Laboratorio de Microeconomía I

Centro de Investigación y Docencia Económicas
Maestría en Economía - 2025
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Tarea 7 Solver

Exercises

Ex. 1.38 Verify that the expenditure function obtained from the CES direct utility function in **Example 1.3** (J&R, 2011; p.39) satisfies all the properties given in **Theorem 1.7**.

Proof. La función de utilidad directa es:

$$u(x_1, x_2) = (x_1^\rho + x_2^\rho)^{1/\rho}$$

donde $0 \neq \rho < 1$.

El problema de minimización de costos es:

$$\begin{aligned} \min_{(x_1,x_2)\in\mathbb{R}^n_+} \quad & p_1x_1 + p_2x_2 \\ \text{s.t.} \quad & (x_1^\rho + x_2^\rho)^{1/\rho} \geq u \end{aligned}$$

El Lagrange asociado es:

$$\mathcal{L}(\mathbf{x}, \lambda) = p_1x_1 + p_2x_2 - \lambda[(x_1^\rho + x_2^\rho)^{1/\rho} - u]$$

Como las preferencias son monotónicas, la restricción se cumple con igualdad y entonces podemos aplicar el método de Lagrange.

Las condiciones de primer orden son:

$$\begin{aligned} \left.\frac{\partial \mathcal{L}}{\partial x_1}\right|_{(\mathbf{x},\lambda)=(\mathbf{x}^*,\lambda^*)} &= p_1 - \lambda(x_1^\rho + x_2^\rho)^{\frac{1-\rho}{\rho}}\ x_1^{\rho-1} = 0 \\ \left.\frac{\partial \mathcal{L}}{\partial x_2}\right|_{(\mathbf{x},\lambda)=(\mathbf{x}^*,\lambda^*)} &= p_2 - \lambda(x_1^\rho + x_2^\rho)^{\frac{1-\rho}{\rho}}\ x_2^{\rho-1} = 0 \\ \left.\frac{\partial \mathcal{L}}{\partial \lambda}\right|_{(\mathbf{x},\lambda)=(\mathbf{x}^*,\lambda^*)} &= (x_1^\rho + x_2^\rho)^{1/\rho} - u = 0 \end{aligned}$$

Resolviendo el sistema de ecuaciones, llegamos a las demandas hicksianas:

$$\boxed{x_1^h(\mathbf{p},u) = u\left(p_1^r + p_2^r\right)^{\frac{1-r}{r}} p_1^{r-1}}$$
$$\boxed{x_2^h(\mathbf{p},u) = u\left(p_1^r + p_2^r\right)^{\frac{1-r}{r}} p_2^{r-1}}$$

con $r := \rho/(\rho-1) < 0$.

La función de gasto es:

$$\begin{aligned}
e(\mathbf{p},u) &= \min_{(x_1,x_2)\in\mathbb{R}^n_+} \{p_1x_1 + p_2x_2 \mid (x_1^\rho + x_2^\rho)^{1/\rho} \leq u\} \\
&= p_1x_1^h(\mathbf{p},u) + p_2x_2^h(\mathbf{p},u) \\
&= p_1\, u\left(p_1^r + p_2^r\right)^{\frac{1-r}{r}} p_1^{r-1} + p_2\, u\left(p_1^r + p_2^r\right)^{\frac{1-r}{r}} p_2^{r-1} \\
&= u\left(p_1^r + p_2^r\right)^{\frac{1-r}{r}}\left(p_1^r + p_2^r\right) \\
&= u\left(p_1^r + p_2^r\right)^{\frac{1}{r}}
\end{aligned}$$

Por lo tanto,

$$\boxed{e(\mathbf{p},u) = u\left(p_1^r + p_2^r\right)^{\frac{1}{r}}} \tag{E.11}$$

Verificaremos que (E.11) satisface las propiedades de una función de gasto (**Theorem 1.7**).

(1). $e(\mathbf{p},u(\mathbf{x}_0)) = 0$ cuando $u(\mathbf{x}_0)$ es el nivel de utilidad mínimo en $\mathcal{U}$.

Sea $\mathcal{U} = \{u(\mathbf{x}) \mid \mathbf{x}\in\mathbb{R}^n_+\}$ el conjunto alcanzable de niveles de utilidad. El dominio de $e(\cdot)$ es $\mathbb{R}^n_{++}\times\mathcal{U}$. Queremos demostrar que la función de gasto $e(\mathbf{p},u)$ evaluada en el nivel de utilidad mínimo factible es igual a cero. Como $u(\cdot)$ es estrictamente creciente en $\mathbf{x}\in\mathbb{R}^n_+$, entonces su valor mínimo en $\mathcal{U}$ es $\mathbf{x}_0 = \mathbf{0}$. Entonces,

$$\begin{aligned}
e(\mathbf{p},u(\mathbf{x}_0)) &= u(\mathbf{x}_0)\left(p_1^r + p_2^r\right)^{\frac{1}{r}} \\
&= \left((0)^\rho + (0)^\rho\right)^{1/\rho}\left(p_1^r + p_2^r\right)^{\frac{1}{r}} \\
&= 0
\end{aligned}$$

(2). Continua en su dominio.

Claramente, $e(\mathbf{p},u) = u\left(p_1^r + p_2^r\right)^{\frac{1}{r}}$ es continua en $\mathbb{R}^n_{++}\times\mathcal{U}$.

(3). Para todo p $>> 0$, estrictamente creciente y no acotada superiormente en u.

Para todo $\mathbf{p} >> 0$:

$$\frac{\partial e}{\partial u} = (p_1^r + p_2^r)^{\frac{1}{r}} > 0,$$

por lo que $e(\mathbf{p}, u)$ es estrictamente creciente en u.

Sea $M > 0$. Si tomamos $u^* := \frac{M+\epsilon}{(p_1^r+p_2^r)^{1/r}}$ con $\epsilon > 0$, entonces

$$e(\mathbf{p}, u^*) = u^*(p_1^r + p_2^r)^{\frac{1}{r}} > M$$

Como esto sucede para cualquier $M > 0$, entonces $e(\mathbf{p}, u)$ no está acotada superiormente en u.

(4). Creciente en p.

$$\frac{\partial e}{\partial p_1} = u\,(p_1^r + p_2^r)^{\frac{1-r}{r}} p_1^{r-1} > 0$$
$$\frac{\partial e}{\partial p_2} = u\,(p_1^r + p_2^r)^{\frac{1-r}{r}} p_2^{r-1} > 0$$

para todo $\mathbf{p} >> 0$.

(5). Homogénea de grado 1 en p.

Sea $t > 0$. Entonces,

$$\begin{aligned} e(t\mathbf{p}, u) &= u\left((tp_1)^r + (tp_2)^r\right)^{\frac{1}{r}} \\ &= u\ \left(t^r(p_1^r + p_2^r)\right)^{\frac{1}{r}} \\ &= t\,u\ \left((p_1^r + p_2^r)\right)^{\frac{1}{r}} \\ &= t\,e(\mathbf{p}, u) \end{aligned}$$

Por lo tanto, $e(\mathbf{p}, u)$ es homogénea de grado 1 en $\mathbf{p}$.

(6). Cóncava en p.

Sea $u > 0$ fijo y $g(\mathbf{p}) := (p_1^r + p_2^r)^{\frac{1}{r}}$, con $\mathbf{p} \gg 0$.

Queremos demostrar que la matriz Hessiana $H_g(\mathbf{p})$ es negativa semidefinida, lo cual garantiza que $g(\mathbf{p})$ es cóncava en $\mathbf{p}$.

Primeras derivadas:

$$\frac{\partial g}{\partial p_i} \ = \ (p_1^r + p_2^r)^{\frac{1}{r}-1} p_i^{r-1}, \qquad i = 1, 2$$

Define $a := \frac{1}{r} - 1$.

Segundas derivadas:

$$\begin{aligned}
\frac{\partial^2 g}{\partial p_1^2} &= \frac{\partial}{\partial p_1}\Big((p_1^r + p_2^r)^a p_1^{r-1}\Big) \\
&= a\,(p_1^r + p_2^r)^{a-1}(r\,p_1^{r-1})\,p_1^{r-1} + (p_1^r + p_2^r)^a (r-1) p_1^{r-2} \\
&= (1-r)\,(p_1^r + p_2^r)^{a-1} p_1^{2r-2} + (r-1)\,(p_1^r + p_2^r)^a p_1^{r-2} \\
&= (r-1)\,(p_1^r + p_2^r)^{a-1} p_1^{r-2}\Big((p_1^r + p_2^r) - p_1^r\Big) \\
&= (r-1)\,(p_1^r + p_2^r)^{a-1} p_1^{r-2}\,p_2^r \\
\frac{\partial^2 g}{\partial p_2^2} &= (r-1)\,(p_1^r + p_2^r)^{a-1} p_2^{r-2}\,p_1^r \\
\frac{\partial^2 g}{\partial p_1 \partial p_2} &= \frac{\partial}{\partial p_2}\Big((p_1^r + p_2^r)^a p_1^{r-1}\Big) \\
&= \ a\,(p_1^r + p_2^r)^{a-1}(r\,p_2^{r-1})\,p_1^{r-1} \\
&= (1-r)\,(p_1^r + p_2^r)^{a-1} p_1^{r-1} p_2^{r-1} \\
&= -\,(r-1)\,(p_1^r + p_2^r)^{a-1} p_1^{r-1} p_2^{r-1}
\end{aligned}$$

Matriz Hessiana:

$$H_g(\mathbf{p}) = \begin{pmatrix} (r-1)\,(p_1^r + p_2^r)^{a-1} p_1^{r-2} p_2^r & -\,(r-1)\,(p_1^r + p_2^r)^{a-1} p_1^{r-1} p_2^{r-1} \\ -\,(r-1)\,(p_1^r + p_2^r)^{a-1} p_1^{r-1} p_2^{r-1} & (r-1)\,(p_1^r + p_2^r)^{a-1} p_2^{r-2} p_1^r \end{pmatrix}$$

Procedemos por el criterio de signos de los menores principales:

$$\Delta_1 = (r-1)\,(p_1^r + p_2^r)^{a-1}\,p_1^{r-2}\,p_2^r < 0$$

porque $r = \rho/(\rho - 1) < 0$.

$$\begin{aligned}
\Delta_2 = &\Big((r-1)\,(p_1^r + p_2^r)^{a-1} p_1^{r-2} p_2^r\Big)\Big((r-1)\,(p_1^r + p_2^r)^{a-1} p_2^{r-2} p_1^r\Big) \\
&- \ \Big(-\,(r-1)\,(p_1^r + p_2^r)^{a-1} p_1^{r-1} p_2^{r-1}\Big)^2 \\
&= (r-1)^2\,(p_1^r + p_2^r)^{2a-2}\,p_1^{2r-2} p_2^{2r-2} \ - \ (r-1)^2\,(p_1^r + p_2^r)^{2a-2}\,p_1^{2r-2} p_2^{2r-2} \\
&= 0
\end{aligned}$$

Como $\Delta_1 < 0$ y $\Delta_2 = 0$, entonces se cumple el criterio de $\Delta_1 \leq 0$ y $\Delta_2 \geq 0$. Por lo tanto, $H_g(\mathbf{p})$ es negativa semidefinida. Luego g es **cóncava** en $\mathbf{p}$ (pero **no estrictamente cóncava**).

Por lo tanto,

$$e(\mathbf{p}, u) = u\, g(\mathbf{p}) \quad \text{es cóncava en } \mathbf{p} \text{ para todo } u > 0$$

(7). Lema de Shepard.

Supongamos que $e(\mathbf{p}, u)$ es diferenciable en un punto $(\mathbf{p}, u) = (\mathbf{p}^0, u^0)$ arbitrario, con $\mathbf{p}^0 >> \mathbf{0}$. Entonces,

$$\begin{aligned}
\frac{\partial e(\mathbf{p}, u)}{\partial p_1} &= u\ (p_1^r + p_2^r)^{\frac{1-r}{r}}\ p_1^{r-1} \\
&= x_1^h(\mathbf{p}, u)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial e(\mathbf{p}, u)}{\partial p_2} &= u\ (p_1^r + p_2^r)^{\frac{1-r}{r}}\ p_2^{r-1} \\
&= x_2^h(\mathbf{p}, u)
\end{aligned}$$

□

Ex. 1.56 What restrictions must the $\alpha_i, f(y), w(p_1, p_2)$, and $z(p_1, p_2)$ satisfy if each of the following is to be a legitimate indirect utility function?

(a) $v(p_1, p_2, p_3, y) = f(y) p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}$

(b) $v(p_1, p_2, y) = w(p_1, p_2) + z(p_1, p_2)/y$

Proof. **(a)** $v(p_1, p_2, p_3, y) = f(y) p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}$

Verifiquemos cada propiedad que tendría que cumplir $v(p_1, p_2, p_3, y)$

1. **Continua en $\mathbb{R}^n_{++} \times \mathbb{R}_+$.**

   Basta con que $f(y)$ sea continua en $y \in \mathbb{R}_+$.

2. **Homogénea de grado 0 en $(\mathbf{p}, y)$.** Sea $t > 0$. Supongamos que $f(y)$ es homogénea de grado uno y que $\alpha_1 + \alpha_2 + \alpha_3 = -1$. Entonces

$$\begin{aligned}
v(tp_1, tp_2, tp_3, ty) &= f(ty)(tp_1)^{\alpha_1}(tp_2)^{\alpha_2}(tp_3)^{\alpha_3} \\
&= t^{\alpha_1+\alpha_2+\alpha_3}\, t\, f(y) p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \\
&= t^0\, f(y) p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \\
&= t^0\, v(p_1, p_2, p_3, y)
\end{aligned}$$

3. **Estrictamente creciente en y.**

   Basta con que $f(y)$ sea estrictamente creciente en $y \in \mathbb{R}_+$.

4. **Decreciente en $\mathbf{p}$.**

   Para p_i:

$$\frac{\partial v}{\partial p_i} = \alpha_i \, f(y) p_i^{\alpha_i - 1} p_j^{\alpha_j} p_l^{\alpha_l}$$

   Como $\mathbf{p} >> 0$, entonces necesitamos que $\alpha_i < 0, \ \forall \, i = 1, 2, 3$.

5. **Cuasiconvexa en $(\mathbf{p}, y)$.**

   Basta con que $f(y)$ sea cuasiconvexa en y, pues $g(\mathbf{p}) = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}$ es cuasiconvexa para $\alpha_i < 0$ y $\alpha_1 + \alpha_2 + \alpha_3 = -1$ (piensen en curvas de nivel en el espacio de precios (p_i, p_j), fijando y y algún p_l: son convexas respecto al origen).

6. **Identidad de Roy.**

   Necesitamos que

$$x_i(\mathbf{p}^0, y^0) = -\frac{\partial v(\mathbf{p}^0, y^0)/\partial p_i}{\partial v(\mathbf{p}^0, y^0)/\partial y}, \quad i = 1, \ldots, n.$$

   Entonces, para un punto $(\mathbf{p}, y) = (\mathbf{p}^0, y^0)$ arbitrario, con $\mathbf{p} >> \mathbf{0}$ y $y > 0$:

$$\begin{aligned} -\frac{\partial v(\mathbf{p}, y)/\partial p_i}{\partial v(\mathbf{p}, y)/\partial y} &= -\frac{\alpha_i \, f(y) p_i^{\alpha_i - 1} p_j^{\alpha_j} p_l^{\alpha_l}}{f_y \, p_i^{\alpha_i} p_j^{\alpha_j} p_l^{\alpha_l}} \\ &= -\frac{\alpha_i \, f(y)}{p_i \, f_y} \end{aligned}$$

   Entonces, para todo $i = 1, 2, 3$, necesitamos que

$$x_i(\mathbf{p}, y) = -\frac{\alpha_i \, f(y)}{p_i \, f_y}$$

(b) $v(p_1, p_2, y) = w(p_1, p_2) + z(p_1, p_2)/y$

Verifiquemos cada propiedad que tendría que cumplir $v(p_1, p_2, y) = w(p_1, p_2) + z(p_1, p_2)/y$.

1. **Continua en $\mathbb{R}^2_{++} \times \mathbb{R}_+$.**

   Basta con que $w(\mathbf{p})$ y $z(\mathbf{p})$ sean continuas en $\mathbf{p} \in \mathbb{R}^2_{++}$.

2. **Homogénea de grado 0 en $(\mathbf{p}, y)$.**

   Sea $t > 0$. Supngamos que $w(p_1, p_2)$ es homogénea de grado 0 y $z(p_1, p_2)$ homogénea de grado 1. Entonces:

$$\begin{aligned} v(tp_1, tp_2, ty) &= w(tp_1, tp_2) + \frac{z(tp_1, tp_2)}{(ty)} \\ &= t^0\, w(p_1, p_2) + \frac{t\, z(p_1, p_2)}{t\, y} \\ &= t^0 \left(w(p_1, p_2) + \frac{z(p_1, p_2)}{y} \right) \\ &= t^0\, v(p_1, p_2, y) \end{aligned}$$

3. **Estrictamente creciente en y.**

$$\frac{\partial v}{\partial y} = -\frac{z(\mathbf{p})}{y^2}.$$

   Como $y > 0$, basta imponer $z(\mathbf{p}) < 0$ para todo $\mathbf{p} >> \mathbf{0}$.

4. **Decreciente en p.**

   Para p_i:

$$\frac{\partial v}{\partial p_i} = \frac{\partial w}{\partial p_i} + \frac{1}{y}\frac{\partial z}{\partial p_i}$$

   Como $y > 0$, basta con que

$$\frac{\partial w}{\partial p_i} \leq 0 \quad \text{y} \quad \frac{\partial z}{\partial p_i} \leq 0$$

   con alguna de las desigualdades estricta.

5. **Cuasiconvexa en $(\mathbf{p}, y)$.**

   Basta con que $w(\mathbf{p})$ y $z(\mathbf{p})$ sean cuasiconvexas en $\mathbf{p}$ y que $z(\mathbf{p}) \leq 0$ (fijando $y > 0$, los conjuntos de contorno inferior $\{\, \mathbf{p} \in \mathbb{R}^2_{++} \mid w(\mathbf{p}) + z(\mathbf{p})/y \leq v_0 \,\}$ son convexos si w y z lo son. Una vez más, piensen geométricamente, las curvas de nivel en (p_i, p_j) se curvan hacia el origen cuando $z(\mathbf{p}) \leq 0$).

6. **Identidad de Roy.**

   Necesitamos que

$$x_i(\mathbf{p}^0, y^0) = -\frac{\partial v(\mathbf{p}^0, y^0)/\partial p_i}{\partial v(\mathbf{p}^0, y^0)/\partial y}, \quad i = 1, 2.$$

Entonces, para un punto $(\mathbf{p}, y) = (\mathbf{p}^0, y^0)$ arbitrario, con $\mathbf{p} >> \mathbf{0}$ y $y > 0$:

$$
\begin{aligned}
-\frac{\partial v(\mathbf{p}, y)/\partial p_i}{\partial v(\mathbf{p}, y)/\partial y} &= -\frac{w_{p_i} + \dfrac{1}{y}\, z_{p_i}}{-\dfrac{z(p_1, p_2)}{y^2}} \\
&= \left(w_{p_i} + \frac{1}{y}\, z_{p_i}\right) \frac{y^2}{z(p_1, p_2)} \\
&= \frac{y^2}{z(p_1, p_2)}\, w_{p_i} \ + \ \frac{y}{z(p_1, p_2)}\, z_{p_i}.
\end{aligned}
$$

Por lo tanto, para $i = 1, 2$, necesitamos que

$$
x_i(\mathbf{p}, y) = \frac{y^2}{z(\mathbf{p})}\, w_{p_i}(\mathbf{p}) \ + \ \frac{y}{z(\mathbf{p})}\, z_{p_i}(\mathbf{p}).
$$

(Noten que con $z(\mathbf{p}) < 0$ y $w_{p_i}, z_{p_i} \leq 0$, se obtiene $x_i(\mathbf{p}, y) \geq 0$).

$\square$

Ex. 1.59 If $e(p, u) = z(p_1, p_2) p_3^m u$, where $m > 0$, what restrictions must $z(p_1, p_2)$ satisfy for this to be a legitimate expenditure function?

Proof. Verifiquemos cada propiedad que tendría que cumplir $e(p, u) = z(p_1, p_2) p_3^m u$.

1. **Igual a cero cuando u toma el nivel más bajo de utilidad en $\mathcal{U}$.**

   Sea $u_0 \in \mathcal{U}$ el nivel mínimo de utilidad factible. Supongamos que $u_0 = 0$, entonces

   $$e(\mathbf{p}, 0) = z(p_1, p_2) p_3^m (0) = 0$$

   Por lo tanto, se cumple si $u_0 = 0$ es el nivel mínimo en $\mathcal{U}$.

2. **Continua en $\mathbb{R}^n_{++} \times \mathcal{U}$.** Basta con que $z(p_1, p_2)$ sea continua en (p_1, p_2).

3. **Para todo $\mathbf{p} \gg 0$, estrictamente creciente y no acotada superiormente en u.** Dado que

   $$\frac{\partial e}{\partial u} = z(p_1, p_2) p_3^m$$

   esta derivada será estrictamente positiva si y solo si $z(p_1, p_2) > 0$ para todo $\mathbf{p} \gg 0$. Con esta condición, e es estrictamente creciente en u y no acotada superiormente.

4. **Creciente en los precios p.**

   Para cada p_i:

$$\frac{\partial e}{\partial p_1} = z_{p_1}\, p_3^m u, \qquad \frac{\partial e}{\partial p_2} = z_{p_2}\, p_3^m u, \qquad \frac{\partial e}{\partial p_3} = m\, z\, p_3^{m-1} u$$

   Dado que $p_3, u > 0$, basta con que $z(p_1, p_2)$ sea no decreciente en p_1 y p_2, y $m > 0$, para que e sea creciente en cada precio.

5. **Homogénea de grado 1 en p.**

   Sea $t > 0$. Entonces

$$\begin{aligned} e(t\mathbf{p}, u) &= z(tp_1, tp_2)(tp_3)^m u \\ &= t^m z(tp_1, tp_2) p_3^m u \end{aligned}$$

   Para que $e(t\mathbf{p}, u) = te(\mathbf{p}, u)$ se necesita que

$$z(tp_1, tp_2) = t^{1-m} z(p_1, p_2),$$

   es decir, que $z(p_1, p_2)$ sea **homogénea de grado** $(1 - m)$ en (p_1, p_2).

6. **Cóncava en p.**

   Si $z(p_1, p_2)$ es cóncava en (p_1, p_2) y $m \in (0, 1]$, el producto $z(p_1, p_2)p_3^m$ es cóncavo en $\mathbf{p}$ (el producto de funciones positivas cóncavas preserva la concavidad).

7. **Lema de Shephard.**

   Necesitamos que

$$x_i^h(\mathbf{p}, u) = \frac{\partial e(\mathbf{p}, u)}{\partial p_i}, \quad i = 1, 2, 3.$$

   Por lo tanto, tiene que cumplirse que

$$x_1^h(\mathbf{p}, u) = z_{p_1}(p_1, p_2)p_3^m u, \qquad x_2^h(\mathbf{p}, u) = z_{p_2}(p_1, p_2)p_3^m u, \qquad x_3^h(\mathbf{p}, u) = m\, z(p_1, p_2)p_3^{m-1} u$$

   Noten que con $z(p_1, p_2) > 0$, $m > 0$ y $z_{p_i} \geq 0$, las demandas hicksianas son no negativas para todo $\mathbf{p} \gg 0$.

$\square$

Ex. 3.E.8^A For the Cobb–Douglas utility function, verify that the relationships in (3.E.1) and (3.E.4) hold. Note that the expenditure function can be derived by simply inverting the indirect utility function, and vice versa.

Proof. Sea $u(x_1, x_2) = x_1^{\alpha_1} x_2^{\alpha_2}$, con $\alpha_1, \alpha_2 > 0$. Sin pérdida de generalidad, supongamos que $\alpha_1 + \alpha_2 = 1$.

Demandas Marshallianas y Función de Utilidad Indirecta.

Del problema de maximización de utilidad (UMP), obtenemos las demandas marshallianas para cada bien:

$$x_1(\mathbf{p}, y) = y\,\frac{\alpha_1}{p_1}, \qquad x_2(\mathbf{p}, y) = y\,\frac{\alpha_2}{p_2}.$$

Entonces,

$$\begin{aligned}
v(\mathbf{p}, y) &= u\Big(x(\mathbf{p}, y)\Big) \\
&= \left(y\frac{\alpha_1}{p_1}\right)^{\alpha_1} \left(y\frac{\alpha_2}{p_2}\right)^{\alpha_2} \\
&= y\left(\frac{\alpha_1}{p_1}\right)^{\alpha_1} \left(\frac{\alpha_2}{p_2}\right)^{\alpha_2}
\end{aligned}$$

Demandas Hicksianas y Función de Gasto.

Para un nivel de utilidad $u > 0$, tenemos que

$$v(\mathbf{p}, y) = y\left(\frac{\alpha_1}{p_1}\right)^{\alpha_1} \left(\frac{\alpha_2}{p_2}\right)^{\alpha_2} = u$$

Invirtiendo $v(\mathbf{p}, y)$ respecto a y, obtenemos la función de gasto:

$$y = e(\mathbf{p}, u) = u\left(\frac{p_1}{\alpha_1}\right)^{\alpha_1} \left(\frac{p_2}{\alpha_2}\right)^{\alpha_2}$$

Por **Lema de Shephard**:

$$x_1^h(\mathbf{p}, u) = \frac{\partial e}{\partial p_1} = \frac{\alpha_1}{p_1}\,u\left(\frac{p_1}{\alpha_1}\right)^{\alpha_1} \left(\frac{p_2}{\alpha_2}\right)^{\alpha_2}$$

$$x_2^h(\mathbf{p}, u) = \frac{\partial e}{\partial p_2} = \frac{\alpha_2}{p_2}\,u\left(\frac{p_1}{\alpha_1}\right)^{\alpha_1} \left(\frac{p_2}{\alpha_2}\right)^{\alpha_2}$$

(1) Verificación de (3.E.1): $e(\mathbf{p}, v(\mathbf{p}, y)) = y$ **y** $v(\mathbf{p}, e(\mathbf{p}, u)) = u$

$$e\Big(\mathbf{p}, v(\mathbf{p}, y)\Big) = \left[y\left(\frac{\alpha_1}{p_1}\right)^{\alpha_1}\left(\frac{\alpha_2}{p_2}\right)^{\alpha_2}\right]\left(\frac{p_1}{\alpha_1}\right)^{\alpha_1}\left(\frac{p_2}{\alpha_2}\right)^{\alpha_2} = y$$

$$v\Big(\mathbf{p}, e(\mathbf{p}, u)\Big) = \left[u\left(\frac{p_1}{\alpha_1}\right)^{\alpha_1}\left(\frac{p_2}{\alpha_2}\right)^{\alpha_2}\right]\left(\frac{\alpha_1}{p_1}\right)^{\alpha_1}\left(\frac{\alpha_2}{p_2}\right)^{\alpha_2} = u$$

(2) Verificación de (3.E.4): $x^h(\mathbf{p}, u) = x(\mathbf{p}, e(\mathbf{p}, u))$ **y** $x(\mathbf{p}, y) = x^h(\mathbf{p}, v(\mathbf{p}, y))$

$$x_i\Big(\mathbf{p}, e(\mathbf{p}, u)\Big) = \frac{\alpha_i}{p_i}\, e(\mathbf{p}, u) = \frac{\alpha_i}{p_i}\, u\left(\frac{p_1}{\alpha_1}\right)^{\alpha_1}\left(\frac{p_2}{\alpha_2}\right)^{\alpha_2} = x_i^h(\mathbf{p}, u), \quad i = 1, 2.$$

$$x_i^h\Big(\mathbf{p}, v(\mathbf{p}, w)\Big) = \frac{\alpha_i}{p_i}\, v(\mathbf{p}, w) = \frac{\alpha_i w}{p_i} = x_i(\mathbf{p}, w), \quad i = 1, 2.$$

Con esto, se verifican (3.E.1) y (3.E.4). □

Ex. 3.G.15^B Consider the utility function

$$u = 2x_1^{1/2} + 4x_2^{1/2}.$$

(a) Find the compensated demand function $h(\cdot)$ (Hicksian demand functions).

(b) Find the expenditure function, and verify that $h(p, u) = \nabla_p e(p, u)$.

Ex. (Opcional) For the utility function in **Exercise 3.G.15^B**, solve the following:

(c) Using **Theorem 1.8**, derive the indirect utility function using the expenditure function from (b).

(d) Using **Theorem 1.9**, derive the Marshallian demand function for good x_1 using the Hicksian demand function you derived in (a) and the indirect utility function you derived in (c).

Proof. Sea $u(x_1, x_2) = 2x_1^{1/2} + 4x_2^{1/2}$, con $\mathbf{p} \gg \mathbf{0}$ y $u > 0$.

(a) Demandas Hicksianas $x_i^h(\mathbf{p}, u)$.

El problema de minimización de gasto (EMP) es:

$$\min_{(x_1, x_2) \in \mathbb{R}_+^n} \; p_1 x_1 + p_2 x_2 \quad \text{s.a.} \quad 2\sqrt{x_1} + 4\sqrt{x_2} \;\geq\; u$$

El Lagrange asociado es:

$$\mathcal{L}(\mathbf{x},\lambda) = p_1x_1 + p_2x_2 - \lambda\Big(2x_1^{1/2} + 4x_2^{1/2} - u\Big).$$

Como las preferencias son monotónicas, la restricción se cumple con igualdad y entonces podemos aplicar el método de Lagrange.

Las condiciones de primer orden son:

$$\left.\frac{\partial \mathcal{L}}{\partial x_1}\right|_{(\mathbf{x},\lambda)=(\mathbf{x}^*,\lambda^*)} = p_1 - \lambda x_1^{-1/2} = 0$$
$$\left.\frac{\partial \mathcal{L}}{\partial x_2}\right|_{(\mathbf{x},\lambda)=(\mathbf{x}^*,\lambda^*)} = p_2 - 2\lambda x_2^{-1/2} = 0$$
$$\left.\frac{\partial \mathcal{L}}{\partial \lambda}\right|_{(\mathbf{x},\lambda)=(\mathbf{x}^*,\lambda^*)} = 2x_1^{1/2} + 4x_2^{1/2} - u = 0$$

Resolviendo el sistema de ecuaciones:

$$p_1 = \lambda\, x_1^{-1/2} \;\Rightarrow\; \sqrt{x_1} = \frac{\lambda}{p_1} \;\Rightarrow\; x_1 = \left(\frac{\lambda}{p_1}\right)^2,$$

$$p_2 = \frac{2\lambda}{\sqrt{x_2}} \;\Rightarrow\; \sqrt{x_2} = \frac{2\lambda}{p_2} \;\Rightarrow\; x_2 = \left(\frac{2\lambda}{p_2}\right)^2$$

Sustituyendo en la restricción: $2\sqrt{x_1} + 4\sqrt{x_2} = u$,

$$2\frac{\lambda}{p_1} + 4\frac{2\lambda}{p_2} = u \implies \boxed{\lambda^* = \frac{u}{\frac{2}{p_1} + \frac{8}{p_2}} = \frac{u\,p_1p_2}{2p_2 + 8p_1}}$$

Por lo tanto,

$$\boxed{x_1^h(\mathbf{p},u) = \left(\frac{\lambda}{p_1}\right)^2 = \frac{u^2\,p_2^2}{4(p_2 + 8p_1)^2}}$$

$$\boxed{x_2^h(\mathbf{p},u) = \left(\frac{2\lambda}{p_2}\right)^2 = \frac{u^2\,p_1^2}{(p_2 + 4p_1)^2}}$$

(b) Función de gasto $e(\mathbf{p}, u)$ y verificación de Lema Shephard

La función de gasto mínimo es

$$\begin{aligned} e(\mathbf{p}, u) &= p_1 x_1^h(\mathbf{p}, u) + p_2 x_2^h(\mathbf{p}, u) \\ &= \frac{u^2}{4} \frac{p_1 p_2}{p_2 + 4p_1} \end{aligned}$$

que es homogénea de grado 1 en $\mathbf{p}$ y estrictamente creciente en u.

Lema de Shephard

Para p_1:

$$\frac{\partial e(\mathbf{p}, u)}{\partial p_1} = \frac{u^2}{4} \frac{p_2^2}{(p_2 + 4p_1)^2} = x_1^h(\mathbf{p}, u)$$

Para p_2:

$$\frac{\partial e(\mathbf{p}, u)}{\partial p_2} = \frac{u^2 \, p_1^2}{(p_2 + 4p_1)^2} = x_2^h(\mathbf{p}, u)$$

Luego $x^h(\mathbf{p}, u) = \nabla_p e(\mathbf{p}, u)$.

(Opcional) (c) Utilidad indirecta vía dualidad

De $e(\mathbf{p}, v(\mathbf{p}, y)) = y$ se obtiene:

$$y = \frac{v(\mathbf{p}, y)^2}{4} \frac{p_1 p_2}{p_2 + 4p_1} \implies v(\mathbf{p}, y)^2 = 4y \, \frac{p_2 + 4p_1}{p_1 p_2}$$

Tomando raíz cuadrada a ambos términos,

$$v(\mathbf{p}, y) = 2\sqrt{y \, \frac{p_2 + 4p_1}{p_1 p_2}} = 2\left(\frac{y}{p_1} + \frac{4y}{p_2}\right)^{1/2}$$

(Opcional) (d) Demanda marshalliana de x_1 vía dualidad

De $x(\mathbf{p}, y) = x^h(\mathbf{p}, v(\mathbf{p}, y))$ se obtiene:

$$x_1(\mathbf{p}, y) = v(\mathbf{p}, y)^2 \, \frac{p_2^2}{4(p_2 + 4p_1)^2} = \left(4y \, \frac{p_2 + 4p_1}{p_1 p_2}\right) \frac{p_2^2}{4(p_2 + 4p_1)^2} = y \, \frac{p_2}{p_1(p_2 + 4p_1)}$$

Análogamente, $x_2(\mathbf{p}, y) = y \dfrac{4\, p_1}{p_2(p_2 + 4p_1)}$ y se puede verificar que $p_1 x_1 + p_2 x_2 = y$. □

Appendix

We define the **expenditure function** as the minimum-value function,

$$e(\mathbf{p}, u) \equiv \min_{\mathbf{x} \in \mathbb{R}^n_+} \{\mathbf{p} \cdot \mathbf{x} \mid u(\mathbf{x}) \geq u\} \tag{1.14}$$

for all $\mathbf{p} \gg \mathbf{0}$ and all attainable utility levels u. For future reference, let $\mathcal{U} = \{u(\mathbf{x}) \mid \mathbf{x} \in \mathbb{R}^n_+\}$ denote the set of attainable utility levels. Thus, the domain of $e(\cdot)$ is $\mathbb{R}^n_{++} \times \mathcal{U}$.

Note that $e(\mathbf{p}, u)$ is well-defined because for $\mathbf{p} \in \mathbb{R}^n_{++}$, $\mathbf{x} \in \mathbb{R}^n_+$, $\mathbf{p} \cdot \mathbf{x} \geq 0$. Hence, the set of numbers $\{e \mid e = \mathbf{p} \cdot \mathbf{x}$ for some $\mathbf{x}$ with $u(\mathbf{x}) \geq u\}$ is bounded below by zero. Moreover because $\mathbf{p} \gg \mathbf{0}$, this set can be shown to be closed. Hence, it contains a smallest number. The value $e(\mathbf{p}, u)$ is precisely this smallest number. Note that any solution vector for this minimisation problem will be non-negative and will depend on the parameters $\mathbf{p}$ and u. Notice also that if $u(\mathbf{x})$ is continuous and strictly quasiconcave, the solution will be unique, so we can denote the solution as the function $\mathbf{x}^h(\mathbf{p}, u) \geq 0$. As we have seen, if $\mathbf{x}^h(\mathbf{p}, u)$ solves this problem, the lowest expenditure necessary to achieve utility u at prices $\mathbf{p}$ will be exactly equal to the cost of the bundle $\mathbf{x}^h(\mathbf{p}, u)$, or

$$e(\mathbf{p}, u) = \mathbf{p} \cdot \mathbf{x}^h(\mathbf{p}, u). \tag{1.15}$$

Theorem 1.7 Properties of the Expenditure Function.

If $u(\cdot)$ is continuous and strictly increasing, then $e(\mathbf{p}, u)$ defined in (1.14) is

1. Zero when u takes on the lowest level of utility in $\mathcal{U}$,
2. Continuous on its domain $\mathbb{R}^n_{++} \times \mathcal{U}$,
3. For all $\mathbf{p} \gg \mathbf{0}$, strictly increasing and unbounded above in u,
4. Increasing in $\mathbf{p}$,
5. Homogeneous of degree 1 in $\mathbf{p}$,
6. Concave in $\mathbf{p}$.

If, in addition, $u(\cdot)$ is strictly quasiconcave, we have

7. *Shephard's lemma*: $e(\mathbf{p}, u)$ is differentiable in $\mathbf{p}$ at $(\mathbf{p}^0, u^0)$ with $\mathbf{p}^0 \gg \mathbf{0}$, and

$$\frac{\partial e(\mathbf{p}^0, u^0)}{\partial p_i} = x_i^h(\mathbf{p}^0, u^0), \quad i = 1, \ldots, n.$$

Theorem 1.8 Relations Between Indirect Utility and Expenditure Functions.

Let $v(\mathbf{p}, y)$ and $e(\mathbf{p}, u)$ be the indirect utility function and expenditure function for some consumer whose utility function is continuous and strictly increasing. Then for all $\mathbf{p} \gg \mathbf{0}$, $y \geq 0$, and $u \in \mathcal{U}$:

1. $e(\mathbf{p}, v(\mathbf{p}, y)) = y$.
2. $v(\mathbf{p}, e(\mathbf{p}, u)) = u$.

Theorem 1.9 Duality Between Marshallian and Hicksian Demand Functions.

Under Assumption 1.2 we have the following relations between the Hicksian and Marshallian demand functions for $\mathbf{p} \gg \mathbf{0}$, $y \geq 0$, $u \in \mathcal{U}$, and $i = 1, \ldots, n$:

1. $x_i(\mathbf{p}, y) = x_i^h(\mathbf{p}, v(\mathbf{p}, y))$.
2. $x_i^h(\mathbf{p}, u) = x_i(\mathbf{p}, e(\mathbf{p}, u))$.

Proposition 3.E.1

Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}_+^L$ and that the price vector is $p \gg 0$. We have

(i) If x^* is optimal in the UMP when wealth is $w > 0$, then x^* is optimal in the EMP when the required utility level is $u(x^*)$. Moreover, the minimized expenditure level in this EMP is exactly w.

(ii) If x^* is optimal in the EMP when the required utility level is $u > u(0)$, then x^* is optimal in the UMP when wealth is $p \cdot x^*$. Moreover, the maximized utility level in this UMP is exactly u.

Proposition 3.E.1 allows us to make an important connection between the expenditure function and the indirect utility function developed in Section 3.D. In particular, for any $p \gg 0$, $w > 0$, and $u > u(0)$ we have

$$e(p, v(p, w)) = w \quad \text{and} \quad v(p, e(p, u)) = u. \tag{3.E.1}$$

These conditions imply that for a fixed price vector $\bar{p}$, $e(\bar{p}, \cdot)$ and $v(\bar{p}, \cdot)$ are inverses to one another (see Exercise 3.E.8) (...) there is a direct correspondence between the properties of the expenditure function and the indirect utility function. They both capture the same underlying features of the consumer's choice problem.

Using Proposition 3.E.1, we can relate the Hicksian and Walrasian demand correspondences as follows:

$$h(p, u) = x(p, e(p, u)) \quad \text{and} \quad x(p, w) = h(p, v(p, w)). \tag{3.E.4}$$