Probabilistic Graphical Models: Problem Set 2

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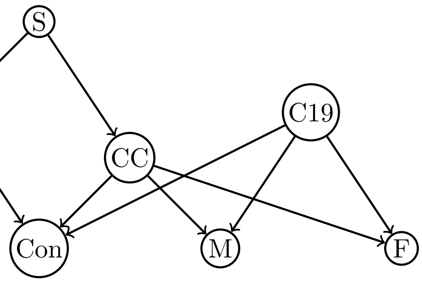
1. (Medical diagnosis revisited) Suppose that the random variables in our medical diagnosis model from the first lesson $\mathbf{X} = [Con, M, F, H, CC, C19, S]^T$ factorizes according to the following DAG $\mathcal{G}$:

$\mathcal{G}$



   (a) What is the relationship between this graph $\mathcal{G}$ and the undirected graph from problem 1 on the first practice problem set?

   (b) Do you think this model is better or worse than our undirected graphical model for the same system of variables on practice problem set 1? Why or why not?

   (c) Suppose we were unsure if $\mathbf{X}$ was in $\mathcal{M}_F(\mathcal{G})$. What is the fewest number of CI relations we would need to check to convince ourselves that it is?

2. There are four DAGs on three nodes whose underlying undirected graph (i.e. **skeleton**) is a path. What are the d-separation statements for each of these four DAGs? What do you notice? What are the implications of what you notice for distributions Markov to anyone of these DAGs?

3. A DAG $\mathcal{G} = ([m], E)$ with linear extension $\pi = 12\cdots m$ is called **perfect** if for all $i \in [m]$, the set of nodes $\mathrm{pa}_{\mathcal{G}}(i)$ form a clique in $\mathcal{G}$. Let $\overline{\mathcal{G}}$ denote the skeleton of $\mathcal{G}$. Show that $\mathcal{M}_{\mathrm{Glo}}(\mathcal{G}) = \mathcal{M}_{\mathrm{Glo}}(\overline{\mathcal{G}})$ whenever $\mathcal{G}$ is perfect.

4. Consider three discrete random variables X_1, X_2, X_3.

   (a) Suppose that X_1 and X_2 are both binary with state space $\{0, 1\}$. Show that if

$$f_{X_1,X_2}(0,0) = f_{X_1}(0) f_{X_2}(0)$$

   then $X_1 \perp\!\!\!\perp X_2$.

   (b) Is it the case that for binary X_1, X_2, and X_3 with state space $\{0, 1\}$ that $X_1 \perp\!\!\!\perp X_2 \mid (X_3 = 0)$ implies $X_1 \perp\!\!\!\perp X_2 \mid X_3$? Provide either a proof or a counterexample. In either case, determine the DAG with the fewest edges with respect to which the distribution(s) in your result satisfy the global Markov property.

5. Let $\mathcal{G} = ([m], E)$ a DAG with linear extension $\pi = 12\cdots m$ and consider the **linear Gaussian DAG model**

$$\mathbf{X} = A\,\mathbf{X} + E$$

   for the lower triangular $m \times m$ matrix $A = [a_{i,j}]_{i,j=0}^m$ in which $a_{i,j} \neq 0$ if and only if $i \in \mathrm{pa}_{\mathcal{G}}(j)$, and for the vector $E = [\varepsilon_1, \ldots, \varepsilon_m]^T$ of independent standard normal random variables $\varepsilon_i \sim \mathcal{N}(\mathbf{0}, 1)$.

(a) Show that $\mathbf{X}$ has a multivariate normal distribution $N(0, \Sigma')$, where $\Sigma' = (1 - A)^{-1}(1 - A)^{-T}$ and 1 denotes the $m \times m$ identity matrix.

(b) Consider a linear Gaussian DAG model for the DAG $\mathcal{G} = ([4], E)$ where

$$E = \{1 \to 2, 1 \to 3, 2 \to 4, 3 \to 4\}.$$

Does $\mathbf{X}$ factorize according to $\mathcal{G}$?