Homotopy invariants of LS-category type for aspherical spaces

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The origin: Lusternik-Schnirelmann category

Lusternik-Schnirelmann category (Lusternik-Schnirelmann '34) $\mathrm{cat}(X) = \min k$ s.t. $\exists \{U_i\}_{0 \leq i \leq k}$ open cover of X with $U_i \simeq * \ \forall 0 \leq i \leq k$.

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Theorem (The Lusternik-Schnirelmann theorem, '34)

Let M be a C^2 Banach manifold and $f\colon M \to \mathbb{R}$ bounded below and s. t. given any $S \subset M$ for which f bounded but for which $||df||$ is not bounded away from 0 on S, $\exists$ a critical point of f on $\overline{S}$. Then

$$\mathit{crit}(f) \geq \mathrm{cat}(M) + 1.$$

Indeed,

$$\mathrm{cat}(M) + 1 \leq \mathit{Crit}(M) = \min |\textit{critical points for any smooth function on } M|.$$

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$$\mathrm{secat}(p) < \frac{d+1}{s+1}.$$

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For $\mathrm{ev}_1 : P_*X \to X$ by $\gamma \mapsto \gamma(1)$ then $\mathrm{secat}(\mathrm{ev}_1) = \mathrm{cat}(X)$.

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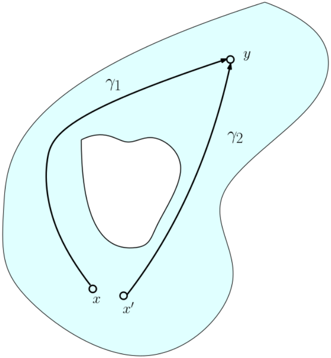

A topological feature of the configuration space inducing instability on the motion planning.

Distinguished case: topological complexity

Topological complexity (Farber '01) $\mathrm{TC}(X)$ = min k s.t. $\exists\ \{U_i\}_{0\le i\le k}$ open cover of X with (cont.) local sections of $\pi\colon PX \to X \times X$.

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- It is homotopy invariant.
- X path connected and paracompact $\mathrm{cat}(X) \leq \mathrm{TC}(X) \leq \mathrm{cat}(X \times X)$.
- $\mathrm{cl}_A \leq \mathrm{cat}(X)$ and $\mathrm{nilker}\left[H^*(X \times X; A) \xrightarrow{\Delta^*} H^*(X : A)\right] \leq \mathrm{TC}(X)$.
- X $(s-1)$-conn. then

$$\mathrm{cat}(X) \leq \frac{\dim(X)}{s} \qquad \text{and} \qquad \mathrm{TC}(X) \leq \frac{2\dim(X)}{s}.$$

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Theorem (Costa-Farber '10)

Let X be a CW-complex with $n = \dim(X) \ge 2$. One has $\mathrm{TC}(X) = 2n$ iff $\mathfrak{v}^{2n} \neq 0$ for a special class

$$\mathfrak{v}^{2n} \in H^{2n}(X\times X; J^{\otimes 2n}) \qquad J = \ker[\mathbb{Z}[\pi_1(X)] \xrightarrow{\varepsilon} \mathbb{Z}]$$

called canonical class.

Basic examples

S^{2n+1}:

$U_0 := \{(x,y) | x, y \in S^{2n+1} \text{ with } x \neq -y\}$ $\quad U_1 := \{(x,y) | x, y \in S^{2n+1} \text{ such that } x \neq y\}$.

$s_0(x,y)$ is the shortest geodesic joining x and y. $s_1(x,y)$ is the map which moves x to $-y$ as before, and then $-y$ to y through non-vanishing continuous tangent vector field v

$$-\cos(\pi t)y + \sin(\pi t)\frac{v(y)}{|v(y)|}.$$

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S^{2n}: $u \in H^{2n}(S^{2n})$ and define

$$v := u \otimes 1 - 1 \otimes u \in H^{2n}(S^{2n} \times S^{2n}).$$

$\Delta^*(u \otimes 1) = u = \Delta^*(1 \otimes u)$ and $\Delta^*(v) = 0$. Observe

$$\begin{aligned} v \cup v &= ((u \otimes 1) - (1 \otimes u)) \cup ((u \otimes 1) - (1 \otimes u)) \\ &= -(u \otimes 1) \cup (1 \otimes u) - (1 \otimes u) \cup (u \otimes 1) \\ &= -2u \otimes u \neq 0. \end{aligned}$$

By cohomological lower bound we have $\mathrm{TC}(S^{2n}) \geq 2$. By the upper dimensional bound $\mathrm{TC}(S^{2n}) \leq 2$.

Basic examples

$X = \underbrace{S^n \times \cdots \times S^n}_{k}$: We have that

$$\mathrm{TC}(X) \leq \begin{cases} k & \text{if } n \text{ is odd} \\ 2k & \text{if } n \text{ is even.} \end{cases}$$

It is an equality. Let $u_i \in H^n(X;\mathbb{Q})$ be the pullback of the fundamental class of S^n via projection onto the i-th factor.

$$\prod_{i=1}^{k}(1 \otimes u_i - u_i \otimes 1) \neq 0 \text{ if n is odd} \qquad \prod_{i=1}^{k}(1 \otimes u_i - u_i \otimes 1)^2 \neq 0 \text{ if n is even.}$$

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Σ_g: Cases $g = 1, 2$ seen. So $g \geq 2$. We find 1-dimensional classes $u_1, u_2, v_1, v_2 \in H^1(\Sigma_g, \mathbb{Q})$ satisfying $u_1 u_2 = v_1 v_2 = u_1 v_2 = u_2 v_1 = u_1^2 = u_2^2 = v_1^2 = v_2^2 = 0$ and $u_1 v_1 = u_2 v_2$ is non trivial in $H^2(\Sigma_g, \mathbb{Q})$.

$$\prod_{i=1}^{2}(u_i \otimes 1 - 1 \otimes u_i) \cup (v_i \otimes 1 - 1 \otimes v_i) \neq 0$$

so $\mathrm{TC}(\Sigma_g) \geq 4$. By the dimension connectivity bound $\mathrm{TC}(\Sigma_g) \leq 2\dim(\Sigma_g) = 4$.

Interesting application: immersion dimensions

The immersion problem for $\mathbb{R}P^n$: what is the least dimension m of an euclidean space $\mathbb{R}^m$ such that there is an immersion $\iota\colon \mathbb{R}P^n \hookrightarrow \mathbb{R}^m$?

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Theorem (Farber-Tabachnikov-Yuzvinsky '02)

$$\mathrm{TC}(\mathbb{R}P^n) = \begin{cases} \mathrm{Imm}(\mathbb{R}P^n) & n \neq 1,3,7 \\ n & n = 1,3,7. \end{cases}$$

Another application: fixed point properties

- X has fixed point property (FPP) if, for every cont. self-map $f : X \to X$ there is a fixed point.
- $((X, \tau), Y)$ triple with $\tau\colon X \to X$ fixed point free involution. $((X, \tau), Y)$ satisfies Borsuk-Ulam property (BUP) if for every cont map $f : X \to Y$ exists $x \in X$ such that $f(\tau(x)) = f(x)$.
- For X, Y and cont $g : X \to Y$ (X, Y, g) has the coincidence property (CP) if for every map $f : X \to Y$ there is $x \in X$ s.t. $f(x) = g(x)$.

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$F(X, k) = \{(x_1, \cdots, x_k) \in X^k \mid x_i \neq x_j \text{ whenever } i \neq j\}$ $\quad \pi^X_{k,r}(x_1, \cdots, x_r, \cdots, x_k) = (x_1, \cdots, x_r)$.

Theorem (Ipanaque-González '21, Ipanaque-Gonçalves '23, Ipanaque-Torres Estrella '24)

- *X has FPP iff* $\mathrm{sec}(\pi^X_{2,1}) = 2$.
- *If* $\mathrm{secat}(X \to X/\tau) > \mathrm{secat}(F(Y, 2) \to F(Y, 2)/\Sigma_2)$ *then* $((X, \tau), Y)$ *satisfies BUP.*
- (X, Y, g) *has CP iff* $\mathrm{sec}_g(\pi^Y_{2,1}) = 2$.

Sequential topological complexities

(Rudyak'10):

$$p_r\colon PX \to X^r \qquad p_r(\gamma) = \left(\gamma(0), \gamma\left(\frac{1}{r-1}\right) \cdots, \gamma\left(\frac{r-2}{r-1}\right), \gamma(1)\right) \qquad r \geq 2.$$

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- For $r = 2$ we recover $\mathrm{TC}_2(X) = \mathrm{TC}(X)$.
- One has $\mathrm{cat}(X^{r-1}) \leq \mathrm{TC}_r(X) \leq \mathrm{cat}(X^r)$.

TC_r and critical point theory

Palais-Smale condition Let (M, g) be complete Riemmanian Banach manifold, $F \in C^1(M)$. Let $||\cdot||$ be the norm induced by g on each tangent space of M. (F, g) satisfies the PS-condition if every sequence $\{x_n\}n \in \mathbb{N}$ for which $\{||F(x_n)||\}_{n\in\mathbb{N}}$ bounded and with $lim_{n\to\infty} \nabla^g F(x_n) = 0$ has a convergent subsequence.

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Let M Riemannian smooth manifold, $F \in C^1(M^r)$ satisfy PS-condition wrt complete Riemannian metric on M^r and bounded from below. Then

$$\mathrm{TC}_{r,M}(F^a) \leq \sum_{\mu\in(-\infty,a]} \mathrm{TC}_{r,M}(\{x \in \ Crit\, F \mid F(x) = \mu\}) \qquad \forall a \in \mathbb{R}.$$

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$F \in C^1(M^r)$ us an r-navigation function if F satisfies PS wrt complete Riem. metric on M^r and $F \geq 0$ and $F^{-1}(\{0\}) = \Delta_{r,M}$.

Corollary (Mescher-Stegemeyer'24)

Let $F : M^r \to \mathbb{R}$ be an r-navigation function, $a \geq 0$. Then

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In particular $\mathrm{TC}_{r,M}(F^a) \leq |\{x \in \ Crit\ F \mid 0 < F(x) \leq a\}|$.

The Eilenberg-Ganea problem

Recall for G a group $K(G,1)$ is a space with $\pi_1(K(G,1)) = G$ and $\pi_k(K(G,1)) = 0$ $\forall k > 1$. G is geometrically finite if there exists a finite CW model for $K(G,1)$.

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- Basabe-González-Rudyak-Tamaki '14: $\mathrm{TC}_r(\mathbb{Z}^n) = (r-1)\mathrm{cd}(\mathbb{Z}^n) = (r-1)n$.
- Farber-Oprea '19: generalize FGLO bounds.
- Hughes-Li '22: G hyperbolic, $G \ncong \mathbb{Z} \Rightarrow \mathrm{TC}_r(G) = r\mathrm{cd}(G)$.

Sectional category of subgroup inclusions

Let $\varphi\colon G_1 \to G_2$ be a group homomorphism. There exists $f_\varphi : K(G_1, 1) \to K(G_2, 1)$ s.t. the induced homomorphism $\pi_1(f_\varphi) = \varphi$.

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General bounds are given by

- $\mathrm{secat}(H \hookrightarrow G) \leq \mathrm{cd}(G)$.
- $\mathrm{secat}(H \hookrightarrow G) \geq \mathrm{nilker}\left(i^*\colon H^*(G, A) \to H^*(H, \mathrm{Res}^G_H(A))\right)$

$\mathrm{secat}(H \hookrightarrow G)$ and classifying spaces of families

Denote $E_{\mathcal{F}}G$ as the classifying space for the family of subgroups wrt $\mathcal{F}$.

- $E_{\mathcal{F}}G$ has all isotropy sbgps on $\mathbb{F}$.
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Theorem (Błaszczyk-Carrasquel-EB'20)

$\mathrm{secat}(H \hookrightarrow G)$ coincides with min. $n \geq 0$ s.t. $\rho\colon EG \to (E_{\langle H\rangle}G)_n$ can be factorized up to G-homotopy as

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Theorem (Błaszczyk-Carrasquel-EB'20)

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The map ρ is induced by the universal property of $E_{\langle H \rangle}G$. To prove it we use

Lemma (Błaszczyk-Carrasquel Vera-EB '20)

$\mathrm{secat}(H \hookrightarrow G) \leq n$ *iff the Borel fibration* $p_n\colon EG \times_G *^{n+1}(G/H) \to EG/G$ *has a section.*

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This generalizes Farber-Grant-Lupton-Oprea '19 and Farber-Oprea '19 for TC and TC^r.

Relationship with Bredon and Adamson cohomological dimension

Theorem (Farber-Grant-Lupton-Oprea'19, Błaszczyk-Carrasquel-EB'20)

$$\max\{n \in \mathbb{N} \mid \exists \alpha \in H^n_{\langle\Delta\rangle}(\pi \times \pi, \underline{M}) \text{ s.t. } \rho^*(\alpha) \neq 0\} \leq \mathrm{TC}(\pi) \leq \max\{3, \mathrm{cd}_{\langle\Delta\rangle}(\pi \times \pi)\}.$$

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$[F_n, F_n] \hookrightarrow F_n$. One has $\mathrm{secat}\big([F_n, F_n] \hookrightarrow F_n\big) = 1$, but

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Grant-Li-Mehir-Patchkoria '23: $G = \langle a, b \mid c := a^2 = b^2\rangle$. They show

$$\mathrm{cd}[G \times G : \Delta_G] = \mathrm{cd}_{\langle \Delta_G\rangle}(G \times G) = \infty$$

but $\mathrm{TC}(G) = 4$ (Cohen-Vandembroucq '17).

The case of normal subgroups

Grant '12: The cohomological dimension of a group homomorphism $\phi\colon G \to H$, $\mathrm{cd}(\phi)$ is the maximum $k \geq 0$ for which exists H-module A so that

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Theorem (EB-Farber-Mescher-Oprea)

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To get significant bounds beyond the normal case we need to use more elaborated constructions in group cohomology.

The Berstein-Schwarz relative class

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The r-th canonical class $\mathfrak{v}_r \in H^1(\pi^r, I_r)$ is the relative Berstein class of π^r wrt $\Delta_{\pi,r}$.

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The r-th canonical class $\mathfrak{v}_r \in H^1(\pi^r, I_r)$ is the relative Berstein class of π^r wrt $\Delta_{\pi,r}$. If $r = 2$, $\mathfrak{v}_2$ is the canonical class of Costa and Farber.

The Berstein-Schwarz relative class

For a subgroup $H \leqslant G$ consider the augmentation ideal

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Błaszczyk, Carrasquel Vera, EB '20: Define the Berstein-Schwarz class of G relative to H as $\omega \in H^1(G, I)$ represented by the cocycle

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New lower bounds for secat and TC_r

Theorem (EB-Farber-Mescher-Oprea'23)

Let G be a geometrically finite group and $H \leq G$. Let

$$\kappa_{G,H} := \max\{\mathrm{cd}(H \cap xHx^{-1}) \mid x \in G \setminus H\}.$$

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If we specialize to the diagonal inclusion $\Delta_{\pi,r} \hookrightarrow \pi^r$ we get

Corollary (EB-Farber-Mescher-Oprea'23)
Let π be a geometrically finite group and let $r \in \mathbb{N}$ with $r \geq 2$. Then

$$\mathrm{TC}_r(K(\pi, 1)) \geq r \cdot \mathrm{cd}(\pi) - k(\pi),$$

where $k(\pi) = \max\{\mathrm{cd}(C(g)) \mid g \in \pi \setminus \{1\}\}$.

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Proposition (EB-Farber-Mescher-Oprea'23)

If $\mathrm{Im}\left[\delta^{n-1,1} \circ \delta^{n-2,2} \circ \cdots \circ \delta^{n-k,k} \colon \mathrm{Ext}^{n-k}_{\mathbb{Z}[G]}(I^{\otimes k}, A) \to H^n(G, A)\right] \neq 0$ *then*

$$\mathrm{secat}(H \hookrightarrow G) \geq k.$$

Forming a spectral sequence

To show that image is non-zero, we can assemble our Ext-sequences into an exact couple

$$\begin{tikzcd} D_0 \arrow[rr, "i_0"] & & D_0 \arrow[dl, "j_0"] \\ & E_0 \arrow[ul, "k_0"] & \end{tikzcd}.$$

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In these terms, the corollary turns to be

Corollary

Let $n, p \in \mathbb{N}$ with $p \leq n$. If $D_p^{n,0} \neq \{0\}$, then $\omega^p \neq 0$ and thus $\mathrm{secat}(H \hookrightarrow G) \geq p$.

The sketch of the proof

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$$H \times (G/H)^s \to (G/H)^s, \quad h \cdot (g_1 H, \ldots, g_s H) = (hg_1 H, hg_2 H, \ldots, hg_s H).$$

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Proposition (EB-Farber-Mescher-Oprea'23)

For each $C \in \mathcal{C}'_s(G/H)$ fix a representative $x_C \in C$ and let $N_C := H_{x_C}$ be the isotropy group of x_C. Then

$$E_0^{r,s} \cong \prod_{C \in \mathcal{C}'_s(G/H)} H^r(N_C; \mathrm{Res}^G_{N_C}(A)) \quad \forall r \in \mathbb{N}$$

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For $u \in D_0^{n,0}$ one identifies obstructions to $u \in D_k^{n,0}$ lying in the groups

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An example of application

Let G be a group. A subgroup $H \leq G$ is *malnormal* if

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Corollary (EB-Farber-Mescher-Oprea'23)

Let G be a geometrically finite group, and $H \leqslant G$ malnormal. Then $\mathrm{secat}(H \hookrightarrow G)$ is maximal, i.e. $\mathrm{secat}(H \hookrightarrow G) = \mathrm{cd}(G)$.

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Corollary (EB-Farber-Mescher-Oprea'23)

Let π_1 and π_2 be geometrically finite groups and consider a free product with amalgamation $\pi_1 *_H \pi_2$, such that H is malnormal in π_1 or malnormal in π_2. Then for each $r \geq 2$

$$\mathrm{TC}_r(\pi_1 *_H \pi_2) \geq r \cdot \mathrm{cd}(\pi_1 *_H \pi_2) - \max\{k(\pi_1), k(\pi_2)\}.$$

Parametrized TC of group epimorphisms

For fibration $p: E \to B$ (with fibre X) set
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$E^I_B := \{\gamma\colon I := [0,1] \to E \mid \exists x_0 \in B \text{ s.t. } p \circ \gamma = c_{x_0}\}$ Define

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Π is a fibration with fibre ΩX.
(Cohen, Farber, Weinberger '21) $\mathrm{TC}[p : E \to B] = \mathrm{secat}(\Pi : E^I_B \to E \times_B E)$ is the parametrized topological complexity of p.
(Grant '22) Let $\rho\colon G \twoheadrightarrow Q$ be a group epimorphism. There is a fibration $f_\rho\colon K(G,1) \to K(Q,1)$ with $\pi_1(f_\rho) = \rho$.
The parametrized TC of epimorphism ρ $\mathrm{TC}[\rho\colon G \twoheadrightarrow Q] := \mathrm{TC}[f_\rho\colon K(G,1) \to K(Q,1)]$
Grant shows $\mathrm{TC}[\rho\colon G \twoheadrightarrow Q] = \mathrm{secat}(\Delta\colon G \hookrightarrow G \times_Q G)$

Corollary (EB-Farber-Mescher-Oprea'23)

Let G and Q be geometrically finite groups and let $\rho : G \twoheadrightarrow Q$ be an epimorphism. Then

$$\mathrm{TC}[\rho : G \twoheadrightarrow Q] \geq \mathrm{cd}(G \times_Q G) - k(\rho),$$

where $k(\rho) = \max\{\mathrm{cd}(C(g)) \mid g \in \ker \rho,\ g \neq 1\}$.

Canonical class for non-aspherical spaces

Let $\pi = \pi_1(X)$ $\qquad I_r := \ker\left[\varepsilon : \mathbb{Z}[\pi^{r-1}] \to \mathbb{Z}\right]$. Define

$$f_r : \pi^r \to I_r, \quad f_r(g_1, g_2, \ldots, g_r) = (g_1 g_2^{-1} - 1, g_2 g_3^{-1} - 1, \ldots, g_{r-1} g_r^{-1} - 1)$$

f_r is a crossed homomorphism. The *r-th canonical class of X* is the class

$$\mathfrak{v}_r \in H^1(X^r; I_r), \quad \mathfrak{v}_r := [f_r]$$

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Corollary (EB-Farber-Mescher-Oprea'23)

X connected CW complex with $\pi_1(X) = \pi$, let $K = K(\pi, 1)$ and $f_X : X \to K$ a classifying map for the universal cover of X. Then

$$\mathfrak{v}_r^X = (f_X^r)^*(\mathfrak{v}_r^K) \in H^1(X^r, I_r).$$

Lower bounds for non-aspherical spaces

Our lower bounds allow to give also bounds for spaces not necessarily aspherical:

Theorem (EB-Farber-Mescher-Oprea'23)

Let π be a geom. fin. group and X a conn. loc. fin. CW-complex with $\pi_1(X) = \pi$ and $\widetilde{X}$ $(k-1)$-connected. If $\mathrm{cd}(\pi) \leq k$, then

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And also to give a sequential version of the Farber-Costa theorem:

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Let $n, r \in \mathbb{N}$ with $r \geq 2$ and let X be an n-dimensional CW complex. It holds that $\mathrm{TC}_r(X) = rn$ if and only if $\mathfrak{v}_r^{r \cdot n} \neq 0$.

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And a nice result on the non-maximality of TC_r for fin. dim. CW complexes with abelian fund. group:

Corollary (EB-Farber-Mescher-Oprea'23)

Let X be a connected n-dimensional finite CW complex whose fundamental group is free abelian of rank at most n. Then

$$\mathrm{TC}_r(X) < rn \qquad \forall r \geq 2.$$

¡Gracias por su atención!
Thank you for your attention!
Dziękuję za uwagę!

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NARODOWE CENTRUM NAUKI