Laboratorio de Microeconomía I

Centro de Investigación y Docencia Económicas
Maestría en Economía - 2025
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Tarea 4

Definition 1. Let a C^2 function f be defined on some domain $X \subset \mathbb{R}^n$, and let $x \in X$. Denote the second partial derivative of f as $f_{ij} := \frac{\partial^2 f(x)}{\partial x_i x_j}$.

For $k = 1, \ldots, n$, let $\bar{H}_k[f(x)]$ be the $(k+1) \times (k+1)$ matrix given by

$$\bar{H}_k[f(x)] := \begin{bmatrix} 0 & f_1(x) & \cdots & f_k(x) \\ f_1(x) & f_{11}(x) & \cdots & f_{1k}(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_k(x) & f_{k1}(x) & \cdots & f_{kk}(x) \end{bmatrix}.$$

The above matrix is the leading principal minor of the **bordered Hessian** of f at x of order k. The following theorem gives the second-derivative characterization of a *quasiconcave* function.

Theorem 1. *Let X be an open and convex set in $\mathbb{R}^n$, and let $f : X \to \mathbb{R}$ be a C^2 function. Then*

(a) If f is quasiconcave on X, then we have $(-1)^k \det\Big(\bar{H}_k[f(x)]\Big) \geq 0$ for $k = 1, \ldots, n$;

(b) If $(-1)^k \det\Big(\bar{H}_k[f(x)]\Big) > 0 \quad$ for $k = 1, \ldots, n$, then f is quasiconcave on X.

Exercises

Ex. 1.12 Suppose $u(x_1, x_2)$ and $v(x_1, x_2)$ are utility functions.

(a) Prove that if $u(x_1, x_2)$ and $v(x_1, x_2)$ are both homogeneous of degree r, then $s(x_1, x_2) \equiv u(x_1, x_2) + v(x_1, x_2)$ is homogeneous of degree r.

(b) Prove that if $u(x_1, x_2)$ and $v(x_1, x_2)$ are quasiconcave, then $m(x_1, x_2) \equiv \min\{u(x_1, x_2),\, v(x_1, x_2)\}$ is also quasiconcave.

Ex. 3.C.6 Suppose that in a two-commodity world, the consumer's utility function takes the form

$$u(x) = [\alpha_1 x_1^{\rho} + \alpha_2 x_2^{\rho}]^{1/\rho}.$$

This utility function is known as the *constant elasticity of substitution* (or *CES*) utility function.

(a) Show that when $\rho = 1$, indifference curves become linear.

(b) Show that as $\rho \to 0$, this utility function comes to represent the same preferences as the (generalized) Cobb–Douglas utility function $u(x) = x_1^{\alpha_1} x_2^{\alpha_2}$.

(c) Show that as $\rho \to -\infty$, indifference curves become "right angles"; that is, this utility function has in the limit the indifference map of the Leontief utility function $u(x_1, x_2) = \min\{x_1, x_2\}$.

Ex. 4 Let $u : \mathbb{R}_+^2 \to \mathbb{R}$ be twice continuously differentiable. Prove that the condition $\frac{\partial}{\partial x_1}\mathrm{MRS}_{12}(x_1, x_2) \leq 0$ is equivalent to the quasiconcavity of u as expressed by the sign criterion on the determinants of the minors of the bordered Hessian (**Theorem 1**).

Exercises

Ex. 1.12 Suppose $u(x_1, x_2)$ and $v(x_1, x_2)$ are utility functions.

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(b) Prove that if $u(x_1, x_2)$ and $v(x_1, x_2)$ are quasiconcave, then $m(x_1, x_2) \equiv \min\{u(x_1, x_2), v(x_1, x_2)\}$ is also quasiconcave.

Definición: Sea $f : X \subset \mathbb{R}^n \to \mathbb{R}$. Decimos que f es homogenea de grado r si

$$f(tx) = t^k f(x) \quad \forall t > 0 \ \& \ \forall x \in X.$$

Las funciones homogeneas muestran un comportamiento regular cuando todas las variables se escalan simultáneamente y en la misma proporción.

(a) P.D. $s(x_1, x_2) = u(x_1, x_2) + v(x_1, x_2)$ homogénea de grado r.

Sean $u(x_1, x_2)$ y $v(x_1, x_2)$ funciones homogéneas de grado r.

Define $s(x_1, x_2) := u(x_1, x_2) + v(x_1, x_2)$. Entonces, $\forall t > 0$:

$$\begin{aligned} s(tx_1, tx_2) &= u(tx_1, tx_2) + v(tx_1, tx_2) \\ &= t^r u(x_1, x_2) + t^r v(x_1, x_2) \qquad (\text{porque } u(\cdot) \ \& \ v(\cdot) \text{ son homogéneas de grado } r) \\ &= t^r \left(u(x_1, x_2) + v(x_1, x_2) \right) \\ &= t^r s(x_1, x_2) \end{aligned}$$

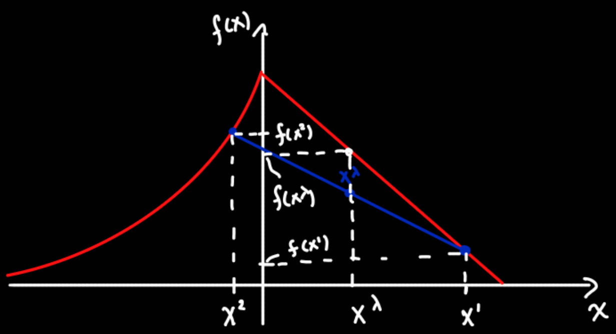
Por lo tanto, $s(x_1, x_2)$ es homogénea de grado r.

Definición: Sea $X \subset \mathbb{R}^n$ conjunto convexo y $f : X \to \mathbb{R}$. Decimos que f es cuasicóncava si, $\forall x^1, x^2 \in X$:

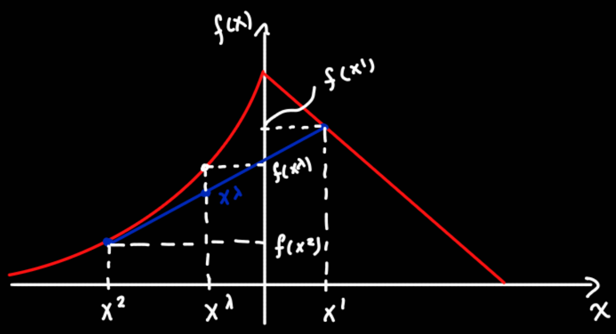
$$f(\lambda x^1 + (1-\lambda)x^2) \geq \min\{f(x^1), f(x^2)\} \quad \forall \lambda \in [0,1]$$

Ejemplo: $f : X \subset \mathbb{R} \to \mathbb{R}$

En este caso, $\min\{f(x^1), f(x^2)\} = f(x^1)$.
Vemos que $f(x^\lambda) \geq f(x^1) \quad \forall \lambda \in [0,1]$



En este caso, $\min\{f(x^1), f(x^2)\} = f(x^2)$
Vemos que $f(x^\lambda) \geq f(x^2) \quad \forall \lambda \in [0,1]$.



Un teorema muy importante (y útil) para funciones cuasicóncavas involucra conjuntos de contorno superior. Demos primero unas definiciones de apoyo.

Definición: (Conjuntos de nivel).

El conjunto $\mathcal{L}(y)$ es un conjunto de nivel de una función $f: \mathcal{X} \subset \mathbb{R}^n \to \mathbb{R}$ ssi:

$$\mathcal{L}(y) = \{ x \in \mathcal{X} \mid f(x) = y \} \quad \text{donde } y \in \mathbb{R}$$

Definición: (Conjuntos de contorno).

Sea $f: \mathcal{X} \subset \mathbb{R}^n \to \mathbb{R}$.

El conjunto $S(y)$ es un conjunto de contorno superior de f para el nivel $y \in \mathbb{R}$ ssi:

$$\boxed{S(y) = \{ x \in \mathcal{X} \mid f(x) \geq y \}}$$

El conjunto $I(y)$ es un conjunto de contorno inferior de f para el nivel $y \in \mathbb{R}$ ssi:

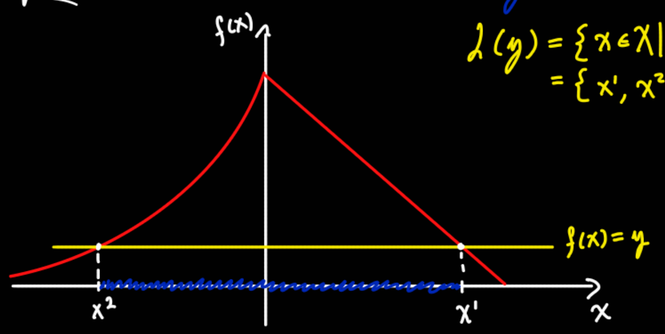
$$\boxed{I(y) = \{ x \in \mathcal{X} \mid f(x) \leq y \}}$$

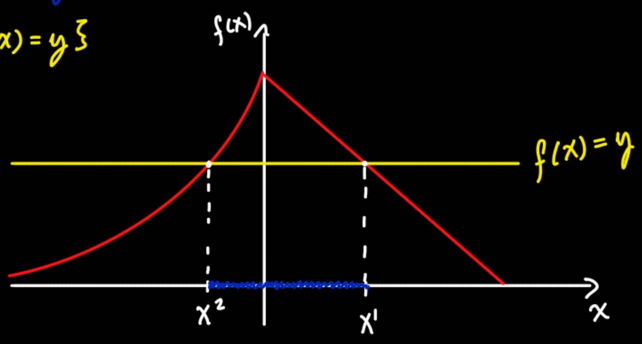
Teorema: Sea $f: \mathcal{X} \subset \mathbb{R}^n \to \mathbb{R}$. f es una función cuasicóncava ssi $S(y)$ es un conjunto convexo para todo $y \in \mathbb{R}$.

Ejemplo: $f: \mathcal{X} \subset \mathbb{R} \to \mathbb{R}$

$S(y) = \{ x \in \mathcal{X} \mid f(x) \geq y \}$

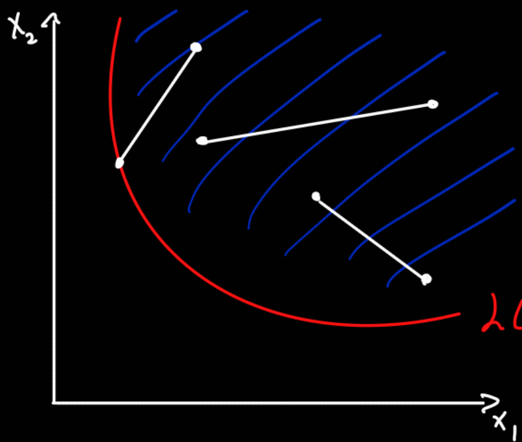
$\mathcal{L}(y) = \{ x \in \mathcal{X} \mid f(x) = y \} = \{ x^1, x^2 \}$





Ejemplo: Cobb-Douglas con $n=2$. Sea $\mathcal{X} = \mathbb{R}^2_+$ y $u: \mathcal{X} \to \mathbb{R}$ definida como $u(x_1, x_2) = x_1^{\alpha} x_2^{\beta}$.

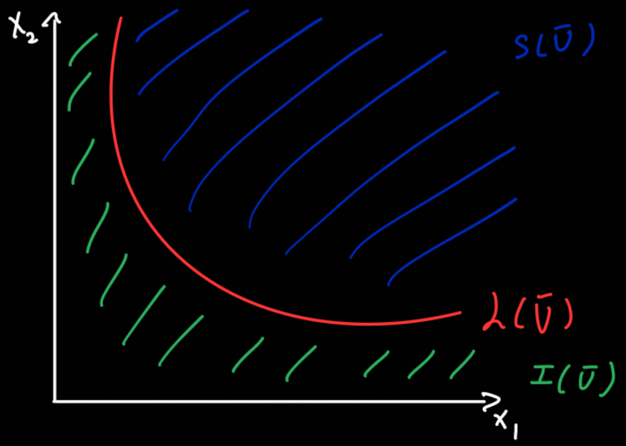
Sea $\bar{U} \in \mathbb{R}$.
$S(\bar{U})$ es convexo.
Entonces, $u(x_1, x_2)$ es cuasicóncava.

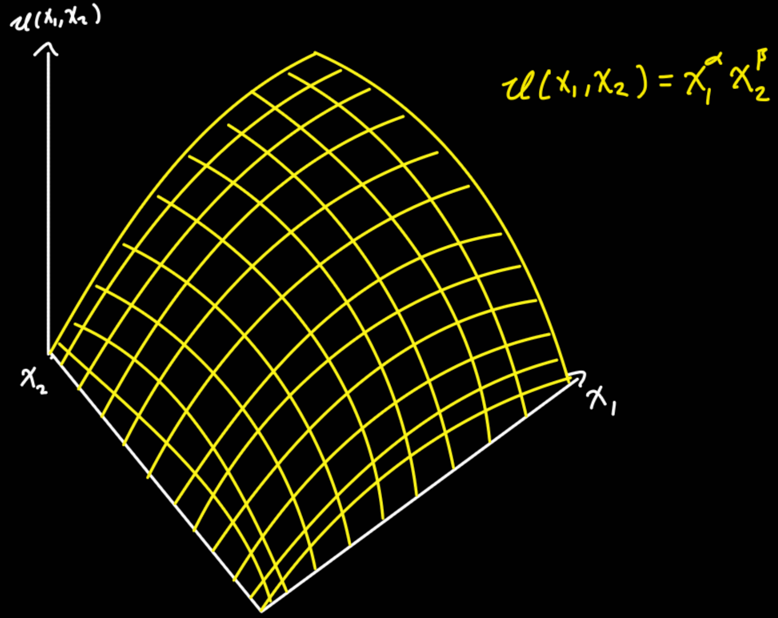


$S(\bar{U}) = \{ (x_1, x_2) \in \mathcal{X} \mid u(x_1, x_2) \geq \bar{U} \}$

$\mathcal{L}(\bar{U}) = \{ (x_1, x_2) \in \mathcal{X} \mid u(x_1, x_2) = \bar{U} \}$

Note que los conjuntos de nivel son un subconjunto del dominio de la función, y no valores en la imagen de la función.





(b) P.D. $m(\lambda x+(1-\lambda)y) \geq \min\{m(x), m(y)\}$

Sea $X=\mathbb{R}^2_+$ y $u: X \to \mathbb{R}$, $v: X \to \mathbb{R}$ funciones cuasicóncavas.

Define $m(x_1,x_2)=\min\{u(x_1,x_2), v(x_1,x_2)\}$ y considera $x=(x_1,x_2)$, $y=(y_1,y_2)\in X$.

Sea $x^{\lambda}=\lambda x+(1-\lambda)y \in X$, con $\lambda\in[0,1]$.

Como $u(x)$ y $v(x)$ son cuasicóncavas, entonces:

$$u(\lambda x+(1-\lambda)y) \geq \min\{u(x), u(y)\},$$

$$v(\lambda x+(1-\lambda)y) \geq \min\{v(x), v(y)\}$$

Si $u(\lambda x+(1-\lambda)y) \leq v(\lambda x+(1-\lambda)y)$, entonces:

$$\begin{aligned}
m(\lambda x+(1-\lambda)y) &= \min\{u(\lambda x+(1-\lambda)y), v(\lambda x+(1-\lambda)y)\} \\
&= u(\lambda x+(1-\lambda)y) \\
&\geq \min\{u(x), u(y)\} \qquad \text{(porque } u(x) \text{ es cuasicóncava)} \\
&\geq \min\{\min\{u(x),u(y)\}, \min\{v(x),v(y)\}\} \\
&= \min\{u(x), u(y), v(x), v(y)\} \\
&= \min\{\min\{u(x),v(x)\}, \min\{u(y),v(y)\}\} \\
&= \min\{m(x), m(y)\}
\end{aligned}$$

Si $u(\lambda x+(1-\lambda)y) > v(\lambda x+(1-\lambda)y)$, entonces:

$$\begin{aligned} m(\lambda x+(1-\lambda)y) &= \min\{u(\lambda x+(1-\lambda)y), v(\lambda x+(1-\lambda)y)\} \\ &= v(\lambda x+(1-\lambda)y) \\ &\geq \min\{v(x), v(y)\} \qquad \text{(porque } v(x) \text{ es cuasicóncava)} \\ &\geq \min\{\min\{u(x), u(y)\}, \min\{v(x), v(y)\}\} \\ &= \min\{u(x), u(y), v(x), v(y)\} \\ &= \min\{\min\{u(x), v(x)\}, \min\{u(y), v(y)\}\} \\ &= \min\{m(x), m(y)\} \end{aligned}$$

En ambos casos se cumple que:

$$m(\lambda x+(1-\lambda)y) \geq \min\{m(x), m(y)\}, \quad \forall \lambda \in [0,1]$$

Por lo tanto, $m(x)$ es cuasicóncava.

This utility function is known as the *constant elasticity of substitution* (or *CES*) utility function.

(a) Show that when $\rho = 1$, indifference curves become linear.

(a) Esta es trivial.

(b) Show that as $\rho \to 0$, this utility function comes to represent the same preferences as the (generalized) Cobb–Douglas utility function $u(x) = x_1^{\alpha_1} x_2^{\alpha_2}$.

(b) CES $\to$ $u(x) = x_1^{\alpha_1} x_2^{\alpha_2}$ cuando $\rho \to 0$.

Sea $u(x) = [\alpha_1 x_1^\rho + \alpha_2 x_2^\rho]^{\frac{1}{\rho}}$ con $\rho \in (-\infty, 1] \setminus \{0\}$ y $\alpha_1 + \alpha_2 = 1$. Entonces:

$$\ln u(x) = \frac{1}{\rho} \ln [\alpha_1 x_1^\rho + \alpha_2 x_2^\rho]$$

Así, definimos $F(\rho) := \ln[\alpha_1 e^{\rho \ln x_1} + \alpha_2 e^{\rho \ln x_2}]$ & $G(\rho) = \rho$. Entonces,

$$\begin{aligned} \lim_{\rho \to 0} \ln u(x) &= \lim_{\rho \to 0} \frac{F(\rho)}{G(\rho)} \\ &= \frac{\ln[\alpha_1 + \alpha_2]}{0} \qquad (\alpha_1 + \alpha_2 = 1) \\ &= \frac{0}{0} \end{aligned}$$

Así, por L'Hôpital sabemos que:

$$\lim_{\rho\to 0} \frac{F(\rho)}{G(\rho)} = \lim_{\rho\to 0} \frac{F'(\rho)}{G'(\rho)}$$

Entonces,

$$\lim_{\rho\to 0} \ln u(x) = \lim_{\rho\to 0}\left[\frac{\alpha_1 e^{\rho\ln x_1}\ln x_1 + \alpha_2 e^{\rho \ln x_2}\ln x_2}{\alpha_1 x_1^{\rho} + \alpha_2 x_2^{\rho}}\right]$$

$$= \lim_{\rho\to 0}\left[\frac{\alpha_1 \cancel{x_1^{\rho}}^{\to 1}\ln x_1 + \alpha_2 \cancel{x_2^{\rho}}^{\to 1}\ln x_2}{\alpha_1 x_1^{\rho} + \alpha_2 x_2^{\rho}}\right]$$

$$= \frac{\alpha_1 \ln x_1 + \alpha_2 \ln x_2}{\cancel{\alpha_1 + \alpha_2}^{\to 1}}$$

$$= \alpha_1 \ln x_1 + \alpha_2 \ln x_2$$

$$= \ln[x_1^{\alpha_1} x_2^{\alpha_2}]$$

De este modo, tenemos:

$$\lim_{\rho\to 0} \ln u(x) = \ln[x_1^{\alpha_1} x_2^{\alpha_2}] \iff \lim_{\rho\to 0} e^{\ln u(x)} = e^{\ln[x_1^{\alpha_1} x_2^{\alpha_2}]}$$

$$\iff \lim_{\rho\to 0} u(x) = x_1^{\alpha_1} x_2^{\alpha_2}$$

Por lo tanto, $\boxed{\lim_{\rho\to 0} u(x) = x_1^{\alpha_1} x_2^{\alpha_2}.}$

Para "n" bienes. Sea $u: X \subset \mathbb{R}^n \to \mathbb{R}$ definida por:

$$u(x) = \left[\sum_{i=1}^{n} \alpha_i x_i^{\rho}\right]^{\frac{1}{\rho}}$$

con $\rho \in (-\infty, 1] \setminus \{0\}$ y $\sum_{i=1}^{n} \alpha_i = 1$.

Entonces,

$$\ln u(x) = \frac{1}{\rho} \ln\left[\sum_{i=1}^{n} \alpha_i x_i^{\rho}\right]$$

Define $F(\rho) := \ln[\sum_{i=1}^{n} \alpha_i x_i^{\rho}]$ & $G(\rho) = \rho$. Entonces

$$\lim_{\rho\to 0} \ln u(x) = \lim_{\rho\to 0}\left[\frac{1}{\rho}\ln\left[\sum_{i=1}^{n}\alpha_i x_i^{\rho}\right]\right]$$

$$= \frac{1}{\rho}\ln\left[\cancel{\sum_{i=1}^{n}\alpha_i}^{1}\right]$$

$$= \frac{0}{0}$$

Por L'Hôpital:

$$\lim_{\rho \to 0} \ln u(x) = \lim_{\rho \to 0} \left[\frac{\sum_{i=1}^{n} \alpha_i x_i^{\rho} \ln x_i}{\sum_{i=1}^{n} \alpha_i x_i^{\rho}} \right]$$

$$= \frac{\sum_{i=1}^{n} \alpha_i \ln x_i}{\cancel{\sum_{i=1}^{n} \alpha_i} \to 1}$$

$$= \sum_{i=1}^{n} \alpha_i \ln x_i$$

$$= \ln \left[\prod_{i=1}^{n} x_i^{\alpha_i} \right]$$

Por lo tanto,

$$\boxed{\lim_{\rho \to 0} \left[\sum_{i=1}^{n} \alpha_i x_i^{\rho} \right]^{\frac{1}{\rho}} = \prod_{i=1}^{n} x_i^{\alpha_i}.}$$

(c) Show that as $\rho \to -\infty$, indifference curves become "right angles"; that is, this utility function has in the limit the indifference map of the Leontief utility function $u(x_1, x_2) = \min\{x_1, x_2\}$.

Procedemos de forma análoga al ejercicio anterior.

Sea $x_m = \min\{x_1, x_2\}$. Definimos $F(\rho) = \ln[\alpha_1 x_1^{\rho} + \alpha_2 x_2^{\rho}]$ y $G(\rho) = \rho$. Entonces:

$$\lim_{\rho \to -\infty} \ln u(x) = \lim_{\rho \to -\infty} \frac{F(\rho)}{G(\rho)}$$

$$= \lim_{\rho \to -\infty} \left[\frac{1}{\rho} \ln [\alpha_1 \cancel{e^{\rho \ln x_1} + \alpha_2 e^{\rho \ln x_2}}] \right] \to 0$$

$$= \frac{-\infty}{-\infty}$$

Así, por L'Hôpital:

$$\lim_{\rho \to -\infty} \ln u(x) = \lim_{\rho \to -\infty} \left[\frac{\alpha_1 x_1^{\rho} \ln x_1 + \alpha_2 x_2^{\rho} \ln x_2}{\alpha_1 x_1^{\rho} + \alpha_2 x_2^{\rho}} \right]$$

$$= \lim_{\rho \to -\infty} \left[\frac{x_m^{-\rho}}{x_m^{-\rho}} \cdot \frac{\alpha_1 x_1^{\rho} \ln x_1 + \alpha_2 x_2^{\rho} \ln x_2}{\alpha_1 x_1^{\rho} + \alpha_2 x_2^{\rho}} \right]$$

$$= \lim_{\rho \to -\infty} \left[\frac{\alpha_1 \left(\frac{x_1}{x_m}\right)^{\rho} \ln x_1 + \alpha_2 \left(\frac{x_2}{x_m}\right)^{\rho} \ln x_2}{\alpha_1 \left(\frac{x_1}{x_m}\right)^{\rho} + \alpha_2 \left(\frac{x_2}{x_m}\right)^{\rho}} \right]$$

$$= \lim_{\rho \to -\infty} \left[\frac{\alpha_1 \left(\frac{x_m}{x_1}\right)^{-\rho} \ln x_1 + \alpha_2 \left(\frac{x_m}{x_2}\right)^{-\rho} \ln x_2}{\alpha_1 \left(\frac{x_m}{x_1}\right)^{-\rho} + \alpha_2 \left(\frac{x_m}{x_2}\right)^{-\rho}} \right]$$

S.P.G., supongamos que $x_m = x_1$. Entonces:

$$\lim_{\rho \to -\infty} \ln u(x) = \lim_{\rho \to -\infty} \left[\frac{\alpha_1 \ln x_1 + \alpha_2 \left(\frac{x_m}{x_2}\right)^{-\rho} \ln x_2}{\alpha_1 + \alpha_2 \left(\frac{x_m}{x_2}\right)^{-\rho}} \right]$$

Por definición, $0 \leq x_m < x_2$. Entonces, $\frac{x_m}{x_2} < 1$

Así, cuando $\rho \to -\infty$, $\left(\frac{x_m}{x_2}\right)^{-\rho} \to 0$. Entonces:

$$\lim_{\rho \to -\infty} \ln u(x) = \lim_{\rho \to -\infty} \left[\frac{\alpha_1 \ln x_1 + \cancelto{0}{\alpha_2 \left(\frac{x_m}{x_2}\right)^{-\rho}} \ln x_2}{\alpha_1 + \cancelto{0}{\alpha_2 \left(\frac{x_m}{x_2}\right)^{-\rho}}} \right]$$

$$= \frac{\alpha_1 \ln x_1}{\alpha_1}$$

$$= \ln x_1 \qquad (x_1 = x_m)$$

$$= \ln x_m$$

De este modo, vemos que

$$\lim_{\rho \to -\infty} \ln u(x) = \ln x_m \iff \lim_{\rho \to -\infty} u(x) = x_m = \min\{x_1, x_2\}$$

Por lo tanto, $\boxed{\lim_{\rho \to -\infty} u(x) = \min\{x_1, x_2\}.}$

$\boxed{\text{Para "n" bienes.}}$ Sea $u: X \subset \mathbb{R}^n \to \mathbb{R}$ definida por: $u(x) = \left[\sum_{i=1}^{n} \alpha_i x_i^{\rho}\right]^{\frac{1}{\rho}}$

Define $F(\rho) := \ln\left[\sum_{i=1}^{n} \alpha_i x_i^{\rho}\right]$ & $G(\rho) = \rho$. Entonces

$$\lim_{\rho \to -\infty} \ln u(x) = \lim_{\rho \to -\infty} \left[\frac{1}{\rho} \ln \left[\cancelto{0}{\sum_{i=1}^{n} \alpha_i e^{\rho \ln x_i}} \right] \right]$$

$$= \frac{-\infty}{-\infty}$$

Por L'Hôpital:

$$\lim_{\rho \to -\infty} \ln u(x) = \lim_{\rho \to -\infty} \left[\frac{\sum_{i=1}^{n} \alpha_i x_i^{\rho} \ln x_i}{\sum_{i=1}^{n} \alpha_i x_i^{\rho}} \right]$$

Sea $x_m = \min\{x_1, \dots, x_n\}$. Entonces:

$$\lim_{\rho \to -\infty} \ln u(x) = \lim_{\rho \to -\infty} \left[\frac{x_m^{-\rho}}{x_m^{-\rho}} \cdot \frac{\sum_{i=1}^{n} \alpha_i x_i^{\rho} \ln x_i}{\sum_{i=1}^{n} \alpha_i x_i^{\rho}} \right]$$

$$= \lim_{\rho \to -\infty} \left[\frac{\sum_{i=1}^{n} \alpha_i \left(\frac{x_i}{x_m}\right)^{\rho} \ln x_i}{\sum_{i=1}^{n} \alpha_i \left(\frac{x_i}{x_m}\right)^{\rho}} \right]$$

$$= \lim_{\rho \to -\infty} \left[\frac{\alpha_m \ln(x_m) + \sum_{\substack{i=1 \\ i \neq m}}^{n-1} \alpha_i \left(\frac{x_i}{x_m}\right)^{\rho} \ln x_i}{\alpha_m + \sum_{\substack{i=1 \\ i \neq m}}^{n-1} \alpha_i \left(\frac{x_i}{x_m}\right)^{\rho}} \right]$$

$$= \lim_{\rho \to -\infty} \left[\frac{\alpha_m \ln(x_m) + \sum_{\substack{i=1 \\ i \neq m}}^{n-1} \alpha_i \left(\frac{x_m}{x_i}\right)^{-\rho} \ln x_i}{\alpha_m + \sum_{\substack{i=1 \\ i \neq m}}^{n-1} \alpha_i \left(\frac{x_m}{x_i}\right)^{-\rho}} \right]$$

Vemos que $x_i \geq x_m \ \forall i = 1, \dots, n-1, \ i \neq m$. Entonces $\frac{x_m}{x_i} \leq 1$.

Esto implica que $\lim_{\rho \to -\infty} \left(\frac{x_m}{x_i}\right)^{-\rho} = 0 \quad \forall i = 1, \dots, n-1, \ i \neq m$. Entonces:

$$\lim_{\rho \to -\infty} \ln u(x) = \lim_{\rho \to -\infty} \left[\frac{\alpha_m \ln(x_m) + \cancelto{0}{\sum_{\substack{i=1 \\ i \neq m}}^{n-1} \alpha_i \left(\frac{x_m}{x_i}\right)^{-\rho}} \ln x_i}{\alpha_m + \cancelto{0}{\sum_{\substack{i=1 \\ i \neq m}}^{n-1} \alpha_i \left(\frac{x_m}{x_i}\right)^{-\rho}}} \right]$$

$$= \frac{\alpha_m \ln(x_m)}{\alpha_m}$$

$$= \ln(x_m)$$

$$= \ln(\min\{x_1, \dots, x_n\})$$

Por lo tanto,

$$\boxed{\lim_{\rho \to -\infty} \left[\sum_{i=1}^{n} \alpha_i x_i^{\rho} \right]^{\frac{1}{\rho}} = \min\{x_1, \dots, x_n\}}$$

Ex. 4 Let $u : \mathbb{R}^2_+ \to \mathbb{R}$ be twice continuously differentiable. Prove that the condition $\frac{\partial}{\partial x_1}\text{MRS}_{12}(x_1, x_2) \leq 0$ is equivalent to the quasiconcavity of u as expressed by the sign criterion on the determinants of the minors of the bordered Hessian (**Theorem 1**).

(4) Para probar la equivalencia, veamos primero los resultados necesarios.

Sea $u: \mathbb{R}^2_+ \to \mathbb{R}$ función C^2. Por definición, la $MRS_{12}(x_1, x_2)$ es la pendiente de la recta tangente al punto (x_1, x_2) en una curva de indiferencia:

$$du(x_1,x_2) = u_1\, dx_1 + u_2\, dx_2 = 0 \iff \frac{dx_2}{dx_1} = -\frac{u_1}{u_2} = -RMS_{12}(x_1, x_2)$$

Entonces, la derivada total de la RMS_{12} respecto a x_1 es:

$$\begin{aligned}
\frac{d}{dx_1} RMS_{12}(x_1,x_2) &= \frac{\partial RMS_{12}}{\partial x_1} + \frac{\partial RMS_{12}}{\partial x_2}\frac{dx_2}{dx_1} \\
&= \frac{u_{11}u_2 - u_{21}u_1}{u_2^2} + \frac{u_{12}u_2 - u_{22}u_1}{u_2^2}\left(-\frac{u_1}{u_2}\right) \\
&= \frac{u_{11}u_2 - u_{21}u_1}{u_2^2} - \frac{u_1u_2u_{12} - u_{22}u_1^2}{u_2^3} \\
&= \frac{1}{u_2^3}\left(u_{11}u_2^2 - u_1u_2u_{21} - u_1u_2u_{12} + u_{22}u_1^2\right) \\
&= \frac{1}{u_2^3}\left(u_{11}u_2^2 - 2u_1u_2u_{12} + u_{22}u_1^2\right)
\end{aligned}$$

donde la penúltima línea usa el hecho de que $u(x_1,x_2)$ es de C^2, lo que implica que $u_{12} = u_{21}$ (Teorema de Young).

$$\boxed{\text{Por lo tanto, } \frac{d}{dx_1}\left[RMS_{12}(x_1,x_2)\right] < 0 \text{ ssi } u_{11}u_2^2 - 2u_1u_2u_{12} + u_{22}u_1^2 < 0} \quad (1)$$

Ahora, la matriz Hessiana Orlada de $u(x_1,x_2)$ es:

$$\bar{H}[u(x_1,x_2)] = \begin{pmatrix} 0 & u_1 & u_2 \\ u_1 & u_{11} & u_{12} \\ u_2 & u_{21} & u_{22} \end{pmatrix}$$

Sus menores principales son:

$$\bar{H}_2 = \begin{pmatrix} 0 & u_1 \\ u_1 & u_{11} \end{pmatrix}, \quad \bar{H}_3 = \begin{pmatrix} 0 & u_1 & u_2 \\ u_1 & u_{11} & u_{12} \\ u_2 & u_{21} & u_{22} \end{pmatrix}$$

Calculamos sus determinantes:

$$|\bar{\mathcal{H}}_2| = \begin{vmatrix} 0 & u_1 \\ u_1 & u_{11} \end{vmatrix} = -u_1^2 < 0 \text{ (porque } u(\cdot) \text{ es creciente)}$$

,

$$|\bar{\mathcal{H}}_3| = \begin{vmatrix} 0 & u_1 & u_2 \\ u_1 & u_{11} & u_{12} \\ u_2 & u_{21} & u_{22} \end{vmatrix}$$

$$= u_1 u_2 u_{12} + u_1 u_2 u_{21} - u_2^2 u_{11} - u_1^2 u_{22}$$

$$= -(u_{11} u_2^2 - 2 u_1 u_2 u_{12} + u_{22} u_1^2)$$

Para que $u(\cdot)$ sea cuasicóncava, necesitamos que $|\bar{\mathcal{H}}_3| > 0$.

$$\boxed{\text{Por lo tanto, } |\bar{\mathcal{H}}_3| > 0 \text{ ssi } u_{11} u_2^2 - 2 u_1 u_2 u_{12} + u_{22} u_1^2 < 0} \quad (2)$$

Las condiciones (1) y (2) son equivalentes. Ambas requieren que:

$$\boxed{u_{11} u_2^2 - 2 u_1 u_2 u_{12} + u_{22} u_1^2 < 0}$$

Es decir:

$$\boxed{\begin{aligned} \frac{d}{dx_1}[RMS_{12}(x_1,x_2)] < 0 &\iff u_{11} u_2^2 - 2 u_1 u_2 u_{12} + u_{22} u_1^2 < 0 \\ &\iff |\bar{\mathcal{H}}_3| > 0 \\ &\iff u(\cdot) \text{ es cuasicóncava} \end{aligned}}$$