Probabilistic Graphical Models: Problem Set 5

Svante Linusson, Liam Solus
KTH Royal Institute of Technology

15 September 2023

1. What is the treewidth of each of the following graphs?

$\mathcal{G}$ $\qquad\qquad$ $\mathcal{H}$

2. Let $\mathcal{G} = ([m], E)$ be a DAG and let $\Phi = \{f_{X_i}(X_i \mid X_{\mathrm{pa}_{\mathcal{G}}(i)})\}_{i=1}^m$ be its set of factors. What is the relationship between $\mathcal{G}$ and $\mathcal{H}_\Phi$?

3. Show that every induced graph $\mathcal{G}_{\Phi,\prec}$ is chordal.

4. Let $\mathcal{G}$ be a connected, undirected graph. Show that $\mathcal{G}$ is a tree if and only if its treewidth is $\omega_{\mathcal{G}} = 1$.

5. (Medical diagnosis again.) Suppose in our medical diagnosis example we assume that the distribution $\mathbf{X} = [Con, M, F, H, CC, C19, S]^T$ is Markov to the DAG presented in the lecture. We assume $\mathbf{X}$ abides by the following hierarchy:

$$\begin{array}{ccc}
S \sim \mathrm{Uniform}(\{0,1,2,3\}) & C19 \sim \mathrm{Ber}(\theta_9) & \\
H|S=0 \sim \mathrm{Ber}(\theta_1) & Con|H=0, CC=0, C19=0 \sim \mathrm{Ber}(\theta_{10}) & M|CC=0, C19=0 \sim \mathrm{Ber}(\theta_{18}) \\
H|S=1 \sim \mathrm{Ber}(\theta_2) & Con|H=0, CC=0, C19=1 \sim \mathrm{Ber}(\theta_{11}) & M|CC=0, C19=1 \sim \mathrm{Ber}(\theta_{19}) \\
H|S=2 \sim \mathrm{Ber}(\theta_3) & Con|H=0, CC=1, C19=0 \sim \mathrm{Ber}(\theta_{12}) & M|CC=1, C19=0 \sim \mathrm{Ber}(\theta_{20}) \\
H|S=3 \sim \mathrm{Ber}(\theta_4) & Con|H=1, CC=0, C19=0 \sim \mathrm{Ber}(\theta_{13}) & M|CC=1, C19=1 \sim \mathrm{Ber}(\theta_{21}) \\
CC|S=0 \sim \mathrm{Ber}(\theta_5) & Con|H=0, CC=1, C19=1 \sim \mathrm{Ber}(\theta_{14}) & F|CC=0, C19=0 \sim \mathrm{Ber}(\theta_{22}) \\
CC|S=1 \sim \mathrm{Ber}(\theta_6) & Con|H=1, CC=0, C19=1 \sim \mathrm{Ber}(\theta_{15}) & F|CC=0, C19=1 \sim \mathrm{Ber}(\theta_{23}) \\
CC|S=2 \sim \mathrm{Ber}(\theta_7) & Con|H=1, CC=1, C19=0 \sim \mathrm{Ber}(\theta_{16}) & F|CC=1, C19=0 \sim \mathrm{Ber}(\theta_{24}) \\
CC|S=3 \sim \mathrm{Ber}(\theta_8) & Con|H=1, CC=1, C19=1 \sim \mathrm{Ber}(\theta_{17}) & F|CC=1, C19=1 \sim \mathrm{Ber}(\theta_{25})
\end{array}$$

We assume that, based on historical data, we have fit the model parameters as

$$\begin{aligned}
[\theta_1, \ldots, \theta_{25}]^T = [&0.03, 0.70, 0.48, 0.65, 0.73,\\
&0.16, 0.22, 0.96, 0.81, 0.57,\\
&0.56, 0.36, 0.01, 0.42, 0.32,\\
&0.64, 0.12, 0.53, 0.57, 0.52,\\
&0.11, 0.53, 0.59, 0.54, 0.06]^T.
\end{aligned}$$

Let $\mathbf{Y} = [S, Con, M, F]^T$. We observe for a new patient with the data $\mathbf{y} = [0, 1, 0, 1]^T$. Use variable elimination to compute the posterior expectations

$$\mathrm{E}[H|\mathbf{Y}=\mathbf{y}], \qquad \mathrm{E}[CC|\mathbf{Y}=\mathbf{y}], \qquad \text{and} \qquad \mathrm{E}[C19|\mathbf{Y}=\mathbf{y}].$$