Laboratorio de Microeconomía I

Centro de Investigación y Docencia Económicas
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Laboratorista: Arturo López
Tarea 8 Solver

Exercises

Ex. 1.60 Suppose $x_1(\mathbf{p}, y)$ and $x_2(\mathbf{p}, y)$ have equal income elasticity at $(\mathbf{p}^0, y^0)$. Show that

$$\frac{\partial x_1}{\partial p_2} = \frac{\partial x_2}{\partial p_1} \text{ at } (\mathbf{p}^0, y^0)$$

Proof. Supongamos que $x_1(\mathbf{p}, y)$ y $x_2(\mathbf{p}, y)$ tienen la misma elasticidad ingreso en un punto $(\mathbf{p^0}, \mathbf{y^0})$. Entonces,

$$\frac{\partial x_1}{\partial y}\frac{y}{x_1} = \frac{\partial x_2}{\partial y}\frac{y}{x_2} \text{ en } (\mathbf{p^0}, \mathbf{y^0})$$

Esto implica que

$$\frac{\partial x_1}{\partial y} = \frac{\partial x_2}{\partial y}\frac{x_1}{x_2}$$

Consideremos ahora la ecuación de Slutsky en $(\mathbf{p}^0, y^0)$:

$$\frac{\partial x_i}{\partial p_j} = \frac{\partial x_i^h}{\partial p_j} - x_j \frac{\partial x_i}{\partial y}, \qquad i, j = 1, 2$$

Por simetría en los términos de sustitución (**Theorem 1.14**, Jehle & Reny, 2011):

$$\frac{\partial x_i^h}{\partial p_j} = \frac{\partial x_j^h}{\partial p_i} \qquad i, j = 1, 2$$

Entonces, a partir de la ecuación de slutsky

$$\begin{aligned}
\frac{\partial x_1}{\partial p_2} - \frac{\partial x_2}{\partial p_1} &= \left(\frac{\partial x_1^h}{\partial p_2} - x_2 \frac{\partial x_1}{\partial y}\right) - \left(\frac{\partial x_2^h}{\partial p_1} - x_1 \frac{\partial x_2}{\partial y}\right) \\
&= \left(\frac{\partial x_1^h}{\partial p_2} - x_2 \frac{\partial x_1}{\partial y}\right) - \left(\frac{\partial x_1^h}{\partial p_2} - x_1 \frac{\partial x_2}{\partial y}\right) \quad \text{(por \textbf{Theorem 1.14})} \\
&= -x_2 \frac{\partial x_1}{\partial y} + x_1 \frac{\partial x_2}{\partial y}
\end{aligned}$$

Dado que en $x_1(\mathbf{p}, y)$ y $x_2(\mathbf{p}, y)$ tienen la misma elasticidad ingreso en $(\mathbf{p^0}, \mathbf{y^0})$, entonces:

$$\begin{aligned}
\frac{\partial x_1}{\partial p_2} - \frac{\partial x_2}{\partial p_1} &= -\, x_2 \,\frac{\partial x_1}{\partial y} + \, x_1 \,\frac{\partial x_2}{\partial y} \\
&= -\, x_2 \,\frac{\partial x_2}{\partial y}\frac{x_1}{x_2} + \, x_1 \,\frac{\partial x_2}{\partial y} \\
&= -\, x_1 \,\frac{\partial x_2}{\partial y} + \, x_1 \,\frac{\partial x_2}{\partial y} \\
&= 0
\end{aligned}$$

Es decir,

$$\frac{\partial x_1}{\partial p_2} - \frac{\partial x_2}{\partial p_1} = 0$$

Luego,

$$\frac{\partial x_1}{\partial p_2} = \frac{\partial x_2}{\partial p_1} \text{ en } (\mathbf{p^0}, \mathbf{y^0})$$

□

Ex. 1.63 The substitution matrix of a utility-maximising consumer's demand system at prices $(8, p)$ is

$$\begin{pmatrix} a & b \\ 2 & -1/2 \end{pmatrix}.$$

Find a, b, and p.

Proof. Sabemos que la matriz de sustitución $\sigma(\mathbf{p}, u)$ es negativa semidefinida (**Theorem 1.15**) y simétrica (**Theorem 1.14**).

Entonces, por simetría tenemos que $b = 2$.

Por otro lado, consideremos la **Tercera Ley de Hicks** (ver Ej. 1.62 en Jehle & Reny, 2011):

$$\sum_{j=1}^{n} \frac{\partial x_i^h(\mathbf{p}, u)}{\partial p_j} p_j = 0, \qquad i = 1, \ldots, n$$

en forma matricial:

$$D_{\mathbf{p}} \mathbf{x}^h(\mathbf{p}, u)\, \mathbf{p} = \mathbf{0}$$

equivalentemente:

$$\sigma(\mathbf{p}, u)\, \mathbf{p} = \mathbf{0}$$

Por lo tanto,

$$\begin{pmatrix} a & 2 \\ 2 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 8 \\ p \end{pmatrix} = \begin{pmatrix} 8a + 2p \\ 16 - \frac{1}{2}p \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

De la segunda ecuación obtenemos:

$$16 - \frac{1}{2}p = 0 \quad \Rightarrow \quad p = 32$$

Sustituyendo $p = 32$ en la primera ecuación:

$$8a + 32 \cdot 2 = 0 \quad \Rightarrow \quad a = -8$$

Luego, la matriz de sustitución es:

$$\sigma(\mathbf{p}, u) = \begin{pmatrix} -8 & 2 \\ 2 & -\frac{1}{2} \end{pmatrix}$$

Ahora, aplicamos el criterio de signos de los menores principales para una matriz negativa semidefinida:

Lemma: Sea A una matriz simétrica. Entonces, A es **negativa semidefinida** si y sólo si sus $2^n - 1$ menores principales alternan en signo, con los de orden impar siendo ≤ 0 y los de orden par siendo ≥ 0.

Por lo tanto:

Primer menor principal

$$\sigma_1 = -8 < 0$$

Segundo menor principal

$$\sigma_2 = \begin{vmatrix} -8 & 2 \\ 2 & -\frac{1}{2} \end{vmatrix} = 4 - 4 = 0$$

Por lo tanto, con $a = -8$ y $b = 2$, $\sigma(\mathbf{p}, u)$ es simétrica y negativa semidefinida. □

Ex. 3.G.3$^{\mathbf{B}}$ Consider the (linear expenditure system) utility function given in Exercise 3.D.6.

(a) Derive the Hicksian demand and expenditure functions. Check the properties listed in Propositions 3.E.2 and 3.E.3.

(b) Show that the derivatives of the expenditure function are the Hicksian demand function you derived in (a).

(c) Verify that the Slutsky equation holds.

(d) Verify that the own-substitution terms are negative and that compensated cross-price effects are symmetric.

(e) Show that $S(p, w)$ is negative semidefinite and has rank 2.

Proof. Partimos de la función de utilidad del sistema de gasto lineal:

$$u(x_1, x_2, x_3) = (x_1 - b_1)^\alpha (x_2 - b_2)^\beta (x_3 - b_3)^\gamma, \quad \text{con } \alpha + \beta + \gamma = 1$$

(a) Funciones de demanda Hicksiana y función de gasto

El problema de minimización de gasto es:

$$\begin{aligned} \min_{x_1,x_2,x_3} \; & p_1 x_1 + p_2 x_2 + p_3 x_3 \\ \text{s.a.} \quad & (x_1 - b_1)^\alpha (x_2 - b_2)^\beta (x_3 - b_3)^\gamma = u \end{aligned}$$

Definimos las siguientes variables:

$$z_i := x_i - b_i, \quad i = 1, 2, 3 \quad \& \quad \alpha_1 = \alpha, \;\; \alpha_2 = \beta, \;\; \alpha_3 = \gamma$$

Así, el problema se vuelve una versión desplazada de una función de tipo Cobb-Douglas.

El problema entonces se vuelve:

$$\begin{aligned} \min_{z_1,z_2,z_3} \; & \sum_{i=1}^{3} p_i (z_i + b_i) \\ \text{s.a.} \quad & z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} = u \end{aligned}$$

El Lagrangiano asociado:

$$\mathcal{L}(\mathbf{p}, \mathbf{z}, \lambda) = \sum_{i=1}^{3} p_i (z_i + b_i) + \lambda \left(u - z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3}\right)$$

Las condiciones de primer orden son simétricas:

$$\left.\frac{\partial \mathcal{L}}{\partial z_i}\right|_{(\mathbf{p},\mathbf{z},\lambda)=(\mathbf{p}^*,\mathbf{z}^*,\lambda^*)} = p_i - \lambda\, \alpha_i z_i^{-1}\, z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} = 0\,, \quad i = 1,2,3 \tag{1}$$

$$\left.\frac{\partial \mathcal{L}}{\partial \lambda}\right|_{(\mathbf{p},\mathbf{z},\lambda)=(\mathbf{p}^*,\mathbf{z}^*,\lambda^*)} = u - z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} = 0 \tag{2}$$

De (1) se obtiene:

$$z_i = \lambda \alpha_i \frac{u}{p_i} \tag{3}$$

Sustituyendo (3) en (2):

$$(\lambda u)^{\alpha_1+\alpha_2+\alpha_3} (\alpha_1^{\alpha_1} \alpha_2^{\alpha_2} \alpha_3^{\alpha_3})(p_1^{-\alpha_1} p_2^{-\alpha_2} p_3^{-\alpha_3}) = u$$

y dado que $\alpha_1 + \alpha_2 + \alpha_3 = 1$:

$$\lambda = \frac{p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}}{\alpha_1^{\alpha_1} \alpha_2^{\alpha_2} \alpha_3^{\alpha_3}} = \frac{G(\mathbf{p})}{A} \tag{4}$$

donde $A := \alpha_1^{\alpha_1} \alpha_2^{\alpha_2} \alpha_3^{\alpha_3}$ y $G(\mathbf{p})$ es un índice de precios geométrico.

en términos de las variables originales:

$$(\lambda u)^{\alpha+\beta+\gamma} (\alpha^\alpha \beta^\beta \gamma^\gamma)(p_1^{-\alpha} p_2^{-\beta} p_3^{-\gamma}) = u \implies \lambda = \frac{p_1^\alpha p_2^\beta p_3^\gamma}{\alpha^\alpha \beta^\beta \gamma^\gamma}$$

Sustituyendo en (4) en (3), y dado que $z_i = x_i - b_i$ para $i = 1, 2, 3$, obtenemos:

$$x_i^h(\mathbf{p}, u) = b_i + \alpha_i\, u\, \frac{G(\mathbf{p})}{A\, p_i}\,, \qquad i = 1,2,3$$

La función de gasto mínimo es:

$$\begin{aligned} e(\mathbf{p},u) &= \sum_{i=1}^{3} p_i\, x_i^h(\mathbf{p},u) \\ &= \sum_{i=1}^{3} p_i \left(b_i + \alpha_i\, u\, \frac{G(\mathbf{p})}{A\, p_i} \right) \\ &= \sum_{i=1}^{3} p_i b_i + \frac{G(\mathbf{p})}{A}\, u \\ &= \mathbf{p}^\top \mathbf{b} + \frac{G(\mathbf{p})}{A}\, u \end{aligned}$$

donde $\mathbf{p} = (p_1, p_2, p_3)^\top$ y $\mathbf{b} = (b_1, b_2, b_3)^\top$.

(c) Verificación de la ecuación de Slutsky

De la función de gasto $e(\mathbf{p},u)$, obtenemos la utilidad indirecta vía dualidad:

$$\begin{aligned} e(\mathbf{p}, v(\mathbf{p},y)) = y \iff & \ \mathbf{p}^\top \mathbf{b} + \frac{G(\mathbf{p})}{A}\, v(\mathbf{p},y) = y \\ \iff & \ v(\mathbf{p},y) = \frac{A}{G(\mathbf{p})}\,(y - \mathbf{p}^\top \mathbf{b}) \end{aligned}$$

Aplicaremos la identidad de Roy:

$$\frac{\partial v}{\partial y} = \frac{A}{G(\mathbf{p})}$$

y

$$\frac{\partial v}{\partial p_i} = -\frac{A}{G(\mathbf{p})}\, b_i - \frac{A}{G(\mathbf{p})}\, \frac{\alpha_i}{p_i} \left(y - \mathbf{p}^\top \mathbf{b} \right)$$

Aplicando la identidad de Roy:

$$x_i(\mathbf{p},y) = -\frac{\dfrac{\partial v}{\partial p_i}}{\dfrac{\partial v}{\partial y}} = -\frac{-\frac{A}{G(\mathbf{p})}\, b_i - \frac{A}{G(\mathbf{p})}\, \frac{\alpha_i}{p_i} \left(y - \mathbf{p}^\top \mathbf{b} \right)}{\dfrac{A}{G(\mathbf{b})}} = b_i + \frac{\alpha_i}{p_i}\,(y - \mathbf{p}^\top \mathbf{b})$$

Por lo tanto, las demandas marshallianas son:

$$x_i(\mathbf{p},y) = b_i + \frac{\alpha_i}{p_i}(y - \mathbf{p}^\top \mathbf{b})\,, \quad i = 1,2,3$$

Entonces, los componentes de la ecuación de Slutsky son:

$$\frac{\partial x_i}{\partial y} = \frac{\alpha_i}{p_i}, \qquad \frac{\partial x_i}{\partial p_j} = \begin{cases} -\dfrac{\alpha_i}{p_i^2}(y - \mathbf{p}^\top \mathbf{b}) - \dfrac{\alpha_i b_i}{p_i} & j = i, \\ -\dfrac{\alpha_i b_j}{p_i} & j \neq i \end{cases}$$

Por otro lado, a partir de la Hicksiana

$$x_i^h(\mathbf{p}, u) = b_i + \alpha_i\, u\, \frac{G(\mathbf{p})}{A\, p_i}$$

derivamos parcialmente respecto al precio $j \neq i$:

$$\frac{\partial x_i^h}{\partial p_j} = \frac{\alpha_i u}{A p_i} \frac{\partial G}{\partial p_j} = \frac{\alpha_i \alpha_j}{p_i p_j} \frac{u\, G(\mathbf{p})}{A}$$

y para $j = i$:

$$\begin{aligned} \frac{\partial x_i^h}{\partial p_i} &= \frac{\alpha_i u}{A} \left(\frac{1}{p_i} \frac{\partial G}{\partial p_i} - \frac{G(\mathbf{p})}{p_i^2} \right) \\ &= \frac{\alpha_i u\, G(\mathbf{p})}{A} \left(\frac{\alpha_i}{p_i^2} - \frac{1}{p_i^2} \right) \\ &= \alpha_i \frac{u\, G(\mathbf{p})}{A} \frac{\alpha_i - 1}{p_i^2} \end{aligned}$$

Evaluando en $u^* = v(\mathbf{p}, y) = \dfrac{A}{G(\mathbf{p})}\,(y - \mathbf{p}^\top \mathbf{b})$, queda:

$$\boxed{\begin{aligned} j \neq i: \quad & \frac{\partial x_i^h(\mathbf{p}, u^*)}{\partial p_j} = \frac{\alpha_i \alpha_j}{p_i p_j}\,(y - \mathbf{p}^\top \mathbf{b}) \\ j = i: \quad & \frac{\partial x_i^h(\mathbf{p}, u^*)}{\partial p_i} = \alpha_i\,(y - \mathbf{p}^\top \mathbf{b}) \left(\frac{\alpha_i - 1}{p_i^2} \right) \end{aligned}}$$

Ahora, verifiquemos la ecuación de Slutsky.

Mostraremos que:

$$\left. \frac{\partial x_i^h}{\partial p_j} \right|_{u^*} - \left(\frac{\partial x_i}{\partial p_j} + x_j \frac{\partial x_i}{\partial y} \right) = 0$$

para $i = j$ y para $i \neq j$.

Caso 1: $j \neq i$

Recordemos que:

$$\frac{\partial x_i}{\partial p_j} = -\frac{\alpha_i b_j}{p_i}, \qquad x_j(\mathbf{p}, y) = b_j + \frac{\alpha_j}{p_j}(y - \mathbf{p}^\top \mathbf{b}), \qquad \frac{\partial x_i}{\partial y} = \frac{\alpha_i}{p_i}$$

Sustituyendo en la diferencia:

$$\begin{aligned}
&\left.\frac{\partial x_i^h}{\partial p_j}\right|_{u^*} - \left(\frac{\partial x_i}{\partial p_j} + x_j \frac{\partial x_i}{\partial y}\right) \\
&= \frac{\alpha_i \alpha_j}{p_i p_j}(y - \mathbf{p}^\top \mathbf{b}) - \left(-\frac{\alpha_i b_j}{p_i} + \left[b_j + \frac{\alpha_j}{p_j}(y - \mathbf{p}^\top \mathbf{b})\right]\frac{\alpha_i}{p_i}\right) \\
&= \frac{\alpha_i \alpha_j}{p_i p_j}(y - \mathbf{p}^\top \mathbf{b}) - \left(-\frac{\alpha_i b_j}{p_i} + \frac{\alpha_i b_j}{p_i} + \frac{\alpha_i \alpha_j}{p_i p_j}(y - \mathbf{p}^\top \mathbf{b})\right) \\
&= \frac{\alpha_i \alpha_j}{p_i p_j}(y - \mathbf{p}^\top \mathbf{b}) - \frac{\alpha_i \alpha_j}{p_i p_j}(y - \mathbf{p}^\top \mathbf{b}) = 0
\end{aligned}$$

Caso 2: $j = i$

Recordemos que:

$$\frac{\partial x_i}{\partial p_i} = -\frac{\alpha_i}{p_i^2}(y - \mathbf{p}^\top \mathbf{b}) - \frac{\alpha_i b_i}{p_i}, \qquad x_i(\mathbf{p}, y) = b_i + \frac{\alpha_i}{p_i}(y - \mathbf{p}^\top \mathbf{b}), \qquad \frac{\partial x_i}{\partial y} = \frac{\alpha_i}{p_i}$$

Sustituyendo en la diferencia:

$$\begin{aligned}
&\left.\frac{\partial x_i^h}{\partial p_i}\right|_{u^*} - \left(\frac{\partial x_i}{\partial p_i} + x_i \frac{\partial x_i}{\partial y}\right) \\
&= \alpha_i \frac{\alpha_i - 1}{p_i^2}(y - \mathbf{p}^\top \mathbf{b}) - \left[-\frac{\alpha_i}{p_i^2}(y - \mathbf{p}^\top \mathbf{b}) - \frac{\alpha_i b_i}{p_i} + \left(b_i + \frac{\alpha_i}{p_i}(y - \mathbf{p}^\top \mathbf{b})\right)\frac{\alpha_i}{p_i}\right] \\
&= \alpha_i \frac{\alpha_i - 1}{p_i^2}(y - \mathbf{p}^\top \mathbf{b}) - \left[-\frac{\alpha_i}{p_i^2}(y - \mathbf{p}^\top \mathbf{b}) - \frac{\alpha_i b_i}{p_i} + \frac{\alpha_i b_i}{p_i} + \frac{\alpha_i^2}{p_i^2}(y - \mathbf{p}^\top \mathbf{b})\right] \\
&= \alpha_i \frac{\alpha_i - 1}{p_i^2}(y - \mathbf{p}^\top \mathbf{b}) - \left[\frac{\alpha_i^2}{p_i^2}(y - \mathbf{p}^\top \mathbf{b}) - \frac{\alpha_i}{p_i^2}(y - \mathbf{p}^\top \mathbf{b})\right] \\
&= \alpha_i \frac{\alpha_i - 1}{p_i^2}(y - \mathbf{p}^\top \mathbf{b}) - \alpha_i \frac{\alpha_i - 1}{p_i^2}(y - \mathbf{p}^\top \mathbf{b}) = 0
\end{aligned}$$

Por lo tanto, en ambos casos se cumple la ecuación de Slutsky.

(d) Signos propios y simetría.

A partir de la demanda hicksiana

$$x_i^h(\mathbf{p},u) = b_i + \alpha_i\, u\, \frac{G(\mathbf{p})}{A\, p_i}$$

obtenemos las derivadas parciales con respecto a los precios.

Efectos propios ($i = j$):

$$\frac{\partial x_i^h}{\partial p_i} = \alpha_i\, \frac{u\, G(\mathbf{p})}{A}\, \frac{\alpha_i - 1}{p_i^2} < 0$$

porque $0 < \alpha_i < 1$.

Por lo tanto, los efectos propios son negativos.

Efectos cruzados ($i \neq j$):

Para i respecto al precio j:

$$\frac{\partial x_i^h}{\partial p_j} = \frac{\alpha_i\, \alpha_j}{p_i\, p_j}\, \frac{u\, G(\mathbf{p})}{A}$$

Para j respecto al precio i:

$$\frac{\partial x_j^h}{\partial p_i} = \frac{\alpha_j\, \alpha_i}{p_j\, p_i}\, \frac{u\, G(\mathbf{p})}{A}$$

Por lo tanto, los efectos cruzados son simétricos.

(e) Negativa semidefinida y rango.

Para determinar la definitud y el rango de $\sigma(\mathbf{p},u)$, basta con examinar los menores principales.

Como la matriz es simétrica, entonces:

$$\sigma(\mathbf{p},u) = \frac{u\, G(\mathbf{p})}{A} \begin{pmatrix} \alpha_1(\alpha_1 - 1)/p_1^2 & \alpha_1\alpha_2/(p_1 p_2) & \alpha_1\alpha_3/(p_1 p_3) \\ \alpha_1\alpha_2/(p_1 p_2) & \alpha_2(\alpha_2 - 1)/p_2^2 & \alpha_2\alpha_3/(p_2 p_3) \\ \alpha_1\alpha_3/(p_1 p_3) & \alpha_2\alpha_3/(p_2 p_3) & \alpha_3(\alpha_3 - 1)/p_3^2 \end{pmatrix}$$

Evaluación de menores principales.

– De orden 1:

$$\Delta_1^{(i)} = \sigma_{ii} = \frac{\alpha_i\, u\, G(\mathbf{p})}{A}\frac{\alpha_i - 1}{p_i^2} < 0 \quad \text{porque } 0 < \alpha_i < 1.$$

– De orden 2 (por ejemplo $\{1, 2\}$):

$$\Delta_2^{(1,2)} = \det \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} = \left(\frac{u\, G(\mathbf{p})}{A} \right)^2 \frac{\alpha_1\alpha_2\alpha_3}{p_1^2 p_2^2} > 0$$

El signo es el mismo para los demás pares $(1,3)$ y $(2,3)$.

– De orden 3:

$$\Delta_3 = \det \sigma(\mathbf{p}, u) = 0$$

porque $\sigma(\mathbf{p}, u)\, \mathbf{p} = \mathbf{0}$ (Tercera Ley de Hicks).

Por lo tanto, los signos alternan de acuerdo con el patrón:

$$\Delta_1 < 0, \qquad \Delta_2 > 0, \qquad \Delta_3 = 0,$$

lo que implica que $\sigma(\mathbf{p}, u)$ es **negativa semidefinida**. Además, como el último determinante es nulo pero los de orden 2 son estrictamente positivos, entonces:

$$\operatorname{rank}(\sigma(\mathbf{p}, u)) = 2$$

□

Ex. 3.G.9$^{\mathbf{C}}$ Compute the Slutsky matrix from the indirect utility function.

Proof. Sea $v(\mathbf{p}, y)$ función de utilidad indirecta.

Por definición, los elementos de la matriz de Slutsky $s(\mathbf{p}, y)$ son:

$$s(\mathbf{p}, y)_{ij} = \frac{\partial x_i(\mathbf{p}, y)}{\partial p_j} + x_j(\mathbf{p}, y)\, \frac{\partial x_i(\mathbf{p}, y)}{\partial y}$$

Procedemos a obtener los términos para la función de utilidad indirecta v.

Por identidad de Roy, sabemos que:

$$x_i(\mathbf{p}, y) = -\frac{\dfrac{\partial v(\mathbf{p}, y)}{\partial p_i}}{\dfrac{\partial v(\mathbf{p}, y)}{\partial y}} = -\frac{v_{p_i}}{v_y}$$

Derivando $x_i(\mathbf{p}, y) = -v_{p_i}/v_y$ respecto a p_j y y:

$$\frac{\partial x_i}{\partial p_j} = -\frac{1}{v_y^2}\Big(v_{p_i p_j}\, v_y - v_{p_j y}\, v_{p_i}\Big), \qquad \frac{\partial x_i}{\partial y} = -\frac{1}{v_y^2}\Big(v_{p_i y}\, v_y - v_{p_i}\, v_{yy}\Big)$$

Sustituyendo en la definición,

$$\begin{aligned} s(\mathbf{p},y)_{ij} &= \frac{\partial x_i}{\partial p_j} + x_j\,\frac{\partial x_i}{\partial y} \\ &= -\frac{1}{v_y^2}\Big(v_{p_ip_j}\,v_y - v_{p_jy}\,v_{p_i}\Big) \;+\; \Big(-\frac{v_{p_j}}{v_y}\Big)\Big(-\frac{1}{v_y^2}\big(v_{p_iy}\,v_y - v_{p_i}\,v_{yy}\big)\Big) \\ &= -\frac{v_{p_ip_j}}{v_y} \;+\; \frac{v_{p_i}\,v_{p_jy}}{v_y^2} + \frac{v_{p_j}}{v_y^3}\Big(v_{p_iy}\,v_y - v_{p_i}\,v_{yy}\Big) \\ &= -\frac{v_{p_ip_j}}{v_y} \;+\; \frac{v_{p_i}\,v_{p_jy} + v_{p_j}\,v_{p_iy}}{v_y^2} \;-\; \frac{v_{yy}\,v_{p_i}\,v_{p_j}}{v_y^3} \end{aligned}$$

Por lo tanto, la matriz de Slutsky para la función de utilidad indirecta es:

$$s(\mathbf{p},y) = [s(\mathbf{p},y)]_{i,j=1}^n = -\frac{v_{p_ip_j}}{v_y} \;+\; \frac{v_{p_i}\,v_{p_jy} + v_{p_j}\,v_{p_iy}}{v_y^2} \;-\; \frac{v_{yy}\,v_{p_i}\,v_{p_j}}{v_y^3}$$

Ejemplo: Función LES.

La utilidad indirecta es:

$$v(\mathbf{p},y) = \frac{A}{G(\mathbf{p})}\Big(y - \mathbf{p}^\top\mathbf{b}\Big), \qquad G(\mathbf{p}) = p_1^\alpha p_2^\beta p_3^\gamma, \quad A = \alpha^\alpha\beta^\beta\gamma^\gamma$$

Derivadas de primer orden:

$$v_y = \frac{A}{G(\mathbf{p})}, \qquad v_{p_i} = -\frac{A}{G(\mathbf{p})}\Big[b_i + \frac{\alpha_i}{p_i}\Big(y - \mathbf{p}^\top\mathbf{b}\Big)\Big]$$

Derivadas mixtas y segundas derivadas:

$$v_{yy} = 0, \qquad v_{p_iy} = -\frac{A}{G(\mathbf{p})}\frac{\alpha_i}{p_i}$$

y para las derivadas segundas respecto a precios:

$$v_{p_ip_j} = \frac{A}{G(\mathbf{p})}\Bigg[\frac{\alpha_i\alpha_j}{p_ip_j}\Big(y - \mathbf{p}^\top\mathbf{b}\Big) + \mathbf{1}\{i=j\}\frac{\alpha_i}{p_i}\Big(b_i + \frac{\alpha_i - 1}{p_i}\Big(y - \mathbf{p}^\top\mathbf{b}\Big)\Big)\Bigg]$$

A partir de la expresión general:

$$s(\mathbf{p},y)_{ij} = -\frac{v_{p_ip_j}}{v_y} + \frac{v_{p_i}v_{p_jy} + v_{p_j}v_{p_iy}}{v_y^2} - \frac{v_{yy}v_{p_i}v_{p_j}}{v_y^3}$$

El último término desaparece porque $v_{yy} = 0$.

Tras sustituir todas las derivadas y simplificar, (háganlo en papel para comprobar) los términos con b_i se cancelan, y el resultado queda:

$$s(\mathbf{p}, y)_{ij} = \left(y - \mathbf{p}^\top \mathbf{b}\right)\left[\frac{\alpha_i(\alpha_i - 1)}{p_i^2}\mathbf{1}\{i = j\} + \frac{\alpha_i\alpha_j}{p_i p_j}(1 - \mathbf{1}\{i = j\})\right]$$

Por lo tanto, la matriz de Slutsky es:

$$s(\mathbf{p}, y) = \left(y - \mathbf{p}^\top \mathbf{b}\right)\begin{pmatrix} \frac{\alpha(\alpha-1)}{p_1^2} & \frac{\alpha\beta}{p_1 p_2} & \frac{\alpha\gamma}{p_1 p_3} \\ \frac{\alpha\beta}{p_1 p_2} & \frac{\beta(\beta-1)}{p_2^2} & \frac{\beta\gamma}{p_2 p_3} \\ \frac{\alpha\gamma}{p_1 p_3} & \frac{\beta\gamma}{p_2 p_3} & \frac{\gamma(\gamma-1)}{p_3^2} \end{pmatrix}$$

Finalmente, si evaluamos en $u^* = v(\mathbf{p}, y)$, se cumple que

$$y - \mathbf{p}^\top \mathbf{b} = \frac{G(\mathbf{p})}{A}\, u^*$$

por lo que $s(\mathbf{p}, y) = \sigma(\mathbf{p}, u^*)$, coincidiendo exactamente con la matriz de sustitución que derivamos en el ejercicio anterior para la función LES. $\square$

Appendix

Ex. 3.D.6$^{\mathrm{B}}$ (Mas Colell et al., 1995). Consider the three-good setting in which the consumer has utility function

$$u(x) = (x_1 - b_1)^{\alpha}(x_2 - b_2)^{\beta}(x_3 - b_3)^{\gamma}.$$

(a) Why can you assume that $\alpha + \beta + \gamma = 1$ without loss of generality? Do so for the rest of the problem.

(b) Write down the first-order conditions for the UMP, and derive the consumer's Walrasian demand and indirect utility functions. This system of demands is known as the *linear expenditure system* and is due to Stone (1954).

(c) Verify that these demand functions satisfy the properties listed in Propositions 3.D.2 and 3.D.3.

Proposition 3.G.3: The Slutsky Equation (Mas Colell et al., 1995).

Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated and strictly convex preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. Then for all (p, w), and $u = v(p, w)$, we have

$$\frac{\partial h_\ell(p,u)}{\partial p_k} = \frac{\partial x_\ell(p,w)}{\partial p_k} + \frac{\partial x_\ell(p,w)}{\partial w} x_k(p,w) \quad \text{for all } \ell, k.$$

or equivalently, in matrix notation,

$$D_p h(p,u) = D_p x(p,w) + D_w x(p,w) x(p,w)^\top.$$

Proposition 3.G.3 implies that the matrix of price derivatives $D_p h(p,u)$ of the Hicksian demand function is equal to the matrix

$$S(p,w) = \begin{bmatrix} s_{11}(p,w) & \cdots & s_{1L}(p,w) \\ \vdots & \ddots & \vdots \\ s_{L1}(p,w) & \cdots & s_{LL}(p,w) \end{bmatrix},$$

with

$$s_{\ell k}(p,w) = \frac{\partial x_\ell(p,w)}{\partial p_k} + \frac{\partial x_\ell(p,w)}{\partial w} x_k(p,w).$$

This matrix is known as the *Slutsky substitution matrix.* Note, in particular, that $S(p,w)$ is directly computable from knowledge of the (observable) Walrasian demand function $x(p,w)$.

Because $S(p,w) = D_p h(p,u)$, Proposition 3.G.2 implies that when demand is generated from preference maximization, $S(p,w)$ must possess the following three properties: it must be *negative semidefinite*, *symmetric*, and satisfy $S(p,w)p = 0$.

Theorem 1.11 The Slutsky Equation (Jehle & Reny, 2011).

Let $x(\mathbf{p}, y)$ be the consumer's Marshallian demand system. Let u^* be the level of utility the consumer achieves at prices $\mathbf{p}$ and income y. Then,

$$\underbrace{\frac{\partial x_i(\mathbf{p}, y)}{\partial p_j}}_{\text{TE}} = \underbrace{\frac{\partial x_i^h(\mathbf{p}, u^*)}{\partial p_j}}_{\text{SE}} \underbrace{- x_j(\mathbf{p}, y)\frac{\partial x_i(\mathbf{p}, y)}{\partial y}}_{\text{IE}}, \qquad i, j = 1, \ldots, n.$$

Theorem 1.15 Negative Semidefinite Substitution Matrix (Jehle & Reny, 2011).

Let $x^h(\mathbf{p}, u)$ be the consumer's system of Hicksian demands, and let

$$\sigma(\mathbf{p}, u) \equiv \begin{pmatrix} \frac{\partial x_1^h(\mathbf{p},u)}{\partial p_1} & \cdots & \frac{\partial x_1^h(\mathbf{p},u)}{\partial p_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n^h(\mathbf{p},u)}{\partial p_1} & \cdots & \frac{\partial x_n^h(\mathbf{p},u)}{\partial p_n} \end{pmatrix},$$

called the *substitution matrix*, containing all the Hicksian substitution terms. Then the matrix $\sigma(\mathbf{p}, u)$ is negative semidefinite.

Theorem 1.16 Symmetric and Negative Semidefinite Slutsky Matrix (Jehle & Reny, 2011).

Let $x(\mathbf{p}, y)$ be the consumer's Marshallian demand system. Define the ijth Slutsky term as

$$\frac{\partial x_i(\mathbf{p}, y)}{\partial p_j} + x_j(\mathbf{p}, y)\frac{\partial x_i(\mathbf{p}, y)}{\partial y},$$

and form the entire $n \times n$ **Slutsky matrix** of price and income responses as follows:

$$s(\mathbf{p}, y) = \begin{pmatrix} \frac{\partial x_1(\mathbf{p},y)}{\partial p_1} + x_1(\mathbf{p}, y)\frac{\partial x_1(\mathbf{p},y)}{\partial y} & \cdots & \frac{\partial x_1(\mathbf{p},y)}{\partial p_n} + x_n(\mathbf{p}, y)\frac{\partial x_1(\mathbf{p},y)}{\partial y} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n(\mathbf{p},y)}{\partial p_1} + x_1(\mathbf{p}, y)\frac{\partial x_n(\mathbf{p},y)}{\partial y} & \cdots & \frac{\partial x_n(\mathbf{p},y)}{\partial p_n} + x_n(\mathbf{p}, y)\frac{\partial x_n(\mathbf{p},y)}{\partial y} \end{pmatrix}.$$

Then $s(\mathbf{p}, y)$ is symmetric and negative semidefinite.

Proposition 3.E.2 (Mas Colell et al., 1995):

Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$.

The expenditure function $e(p, u)$ is

(i) Homogeneous of degree one in p.

(ii) Strictly increasing in u and nondecreasing in p_ℓ for any ℓ.

(iii) Concave in p.

(iv) Continuous in p and u.

Proposition 3.E.3 (Mas Colell et al., 1995):

Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$.

Then for any $p \gg 0$, the Hicksian demand correspondence $h(p, u)$ possesses the following properties:

(i) *Homogeneity of degree zero in p:* $h(\alpha p, u) = h(p, u)$ for any p, u and $\alpha > 0$.

(ii) *No excess utility:* For any $x \in h(p, u)$, $u(x) = u$.

(iii) *Convexity/uniqueness:* If $\succsim$ is convex, then $h(p, u)$ is a convex set; and if $\succsim$ is strictly convex, so that $u(\cdot)$ is strictly quasiconcave, then there is a unique element in $h(p, u)$.