinal utility functions in economics.

⊛ As a first example let us take our last example $f_4(x)$. Transforming this with the strictly increasing function $\Psi(y) = y^3$ leads to the quasi-concave function $F_4(x) = f_4(x)^3$, as shown in Figure 5.

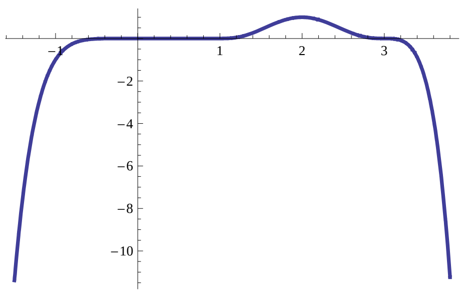

Figure 5: The function $(\, f_4(x)\,)^3$

⊛ Transforming f_4 with another strictly increasing function $\Psi(y) = e^y$, leads to another quasi-concave function $\hat{F}_4(x) = e^{f_4(x)}$, displayed in Figure 6.

Recall that, for concave functions, a strictly monotonically increasing transformation does not lead to concave functions, in general. For obtaining concave functions, further properties, like the concavity of Ψ are needed. Therefore, considering concavity alone, this is not really well–suited for ordinal utility theory.

⊛ We consider the function $f_5(x_1, x_2) = -x_1^2 - x_2^2$. This is a concave function, therefore it is also quasi–concave. Let us choose $\Psi(y) = e^y$ then, due to Proposition 2, $F_5(x_1, x_2) = e^{f_5(x_1,x_2)}$ will be quasi–concave. The graph of

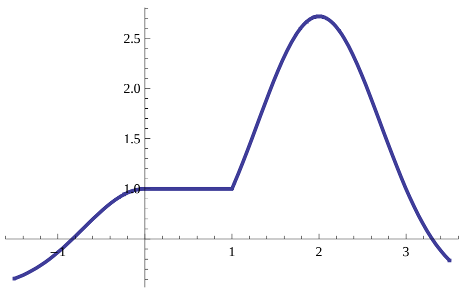

Figure 6: The function $e^{f_4(x)}$

this function is shown in Figure 7.

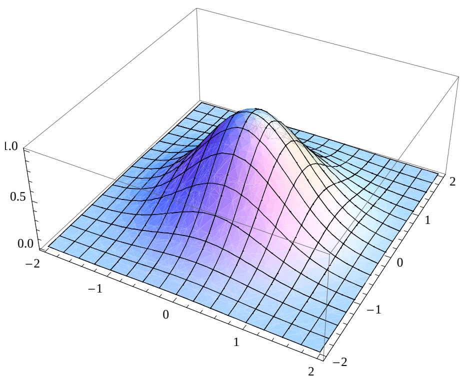

Figure 7: The bell–shaped function $e^{-x_1^2-x_2^2}$

Notice that, although F_5 has been obtained from a concave function via a strictly monotonically increasing transformation, F_5 is by no means concave.

2.2.3 Warning: sums of quasi–concave functions

Recall that concave functions have the following property: if $g_1, \ldots, g_m$ are concave functions and $\lambda_i \geq 0$, $i = 1, \ldots, m$ then $F(x) := \lambda_1 f_1(x) + \ldots + \lambda_m f_m(x)$ is a concave function.

This property does not hold for quasi–concave functions, in general. This is a drawback concerning economic theory, because nonnegative linear combinations of utility functions appear, for instance, in expected utility theory. Below we consider a series of examples for this phenomenon.

⊛ $g_1(x) := e^x$ is strictly monotonically increasing therefore it is quasi–concave. $g_2(x) := e^{-x}$ is strictly monotonically decreasing therefore it is quasi–concave, too. However, $g(x) := g_1(x) + g_2(x) = e^x + e^{-x}$ is strictly convex; it is by no means quasi–concave, see Figure 8.

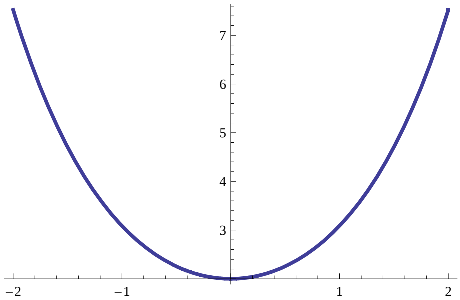

Figure 8: The function $e^x + e^{-x}$

⊛ Let us consider next our very first example $f_1(x) = x$, see Figure 1. Taking $G_1(x_1, x_2) := f_1(x_1) + f_1(x_2) = x_1 + x_2$ everything is fine, G being a linear function, it is quasi–concave; see Figure 9 for the graph, indifference curves (contour lines), and an upper level set to $\alpha = 2$ of this function.

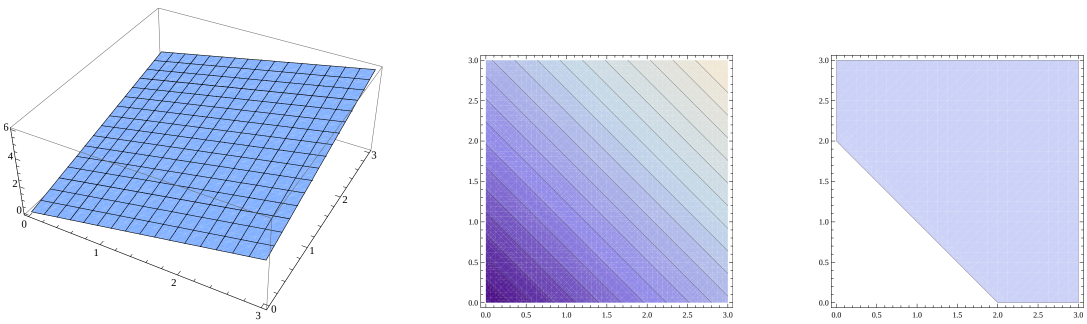

Figure 9: Graph, indifference curves, and an upper level set of $G_1(x_1, x_2) = x_1 + x_2$

⊛ Next let us take our second example f_2, see Figure 2. We construct the additively separable function of two variables according to $G_2(x_1, x_2) := f_2(x_1) + f_2(x_2)$. Graph, contour lines, and the upper level set to $\alpha = 1$ are displayed in Figure 10.

Notice that the indifference curves, displayed in the middle of Figure 10 might suggest that the level sets are convex. It is important to see that con-

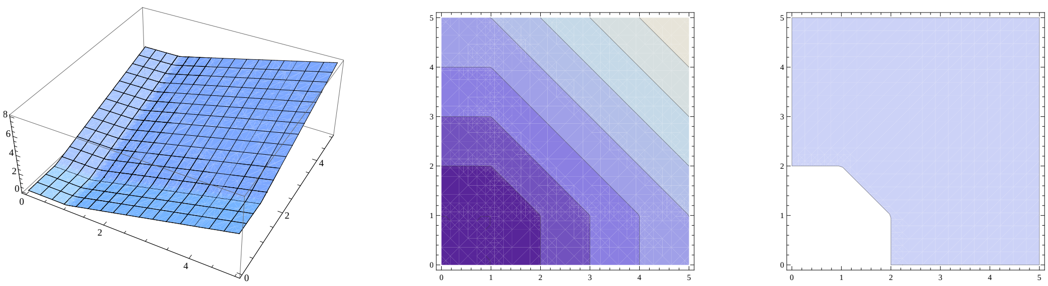

Figure 10: Graph, indifference curves, and an upper level set of $G_2(x_1, x_2) = f_2(x_1) + f_2(x_2)$

sidering just the indifference curves might be quite deceiving, as illustrated in the figure at the right–hand–side: the upper level set (shaded region) is <u>non–convex</u> for $\alpha = 1$! Therefore, G_2 is not a quasi–concave function of two variables, despite of its additively separable structure.

⊛ Analogously, consider $G_3(x_1, x_2) := f_3(x_1) + f_3(x_2) = x_1^3 + x_2^3$ (see Figure 3), which is not quasi–concave either, as one can see by inspecting the indifference curves or the upper level set to $\alpha = 0.5$ in Figure 11.

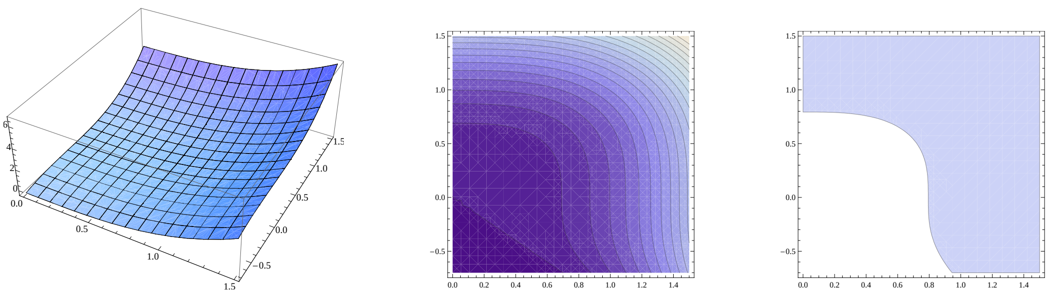

Figure 11: Graph, indifference curves, and an upper level set of $G_3(x_1, x_2) = f_3(x_1) + f_3(x_2) = x_1^3 + x_2^3$

⊛ Taking $G_4(x_1, x_2) := f_4(x_1) + f_4(x_2)$, (see Figure 4), the resulting function will not be quasi–concave, either, see Figure 12.

⊛ Finally, let us take the bell-shaped curves $F_{5,1}(x_1, x_2) = e^{-x_1^2 - x_2^2}$ and $F_{5,2}(x_1, x_2) = e^{-(x_1-1.5)^2-(x_2-1.5)^2}$, which are both quasi–concave, see page 8. Let their sum be $G_5(x_1, x_2) := F_{5,1}(x_1, x_2) + F_{5,2}(x_1, x_2)$. Again, a non–quasi–concave function arises, see Figure 13, where the upper level set (shaded region) corresponds to $\alpha = 0.7$ and it is obviously not convex because it

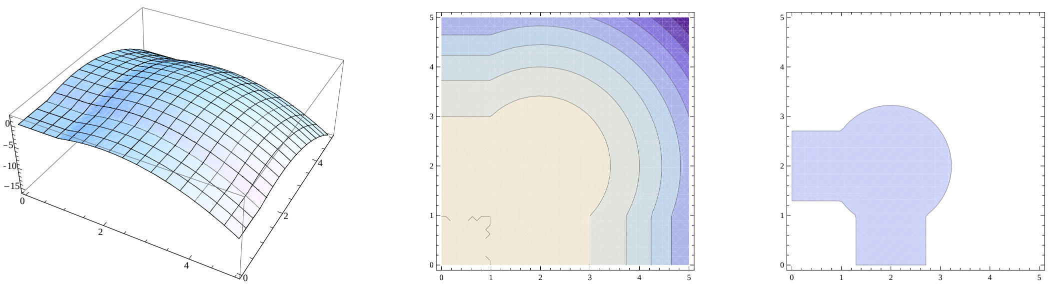

Figure 12: Graph, indifference curves, and an upper level set of $G_4(x_1, x_2) = f_4(x_1) + f_4(x_2)$

consists of two disjoint parts.

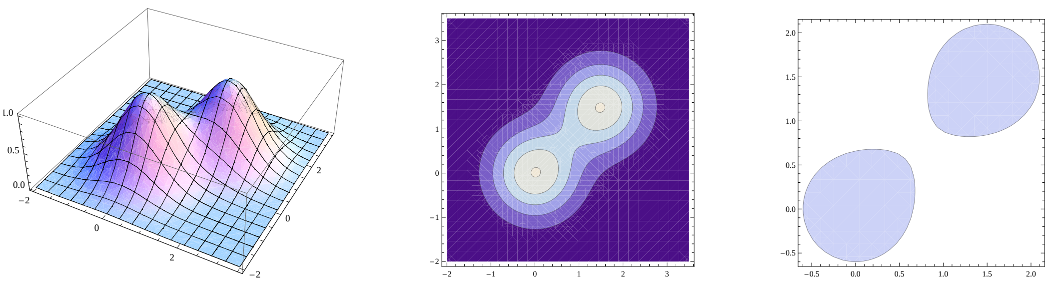

Figure 13: Graph, indifference curves, and an upper level set of $G_5(x_1, x_2) = F_{5,1}(x_1, x_2) + F_{5,2}(x_1, x_2)$

Summary: the sum of quasi–concave functions is generally not a quasi–concave function, as the examples above show.

3 Strictly quasi–concave functions

3.1 Mathematical definitions and properties

Recall that f is a strictly concave function over X, if for any $x, y \in X$, $x \neq y$, and for any λ with $0 < \lambda < 1$ we have the inequality

$$f(\,\lambda x + (1-\lambda)y\,) > \lambda f(x) + (1-\lambda)f(y).$$

from which we readily obtain

$$f(\,\lambda x + (1-\lambda)y\,) > \lambda f(x) + (1-\lambda)f(y) \;\geq\; \min\{f(x), f(y)\}$$

and which suggests the following generalization of concavity:

Definition 4 *(strict quasi–concavity)*
$f(x)$ is strictly quasi–concave on X, if $\forall x, y \in X$, $x \neq y$, $0 < \lambda < 1$ we have

$$\min\{f(x), f(y)\} < f(\,\lambda x + (1-\lambda)y\,).$$

$f(x)$ is strictly quasi–convex on $X \iff -f$ is strictly quasi–concave on X.

From the discussion above immediately follows:

- $\triangleright$ f is strictly concave over $X \Rightarrow f$ is strictly quasi–concave over X;
- $\triangleright$ f is strictly quasi–concave over $X \Rightarrow f$ is quasi–concave over X.

Definition 5 *(strictly convex sets)*
A set $C \subset \mathbb{R}^n$ is called strictly convex if $x, y \in C$, $0 < \lambda < 1$ imply that $\lambda x + (1-\lambda)y \in Int(C)$, where Int$(C)$ denotes the set of interior points of C.

Next we consider the upper level sets of strictly quasi–concave functions. It is tempting to think that strict quasi–concavity is equivalent with strict convexity of the upper level sets. This is partially true; we have namely:

Proposition 3 *If f is a continuous strictly quasi–concave function then its upper level sets are either empty or otherwise they are strictly convex sets.*

Proof:
Let $\mathcal{L}_\alpha^U := \{x \mid f(x) \geq \alpha\} \neq \emptyset$. Due to the continuity of f, any point y with $f(y) > \alpha$ must belong to the interior of $\mathcal{L}_\alpha^U$. Since f is quasi–concave, $\mathcal{L}_\alpha^U$ is a convex set. If this set is not strictly convex then $\exists\ x^{(1)}, x^{(2)} \in \mathcal{L}_\alpha^U$ and $0 < \lambda < 1$ such that $x_\lambda := \lambda x^{(1)} + (1-\lambda)x^{(2)} \notin Int\mathcal{L}_\alpha^U$. Consequently, due to our previous remark concerning the implications of continuity of f, $f(x_\lambda) = \alpha$ must hold. On the other hand, quasi–concavity of f implies that $\min\{f(x^{(1)}), f(x^{(2)})\} \leq f(x_\lambda) = \alpha$ implying $\min\{f(x^{(1)}), f(x^{(2)})\} = f(x_\lambda)$ which clearly contradicts the strict quasi–concavity of f. $\boxtimes$

Note that the converse implication does not hold: there exist functions with all of their nonempty upper level sets being strictly convex which are, nevertheless, not strictly quasi–concave. An example for this can be seen in the next section.

Analogously as in the quasi–concave case we have:

Proposition 4 *Let f be a strictly quasi–concave function and Ψ a strictly increasing function, defined on the range of f. Then $F(x) := \Psi(\,f(x)\,)$ is strictly quasi–concave.*

Proof: We have for any $x^{(1)}, x^{(1)} \in X$, $0 < \lambda < 1$, and $x_\lambda := \lambda x^{(1)} + (1-\lambda)x^{(2)}$

$$\begin{aligned} F(x_\lambda) &= \Psi(f(x_\lambda)) = \Psi(f(\lambda x^{(1)} + (1-\lambda)x^{(2)})) \\ &> \Psi(\min\{f(x^{(1)}), f(x^{(2)})\}) = \min\{\Psi(f(x^{(1)})), \Psi(f(x^{(2)}))\}) \\ &= \min\{F(x^{(1)}), F(x^{(2)})\} \end{aligned}$$

proving the strict quasi–concavity of F. ⊠

3.2 Examples and intuition

For functions of a single variable it is clear that strictly monotone (increasing or decreasing) functions are both strictly quasi–concave and strictly quasi–convex. This is in contrast with strictly concave functions which can not be also strictly convex.

From the examples in Section 2.2, f_1, f_3, and f_5 are strictly quasi–concave (see Figures 1, 3, and 7). The functions f_2 and f_4 with horizontal pieces on their graph are not strictly quasi–concave (see Figures 2 and 4).

Thus, intuitively, strict quasi–concavity rules out horizontal pieces (plateaus) on the graph of the function.

⊛ In fact, changing f_4 as

$$\hat{f}_4(x) := \begin{cases} -x^2 & \text{if } x \leq 0 \\ -(x-1)^2 + 1 & \text{if } x \geq 0 \end{cases}$$

results in a strictly quasi–concave function as shown in Figure 14.

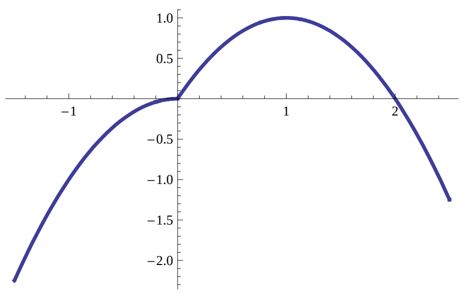

Figure 14: The function $\hat{f}_4(x)$

⊛ The above considerations suggest the intuitive interpretation that graphs of strictly quasi–concave functions need to be "curved". This is not so in general, just take f_1 (Figure 1) which is a strictly quasi–concave linear function.

We know that strictly concave functions are also strictly quasi–concave. The example of f_1 shows that there are strictly quasi–concave functions which are concave, but not strictly concave.

⊛ According to Proposition 3, the nonempty upper level sets of strictly quasi–concave functions are strictly convex sets. The converse is not true. As an example we consider

$$F_6(x_1, x_2) := \begin{cases} 0 & \text{if } x_1^2 + x_2^2 \le 1 \\ 1 - x_1^2 - x_2^2 & \text{if } x_1^2 + x_2^2 > 1. \end{cases}$$

displayed in Figure 15.

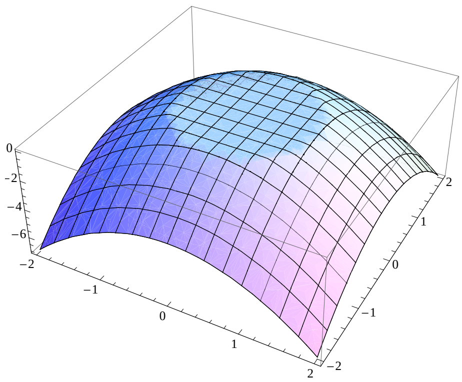

Figure 15: The function $F_6(x_1, x_2)$

The nonempty upper level sets of this functions are circular domains centered at the origin, therefore these are strictly convex sets. Nevertheless, F_5 is clearly not strictly quasi–concave, its graph having a plateau at the top.

3.2.1 Strictly monotone transformations

According to Proposition 4, transforming strictly quasi–concave functions via strictly monotonically increasing functions results in strictly quasi–concave functions.

⊛ As a first example consider $\hat{f}_4(x)$, transformed via $\Psi(y) = e^y$, resulting in $\hat{F}_4(x) = e^{\hat{f}_4(x)}$ displayed in Figure 16, which is clearly strictly quasi–concave.

⊛ As a second example we may take the bell–shaped function on page 8 shown in Figure 7. This is a strictly monotone transformation of the strictly

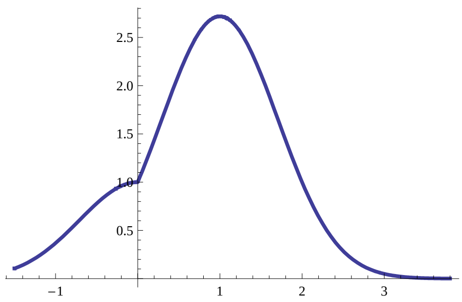

Figure 16: The function $\hat{F}_4(x)$

concave function $f_5(x_1, x_2) = -x_1^2 - x_2^2$, therefore it is strictly quasi–concave.

3.2.2 Warning: sums of strictly quasi–concave functions

Recall that for strictly concave functions holds: if $g_1, \ldots, g_m$ are strictly concave functions and $\lambda_i \geq 0$, $i = 1, \ldots, m$, $\exists i$; $\lambda_i > 0$ then $F(x) := \lambda_1 f_1(x) + \ldots + \lambda_m f_m(x)$ is a strictly concave function.

This property does not hold for strictly quasi–concave functions, in general.

⊛ As a first example let us consider $g(x) := g_1(x) + g_2(x) = e^x + e^{-x}$ which is the sum of two strictly monotone – and therefore strictly quasi–concave – functions. $g(x)$ is not strictly quasi–concave; it is not even quasi–concave (it is strictly convex), see Figure 8.

⊛ As a second example we take the sum of two bell–shaped functions (see page 10), which are both strictly quasi–concave. Their sum is, however not even quasi–concave, see Figure 13, where the upper level set (shaded region) corresponds to $\alpha = 0.5$ and it is obviously not a convex set.

4 Quasi–concave programming problems

We will consider the following optimization problem:

$$\begin{aligned} \max \quad & f(x) \\ & x \in X. \end{aligned} \tag{1}$$

First let us assume that f is a quasi–concave function. Then the above problem may have local maxima which are not global. As an example consider the function f_4 displayed in Figure 4. For this function, any $x \in (0,1)$ is a local maximum whereas the global maximum is at $x^* = 2$.

The difficulty is caused by horizontal pieces on the graph of the function which is ruled out by requiring strict quasi–concavity. In fact, we have

Proposition 5 *Let X be a nonempty convex set and assume that f is strictly quasi–concave. Then any local maximum is a global solution of (1).*

Proof: Assume that $\bar{x} \in X$ is a local maximum, that is, exists $\varepsilon > 0$ such that $f(x) \leq f(\bar{x})$ holds for any $x \in X$ with $\parallel x - \bar{x} \parallel \leq \varepsilon$. Assume that $\bar{x}$ is not a global maximum. Consequently, $\exists y \in X$ such that $f(y) > f(\bar{x})$ holds. Let $x_\lambda := \lambda y + (1-\lambda)\bar{x}$, $0 \leq \lambda \leq 1$. The convexity of X implies $x_\lambda \in X \ \forall \lambda \in [0,1]$. On the other hand, if λ is small enough, we clearly have $\parallel x_\lambda - \bar{x} \parallel \leq \varepsilon$. Utilizing the strict quasi–concavity of f we get:

$$f(x_\lambda) = f(\lambda y + (1-\lambda)\bar{x}) > \min\{f(\bar{x}), f(y)\} = f(\bar{x})$$

which holds for $\lambda > 0$ and λ small enough. This is in contradiction with the assumption that $\bar{x}$ is a local maximum. ⊠

The strict quasi–concavity of f also implies that optimal solutions of (1) are unique, provided that they exist. This has important implications in economics; having strictly quasi–concave utility functions implies, for instance, that the solution of the consumer's optimization problem will be unique.

Proposition 6 *Let X be a nonempty convex set and assume that f is strictly quasi–concave. Provided that an optimal solution of (1) exists, the solution is unique.*

Proof: Assume that $x^* \in X$ is an optimal solution of (1) and $y \in X$, $y \neq x^*$ is an alternative maximum. The we have $f(x^*) = f(y)$ and the strict quasi–concavity of f implies that

$$f(\lambda x^* + (1-\lambda)y) > min\{f(x^*), f(y)\} = f(x^*)$$

holds for any $0 < \lambda < 1$. This contradicts the optimality of x^* since we have $\lambda x^* + (1-\lambda)y \in X$ for $\lambda \in [0,1]$, due to the convexity of X. ⊠

Next we assume that

$$X = \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \ldots, g_m(x) \geq 0\}$$

where g_i is quasi–concave, $i = 1, \dots, m$. Then X is a convex set. In fact, $\{x \mid g_i(x) \geq 0\}$ is an upper level set of g_i, corresponding to $\alpha = 0$, therefore it is convex, $\forall i$. We have

$$X = \bigcap_{i=1}^{m} \{x \mid g_i(x) \geq 0\}$$

therefore X is a convex set.

Now we are ready to formulate the general quasi–concave optimization problem:

$$\begin{aligned} \max \quad & f(x) \\ g_i(x) \; & \geq \; 0, \; i = 1, \dots, m \end{aligned} \tag{2}$$

where $f, g_1, \dots, g_m$ are quasi–concave functions.

According to the discussion above, the set of feasible solutions of (2) is a convex set. For the objective function we have assumed just quasi–concavity, consequently (2) may have local solutions which are not global.

Assuming instead that f is strictly quasi–concave implies that (2) has a unique optimal solution and that no local maxima exist.

Assuming continuous differentiability, the Kuhn–Tucker conditions (FOC) for (2) are

$$\begin{aligned} \nabla f(x) \quad + \sum_{i=1}^{m} \lambda_i \nabla g_i(x) \; & = 0 \\ g_i(x) \; & \geq \; 0, \; i = 1, \dots, m \\ \lambda_i \, g_i(x) \; & = \; 0, \; i = 1, \dots, m \\ \lambda_i \qquad & \geq \; 0, \; i = 1, \dots, m. \end{aligned} \tag{3}$$

These are necessary conditions of optimality under the usual regularity conditions for general nonlinear programming problems. Besides these, especially for quasi–concave optimization problems we have:

Proposition 7 *(Arrow and Enthoven)*
Assume that $g_1, \dots, g_m$ are quasi–concave functions and that the following regularity conditions hold:

(a) $\exists \bar{x} \in \mathbb{R}^n$ such that $g_i(\bar{x}) > 0$ for all $i = 1, \dots, m$ (Slater condition) and

(b) for each i either g_i is concave or otherwise $\nabla g_i(x) \neq 0$ for each feasible solution of (2).

Then we have: if x^* is a locally optimal solution of (2) then there exists λ^* such that with (x^*, λ^*) the Kuhn–Tucker conditions (3) hold.

Proof: see Arrow and Enthoven [1]. ⊠

The Kuhn–Tucker conditions are also sufficient optimality conditions under appropriate assumptions.

Proposition 8 *(Arrow and Enthoven)*
Assume that $f, g_1, \ldots, g_m$ are quasi–concave functions and that for (x^*, λ^*) the Kuhn–Tucker conditions (3) hold. If f is twice continuously differentiable on the feasible set and additionally $\nabla f(x^*) \neq 0$ holds then x^* is an optimal solution of (2).

Proof: see Arrow and Enthoven [1]. ⊠

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