Probabilistic Graphical Models: Problem Set 3 (Solutions)

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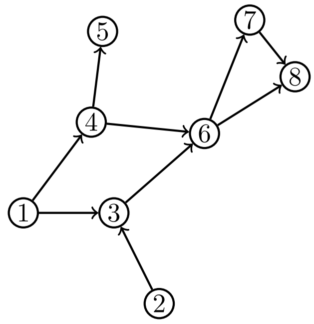
1. Consider the DAG $\mathcal{G} = ([8], E)$ depicted below:

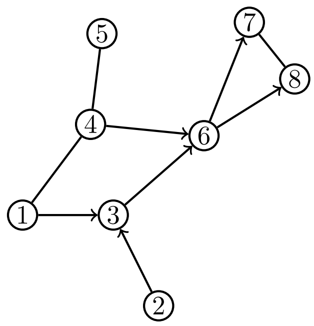


   (a) What is the essential graph of $\mathcal{G}$?

   (b) How many DAGs are in the Markov equivalence class of $\mathcal{G}$?

Solution:

a) We note that the v-structures are $1 \to 2 \leftarrow 3$ and $3 \to 6 \leftarrow 4$ so those four edges cannot change direction. Then we notice that $6 \to 8$ and $6 \to 7$ cannot be reversed since that would cause a new v-structure with the edge $3 \to 6$. The other edges can be be given different directions however. The answer is thus



b) The edges $1 - 4 - 5$ can be directed in three ways without creating a v-structure and the edge $7 - 8$ can be directed either way without creating a v-structure. And there is no risk of creating a cycle in either case. They are independent so the total number is $2 \cdot 3 = 6$.

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2. Explain why the edge $a \to b$ in an essential graph $\mathcal{D}$ cannot be reversed if it is strongly protected.

Solution:

As explained at the end of lecture 3, we look at the four cases in the definition of strongly protected. Note that they are **induced** subgraphs. For the first two a v-structure is formed or broken if $a \to b$ is reversed, which would chagne the MEC. For the third case it would create a cycle. The fourth case is slightly more complicated. If $a \to b$ is reversed then we must direct $c_1 \to a$ to avoid a cycle with b, a, c_1 and similarly we must have $c_2 \to a$. But this now creates a v-structure $c_1 \to a \leftarrow c_2$.

3. For a DAG $\mathcal{G} = (V, E)$, $u, v \in V$. Prove that if u, v are not adjacent then for either $C = \mathrm{pa}_{\mathcal{G}}(u)$ or $C = \mathrm{pa}_{\mathcal{G}}(v)$, there is no d-connecting path between u and v given C in $\mathcal{G}$.

Solution:

Let first $C = \mathrm{pa}_{\mathcal{G}}(u)$ and assume that there is a d-connecting path between u and v. That path must start from u with an edge directed away u and to be d-connecting it must coninue to be a directed path until it meets an ancestor of some node in C but that would create a cycle, so it cannot happen. Thus the path must be a directed path from u to v.

If we instead let $C = \mathrm{pa}_{\mathcal{G}}(v)$, then we can similarly conclude that the only possible d-connecting path would be a directed path from v to u. Since it would create a cycle to have both a directed path from u to v and another from v to u we can conclude that for at least one of $C = \mathrm{pa}_{\mathcal{G}}(u)$ or $C = \mathrm{pa}_{\mathcal{G}}(v)$, there is no d-connecting path between u and v given C in $\mathcal{G}$.

4. Let $\mathcal{G}$ and $\mathcal{H}$ be two DAGs on node set V such that $\mathcal{G} \leq \mathcal{H}$ (i.e. $\mathcal{CI}(\mathcal{H}) \subseteq \mathcal{CI}(\mathcal{G})$). Prove that if $\mathcal{G}$ contains the v-structure $x \to z \leftarrow y$ then either $\mathcal{H}$ contains the same v-structure or x and y are adjacent in $\mathcal{H}$.

Solution:

Suppose this is not the case and $\mathcal{H}$ does not contain the v-structure and X and Y are not adjacent in $\mathcal{H}$. From Exercise 3 we can conclude that if there is an edge between to vertices in $\mathcal{G}$ then they are adjacent also in $\mathcal{H}$, thus Z must be adjacent to both X and Y in $\mathcal{H}$. By our assumption, Z is a parent of either X or Y in $\mathcal{H}$. This implies by Exercise 3, that there exists a conditioning set C that includes node Z (and does not include either node X or node Y) such that there is no d-connecting path between X and Y given C in $\mathcal{H}$. But the path $X \to Z \leftarrow Y$ in G is d-connecting given any set that includes Z (and excludes X and Y), including the set C, which contradicts the fact that $\mathcal{G} \leq \mathcal{H}$.

5. An edge $i \to j$ in a DAG $\mathcal{G} = ([m], E)$ is called **covered** if $\mathrm{pa}_{\mathcal{G}}(j) = \mathrm{pa}_{\mathcal{G}}(i) \cup \{i\}$. In this problem we will show that two DAGs $\mathcal{G} = ([m], E)$ and $\mathcal{G}' = ([m], E')$ are Markov equivalent if and only if there exists a sequence of DAGs $\mathcal{G}_1 := \mathcal{G}, \ldots, \mathcal{G}_M := \mathcal{G}'$ such that the only difference between $\mathcal{G}_i$ and $\mathcal{G}_{i+1}$ for all $i \in [M-1]$ is the reversal of a single covered edge.

   (a) Let $\mathcal{G} = ([m], E)$ be a DAG containing the edge $i \to j$ and let $\mathcal{G}' = ([m], E')$ be the directed graph produced by reversing the edge $i \to j$ in $\mathcal{G}$. Show that $\mathcal{G}'$ is a DAG that is Markov equivalent to $\mathcal{G}$ if and only if $i \to j$ is a covered edge in $\mathcal{G}$.

   (b) Consider two Markov equivalent DAGs $\mathcal{G} = ([m], E)$ and $\mathcal{G}' = ([m], E')$. Fix a linear extension $\pi = \pi_1 \cdots \pi_m$ of $\mathcal{G}$ and for $i \in [m]$ define

   $$P_i = \{j \in [m] : j \to i \in \Delta(\mathcal{G}, \mathcal{G}')\},$$

   where

   $$\Delta(\mathcal{G}, \mathcal{G}') = \{j \to i \in E : j \leftarrow i \in E'\}.$$

   Let k be the smallest number such that $P_{\pi_k} \neq \emptyset$ and let s be the largest number such that $\pi_s \in P_{\pi_k}$. Prove that $\pi_s \to \pi_k$ is a covered edge in $\mathcal{G}$.

   (c) Prove the theorem stated at the start of the problem.

   (d) Implement an algorithm that takes in a DAG $\mathcal{G}$ and computes all elements of its Markov equivalence class.

Solution:

a) We use the theorem by Verma and Pearl that two DAGs are Markov equivalent (MEQ) iff they have the same skeleton and the same v-structures. Assume first that the graphs are MEQ. If $\exists x \in \mathrm{pa}_{\mathcal{G}}(j) \setminus (\mathrm{pa}_{\mathcal{G}}(i) \cup \{i\})$ then also $x \notin \mathrm{ch}_{\mathcal{G}}(i)$ since that would mean that $i \to x \to j \to i$ is a cycle in $\mathcal{G}'$. Since $\mathcal{G}$ and $\mathcal{G}'$ have the same skeleton, there is no edge between i and x in $\mathcal{G}$ either so $i \to j \leftarrow x$ is a v-structure in $\mathcal{G}$ which would not be in $\mathcal{G}'$. A contradiction and thus no such x exists. Similarly, if $\exists x \in \mathrm{pa}_{\mathcal{G}}(i) \setminus \mathrm{pa}_{\mathcal{G}}(j)$ we use the same reasoning with $\mathcal{G}$ and $\mathcal{G}'$ interchanged.

The other direction is easier. If $i \to j$ is a covered edge in $\mathcal{G}$ then it cannot be part of a v-structure so neither skeleton nor v-structures will change if we reverse it. Remains to show that $\mathcal{G}'$ is a DAG. But if we created a cycle including the edges $x \to j \to i$ for some $x \in \mathrm{pa}_{\mathcal{G}}(j)$, then it would already have been a cycle in $\mathcal{G}$ by the edge $x \to i$, which we know it is not.

b) Recall that we have defined the edges to point from the smaller number to the larger number in a linear extension. Since $\mathcal{G}$ and $\mathcal{G}'$ are MEQ they have the same skeleton. First, if there is a node x, $\pi_x < \pi_s$ with $\pi_x \to \pi_s$ in $\mathcal{G}$ then by choice of k, $\pi_x \to \pi_s$ also in $\mathcal{G}'$. That means there has to be an edge $\pi_x \to \pi_k$ in $\mathcal{G}'$ to avoid making $\pi_x \to \pi_s \leftarrow \pi_k$ a v-structure. By choice of s $\pi_x \to \pi_k$ also in $\mathcal{G}$. Second, if there is a node x, $\pi_x < \pi_s$ with $\pi_x \to \pi_k$ in $\mathcal{G}$, we would with a similar reasoning obtain that $\pi_x \to \pi_s$ in both graphs. Finally, if there is a node x, $\pi_s \pi_x < \pi_k$ with $\pi_x \to \pi_k$ in $\mathcal{G}$, then by choice of s, $\pi_x \to \pi_k$ also in $\mathcal{G}'$. To not have different v-structures both graphs must have an edge $\pi_s \to \pi_x$, but this would create a cycle in $\mathcal{G}'$, a contradiction. Thus $\pi_s \to \pi_k$ is covered.

c) Given two MEQ graphs $\mathcal{G}$, $\mathcal{G}'$ having n edges with different orientation. Part b) shows how to find a covered edge and a) shows we can reverse it and obtain new DAG still MEQ and having different orientation for $n-1$ edges compared to $\mathcal{G}'$. Induction over n proves that direction. The other direction follows from a).