Lesson 3: Markov equivalence

~~Want~~ Goal: Find directed graph that explains correlation & causality.

Approaches

- Prior knowledge   ~~Summer school temperature height above sea level~~
- Guess & check
- Algorithms   ← Data-driven

Input: Maybe Data as

- sample covariance matrix
- list of observed CI relations $\mathcal{CI}$

We will use properties of DAGs and d-separation

CI relations are $A \perp\!\!\!\perp B \mid C$
but do not give unique DAG.

Example:

$X \to Y \to Z$

$X \leftarrow Y \leftarrow Z$

$X \leftarrow Y \to Z$

$\left.\right\}$ $X \not\!\perp\!\!\!\perp Z$, $X \perp\!\!\!\perp Z \mid Y$

$X \to Y \leftarrow Z$ $X \not\!\perp\!\!\!\perp Z \mid Y$, $X \perp\!\!\!\perp Z$

G, H DAGs having the same CI-relations are called Markov equivalent i.e. $\mathcal{CI}(G) = \mathcal{CI}(H)$

Will give two important theorems this hour.
Next hour: Two algorithms using these theorems.
Last lecture: How to distinguish within a Markov Equivalence class

$\mathcal{CI}(\mathbb{P}) := \{ A \perp\!\!\!\perp B \mid C : A \perp\!\!\!\perp B \mid C \text{ holds in } \mathbb{P} \}$

$\mathcal{CI}(G) := \{ A \perp\!\!\!\perp B \mid C : A \,\&\, B \text{ are d-separated given } C \text{ in } G \}$

G is an I-map of $\mathbb{P}$ if $\mathcal{CI}(G) \subset \mathcal{CI}(\mathbb{P})$

G is a minimal I-map of $\mathbb{P}$ if no subgraph of G is an I-map of $\mathbb{P}$.

G is ~~faithfull~~ a perfect I-map if $\mathcal{CI}(G) = \mathcal{CI}(\mathbb{P})$

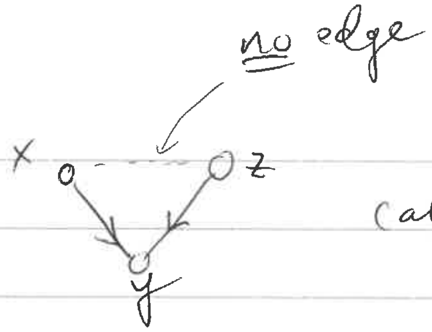
$\mathbb{P}$ is then faithfull to G.

Note $\mathcal{CI}(\vec{K}_n) = \emptyset$ so I-map for every $\mathbb{P}$.

"More edges $\Longleftrightarrow$ fewer CI relations"

Want: Given $\mathbb{P}$, or data ~~$\mathcal{D}$~~ sampled from $\mathbb{P}$
find 'minimal' I-map G.

$\lambda \quad \pi$

Recall 

called <u>v-structure</u>

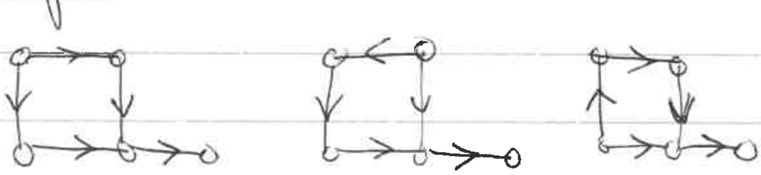
<u>Theorem 1</u> (Verma, Pearl '89) G_1, G_2 DAGs

G_1 & G_2 are Markov equivalent

$\Longleftrightarrow$

Same underlying graph (skeleton) & same v-structures

<u>Example:</u>  three possible DAGs



Recall $x, y \in V(G)$ are <u>d-separated</u> by C if $\forall$ path
$x = x_0 - x_1 - x_2 - \cdots - x_k = y$ $\exists\, x_i$ s.t.

I $\begin{cases} x_{i-1} \to x_i \to x_{i+1} \\ x_{i-1} \leftarrow x_i \leftarrow x_{i+1} \end{cases}$ $x_i \in C$

II $x_{i-1} \leftarrow x_i \to x_{i+1}$ $x_i \in C$

or

III $x_{i-1} \to x_i \leftarrow x_{i+1}$ $x_i \notin C \cup an(C)$

<u>A path avoiding all these is called <u>d-connecting</u></u>

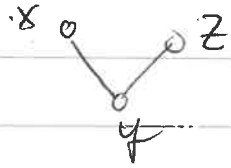
Lemma 1: G DAG, $x, y \in V(G)$

x, y adjacent $\iff$ x, y not d-separated by any set in G.

Proof: $\Rightarrow$ $x \overset{\text{edge}}{-} y$, then clearly not d-separated.

$\Leftarrow$ Problem 1 ~~after the lecture~~

in problemsession

Lemma 2: $G = (V, E)$ DAG  $xz \notin E$ induced subgraph of skeleton of G.

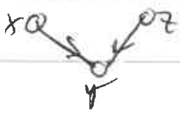
TFAE

1) ~~in~~ $x - y - z$ a v-structure.

2) $\exists C \subseteq V$, $y \notin C$ s.t. C d-separates x from z.

3) $\forall A \subseteq V$, $y \in A$ $\Rightarrow$ A do not d-separate x from z.

Proof: $1 \Rightarrow 2$) Lemma 1 since $xz \notin E$ $\exists$ C d-separating x from y.

if $y \in C$ then  would be d-connecting so $y \notin C$.

~~ELEUR~~ Assume ~~$\not\in$~~ $C \not\ni y$ d-separates x from z

$2 \Rightarrow 1$)

$\Rightarrow$ $x \to y \leftarrow z$ 1)

$1 \Rightarrow 3$

~~3)~~ $\Rightarrow$ $y \in A$ then A does not d-separate x & z

$3) \Rightarrow 1)$ ~~if~~ $x - y - z$ but a v-structure

$xz \notin E$ $\underset{\text{Lemma 1}}{\Rightarrow}$ $\exists A$ that d-sep x & z

by 3) $y \notin A$ but then $x - y - z$ a d-connecting path ↯

~~if $y \in A$~~

$\Rightarrow 1)$

$\square$

Proof of Verma-Pearl.

$\Rightarrow$ Assume G_1 & G_2 Mark. equiv. (M.E)

$uw \in E(G_1) \overset{\text{Lem 1}}{\Leftrightarrow} u,w$ not d-sep by any set in G_1

$\Updownarrow$ M. eq

$uw \in E(G_2) \overset{\text{Lem 1}}{\Leftrightarrow} u,w$ not d-sep by any set in G_2

Same skeleton ok.

$x \to y \leftarrow z$ v-structure in $G_1 \overset{\text{Lem 2}}{\Leftrightarrow} \exists\ \underset{\not\ni y}{Z} \subseteq V$ that d-sep x,z in G_1

$\Updownarrow$ Mar. eq.

$x \to y \leftarrow z$ v-structure in $G_2 \overset{\text{Lem 2}}{\Leftrightarrow} \exists\ \underset{\not\ni y}{Z} \subseteq V$ that d-sep x,z in G_2

ok.

$\Leftarrow$ difficult direction.

As we saw in the earlier example

one of the three remaining edges was forced a direction two where not.

"This is called the essential graph of the Markov equivalence class, a mixed graph."

If G DAG, let G^* be the smallest (mixed) graph larger than all $G' \in [G]$.
$[G]$ M. eq. class

G^* is the essential graph of G. (take union of all $G' \in [G]$)

$G' \subseteq G^*$ if
1) $V_{G'} \subseteq V_{G^*}$
2) $\forall\, a \to b \in G' \Rightarrow \begin{cases} a \to b \in G^* \\ \text{or} \\ a - b \in G^* \end{cases}$
3) $\forall\, a - b \in G' \Rightarrow a - b \in G^*$

"Def: $a \to b \in G$ is an essential edge if $a \to b \in G'$ $\forall G' \in [G]$
Equivalent def: G^* has same skeleton as G & $a \to b \in G^*$ iff $a \to b$ essential edge in G."

D mixed graph.

Def - An arrow free (chain) component in D is a max'l undirected connected subgraph.

- mixed directed cycle $(a_0, \dots, a_n)$ $a_0 = a_n$ s.t. $a_i - a_{i+1}$ or $a_i \to a_{i+1}$ $\forall i$
at least one directed edge.
w/o undirected cycle.

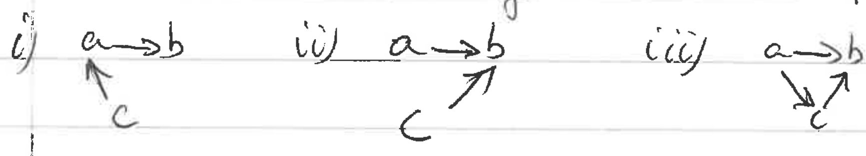
- D is a chain graph if it has no mixed directed cycle.

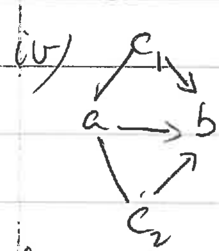
Theorem 2 (Andersson, Madigan, Perlman '97)

$D = G^*$ for some DAG G iff

1) D is a chain graph
2) every arrow free component of D is chordal.
3) no $a \to b - c$ as induced subgraph.
4) All arrows are strongly protected.
(edges cannot be reversed)

Def: $a \to b$ is strongly protected if in one of the following induced config.





In these $a \to b$ cannot be reversed

Problem 4a

Proof: ~~$\Leftarrow$ Problem 4b~~ $\Rightarrow$ 1, 2, 3 explain

Lemma 1: $D = G^*$ then $a \to b - c$ cannot be an induced subgraph.

Proof: $a \to b \in G^* \Leftrightarrow a \to b$ essential

$\exists\; G_1, G_2 \in [G]$

$G_1 \quad a \to b \to c$
$G_2 \quad a \to b \leftarrow c$

} not same v-structure so cannot be M. equiv. $\blacksquare$

$\Rightarrow$ 4) in problem sheet