# Laboratory of Microeconomics I

## Fundamentals of Real Analysis

Centro de Investigación y Docencia Económicas
Maestría en Economía
Arturo López

# Contents

# Introduction

This document serves as a mathematical foundation review for Microeconomics I. It's designed to cover the essential mathematical tools we will need at the beginning of the semester. While it aims to be broad and comprehensive, it is by no means exhaustive. Most of the material presented here will be studied in greater depth during your Mathematics course.

The material draws heavily from Lebl (2023), Dam (2023), Ok (2007), Kumaresan (2005) and De la Fuente (2000).

For brevity, we will omit basic notions of set theory and functions. For a concrete review, see De la Fuente (2000, sections 1.1 and 1.4).

# 1 Binary Relations

Consider a set $X$. That is, a collection of objects which we call elements of $X$. The set $X$ is just a collection of objects, with no additional structure whatsoever. We want to endow $X$ with some structure. We will begin with the concept of a *binary relation*.

## 1.1 Definition and Examples

**Definition 1.1** (Binary Relation)**.** Let $X$ and $Y$ be two sets. A subset $R$ of the Cartesian product $X \times Y$ is called a *binary relation* from $X$ to $Y$. If $(x, y) \in R$, we write $xRy$.

Intuitively, the concept of a *binary relation* provides a way to formalize the idea that two objects, typically two elements of the same set, stand in a certain relationship to each other.

---

**Example 1.1.** Let $X = \{1, 2, 4\}$ and $Y = \{2, 3\}$. Then $R = \{(1, 2), (1, 3), (2, 3)\}$ defines a binary relation from $X$ to $Y$ that we might interpret as "$<$" (less than). Note that $(2, 4)$ is not a binary relation from $X$ to $Y$ (why?).

---

The following types of binary relations are of significant importance for microeconomic theory.

### 1.1.1 Preorderings

A class of binary relations of particular interest allows us to formalize the idea that some elements of a set "dominate" others in a specific sense. Such relations are called order relations, and a set endowed with an order relation is called an ordered set.

**Definition 1.2** (Preorder)**.** A binary relation $\succsim$ on a non-empty set $X$ is called a *preorder* if it satisfies the following properties:

(i) **Reflexivity**: $x \succsim x$ for all $x \in X$.

(ii) **Transitivity**: If $x \succsim y$ and $y \succsim z$, then $x \succsim z$ for all $x, y, z \in X$.

The pair $(X, \succsim)$ is called a *preordered set.*

**Definition 1.3** (Total Preorder)**.** A preorder $\succsim$ on a non-empty set $X$ is called a *total preorder* (or *complete preorder*) if it additionally satisfies:

(iii) **Completeness**: For all $x, y \in X$, either $x \succsim y$ or $y \succsim x$ (or both).

The pair $(X, \succsim)$ is called a *totally preordered set.*

Intuitively, we interpret the relation $x \succsim y$ as "$x$ dominates $y$" in some relevant sense (e.g., "$x$ is at least as large as $y$", "$x$ is at least as preferred as $y$", and so on).

**Definition 1.4** (Equivalence Relation)**.** A binary relation $\sim$ on a non-empty set $X$ is called an *equivalence relation* if it satisfies the following properties:

(i) **Reflexivity**: $x \sim x$ for all $x \in X$.

(ii) **Symmetry**: If $x \sim y$, then $y \sim x$ for all $x, y \in X$.

(iii) **Transitivity**: If $x \sim y$ and $y \sim z$, then $x \sim z$ for all $x, y, z \in X$.

For any $x \in X$, the *equivalence class* of $x$ is defined as the set

$$[x]_\sim := \{y \in X \mid y \sim x\}.$$

An equivalence relation formalizes the notion of identifying two distinct objects when they share a particular property of interest. Naturally, such identification should satisfy certain consistency requirements. Intuitively, any object $x$ must be considered identical to itself. If $x$ is identified with $y$, then $y$ must be identified with $x$. Finally, if $x$ and $y$ are regarded as identical, and $y$ and $z$ are also regarded as identical, then $x$ should be considered identical to $z$. The properties of *reflexivity*, *symmetry*, and *transitivity* formalize these intuitive notions, respectively. A binary relation satisfying all these three properties is called an **equivalence relation**.

**Example 1.2 (Consumer Preferences).** Let $x \in X \subseteq \mathbb{R}^n_+$ represent the vector of quantities of $n$ goods consumed by an agent, where $X \subseteq \mathbb{R}^n_+$ is called the **consumption set**. Define a **preference relation** $\succsim$ on $X$ as follows: we read $x \succsim y$ as "*$x$ is at least as good as $y$*". From the preference relation $\succsim$, we can derive two other important relations on $X$:

(a) The **strict preference relation**, $\succ$, defined by

$$x \succ y \iff x \succsim y \text{ and } y \not\succsim x$$

and read as "*$x$ is (strictly) preferred to $y$*".

(b) The **indifference relation**, $\sim$, defined by

$$x \sim y \iff x \succsim y \text{ and } y \succsim x$$

and read as "*$x$ is indifferent to $y$*".

The relation $\succsim$ is called *rational* [a] if it satisfies:

(i) **Completeness**: For all $x, y \in X$, either $x \succsim y$ or $y \succsim x$ or both.

(ii) **Transitivity**: For all $x, y, z \in X$, if $x \succsim y$ and $y \succsim z$, then $x \succsim z$.

[a] Note that a rational relation is a complete preorder.

The following result characterizes the properties of the derived relations when preferences are rational.

**Lemma 1.1.** *If $\succsim$ is rational, then the indifference relation $\sim$ is an equivalence relation, but the strict preference relation $\succ$ is not.*

**Exercise 1.1.** Prove **Lemma 1.1**.

*Proof.* We prove each claim separately:

For the first implication, we want to prove that $\sim$ is an equivalence relation. So, we need to verify reflexivity, symmetry, and transitivity.

- *Reflexivity*: For any $x \in X$, completeness of $\succsim$ implies that $x \succsim x$ or $x \succsim x$, then $x \succsim x$. Thus $x \sim x$.
- *Symmetry*: If $x \sim y$, then by definition $x \succsim y$ and $y \succsim x$, which clearly gives $y \sim x$.

- *Transitivity*: Let $x \sim y$ and $y \sim z$. Then $x \succsim y$, $y \succsim x$, $y \succsim z$, and $z \succsim y$. By transitivity of $\succsim$: $x \succsim y \succsim z$ implies $x \succsim z$, and $z \succsim y \succsim x$ implies $z \succsim x$. Hence $x \sim z$.

For the second implication, we want to prove that $\succ$ is not an equivalence relation. It is enough to show that $\succ$ is not reflexive. For any $x \in X$, we have $x \succsim x$ by reflexivity of $\succsim$. But then $x \not\succ x$ since the definition of $\succ$ requires $x \not\precsim x$, which contradicts $x \succsim x$. Therefore, $\succ$ is not reflexive, hence not an equivalence relation. $\square$

# 2 Metric Spaces

In this section, we endow a set $X$ with a different kind of structure that allows us to formalize the notion of the "distance" between two elements of $X$.

## 2.1 Definition and Examples

A *metric space* is a set equipped with a function that quantifies the distance between any two of its elements. This distance function (or *metric*) must satisfy certain natural and intuitive conditions that capture our geometric understanding of a distance between two points. The definition below formalizes these requirements.

**Definition 2.1** (Metric)**.** Let $X$ be a nonempty set. A function $d : X \times X \to \mathbb{R}$ is said to be a *metric* or a *distance function* on $X$ if it satisfies the following properties:

(i) **Non-negativity**: $d(x, y) \geq 0$ for all $x, y \in X$ and,

(ii) **Identity of Indiscernibles**: $d(x, y) = 0$ iff $x = y$.

(iii) **Symmetry**: $d(x, y) = d(y, x)$ for all $x, y \in X$.

(iv) **Triangle Inequality**: $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

The pair $(X, d)$ is then called a *metric space*.

---

**Example 2.1** (Standard Metric on $\mathbb{R}$)**.** The most known example is the set $\mathbb{R}$ of real numbers endowed with the metric $d(x, y) := |x - y|$. This metric is induced by the absolute value function (which is the Euclidean norm in $\mathbb{R}$) and corresponds to our usual notion of distance on the real line.

---

**Example 2.2** (Discrete Metric)**.** Let $X$ be a nonempty set. For any $x, y \in X$, define the *discrete metric* $d : X \times X \to \mathbb{R}$ by:

$$d(x, y) := \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases} \tag{2.1}$$

The pair $(X, d)$ is a metric space.

The discrete metric turns any set into a metric space where distinct points are always at a unit distance from each other.

**Example 2.3.** Consider the set $\mathcal{C}([a, b])$ of all continuous functions $f : [a, b] \to \mathbb{R}$. For all $1 \leq p < \infty$, define the function $d_p : \mathcal{C}([a, b]) \times \mathcal{C}([a, b]) \to \mathbb{R}$ by:

$$d_p(f, g) := \left[\int_a^b |f(x) - g(x)|^p dx\right]^{1/p}, \quad \forall f, g \in \mathcal{C}([a, b]) \tag{2.2}$$

The pair $(\mathcal{C}([a, b]), d_p)$ is a metric space.

Moreover, define the function $d_\infty : \mathcal{C}([a, b]) \times \mathcal{C}([a, b]) \to \mathbb{R}$ by:

$$d_\infty(f, g) := \sup_{x \in [a,b]} |f(x) - g(x)|, \quad \forall f, g \in \mathcal{C}([a, b]) \tag{2.3}$$

The pair $(\mathcal{C}([a, b]), d_\infty)$ is a metric space.

**Example 2.4.** Let $(X, d_X)$ and $(Y, d_y)$ be metric spaces. Then, for all $1 \leq p < \infty$, define the function $d : X \times Y \to \mathbb{R}$ by:

$$d((x_1, y_1), (x_2, y_2)) := [d_X(x_1, x_2)^p + d_Y(y_1, y_2)^p]^{1/p} \quad \forall (x_1, y_1), (x_2, y_2) \in X \times Y \tag{2.4}$$

The pair $(X \times Y, d)$ is a metric space.

We now introduce the *standard* or *Euclidean metric* on $\mathbb{R}^n$, which is the metric induced by the Euclidean norm on $\mathbb{R}^n$.

*Recall*: Every inner product $\langle \cdot, \cdot \rangle$ induces a norm via $\|x\| = \sqrt{\langle x, x \rangle}$. Moreover, every norm on a vector space induces a metric via $d(x, y) = \|x - y\|$. Therefore, from any inner product space $(X, \langle \cdot, \cdot \rangle)$, we can induce a normed vector space $(X, || \cdot ||)$ and then induce a metric space $(X, d)$.

**Example 2.5** (Euclidean Metric on $\mathbb{R}^n$)**.** Consider $\mathbb{R}^n$ equipped with the standard inner product (dot product):

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$$

for $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ in $\mathbb{R}^n$.

This inner product induces the *Euclidean norm*:

$$\|x\|_2 = \sqrt{\langle x, x\rangle} = \sqrt{\sum_{i=1}^{n} x_i^2}$$

The pair $(\mathbb{R}^n, \|\cdot\|)$ is called the *$n$-dimensional Euclidean space*.

The Euclidean norm, in turn, induces the *Euclidean metric* on $\mathbb{R}^n$ defined by:

$$d_2(x, y) = \|x - y\|_2 = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

It is straightforward to verify that $d_2$ satisfies all the metric axioms, making $(\mathbb{R}^n, d_2)$ a metric space (for the triangle inequality, you may use the Minkowski's Inequality for $\ell^p$ norms).

**Example 2.6** ($p$-Metrics on $\mathbb{R}^n$)**.** Now that we understand how norms induce metrics, we can introduce a family of important metrics on $\mathbb{R}^n$. For each $1 \le p \le \infty$, $(\mathbb{R}^n, d_p)$ is a metric space, where $d_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ is defined by:

$$d_p(x, y) := \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{\frac{1}{p}} \quad \text{for } 1 \le p < \infty$$

and

$$d_\infty(x, y) := \max\{|x_i - y_i| : i = 1, \dots, n\} \quad \text{for } p = \infty$$

These metrics are induced by the corresponding $\ell^p$ norms $\|x\|_p := (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}$ via $d_p(x, y) = \|x - y\|_p$.

**Exercise 2.1.** Let $X = \mathbb{R}$ and define $d : X \times X \to [0, \infty)$ by:

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ |x| + |y| & \text{if } x \neq y \end{cases}$$

Prove that $d$ is a metric on $\mathbb{R}$.

(*Hint:* Verify the properties of a metric (**Definition 2.1**). For the triangle inequality, remember the triangle inequality of the euclidean norm: $|x + y| \leq |x| + |y|$.)

*Proof.* We need to verify the properties of a metric (**Definition 2.1**).

*(i) Non-negativity*: For all $x, y \in \mathbb{R}$:

- If $x = y$, then $d(x, y) = 0$ by definition of $d$.
- If $x \neq y$, then $d(x, y) = |x| + |y| \geq 0$ since absolute values are non-negative.

Therefore, $d(x, y) \geq 0$ for all $x, y \in \mathbb{R}$.

*(ii) Identity of Indiscernibles:* $d(x, y) = 0 \iff x = y$

- $\boxed{\Rightarrow}$ By contradiction assume that $x \neq y$. Then $d(x, y) = |x| + |y|$ by definition of $d$.

  If $d(x, y) = 0$, we have that $|x| + |y| = 0$. Since absolute values are non-negative, this implies that $|x| = |y| = 0$, which means $x = y = 0$, contradicting our assumption that $x \neq y$. Therefore, we must have that if $d(x, y) = 0$, then $x = y$.
- $\boxed{\Leftarrow}$ If $x = y$, then $d(x, y) = 0$ by definition of $d$.

*(iii) Symmetry:* $d(x, y) = d(y, x)$ For all $x, y \in \mathbb{R}$:

- If $x = y$, then $y = x$, so $d(x, y) = 0 = d(y, x)$ by definition of $d$.
- If $x \neq y$, then $y \neq x$, and:

$$\begin{aligned} d(x, y) &= |x| + |y| \\ &= |y| + |x| \\ &= d(y, x) \end{aligned}$$

*(iv) Triangle inequality:* $d(x, z) \leq d(x, y) + d(y, z)$

Let $x, y, z \in \mathbb{R}$. We consider two cases:

**C1:** If $x = z$, then $d(x, z) = 0 \leq d(x, y) + d(y, z)$ by non-negativity.

**C2:** If $x \neq z$, then $d(x, z) = |x| + |z|$. We need to show:

$$|x| + |z| \leq d(x, y) + d(y, z)$$

If $x = y$, then $d(x, y) = 0$ and $d(y, z) = |y| + |z| = |x| + |z|$ (since $y \neq z$, otherwise $x = z$).

If $y = z$, then $d(y, z) = 0$ and $d(x, y) = |x| + |y| = |x| + |z|$ (since $x \neq y$, otherwise $x = z$).

If $x \neq y$ and $y \neq z$, then:

$$\begin{aligned} d(x, y) + d(y, z) &= (|x| + |y|) + (|y| + |z|) \\ &= |x| + 2|y| + |z| \\ &\geq |x| + |z| \\ &= d(x, z) \end{aligned}$$

In all cases, the triangle inequality holds.

Therefore, $d$ is a metric on $\mathbb{R}$. □

---

**Exercise 2.2.** Let $(X, d_1)$ and $(Y, d_2)$ be metric spaces and the function $d_3 : (X \times Y) \times (X \times Y) \to \mathbb{R}$ defined as follows:

$$d_3((x_1, y_1), (x_2, y_2)) = d_1(x_1, x_2) + d_2(y_1, y_2), \quad \forall (x_1, y_1), (x_2, y_2) \in X \times Y$$

Show that $(X \times Y, d_3)$ is a metric space.

---

*Proof.* We need to verify the properties of a metric (**Definition 2.1**).

*(i) Non-negativity*: Since $d_1$ and $d_2$ are metrics, we have $d_1(x_1, x_2) \geq 0$ and $d_2(y_1, y_2) \geq 0$ for all $(x_1, y_1), (x_2, y_2) \in X \times Y$. Therefore:

$$d_3((x_1, y_1), (x_2, y_2)) = d_1(x_1, x_2) + d_2(y_1, y_2) \geq 0$$

*(ii) Identity of Indiscernibles:* $d_3((x_1, y_1), (x_2, y_2)) = 0 \iff (x_1, y_1) = (x_2, y_2)$

- $\boxed{\Rightarrow}$ If $d_3((x_1, y_1), (x_2, y_2)) = 0$, then:

  $$d_1(x_1, x_2) + d_2(y_1, y_2) = 0$$

  Since $d_1(x_1, x_2) \geq 0$ and $d_2(y_1, y_2) \geq 0$, the only way their sum equals zero is if both terms are zero:

  $$d_1(x_1, x_2) = 0 \quad \text{and} \quad d_2(y_1, y_2) = 0$$

  Since $d_1$ and $d_2$ are metrics, this implies:

  $$x_1 = x_2 \quad \text{and} \quad y_1 = y_2$$

Therefore, $(x_1, y_1) = (x_2, y_2)$.

- $\boxed{\Leftarrow}$ If $(x_1, y_1) = (x_2, y_2)$, then $x_1 = x_2$ and $y_1 = y_2$.

  Since $d_1$ and $d_2$ are metrics:

$$\begin{aligned} d_3((x_1, y_1), (x_2, y_2)) &= d_1(x_1, x_2) + d_2(y_1, y_2) \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

*(iii) Symmetry:* $d_3((x_1, y_1), (x_2, y_2)) = d_3((x_2, y_2), (x_1, y_1))$

Since $d_1$ and $d_2$ are metrics:

$$\begin{aligned} d_3((x_1, y_1), (x_2, y_2)) &= d_1(x_1, x_2) + d_2(y_1, y_2) \\ &= d_1(x_2, x_1) + d_2(y_2, y_1) \\ &= d_3((x_2, y_2), (x_1, y_1)) \end{aligned}$$

*(iv) Triangle inequality:* $d_3((x_1, y_1), (x_3, y_3)) \leq d_3((x_1, y_1), (x_2, y_2)) + d_3((x_2, y_2), (x_3, y_3))$

Let $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X \times Y$.

Since $d_1$ and $d_2$ are metrics, they satisfy the triangle inequality:

$$d_1(x_1, x_3) \leq d_1(x_1, x_2) + d_1(x_2, x_3)$$

$$d_2(y_1, y_3) \leq d_2(y_1, y_2) + d_2(y_2, y_3)$$

Then:

$$\begin{aligned} d_3((x_1, y_1), (x_3, y_3)) &= d_1(x_1, x_3) + d_2(y_1, y_3) \\ &\leq d_1(x_1, x_2) + d_1(x_2, x_3) + d_2(y_1, y_2) + d_2(y_2, y_3) \\ &= [d_1(x_1, x_2) + d_2(y_1, y_2)] + [d_1(x_2, x_3) + d_2(y_2, y_3)] \\ &= d_3((x_1, y_1), (x_2, y_2)) + d_3((x_2, y_2), (x_3, y_3)) \end{aligned}$$

Which proves the triangle inequality for $d_3$.

Therefore, $(X \times Y, d_3)$ is a metric space. $\square$

**Exercise 2.3.** Let $X = \mathbb{R}$ and define $d : X \times X \to [0, \infty)$ by:

$$d(x, y) = \frac{|x - y|}{1 + |x - y|}$$

Prove that $d$ is a metric on $\mathbb{R}$. (*Hint: For the triangle inequality, note that* $d(x, y) = f(|x - y|)$ *and* $f(u) = \frac{u}{1+u}$ *is an increasing function.*)

*Proof.* We need to verify the properties of a metric (**Definition 2.1**).

*(i) Non-negativity*: If $x = y$, then $d(x, y) = 0$. Other case, if $x \neq y$, then $d(x, y) = \frac{|x-y|}{1+|x-y|}$. Clearly, $|x - y| > 0$ and $1 + |x - y| > 0$, which implies that $d(x, y) > 0$. Therefore $d(x, y) \geq 0$.

*(ii) Identity of Indiscernibles:* $d(x, y) = 0 \iff x = y \quad \forall x, y \in X$

- $\boxed{\Rightarrow}$ If $d(x, y) = 0$, then $\frac{|x-y|}{1+|x-y|} = 0$, which implies $|x - y| = 0$. We know that this is true iff $x = y$.
- $\boxed{\Leftarrow}$ If $x = y$, then $|x - y| = |x - x| = 0$. Hence, $d(x, y) = \frac{|x-y|}{1+|x-y|} = \frac{|x-x|}{1+|x-x|} = \frac{0}{1} = 0$

*(iii) Symmetry:* $d(x, y) = d(y, x) \quad \forall x, y \in X$

- If $x = y$, then $d(x, y) = 0 = d(y, x)$.
- If $x \neq y$, then $y \neq x$ and, by properties of the absolute value:

$$d(x, y) = \frac{|x - y|}{1 + |x - y|} = \frac{|y - x|}{1 + |y - x|} = d(y, x)$$

*(iv) Triangle inequality:* $d(x, z) \leq d(x, y) + d(y, z) \quad \forall x, y, z \in X$.

If any two of $x, y, z$ are equal, the inequality holds trivially:

- If $x = z$, $d(x, z) = 0 \leq d(x, y) + d(y, z)$ because of non-negativity.
- If $x = y$, then $d(x, y) = 0$ and $d(x, z) = d(y, z)$, so:

$$d(x, z) = 0 + d(y, z) = d(x, y) + d(y, z)$$

- If $y = z$, then $d(y, z) = 0$ and $d(x, y) = d(x, z)$, so:

$$d(x, z) = d(x, y) + 0 = d(x, y) + d(y, z)$$

Now, assume $x, y, z$ are all distinct. We need to show:

$$\frac{|x-z|}{1+|x-z|} \leq \frac{|x-y|}{1+|x-y|} + \frac{|y-z|}{1+|y-z|}$$

Note that $d(x,y) = f(|x-y|)$, where $f(u) = \frac{u}{u+1}$ is an increasing function in $u$: $f'(u) = \frac{1}{(u+1)^2} > 0 \quad \forall u \geq 0$.

Then:

$$\begin{aligned}
d(x,z) &= f(|x-z|) \\
&= f(|x-y+y-z|) \\
&\leq f(|x-y|+|y-z|) && \text{(by the triangle inequality of } |\cdot|) \\
&= \frac{|x-y|+|y-z|}{1+|x-y|+|y-z|} \\
&= \frac{|x-y|}{1+|x-y|+|y-z|} + \frac{|y-z|}{1+|x-y|+|y-z|} \\
&< \frac{|x-y|}{1+|x-y|} + \frac{|y-z|}{1+|y-z|} && \text{(because } f \text{ is increasing)} \\
&= d(x,y) + d(y,z)
\end{aligned}$$

Hence, $d(x,z) \leq d(x,y) + d(y,z)$.

Therefore, $d$ is a metric on $\mathbb{R}$. □

**Exercise 2.4.** Let $X = \mathbb{R}$ and define $d : X \times X \to [0, \infty)$ by:

$$d(x,y) := |x^n - y^n|$$

Show for which $n \in \mathbb{N}$ $d$ is a metric on $\mathbb{R}$.

*Proof.* We need to verify the properties of a metric (**Definition 2.1**).

*(i) Non-negativity*: Since $d(x,y) = |x^n - y^n|$ and the absolute value is always non-negative, we have $d(x,y) \geq 0$ for all $x, y \in \mathbb{R}$.

*(ii) Identity of Indiscernibles:* $d(x,y) = 0 \iff x = y \quad \forall x, y \in X$

- $\boxed{\Rightarrow}$ If $d(x,y) = 0$, then $|x^n - y^n| = 0$, which implies $x^n = y^n$. So, we have two special cases:
  - If $n$ is even, then from $x^n = y^n$ we get $x^n - y^n = 0$. There exist solutions where $x \neq y$. For example, if $n = 2$, then $x = 2$ and $y = -2$ give $x^2 = y^2 = 4$, but $x \neq y$. Therefore,

for even $n$, this property fails.

- If $n$ is odd, the function $f(x) = x^n$ is strictly increasing and hence injective on $\mathbb{R}$, so $x^n = y^n$ implies $x = y$.

- $\boxed{\Leftarrow}$ If $x = y$, then $d(x, y) = |x^n - x^n| = |0| = 0$.

*(iii) Symmetry:* $d(x, y) = d(y, x) \quad \forall x, y \in X$

$$d(x, y) = |x^n - y^n| = |-(y^n - x^n)| = |y^n - x^n| = d(y, x)$$

*(iv) Triangle inequality:* $d(x, z) \leq d(x, y) + d(y, z) \quad \forall x, y, z \in X$

Note that $x^n, y^n, z^n \in \mathbb{R}\ \forall n \in \mathbb{N}$. Thus, we can apply the triangle inequality of the standard euclidean metric in $\mathbb{R}$:

$$|a - c| \leq |a - b| + |b - c| \quad \forall a, b, c \in \mathbb{R}$$

Setting $a = x^n$, $b = y^n$, and $c = z^n$, we get:

$$|x^n - z^n| \leq |x^n - y^n| + |y^n - z^n|$$

Which confirms that $d(x, z) \leq d(x, y) + d(y, z)$.

Therefore, $d$ is a metric on $\mathbb{R}$ iff $n$ is odd. □

**Exercise 2.5.** Prove that the pair $(\mathcal{C}([a, b]), d_\infty)$ in **Example 2.3** is a metric space.

*Proof.* We need to verify the properties of a metric (**Definition 2.1**).

*(i) Non-negativity*: Since $|f(x) - g(x)| \geq 0$ for all $x \in [a, b]$, by definition of the supremum we have:

$$\sup_{u \in [a,b]} |f(u) - g(u)| \geq |f(x) - g(x)| \geq 0$$

for all $f, g \in \mathcal{C}[a, b]$. Therefore, $d(f, g) \geq 0$ for all $f, g \in \mathcal{C}([a, b])$.

*(ii) Identity of Indiscernibles:* $d(f, g) = 0 \iff f = g \quad \forall f, g \in \mathcal{C}([a, b])$

- $\boxed{\Rightarrow}$ If $d(f, g) = 0$, then $\sup\limits_{u \in [a,b]} |f(u) - g(u)| = 0$.

  Since $|f(x) - g(x)| \geq 0$ for all $x \in [a, b]$ and the supremum is 0, we must have $|f(x) - g(x)| = 0$ for all $x \in [a, b]$.

  This implies $f(x) = g(x)$ for all $x \in [a, b]$, so $f = g$.

- $\boxed{\Leftarrow}$ If $f = g$, then $f(x) = g(x)$ for all $x \in [a, b]$.

Therefore, $|f(x) - g(x)| = 0$ for all $x \in [a, b]$, which gives $d(f, g) = \sup_{x\in[a,b]} |f(x) - g(x)| = \sup_{x\in[a,b]} 0 = 0$.

*(iii) Symmetry:* $d(f, g) = d(g, f) \quad \forall f, g \in \mathcal{C}([a, b])$

$$\begin{aligned} d(f,g) &= \sup_{x\in[a,b]} |f(x) - g(x)| \\ &= \sup_{x\in[a,b]} |-(g(x) - f(x))| \\ &= \sup_{x\in[a,b]} |g(x) - f(x)| \\ &= d(g, f) \end{aligned}$$

*(iv) Triangle inequality:* $d(f, h) \leq d(f, g) + d(g, h) \quad \forall f, g, h \in \mathcal{C}([a, b])$

For any $x \in [a, b]$, by the triangle inequality of the euclidean metric:

$$\begin{aligned} |f(x) - h(x)| &= |(f(x) - g(x)) + (g(x) - h(x))| \\ &\leq |f(x) - g(x)| + |g(x) - h(x)| \end{aligned}$$

Since this holds for all $x \in [a, b]$, by the definition of the supremum:

$$\begin{aligned} |f(x) - h(x)| &\leq |f(x) - g(x)| + |g(x) - h(x)| \\ &\leq \sup_{u\in[a,b]} |f(u) - g(u)| + \sup_{u\in[a,b]} |g(u) - h(u)| \\ &= d(f, g) + d(g, h) \end{aligned}$$

Taking the supremum over all $x \in [a, b]$ on the left side:

$$d(f, h) = \sup_{x\in[a,b]} |f(x) - h(x)| \leq d(f, g) + d(g, h)$$

Therefore, $d(f, h) \leq d(f, g) + d(g, h)$.

Hence, $(\mathcal{C}([a, b]), d)$ is a metric space. $\square$

## 2.2 Analysis of Metric Spaces

The topology of a metric space is fundamentally determined by specifying which subsets are considered *open* and which are *closed.* These designations capture essential properties needed for rigorous definitions of limits, continuity, and other fundamental concepts in analysis.

Now, we define two special sets.

**Definition 2.2** (Open and Closed Balls)**.** Let $(X, d)$ be a metric space. For $x \in X$ and $r > 0$, the *open ball* with center $x$ and radius $r$ is given by

$$B_r(x) = \{y \in X \mid d(x, y) < r\},$$

and the *closed ball* with center $x$ and radius $r$ is given by

$$\overline{B}_r(x) = \{y \in X \mid d(x, y) \leq r\}.$$

We now define open and closed sets.

**Definition 2.3** (Open and Closed Sets)**.** Let $(X, d)$ be a metric space. The set $A \subseteq X$ is *open* if for every $x \in A$, there exists $\epsilon > 0$ such that $B_\epsilon(x) \subseteq A$. The set $A$ is *closed* if the complement $A^c = X \setminus A$ is open.

**Definition 2.4** (Interior, Exterior, Boundary, and Closure)**.** Let $(X, d)$ be a metric space, $A \subseteq X$ and $x \in X$.

(a) A point $x$ is an *interior point* of $A$ if there exists $\epsilon > 0$ such that $B_\epsilon(x) \subseteq A$. The set of all interior points of $A$ is denoted by *int*$(A)$.

(b) A point $x$ is an *exterior point* of $A$ if there exists $\epsilon > 0$ such that $B_\epsilon(x) \subseteq A^c$. The set of all exterior points of $A$ is denoted by *ext*$(A)$.

(c) A point $x$ is a *boundary point* of $A$ if for any $\epsilon > 0$, the open ball $B_\epsilon(x)$ has non-empty intersection with both $A$ and $A^c$, i.e., $B_\epsilon(x) \cap A \neq \emptyset$ and $B_\epsilon(x) \cap A^c \neq \emptyset$. The set of all boundary points of $A$ is denoted by *bd*$(A)$.

(d) A point $x$ is a *closure point* of $A$ if for any $\epsilon > 0$, the open ball $B_\epsilon(x)$ has a non-empty intersection with $A$, i.e., $B_\epsilon(x) \cap A \neq \emptyset$. The set of all closure points of $A$ is denoted by $\overline{A}$.

### 2.2.1 Open Sets

We will now practice how to prove set structures using the previous definitions.

**Exercise 2.6.** Let $(X, d)$ be a metric space, $x \in X$ and $r > 0$. Show that the open ball $B_r(x)$ is an open set.
(*Hint*: For any $y \in B_r(x)$, you need to find $s > 0$ such that $B_s(y) \subset B_r(x)$. Consider the geometric relationship between the distances $d(x, y)$ and $r$ to found a proper $s > 0$, and

use the triangle inequality to prove it.)

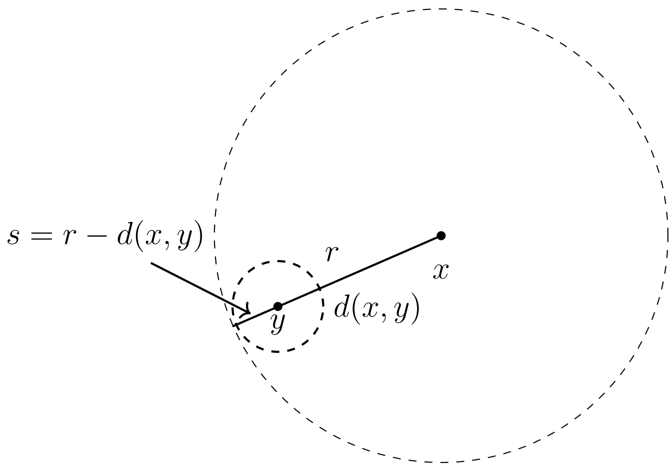



Figure 1: $B_r(x)$

*Proof Strategy:* In order to prove that $B_r(x)$ is open, we need to prove that, for any $y \in B_r(x)$, there exists some $s > 0$ such that $B_s(y) \subset B_r(x)$. Recall that, for arbitrary sets $A$ and $B$, we have:

$$A \subset B \iff (x \in A \implies x \in B) \text{ for all } x \in A$$

Therefore, we need to prove that given $y \in B_r(x)$, there exists some $s > 0$ such that $z \in B_s(y) \implies z \in B_r(x)$ for all $z$.

Figure 1 suggests that $s = r - d(x, y)$ is a good candidate. We will use the triangle inequality to verify that this choice works.

*Proof.* Let $y \in B_r(x)$ and $s := r - d(x, y)$. Since $y \in B_r(x)$, we have $d(x, y) < r$, so clearly $s > 0$.

Consider the open ball $B_s(y)$ with center at $y$ and radius $s$, and let $z \in B_s(y)$. Then, by the triangle inequality:

$$\begin{aligned} d(x, z) &\leq d(x, y) + d(y, z) \\ &< d(x, y) + s \\ &= d(x, y) + r - d(x, y) \\ &= r \end{aligned}$$

Given that $d(x, z) < r$, we have $z \in B_r(x)$. Hence, $z \in B_s(y) \implies z \in B_r(x)$, which verifies that there exists $s > 0$ such that $B_s(y) \subset B_r(x)$ for all $y \in B_r(x)$. Therefore, $B_r(x)$ is open. $\square$

**Exercise 2.7.** Show that $A := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1x_2 \neq 0\}$ is open in $\mathbb{R}^2$. (*Hint*: Think geometrically to find a proper $r > 0$ and then prove this claim.)

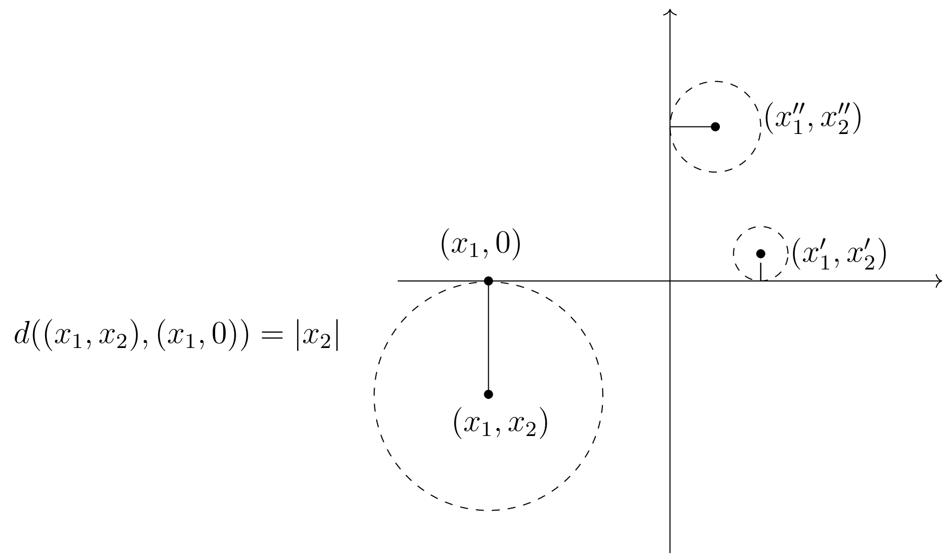



Figure 2: $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1x_2 \neq 0\}$ is open

*Proof Strategy:* Figure 2 suggests that, in order to get an open ball fully contained in $A$ for all $(x_1, x_2) \in A$, which means that for all $(x_1', x_2') \in B_r((x_1, x_2))$, we have that $x_1'x_2' \neq 0$, we need to take a radius less than the distance from $(x_1, x_2)$ to both coordinate axes. The distance from the point $(x_1, x_2)$ to the $y$-axis is $|x_1|$, and the distance to the $x$-axis is $|x_2|$. On the $y$-axis, $x_1 = 0$ and on the $x$-axis, $x_2 = 0$. So, we want to be far enough from both axes. In order to stay away from both axes, we need $r > 0$ less than the minimum of these distances, that is, less than $\min\{|x_1|, |x_2|\}$. Therefore, a good candidate is $r = \frac{1}{2}\min\{|x_1|, |x_2|\}$.

*Proof.* Let $(x_1, x_2) \in A$ and $r := \frac{1}{2}\min\{|x_1|, |x_2|\}$. Clearly, $r > 0$.

Consider the open ball $B_r((x_1, x_2))$ with center at $(x_1, x_2)$ and radius $r$. Let $(x_1', x_2') \in B_r((x_1, x_2))$, so $d((x_1, x_2), (x_1', x_2')) < r$. We need to prove that $(x_1', x_2') \in A$, which means that $x_1' \neq 0$ and $x_2' \neq 0$.

- To prove that $x_1' \neq 0$, we have two cases:
  - If $|x_1| \leq |x_2|$, then $r = \frac{1}{2}|x_1|$ and $d((x_1, x_2), (x_1', x_2')) < \frac{1}{2}|x_1|$. The distance from $(x_1, x_2)$ to the $y$-axis (where $x_1 = 0$) is $|x_1|$. Given that

$$d((x_1, x_2), (x_1', x_2')) < \frac{1}{2}|x_1| < |x_1|$$

    the point $(x_1', x_2')$ cannot reach the $y$-axis. Therefore, $x_1' \neq 0$.

  - If $|x_2| < |x_1|$, then $r = \frac{1}{2}|x_2|$ and $d((x_1, x_2), (x_1', x_2')) < \frac{1}{2}|x_2|$. The distance from $(x_1, x_2)$ to the $y$-axis (where $x_1 = 0$) is $|x_1|$. Given that

$$d((x_1, x_2), (x_1', x_2')) < \frac{1}{2}|x_2| < \frac{1}{2}|x_1| < |x_1|$$

    the point $(x_1', x_2')$ cannot reach the $y$-axis. Therefore, $x_1' \neq 0$.

- To prove that $x_2' \neq 0$, we have analogous two cases:
  - If $|x_2| \leq |x_1|$, then $r = \frac{1}{2}|x_2|$ and $d((x_1, x_2), (x_1', x_2')) < \frac{1}{2}|x_2|$. The distance from $(x_1, x_2)$ to the $x$-axis (where $x_2 = 0$) is $|x_2|$. Given that

$$d((x_1, x_2), (x_1', x_2')) < \frac{1}{2}|x_2| < |x_2|$$

    the point $(x_1', x_2')$ cannot reach the $x$-axis. Therefore, $x_2' \neq 0$.
  - If $|x_1| < |x_2|$, then $r = \frac{1}{2}|x_1|$ and $d((x_1, x_2), (x_1', x_2')) < \frac{1}{2}|x_1|$. The distance from $x$ to the $x$-axis (where $x_2 = 0$) is $|x_2|$. Given that

$$d((x_1, x_2), (x_1', x_2')) < \frac{1}{2}|x_1| < \frac{1}{2}|x_2| < |x_2|$$

    the point $(x_1', x_2')$ cannot reach the $x$-axis. Therefore, $x_2' \neq 0$.

Hence, for any $(x_1, x_2) \in A$ there exists some $r > 0$ such that $B_r((x_1, x_2)) \subset A$. Therefore, $A$ is open. ☐

---

**Exercise 2.8.** Let $a > 0$. Prove that $A := \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, 0 \leq y \leq a\}$ is closed in $\mathbb{R}^2$. (*Hint*: Think geometrically. You need to prove that $A^c$ is open.)

---

*Proof.* In order to show that $A$ is closed, we will show that $A^c$ is open. That is, we will prove the claim that the set

$$A^c := \{(x, y) \in \mathbb{R}^2 \mid x < 0 \text{ or } y < 0 \text{ or } y > a\}$$

is open.

Let $(x_0, y_0) \in A^c$. Then, one of the following cases hold: $x_0 < 0$ or $y_0 < 0$ or $y_0 > a$. We consider each case in order to find a proper $r > 0$:

**C1:** If $x_0 < 0$, then the distance from $(x_0, y_0)$ to the $y$-axis, which is where $x_0 \geq 0$, is $|x_0|$. Therefore, in order to get far enough, choose $r_1 := \frac{1}{2}|x_0|$. Clearly, $r_1 > 0$.

Now, consider the open ball $B_{r_1}((x_0, y_0))$ and take $(x_1, y_1) \in B_{r_1}((x_0, y_0))$. Then, $||(x_1, y_1) - (x_0, y_0)|| < r_1$. We need to prove that $x_1 < 0$.

Note that:

$$(x_1 - x_0)^2 \leq (x_1 - x_0)^2 + (y_1 - y_0)^2 \quad \Longleftrightarrow \quad |x_1 - x_0| \leq \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$$

Then:

$$\begin{aligned} |x_1 - x_0| &\leq \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} \\ &= ||(x_1, y_1) - (x_0, y_0)|| \\ &< r_1 \\ &= \frac{1}{2}|x_0| \end{aligned}$$

Given that $x_0 < 0$, then $\frac{1}{2}|x_0| = \frac{1}{2}(-x_0) > 0$. Therefore:

$$\begin{aligned} x_1 &< r_1 + x_0 \\ &= \frac{1}{2}(-x_0) + x_0 \\ &= \frac{1}{2}(x_0) \\ &< 0 \end{aligned}$$

That is, $x_1 < 0$. Therefore, $(x_1, y_1) \in A^c$ which implies $B_{r_1}((x_0, y_0)) \subset A^c$.

**C2:** If $y_0 < 0$, then the distance from $(x_0, y_0)$ to the $x$-axis, which is where $y_0 \geq 0$, is $|y_0|$. Therefore, in order to get far enough, choose $r_2 := \frac{1}{2}|y_0|$. Clearly, $r_2 > 0$.

Now, consider the open ball $B_{r_2}((x_0, y_0))$ and take $(x_1, y_1) \in B_{r_2}((x_0, y_0))$. Then, $||(x_1, y_1) - (x_0, y_0)|| < r_2$. We need to prove that $y_1 < 0$.

Using the same logic as before,

$$\begin{aligned} |y_1 - y_0| &\leq \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} \\ &= ||(x_1, y_1) - (x_0, y_0)|| \\ &< r_2 \\ &= \frac{1}{2}|y_0| \end{aligned}$$

Given that $y_0 < 0$, then $\frac{1}{2}|y_0| = \frac{1}{2}(-y_0) > 0$. Therefore:

$$\begin{aligned}
y_1 &< r_2 + y_0 \\
&= \frac{1}{2}(-y_0) + y_0 \\
&= \frac{1}{2}(y_0) \\
&< 0
\end{aligned}$$

That is, $y_1 < 0$. Therefore, $(x_1, y_1) \in A^c$ which implies $B_{r_2}((x_0, y_0)) \subset A^c$.

**C3:** If $y_0 > a$, then the distance from $(x_0, y_0)$ to the $y_0 \leq a$ is $y_0 - a$. Therefore, in order to get far enough, choose $r_3 := \frac{1}{2}(y_0 - a)$. Given that $y_0 > a$, then $r_3 > 0$.

Now, consider the open ball $B_{r_3}((x_0, y_0))$ and take $(x_1, y_1) \in B_{r_3}((x_0, y_0))$. Then, $||(x_1, y_1) - (x_0, y_0)|| < r_3$. We need to prove that $y_1 > a$.

Using the same logic as before,

$$\begin{aligned}
|y_1 - y_0| &\leq \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} \\
&= ||(x_1, y_1) - (x_0, y_0)|| \\
&< r_3 \\
&= \frac{1}{2}(y_0 - a)
\end{aligned}$$

Therefore:

$$\begin{aligned}
y_1 &> -\frac{1}{2}(y_0 - a) + y_0 \\
&= \frac{1}{2}(y_0) + \frac{1}{2}a \\
&> \frac{1}{2}(a) + \frac{1}{2}a \\
&= a
\end{aligned}$$

That is, $y_1 > a$. Therefore, $(x_1, y_1) \in A^c$ which implies $B_{r_3}((x_0, y_0)) \subset A^c$.

Therefore, $A^c$ is open, which implies that $A$ is closed. □

**Exercise 2.9.** Prove that $A := \mathbb{R} \times \mathbb{Z}$ is closed in $\mathbb{R}^2$. (*Hint:* Think geometrically. Show that $A^c$ is open by writing it as an infinite union of open sets (which sets?) and use the fact that a union of open sets is open.)

*Proof.* Note that $A = \mathbb{R} \times \mathbb{Z} = \{(x, y) \in \mathbb{R}^2 \mid x \in \mathbb{R},\ y \in \mathbb{Z}\}$ is the set of all horizontal lines at

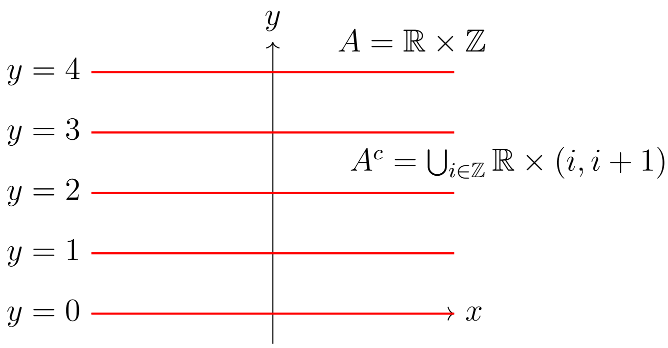



Figure 1: The set $A$ (red horizontal lines) and its complement $A^c$ (shaded strips).

integer heights.

We will prove that $A$ is closed by proving that $A^c$ is open.

For each integer $i \in \mathbb{Z}$ define the set:

$$X_i := \{(x, y) \in \mathbb{R}^2 \mid x \in \mathbb{R},\ i < y < i + 1\} = \mathbb{R} \times (i, i + 1).$$

Then, $A^c = \bigcup_{i\in\mathbb{Z}} X_i$ We need to prove that $X_i$ is open for each integer $i \in \mathbb{Z}$.

To see that $X_i$ is open, let $(x_0, y_0) \in X_i$. Then, $i < y_0 < i + 1$

Choose $r := \frac{1}{2} \min\{y_0 - i, (i + 1) - y_0\} > 0$.

Consider the open ball $B_r((x_0, y_0))$ in the euclidean metric. Let $(x_1, y_1) \in B_r((x_0, y_0))$. We need to prove that $i < y_1 < i + 1$.

We've seen in previous examples that, from the euclidean metric, $\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} < r$ we have: $|y_1 - y_0| < r$.

Then, for any $(x_1, y_1) \in B_r((x_0, y_0))$, we have $|y_1 - y_0| < r$, which implies

$$\begin{aligned}
y_1 &< y_0 + r \\
&= y_0 + \frac{1}{2}\min\{y_0 - i, (i + 1) - y_0\} \\
&\le y_0 + \frac{1}{2}((i + 1) - y_0) \\
&= \frac{1}{2}(y_0 + (i + 1)) \\
&< \frac{1}{2}((i + 1) + (i + 1)) \\
&= i + 1.
\end{aligned}$$

Similarly,

$$\begin{aligned}
y_1 &> y_0 - r \\
&= y_0 - \frac{1}{2}\min\{y_0 - i, (i+1) - y_0\} \\
&\geq y_0 - \frac{1}{2}(y_0 - i) \\
&= \frac{1}{2}(y_0 + i) \\
&> \frac{1}{2}(i + i) \\
&= i
\end{aligned}$$

Therefore, $i < y_1 < i + 1$, which shows that $(x_1, y_1) \in X_i$. Hence, $B_r((x_0, y_0)) \subset X_i$, and $X_i$ is open.

Since $A^c$ is a union of open sets, it is open. Hence, $A$ is closed. $\square$

---

**Exercise 2.10.** Let $A := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \neq 1\}$. Show that $A$ is open in $\mathbb{R}^2$. (*Hint:* Think geometrically. For the formal proof, remember the reverse triangle inequality of the euclidean norm: $|||x|| - ||y||| \leq \ ||x - y||$ for some $x, y \in \mathbb{R}^n$)

---

*Proof.* Let $x = (x_1, x_2) \in A$. Then $x_1^2 + x_2^2 \neq 1$.

Define $r := \frac{1}{2} \mid ||x|| - 1 \mid$. Given that $x_1^2 + x_2^2 \neq 1$, then $(x_1^2 + x_2^2)^{\frac{1}{2}} \neq 1$. Hence, $||x|| \neq 1$, which implies that $r > 0$.

Consider the open ball $B_r(x)$ with center at $x$ and radius $r$. Let $x' = (x_1', x_2') \in B_r(x)$, so $d(x, x') < r$. We need to prove that $x' \in A$, which means that $x_1^2 + x_2^2 \neq 1$. For doing so, consider the next partition of $A$:

$$A_1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$$

$$A_2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 > 1\}$$

We will prove that $x' \in A_i$ for each $i = 1, 2$ that $x \in A_i$.

**C1:** If $x \in A_1$, then $||x|| < 1$.

Then $r = \frac{1}{2}(1 - ||x||)$ and $d(x, x') < \frac{1}{2}(1 - ||x||)$.

Let $x' = (x_1', x_2') \in B_r(x)$, so $d(x, x') < r$. We need to show that $||x'|| < 1$, wich implies that $x' \in A_1$.

By the reverse triangle inequality:

$$\begin{aligned} |||x'|| - ||x||| \leq \ & ||x' - x|| \\ &= d(x, x') \\ &< \frac{1}{2}(1 - ||x||) \end{aligned}$$

By the definition of the absolute value, we have that:

$$|||x'|| - ||x||| < \frac{1}{2}(1 - ||x||) \implies -\frac{1}{2}(1 - ||x||) < ||x'|| - ||x|| < \frac{1}{2}(1 - ||x||)$$

This gives us:

$$\begin{aligned} ||x'|| &< ||x|| + \frac{1}{2}(1 - ||x||) \\ &= \frac{1}{2}||x|| + \frac{1}{2} \\ &= \frac{1}{2}(||x|| + 1) \end{aligned}$$

Since $||x|| < 1$, we have $||x|| + 1 < 2$, so:

$$\begin{aligned} ||x'|| &< \frac{1}{2}(||x|| + 1) \\ &< \frac{1}{2} \cdot 2 \\ &= 1 \end{aligned}$$

Therefore, $||x'||^2 = (x_1')^2 + (x_2')^2 < 1$, which means that $x' \in A_1 \subset A$.

**C2:** If $x \in A_2$, then $||x|| > 1$.

Then $r = \frac{1}{2}(||x|| - 1)$ and $d(x, x') < \frac{1}{2}(||x|| - 1)$.

Let $x' = (x_1', x_2') \in B_r(x)$, so $d(x, x') < r$. We need to show that $||x'|| > 1$, wich implies that $x' \in A_2$.

By the reverse triangle inequality:

$$\begin{aligned} |||x'|| - ||x||| \leq \ & ||x' - x|| \\ &= d(x, x') \\ &< \frac{1}{2}(||x|| - 1) \end{aligned}$$

By the definition of the absolute value, we have that:

$$||x'|| - ||x||| < \frac{1}{2}(||x|| - 1) \implies -\frac{1}{2}(||x|| - 1) < ||x'|| - ||x|| < \frac{1}{2}(||x|| - 1)$$

This gives us:

$$\begin{aligned} ||x'|| &> ||x|| - \frac{1}{2}(||x|| - 1) \\ &= \frac{1}{2}||x|| + \frac{1}{2} \\ &= \frac{1}{2}(||x|| + 1) \end{aligned}$$

Since $||x|| > 1$, we have $||x|| + 1 > 2$, so:

$$\begin{aligned} ||x'|| &> \frac{1}{2}(||x|| + 1) \\ &> \frac{1}{2} \cdot 2 \\ &= 1 \end{aligned}$$

Therefore, $||x'||^2 = (x_1')^2 + (x_2')^2 > 1$ , which means $x' \in A_2 \subset A$.

In both cases, $x' \in A$.

Hence, for any $x \in A$ there exists some $r > 0$ such that $B_r(x) \subset A$. Therefore, $A$ is open. □

---

**Exercise 2.11.** Let $A := \{(x, y) \in \mathbb{R}^2 : 0 \leq x < 1, 0 \leq y < 1\}$. Is $A$ open? (*Hint:* Again, think geometrically. Will be pretty clear.)

---

*Proof.* We will show that $A$ is not open.

To prove this, we need to find a point in $A$ for which no open ball is entirely contained in $A$.

Consider the point $x = (\frac{1}{2}, 0) \in A$. We will show that for any $r > 0$, the ball $B_r(x)$ contains points not in $A$.

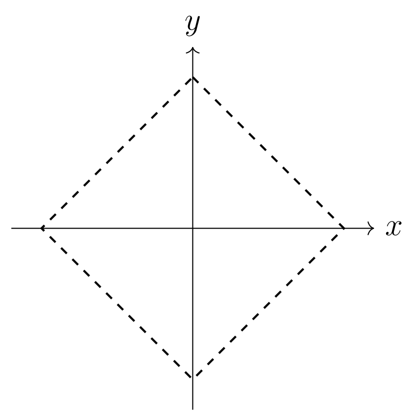
Let $r > 0$ be arbitrary. For any $r > 0$, choose $\epsilon = \min\left\{\frac{r}{2}, \frac{1}{4}\right\} > 0$. Consider the point

$x' = (\frac{1}{2}, -\epsilon)$. We have:

$$\begin{aligned} d(x', x) &= d\left((\frac{1}{2}, -\epsilon), (\frac{1}{2}, 0)\right) \\ &= \sqrt{\left(\frac{1}{2} - \frac{1}{2}\right)^2 + (-\epsilon - 0)^2} \\ &= \sqrt{\epsilon^2} \\ &= \epsilon \\ &< r \end{aligned}$$

That is, $x' \in B_r(x)$. However, since $-\epsilon < 0$, we have $x' \notin A$. This means that $x' \in B_r(x)$ but $x' \notin A$, so $B_r(x) \not\subset A$. Hence, there is no open ball centered at $(\frac{1}{2}, 0)$ that is entirely contained in $A$. Therefore, $A$ is not open. □

**Exercise 2.12.** Let $A := \{(x, y) \in \mathbb{R}^2 : |x| + |y| < 1\}$. Is $A$ open? (*Hint:* Think geometrically.).




*Proof.* Let $(x_0, y_0) \in A$. Then $|x_0| + |y_0| < 1$.

Define $r := \frac{1}{2}(1 - (|x_0| + |y_0|))$. Given that $|x_0| + |y_0| < 1$, we have $1 - (|x_0| + |y_0|) > 0$, which implies that $r > 0$.

Consider the open ball $B_r((x_0, y_0))$.

Let $(x_1, y_1) \in B_r((x_0, y_0))$, so $d((x_1, y_1), (x_0, y_0)) < r$.

We need to prove that $(x_1, y_1) \in A$, which means that $|x_1| + |y_1| < 1$.

By the reverse triangle inequality $||x_1| - |x_0|| \leq |x_1 - x_0|$, we have that $-|x_1 - x_0| \leq |x_1| - |x_0| \leq |x_1 - x_0|$. Then:

$$|x_1| \le |x_0| + |x_1 - x_0| \tag{2.5}$$
$$|y_1| \le |y_0| + |y_1 - y_0| \tag{2.6}$$

Therefore:

$$\begin{aligned}
|x_1| + |y_1| &\le |x_0| + |x_1 - x_0| + |y_0| + |y_1 - y_0| \\
&= (|x_0| + |y_0|) + (|x_0 - x_1| + |y_1 - y_0|) \\
&\le (|x_0| + |y_0|) + (\sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2} + \sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2}) \\
&= (|x_0| + |y_0|) + 2||(x_0, y_0) - (x_1, y_1)|| \\
&= (|x| + |y|) + 2d((x_0, y_0), (x_1, y_1)) \\
&< (|x| + |y|) + 2r \\
&= (|x| + |y|) + 2 \cdot \frac{1}{2}(1 - (|x| + |y|)) \\
&= (|x| + |y|) + (1 - (|x| + |y|)) \\
&= 1
\end{aligned}$$

Thus $(x_1, y_1) \in A$, which shows that $B_r((x_0, y_0)) \subset A$.

Therefore, $A$ is open. $\square$

### 2.2.2 Convergence in Metric Spaces

Before we jump into closed sets, we need to mention some significant definitions regarding convergence in metric spaces.

**Definition 2.5** (Convergence in Metric Spaces)**.** Let $(X, d)$ be a metric space, and $\{x_n\}$ a sequence in $X$. We say that $\{x_n\}$ converges to $x^* \in X$, or that the sequence has limit $x^*$, if:

$$\forall \epsilon > 0, \exists N_\epsilon \text{ such that } d(x_n, x^*) < \epsilon \quad \forall n > N_\epsilon$$

**Definition 2.6** (Limit points)**.** Let $(X, d)$ be a metric space and $A$ be a set in $X$. A point $x \in X$ is a *limit point* (or *cluster point*) of $A$ if every open ball around it contains at least one point of $A$, which is distinct from $x$. Formally, $x$ is a limit point if and only if for each $\varepsilon > 0$ we have $B_\varepsilon(x) \cap (A \setminus \{x\}) \neq \emptyset$.

The set of all limit points of $A$ is called its **derived set** and is denoted by $D(A)$.

**Proposition 2.1.** *Let $(X, d)$ be a metric space and $A$ be a set in $X$. A point $x \in X$ is in $D(A)$*

*if and only if there exists a sequence in $A \setminus \{x\}$ that converges to $x$.*

### 2.2.3 Closed Sets

We have seen in **Definition 2.3** that a set $A \subseteq X$ in the metric space $(X, d)$ is closed if its complement is open. Therefore, in order to prove that $A$ is closed, it is enough to prove that $A^c$ is open, which we have done numerous times above.

Nonetheless, the following results are useful as well (especially **Theorem 2.2 (c)**):

**Theorem 2.1.** *Let $(X, d)$ be a metric space and $A$ be a set in $X$. Then the following three statements are equivalent:*

*(a) $A$ is closed.*

*(b) $A$ contains all its limit points, i.e., $D(A) \subseteq A$.*

*(c) Every convergent sequence in $A$ has its limit in $A$, i.e., if $x_n \in A$ for all $n$ and $\{x_n\}$ converges to $x$, then $x \in A$.*

That is, in order to prove that $A \subseteq X$ is closed, we can prove that every convergent sequence contained in $A$ has its limit in $A$, i.e.: if $\{x_n\}$ is a convergent sequence such that $x_n \in A \quad \forall n \in \mathbb{N}$ and $\{x_n\} \to x^*$, then $x^* \in A$.

### 2.2.4 Compact Sets

**Definition 2.7** (Bounded Set)**.** A subset $S$ of $X$ is *bounded* if there exist $x \in X$ and $r > 0$ such that $S \subset B_r(x)$.

**Theorem 2.2** (Heine-Borel)**.** *Consider the metric space $(\mathbb{R}^n, d)$. A set $A \subseteq \mathbb{R}^n$ is compact iff it is closed and bounded.*

---

**Exercise 2.13.** Let $H_+(x) := \{x \in \mathbb{R}^n_+ \mid a \cdot x \leq b\}$ with $a \in \mathbb{R}^n_{++}$ and $b > 0$. Prove that $H_+(x)$ is compact.

(*Hint:* For closedness, use convergent sequences).

---

*Proof.* From Heine-Borel, we know that a set $A \subseteq \mathbb{R}^n$ is *compact* iff it is closed and bounded.

Let $\{x_k\}_{k \in \mathbb{N}} \subset H_+(x)$ be a sequence that converges to some $x^* \in \mathbb{R}^n$. Then we have that $a \cdot x_k \leq b$ for all $k \in \mathbb{N}$. Taking the limit in both sides gives:

$$\lim_{k \to \infty} a \cdot x_k \leq \lim_{k \to \infty} b \implies a \cdot x^* \leq b$$

Therefore, $x^* \in H_+(x)$, so $H_+(x)$ contains all its limit points and it is closed.

Now, given that $a >> 0$, note that

$$a \cdot x = \sum_{i=1}^{n} a_i x_i \leq b \implies x_i \leq \frac{b}{a_i} \ \forall i = 1, \cdots, n$$

Therefore, consider the euclidean open ball $B_r(0)$ with $r > 0$ defined by:

$$r := 2\sqrt{n}\ \max\{\frac{b}{a_i} \mid i = 1, \ldots, n\}.$$

We want to prove that, for any $x \in H_+(x)$, we have $x \in B_r(0)$, i.e, $\|x - 0\| < r$.

So, for any $x \in H_+(x)$,

$$\begin{aligned}
\|x - 0\| &= \sqrt{\sum_{i=1}^{n} x_i^2} \\
&\leq \sqrt{\sum_{i=1}^{n} \left(\frac{b}{a_i}\right)^2} \\
&\leq \sqrt{\sum_{i=1}^{n} \left(\max_{j=1,\ldots,n} \{\frac{b}{a_j}\}\right)^2} \\
&= \sqrt{n\ \left(\max_{j=1,\ldots,n} \{\frac{b}{a_j}\}\right)^2} \\
&= \sqrt{n}\ \max_{j=1,\ldots,n} \{\frac{b}{a_j}\} \\
&< 2\sqrt{n}\ \max_{j=1,\ldots,n} \{\frac{b}{a_j}\} \\
&= r
\end{aligned}$$

Which means that $H_+(x) \subset B_r(0)$, so it is bounded.

Therefore, $H_+(x)$ is compact. $\square$

# 3 Convex Analysis

## 3.1 Convex Sets

**Definition 3.1** (Convex Set)**.** A set $X$ is **convex** if given any two points $x'$ and $x''$ in $X$, the point

$$x^\lambda = (1-\lambda)x' + \lambda x''$$

is also in $X$ for every $\lambda \in [0,1]$.

That is, a set $X$ is convex if all the points in the line segment between any two points in $X$ lie in $X$.

**Example 3.1.** Following are examples of convex sets:

(a) The disc $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq r\}$ is a convex set.

(b) The circle $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = r\}$ is **not** a convex set.

(c) The set $H(p,\alpha) = \{x \in \mathbb{R}^n : p \cdot x = \alpha \text{ for } p \in \mathbb{R}^n \text{ and } \alpha \in \mathbb{R}\}$ is a convex set.

**Exercise 3.1** (Solution set of linear equations)**.** Let $C := \{x \in \mathbb{R}^n \mid Ax = b\}$ be the set of all solutions to the system of linear equations $Ax = b$, where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Show that $C$ is convex. (*Hint: Take two elements from $C$ and show that $x^\lambda \in C$ algebraically.*)

*Proof.* We need to show that for any two points $x_1, x_2 \in C$ and any $\lambda \in [0,1]$, the convex combination $x^\lambda = (1-\lambda)x_1 + \lambda x_2$ is also in $C$.

Let $x_1, x_2 \in C$. Then by definition of $C$:

$$Ax_1 = b \quad \text{and} \quad Ax_2 = b$$

Consider the convex combination $x^\lambda = (1-\lambda)x_1 + \lambda x_2$ for any $\lambda \in [0,1]$.

We need to show that $x^\lambda \in C$, i.e., that $Ax^\lambda = b$:

$$\begin{aligned}
Ax^{\lambda} &= A[(1-\lambda)x_1 + \lambda x_2] \\
&= A(1-\lambda)x_1 + A\lambda x_2 && \text{(linearity of matrix multiplication)} \\
&= (1-\lambda)Ax_1 + \lambda Ax_2 && \text{(distributivity)} \\
&= (1-\lambda)b + \lambda b && \text{(since } Ax_1 = b \text{ and } Ax_2 = b\text{)} \\
&= b(1-\lambda) + b\lambda \\
&= b[(1-\lambda) + \lambda] \\
&= b
\end{aligned}$$

Therefore, $Ax^{\lambda} = b$, which means $x^{\lambda} \in C$.

Since this holds for any $x_1, x_2 \in C$ and any $\lambda \in [0,1]$, the set $C$ is convex. □

---

**Exercise 3.2.** Let $a > 0$. Prove that $A := \{(x,y) \in \mathbb{R}^2 \mid x \geq 0, 0 \leq y \leq a\}$ is convex. (*Hint: Take two elements from $A$ and show that $x^{\lambda} \in A$ algebraically.*)

---

*Proof.* Let $(x_1, y_1), (x_2, y_2) \in A$ and $\lambda \in [0,1]$.

Define $x^{\lambda} = \lambda(x_1, y_1) + (1-\lambda)(x_2, y_2) = (\lambda x_1 + (1-\lambda)x_2\ ,\ \lambda y_1 + (1-\lambda)y_2)$.

Given that $x_1, x_2 \geq 0,\ y_1, y_2 \geq 0$, then $(\lambda x_1 + (1-\lambda)x_2) \geq 0$ and $(\lambda y_1 + (1-\lambda)y_2) \geq 0$.

Moreover, given that $y_1, y_2 \leq a$, we have:

$$\begin{aligned}
\lambda y_1 + (1-\lambda)y_2 &\leq \lambda a + (1-\lambda)a \\
&= a
\end{aligned}$$

Therefore, $x^{\lambda} \in A$, so $A$ is convex. □

---

**Exercise 3.3.** Let $X$ be a convex set in $\mathbb{R}^n$, and let $\alpha$ be a real number. Show that the set

$$\alpha X = \{z \in \mathbb{R}^n : z = \alpha x \text{ for some } x \in X\}$$

is convex. (*Hint: Take two elements from $\alpha X$ and show that $x^{\lambda} \in \alpha X$ algebraically.*)

---

*Proof.* We need to show that for any two points $z_1, z_2 \in \alpha X$ and any $\lambda \in [0,1]$, the convex combination $(1-\lambda)z_1 + \lambda z_2$ is also in $\alpha X$.

Let $z_1, z_2 \in \alpha X$. Then by definition, there exist $x_1, x_2 \in X$ such that:

$$z_1 = \alpha x_1 \quad \text{and} \quad z_2 = \alpha x_2$$

Consider the convex combination $(1-\lambda)z_1 + \lambda z_2$ for any $\lambda \in [0,1]$:

$$\begin{aligned}(1-\lambda)z_1 + \lambda z_2 &= (1-\lambda)(\alpha x_1) + \lambda(\alpha x_2) \\ &= \alpha[(1-\lambda)x_1 + \lambda x_2]\end{aligned}$$

Since $X$ is convex and $x_1, x_2 \in X$, we have $(1-\lambda)x_1 + \lambda x_2 \in X$ for any $\lambda \in [0,1]$.

Let $x^* = (1-\lambda)x_1 + \lambda x_2 \in X$. Then:

$$(1-\lambda)z_1 + \lambda z_2 = \alpha x^*$$

Since $x^* \in X$, we have $\alpha x^* \in \alpha X$. Therefore, $(1-\lambda)z_1 + \lambda z_2 \in \alpha X$, which proves that $\alpha X$ is convex. □

---

**Exercise 3.4.** Let $X$ and $Y$ be convex sets in $\mathbb{R}^n$. Show that the set

$$X + Y = \{z \in \mathbb{R}^n : z = x + y \text{ for some } x \in X \text{ and } y \in Y\}$$

is convex. (*Hint: Take two elements from $X+Y$ and show that $x^\lambda \in X+Y$ algebraically.*)

---

*Proof.* We need to show that for any two points $z_1, z_2 \in X+Y$ and any $\lambda \in [0,1]$, the convex combination $(1-\lambda)z_1 + \lambda z_2$ is also in $X+Y$.

Let $z_1, z_2 \in X+Y$. Then by definition, there exist $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ such that:

$$z_1 = x_1 + y_1 \quad \text{and} \quad z_2 = x_2 + y_2$$

Consider the convex combination $(1-\lambda)z_1 + \lambda z_2$ for any $\lambda \in [0,1]$:

$$\begin{aligned}(1-\lambda)z_1 + \lambda z_2 &= (1-\lambda)(x_1 + y_1) + \lambda(x_2 + y_2) \\ &= (1-\lambda)x_1 + (1-\lambda)y_1 + \lambda x_2 + \lambda y_2 \\ &= [(1-\lambda)x_1 + \lambda x_2] + [(1-\lambda)y_1 + \lambda y_2]\end{aligned}$$

Since $X$ is convex and $x_1, x_2 \in X$, we have:

$$x^* := (1-\lambda)x_1 + \lambda x_2 \in X$$

Since $Y$ is convex and $y_1, y_2 \in Y$, we have:

$$y^* := (1-\lambda)y_1 + \lambda y_2 \in Y$$

Therefore:

$$(1-\lambda)z_1 + \lambda z_2 = x^* + y^*$$

Since $x^* \in X$ and $y^* \in Y$, we have $x^* + y^* \in X + Y$ by definition.

Hence, $(1-\lambda)z_1 + \lambda z_2 \in X + Y$, which proves that $X + Y$ is convex. $\square$

---

**Exercise 3.5** (Open ball)**.** Consider the Euclidean metric space $(\mathbb{R}^n, \|\cdot\|)$. Let $x \in \mathbb{R}^n$ and $r > 0$. Prove that the open ball $B_r(x) = \{y \in \mathbb{R}^n : \|x - y\| < r\}$ is convex.

---

*Proof.* We need to show that for any two points $y_1, y_2 \in B_r(x)$ and any $\lambda \in [0,1]$, the convex combination $y^\lambda = (1-\lambda)y_1 + \lambda y_2$ is also in $B_r(x)$.

Let $y_1, y_2 \in B_r(x)$. Then by definition:

$$\|x - y_1\| < r \quad \text{and} \quad \|x - y_2\| < r$$

Consider the convex combination $y^\lambda = (1-\lambda)y_1 + \lambda y_2$ for any $\lambda \in [0,1]$.

We need to show that $y^\lambda \in B_r(x)$, i.e., that $\|x - y^\lambda\| < r$:

$$\begin{aligned}
\|x - y^\lambda\| &= \|x - ((1-\lambda)y_1 + \lambda y_2)\,\| \\
&= \|x - (1-\lambda)y_1 - \lambda y_2\| \\
&= \|\lambda x - \lambda x \; + x - (1-\lambda)y_1 - \lambda y_2\| \\
&= \|(1-\lambda)x + \lambda x - (1-\lambda)y_1 - \lambda y_2\| \\
&= \|(1-\lambda)(x - y_1) + \lambda(x - y_2)\| \\
&\leq \|(1-\lambda)(x - y_1)\| + \|\lambda(x - y_2)\| \quad \text{(triangle inequality of norms)} \\
&= (1-\lambda)\|x - y_1\| + \lambda\|x - y_2\| \qquad \text{(homogeneity of norms)}
\end{aligned}$$

Since $y_1, y_2 \in B_r(x)$, we have $\|x - y_1\| < r$ and $\|x - y_2\| < r$. Then:

$$\begin{aligned}\|x - y^\lambda\| &\leq (1-\lambda)\|x - y_1\| + \lambda\|x - y_2\| \\ &< (1-\lambda)r + \lambda r \\ &= r\end{aligned}$$

Therefore, $\|x - y^\lambda\| < r$, which means $y^\lambda \in B_r(x)$. Hence, $B_r(x)$ is convex. $\square$

---

**Exercise 3.6.** Prove **Example 3.1 (a)**. (*Hint: Think geometrically, then take two points from $D$ and show that $x^\lambda \in D$ algebraically. At some point, the Cauchy-Schwarz inequality maybe useful: $x_1x_2 + y_1y_2 \leq \sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2}$*)

---

*Proof.* We need to prove that the disc $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq r\}$ is convex.

Let $(x_1, y_1), (x_2, y_2) \in D$. Then $x_1^2 + y_1^2 \leq r$ and $x_2^2 + y_2^2 \leq r$.

For any $\lambda \in [0, 1]$, consider the point:

$$\begin{aligned}x^\lambda &= (1-\lambda)(x_1, y_1) + \lambda(x_2, y_2) \\ &= ((1-\lambda)x_1 + \lambda x_2, (1-\lambda)y_1 + \lambda y_2)\end{aligned}$$

We need to show that $x^\lambda \in D$, i.e., $((1-\lambda)x_1 + \lambda x_2)^2 + ((1-\lambda)y_1 + \lambda y_2)^2 \leq r$.

Expanding:

$$\begin{aligned}&((1-\lambda)x_1 + \lambda x_2)^2 + ((1-\lambda)y_1 + \lambda y_2)^2 \\ &= (1-\lambda)^2 x_1^2 + 2\lambda(1-\lambda)x_1x_2 + \lambda^2 x_2^2 + (1-\lambda)^2 y_1^2 + 2\lambda(1-\lambda)y_1y_2 + \lambda^2 y_2^2 \\ &= (1-\lambda)^2(x_1^2 + y_1^2) + \lambda^2(x_2^2 + y_2^2) + 2\lambda(1-\lambda)(x_1x_2 + y_1y_2)\end{aligned}$$

Since $(x_1, y_1), (x_2, y_2) \in D$:

$$(1-\lambda)^2(x_1^2 + y_1^2) + \lambda^2(x_2^2 + y_2^2) \leq (1-\lambda)^2 r + \lambda^2 r$$

By Cauchy-Schwarz: $x_1x_2 + y_1y_2 \leq \sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2} \leq \sqrt{r}\sqrt{r} = r$.

Therefore:

$$\begin{aligned}((1-\lambda)x_1+\lambda x_2)^2+((1-\lambda)y_1+\lambda y_2)^2 &\leq (1-\lambda)^2 r+\lambda^2 r+2\lambda(1-\lambda)r\\ &= r[(1-\lambda)^2+\lambda^2+2\lambda(1-\lambda)]\\ &= r[(1-\lambda)+\lambda]^2 = r\end{aligned}$$

Hence $x^\lambda \in D$, so $D$ is convex. □

**Exercise 3.7.** Prove **Example 3.1 (b)**. (*Hint: Think geometrically and find a counterexample.*)

*Proof.* We show that the circle $C = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = r\}$ is not convex by finding a counterexample.

Take $(r,0),(0,r) \in C$. Let $\lambda = \frac{1}{2}$. Then:

$$\begin{aligned}x^{\frac{1}{2}} &= \frac{1}{2}(r,0)+\frac{1}{2}(0,r)\\ &= \left(\frac{r}{2},\frac{r}{2}\right)\end{aligned}$$

We verify if $x^{\frac{1}{2}} \in C$:

$$\begin{aligned}\left(\frac{r}{2}\right)^2+\left(\frac{r}{2}\right)^2 &= \frac{r^2}{4}+\frac{r^2}{4}\\ &= \frac{r^2}{2}\\ &\neq r\end{aligned}$$

Since $r > 0$, then $x^{\frac{1}{2}} \notin C$.

Therefore, $C$ is not convex. □

**Exercise 3.8.** Let $C(\mathbb{R})$ be the vector space of continuous real-valued functions. Let $\mathcal{D} \subset C(\mathbb{R})$ be the set of those $f$ such that

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \quad \text{and} \quad f(x) \geq 0 \text{ for all } x \in \mathbb{R}$$

Show that $\mathcal{D}$ is convex.

(*Hint: Use the linearity of integration to show that convex combinations of this type of* *functiones are also in $\mathcal{D}$.)*

*Proof.* We need to show that for any two functions $f, g \in \mathcal{D}$ and any $\lambda \in [0,1]$, the convex combination $h = (1-\lambda)f + \lambda g$ is also in $\mathcal{D}$.

Let $f, g \in \mathcal{D}$ and $\lambda \in [0,1]$. Define $h(x) = (1-\lambda)f(x) + \lambda g(x)$ for all $x \in \mathbb{R}$.

We need to verify that $h \in \mathcal{D}$, which requires showing three properties:

*Continuity:* Since $f$ and $g$ are continuous functions and $\lambda, (1-\lambda)$ are constants, the linear combination $h(x) = (1-\lambda)f(x) + \lambda g(x)$ is also continuous.

*Non-negativity:* For any $x \in \mathbb{R}$, since $f(x) \geq 0$, $g(x) \geq 0$, $\lambda \geq 0$, and $(1-\lambda) \geq 0$, we have:

$$h(x) = (1-\lambda)f(x) + \lambda g(x) \geq 0$$

Therefore, $h : \mathbb{R} \to [0, \infty)$.

*Integration:* By the linearity of integration:

$$\begin{aligned}
\int_{-\infty}^{\infty} h(x)\,dx &= \int_{-\infty}^{\infty} [(1-\lambda)f(x) + \lambda g(x)]\,dx \\
&= \int_{-\infty}^{\infty} (1-\lambda)f(x)\,dx + \int_{-\infty}^{\infty} \lambda g(x)\,dx \\
&= (1-\lambda)\int_{-\infty}^{\infty} f(x)\,dx + \lambda \int_{-\infty}^{\infty} g(x)\,dx
\end{aligned}$$

Since $f, g \in \mathcal{D}$, we have $\int_{-\infty}^{\infty} f(x)\,dx = 1$ and $\int_{-\infty}^{\infty} g(x)\,dx = 1$. Therefore:

$$\begin{aligned}
\int_{-\infty}^{\infty} h(x)\,dx &= (1-\lambda)\cdot 1 + \lambda \cdot 1 \\
&= (1-\lambda) + \lambda \\
&= 1
\end{aligned}$$

Since $h$ satisfies all three required properties, we have $h \in \mathcal{D}$.

Therefore, $\mathcal{D}$ is convex. $\square$

# 4 Concave and Quasiconcave Functions

**Theorem 4.1.** *A function $f : X \to \mathbb{R}$ is concave (convex) on $X$ if and only if for all $x, y \in X$ and for all $\lambda \in [0, 1]$, it is the case that*

$$f(\lambda x + (1-\lambda)y) \geq (\leq) \lambda f(x) + (1-\lambda) f(y)$$

**Theorem 4.2.** *Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be two concave functions. Then*

*(a) the function $\alpha f + \beta g : \mathbb{R}^n \to \mathbb{R}$ is concave for $\alpha, \beta \geq 0$,*

*(b) the function $\min\{f, g\} : \mathbb{R}^n \to \mathbb{R}$ is concave.*

**Exercise 4.1.** Prove **Theorem 4.2**.

*Proof.* We prove both parts separately.

**(a):** Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be concave functions and $\alpha, \beta \geq 0$. Define $h(x) := \alpha f(x) + \beta g(x)$. Let $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$.

Then

$$h(\lambda x + (1-\lambda)y) = \alpha f(\lambda x + (1-\lambda)y) + \beta g(\lambda x + (1-\lambda)y)$$

Since $f$ and $g$ are concave:

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &\geq \lambda f(x) + (1-\lambda) f(y) \\ g(\lambda x + (1-\lambda)y) &\geq \lambda g(x) + (1-\lambda) g(y) \end{aligned}$$

And given that $\alpha, \beta \geq 0$, we have then:

$$\begin{aligned} h(\lambda x + (1-\lambda)y) &= \alpha f(\lambda x + (1-\lambda)y) + \beta g(\lambda x + (1-\lambda)y) \\ &\geq \alpha[\lambda f(x) + (1-\lambda) f(y)] + \beta[\lambda g(x) + (1-\lambda) g(y)] \\ &= \lambda[\alpha f(x) + \beta g(x)] + (1-\lambda)[\alpha f(y) + \beta g(y)] \\ &= \lambda h(x) + (1-\lambda) h(y) \end{aligned}$$

Therefore, $h$ is concave.

**(b):** Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be concave functions. Define $h(x) := \min\{f(x), g(x)\}$. Let $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$.

Since $f$ and $g$ are concave:

$$f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y)$$
$$g(\lambda x + (1-\lambda)y) \geq \lambda g(x) + (1-\lambda)g(y)$$

Therefore:

$$\begin{aligned} h(\lambda x + (1-\lambda)y) &= \min\{f(\lambda x + (1-\lambda)y), g(\lambda x + (1-\lambda)y)\} \\ &\geq \min\{\lambda f(x) + (1-\lambda)f(y), \lambda g(x) + (1-\lambda)g(y)\} \end{aligned}$$

Now, we use the fact that for any real numbers $a_1, a_2, b_1, b_2$ and $\lambda \in [0,1]$:

$$\min\{\lambda a_1 + (1-\lambda)b_1, \lambda a_2 + (1-\lambda)b_2\} \geq \lambda \min\{a_1, a_2\} + (1-\lambda)\min\{b_1, b_2\}$$

Which in turn gives us:

$$\begin{aligned} h(\lambda x + (1-\lambda)y) &\geq \lambda \min\{f(x), g(x)\} + (1-\lambda)\min\{f(y), g(y)\} \\ &= \lambda h(x) + (1-\lambda)h(y) \end{aligned}$$

Therefore, $h$ is concave. □

**Theorem 4.3.** *A function $f : X \to \mathbb{R}$ is quasiconcave on $X$ if and only if for all $x, y \in X$ and for all $\lambda \in [0,1]$, it is the case that*

$$f(\lambda x + (1-\lambda)y) \geq \min\{f(x), f(y)\}.$$

Similarly, a function $f : X \to \mathbb{R}$ is quasiconvex on $X$ if and only if for all $x, y \in X$ and for all $\lambda \in [0,1]$, it is the case that

$$f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\}.$$

**Theorem 4.4.** *Let $f : X \to \mathbb{R}$ be concave on $X$, then it is also quasiconcave. Similarly, if $f : X \to \mathbb{R}$ is convex on $X$, then it is also quasiconvex.*

**Theorem 4.5.** *Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be two quasiconcave functions. Then*

*(a) the function $\alpha f : \mathbb{R}^n \to \mathbb{R}$ is quasiconcave for $\alpha \geq 0$,*

*(b) the function $\min\{f, g\} : \mathbb{R}^n \to \mathbb{R}$ is quasiconcave,*

*(c) if $h : \mathbb{R} \to \mathbb{R}$ is non-decreasing, then $h \circ f : \mathbb{R}^n \to \mathbb{R}$ is quasiconcave.*

**Exercise 4.2.** Prove **Theorem 4.5**.

*Proof.* **(a):** Let $f : \mathbb{R}^n \to \mathbb{R}$ be quasiconcave and $\alpha \geq 0$. Define $h(x) := \alpha f(x)$. Let $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$.

Since $f$ is quasiconcave and $\alpha \geq 0$:

$$\begin{aligned} h(\lambda x + (1-\lambda)y) &= \alpha f(\lambda x + (1-\lambda)y) \\ &\geq \alpha \min\{f(x), f(y)\} \\ &= \min\{\alpha f(x), \alpha f(y)\} \\ &= \min\{h(x), h(y)\} \end{aligned}$$

Therefore, $h$ is quasiconcave.

**(b):** Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be quasiconcave. Define $h(x) := \min\{f(x), g(x)\}$. Let $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$.

Since $f$ and $g$ are quasiconcave:

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &\geq \min\{f(x), f(y)\} \\ g(\lambda x + (1-\lambda)y) &\geq \min\{g(x), g(y)\} \end{aligned}$$

Therefore:

$$\begin{aligned} h(\lambda x + (1-\lambda)y) &= \min\{f(\lambda x + (1-\lambda)y), g(\lambda x + (1-\lambda)y)\} \\ &\geq \min\{\min\{f(x), f(y)\}, \min\{g(x), g(y)\}\} \\ &= \min\{f(x), f(y), g(x), g(y)\} \\ &= \min\{\min\{f(x), g(x)\}, \min\{f(y), g(y)\}\} \\ &= \min\{h(x), h(y)\} \end{aligned}$$

Therefore, $h$ is quasiconcave.

**(c):** Let $f : \mathbb{R}^n \to \mathbb{R}$ be quasiconcave and $h : \mathbb{R} \to \mathbb{R}$ be non-decreasing. We need to show that $(h \circ f)(x) = h(f(x))$ is quasiconcave.

Let $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$. Since $f$ is quasiconcave:

$$f(\lambda x + (1-\lambda)y) \geq \min\{f(x), f(y)\}$$

Since $h$ is non-decreasing:

$$\begin{aligned} h(f(\lambda x + (1-\lambda)y)) &\geq h(\min\{f(x), f(y)\}) \\ &= \min\{h(f(x)), h(f(y))\} \end{aligned}$$

Hence:

$$(h \circ f)(\lambda x + (1-\lambda)y) \geq \min\{(h \circ f)(x), (h \circ f)(y)\}$$

Therefore, $h \circ f$ is quasiconcave. $\square$

# References

Dam, K. (2023). *Introduction to static optimization.* Centro de Investigación y Docencia Económicas, Mexico City, Mx. https://kaniskadam.weebly.com/uploads/4/2/7/7/42773651/lecturenotesopt.pdf.

De la Fuente, A. (2000). *Mathematical Methods and Models for Economists.* Cambridge University Press, Cambridge, U.K.

Kumaresan, S. (2005). *Topology of Metric Spaces.* Alpha Science International Ltd., Harrow, U.K. University of Mumbai, Mumbai, India.

Lebl, J. (2023). *Basic Analysis I: Introduction to Real Analysis, Volume I*, volume 1 of *Basic Analysis: Introduction to Real Analysis.* https://ocw.mit.edu/courses/18-100a-real-analysis-fall-2020/pages/lecture-notes-and-readings/.

Ok, E. A. (2007). *Real Analysis with Economic Applications.* Princeton University Press, New Jersey, USA.