Probabilistic Graphical Models: Problem Set 3

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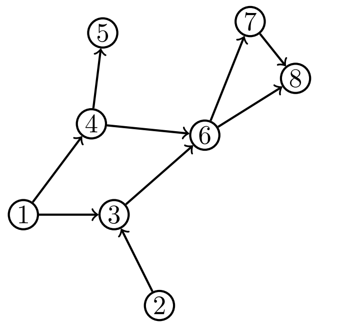
1. Consider the DAG $\mathcal{G} = ([8], E)$ depicted below:



   (a) What is the essential graph of $\mathcal{G}$?

   (b) How many DAGs are in the Markov equivalence class of $\mathcal{G}$?

2. Explain why the edge $a \to b$ in an essential graph $\mathcal{D}$ cannot be reversed if it is strongly protected.

3. For a DAG $\mathcal{G} = (V, E)$, $u, v \in V$. Prove that if u, v are not adjacent then for either $C = \mathrm{pa}_{\mathcal{G}}(u)$ or $C = \mathrm{pa}_{\mathcal{G}}(v)$, there is no d-connecting path between u and v given C in $\mathcal{G}$.

4. Let $\mathcal{G}$ and $\mathcal{H}$ be two DAGs on node set V such that $\mathcal{G} \leq \mathcal{H}$ (i.e. $\mathcal{CI}(\mathcal{H}) \subseteq \mathcal{CI}(\mathcal{G})$). Prove that if $\mathcal{G}$ contains the v-structure $x \to z \leftarrow y$ then either $\mathcal{H}$ contains the same v-structure or x and y are adjacent in $\mathcal{H}$.

5. An edge $i \to j$ in a DAG $\mathcal{G} = ([m], E)$ is called **covered** if $\mathrm{pa}_{\mathcal{G}}(j) = \mathrm{pa}_{\mathcal{G}}(i) \cup \{i\}$. In this problem we will show that two DAGs $\mathcal{G} = ([m], E)$ and $\mathcal{G}' = ([m], E')$ are Markov equivalent if and only if there exists a sequence of DAGs $\mathcal{G}_1 := \mathcal{G}, \dots, \mathcal{G}_M := \mathcal{G}'$ such that the only difference between $\mathcal{G}_i$ and $\mathcal{G}_{i+1}$ for all $i \in [M-1]$ is the reversal of a single covered edge.

   (a) Let $\mathcal{G} = ([m], E)$ be a DAG containing the edge $i \to j$ and let $\mathcal{G}' = ([m], E')$ be the directed graph produced by reversing the edge $i \to j$ in $\mathcal{G}$. Show that $\mathcal{G}'$ is a DAG that is Markov equivalent to $\mathcal{G}$ if and only if $i \to j$ is a covered edge in $\mathcal{G}$.

   (b) Consider two Markov equivalent DAGs $\mathcal{G} = ([m], E)$ and $\mathcal{G} = ([m], E')$. Fix a linear extension $\pi = \pi_1 \cdots \pi_m$ of $\mathcal{G}$ and for $i \in [m]$ define

   $$P_i = \{j \in [m] : i \to j \in \Delta(\mathcal{G}, \mathcal{G}')\},$$

   where

   $$\Delta(\mathcal{G}, \mathcal{G}') = \{i \to j \in E : i \leftarrow j \in E'\}.$$

   Let k be the smallest number such that $P_{\pi_k} \neq \emptyset$ and let s be the largest number such that $\pi_s \in P_{\pi_k}$. Prove that $\pi_s \to \pi_k$ is a covered edge in $\mathcal{G}$.

   (c) Prove the theorem stated at the start of the problem.

   (d) Implement an algorithm that takes in a DAG $\mathcal{G}$ and computes all elements of its Markov equivalence class.