

# Nine Kinds of Quasiconcavity and Concavity\*

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## 1. INTRODUCTION

There is no need to emphasize the importance of quasiconcave (and concave) functions in economics and operations research. However, in view of the many excellent papers and books characterizing quasiconcave functions (e.g., [11; 1; 29; 24, pp. 131-150]), it is perhaps necessary to justify yet another paper on the subject.

It is well known (cf. [11, pp. 87-88] or [30, 27]) that a twice continuously differentiable function  $f$  of  $N$  variables  $x \equiv (x_1, x_2, \dots, x_N)^T$  defined over an open convex set  $S$  is concave if and only if the function's Hessian matrix of second order partial derivatives,  $\nabla^2 f(x)$ , is negative semidefinite for every  $x \in S$ . This is an extremely useful characterization of a concave function in practice. However, a similar simple characterization for twice differentiable quasiconcave functions had not been reported in the literature. In Section 2 below, we provide relatively simple necessary and sufficient conditions for a twice continuously differentiable function to be quasiconcave.

Our criterion for quasiconcavity in the twice continuously differentiable case can be derived from a criterion involving only local properties of the function. It turns out to be possible to characterize all seven varieties of

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concave and quasiconcave functions which were discussed by Ponstein [29]<sup>1</sup> (plus two additional types of concave functions) in a similar manner, using only local properties of the function to be characterized. In Sections 2 to 10 below, we report these "new" characterizations, although in some cases, the "new" characterizations turn out to be fairly close to known characterizations.

To summarize, we hope that the present will serve two purposes: (i) to acquaint economists with the rather extensive operations research literature on quasiconcavity; and (ii) to provide alternative characterizations of the various kinds of concavity in general as well as in the once and twice differentiable cases.

Proofs of new theorems are collected in an appendix.

## 2. QUASICONCAVITY

**DEFINITION 1** (Fenchel [11, p. 117]). A function  $f$  of  $N$  real variables defined over a convex subset  $S$  of  $R^N$  is quasiconcave iff the upper level sets  $L(\alpha) \equiv \{x: f(x) \geq \alpha\}$  are convex sets<sup>2</sup> for all real numbers  $\alpha$ .

In the remainder of the paper, we assume that  $S$  is a convex subset of  $R^N$ . If we require *in addition* that  $S$  be an open set, we shall say "open  $S$ ."

An alternative useful characterization of quasiconcavity is provided by the following result.

**THEOREM 1** (Fenchel [11, p. 118]). *f is a quasiconcave function defined over S iff*

$$x^1 \in S, \quad x^2 \in S, \quad f(x^1) \leq f(x^2), \quad 0 \leq \lambda \leq 1,$$

*implies*

$$f(x^1) \leq f(\lambda x^1 + (1 - \lambda) x^2).$$

Let  $\nabla f(x^0) \equiv (\partial f(x^0)/\partial x_1, \dots, \partial f(x^0)/\partial x_N)^T$  denote the column vector of first order partial derivatives of  $f$  evaluated at the point  $x^0 \equiv (x_1^0, \dots, x_N^0)^T$ . Let  $x^T y$  denote the inner product of the vectors  $x$  and  $y$ . If the first (second) order partial derivatives of  $f$  exist and are continuous functions, we say that  $f$  is

<sup>1</sup> Ponstein actually provided characterizations of convex and quasiconvex functions. However, his results are immediately applicable to concave functions since  $f$  is a concave (quasiconcave) function if and only if  $-f$  is a convex (quasiconvex) function.

<sup>2</sup> A set  $S$  is convex iff for every  $x, y \in S$  and scalar  $\lambda$  such that  $0 \leq \lambda \leq 1$ , we have  $\lambda x + (1 - \lambda) y \in S$ .

once (twice) continuously differentiable. If the (two-sided) directional derivative

$$D_v f(x^0) \equiv \lim_{t \rightarrow 0} [f(x^0 + tv) - f(x^0)]/t$$

exists for all feasible directions  $v$  (i.e.,  $v^T v = 1$  and  $x^0 + tv \in S$  for some  $t > 0$ ), then  $f$  is *directionally* differentiable at  $x \in S$ . If in addition,  $D_v f(x^0) = v^T \nabla f(x^0)$  for all feasible directions  $v$ , then  $f$  is (Gateaux) *differentiable* at  $x^0$  (cf. Avriel [2, p. 84]).

**THEOREM 2** (Arrow and Enthoven [1, p. 780]). *Let  $f$  be a differentiable function defined over an open  $S$ . Then  $f$  is quasiconcave over  $S$  iff  $x^1 \in S$ ,  $x^2 \in S$ ,  $f(x^2) \geq f(x^1)$ , implies  $(x^2 - x^1)^T \nabla f(x^1) \geq 0$ .*

If  $f$  is twice continuously differentiable, the existing literature does not provide us with a simple characterization of quasiconcave functions comparable to the characterization given by Theorem 2 above for the once differentiable case. In order to provide a characterization in the twice differentiable case, it proves to be convenient to first develop a criterion for quasiconcavity in general and then utilize this criterion in the twice differentiable case. However, first we require a preliminary result.

**THEOREM 3.** *Let  $f$  be a quasiconcave function defined over  $S$ . Then the minimum of  $f$  over any compact (i.e., closed and bounded) line segment contained in  $S$  is attained on that line segment; i.e., if we let  $x^0 \in S$ ,  $v \in R^N$  with  $v^T v = 1$ ,  $\bar{t}v \in S$ , then*

$$\begin{aligned} \alpha \equiv \inf_t \{f(x^0 + tv): t \in [0, \bar{t}]\} &= \min_t \{f(x^0 + tv): t \in [0, \bar{t}]\} \\ &= f(x^0 + t^*v), \end{aligned} \tag{1}$$

where  $t^* \in [0, \bar{t}]$  and  $[0, \bar{t}]$  denotes the closed line segment joining 0 and  $\bar{t}$ .

If (1) is true, then we say that  $f$  has the *line segment minimum property*. Theorem 3 says that a quasiconcave function has this property.

Before we can provide our third characterization of quasiconcavity (Theorems 1 and 2 provide the first two characterizations), we need to introduce a special type of local minimum that is stronger than the concept of a local minimum but weaker than a strict local minimal. Let  $g$  be a function of one variable defined over the closed interval  $[b, c]$  in the following three definitions, with  $b < c$ .

**DEFINITION 2.**  $g$  attains a *one-sided semistrict local minimum from above* at  $t_0 \in [b, c)$  (the line segment joining  $b$  and  $c$  excluding  $c$ ) iff there

exists  $\varepsilon > 0$  such that  $g(t_0) \leq g(t_0 + h)$  for all  $h$  such that  $0 < h \leq \varepsilon \leq c - t_0$  and  $g(t_0) < g(t_0 + \varepsilon)$ .

**DEFINITION 3.**  $g$  attains a *one-sided semistrict local minimum from below* at  $t_0 \in (b, c]$  (the line segment joining  $b$  and  $c$  excluding  $b$ ) iff there exists  $\varepsilon > 0$  such that  $g(t_0) \leq g(t_0 - h)$  for all  $h$  such that  $0 < h \leq \varepsilon \leq t_0 - b$  and  $g(t_0) < g(t_0 - \varepsilon)$ .

Note that a one-sided semistrict local minimum is not a local minimum since the usual definition of a local minimum at  $t_0$  requires that  $g$  be locally minimized in *both* directions at  $t_0$ .

**DEFINITION 4.**  $g$  attains a *semistrict local minimum* at  $t^0 \in (b, c)$  (the line segment joining  $b$  and  $c$  excluding both end points) iff  $g$  attains a one-sided semistrict local minimum from above above and below at  $t_0$ .

Thus  $g$  attains a semistrict local minimum at  $t_0$  iff it attains a local minimum at  $t_0$  and the function eventually increases as we travel away from  $t_0$  in either direction. Note that a strict local minimum is a semistrict local minimum which in turn is a local minimum.

**THEOREM 4.** A function  $f$  defined over  $S$  is quasiconcave iff  $f$  satisfies the line segment minimum property (1) over  $S$  and the following property:

$$x^0 \in S, \quad v^T v = 1, \quad \bar{t} > 0, \quad x^0 + \bar{t}v \in S \text{ implies that}$$

$$g(t) \equiv f(x^0 + tv)$$

does not attain a semistrict local minimum at any  $t \in (0, \bar{t})$ . (2)

**COROLLARY 4.1.** A continuous from below (lower semicontinuous) or a continuous function  $f$  defined over  $S$  is quasiconcave iff  $f$  satisfies (2).

A function  $f$  is continuous from below over  $S$  iff  $\{x: f(x) \leq \alpha\}$  is a closed set for each real number  $\alpha$ .

The corollary follows from the theorem since a continuous from below or continuous function satisfies the line segment minimum property (1).

**COROLLARY 4.2.** A directionally differentiable function  $f$  defined over  $S$  is quasiconcave iff

$$\begin{aligned} x^0 \in S, v^T v = 1, \bar{t} > 0, x^0 + \bar{t}v \in S, x^0 - \bar{t}v \in S, D_v f(x^0) \\ \equiv \lim_{h \rightarrow 0} [f(x^0 + hv) - f(x^0)]/h = 0 \text{ implies } g(t) \\ \equiv f(x^0 + tv) \text{ defined for } x^0 + tv \in S \text{ does not attain a semistrict} \\ \text{local minimum at } t = 0. \end{aligned} \quad (3)$$

The corollary follows from the theorem since the directional differentiability of  $f$  implies that  $f$  is also continuous along any line segment contained in  $S$ . Also if  $g(t) \equiv f(x^0 + tv)$  attains a semistrict local minimum at  $t = 0$ , then  $g$  must attain a local minimum at  $t = 0$ . Since  $f$  is directionally differentiable, we must have  $g'(0) = D_v f(x^0) = 0$ .

**COROLLARY 4.3.** *A differentiable function  $f$  defined over an open  $S$  is quasiconcave iff*

$$x^0 \in S, v^T v = 1, v^T \nabla f(x^0) = 0 \text{ implies } g(t) \equiv f(x^0 + tv) \text{ does not attain a semistrict local minimum at } t = 0. \quad (4)$$

**COROLLARY 4.4.** *A twice continuously differentiable function  $f$  defined over an open  $S$  is quasiconcave iff*

$$x^0 \in S, v^T v = 1, v^T \nabla f(x^0) = 0 \text{ implies (i) } v^T \nabla^2 f(x^0) v < 0 \text{ or (ii) } v^T \nabla^2 f(x^0) v = 0 \text{ and } g(t) \equiv f(x^0 + tv) \text{ does not attain a semistrict local minimum at } t = 0. \quad (5)$$

Crouzeix [5] has shown that another characterization of quasiconcavity can be obtained if (5)(ii) is replaced by:  $v^T \nabla^2 f(x^0) v = 0$  and  $g(t) \equiv f(x^0 + tv)$  is a quasiconcave function of  $t$  for all  $t$  such that  $x^0 + tv \in S$ .

It is possible to rewrite (5) in a format which is more readily checked if we make use of eigenvalues and restricted eigenvalues (see Samuelson [33, pp. 373–377] or Diewert and Woodland [10, pp. 393–394]). Conditions (6) and (7) together are equivalent to (5) above.

$$x^0 \in S, \nabla f(x^0) = 0_N, \nabla^2 f(x^0) v^i = \lambda_i v^i \text{ for } i = 1, 2, \dots, N \text{ with } v^{iT} v^j = \begin{cases} 1 & \text{if } i=j, \\ 0 & \text{if } i \neq j \end{cases} \text{ implies for each } i, \text{ either (i) } \lambda_i < 0 \text{ or (ii) } \lambda_i = 0 \text{ and } g^i(t) \equiv f(x^0 + tv^i) \text{ does not attain a semistrict local minimum at } t = 0; \quad (6)$$

$$x^0 \in S, \nabla f(x^0) \neq 0_N, \nabla^2 f(x^0) v^i + \nabla f(x^0) k_i = \lambda_i v^i \text{ for } i = 1, 2, \dots, N-1 \text{ with } v^{iT} \nabla f(x^0) = 0 \text{ for } i = 1, 2, \dots, N-1 \text{ and } v^{iT} v^j = \begin{cases} 1 & \text{if } i=j, \\ 0 & \text{if } i \neq j \end{cases} \text{ implies for each } i, i = 1, 2, \dots, N-1, \text{ either (i) } \lambda_i < 0 \text{ or (ii) } \lambda_i = 0 \text{ and } g^i(t) \equiv f(x^0 + tv^i) \text{ does not attain a semistrict local minimum at } t = 0. \quad (7)$$

Note that  $0_N$  is a vector of zeroes and the  $k_i$ 's are scalars. The  $N \lambda_i$ 's appearing in (6) are the eigenvalues of  $\nabla^2 f(x^0)$ ; i.e., they are the  $N$  roots of the determinantal equation  $|\nabla^2 f(x^0) - \lambda I_N| = 0$  and the  $v^i$  are corresponding eigenvectors. The  $N-1 \lambda_i$ 's appearing in (7) are the eigenvalues of  $\nabla^2 f(x^0)$ .

in the subspace orthogonal to the nonzero gradient vector  $\nabla f(x^0)$ ; i.e., the  $\lambda_i$ 's are the  $N - 1$  roots of the determinantal equation

$$\begin{vmatrix} \nabla^2 f(x^0) - \lambda I_N, & \nabla f(x^0) \\ \nabla^T f(x^0), & 0 \end{vmatrix} = 0 \quad (8)$$

and the  $v^i$  appearing in (7) are corresponding restricted eigenvectors.

We turn now to sufficient conditions for quasiconcavity. Obviously, conditions which are sufficient to imply (5) can be obtained by dropping (ii) from (6) and (7). Call the resulting conditions (9) and (10). A condition which is equivalent to (9) is that the Hessian matrix  $\nabla^2 f(x^0)$  be negative definite and this latter condition can be translated into well known determinantal conditions (e.g., Bellman [3, p. 74]). Determinantal conditions which are *necessary and sufficient* for (10) (i.e., for  $\nabla^2 f(x^0)$  to be negative definite in the subspace orthogonal to  $\nabla f(x^0)$ ) are due to Bellman [3, pp. 77-78], but are much less well known. Finally, the following determinantal conditions are *sufficient* to imply (10) (and hence also (7)):

$$x^0 \in S \text{ implies } (-1)^1 B(1, N+1) > 0, (-1)^2 B(1, 2, N+1) > 0, \dots, (-1)^N B(1, 2, \dots, N, N+1) > 0, \text{ where}$$

$$B \equiv \begin{bmatrix} \nabla^2 f(x^0), & \nabla f(x^0) \\ \nabla^T f(x^0), & 0 \end{bmatrix}$$

and  $B(i_1, i_2, \dots, i_n)$  denotes the determinant of the submatrix of  $B$  consisting of rows and columns  $i_1, i_2, \dots, i_n$ . (11)

The restriction  $\nabla f(x^0) \neq 0_N$  does not appear in (11) since if  $\nabla f(x^0) = 0_N$ , then all of the determinants of the submatrices of the bordered Hessian matrix  $B$  which include the last column (of zeroes) will be zero. Conditions (11) are the usual (cf. Arrow and Enthoven [1, p. 782]) sufficient conditions for  $f$  to be quasiconcave over  $S$ . Note that the first inequality in (11) requires that  $\partial f(x^0)/\partial x_1 \neq 0$ . Note also how much stronger conditions (11) are than the necessary and sufficient conditions (5).

It is obvious that the following condition (compare it with (5)) is a necessary condition for  $f$  to be quasiconcave (but it is not in general sufficient):

$$x^0 \in S, v^T v = 1, v^T \nabla f(x^0) = 0 \text{ implies } v^T \nabla^2 f(x^0) v \leq 0. \quad (12)$$

It can be shown that condition (12) is equivalent to conditions (13) and (14), where (13) is (6), except that (ii) is dropped and (i) is replaced by the condition  $\lambda_i \leq 0$  (i.e., the eigenvalues of  $\nabla^2 f(x^0)$  are nonpositive), and (14) is (7), except that (ii) is dropped and (i) is replaced by  $\lambda_i \leq 0$  (i.e., the

restricted eigenvalues of  $\nabla^2 f(x^0)$  in the subspace orthogonal to  $\nabla f(x^0) \neq 0_N$  are nonpositive).

Necessary and sufficient conditions for (13) (i.e., that  $\nabla^2 f(x^0)$  be negative semidefinite) are the following determinantal conditions (e.g., Debreu [7]):

$$\begin{aligned} x^0 \in S, \nabla f(x^0) = 0_N, A \equiv \nabla^2 f(x^0) \text{ implies } (-1)^i A(i_1) \geq 0, \\ i_1 = 1, 2, \dots, N; (-1)^j A(i_1, i_2) > 0, 1 \leq i_1 < i_2 \leq N; \dots; \\ (-1)^N A(i_1, i_2, \dots, i_N) \geq 0, 1 \leq i_1 < i_2 < \dots < i_N \leq N, \end{aligned} \quad (15)$$

where  $A(i_1, i_2, \dots, i_n)$  denotes the determinant of the submatrix of  $A$  consisting of rows and columns  $i_1, i_2, \dots, i_n$ . Note that the determinantal conditions (15) must be checked for every distinct subset of the rows and columns (where the column indices equal the row indices) of  $\nabla^2 f(x^0)$ . Many economists do not seem to be aware that conditions (15) are necessary and sufficient for the symmetric matrix  $A$  to be negative semidefinite, and mistakes are repeatedly made with respect to this point (e.g., Proposition I in [15, p. 597] is false).

Necessary and sufficient conditions for (14) (i.e., that  $\nabla^2 f(x^0)$  be negative semidefinite in the subspace orthogonal to  $\nabla f(x^0) \neq 0_N$ ) were derived by Debreu (5) and are equivalent to the following conditions:

$$\begin{aligned} x^0 \in S, \nabla f(x^0) \neq 0_N \text{ implies } (-1)^2 B(i_1, i_2, N+1) \geq 0, \\ 1 \leq i_1 < i_2 \leq N; \\ (-1)^3 B(i_1, i_2, i_3, N+1) \geq 0, 1 \leq i_1 < i_2 < i_3 \leq N; \dots; \\ (-1)^N B(i_1, i_2, \dots, i_N, N+1) \geq 0, 1 \leq i_1 < i_2 < \dots < i_N \leq N. \end{aligned} \quad (16)$$

The bordered Hessian matrix  $B$  and  $B(i_1, i_2, \dots, i_n)$  are defined in (11) above. Note that the conditions  $(-1)^i B(i_1, N+1) = (-1)^i [\partial f(x^0)/\partial x_1]^i \geq 0$  are automatically satisfied and need not be checked.

Finally, to complete our discussion of determinantal conditions appearing in the literature on quasiconcavity, we note that the following conditions are necessary for conditions (16):

$$\begin{aligned} x^0 \in S \text{ implies } (-1)^1 B(1, N+1) \geq 0, (-1)^2 B(1, 2, N+1) \geq 0, \dots, \\ (-1)^N B(1, 2, \dots, N, N+1) \geq 0. \end{aligned} \quad (17)$$

Conditions (17) are the usual necessary conditions for  $f$  to be a twice continuously differentiable quasiconcave function (cf. [1, p. 781; 27, p. 307; 15, p. 600]).<sup>3</sup>

It is natural to ask what additional conditions can be added to the

<sup>3</sup> Newman [27, p. 307] and Ginsberg [15, p. 600] call a function  $f$  which satisfies conditions (17) a weakly quasiconcave function. This does not appear to be very appropriate terminology, since if  $g(x_2, x_3, \dots, x_N)$  is any twice continuously differentiable function of  $N-1$  variables, then  $f(x_1, x_2, \dots, x_N) \equiv g(x_2, \dots, x_N)$  will be a weakly quasiconcave function of the  $N$  variables  $x_1, x_2, \dots, x_N$ .

necessary condition for quasiconcavity (12) in order to obtain sufficiency. Katzner [21, p. 211] shows that (12) is necessary and sufficient for quasiconcavity for the class of functions which have positive first order partial derivatives. The following theorem generalizes Katzner's result.

**THEOREM 5.** *Let  $f$  be a twice continuously differentiable function of  $N$  variables defined over the open convex set  $S$  which satisfies the assumption  $\nabla f(x^0) \neq 0_N$  for every  $x^0 \in S$ . Then  $f$  is quasiconcave if and only if condition (12) is satisfied.*

Thus if the twice continuously differentiable function  $f$  defined over the open convex set  $S$  has a nonzero gradient vector over  $S$ , then the determinantal conditions (16) are necessary and sufficient for quasiconcavity.

Crouzeix and Ferland [6] and Ferland [13] have recently obtained additional criteria for quasiconcavity in the twice differentiable case.

### 3. SEMISTRRICT QUASICONCAVITY

**DEFINITION 5** (Elkin [14], Ginsberg [15]<sup>4</sup>).  *$f$  is a semistrictly quasiconcave function defined over  $S$  iff*

$$x^1 \in S, \quad x^2 \in S, \quad f(x^2) > f(x^1), \quad 0 < \lambda < 1$$

implies

$$f(\lambda x^1 + (1 - \lambda) x^2) > f(x^1). \quad (18)$$

Semistrict quasiconcavity is important, mainly because of the following theorem.

**THEOREM 6** (Martos [25, p. 244], Mangasarian [23, p. 284]). *Let  $f$  be a semistrictly quasiconcave function defined over  $S$  and suppose  $f$  attains a local maximum at  $x^0 \in S$ . Then  $f$  attains a global maximum over  $S$  at  $x^0$ . Moreover, if  $f$  is also continuous from above, then the set of global maximizers is a convex set.*

<sup>4</sup> Mangasarian [23] Ponstein [29], and Thompson and Parke [35] define  $f$  to be strictly quasiconcave iff  $f$  satisfies (18). However, Karamardian [20] showed that such a strictly quasiconcave function was not necessarily quasiconcave unless the function was also continuous from above (i.e., unless the upper level sets  $L(u) = \{x: f(x) \geq u, x \in S\}$  are closed for each  $u$ ); e.g., consider  $f(t) = \{1 \text{ if } 0 \leq t < 1 \text{ or } 1 < t \leq \infty, 0 \text{ if } t = 1\}$ . Martos [25] called functions which satisfied (18) "explicitly quasiconcave" but he also assumed that  $f$  was continuous in which case (18) implies that  $f$  is also quasiconcave. Hanson [16] calls a function which satisfies (18) "functionally concave."

The following characterization of semistrict quasiconcavity is analogous to our third characterization of quasiconcavity.

**THEOREM 7.** *A continuous from above function  $f$  defined over  $S$  is semistrictly quasiconcave iff  $f$  satisfies the line segment minimum property (1) and (19) below:*

$$\begin{aligned} x^0 \in S, v^T v = 1, \bar{t} > 0, x^0 + \bar{t}v \in S, x^0 - \bar{t}v \in S, g(t) \\ \equiv f(x^0 + tv), g \text{ attains a local minimum at } 0 \text{ implies } g \\ \text{does not attain a one sided semistrict local minimum at } 0. \end{aligned} \quad (19)$$

**COROLLARY 7.1.** *A continuous function  $f$  defined over  $S$  is semistrictly quasiconcave iff  $f$  satisfies (19).*

**COROLLARY 7.2.** *A directionally differentiable function  $f$  defined over  $S$  is semistrictly quasiconcave iff*

$$\begin{aligned} x^0 \in S, v^T v = 1, \bar{t} > 0, x^0 + \bar{t}v \in S, x^0 - \bar{t}v \in S, D_v f(x^0) = 0 \\ \text{implies } g(t) \equiv f(x^0 + tv) \text{ defined for } x^0 + tv \in S \text{ does not} \\ \text{attain a local minimum at } t = 0 \text{ that is also a one-sided} \\ \text{semistrict local minimum.} \end{aligned} \quad (20)$$

**COROLLARY 7.3.** *A twice continuously differentiable function  $f$  defined over an open  $S$  is semistrictly quasiconcave iff*

$$x^0 \in S, \quad v^T v = 1, \quad v^T \nabla f(x^0) = 0$$

*implies*

$$\begin{aligned} \text{(i) } v^T \nabla^2 f(x^0) v < 0 \text{ or (ii) } v^T \nabla^2 f(x^0) v = 0 \text{ and } g(t) \equiv f(x^0 + tv) \\ \text{does not attain a local minimum at } t = 0 \text{ that is also a} \\ \text{one sided semistrict local minimum.} \end{aligned} \quad (21)$$

#### 4. STRICT QUASICONCAVITY

**DEFINITION 6** (Ponstein [29, p. 118]<sup>5</sup>).  *$f$  is a strictly quasiconcave function defined over  $S$  iff*

$$x^1 \in S, \quad x^2 \in S, \quad x^1 \neq x^2, \quad f(x^1) \leq f(x^2), \quad 0 < \lambda < 1$$

<sup>5</sup> Ponstein uses the term “unnamed concavity” in place of strict quasiconcavity while Thompson and Parke [35] use the term “ $X$  concavity.” However, the term strict quasiconcavity seems to be consistent with the terminology used in economics; e.g. Katzner [21, p. 210] or Ginsberg [15, p. 598].

implies

$$f(x^1) < f(\lambda x^1 + (1 - \lambda) x^2). \quad (22)$$

It is easy to see (cf. Ponstein [29, p. 118] and Thompson and Parke [35, p. 309]) that a strictly quasiconcave function is semistrictly quasiconcave and also quasiconcave. Moreover, the indifference sets (level curves)  $I(u) \equiv \{x: f(x) = u, x \in S\}$  of a strictly quasiconcave function do not contain any straight line segments.

The following theorem is a slight extension of Theorem 6.

**THEOREM 8.** *If  $f$  is a strictly quasiconcave function defined over  $S$  and  $f$  attains a local maximum at  $x^0 \in S$ , then  $x^0$  is the unique global maximizer of  $f$  over  $S$ .*

The following characterization of strict quasiconcavity is analogous to our third characterization of quasiconcavity.

**THEOREM 9.** *A function  $f$  defined over  $S$  is strictly quasiconcave iff  $f$  has the line segment minimum property (1) and (23) below:*

$$\begin{aligned} x^0 \in S, v^T v = 1, \bar{t} > 0, x^0 + \bar{t}v \in S, x^0 - \bar{t}v \in S \text{ implies } g(t) \\ \equiv f(x^0 + tv) \text{ does not attain a local minimum at } t = 0. \end{aligned} \quad (23)$$

**COROLLARY 9.1.** *A continuous (or continuous from below) function  $f$  defined over  $S$  is strictly quasiconcave iff  $f$  satisfies (23).*

**COROLLARY 9.2.** *A directionally differentiable function  $f$  defined over  $S$  is strictly quasiconcave iff the following condition holds:*

$$\begin{aligned} x^0 \in S, v^T v = 1, \bar{t} > 0, x^0 + \bar{t}v \in S, x^0 - \bar{t}v \in S, D_v f(x^0) = 0 \\ \text{implies } g(t) \equiv f(x^0 + tv) \text{ does not attain a local} \\ \text{minimum at } t = 0. \end{aligned} \quad (24)$$

**COROLLARY 9.3.** *A twice continuously differentiable function  $f$  defined over an open  $S$  is strictly quasiconcave iff*

$$\begin{aligned} x^0 \in S, v^T v = 1, v^T \nabla f(x^0) = 0 \text{ implies (i) } v^T \nabla^2 f(x^0) v < 0 \\ \text{or (ii) } v^T \nabla^2 f(x^0) v = 0 \text{ and } g(t) \equiv f(x^0 + tv) \text{ does not} \\ \text{attain a local minimum at } t = 0. \end{aligned} \quad (25)$$

### 5. PSEUDOCONCAVITY

**DEFINITION 7** (Mangasarian<sup>6</sup> [23, p. 281]). *A directionally differentiable function  $f$  defined over  $S$  is pseudoconcave iff  $f$  satisfies:*

$$x^0 \in S, v^T v = 1, t > 0, D_v f(x^0) \leq 0 \text{ implies } f(x^0 + tv) \leq f(x^0). \quad (26)$$

Mangasarian [24, pp. 114–145] shows that pseudoconcave functions are also quasiconcave and semistrictly quasiconcave and have the following additional useful properties: (i) if  $x^0$  is a local maximizer for the pseudoconcave function  $f$ , then it is also a global maximizer over  $S$  and (ii) if  $f$  is differentiable,  $S$  is an open convex set and  $\nabla f(x^0) = 0_N$ , then  $x^0$  is a global maximizer for  $f$ . Ponstein [29] shows, however, that pseudoconcave functions are not necessarily strictly quasiconcave.

**THEOREM 10.** *Let  $f$  be a directionally differentiable function defined over  $S$ . Then  $f$  is pseudoconcave over  $S$  iff*

$$\begin{aligned} x^0 \in S, v^T v = 1, \bar{t} > 0, x^0 + \bar{t}v \in S, D_v f(x^0) = 0 \text{ implies} \\ g(t) \equiv f(x^0 + tv) \text{ (defined for all } t \text{ such that)} \\ x^0 + tv \in S \text{ attains a local maximum at } t = 0. \end{aligned} \quad (27)$$

**COROLLARY 10.1.** *A twice continuously differentiable function  $f$  defined over an open  $S$  is pseudoconcave iff*

$$\begin{aligned} x^0 \in S, v^T v = 1, v^T \nabla f(x^0) = 0 \text{ implies (i) } v^T \nabla^2 f(x^0) v < 0 \\ \text{or (ii) } v^T \nabla^2 f(x^0) v = 0 \text{ and } g(t) \equiv f(x^0 + tv) \text{ attains} \\ \text{a local maximum at } t = 0. \end{aligned} \quad (28)$$

Thus a one dimensional function having the property that all of its critical points are local maximums is pseudoconcave, and this fact can serve to characterize the class of pseudoconcave functions.

Additional conditions for pseudoconcavity in the twice differentiable case are given in Crouzeix and Ferland [6]. Finally, using a result due to Ferland [12, p. 300], it can be shown that the word “quasiconcave” in Theorem 5 above can be replaced by the word “pseudoconcave.”

<sup>6</sup> Actually Mangasarian assumed that  $S$  is an open convex set and that  $f$  is differentiable, in which case (26) becomes:  $x^0 \in S, x^1 \in S, (x^1 - x^0)^T \nabla f(x^0) \leq 0$  implies  $f(x^1) \leq f(x^0)$ . Thompson and Parke [35, p. 311] and Diewert [9] provide more general definitions of pseudoconcavity for the case where  $f$  is not necessarily differentiable.

## 6. STRICT PSEUDOCONCAVITY

**DEFINITION 8** (Ponstein<sup>7</sup> [29, p. 117]). Let  $f$  be a directionally differentiable function defined over  $S$ . Then  $f$  is *strictly pseudoconcave* over  $S$  iff

$$x^0 \in S, v^T v = 1, t > 0, x^0 + tv \in S, D_v f(x^0) \leq 0 \text{ implies} \\ f(x^0 + tv) < f(x^0). \quad (29)$$

Comparison of Definitions 7 and 8 shows that strict pseudoconcavity implies pseudoconcavity. Ponstein [29, p. 118] shows that pseudoconcavity plus strict quasiconcavity implies strict pseudoconcavity.

**THEOREM 11.** Let  $f$  be a directionally differentiable function defined over  $S$ . Then  $f$  is strictly pseudoconcave over  $S$  iff

$$x^0 \in S, v^T v = 1, t > 0, x^0 + tv \in S, D_v f(x^0) = 0 \text{ implies} \\ g(t) \equiv f(x^0 + tv) \text{ attains a strict local maximum} \\ \text{at } t = 0. \quad (30)$$

**COROLLARY 11.1.** A twice continuously differentiable function  $f$  defined over an open  $S$  is strictly pseudoconcave iff

$$x^0 \in S, v^T v = 1, v^T \nabla f(x^0) = 0 \text{ implies (i) } v^T \nabla^2 f(x^0) v < 0 \\ \text{or (ii) } v^T \nabla^2 f(x^0) v = 0 \text{ and } g(t) \equiv f(x^0 + tv) \text{ attains} \\ \text{a strict local maximum at } t = 0. \quad (31)$$

Comparison of properties (24) and (30) shows that a strictly pseudoconcave function is also strictly quasiconcave (a result which can be found in Ponstein [29]). Both of these types of functions share the following properties: (i) every local maximum is a unique global maximum and (ii) the indifference sets (level curves)  $I(u) \equiv \{x: f(x) = u, x \in S\}$  do not contain any line segments. A strictly pseudoconcave function has the additional property that  $D_v f(x^0) = 0$  for every feasible direction  $v$  implies  $x^0$  is the unique strict global maximizer of  $f$  over  $S$ . Thus a strictly pseudoconcave function of one variable cannot have a point of inflection whereas a strictly quasiconcave function can.

<sup>7</sup> Ponstein's definition assumed that  $f$  had continuous first order partial derivatives over a convex set  $S$  and his definition was:  $x^1 \in S, x^2 \in S, x^1 \neq x^2, f(x^2) \geq f(x^1)$  implies  $(x^2 - x^1)^T \nabla f(x^1) > 0$ . Definition 8 is a straightforward generalization of the contrapositive of Ponstein's definition.

## 7. STRONG QUASICONCAVITY OR STRONG PSEUDOCONCAVITY

**DEFINITION 9.** Let  $f$  be a directionally differentiable function defined over  $S$ . Then  $f$  is *strongly quasiconcave*<sup>8</sup> over  $S$  iff  $f$  is strictly quasiconcave over  $S$  and in addition has the following property:

$$\begin{aligned} x^0 \in S, v^T v = 1, \bar{t} > 0, x^0 + \bar{t}v \in S, D_v f(x^0) = 0 \text{ implies there} \\ \text{exist } \varepsilon \text{ and } \alpha \text{ such that } 0 < \varepsilon < \bar{t}, 0 < \alpha \text{ and} \\ f(x^0 + tv) < f(x^0) - \alpha t^2 \text{ for } t \in [0, \varepsilon]. \end{aligned} \quad (32)$$

The above definition implies that a strongly quasiconcave function is a strictly quasiconcave function that has the additional property that if the directional derivative of the function is zero in a certain direction  $v$  at a point  $x^0 \in S$  then the function decreases quadratically in a neighbourhood of  $x^0$  along  $v$ .

Note that it is possible to characterize strong quasiconcavity using (32) and (19). However, it is simpler to characterize strongly quasiconcave functions using the concept of a strong local maximum

**DEFINITION 10.** A function of one variable  $g$  defined over a set  $S$  attains a *strong local maximum* at  $t_0 \in S$  iff there exist  $\alpha > 0$  and  $\varepsilon > 0$  such that  $g(t) \leq g(t_0) - \alpha(t - t_0)^2$  for  $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$  and  $t \in S$ .

**THEOREM 12.** Let  $f$  be a directionally differentiable function defined over  $S$ . Then  $f$  is strongly quasiconcave over  $S$  iff  $f$  satisfies:

$$\begin{aligned} x^0 \in S, v^T v = 1, \bar{t} > 0, x^0 + \bar{t}v \in S, D_v f(x^0) = 0 \text{ implies } g(t) \\ \equiv f(x^0 + tv) \text{ attains a strong local maximum at } t = 0. \end{aligned} \quad (33)$$

**COROLLARY 12.1.** A twice continuously differentiable function  $f$  defined over an open  $S$  is strongly quasiconcave iff

$$x^0 \in S, v^T v = 1, v^T \nabla f(x^0) = 0 \text{ implies } v^T \nabla^2 f(x^0) v < 0. \quad (34)$$

If  $\nabla f(x^0) = 0_N$ , then (34) is equivalent to negative definiteness of the Hessian matrix  $\nabla^2 f(x^0)$  (recall conditions (9)), while if  $\nabla f(x^0) \neq 0_N$ , then (34) is equivalent to negative definiteness of  $\nabla^2 f(x^0)$  in the subspace orthogonal to the gradient vector  $\nabla f(x^0)$  (recall conditions (10)). Thus

<sup>8</sup> In a corollary to Theorem 12 below, we show that our definition of strong quasiconcavity is equivalent to the definition used by Newman [27, p. 307] and Ginsberg [15, p. 600], assuming that  $f$  is twice continuously differentiable and is defined over an open convex set. In place of the term "strong quasiconcavity" McFadden [26, p. 403] uses the term "differential strict quasiconcavity." Diewert [9] provides a definition of strong quasiconcavity that does not require the existence of directional derivatives.

strong quasiconcavity in the twice differentiable case can be characterized by conditions (9) and (10), conditions which occur frequently in the economics literature.

Comparison of Theorems 11 and 12 shows that strong quasiconcavity implies strict pseudoconcavity (the reverse implication is not true). Hence we could also call the class of strongly quasiconcave functions the class of strongly pseudoconcave functions.

## 8. CONCAVITY

**DEFINITION 11.**  $f$  is a concave<sup>9</sup> function over  $S$  iff

$$\begin{aligned} x^1 \in S, x^2 \in S, 0 \leq \lambda \leq 1 \text{ implies } f(\lambda x^1 + (1 - \lambda)x^2) \\ \geq f(x^1) + (1 - \lambda)f(x^2). \end{aligned} \quad (35)$$

Ponstein [29] shows that a differentiable concave function is necessarily quasiconcave, semistrictly quasiconcave and pseudoconcave, but it need not be strictly quasiconcave, strictly pseudoconcave or strongly quasiconcave.

**THEOREM 13** (Fenchel [11, p. 81], Roberts and Varberg [32, p. 12]). *Let  $f$  be a directionally differentiable function defined over  $S$ . Then  $f$  is concave over  $S$  iff*

$$\begin{aligned} x^0 \in S, v^T v = 1, t > 0, x^0 + tv \in S \text{ implies} \\ f(x^0 + tv) \leq f(x^0) + t D_v f(x^0). \end{aligned} \quad (36)$$

**COROLLARY 13.1** (Mangasarian [24, p. 84]). *Let  $f$  be a differentiable function defined over an open  $S$ . Then  $f$  is concave over  $S$  iff*

$$x^0 \in S, x^1 \in S \text{ implies } f(x^1) \leq f(x^0) + (x^1 - x^0)^T \nabla f(x^0). \quad (37)$$

Thus every linear approximation (tangent line) to a concave function is coincident with or lies above the function. Fenchel [11, pp. 87–89] and Mangasarian [24, p. 89] also show that if  $f$  is a twice continuously differentiable function defined over an open convex set  $S$ , then the following condition is necessary and sufficient for  $f$  to be concave.

$$x^0 \in S, v^T v = 1 \text{ implies } v^T \nabla^2 f(x^0) v \leq 0. \quad (38)$$

In other words, negative semidefiniteness of the Hessian matrix of  $f$  over  $S$  is necessary and sufficient for  $f$  to be concave over  $S$ .

<sup>9</sup> See the historical references in Fenchel [11] or Rockafellar [30].

The following two theorems provide alternative local characterizations of concavity, without necessarily assuming twice differentiability.

**THEOREM 14.** *f is concave over S iff*

$$\begin{aligned} x^0 \in S, v^T v = 1, t > 0, x^0 + tv \in S, \alpha \in R^1, g(t) \equiv f(x^0 + tv) \\ \text{implies } h(t) \equiv g(t) - \alpha t \text{ is a quasiconcave function of} \\ \text{one variable where } h(t) \text{ is defined for all } t \text{ such that} \\ x^0 + tv \in S, \end{aligned} \quad (39)$$

*or equivalently (using Theorem 4) iff*

$$\begin{aligned} x^0 \in S, v^T v = 1, \bar{t} > 0, x^0 + \bar{t}v \in S, \alpha \in R^1, h(t) \equiv f(x^0 + tv) - \alpha t \\ (\text{defined for all } t \in [0, \bar{t}]) \text{ implies } h \text{ has the line segment} \\ \text{minimum property (1) and } h \text{ does not attain a semistrict} \\ \text{local minimum at any } t_0 \in (0, \bar{t}). \end{aligned} \quad (40)$$

**THEOREM 15.** *Let f be a directionally differentiable function defined over S. Then f is concave over S iff*

$$\begin{aligned} x^0 \in S, v^T v = 1, t > 0, x^0 + tv \in S, \alpha \in R^1, g(t) \equiv f(x^0 + tv) \\ \text{implies } h(t) \equiv g(t) - \alpha t \text{ is a pseudoconcave function} \\ \text{of one variable,} \end{aligned} \quad (41)$$

*or equivalently (using Theorem 10) iff*

$$\begin{aligned} x^0 \in S, v^T v = 1, \bar{t} > 0, x^0 + \bar{t}v \in S, g(t) \equiv f(x^0 + tv), \\ t_0 \in [0, \bar{t}], g'(t_0) \equiv \alpha \text{ implies } h(t) \equiv g(t) - \alpha t \\ \text{attains a local maximum over } [0, \bar{t}] \text{ at } t_0. \end{aligned} \quad (42)$$

**COROLLARY 15.1.** *Let f be a differentiable function defined over an open S. Then f is concave over S iff*

$$\begin{aligned} x^0 \in S, v^T v = 1 \text{ implies } h(t) \equiv f(x^0 + tv) - tv^T \nabla f(x^0) \\ \text{attains a local maximum at } t = 0. \end{aligned} \quad (43)$$

**COROLLARY 15.2.** *Let f be a twice continuously differentiable function defined over an open S. Then f is concave over S iff*

$$\begin{aligned} x^0 \in S, v^T v = 1 \text{ implies (i) } v^T \nabla^2 f(x^0) v < 0 \text{ or} \\ (\text{ii) } v^T \nabla^2 f(x^0) v = 0 \text{ and } h(t) \equiv f(x^0 + tv) - tv^T \nabla f(x^0) \\ \text{attains a local maximum at } t = 0. \end{aligned} \quad (44)$$

The characterization of concavity given by Corollary 15.2 is equivalent to (38), but it is not quite as simple as (38). However, the characterization of

concavity in the once differentiable case given by Corollary 15.1 above involves only the local properties of the function and hence will often be easier to check than the characterization given by Corollary 13.1.

### 9. STRICT CONCAVITY

**DEFINITION 12** (Fenchel [11, p. 57]).  $f$  is *strictly concave* over  $S$  iff

$$\begin{aligned} x^1 \in S, x^2 \in S, x^1 \neq x^2, 0 < \lambda < 1 \text{ implies} \\ f(\lambda x^1 + (1 - \lambda) x^2) > \lambda f(x^1) + (1 - \lambda) f(x^2). \end{aligned} \quad (45)$$

Ponstein [29] shows that a differentiable strictly concave function is quasiconcave, semistrictly quasiconcave, strictly quasiconcave, pseudoconcave and strictly pseudoconcave. However, a strictly concave function need not be strongly quasiconcave.

**THEOREM 16** (Roberts and Varberg [32, p. 14]). *Let  $f$  be a directionally differentiable function defined over  $S$ . Then  $f$  is strictly concave over  $S$  iff*

$$\begin{aligned} x^0 \in S, v^T v = 1, t > 0, x^0 + tv \in S \text{ implies} \\ f(x^0 + tv) < f(x^0) + t D_v f(x^0). \end{aligned} \quad (46)$$

**COROLLARY 16.1** (Ponstein [29, p. 115], Mangasarian [24, p. 87]). *Let  $f$  be a differentiable function defined over an open  $S$ . Then  $f$  is strictly concave over  $S$  iff*

$$\begin{aligned} x^0 \in S, x^1 \in S, x^0 \neq x^1 \text{ implies} \\ f(x^1) < f(x^0) + (x^1 - x^0)^T \nabla f(x^0). \end{aligned} \quad (47)$$

Thus every linear approximation (tangent line) to a strictly concave function lies above the function (except at the point of tangency). Fenchel [11, pp. 73–88] and Bernstein and Toupin [4, p. 72] show that if  $f$  is a twice continuously differentiable function defined over an open convex set  $S$ , then the following condition is necessary and sufficient for  $f$  to be strictly concave:

$$\begin{aligned} x^0 \in S, v^T v = 1, \bar{t} > 0, x^0 + \bar{t}v \in S \text{ implies} \\ (\text{i}) v^T \nabla^2 f(x^0) v \leq 0 \text{ and } (\text{ii}) v^T \nabla^2 f(x^0 + \bar{t}v) v \neq 0 \\ \text{for almost all } \bar{t} \in [0, \bar{t}]. \end{aligned} \quad (48)$$

The following two theorems provide alternative local characterizations of strict concavity, without necessarily assuming twice differentiability.

**THEOREM 17.** *f is strictly concave over S iff*

$$\begin{aligned} x^0 \in S, v^T v = 1, t > 0, x^0 + tv \in S, \alpha \in R^1, g(t) \equiv f(x^0 + tv) \\ \text{implies } h(t) \equiv g(t) - \alpha t \text{ is a strictly quasiconcave} \\ \text{function of one variable,} \end{aligned} \quad (49)$$

*or equivalently (using Theorem 9) iff*

$$\begin{aligned} x^0 \in S, v^T v = 1, \bar{t} > 0, x^0 + \bar{t}v \in S, \alpha \in R^1, h(t) \equiv f(x^0 + tv) - \alpha t \\ \text{implies } h \text{ has the line segment minimum property (1) over} \\ [0, \bar{t}] \text{ and } h \text{ does not attain a local minimum at any } t_0 \in (0, \bar{t}). \end{aligned} \quad (50)$$

**THEOREM 18.** *Let f be a directionally differentiable function defined over S. Then f is strictly concave over S iff*

$$\begin{aligned} x^0 \in S, v^T v = 1, t > 0, x^0 + tv \in S, \alpha \in R^1, g(t) \equiv f(x^0 + tv) \text{ implies} \\ h(t) \equiv g(t) - \alpha t \text{ is a strictly pseudoconcave function,} \end{aligned} \quad (51)$$

*or equivalently (using Theorem 11) iff*

$$\begin{aligned} x^0 \in S, v^T v = 1, \bar{t} > 0, x^0 + \bar{t}v \in S, g(t) \equiv f(x^0 + tv), \\ t_0 \in [0, \bar{t}], g'(t_0) \equiv \alpha \text{ implies } h(t) \equiv g(t) - \alpha t \\ \text{attains a strict local maximum over } [0, \bar{t}] \text{ at } t_0. \end{aligned} \quad (52)$$

**COROLLARY 18.1.** *Let f be a differentiable function defined over an open S. Then f is strictly concave over S iff*

$$\begin{aligned} x^0 \in S, v^T v = 1 \text{ implies } h(t) \equiv f(x^0 + tv) - tv^T \nabla f(x^0) \\ \text{attains a strict local maximum at } t = 0. \end{aligned} \quad (53)$$

**COROLLARY 18.2.** *Let f be a twice continuously differentiable function defined over an open S. Then f is strictly concave over S iff*

$$\begin{aligned} x^0 \in S, v^T v = 1 \text{ implies (i) } v^T \nabla^2 f(x^0) v < 0 \text{ or (ii) } v^T \nabla^2 f(x^0) v = 0 \\ \text{and } h(t) = f(x^0 + tv) - tv^T \nabla f(x^0) \text{ attains a strict local max at} \\ t = 0. \end{aligned} \quad (54)$$

We note that (54) is equivalent to (48).

## 10. STRONG CONCAVITY

**DEFINITION 13** (Poljak [28, p. 73]. *f is strongly concave over S iff there exists  $\alpha > 0$  such that for every  $x^1 \in S, x^2 \in S, 0 \leq \lambda \leq 1$ ,*

$$\begin{aligned} f(\lambda x^1 + (1 - \lambda) x^2) \\ \geq \lambda f(x^1) + (1 - \lambda) f(x^2) + \lambda(1 - \lambda) \alpha(x^1 - x^2)^T (x^1 - x^2). \end{aligned} \quad (55)$$

It is obvious from the above definition that a strongly concave function is strictly concave. The following theorems provide alternative characterizations of strong concavity.

**THEOREM 19** (Rockafellar [31, p. 75]). *f is strongly concave over S iff there exists  $\alpha > 0$  such that g defined by  $g(x) \equiv f(x) + \alpha x^T x$  is a concave function over S.*

**THEOREM 20** (Poljak [28, p. 73]). *Let f be a differentiable function defined over an open S. Then f is strongly concave over S iff there exists an  $\alpha > 0$  such that for every  $x^1 \in S$ ,  $x^2 \in S$ ,*

$$f(x^2) \leq f(x^1) + (x^2 - x^1)^T \nabla f(x^1) - \alpha(x^2 - x^1)^T (x^2 - x^1). \quad (56)$$

**THEOREM 21** (Poljak [28, p. 73]). *Let f be a twice continuously differentiable function defined over an open S. Then f is strongly concave over S iff there exists an  $\alpha > 0$  such that*

$$x^0 \in S, v^T v = 1 \text{ implies } v^T \nabla^2 f(x^0) v \leq -2\alpha. \quad (57)$$

Note that (57) implies the following condition:

$$x^0 \in S, v^T v = 1 \text{ implies } v^T \nabla^2 f(x^0) v < 0, \quad (58)$$

i.e., that the Hessian matrix of f be negative definite over S. Ginsberg [15, p. 597], Jorgenson and Lau [17, p. 40] and Lau [22, p. 135] use property (58) in order to define strong concavity in the twice differentiable case. We note that (58) is equivalent to (57) if f is a twice continuously differentiable function defined over a compact convex set S with a nonempty interior. Finally, note that (58) implies (54) and (48).

It can be shown that if a function is strongly concave, then it also satisfies each of the other eight types of quasiconcavity and concavity.

## 11. CONCLUSION

We have provided definitions and alternative characterizations of nine kinds of quasiconcavity and concavity that occur frequently in applied work in economics and management science.<sup>10</sup> In particular, we have provided characterizations involving the local behaviour of a function along line segments in the general case as well as in the once and twice differentiable case. These alternative characterizations should prove to be useful in applied work.

<sup>10</sup> See Diewert [8] for specific applications of the nine kinds of concavity in economics.

We conclude by noting that there are additional types of generalized concavity that we have not studied. For example, the class of quasiconcave functions that can be transformed by means of a monotonically increasing function of one variable into a concave function defines yet another type of quasiconcavity. This class of functions was studied by Fenchel [11] and more recently by Kannai [18, 19]. Avriel [2, pp. 160–172] and Zang [36] also study other classes of concave transformable functions and give extensive references to the literature.

#### APPENDIX: PROOFS OF THEOREMS

*Proof of Theorem 3.* The proof of this theorem follows readily from some monotonicity properties of quasiconcave functions developed by Stoer and Witzgall [34, p. 172].

*Proof of Theorem 4.* *Quasiconcavity implies (1) and (2).* Property (1) follows from Theorem 3. Hence we need only show that (1) and not (2) imply that  $f$  is not quasiconcave. This is straightforward.

*Properties (1) and (2) imply  $f$  is quasiconcave.* Let  $x^0 \in S$ ,  $x^1 \in S$ ,  $x^0 \neq x^1$  and  $0 < \lambda^* < 1$ . Suppose that  $f(\lambda^*x^0 + (1 - \lambda^*)x^1) < \min\{f(x^0), f(x^1)\}$ . Define  $t_1 \equiv [(x^1 - x^0)^T(x^1 - x^0)]^{1/2} > 0$ ,  $v \equiv (x^1 - x^0)/t_1$ , and  $g(t) \equiv f(x^0 + tv)$  for  $t \in [0, t_1]$ . By (1),

$$\begin{aligned} \inf_t \{g(t): t \in [0, t_1]\} &= \min_t \{g(t): t \in [0, t_1]\} \\ &= \min_{\lambda} \{f(\lambda x^0 + (1 - \lambda)x^1): \lambda \in [0, 1]\} \\ &\leq f(\lambda^*x^0 + (1 - \lambda^*)x^1) \quad \text{since } \lambda^* \text{ is feasible} \\ &< \min\{f(x^0), f(x^1)\} \quad \text{by our supposition} \\ &= \min\{g(0), g(t_1)\} \quad \text{by the definition of } g. \end{aligned}$$

The above inequality shows that  $g$  attains a semistrict local minimum at some  $t_0 \in (0, t_1)$  which contradicts (2). Thus our *supposition* is false and  $f$  is quasiconcave over  $S$ .

*Proof of Theorem 5.* A rather lengthy proof is available from the first author upon request.

*Proof of Theorem 7.* *Semistrict quasiconcavity implies (1) and (19).* Semistrict quasiconcavity implies quasiconcavity by a result of Karamardian [20] and hence by Theorem 3, (1) holds. Hence we need only show that (1) and not (19) imply that  $f$  is not semistrictly quasiconcave. This is straightforward.

*Properties (1) and (19) imply  $f$  is semistrictly quasiconcave.* Let  $x^1 \in S$ ,  $x^2 \in S$  and  $f(x^1) < f(x^2)$ . Suppose there exists  $\lambda^*$  such that  $0 < \lambda^* < 1$  and  $f(x^1) \geq f(\lambda^*x^1 + (1 - \lambda^*)x^2)$ . These inequalities imply that  $f$  attains a local minimum on the interior of the line segment joining  $x^1$  to  $x^2$  that is also a one sided semistrict local minimum. This contradicts (19). Thus our supposition is false and semistrict quasiconcavity of  $f$  follows.

*Proof of Theorem 8.* Suitably modify the proof of Theorem 6.

*Proof of Theorem 9.* Strict quasiconcavity implies (1) and (23). Since strict quasiconcavity implies quasiconcavity and quasiconcavity implies (1), we need only show that (1) and not (23) imply that  $f$  is not strictly quasiconcave. This is straightforward.

*Properties (1) and (23) imply  $f$  is strictly quasiconcave.* Let  $x^1 \in S$ ,  $x^2 \in S$ ,  $x^1 \neq x^2$  and  $f(x^1) \leq f(x^2)$ . Define the set of minimizers of  $f$  over the closed interval joining  $x^1$  and  $x^2$  as  $M$ . If  $M \equiv \{x^1\}$  or  $M \equiv \{x^1, x^2\}$ , then  $f(x^1) < f(\lambda x^1 + (1 - \lambda)x^2)$  for  $0 < \lambda < 1$  and  $f$  is strictly quasiconcave.

Suppose  $M \neq \{x^1\}$  and  $M \neq \{x^1, x^2\}$ . Then there exists a  $\lambda^0$  such that  $0 < \lambda^0 < 1$  and  $f(\lambda^0 x^1 + (1 - \lambda^0)x^2) \leq f(\lambda x^1 + (1 - \lambda)x^2)$  for all  $\lambda$  such that  $0 \leq \lambda \leq 1$ . Thus  $f$  must attain a local minimum on the line segment joining  $x^1$  and  $x^2$  which contradicts (23). Thus our supposition is false and  $f$  is strictly quasiconcave.

*Proof of Theorem 10. Pseudoconcavity implies (27).* Obvious.

*Property (27) implies  $f$  is pseudoconcave.* Let  $x^0 \in S$ ,  $v^T v = 1$ ,  $i > 0$ ,  $D_v f(x^0) \leq 0$ .

Suppose  $f(x^0 + iv) > f(x^0)$ . Case (i):  $D_v f(x^0) < 0$ . In this case, it is clear that  $g(t) \equiv f(x^0 + tv)$  attains at least one local minimum over the open interval  $(0, i)$ . Let  $t_0$  be the largest such local minimizer. Then  $g'(t_0) = D_v f(x^0 + t_0 v) = 0$  but  $t_0$  is not a local maximizer for  $g$ , which contradicts property (27). Case (ii):  $D_v f(x^0) = 0$ . By (27),  $g(t) \equiv f(x^0 + tv)$  attains a local maximum at  $t = 0$ . Thus there exists  $\varepsilon > 0$  such that  $g(t) \leq g(0)$  for  $0 \leq t \leq \varepsilon$ . Since  $g(i) \equiv f(x^0 + iv) > g(0)$  by our supposition, it can be seen that  $g(t)$  must attain at least one local minimum over the open interval  $(0, i)$ . Let  $t_1$  be the largest such local minimizer. Then  $g'(t_1) = D_v f(x^0 + t_1 v) = 0$  but  $t_1$  is not a local maximizer for  $g$ , which contradicts property (27). Thus in both cases, we have a contradiction and thus our supposition is false. Therefore  $f(x^0 + iv) \leq f(x^0)$  and pseudoconcavity of  $f$  follows.

*Proof of Theorem 11.* Similarly to the proof of Theorem 10.

*Proof of Theorem 12. Strong quasiconcavity implies (33).* Obvious.

*Property (33) implies  $f$  is strongly quasiconcave.* Property (33) implies

(24), and hence by Corollary 9.2,  $f$  is strictly quasiconcave. Using Definition 10, it is clear that (33) implies (32). Hence  $f$  is strongly quasiconcave by Definition 9.

*Proof of Corollary 12.1. Strong quasiconcavity implies (34).* Let  $x^0 \in S$ ,  $v^T v = 1$ , and  $D_v f(x^0) = 0$ . By (32) and the openness of  $S$ , there exist  $\alpha > 0$  and  $\varepsilon > 0$  such that  $g(t) \leq g(0) - \alpha t^2$  for  $t \in [-\varepsilon, \varepsilon]$  where  $g(t) \equiv f(x^0 + tv)$ . By Taylor's Theorem, for  $t \in (0, \varepsilon]$ .

$$\begin{aligned} g(t) &= g(0) + g'(0)t + (1/2)g''(t^*)t^2 \quad \text{for some } t^* \in (0, t) \\ &= g(0) + g''(t^*)t^2/2 \quad \text{since } g'(0) = D_v f(x^0) = 0 \\ &\leq g(0) - \alpha t^2 \quad \text{since } g(t) \leq g(0) - \alpha t^2. \end{aligned}$$

The above inequality implies  $g''(t^*) \leq -2\alpha$ , where  $0 < t^* < t$ . Now let  $t$  tend to zero. Since  $f$  is twice continuously differentiable, so is  $g$  and we obtain the following inequality:  $g''(0) = v^T \nabla^2 f(x^0) v \leq -2\alpha < 0$  and thus (34) is true.

*Property (34) implies  $f$  is strongly quasiconcave.* Let  $x^0 \in S$ ,  $v^T v = 1$ ,  $\bar{t} > 0$ ,  $x^0 + \bar{t}v \in S$ ,  $D_v f(x^0) = v^T \nabla f(x^0) = 0$ . By (23), we need only show  $g(t) \equiv f(x^0 + tv)$  attains a strong local maximum at  $t = 0$ . Let  $\delta > 0$  be small enough so that  $x^0 + \delta v \in S$  and define  $\alpha$  by  $\max_{t \in [0, \delta]} \{v^T \nabla^2 f(x^0 + tv) v : 0 \leq t \leq \delta\} \equiv -2\alpha$ . Property (34) implies that  $\alpha > 0$ . By Taylor's Theorem, for  $t \in (0, \delta)$ .

$$\begin{aligned} g(t) &= g(0) + g'(0)t + g''(t^*)t^2/2 \quad \text{for some } t^* \in (0, t) \\ &= g(0) + g''(t^*)t^2/2 \quad \text{since } g'(0) = D_v f(x^0) = 0 \\ &\leq g(0) - \alpha t^2 \end{aligned}$$

since  $g''(t^*) \leq -2\alpha$  by the definition of  $\alpha$ . Thus  $g$  attains a strong local maximum over the interval  $[0, \delta]$  at  $t = 0$ .

*Proof of Theorem 14.  $f$  concave implies (40).* Since  $f$  is concave, so are  $g$  and  $h$ . But  $h$  concave implies that  $h$  is also quasiconcave (cf. Fenchel [11; 117]) and thus conditions (39) and (40) are satisfied.

*Property (40) implies that  $f$  is concave.* Suppose that  $f$  is not concave. Then there exist  $x^0 \in S$ ,  $v^T v = 1$ ,  $\bar{t} > 0$ ,  $0 < \lambda < 1$  such that  $x^0 + \bar{t}v \in S$  and  $f(\lambda x^0 + (1 - \lambda)(x^0 + \bar{t}v)) < \lambda f(x^0) + (1 - \lambda)f(x^0 + \bar{t}v)$ . Define  $g(t) \equiv f(x^0 + tv)$  for  $0 \leq t \leq \bar{t}$  and define the average slope of the function  $g$  between 0 and  $\bar{t}$  as  $\alpha^* \equiv [g(\bar{t}) - g(0)]/[\bar{t} - 0]$ . Define  $h(t) \equiv g(t) - \alpha^*t$  for  $t \in [0, \bar{t}]$ . If  $h$  does not have the line segment minimum property, then  $h$  is not quasiconcave by Theorem 3 and (40) does not hold. If  $h$  does have the line segment minimum property, then using our definition of  $\alpha^*$  and the inequality  $g(\lambda 0 + (1 - \lambda)\bar{t}) < \lambda g(0) + (1 - \lambda)g(\bar{t})$ , it can be seen that  $h$

attains a semistrict local minimum for some  $t_0 \in (0, \bar{t})$ . Hence again (40) does not hold.

*Proof of Theorem 15.*  $f$  concave implies (42). Let  $x^0 \in S$ ,  $v^T v = 1$ ,  $\bar{t} > 0$ ,  $x^0 + \bar{t}v \in S$ ,  $g(t) \equiv f(x^0 + tv)$ ,  $t_0 \in [0, \bar{t}]$ ,  $g'(t_0) \equiv \alpha$  and  $h(t) \equiv g(t) - at$ . Since  $f$  is concave, so are  $g$  and  $h$ . Thus by Theorem 13,  $h(t) \leq h(t_0) + h'(t_0)(t - t_0) = h(t_0) + 0(t - t_0) = h(t_0)$  for all  $t \in [0, \bar{t}]$ . Thus  $h$  attains a local maximum in  $[0, \bar{t}]$  at  $t_0$ .

*Property (42) implies  $f$  is concave.* Let  $x^0 \in S$ ,  $v^T v = 1$ ,  $\bar{t} > 0$ ,  $x^0 + \bar{t}v \in S$ ,  $g(t) \equiv f(x^0 + tv)$ ,  $\alpha \equiv g'(0) = D_v f(x^0)$  and  $h(t) \equiv g(t) - at$  for  $t \in [0, \bar{t}]$ . Note that  $h'(0) = 0$ . Since (42) is equivalent to (41)  $h$  is pseudoconcave over  $[0, \bar{t}]$ . Thus by (26),  $h(\bar{t}) \leq h(0)$  since  $h'(0) = 0$  or  $g(\bar{t}) - a\bar{t} \leq g(0) - a0$  by the definition of  $h$ , or  $g(\bar{t}) \leq g(0) + g'(0)\bar{t}$  by the definition of  $\alpha$ , or  $f(x^0 + \bar{t}v) \leq f(x^0) + \bar{t}D_v f(x^0)$  by the definition of  $g$ . The last inequality implies  $f$  is concave by Theorem 13.

*Proof of Theorem 17.* Similar to the proof of Theorem 14.

*Proof of Theorem 18.* Similar to the proof of Theorem 15.

*Proof of Theorem 19.* If  $g$  defined by  $g(x) \equiv f(x) + ax^T x$  is concave over  $S$  for some  $a > 0$ , then for every  $x^1 \in S$ ,  $x^2 \in S$ ,  $0 \leq \lambda \leq 1$ , we have

$$\begin{aligned} & f(\lambda x^1 + (1 - \lambda)x^2) + a[\lambda x^1 + (1 - \lambda)x^2]^T [\lambda x^1 + (1 - \lambda)x^2] \\ & \geq \lambda f(x^1) + (1 - \lambda)f(x^2) + \lambda a x^{1T} x^1 + (1 - \lambda)a x^{2T} x^2. \end{aligned}$$

A straightforward calculation shows that the above inequality is equivalent to the inequality (55).

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