Stochastic, Robust and Intrinsically Stable MPC-based Controllers for Humanoid Gait Generation

Federico Baldisseri Arturo Maiani Giacomo Ripamonti

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Abstract

In presence of disturbances, the performance of MPC-based controllers typically deteriorates, possibly leading to constraint violation. In the case of humanoid gait generation, this can cause the loss of dynamic balance and internal instability with the subsequent fall of the robot.
The objective of this report is to implement a Stochastic-MPC-based gait generation algorithm for humanoids which is able to guarantee robustness in presence of disturbances.
Such algorithm is compared with a nominal IS-MPC and a Robust MPC.
The designed method is validated through dynamic simulations on the ROBOTIS OP-3 humanoid in Matlab.

1 Physical model

Most techniques for humanoid gait generation are based on the Zero Moment Point (ZMP, the point on the ground for which the horizontal components of the contact moments become zero), which ensures dynamic balance if it is kept inside the support polygon of the robot. The ZMP can be indirectly controlled by generating an appropriate Center of Mass (CoM) trajectory, which is then kinematically tracked. Denote the position of the humanoid CoM and ZMP as (x_c, y_c, z_c) and $(x_z, y_z, 0)$, respectively. Assuming that motion takes place on flat horizontal ground with the CoM traveling at constant height, and neglecting angular momentum contributions around the CoM, it is obtained the simplified model known as Linear Inverted Pendulum (LIP), which, assuming that the robot walks maintaining a fixed orientation, has identical and decoupled x-axis (sagittal) and y-axis (coronal) dynamics.

2 ZMP as a virtual control signal

The optimization problem that is solved by the MPC framework aims at defining the desired ZMP position which will guarantee dynamic balance for the robot. Actually, it is not possible to have direct control over the physical position of the ZMP: it is only possible to measure its position by force sensors under the feet of the robot. In this particular framework, the numeric signal obtained from the Quadratic Programming problem (QP) is used in order to generate a desired trajectory for the CoM by integration of the discretized model of the system. Such reference represents the position that the CoM would acquire if the ZMP was exactly in the position computed by the QP. The desired CoM reference is given to a kinematic controller such as:

$$\dot{q} = J^{\#}_{CoM}(\dot{c}_d + k(c_d - c)).$$

As a matter of fact, the measurement of the actual ZMP is similar to the virtual signal generated from the QP as shown in [2].

3 Stocastic Model Predictive Control SMPC

This section concernes an Optimal Control Problem OCP regarding a LTI system with additive noise subject to State and Control Chance constraints:

$$x_{t+i+1|t} = Ax_{t+i|t} + Bu_{t+i|t} + w_{t+i} \tag{1a}$$
$$\Pr\left\{H_j\ x_{t+i+1|t} \le h_j\right\} \ge 1 - \beta_{x_j} \tag{1b}$$
$$j = 1, 2, ..., n_x$$
$$\Pr\left\{G_j\ u_{t+i|t} \le g_j\right\} \ge 1 - \beta_{u_j} \tag{1c}$$
$$j = 1, 2, ..., n_u.$$

In order to solve this problem, the state equation is split into its deterministic and stochastic part as follows:

$$s_{t+i+1|t} = As_{t+i|t} + Bv_{t+i|t} \tag{2a}$$
$$s_{t|t} = x_{t|t},$$
$$e_{t+i+1|t} = A_K e_{t+i|t} + w_{t+i} \tag{2b}$$
$$e_{t|t} = 0.$$

This division requires the knowledge of the error i.e. full state knowledge, and in addition it makes use of a classic feedforward plus feedback parametrization for the control input $u_{t+i|t}$:

$$u_{t+i|t} = v_{t+i|t} + K(x_{t+i|t} - s_{t+i|t}). \tag{3}$$

The matrix K can be selected in order to render the matrix $A_K \triangleq A + BK$ Shur.
Moreover note that inequality (1b) can be rewritten as follows:

$$\Pr\left\{H_j\ e_{t+i+1|t} \le h_j - H_j\ s_{t+i+1|t}\right\} \ge 1 - \beta_{x_j}.$$

Since the probability distribution of the noise w is assumed gaussian, i.e. $w \in \mathcal{N}(0, \Sigma_w)$, the mean value and covariance matrix of $e_{t+i+1|t}$ take value:

$$\mathbb{E}\left\{H_j e_{t+i+1|t}\right\} = 0 \tag{4}$$
$$\Sigma_{e_{t+i+1|t}} = A_K \Sigma_{e_{t+i|t}} A_K^T + \Sigma_w. \tag{5}$$

Therefore the variance of $H_j e_{t+i+1}$ is equal to:

$$\sigma^2_{t+i+1|t} = \mathbb{E}\left\{H_j e_{t+i+1|t} e^T_{t+i+1|t} H_j^T\right\} =$$
$$= H_j \Sigma_{e_{t+i+1|t}} H_j^T$$
$$\text{with} \quad \Sigma_{e_{t|t}} = 0.$$

In [1] the expression for $\sigma^2_{t+i+1|t}$ is is recursively generated as:

$$\sigma^2_{t+i+1|t} = H_j \Big[\sum_{j=0}^{i} A_K^j \Sigma_w (A_K^j)^T\Big] H_j^T.$$

Then the expression for the probability density function for $\xi \triangleq H_j e_{t+i+1}$ is:

$$p(\xi) = \frac{e^{-\frac{\xi^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}.$$

Therefore the contraint is imposing to find the value of the random variable $H_j e_{t+i+1}$ such that the Cumulative Density Function $\Phi(\eta_{x_j})$ assumes the value $1 - \beta_{x_j}$:

$$\Phi(\eta_{x_j}) = \int_{-\infty}^{\eta_{x_j}} p(\xi) d\xi = 1 - \beta_{x_j}.$$

Since $w \in \mathcal{N}(0, \Sigma_w)$, the closed form expression for $\Phi(\eta_{x_j})$ does not exist, still a numerical approximation can be used in order to compute $\Phi^{-1}(1 - \beta_{x_j})$.
The quantity η_{x_j} represents the lower bound for the possible values assumed by $h_j - H_j s_{t+i+1|t}$:

$$H_j\ e_{t+i+1|t} \le \eta_{x_j} \le h_j - H_j\ s_{t+i+1|t},$$

from which the constraint acting on $s_{t+i+1|t}$ takes the form:

$$H_j\ s_{t+i+1|t} \le h_j - \eta_x. \tag{6}$$

Similarly, such a constraint can be derived for the nominal control input $v_{t+i|t}$. Let us start by rewriting (1c) as:

$$\Pr\left\{G_j K e_{t+i|t} \le g_j - G_j\ v_{t+i|t}\right\} \ge 1 - \beta_{u_j}$$
$$j = 1, 2, ..., n_u.$$

The lower bound for $g_j - G_j\ v_{t+i|t}$ is computed according to $\eta_{u_j} = \tilde{\Phi}^{-1}(1 - \beta_{u_j})$. As previously shown, an inequality constraint for $v_{t+i|t}$ can be derived:

$$G_j\ v_{t+i|t} \le g_j - \eta_{u_j}. \tag{7}$$

The SMPC can finally be formulated as:

$$\begin{aligned}
&\min_v \mathbb{E}\left\{J(x_t, v)\right\} \\
\textbf{s.t.}\quad & s_{t+i+1|t} = As_{t+i|t} + Bv_{t+i|t} \quad i = 0, 1, ..., N-1 \\
& H_j\ s_{t+i+1|t} \le h_j - \eta_x \quad j = 1, 2, ..., n_x \\
& G_j\ v_{t+i|t} \le g_j - \eta_{u_j} \quad j = 1, 2, ..., n_u \\
& s_{t|t} = x_t,
\end{aligned}$$

where the cost index has been defined in order to perform a trajectory tracking:

$$J(x_t, v) = \sum_{i=0}^{N-1} \alpha(\dot{c}_t^d - \dot{c}_{t+i|t})^2 +$$
$$+2\beta(c_t^d - c_{t+i|t}) + \gamma(p_t^d - p_{t+i|t})^2$$

4 RMPC Tube Based Robust MPC

This section considers an OCP regarding a LTI system with additive noise subject to State and Control constraints:

$$x_{t+i+1|t} = Ax_{t+i|t} + Bu_{t+i|t} + w_{t+i} \tag{8a}$$
$$x_{t+i+1|t} \in \mathcal{X} \tag{8b}$$
$$u_{t+i|t} \in \mathcal{U}. \tag{8c}$$

An important assumption resides in the fact that $w_{t+i} \in \mathcal{W}$, where $\mathcal{W}$ is a polytopic compact set i.e. a closed and bounded delimited by hyperplanar surfaces, which includes the origin. An important class of polytopes are the convex polytopes, which can be enclosed in an hypersphere.
Also in this case it is desired to constrain the nominal evolution of the system, which has the same structure as in the SMPC:

$$s_{t+i+1|t} = As_{t+i|t} + Bv_{t+i|t}$$
$$u_{t+i|t} = v_{t+i|t} + K(x_{t+i|t} - s_{t+i|t}).$$

As before this feedback structure aims at stabilizing the error dynamics:

$$e_{t+i+1|t} = A_K e_{t+i|t} + w_{t+i}. \tag{9}$$

This dynamics can be analyzed according to the knowledge of the disturbance set $\mathcal{W}$. The concept of Robust Positive Invarianct set is introduced as follows:
A set $\mathcal{Z}$ is said to be a RPI set if and only if:

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which means that if $e_t \in \mathcal{Z}$ then $\forall \bar{t} > t$, $e_{\bar{t}} \in \mathcal{Z}$.
The symbol $\oplus$ represents the Minkowsky set sum:

$$\mathcal{A} \oplus \mathcal{B} = \{a + b | a \in \mathcal{A}, b \in \mathcal{B}\}.$$

An interesting visualization of such concept can be performed in two dimensions. Suppose $\mathcal{W}$ equal to a circle of radius 0.5 and $A_K = 0.5I$. Let us now verify that the set $\mathcal{Z}$ equal to a circle of radius 1 represents a RPI.

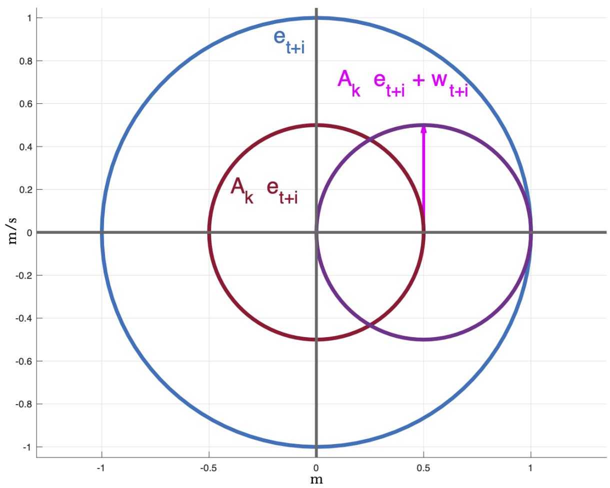

Figure 1: RPI set visualization

In this case the blue circle, which represents the boundary region for $\mathcal{Z}$, is transformed into the red circle by matrix A_K. Then, taking for instance the point $P = (0.5, 0)$ it is applied the disturbance vector $w_{t+i} = (0, 0.5)$. Even if the disturbance vector happened to be $w_{t+i} = (0.5, 0)$, the new error $e_{t+i+1|t}$ would still be contained into $\mathcal{Z}$. Furthermore $\mathcal{Z}$ represents also a minimal Robust Positive Invariant mRPI set: consider $\tilde{\mathcal{Z}}$ a circle of radius 0.8 , $e_{t+i|t} = (0.8, 0)$ and $w_{t+i} = (0.5, 0)$. Then $e_{t+i+1|t} = (0.9, 0)$, which falls out of $\tilde{\mathcal{Z}}$.
In general, the mRPI set associated is represented by:

$$\Omega = \bigoplus_{t=0}^{\infty} A_K^t \mathcal{W}. \tag{10}$$

The intuition behind this result is that:

$$e_{t+i+1|t} \in A_K\big(\bigoplus_{t=0}^{\infty} A_K^t \mathcal{W}\big) \oplus \mathcal{W} =$$
$$= \bigoplus_{t=0}^{\infty} A_K^{t+1} \mathcal{W} \oplus \mathcal{W} = \bigoplus_{t=0}^{\infty} A_K^t \mathcal{W}.$$

As a consequence, the admissible set where $s_{t+i+1|t}$ can take values is defined as:

$$s_{t+i|t} \in \mathcal{X} \ominus \Omega,$$

where $\ominus$ is the Pontryagin set difference and is defined as:

$$\mathcal{A} \ominus \mathcal{B} = \{a \in \mathcal{A} | a + b \in \mathcal{A}, \forall b \in \mathcal{B}\}.$$

Indeed in this case the set difference means that for every $e_{t+i+1|t} \in \Omega$ the sum between $e_{t+i+1|t}$ and $s_{t+i+1|t}$ must be contained in $\mathcal{X}$.
In addition, the constraint for $v_{t+i+1|t}$ can be derived from:

$$v_{t+i|t} \in \mathcal{U} \ominus K\Omega.$$

The RMPC can finally be formulated as a 2-part algorithm: the first part consists in solving the OC problem for the feedforward nominal control action $v_{t+i|t}$, while the second part is represented by the feedback action which is necessary in order to constrain the error in Ω:

$$\begin{aligned}
&\min_{v} \left\{ J(s_t, v) \right\} \\
\textbf{s.t.} \quad & s_{t+i+1|t} = As_{t+i|t} + Bv_{t+i|t} \quad i = 0, 1, ..., N-1 \\
& s_{t+i+1|t} \in \mathcal{X} \ominus \Omega \\
& v_{t+i|t} \in \mathcal{U} \ominus K\Omega \\
& s_{t|t} = x_t.
\end{aligned}$$

5 Intrinsically stable MPC

Similarly to SMPC and RMPC, this approach starts from the state space representation of the LIP dynamics:

$$\ddot{x_c} = \eta^2 (x_c - x_z) + w \tag{11}$$

$$\begin{pmatrix} \dot{x_c} \\ \ddot{x_c} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \eta^2 & 0 \end{pmatrix} \begin{pmatrix} x_c \\ \dot{x_c} \end{pmatrix} + \begin{pmatrix} 0 \\ -\eta^2 \end{pmatrix} z + \begin{pmatrix} 0 \\ 1 \end{pmatrix} w. \tag{12}$$

This system can be transformed in such a way that the matrix $\tilde{A}$ is diagonal:

$$\begin{pmatrix} x_u \\ x_s \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{\eta} \\ 1 & -\frac{1}{\eta} \end{pmatrix} \begin{pmatrix} x_c \\ \dot{x_c} \end{pmatrix}.$$

The corresponding dynamics are:

$$\begin{cases} \dot{x_u} = \eta \ x_u - \eta \ z + \frac{w}{\eta} \\ \dot{x_s} = -\eta \ x_s + \eta \ z - \frac{w}{\eta}. \end{cases}$$

As proven in [2], the explicit representation of the unstable dynamics is:

$$x_u(t, z) = e^{\eta(t-t_0)} x_u(t_0) - \eta \int_{t_0}^{t} e^{\eta(t-\tau)} z(\tau) d\tau + \\ + \frac{1}{\eta} \int_{t_0}^{t} e^{\eta(t-\tau)} w(\tau) d\tau.$$

The evolution of x_u remains bounded provided that a unique initial condition is selected:

$$x_u^*(t_0, z) \triangleq \eta \int_{t_0}^{\infty} e^{-\eta(\tau - t_0)} z(\tau) d\tau - \\ - \frac{1}{\eta} \int_{t_0}^{\infty} e^{\eta(t-\tau)} w(\tau) d\tau$$

Indeed by substituting (16) in (15) it is derived:

$$x_u(t_k, z) = \eta \int_{t_k}^{\infty} e^{\eta(t_k - \tau)} z(\tau) d\tau - \\ - \frac{1}{\eta} \int_{t_k}^{\infty} e^{\eta(t_k - \tau)} w(\tau) d\tau.$$

The value for $x_u(t_k, z)$ is bounded under mild boundedness conditions on the input $z(t)$. Note that such condition is not causal, since it depends on future values assumed by the control.
Before performing the discretization, an extended version for the LIP where the control input is the derivative of z is presented:

$$\begin{pmatrix} \dot{x_c} \\ \ddot{x_c} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \eta^2 & 0 & -\eta^2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_c \\ \dot{x_c} \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \dot{z}. \tag{13}$$

Assuming a control input piece-wise constant the integral becomes:

$$\eta \int_{t}^{\infty} e^{\eta(t-\tau)} z(\tau) d\tau = \eta \sum_{i=0}^{\infty} e^{-\eta \delta i} z(t_k + \delta i) \ \delta,$$

where δ is the integration step. Now an integration method for $z(t_k + \delta i)$ is used, e.g. Euler's method, Tustin's Method etc. Let us use Euler's method:

$$\mathrm{z}_{k+i} = z_{k+i-1} + \delta \dot{z}_{k+i-1}.$$

It is obtained:

$$\sum_{i=0}^{\infty} \dot{z}_{k+i} \ e^{-\eta \delta (i+1)} = \frac{\eta}{1 - e^{-\eta \delta}} (x_u^k - z_k).$$

In order to realize a causal version of this constraint, [3] considers both a Control Horizon $T_c = C\delta$ and a Preview Horizon $T_p = P\delta$, where $C < P$. The values of $\dot{z}$ for $i = 0, .., C-1$ must be chosen according to an optimization problem, while in $i = C, .., P-1$ they must be conjectured and are indicated as $\tilde{\dot{z}}$.
If a perturbed model is considered, the constraint equation becomes:

$$\sum_{i=0}^{C-1} \dot{z}_{k+i}\ e^{-\eta\delta i} = -\sum_{i=C}^{P-1} \tilde{\dot{z}}_{k+i}\ e^{-\eta\delta i} + \\ + \quad \frac{\eta}{1-e^{-\eta\delta}}(x_u^k - z_k + \frac{\tilde{w}_n}{\eta^2}). \qquad (14)$$

A fundamental difference between RMPC, SMPC and IS-MPC is that in the latter no explicit reference signal to track is available (a cost index such as eq. (8) is not present). The walking motion is performed by a time varying constraint for the ZMP, defined for every samling instant over the control horizon. This approach guarantees that if the ZMP is contained in the desired region, the CoM position will not be distant from there since an IS contraint has been enforced. In order to complete the structure of the optimization problem, the constraints for the ZMP must be added:

$$z_{min}(t) \leq z(t) \leq z_{max}(t). \qquad (15)$$

The optimization variables are collected in a vector:

$$\dot{Z}_k = (\dot{z}_k\ ,\ ...\ ,\ \dot{z}_{k+C-1})^T.$$

Then the Quadratic Programming Problem is set as follows:

$$\min_{\dot{Z}_k} \parallel \dot{Z}_k \parallel^2$$
$$\textbf{s.t.} \quad (14), (15).$$

The first sample of the previous OC problem is then fed into (17), which outputs a one step reference value for $x_c, \dot{x}_c$. This reference is tracked by means of a standard kinematic controller.

6 Recursive Feasibility

Recursive feasibility is a challenging aspect in the RMPC. From a theoretical point of view, it is difficult to guarantee recursive feasibility as the stochastic disturbance is possibly unbounded. However, the chance of having multiple high disturbances (with respect to the considered variance) is very low. Nevertheless, in multiple occasions disturbances do not allow the MATLAB function QUADPROG to obtain a solution.
In such scenario a back-up control strategy is used as follows: suppose that at time t the constrained optimization problem results infeasible along the prediction horizon, because of an expected constraint violation at time $t+i$, while at time $t-1$ feasibility was preserved along the entire horizon. At time t it is applied the following control input, which tries to recover the optimal trajectory computed in the previous time instant through a proportional controller:

$$u_{t+i} = v_{t+i|t-1} + K(x_{t+i|t} - s_{t+i|t-1}).$$

Where u is the actual control applied to the system, and v is the nominal control computed by the MPC. In the same fashion, x is the actual state, while s is the forseen state.
Since the results provided by this method showed to be beneficial for the RMPC, it has been implemented also in the other controllers.

7 Simulations

7.1 Parameters

The following numerical parameters have been used for the simulations:

- height of CoM= 0.33 cm
- footstep distance in x direction=6 cm
- footstep distance in y direction=9,8 cm
- foot width, a=5 cm
- sampling time δ=0.008 s
- $c^y_{max} = 5\ cm$
- constant speed in x direction=2 cm/s
- $\alpha = 0.5$, $\beta = 1$, $\gamma = 7$
- $\Sigma_w =\ diag(0,\ 0.0015^2)$
- $|w^{rob}_{max}| = [0,\ 0.003]$.

The disturbance used in [1] is characterized by $\Sigma_w = diag(0.008m,\ 0.008m/s)$ and $|w^{rob}_{max}| = [0.0016,\ 0.016]$. Such values correspond to a robot 5 times higher and with a foot area 6 times larger. Therefore in this report the disturbance has been diminished accordingly, of about 5 times.
Starting from the continuous time system, the discrete time one used for simulations is derived as follows:

$$A_d = exp(A\delta) \quad B_d = \int_0^{\delta} exp(A\psi)B\ d\psi. \tag{16}$$

In both Nominal and Stochastic MPCs, the feedback matrix K is chosen such that the eigenvalues of $A+BK$ are $[0.9\ ,\ 0.8]$, while in Robust MPC is:

$$K_{robust} = \frac{e^{\omega\delta}}{e^{\omega\delta}-1}\begin{pmatrix}1 & \frac{1}{\omega}\end{pmatrix}.$$

The problem needs to be formulated in such a way that it is compatible with the Matlab function **quadprog**. Indeed all optimization variables need to be stacked into a single vector:

$$J = \sum_{i=0}^{N-1} ||r_{t+1+i} - s_{t+1+i}||^2_Q + ||v^r_{t+i} - v_{t+i}||^2_\gamma$$
$$s_{t+1+i} = (c_{t+1+i}, \dot{c}_{t+1+i})^T \quad r_{t+1+i} = (c^d_{t+1+i}, \dot{c}^d_{t+1+i})^T$$
$$Q = diag(\alpha, \beta).$$

Stack the variables into a single vector:

$$\tilde{r} = (c^d_{t+1}, \dot{c}^d_{t+1}\ ...\ c^d_{t+1}, \dot{c}^d_{t+1})^T$$
$$\tilde{X} = (c_{t+1}, \dot{c}_{t+1}, c_{t+2}, \dot{c}_{t+2}\ ...\ , c_{t+N}, \dot{c}_{t+N})^T$$
$$\tilde{v}^r = (v^r_t, v^r_{t+1}\ ...\ v^r_{t+N-1})^T$$
$$\tilde{v} = (v_t, v_{t+1}\ ...\ v_{t+N-1})^T.$$

Now, the cost index can be rewritten as:

$$J = ||\tilde{r} - \tilde{X}||^2_{\tilde{Q}} + \gamma||\tilde{v}^r - \tilde{v}||^2.$$

Define $\tilde{Q} = diag(Q,\ ...\ Q)$:

$$J = ||\tilde{r}||^2_{\tilde{Q}} + ||\tilde{X}||^2_{\tilde{Q}} - 2\tilde{r}^T\tilde{Q}\tilde{X} + \gamma||\tilde{v}^r||^2 - 2\gamma(\tilde{v}^r)^T\tilde{v}.$$

Now if all variables are stacked into vector X, and all constant terms are canceled from the cost index, it is obtained:

$$J^{'} = X^T\begin{pmatrix}\tilde{Q} & 0\\ 0 & \gamma I_{N\times N}\end{pmatrix}X - 2\begin{pmatrix}\tilde{r}^T\tilde{Q} & \gamma(\tilde{v}^r)^T\end{pmatrix}X,$$

where:

$$X = (\tilde{X}^T, \tilde{v}^T)^T = (c_{t+1}, \dot{c}_{t+1}, c_{t+2}, \dot{c}_{t+2}\ ...\ , c_{t+N}, \dot{c}_{t+N}, v_t, v_{t+1}\ ...\ v_{t+N-1})^T.$$

The same structure for optimization varables is used in the dynamic constraints of the problem:

$$\begin{pmatrix}s_{t+1}\\ s_{t+2}\\ ...\\ s_{t+N}\end{pmatrix} = \begin{pmatrix}B & 0 & ... & 0\\ AB & B & ... & 0\\ & & ... & \\ A^{N-1}B & A^{N-2}B & ... & B\end{pmatrix}\begin{pmatrix}v_t\\ v_{t+1}\\ ...\\ v_{t+N-1}\end{pmatrix} + \begin{pmatrix}A\\ A^2\\ ...\\ A^N\end{pmatrix}s_t.$$

The constraint is reduced to:

$$\begin{pmatrix}I_{2N\times 2N} & -S\end{pmatrix}X = Ts_t.$$

The inequality constraint acting on the ZMP can be put in this form too (a is half width of the foot):

$$\begin{cases} v_t \le v^r_t + a - \eta_1\\ v_{t+1} \le v^r_{t+1} + a - \eta_2\\ ...\\ v_{t+N-1} \le v^r_{t+N-1} + a - \eta_{N-1}\\ -v_t \le -v^r_t + a - \eta_1\\ -v_{t+1} \le -v^r_{t+1} + a - \eta_2\\ ...\\ -v_{t+N-1} \le -v^r_{t+N-1} + a - \eta_{N-1}\end{cases}$$

$$\begin{pmatrix}0_{N\times 2N} & I\\ 0_{N\times 2N} & -I\end{pmatrix}X \le b_t,$$

where:

$$b_t = \begin{pmatrix}v^r_t + a - \eta_1\\ ...\\ v^r_{t+N-1} + a - \eta_{t+N-1}\\ -v^r_t + a - \eta_1\\ ...\\ -v^r_{t+N-1} + a - \eta_{t+N-1}\end{pmatrix}.$$

Recall that :

$$\eta_i = \Phi_i^{-1}(1-b)$$
$$\Phi_i = \mathcal{N}(0, \sigma_i)$$
$$\sigma_i^2 = K\Big[\sum_{j=0}^{i-1} A^j_K \Sigma_w (A^j_K)^T\Big]K^T.$$

Now add an inequality constraint on the y coordinate of the CoM in such a way that the robot does not oscillate too much in the coronal plane:

$$|c^y_{t+i}| < c^y_{max} \quad i = 1, .., N.$$

In figure 2 it is shown the plot of η in the case of the probabilistic controller. The x-axis represents the receding horizon of the controller, while the y-axis represents the reduction of the ZMP foot constraint for each side of the foot. In the case of a robust controller this value is fixed at $0.0096m$ for all the prediction horizon.

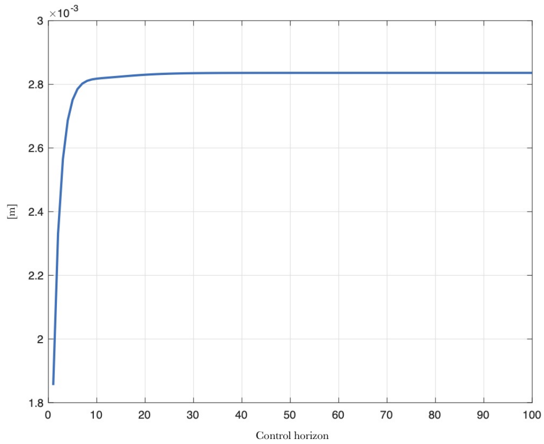

Figure 2: η in probabilistic case

7.2 Constant disturbance

In the set of simulations discussed in this section, a constant disturbance of $0.9 \cdot w^{rob}_{max}$ has been applied to the robot system on both the velocity values. These noises could represent the presence of uneven ground, external forces acting on the robot such as wind, or the robot pushing some objects.
The disturbance is plotted in figure 3.

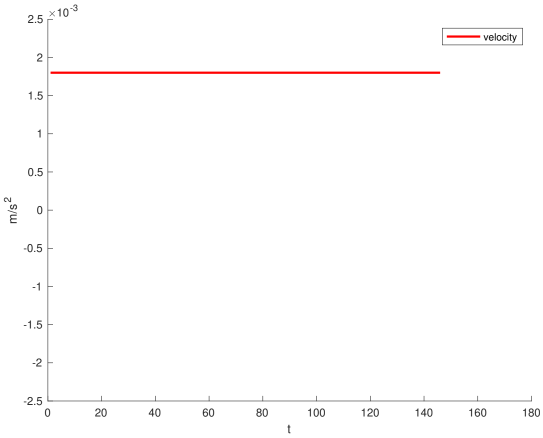

Figure 3: Constant disturbance

Moreover, a constraint on the y-axis position of the CoM is present, as described in section 7.1. Such constraint could stand for the presence of an obstacle, or a narrow passage.
In case of unfeasibility, a back-up controller is used in accordance with section 6.
For the probabilistc controller a value of $\beta_x = 0.05$ is used.
Figure 4, 5, and 6 show the simulation plots.

- The red line represents the ZMP, the blue one its reference, and the green one represents the CoM.
- It is possible to notice that the robot falls in the nominal case.
- As expected, the robust method performs better than the probabilistic one, because of the disturbance being constant: the CoM is kept more centered.
- However, in the probabilistic controller some occasional violation of the constraint on the CoM position occurred.

7.3 Stochastic disturbance

The set of simulations discussed in this section tackles more realistic scenarios, in which the disturbance is not usually constant.

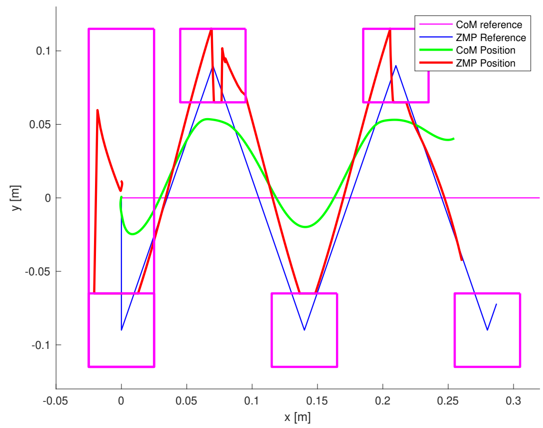

Figure 4: Nominal case

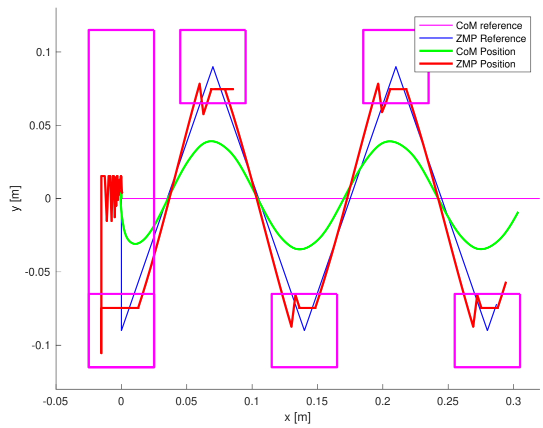

Figure 5: Robust case

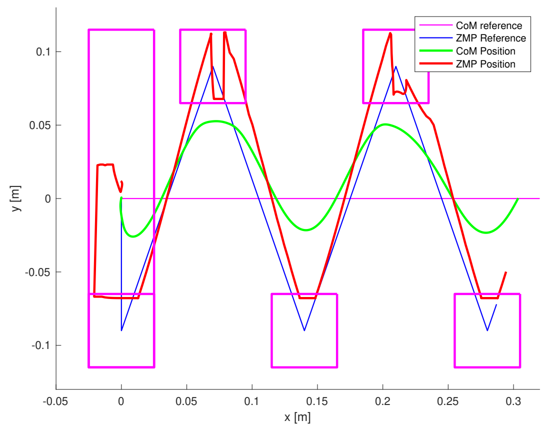

Figure 6: Probabilistic case

A Gaussian with zero mean and covariance matrix Σ_w is used in order to sample the actual disturbances in a random way, coherently with the magnitude of the disturbances used in the previous section. The Gaussian is then truncated to w_{max}^{rob} which is equal to two times the standard deviation of the disturbance distribution. This means that 33% of the times the actual disturbance will be saturated at $\pm\, w_{max}^{rob}$.
The resulting disturbance probability distribution is that of a Gaussian with two Dirac delta functions at w_{max}^{rob}, whose areas are equal to 0.33/2.
The disturbance is plotted in figure 8.

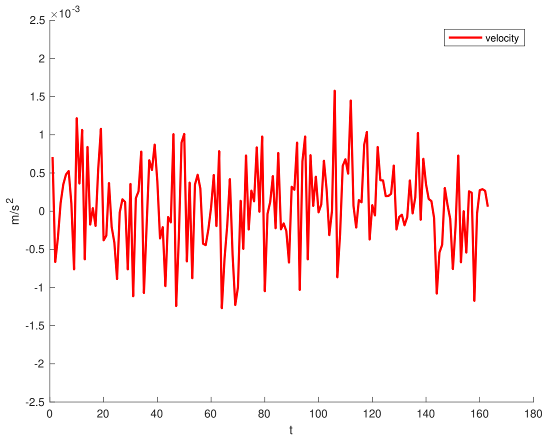

Figure 7: Stochastic disturbance

The sequence of random noises is the same among all the simulations.
Figure 8, 9, 10 and 11 show the simulation plots. The following points summarize the results of the simulations.

- None of the controllers violates any constraint.
- The nominal one performs the best, since it keeps the CoM closest to the reference, with a peak of $0.03m$ in the y-axis.
- The probabilistic controller performs better than the robust one, with peaks respectively of $0.032m$ and $0.037m$ and an improvement of 14%.
- In order to evaluate the quality of the proposed algorithm, it is performed a comparison with the

IS-MPC controller from [3]. As expected, the evolution of the ZMP is smoother in figure 11 with respect to the previous ones, since it is a dynamically extended model. In addition, taking into account the oscillations from the x-axis of the CoM, figure 10 and 11 show similar behaviours.

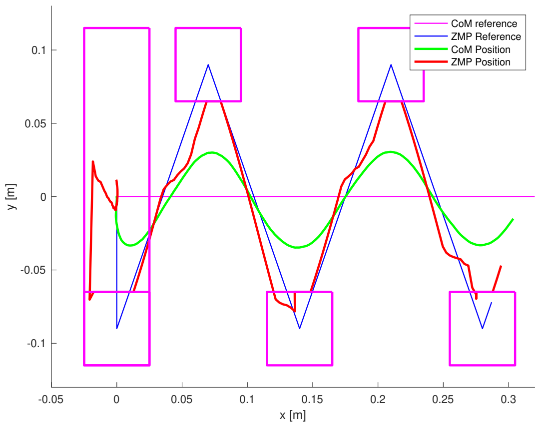

Figure 8: Nominal case [3]

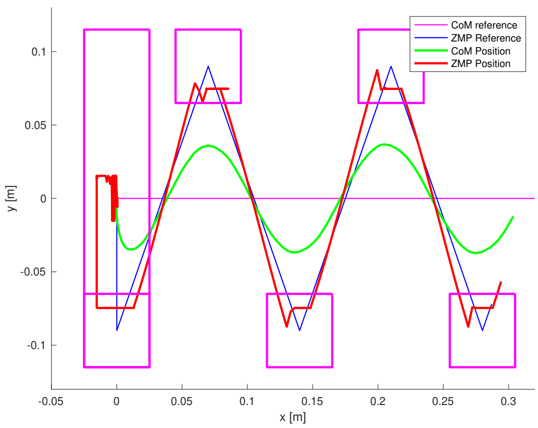

Figure 9: Robust case [3]

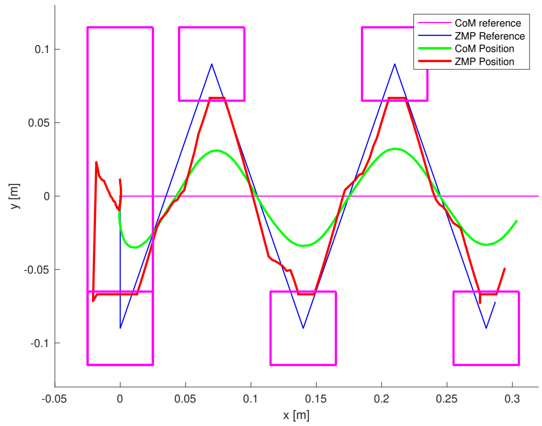

Figure 10: Probabilistic case [3]

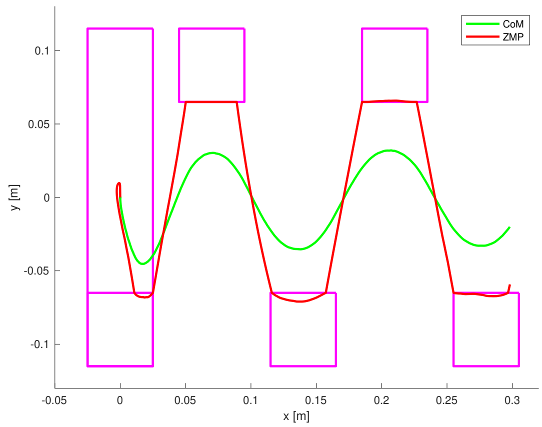

Figure 11: IS-MPC case [3]

7.4 Disturbance rejection

The set of simulations discussed in this section aims at identifying the controller which tolerates the strongest disturbance.
A parameter called $disturb_{multiplier}$ was used in order to incrementally augment the injected disturbance.

- A maximum value of $dist_{multiplier} = 7.5$, yielding an upper bound for the disturbance of $0.0225m/s$ was tolerated by the probabilistic and nominal controllers, as shown in figures 12 and 13.

- Even though the nominal one is still able not to fall, the CoM oscillates quite more.

- On the other hand, the robust controller was not able to find a solution, due to unfeasibility of the QP: it does not work with expected disturbances which are too strong, since the available foot area is restricted too much.

- As opposed to the robust controller, the probabilistic one can still be used when the expected disturbance is significantly strong.

- Moreover, the nominal IS-MPC controller is not able to stabilize the system subject to such disturbance. This has led [3] to develop a Robust IS controller which is able to handle disturbances by introducing a restriction function on the admissibility region for the ZMP.

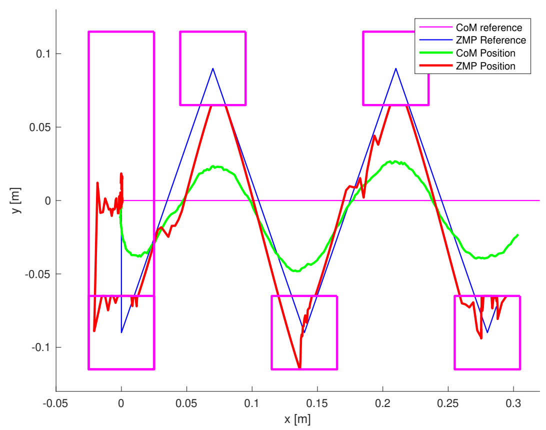

Figure 12: Nom. controller with $dist_{multiplier} = 7.5$ [3]

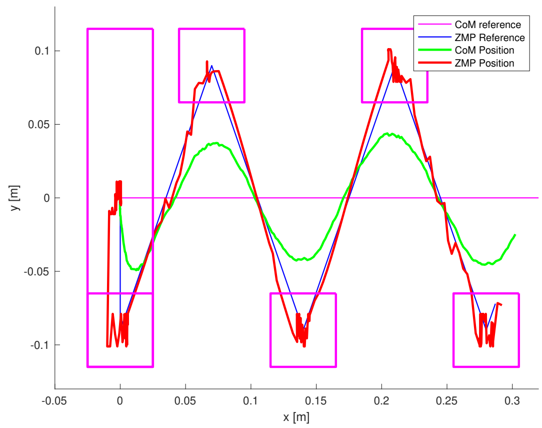

Figure 13: Prob. controller with $dist_{multiplier} = 7.5$ [3]

8 Conclusions

This paper has presented a Stochastic-MPC-based gait generation algorithm for humanoids which is able to guarantee robustness in presence of disturbances. Such algorithm has been compared with a nominal IS-MPC and a Robust MPC. Note that in particular the IS-MPC controller is able to consider disturbance signals which vary around a mid-range value that can change at each sampling time, yielding a bias in the disturbance range.
Finally, the designed method has been validated through dynamic simulations on the OP-3 humanoid. In particular, the probabilistic controller has shown to be the most convenient choice. As a matter of fact, it yields more guarantees with respect to the nominal controller: in the case of a constant disturbance - or sequences of disturbances acting in the same directions - it is able to let the robot maintain balance, whereas the nominal controller fails.
Moreover, it shows better performances with respect to the robust controller in the cases of random disturbances: the extremely conservative behavior of the latter prevents it from reaching good performances in terms of CoM oscillations.
The RMPC is also unusable in the case of large expected disturbances, as the available support polygon becomes too small. On the other hand, the Stochastic-MPC is able to cope with them.

Future work may consist in performing simulations on a real OP-3 robot.

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