On a New Paradigm for Stock Trading Via a Model-Free Feedback Controller

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***Abstract*—This paper describes a new paradigm for stock trading involving the use of classical feedback controllers which are "model free" in that they use neither parameterization nor estimation of stock price dynamics. At time t, the control signal is the investment level $I(t)$, obtained via a mapping on the so-called gain-loss function $g(t)$. While such strategies fall under the umbrella of *technical analysis*, our approach differs from the literature in a fundamental way: Whereas existing work in finance involves statistical analysis via historical back-testing, our new control-theoretic paradigm aims to provide "certification theorems" giving conditions under which certain robustness properties are guaranteed with respect to benchmark classes for the time-varying stock price $p(t)$. We demonstrate our ideas using a linear feedback implementation of a new stock-trading scheme called *Simultaneous Long-Short*. The analysis is carried out in a so-called *idealized frictionless market* first using smooth prices for pedagogical purposes and then using a more realistic benchmark involving Geometric Brownian Motion. Finally, simulations are given which include real-world implementation issues.**

***Index Terms*—Feedback controller.**

I. Introduction

THIS paper describes a new paradigm for stock trading which involves a control-theoretic point of view. Existing results driving this work are unified in the current paper and given in [1]–[12], and, in contrast to literature such as [13]–[41], the control strategies we develop are model free.[1] That is, a parameterized model of the stock price dynamics is not used to determine the investment level, $I(t)$. We treat the stock price $p(t)$ in much the same way as an external disturbance is handled in the control literature; i.e., we seek to establish certain robustness properties of the closed loop system against various benchmark price classes.

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[1]Some of these papers are only tangentially related to the research described here in that they deal with issues in financial markets other than the narrow focus here: trading a single stock.

In this paper, attention is restricted to trading a single stock rather than a portfolio. For a specific stock under consideration, at time t, the investment level $I(t)$ is adjusted via a feedback law which is essentially a reactive response to the cumulative gains or losses $g(t)$ over $[0, t]$. The takeoff point for the analysis to follow is the incremental relationship involving p, I, and g; i.e., multiplying the percentage change in price by the investment, we obtain $dg = (dp/p)I$. Note that this equation may be equivalently interpreted as multiplying the change in price dp by the number of shares $N \doteq I/p$.

Given the fact that our control algorithm is based on variations of the gain-loss function $g(t)$ rather than any type of model, this work falls under the umbrella of "technical analysis" in nearly its purest form. That is, technical analysis generally amounts to processing of price patterns and other quantities which are derived from it. In the case of this paper, we use $g(t)$ as the derived quantity; see [42] and [43] for further details.[2] For completeness of coverage, we also mention another flavor of technical analysis which includes prices modeled in terms of an equilibrium resulting from behaviors and preferences of agents in the marketplace; see [58]–[60]. In the finance literature, technical analysis is strongly differentiated from "fundamental analysis" which makes heavy use of specifics about a company such as earnings, dividends, price-to-earnings ratios and many other items on the balance sheet such as debt and available cash; e.g., see the classic books [61] and [62] on this subject.

1.1 Additional Motivation for New Paradigm: While technical analysis is widely used in practice and is being studied in academia; for example, see [42] and [43], there is no formal theory which provides a sound theoretical foundation explaining why it works; e.g., see [44]–[52] where this issue is brought to the fore. Instead of a formal theory, in the literature, beginning around 1992 with [53], the case for the efficacy of technical analysis is made via a number of significant empirical studies involving the use of statistics and historical data under a number of market conditions; e.g., see [54]–[57]. In view of this existing literature, our main objective in this paper is to "open the door" to the development of a formal explanatory theory for technical analysis—one which relies on control-theoretic concepts and includes performance certification theorems rather than statistical analysis.

1.2 Plan for Remainder of Paper: As a first step, in Section II, we show how the stock-trading problem can be reformulated in a classical control-theoretic setting; i.e., we

[2]Perhaps the most famous example of technical analysis is the so-called "moving-average rule" which involves triggering of buys and sells of stock based on the crossings of various moving averages by the stock price $p(t)$; e.g., see [53].

consider a standard linear feedback implementation of a specific technical analysis stock-trading strategy which is a form of *trend following*; e.g., see [69] and [70]. Then, in Section III, we describe the assumptions on the market in which the analysis of various control schemes are carried out; i.e., a so-called *idealized frictionless market* is defined. Our point of view is that good performance in this idealized setting should accompany back-testing using historical data from a real market. Section IV turns to the issue of "benchmark classes" of stock prices. That is, the performance certification theorems in this paper are proven with respect to price paths satisfying certain regularity conditions. To this end, two benchmark classes are used in this paper: smooth prices and Geometric Brownian Motion.

In Section V, to evaluate the performance of the linear feedback system, the first of these classes, smooth prices, is used. Our main reason for using this class is *not* to demonstrate "real market" price variations. This simplification enables us to explain the key ideas in our theory as a preamble to the more realistic stochastic analysis in Section VI. In this mathematically simple setting, it is easy to demonstrate how "arbitrage" can be assured despite the trader's ignorance of the price process. This is accomplished via a modified linear feedback scheme called *Simultaneous Long-Short*. In Section VI, we see that the "secrets" exposed in the smooth-price case are generalizable to the case when prices are driven by Geometric Brownian Motion. Section VII is dedicated to an important practical extension of the Simultaneous Long-Short scheme in which a saturation limits the size of trades. It is shown that the arbitrage property still holds over smooth price paths. Section VIII is devoted to numerical simulations which incorporate real-world implementation issues, and in Section IX conclusions are provided and directions for future research are indicated.

II. Feedback Loop Implementation

Our goal in this section is to show how the stock-trading process can easily be reformulated in a classical feedback control setting. Throughout the paper, the basic paradigm is illustrated using trading strategies obtained via static linear feedback.

2.1 Control, Output, and Disturbance Variables: When trading a single stock, we take $I(t)$ to be the time-varying investment level, say in dollars, as the control input variable and the cumulative trading gains or losses $g(t)$ over $[0, t]$ as the output to be regulated; we call $g(t)$ the *gain-loss function*. In addition, this open-loop description includes the unmodelled price process $p(t)$, which can be viewed as a disturbance, intervening between $I(t)$ and $g(t)$.

2.2 Closing the Loop: The static model-free trading strategies which we consider are seen to be mappings from $g(t)$ to $I(t)$. That is, the paper begins with consideration of the linear feedback control case; we define this mapping via

$$I(t) = I_0 + Kg(t)$$

where K is the controller gain, $I_0 = I(0)$ is the initial investment and $g(0) = 0$. While not covered here, we note that many of the ideas to follow can be generalized to address classes of dynamic controllers as well; e.g., see [8] where integral action $\int_0^t g(\tau)d\tau$ is also included. In view of the fact that typical price

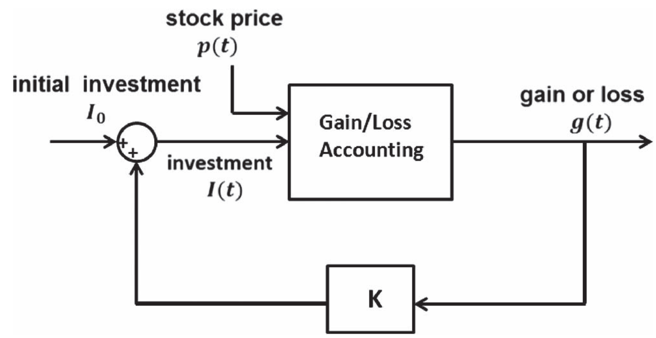

Fig. 1. Feedback control configuration for trading.

paths have high frequency components, consistent with classical feedback theory, we restrict attention to differentiator-free control dynamics. Furthermore, since this controller is "hard wired," there is no parameter estimation and no prediction of $p(t)$. This reactive controller simply adapts the investment level to variations in the gain-loss function $g(t)$; see Fig. 1.

2.3 Trend-Following Aspects: Viewing the controller above in the context of the financial literature, it would be called a "trend follower." To elaborate, suppose $K > 0$ and $I_0 > 0$ above. Then, initially, as the trade begins at $t = 0$, we have $I(0) = I_0 > 0$. This positive investment is said to be "long" and reflects the desire of a rational trader for the stock price to increase. That is, if the stock price $p(t)$ increases over some suitably small time interval $[0, t_1]$ from $p(0)$ to $p(t_1)$, the gain-loss function $g(t)$ will correspondingly increase from $g(0) = 0$ to some level $g(t_1) > 0$. Furthermore, as the increase of $g(t)$ is occurring, the linear feedback strategy forces the investment level $I(t)$ to increase too. Simply put, the investment level is following the gain-loss trend. Hence, the name trend follower is used. By a similar argument, for the case when the price is falling over $[0, t_1]$, we have $g(t)$ becoming increasingly negative and the linear control law now follows the downward trend by *decreasing* $I(t)$.

2.4 On Short-Selling: It should also be noted that the setup above readily handles the case when $I(t) < 0$. Such an investment is said to be "short" and the trader is called a *short seller*. In this case, a rational short seller is hoping that the stock price decreases. For example, if a trader has an initial investment $I(0) = -1000$ dollars and the stock price decreases 10% over the time period $[0, t_1]$, the gross profit resulting from this short trade is \$100. From a practical implementation point of view, short selling is accomplished as follows: Shares are borrowed from the broker and immediately sold in the open market. At some future point, $t = t_1$, the trader "covers" the short position by going into the market, buying back the shares and returning them to the broker. If $p(t_1) < p(0)$, a profit is realized. Similarly, if $p(t_1) > p(0)$, the trader incurs a loss. Analogous to the long trading case, a very similar trend-following interpretation is possible; e.g., when $I(0) < 0$ and $K < 0$, as the trade progresses, a trend in the gain-loss function $g(t)$ is echoed in the size of the negative investment $I(t)$.

III. Idealized Frictionless Markets

In this section, we describe the next important ingredient of the new paradigm, the *idealized frictionless market*. Per discussion in the introduction, we develop performance certification theorems in this idealized setting with our point of view being as follows: Analysis of robust performance of the model-free controller in this idealized market should accompany back-testing using historical stock-price data for real markets; i.e., the designer of a stock-trading controller uses this market to gain credibility. This type of market is central to a number of major results in continuous-time finance including the celebrated Black-Scholes model; e.g., see [63]–[65] and [67]. The assumptions associated with this market are described in the subsections to follow.

3.1 Continuous Trading Assumption: It is assumed that the trader can continuously adjust the investment level $I(t)$. This implies that we allow a fractional number of shares to be held. It is also noted that this assumption has the flavor of what is encountered in the world of high-frequency trading where the ability exists to execute many thousands of trades per second.

3.2 Perfect Liquidity and Price-Taker Assumption: It is assumed that the trader can transact as many shares as desired, at the instantaneous price $p(t)$. In practical terms, this means the volume of shares being traded leads to no "gap" between the bid and ask prices. It also means that the trader is a price taker in the sense that the stock price remains constant at $p(t)$ during the course of the transaction. This would be the case if the trader, at time t, is not buying or selling sufficiently large blocks of stock so as to have an influence on the price. Note that this assumption would be faulty in the case of a "large" hedge or mutual fund buying or selling millions of shares of a stock that typically trades only a few hundred thousand shares per day. For example, if this large trader is a buyer, the price typically increases during the course of the transaction. As the demand-supply gap grows, the "last" of these purchased shares becomes more costly to acquire than the earlier shares.

3.3 No Transaction Cost Assumption: It is assumed that no transaction costs such as brokerage commissions and exchange fees are imposed on the trader. Whereas an assumption of this sort was viewed as a serious impediment a decade or more ago, this is no longer the case. Active traders who are frequently placing orders of significant size, say tens of thousands of dollars, typically have commission structures which are sufficiently low so as to be, more or less, a non-issue. For the small trader carrying out a large number of trades per day, transaction costs would be an issue.

3.4 Adequacy of Resources Assumption: It is assumed that the trading strategy meets the so-called collateral requirements of the broker so that all trades are admissible; i.e., no transactions are "stopped" and no liquidation occurs. For example, this requirement can be satisfied if the account has a suitably large cash balance or if other securities in the account purchased with cash provide adequate collateral. Letting $V(t)$ denote the brokerage *account value*, it is typical that trades will be permitted if $|I(t)| \leq 2V(t)$. Later in this paper, we discuss the case when this inequality is in play.

3.5 Interest and Margin Assumption: In practice, when $V(t) - I(t) > 0$, this cash balance in an investor's account accrues interest[3] at the so-called *risk-free rate of return* $r \geq 0$. On the other hand, when $V(t) - I(t) < 0$, *margin interest* is owed to the broker. These considerations can all readily be captured by the equation updating the account value $V(t)$. For example, to illustrate for the case when both idle cash and margin debt have interest rate r, beginning from initial condition $V(0) = V_0$, we have

$$V(t) = V_0 + g(t) + r \int_0^t (V(\tau) - I(\tau))\, d\tau.$$

Finally, it is noted that the equation above can be readily modified for the more general case when the two interest rates are different. In practice, the margin interest rate is generally larger than the interest rate on idle cash.

For simplicity of presentation, in the sequel, an interest rate of $r = 0$ is assumed for both idle cash in the trader's account and borrowed funds on "margin" from the broker. Hence, in this idealized setting, changes in the trader's account value $V(t)$ correspond to changes in the profit or loss level $g(t)$; e.g., see [4]. That is, since $g(0) = 0$, we have

$$V(t) = V_0 + g(t).$$

Many of the results to follow can readily be modified to address non-zero interest rates.[4]

IV. Benchmark Price Classes

To study the performance of our stock-trading controllers to follow, we use two benchmark price classes to drive the system. Each such class is a collection of "synthetic" price variations which does not correspond to real stock data. Per discussion in the introduction, we consider smooth functions and Geometric Brownian Motion. However, one might easily entertain other classes such as a discrete-time system with stock prices going either up or down a fixed percentage with probabilities p and $1 - p$, respectively, at each time step; this is a so-called binomial lattice; e.g., see [65]. There is an analogy to be drawn between our view of trading and a widely held view by researchers in optimization theory: If an algorithm is proposed to optimize a high-dimensional nonlinear function, its efficacy is often established by considering various benchmarks. For example, if a general-purpose nonlinear optimization algorithm fails to perform on the "nice" benchmark class of convex functions, many would argue that it is not to be trusted for more general problems where no *a priori* model of the function being optimized is assumed.

Given the "hardness" associated with analysis of real-world markets, we use a similar standard in our theory. That is, a technically-based trading rule gains credibility via a demonstration of robust performance on various benchmark price classes. In this regard, perhaps the most famous class of prices in finance are those which are obtained as sample paths of a Geometric Brownian Motion. In summary, the idealized market in combination with some benchmark price class serves as a "proving ground" within which theoretical performance certifications are obtained.

[3] For a short sale with $I(t) < 0$, it is typically the case that small investors cannot obtain interest on the proceeds.

[4] One of the standard methods used to account for interest involves bond-denominated repricing of the stock; i.e., one works with modified prices $p_B(t) \doteq p(t)e^{-rt}$.

4.1 Two Benchmark Classes: Prior to demonstration of our paradigm using Geometric Brownian Motion, we first consider a much simpler price class purely for pedagogical purposes. This first class, which we call *smooth prices*, consists of non-negative continuously differentiable functions of time. While this class is "too simple," it serves the useful purpose of providing the reader with the "secrets" of our methods without undue mathematical complication. For this "overly-idealized" case, we describe a linear feedback based scenario in which an arbitrage can be robustly certified. That is, in an idealized market with non-trivial stock price variations, for this smooth price case, surprisingly, the controller guarantees at future times $t>0$ that we have $g(t)>0$.

Armed with the key ideas for the case of smooth prices, we then consider a collection of Geometric Brownian Motions (GBM) following the stochastic differential equation:

$$\frac{dp}{p} = \mu dt + \sigma dZ$$

where, the parameter μ, often called the *drift*, captures the annualized expected return, and the parameter σ, often called the *volatility*, represents the annualized standard deviation associated with the underlying process. Finally, we take $Z(t)$ to be a standard Wiener process resulting from dZ, which can be viewed as a normal random variable with zero mean and variance dt; see [65] for further detail.

When we study the trading performance over the GBM price class, the results are given in terms of μ and σ which are unknown to the model-free feedback controller. Hence we seek robust performance certification theorems. Most notably, since the sign of the drift μ is unknown, the controller should perform well in both upward and downward trending markets. To hedge the trade so as to be robust against non-zero μ of unknown sign, we introduce the so-called *Simultaneous Long-Short* strategy, a combination of two feedbacks which is seen to have the *Robust Positive Expectation* property

$$\mathbb{E}\left[g(t)\right] > 0.$$

Furthermore, in Section VI, to provide more detail about the risk versus return properties of this strategy, we provide a closed-form expression for the probability density function of $g(t)$. Finally, to conclude this section, we note an important salient feature of both benchmark price classes above. Namely, continuity of prices rules out price jumps which may occur in real markets; e.g., say a company announces earnings after the market closes, the following day, the stock's opening price can be significantly different from its previous close.

V. Stock Trading With Smooth Prices

This section considers some properties of feedback trading strategies for the simplistic smooth price case. That is, in the idealized frictionless market within which trading occurs, we initially assume that the stock price $p(t)$ is an arbitrary continuously differentiable function on $[0, T]$ which is unknown to the trader. We show that a positive gain-loss function $g(t)$ can be guaranteed with a linear feedback controller. This provides motivation for the analysis of the GBM case to follow.

5.1 A Static Linear Feedback Controller: We begin by analyzing the simple linear feedback trading strategy described previously in Section II-2.2. That is, we take the investment function to be $I(t) = I_0 + Kg(t)$ with $I_0 > 0$ and $K > 0$. This strategy is said to be "long" in that it performs well in a bull market with rising prices, but loses when prices fall in a bear market. Now, recalling the discussion in the introduction, beginning with the incremental gain-loss equation

$$dg = \frac{dp}{p} I$$

over time period $[t, t + dt]$, the differential equation for the gain-loss $g(t)$ is

$$\frac{dg}{dt} = \rho(t)\left(I_0 + Kg(t)\right)$$

where

$$\rho(t) \doteq \frac{1}{p}\frac{dp}{dt}$$

is the so-called *return function*. Straightforward integration of this scalar linear time-varying system leads to the gain-loss

$$g(t) = \frac{I_0}{K}\left[\left(\frac{p(t)}{p(0)}\right)^K - 1\right]$$

with the following properties: First, the gain-loss $g(t)$ at time t is independent of the path that the price takes in moving from $p(0)$ to $p(t)$; i.e., all smooth price paths that begin at $p(0)$ and arrive at $p(t)$ lead to the same result. Second, letting $p(t)$ tend to zero, it is easily seen that the maximum loss can never exceed $g^*(t) \doteq -I_0/K$ for all $t > 0$. On the other hand, positive gains can be unbounded as the price of the stock rises. Finally, consistent with common sense, we obtain $g(t) > 0$ if and only if $p(t) > p(0)$; i.e., the long strategy makes money on price increases.

5.2 The Simultaneous Long-Short (SLS) Controller: The long linear feedback strategy introduced above with $I_0 > 0$ and $K > 0$ is clearly not robust to market declines. To hedge against the direction of the market, in this section, we consider an investment strategy with the potential to perform well in both bull and bear markets. This is accomplished by implementing a short linear feedback in "parallel" with the long controller above; see details below. Subsequently, the overall investment level is obtained as the superposition of two linear feedback strategies. That is, with $I_L(t)$ and $I_S(t)$ denoting the levels of the long and short investments respectively, the overall investment is taken to be $I(t) = I_L(t) + I_S(t)$. Intuitively speaking, the two simultaneous linear feedback strategies continuously adapt the investment amount in favor of the winning side of the trade. As shown in the more formal analysis below, for this simplistic smooth price case, irrespective of market direction, this Simultaneous Long-Short (SLS) control can be rigorously proven to always "win" with one exception: the trivial break-even case when the stock does a round trip characterized by a final price $p(T)$ which is the same as the initial price $p(0)$.

Before proceeding, we add a note of clarification regarding what is meant when we say that the trade "wins" above. Winning should only be viewed as a property of the trade rather

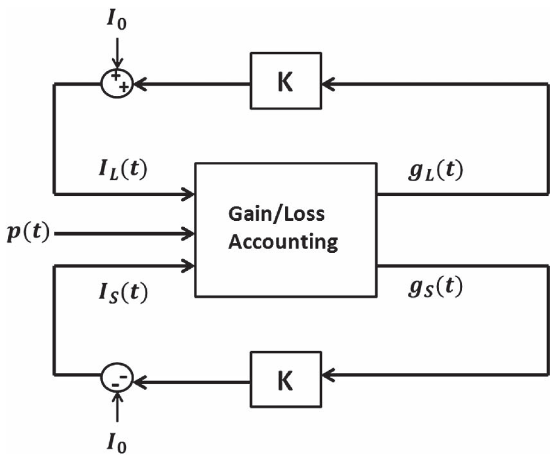

Fig. 2. Feedback configuration for SLS trading.

than an indicator of "goodness" of the trade. The reason for saying this is that it may well be the case that on the way to $g(T) > 0$, the dynamically varying gain-loss function $g(t)$ may behave in a number of ways which suggest that excessive risk was assumed. For example, if $g(t)$ undergoes "roller coaster" type behavior, the final result, while strong, may not be viewed as skillfully obtained.

A second consideration related to the characterization of a trade as winning might be that the return on investment may be unacceptably small; e.g., the profit $g(T)$ relative to some integrated average investment level over $[0, T]$ may not be acceptable versus other opportunities in the marketplace. Yet a third consideration when a "win" is claimed has to do with leverage; i.e., whether the trade requires investment levels which far exceed the value of the account. An example of such a trade is the so-called "doubling strategy," which can be found in the finance literature [68]. In contrast to this literature, as shown in the sequel, a second nice property of the SLS strategy is that the resulting leverage is readily characterized; see Theorem 6.14. Such objections notwithstanding, the guarantee of a "win" remains an important theoretical property of a trading strategy. In the terminology of finance, the guarantee that $g(t) > 0$ is called an "arbitrage."

To construct the arbitrage strategy, recall that $g_L(t)$ and $g_S(t)$ denote the cumulative trading gain or loss from the long and short linear feedback strategies, respectively, and, with $I_0 > 0$ and $K > 0$, the investments are taken to be simple linear time-invariant feedbacks

$$I_L(t) \doteq I_0 + K g_L(t); \quad I_S(t) \doteq -I_0 - K g_S(t).$$

This leads to overall investment

$$I(t) = I_L(t) + I_S(t) = K\left(g_L(t) - g_S(t)\right)$$

and overall trading gain-loss function

$$g(t) = g_L(t) + g_S(t)$$

with initial conditions $g(0) = g_L(0) = g_S(0) = 0$. The feedback control loop associated with this Simultaneous Long-Short (SLS) controller is seen in Fig. 2.

Now, using the solution for the long side given in Section V-5.1, the short side solution is obtained by simply replacing K by $-K$ and I_0 by $-I_0$; i.e.,

$$g_S(t) = \frac{I_0}{K}\left[\left(\frac{p(t)}{p(0)}\right)^{-K} - 1\right].$$

By combining the long and short trades, in the theorem below, we provide an arbitrage result; i.e., using the controller above, a positive trading gain is guaranteed without any model for the smoothly varying stock price. That is, feedback alone is able to continuously adapt the investment amount and ensure a win.

5.3 Arbitrage Theorem: In an idealized frictionless market with smooth prices, for $t \geq 0$, the trading gain-loss function resulting from the SLS controller is

$$g(t) = \frac{I_0}{K}\left[\left(\frac{p(t)}{p(0)}\right)^{K} + \left(\frac{p(t)}{p(0)}\right)^{-K} - 2\right].$$

Moreover, except for the trivial break-even case with price $p(t) = p(0)$, the gain-loss function satisfies

$$g(t) > 0.$$

Proof: Adding the trading gain-loss functions given above

$$g(t) = g_L(t) + g_S(t) = \frac{I_0}{K}\left[\left(\frac{p(t)}{p(0)}\right)^{K} + \left(\frac{p(t)}{p(0)}\right)^{-K} - 2\right].$$

The theorem then follows from the fact that $X^K + X^{-K} - 2 > 0$ for all $X \neq 1$. ■

5.4 Use of Leverage: Associated with the trading gain in the theorem above is the investment level utilized by each side of the trade. For example, on the long side of the trade, using the formula $I(t) = I_0 + Kg(t)$ for the investment, it is readily seen that

$$I_L(t) = I_0\left(\frac{p(t)}{p(0)}\right)^{K}.$$

Hence, for large price increases, the possibility arises that the demanded investment level will exceed the account value. This is a so-called *leveraged* condition, and correspondingly the issue of margin arises; see Section VI-6.13 for details.

VI. Trading Against Brownian Motion

Armed with the key ideas for the simplistic smooth-price case above, we now consider a more realistic scenario involving Geometric Brownian Motion (GBM) as described in Section IV. In addition to providing sample path formulae and the probability density function for the gain-loss function $g(t)$, we explore some of the more surprising robustness properties of the Simultaneous Long-Short strategy, including the certification of bounded worst-case loss and a robust positive expected gain property that is the analog of the smooth price arbitrage. We begin our analysis with the long linear feedback strategy.

6.1 Simple Linear Feedback With GBM Prices: Once again, we consider the static linear feedback investment function $I(t) = I_0 + Kg(t)$ with $I_0 > 0$ and $K > 0$. Under GBM prices as described in Section IV-4.1 with drift μ and volatility σ, combining the formula for the investment, $I(t)$, and the GBM price model leads to a stochastic differential equation describing the incremental change in the gain-loss function

$$dg = \frac{dp}{p} I(t) = (\mu dt + \sigma dZ)(I_0 + Kg).$$

As seen in the theorem to follow, it is possible to solve this stochastic equation along sample paths for the price to obtain $g(t)$ and subsequently $I(t)$.

6.2 Theorem: For $t \geq 0$, in an idealized frictionless market with GBM prices, along a price path $p(\cdot)$, the linear feedback controller $I(t) = I_0 + Kg(t)$ with $I_0 > 0$ and $K > 0$ leads to the gain-loss function

$$g(t) = \frac{I_0}{K}\left[\left(\frac{p(t)}{p(0)}\right)^K e^{\frac{1}{2}\sigma^2(K-K^2)t} - 1\right]$$

and associated investment function

$$I(t) = I_0 \left(\frac{p(t)}{p(0)}\right)^K e^{\frac{1}{2}\sigma^2(K-K^2)t}.$$

Proof: Making the substitution $f = I_0 + Kg$ into the expression for dg above leads to the equation

$$df = K(\mu dt + \sigma dZ)f$$

which itself is of the form of GBM. Thus, using Ito's rule [65], and integrating, its solution is given in closed form as

$$f(t) = f(0)e^{\left(\mu K - \frac{1}{2}\sigma^2 K^2\right)t + \sigma K Z(t)}.$$

By substituting back into $f = I_0 + Kg$, solving for $g(t)$, recognizing that

$$\frac{p(t)}{p(0)} = e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma Z(t)}$$

and recalling that $g(0) = 0$, we obtain

$$g(t) = \frac{I_0}{K}\left[\left(\frac{p(t)}{p(0)}\right)^K e^{\left(\frac{1}{2}\sigma^2(K-K^2)\right)t} - 1\right].$$

The formula for $I(t)$ follows immediately upon substitution of $g(t)$ into $I(t) = I_0 + Kg(t)$. ■

6.3 Remarks on Theorem and SLS Trading: We now note some properties of the sample path gain-loss formula for $g(t)$ in the theorem above. First, the gain-loss is explicitly a function of the price, $p(t)$, and volatility, σ, but not of the drift μ. Furthermore, for $K > 1$, the trade is most profitable if the price rises significantly with little volatility so that $p(t)/p(0)$ is large and σ is small. Additionally, note that regardless of the parameters governing the GBM, the worst-case loss satisfies $g^*(t) > -I_0/K$, identical to the result in the smooth price case in Section V-5.1.

We now turn our attention to the analysis of the Simultaneous Long-Short (SLS) controller that is the superposition of a long and short linear feedback. The first main result, a closed-form sample path formula for the gain-loss function, has two salient features. First, as in Theorem 6.2, $g(t)$ is independent of the drift μ and the price path $p(\cdot)$ over $[0, t]$; only the final value

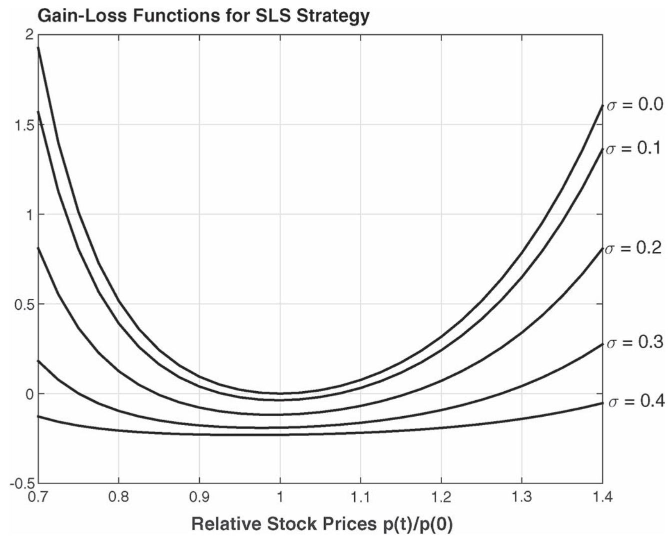

Fig. 3. Gain-loss for $K = 8$, $t = 0.5$, and $I_0 = 1$.

$p(t)$ and volatility σ come into play. Second, this formula for $g(t)$ is sufficiently simple to allow development of a closed-form for its probability density function; see Section VI-6.10.

6.4 Theorem: In an idealized frictionless market with GBM prices, along a price path $p(\cdot)$, for $t \geq 0$, the SLS feedback controller leads to the gain-loss function

$$g(t) = \frac{I_0}{K}\left[\left(\frac{p(t)}{p(0)}\right)^K e^{\frac{1}{2}\sigma^2(K-K^2)t} + \left(\frac{p(t)}{p(0)}\right)^{-K} e^{-\frac{1}{2}\sigma^2(K+K^2)t} - 2\right]$$

and associated investment

$$I(t) = I_0\left[\left(\frac{p(t)}{p(0)}\right)^K e^{\frac{1}{2}\sigma^2(K-K^2)t} - \left(\frac{p(t)}{p(0)}\right)^{-K} e^{-\frac{1}{2}\sigma^2(K+K^2)t}\right].$$

Proof: Using the notation $g_L(t)$ for the long trade gain-loss formula from Theorem 6.2, the gain-loss on the short trade, denoted $g_S(t)$, is similarly obtained by replacing I_0 and K with $-I_0$ and $-K$; i.e.,

$$g_S(t) = \frac{I_0}{K}\left[\left(\frac{p(t)}{p(0)}\right)^{-K} e^{-\frac{1}{2}\sigma^2(K+K^2)t} - 1\right].$$

Thus, the formula for $g(t)$ above is obtained as the sum of $g_L(t)$ and $g_S(t)$. The investment formula follows immediately using the fact that $I(t) = K(g_L(t) - g_S(t))$. ■

6.5 Remarks on Theorem: It is of interest to understand how the gain-loss $g(t)$ depends on the feedback gain K. As seen in Theorem 6.7 to follow, its expected value, $\mathbb{E}[g(t)]$, increases monotonically with respect to K. While this makes "high gain" attractive in an idealized market, it may not be the case in a real market where price jumps can occur; e.g., if a company announces a negative earnings surprise after the market closes, the stock may "gap" down at the open the following morning. On an intuitive level, in a real market, K should be set as a function of the underlying volatility; this is relegated to future research.

Fig. 3 provides a plot of $g(t)$ versus $p(t)/p(0)$ for various values of the volatility σ. The figure indicates that if the stock price $p(t)$ is within an interval about the initial price $p(0)$, the feedback controller will incur a loss. Beginning with the formula above, in the sections to follow, we derive the statistics of $g(t)$ and quantify its win-loss boundary.

6.6 Positivity of the SLS Trading Gain: Since positivity of the gain-loss function is not guaranteed in a market with GBM prices, it is natural to ask: When will the expected value of the trading gain be positive? Remarkably, the following theorem indicates that the answer to this question is "always."

6.7 Robust Positive Expectation Theorem: In an idealized frictionless market with GBM prices, for $t \geq 0$, the expectation, variance and worst-case loss resulting from the SLS feedback control are given, respectively, by

$$\mathbb{E}\left[g(t)\right] = \frac{I_0}{K}\left[e^{\mu K t} + e^{-\mu K t} - 2\right]$$

$$\mathrm{Var}\left[g(t)\right] = \frac{I_0^2}{K^2}\left(e^{\sigma^2 K^2 t} - 1\right)\left(e^{2\mu K t} + e^{-2\mu K t} + e^{-\sigma^2 K^2 t}\right)$$

$$g^*(t) = \frac{2I_0}{K}\left[e^{-\frac{1}{2}\sigma^2 K^2 t} - 1\right].$$

Moreover, except for the trivial break-even case when $\mu = 0$

$$\mathbb{E}\left[g(t)\right] > 0.$$

Proof: Noting that the k-th moment of a log-normal random variable X with $\log X \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$ is

$$\mathbb{E}[X^k] = e^{k\mu_Y + \frac{1}{2}k^2\sigma_Y^2}$$

using the notation $X \doteq p(t)/p(0)$, $\mu_Y \doteq (\mu - (1/2)\sigma^2)t$, $\sigma_Y^2 \doteq \sigma^2 t$ and

$$a = e^{\frac{1}{2}\sigma^2(K-K^2)t}; \quad c = e^{-\frac{1}{2}\sigma^2(K+K^2)t}$$

the expected value of the trading gain is given by

$$\mathbb{E}\left[g(t)\right] = \frac{I_0}{K}\left(a\mathbb{E}(X^K) + c\mathbb{E}(X^{-K}) - 2\right) = \frac{I_0}{K}\left(ae^{K\mu_Y + \frac{1}{2}K^2\sigma_Y^2} + ce^{-K\mu_Y + \frac{1}{2}K^2\sigma_Y^2} - 2\right)$$

which, upon substitution for μ_Y, σ_Y, a, and c, leads to the formula for $\mathbb{E}[g(t)]$. Now, to complete the proof for expectation, we simply note that $\mathbb{E}[g(t)] > 0$ follows immediately from the fact that the function $e^X + e^{-X} - 2$ is positive for $X \neq 0$. For the case of the variance, using $\mathrm{Var}[g(t)] = \mathbb{E}[g^2(t)] - \mathbb{E}^2[g(t)]$, we again obtain a linear combination of the moments of X, and, a straightforward substitution for μ_Y, σ_Y, a, and c leads to the result. Finally, the formula for the worst-case loss $g^*(t)$ comes from re-writing the formula for $g(t)$ in Theorem 6.4 as

$$g(t) = \frac{I_0}{K}e^{-\frac{1}{2}\sigma^2 K^2 t} \times \left[\left(\frac{p(t)e^{\frac{1}{2}\sigma^2 t}}{p(0)}\right)^K + \left(\frac{p(t)e^{\frac{1}{2}\sigma^2 t}}{p(0)}\right)^{-K} - 2e^{\frac{1}{2}\sigma^2 K^2 t}\right].$$

Using the fact $X^K + X^{-K} \geq 2$ gives the result for $g^*(t)$. ■

6.8 Quantifying the Win/Loss Boundary: In the next subsection, we estimate the range of prices at time t for which the trade is winning or losing. It is seen below that there is an open interval of prices $(p_-(\sigma, K, t), p_+(\sigma, K, t))$, about $p(0)$, in which the SLS feedback controller results in a losing trade.

6.9 Lemma: In an idealized frictionless market with GBM prices, for $t \geq 0$, the SLS feedback controller results in a losing trade $g(t) < 0$ if and only if

$$p(t) \in (p_-(\sigma, K, t), p_+(\sigma, K, t))$$

where

$$p_\pm(\sigma, K, t) \doteq \left[e^{-\frac{1}{2}\sigma^2(K-K^2)t}\left(1 \pm \sqrt{1 - e^{-\sigma^2 K^2 t}}\right)\right]^{\frac{1}{K}} p(0).$$

Additionally, the probability of a loss is

$$P\left(g(t) < 0\right) = \Phi\left(\frac{y_+ - \mu_Y}{\sigma_Y}\right) - \Phi\left(\frac{y_- - \mu_Y}{\sigma_Y}\right)$$

where, as in the proof of Theorem 6.7, $\mu_Y \doteq (\mu - \frac{1}{2}\sigma^2)t$

$$y_\pm = \log\left[\frac{1}{a}\left(1 \pm \sqrt{1 - ac}\right)\right]^{\frac{1}{K}}$$

$$a \doteq e^{\frac{1}{2}\sigma^2(K-K^2)t}, \quad c \doteq e^{-\frac{1}{2}\sigma^2(K+K^2)t}$$

and $\Phi(\cdot)$ is the cumulative distribution function of the standard normal random variable.

Proof: Substituting new variable $X \doteq (p(t)/p(0))^K$ into the gain-loss equation and setting it equal to zero gives $aX^2 - 2X + c = 0$. Then, solving this quadratic equation and substituting back in for a, c and X leads to lower and upper loss boundary limits $p_\pm(\sigma, K, t)$. Using the simplifying notation a and c introduced above, the probability of a loss becomes

$$P\left(g(t) < 0\right) = P\left(\left[\frac{1}{a}(1 - \sqrt{1 - ac})\right]^{\frac{1}{K}} < \frac{p(t)}{p(0)} < \left[\frac{1}{a}(1 + \sqrt{1 - ac})\right]^{\frac{1}{K}}\right).$$

Recalling that $p(t)/p(0)$ is generated via Geometric Brownian Motion, it is log-normally distributed with $Y = \log(p(t)/p(0))$ having mean $\mu_Y = (\mu - (1/2)\sigma^2)t$ and standard deviation given by $\sigma_Y = \sigma\sqrt{t}$; e.g., see [65]. Now introducing the classical normalization, $Z \doteq (Y - \mu_Y)/\sigma_Y$, for this standard normal random variable, defining

$$y_\pm \doteq \log\left[\frac{1}{a}(1 \pm \sqrt{1 - ac})\right]^{\frac{1}{K}}$$

the probability of loss is obtained as

$$P\left(g(t) < 0\right) = \Phi\left(\frac{y_+ - \mu_Y}{\sigma_Y}\right) - \Phi\left(\frac{y_- - \mu_Y}{\sigma_Y}\right).$$

■

6.10 Probability Density Function for the SLS Gain-Loss Function: The following notation facilitates presentation of results in the theorem to follow:

$$X_{\pm}(x,t) \doteq \frac{1}{2}e^{-\frac{1}{2}\sigma^2(K-K^2)t} \times \left[\left(\frac{K}{I_0}x+2\right) \pm \sqrt{\left(\frac{K}{I_0}x+2\right)^2 - 4e^{-\sigma^2 K^2 t}}\right]$$

$$\nu \doteq \mu - \frac{\sigma^2}{2}; \quad Z_{\pm}(x,t) \doteq \frac{\log X_{\pm}^{\frac{1}{K}}(x,t) - \nu t}{\sigma\sqrt{t}}$$

$$A(x,t) \doteq \frac{1}{\sigma I_0 \sqrt{2\pi t}\sqrt{\left(\frac{K}{I_0}x+2\right)^2 - 4e^{-\sigma^2 K^2 t}}}.$$

6.11 Theorem: In an idealized frictionless market with GBM prices, for $t > 0$, the probability density function $f(x,t)$ for $g(t)$ is as follows: For $x \le g^*(t)$, the worst-case loss given in Theorem 6.7, $f(x,t) \equiv 0$, and, for $x > g^*(t)$

$$f(x,t) = A(x,t)\left(e^{-\frac{1}{2}Z_+^2(x,t)} + e^{-\frac{1}{2}Z_-^2(x,t)}\right).$$

Proof: Beginning with the formula

$$g(t) = \frac{I_0}{K}\left[\left(\frac{p(t)}{p(0)}\right)^K e^{\frac{1}{2}\sigma^2(K-K^2)t} + \left(\frac{p(t)}{p(0)}\right)^{-K} e^{-\frac{1}{2}\sigma^2(K+K^2)t} - 2\right]$$

using shorthand notation $X \doteq (p(t)/p(0))^K$, we break the proof into two cases:

The Zero Probability Density Function (PDF) Case: To minimize $g(t)$ with respect to X, we set the derivative of the convex function

$$G(X) \doteq aX + cX^{-1} - 2$$

to zero and obtain minimizer $X^* = \sqrt{c/a} = e^{-(1/2)\sigma^2 Kt}$ and associated price ratio $p^*(t)/p(0) = e^{-(1/2)\sigma^2 t}$. Hence, the trading gain corresponds to a loss given by $g^*(t)$. It follows that for $x \le g^*(t)$, the Cumulative Distribution Function (CDF) is given by:

$$F(x,t) \doteq P(g(t) \le x) = 0$$

and the corresponding PDF is $f(x,t) = (\partial F(x,t)/\partial x) = 0$.

The Positive PDF Case: We first calculate the (CDF)

$$F(x,t) = P(g(t) \le x) = P\left(\frac{I_0}{K}(aX + cX^{-1} - 2) \le x\right).$$

Letting $b \doteq (K/I_0)x + 2$, the CDF above reduces to

$$F(x,t) = P(aX^2 - bX + c \le 0)$$
$$= P(X_-(x,t) \le X \le X_+(x,t))$$

where

$$X_{\pm} \doteq \frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}.$$

Now expressing the CDF in terms of the underlying prices and noting that the random variable

$$Y \doteq \log\frac{p(t)}{p(0)} = \log X^{\frac{1}{K}}$$

is $\mathcal{N}(\nu t, \sigma^2 t)$, using the classical normalization, it follows that:

$$Z \doteq \frac{\log X^{\frac{1}{K}} - \nu t}{\sigma\sqrt{t}}$$

is $\mathcal{N}(0,1)$ and

$$F(x,t) = P(Z_- \le Z \le Z_+) = \frac{1}{\sqrt{2\pi}}\int_{Z_-}^{Z_+} e^{-\frac{\zeta^2}{2}}\,d\zeta.$$

To obtain the PDF, we use Leibnitz rule to differentiate the integral above. That is, we obtain the density $f(x,t) = f_+(x,t) - f_-(x,t)$ where

$$f_{\pm}(x,t) \doteq \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}Z_{\pm}^2}\frac{\partial Z_{\pm}}{\partial x}.$$

Now calculating the partial derivative

$$\frac{\partial Z_+}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\log X_+^{\frac{1}{K}} - \nu t}{\sigma\sqrt{t}}\right) = \frac{1}{KX_+\sigma\sqrt{t}}\frac{\partial X_+}{\partial x}$$

and substituting into $f_+(x,t)$ and $f_-(x,t)$, we obtain

$$f_{\pm}(x,t) = \frac{1}{KX_{\pm}\sigma\sqrt{2\pi t}}e^{-\frac{1}{2}Z_{\pm}^2}\frac{\partial X_{\pm}}{\partial x}.$$

To complete the calculation of $f(x,t)$, we note that it is straightforward to verify that

$$\frac{1}{X_+}\frac{\partial X_+}{\partial x} = \frac{K}{I_0\sqrt{b^2-4ac}}$$
$$= \frac{K}{I_0\sqrt{\left(\frac{K}{I_0}x+2\right)^2 - 4e^{-\sigma^2 K^2 t}}}$$
$$= -\frac{1}{X_-}\frac{\partial X_-}{\partial x}.$$

Upon substituting into $f_+(x,t)$, $f_-(x,t)$, and $f(x,t)$ and recalling the definition of $A(x,t)$, we obtain the formula for the PDF. ■

6.12 Remarks: Fig. 4 includes illustrative plots of the probability density function $f(x,t)$ for the gain-loss $g(t)$ which were obtained via the theorem with $I_0 = 1$, $K = 4$, $t = 0.5$, and $\sigma = 0.2$. One salient feature of these plots, consistent with common sense, is that the larger the realized value of the ratio $|\mu|/\sigma$, the more attractive the trade becomes. For example, in the figure, for those plots when $|\mu|/\sigma > 2$, it is obvious by inspection that the probability of a significant rate of return is quite high; e.g., when $|\mu| = 1$, by performing a straightforward integration, we conclude that the probability that the raw rate of return is 100% or greater is about 0.55. For this same scenario, the probability of a 25% or greater return is about 0.94. On the other hand, when the realized value of this ratio is small, the likelihood

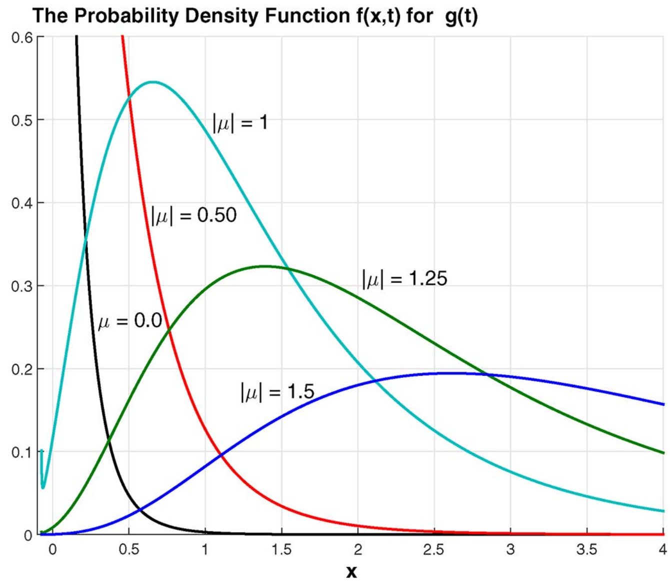

Fig. 4. PDF with $\sigma = 0.2$, $K = 4$, $t = 0.5$, and $I_0 = 1$.

of a loss can be quite high. For example, for the "driftless" case corresponding to $\mu = 0$, a straightforward integration of the density function results in a probability $p \approx 0.62$ of a loss as compared to a probability $p \approx 0.2$ that a raw return of 10% or more will result. However, note that the maximum loss cannot exceed 7% as $g^*(0.5) \approx -0.07$.

6.13 Leverage Considerations: The motivation for this section is derived from the adequacy of resources assumption provided in Section III-3.4. Our goal here is to quantify the so-called leverage which can result from SLS trading. That is, we study the extent to which the trader relies on "credit" from the broker as a consequence of investing at levels $I(t)$ which exceed the account value $V(t)$. This is important because a typical broker, for risk mitigation purposes, will not allow this leverage to get too high. When the broker's leverage limit is exceeded, the trader is either forced to deposit additional assets in the account to serve as collateral, or to have some of the stock positions liquidated to bring the account into compliance.

Proceeding more formally, for a given (μ, σ), we let $\mathcal{P}$ denote the set of admissible GBM sample paths. Then, given a sample path $p \in \mathcal{P}$ and a controller parameterized by its initial investment $I_0 > 0$ and feedback gain $K > 0$, for the resulting investment level $I(t)$ and account value $V(t)$, at time $t \geq 0$, the *pointwise leverage* $L(t)$ is the smallest number $\gamma \geq 0$ such that the condition

$$|I(t)| \leq \gamma V(t)$$

is satisfied. Equivalently

$$L(t) \doteq \inf \left\{ \gamma \geq 0 : |I(t)| \leq \gamma V(t) \right\}.$$

Since leveraged trading with GBM prices includes the theoretical possibility that $V(t) \leq 0$, when the set of γ above is empty, we use the standard mathematical convention that the infimum over an empty set is $+\infty$.

To explore the leverage requirements of the SLS strategy, we begin by assuming that at time $t = 0$, the account value is

$$V(0) = \eta I_0$$

with $\eta > 0$. Then, using the formulae for $g(t)$ and $I(t)$ given in the theorem and noting that the account value is $V(t) = \eta I_0 + g(t)$, along price trajectories of practical interest for which $V(t) > 0$, a lengthy but straightforward computation leads to

$$L(t) = \frac{K \left| \left(\frac{p(t)}{p(0)} \right)^{2K} e^{K\sigma^2 t} - 1 \right|}{\left(\frac{p(t)}{p(0)} \right)^{2K} e^{K\sigma^2 t} + (K\eta - 2) \left(\frac{p(t)}{p(0)} \right)^{K} e^{\frac{1}{2}(K+K^2)\sigma^2 t} + 1}.$$

Given that the sample path $p \in \mathcal{P}$ above is not known in advance, the following question presents itself: Given an unknown sample path $p \in \mathcal{P}$, how large might $L(t)$ potentially become over the course of a trade? To answer this question, by emphasizing the dependence of $L(t)$ on the pair (p, K) by writing $L(t) = L(p, t, K)$, we define the *maximum possible leverage*

$$L_{\max}(K) \doteq \sup_{p \in \mathcal{P}, t \geq 0} L(p, t, K).$$

We see in the theorem below that if the feedback gain K is not suitably large, that is, if $K < 2/\eta$, the potential exists that some sample paths drive the account value negative, thus leading to $L_{\max} = \infty$. On the other hand for larger gains $K \geq 2/\eta$, we obtain $L_{\max} = K$. Furthermore, it is noted that this result is "universal" in the sense that it does not depend on the Brownian motion parameters μ and σ.

6.14 Leverage Theorem: In an idealized frictionless market with GBM prices, if $K \geq 2/\eta$, it follows that:

$$L_{\max}(K) = K.$$

If $K < 2/\eta$, then

$$L_{\max}(K) = +\infty.$$

Proof: First, note that at time $t = 0$, we have $I(0) = 0$ so $L(0) = 0$. Next, for $t > 0$, we consider two cases.

Case 1—$K < 2/\eta$: For convenience we first introduce the shorthand notation

$$X \doteq \left(\frac{p(t)}{p(0)} \right)^{K} e^{\frac{1}{2}K\sigma^2 t}; \; a \doteq e^{-\frac{1}{2}K^2\sigma^2 t}; \; \zeta(a) \doteq \frac{K\eta - 2}{a}$$

and use Theorem 6.4 to obtain the account value as

$$V = \frac{I_0 a}{KX} \left(X^2 + \zeta(a) X + 1 \right).$$

To dispense with this infinite leverage case, it suffices to show that there exists a sample path $p \in \mathcal{P}$ and an associated time $t > 0$ which leads to $V(t) < 0$. Note that because p is a GBM, for any $t > 0$, $p(t)$ is log-normally distributed with support $p(t) > 0$. Thus, any pair (a, X) with $a \in (0, 1)$ and $X > 0$ is realizable via some sample path $p \in \mathcal{P}$ and some $t > 0$. Therefore, it suffices to make a selection of $X > 0$ and $a \in (0, 1)$ such that

$$X^2 + \zeta(a) X + 1 < 0.$$

This is simply accomplished by taking $X = 2$, noting that $\zeta(a) < 0$ and that $\zeta(a) \to -\infty$ as $a \to 0$.

Case 2: $K \geq \frac{2}{\eta}$*:* In this case, noting that $\zeta(a) \geq 0$, the account value cannot vanish and again using Theorem 6.4, the pointwise leverage function is readily calculated to be

$$L(t) = \frac{K|X^2 - 1|}{X^2 + \zeta(a)X + 1} \doteq f(a, X, K).$$

We call $f(a, X, K)$ the "surrogate" for the leverage. Now, to obtain $L_{\max}(K)$ we reduce the optimization over the GBM paths to a static problem by noting that

$$\begin{aligned} L_{\max}(K) &= \sup_{p \in \mathcal{P}, t \geq 0} L(p, t, K) \\ &= \sup_{X > 0, a \in (0,1)} f(a, X, K). \end{aligned}$$

Holding parameter $a \in (0, 1)$ fixed and noting that $f(a, 1, K) = 0$ and

$$\lim_{X \to \infty} f(a, X, K) = K$$

to establish that the supremum of $f(a, X, K)$ with respect to $X \geq 1$ is K, it suffices to prove that $f(a, X, K)$ is monotonically increasing for $X > 1$. Indeed, partial differentiation yields

$$\frac{\partial f}{\partial X} = K \frac{\zeta(a)X^2 + 4X + \zeta(a)}{(X^2 + \zeta(a)X + 1)^2}.$$

Now, since $\zeta(a) \geq 0$, it follows that:

$$\frac{\partial f}{\partial X} > 0.$$

Next, to complete the proof for this case, we show that $f(a, X, K) \leq K$ for $X < 1$. In this case, noting that $f(a, 0, K) = K$ and that $\partial f / \partial X < 0$, for $X \in (0, 1)$, $f(a, X, K)$ decreases from K to 0, which implies that K is the supremum on this interval too. ■

6.15 Remark: While $K \geq 2/\eta$ assures that leverage is bounded, in practice, the broker's restrictions can lead to saturation on the investment function $I(t)$ when $L_{\max}(K) \geq 2$; see Section VII for further analysis.

6.16 GBM Numerical Simulation: To conclude this section, we provide a demonstration of the mechanics of Simultaneous Long-Short trading using Geometric Brownian Motion as a benchmark. In particular, we consider parameter values of $I_0 = 1$, $V(0) = 1$, and $K = 4$, and allow the investment amount $I(t)$ to be updated daily over a six-month time period. Our discrete-time implementation is now described. First, the price process is simulated with each time-step representing a single day. Using an annualized drift of $\mu = -0.45$ and volatility $\sigma = 0.15$, we are emulating a downward trending stock. Now, the prices are updated using the discretized version of GBM

$$p(k+1) = \left(1 + \frac{\mu}{252} + \frac{\sigma}{\sqrt{252}} n(k)\right) p(k)$$

where $n(k)$ is a standard normal random variable with mean zero and variance one.

As far as updating the gain-loss function is concerned, profits and losses are similarly considered on a daily basis. Next, to implement the trading dynamics, we use the discrete-time counterparts at day k for p, I_L, I_S, g_L, g_S, and V. Using daily returns $\rho(k) \doteq (p(k+1) - p(k))/p(k)$, the trading and accounting dynamics are governed by the update equations: $I_L(k) = I_0 + Kg(k)$, $I_S(k) = -I_0 - Kg_S(k)$, $g_L(k+1) = g_L(k) + \rho(k) I_L(k)$, $g_S(k+1) = g_S(k) + \rho(k) I_S(k)$, $g(k+1) = g_L(k) + g_S(k)$ and $V(t) = V(0) + g(t)$. The results for a typical sample path are shown in Figs. 5–7. The price path, shown in Fig. 5, experiences a significant downward trend. Fig. 6 displays the behavior of the gain-loss functions, $g_L(k)$ and $g_S(k)$. Consistent with the market decline, $g_L(k)$ becomes negative during the downtrend, indicating a loss on the long side of the trade, while $g_S(k)$ grows increasingly positive, reflecting the profitability of the short side. The positivity of the overall gain-loss function $g(k) = g_L(k) + g_S(k)$ is consistent with the fact that the controller reacts adaptively to the declining stock price by "amplifying" the short side of the trade and "attenuating" the long side. This adaptive nature of the SLS approach can also be seen in Fig. 7, where the investment functions are plotted. By the end of the six-month period, $I_L(k)$ has been attenuated nearly to zero and $I_S(k)$ is dominating the trade.

Fig. 5. Prices for GBM simulation.

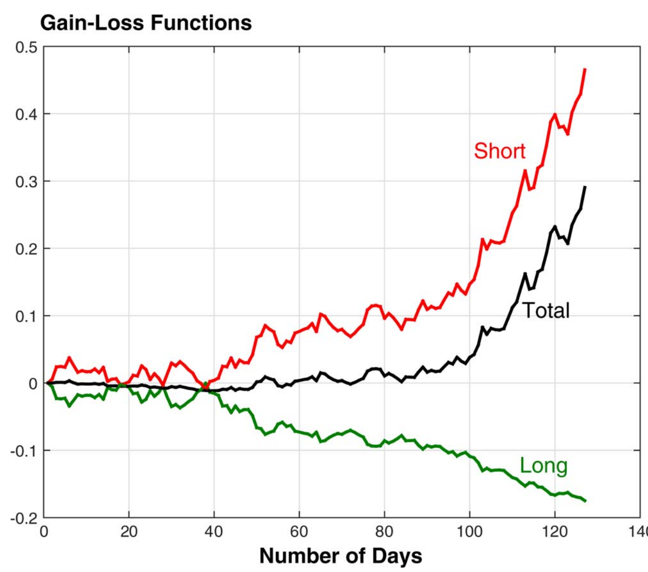

Fig. 6. Trading gains or losses.

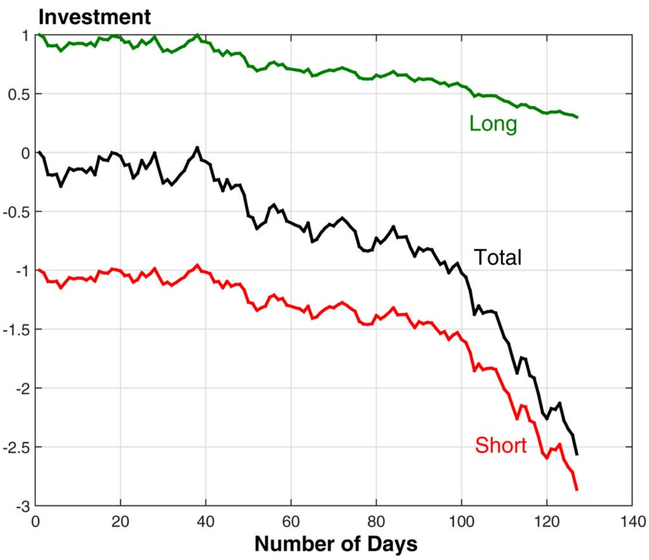

Fig. 7. Investment levels.

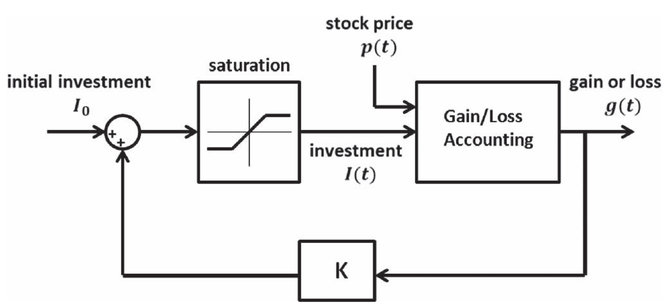

Fig. 8. Saturation for each side of trade.

VII. Extension: Saturation Control

In this section, we sketch the key ideas associated with saturation in SLS trading. That is, in practice, when the trader begins with $I(0) = I_0$ as the investment, it is typically the case, be it for reasons of risk aversion, diversification or collateral constraints imposed by the broker, that the investment is limited to some level which we denote by $I_{\max} = \theta I_0$ with $\theta > 1$. That is, for the long and short sides of a trade, we impose the constraints

$$I_L(t) = \min\{I_0 + Kg_L(t), \theta I_0\}$$
$$I_S(t) = -\min\{I_0 + Kg_S(t), \theta I_0\}.$$

The formulation above introduces a classical nonlinearity into the state equation. As seen in Fig. 8 for the long side of the trade, the short side is similarly modified to accommodate this constraint. For this new system, our main objective is to demonstrate the type of analysis which is possible for this saturation case. To this end, we restrict ourselves to smooth prices and relegate the GBM case to further research.

Theorem 7.1: In an idealized frictionless market with smooth prices, consider the Simultaneous Long-Short controller with investment saturation level $I_{\max} = \theta I_0$ with $\theta > 1$. Then, for $t \geq 0$, except for the trivial break-even case when $p(t) = p(0)$, it follows that the arbitrage condition

$$g(t) > 0$$

is satisfied. Moreover, an explicit formula for the gain-loss under saturation is $g(t) = g_L(t) + g_S(t)$ where

$$g_L(t) = \begin{cases} \frac{I_0}{K}\left(\left(\frac{p(t)}{p(0)}\right)^K - 1\right) & \frac{p(t)}{p(0)} < \theta^{1/K} \\ \frac{I_0}{K}(\theta - 1) + \theta I_0 \log\left(\theta^{-1/K}\frac{p(t)}{p(0)}\right) & \frac{p(t)}{p(0)} \geq \theta^{1/K} \end{cases}$$

$$g_S(t) = \begin{cases} \frac{I_0}{K}\left(\left(\frac{p(t)}{p(0)}\right)^{-K} - 1\right) & \frac{p(t)}{p(0)} > \theta^{-1/K} \\ \frac{I_0}{K}(\theta - 1) - \theta I_0 \log\left(\theta^{1/K}\frac{p(t)}{p(0)}\right) & \frac{p(t)}{p(0)} \leq \theta^{-1/K}. \end{cases}$$

Proof: The formulae for $g_L(t)$ and $g_S(t)$ are verified by direct substitution into the saturated differential equations

$$\frac{dg_L}{dt} = \rho(t)\min\{I_0 + Kg_L(t), \theta I_0\}$$
$$\frac{dg_S}{dt} = -\rho(t)\min\{I_0 + Kg_S(t), \theta I_0\}$$

with initial conditions $g_L(0) = g_S(0) = 0$. In the region $\theta^{-1/K} < p(t)/p(0) < \theta^{1/K}$, the positivity of $g(t)$ follows from Theorem 5.3 as the gain-loss formulae are identical in that region.

For $p(t)/p(0) \geq \theta^{1/K}$, positivity follows by showing that the gain-loss function $g(t)$ is an increasing function of the price $p(t)$ in that region. Indeed, note that $g(t) = g_L(t) + g_S(t) > 0$ at $p(t)/p(0) = \theta^{1/K}$ and

$$\frac{dg(t)}{dp(t)} = \frac{I_0}{p(t)}\left[\theta - \left(\frac{p(t)}{p(0)}\right)^{-K}\right] > 0$$

for $p(t)/p(0) \geq \theta^{1/K}$. A similar argument shows that $g(t) > 0$ in the region $p(t)/p(0) \leq \theta^{-1/K}$ by noting that $g(t) > 0$ for $p(t)/p(0) = \theta^{-1/K}$ and that

$$\frac{dg(t)}{dp(t)} = \frac{I_0}{p(t)}\left[\left(\frac{p(t)}{p(0)}\right)^{K} - \theta\right] < 0$$

for $p(t)/p(0) \leq \theta^{-1/K}$. ■

VIII. Simulation: Trading Facebook

In this section, the main objective is to demonstrate how the SLS controller behaves with real data in a practical setting. In accordance with the long-short adaptive mechanism underlying the controller, if the stock is trending upwards, one expects to see amplification of the long investment $I_L(t)$ and attenuation of the short. Similarly, the reverse holds true when the price is trending downward. With these considerations in mind, we ran two back-tests using daily closing stock prices for Facebook (FB) covering the two years 2013 and 2014. The first simulation was carried out using the "realized" forward prices, and in the second simulation, we used the reverse-order prices. Both of these price paths are shown in Fig. 9.

As described in Section VI-6.16, a discrete-time implementation of the controller with feedback gain $K = 2$ is used and profits and losses are tallied on a daily basis. The initial account value is assumed to be $V_0 = 10\,000$ and the trader is initially 50% invested; i.e., $I_0 = 5000$. Finally, per analysis in Section VII, we include the saturation constraint $I_{\max} = \theta I_0$ with $\theta = 4$. Hence, we limit the investment level to \$20 000.

For the first test with forward prices, following the initial period of decline corresponding to the time period immediately

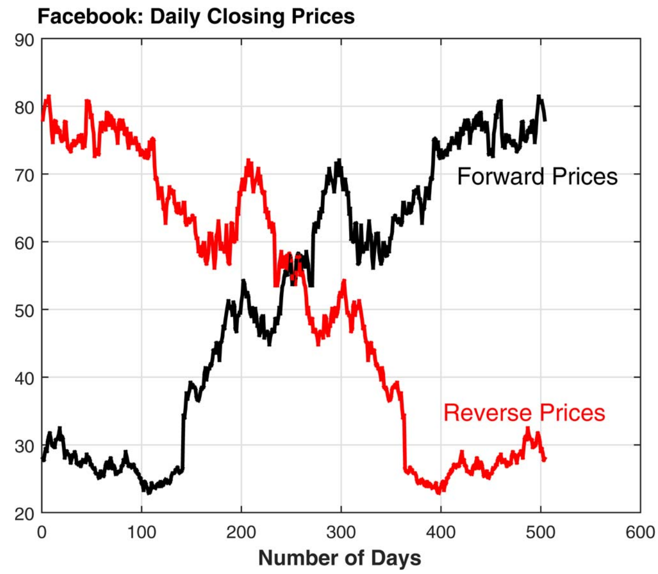

Fig. 9. FB daily closing prices.

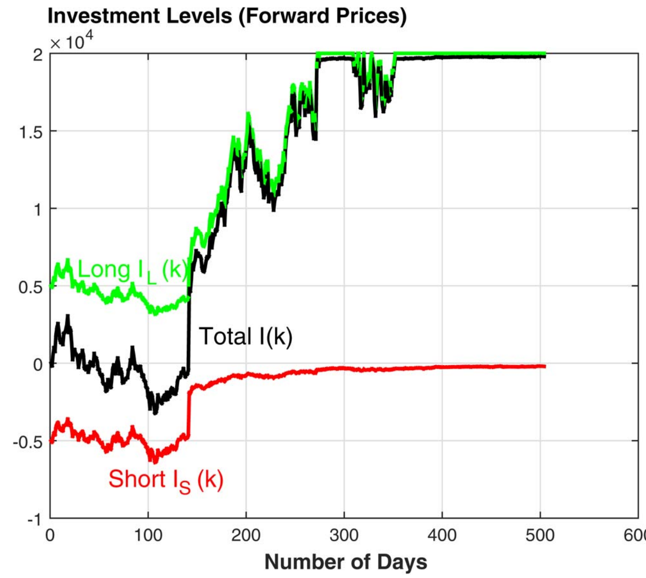

Fig. 11. Investment levels with forward prices.

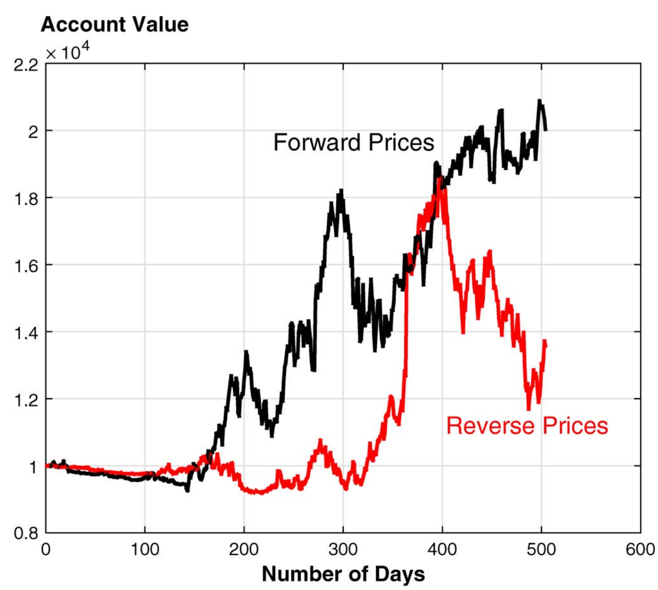

Fig. 10. Account value $V(k)$.

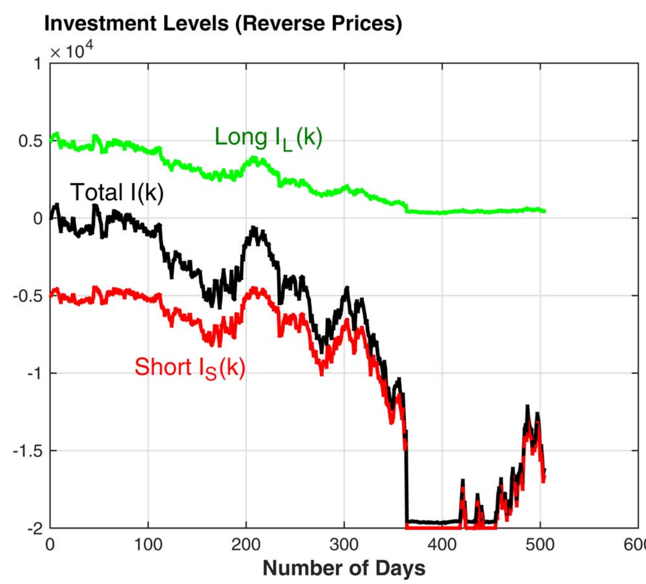

Fig. 12. Investment levels with reverse prices.

after the initial public offering, we largely encounter an upward trending market. For the second test, as mentioned above, we use the reverse-order prices which are seen to be downward trending. One of our objectives here is to demonstrate that the controller is "smart enough" to adapt to either market direction. For these two cases, Fig. 10 shows evolution of the account value $V(k)$. For the forward-price case, Fig. 11 shows the corresponding investment levels, $I_L(k)$ and $I_S(k)$, as well as the net investment level $I(k)$. As seen in the figure, the SLS controller initially favors the short side of the trade as the stock price decreases slightly. However, as the stock price begins to increase, the controller adapts the investment levels and latches onto the long side of the trade $I_L(k)$ while attenuating the short side $I_S(k)$ towards zero. As the price of Facebook further increases, the long investment level $I_L(k)$ eventually saturates. With reverse-order prices, as seen in Fig. 12, the situation is reversed. That is, the short investment $I_S(k)$ eventually dominates and becomes saturated as $I_L(k)$ tends to zero.

IX. Conclusion

In this paper, we described a new feedback control paradigm for stock trading which uses no *a priori* model for the evolution of the stock price. Motivated by the fact that existing work in such a setting involves extensive statistical analysis of a trading algorithm via back-testing with historical data, one main goal in this line of research is to develop a theoretical framework aimed at an explanation of successes and failures. To this end, we provide conditions under which certain robust performance properties are guaranteed. Our point of view is that theoretical performance guarantees in an idealized market with benchmark prices is a necessary condition for credibility of a trading scheme. That is, if a trading algorithm performs well in this idealized setting, it lends credence to back-tested results obtained using real market data which are more complex. Given that this line of research is in its infancy, there are literally dozens of directions for the continuation of this work. Below, we provide a "sampling" of issues which one might consider.

9.1 More General Portfolio Considerations: The results given in this paper involve trading a single stock. Going forward, it would be important to consider the extent to which the results given here can be extended to address trading a portfolio of n stocks. Instead of using a covariance model to weight assets as in classical finance [66], it would be of interest to consider an adaptive scheme possibly along the lines of the Simultaneous Long-Short algorithm. We envision dynamic adaptation of weights $w_1(t), w_2(t), \ldots, w_n(t)$ which are used to determine how the investor should "divvy up" the total investment $I(t)$ among the n stocks.

9.2 Controller Restart: In future work, it would be interesting to study the issue of *controller restart*. By this we mean the following: The controller could include a triggering mechanism to reset the long and short investment levels back to I_0 at some strategically chosen time $t_* > 0$. To understand the potential advantages associated with such a scheme, imagine that this reset is triggered if

$$\min\{I_L(t), |I_S(t)|\} \le I_{\min}$$

where $I_{\min} > 0$ is specified in advance; e.g., say $I_{\min} = 0.1 I_0$. In this case, when one side of the trade is at the 10% level, it is viewed as "played out." Subsequently, a restart of the trade with $I_L(t^*) = I_0$ and $I_S(t^*) = -I_0$ opens the door to adaptation to a market reversal. However, there are also possible negatives associated with such a scheme. Namely, by disallowing either the long or short side to be totally dominant, when the stock price continues moving in the same direction as it did for $t < t^*$, the gains with restart are less than they otherwise would be by leaving well enough alone. Tradeoffs associated with this scenario would be of interest to study.

9.3 Time-Varying Extension: The Robust Positive Expectation Theorem given in Section VI-6.7 may be extended to accommodate Geometric Brownian Motion with time-varying drift $\mu(t)$ and volatility $\sigma(t)$. For this case, our recent research [10] indicates that if these time variations are continuous, a more general positive expectation result is obtained with

$$\mathbb{E}\left[g(t)\right] = \frac{I_0}{K}\left[e^{K\int_0^t \mu(\tau)d\tau} + e^{-K\int_0^t \mu(\tau)d\tau} - 2\right].$$

That is, for $K > 0$, $\mathbb{E}[g(t)]$ above is provably positive except for the trivial break-even case when $\mu(t)$ has zero integral.

9.4 Dynamic Feedback: To simplify the exposition of the new paradigm, we restricted attention to the case of a static feedback controller. However, it is worth noting that some initial research has also been conducted for the case of a dynamic controller. For example in [8], we consider the proportional plus integral (PI) controller

$$I(t) = I_0 + K_P g(t) + K_I \int_0^t g(\tau)d\tau$$

and perform analysis similar to that carried out in Section VI. In the PI case, the dynamics for the expectation of the gain-loss turn out to be described by a second order linear system. This leads to a number of possible new research problems involving exploration of the modes of the system.

9.5 Saturation Controller With Reset: The saturation controller in Section VII exhibits a phenomenon which is closely related to the so-called "anti-reset windup" problem in classical control. To elaborate, we consider the long linear controller $I(t) = I_0 + Kg(t)$ with saturation level $I_{\max}$ as described in Section VII. Now, following a period of large trading gains, if a stock price reversal occurs, the controller may remain saturated for a long period of time because $I_0 + Kg(t)$ might be much larger than $I_{\max}$. To make the controller more responsive to market reversals, in [1], a saturation-reset idea is introduced which adjusts the investment level via the increment

$$dI = \begin{cases} K\min\{dg, 0\} & I(t) \ge I_{\max} \\ Kdg & I(t) < I_{\max}. \end{cases}$$

In this case, no money is added to the long investment if $dg > 0$. However, if the long controller calls for a decrease in the investment level due to $dg(t) < 0$, this action is always taken. Such a "saturation-reset" strategy will automatically lock in gains if a trend reversal occurs. Our view is that this type of control scheme merits further research.

9.6 Selection of Feedback Gains: Throughout this paper, the feedback gain K was taken as given and certain properties such as robust positive expectation were invariant to K. However, when one gets into the detailed quantification of performance, it is immediately apparent that the size of K has a large effect; e.g., the mean and variance of $g(t)$ depend strongly on K. Given these considerations it is arguable that the most important new class of research problems emanating from our work involves "optimal gain selection." There are many directions to pursue. For example, one natural continuation of this work involves the development of new algorithms which dynamically adjust the feedback gain K based on considerations such as the "realized" drift, volatility and various risk metrics.

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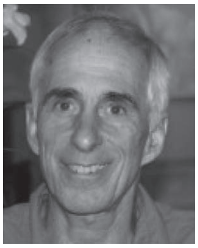

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