Probabilistic Graphical Models: Problem Set 6

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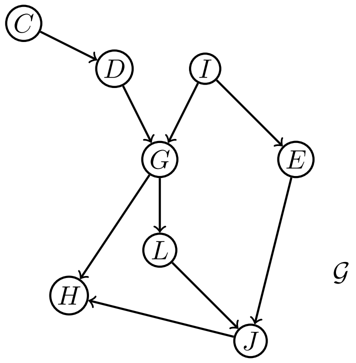
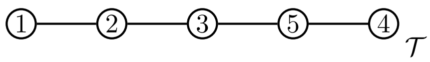
1. Consider the DAG $\mathcal{G}$ associated with a student's performance in some course:



$C =$	Coherence
$D =$	Difficulty
$I =$	Intelligence
$G =$	Grade
$E =$	Exam score
$L =$	Letter of Recommendation
$J =$	Job
$H =$	Happy

Consider an instance of variable elimination where Φ is the set of conditional distributions in the factorization of $\mathbf{X} = [C, D, I, G, E, L, J, H]^T$ with respect to $\mathcal{G}$, $\mathbf{Z} = \{C, D, I, G, E, L, H\}$ and $\prec$ is the ordering (C, D, I, H, G, E, L). The set of intermediate factors $\psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6, \psi_7$ produced by VE specify a clique tree for $\mathcal{G}$. Draw this clique tree.

2. Let Φ denote the set of factors used in problem 1. Consider the clique tree $\mathcal{T}$



where

$$C_1 = \{C, D\}, C_2 = \{D, I, G\}, C_3 = \{G, I, S\}, C_4 = \{G, H, J\}, C_5 = \{G, J, L, S\},$$

and

$$\alpha(f_C(c)) = C_1, \alpha(f_{D|C}(d|c)) = C_1, \alpha(f_{G|I,D}(g|i, d)) = C_2, \alpha(f_I(i)) = C_3, \alpha(f_{E|I}(e|i)) = C_3,$$
$$\alpha(f_{H|G,J}(h|g, j)) = C_4, \alpha(f_{L|G}(\ell|g)) = C_5, \text{ and } \alpha(f_{J|L,E}(j|\ell, e)) = C_5.$$

(a) What are the initial potentials of $\mathcal{T}$?

(b) Fix the root C_r to be C_5 and perform sum-product message passing to compute $\beta_r(C_r)$.

(c) Fix the root C_r to be C_3 and perform sum-product message passing to compute $\beta_r(C_r)$. Were you able to reuse anything from parts (a) or (b) in this computation?

3. The clique tree we constructed in problem (1) has more nodes than the clique tree we used for inference in problem (2). In particular, the clique tree in problem (2) used only the maximal cliques from the tree in problem (1). We now show that this is always possible:

Let $\mathcal{T}$ be a clique tree over a set of factors Φ. Show that there exists a clique tree $\mathcal{T}'$ such that

(1) each clique in $\mathcal{T}'$ is also a clique in $\mathcal{T}$, and

(2) there is no pair of distinct cliques C_i, C_j in $\mathcal{T}'$ such that $C_j \subset C_j$.

4. Consider the Hidden Markov Model from Lecture 6 where $m = 6$ and $n = 4$.

   (a) Use the clique-tree algorithm to compute the marginal distributions

   $$f_{Y_1}(y_1), \quad f_{Y_2}(y_2), \quad f_{Y_3}(y_3), \quad f_{Y_4}(y_4), \quad \text{and } f_{Y_5}(y_5).$$

   (b) We observe the random sample $\mathbf{z} = [\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4]^T$ from $\mathbf{Y} = [Y_1, Y_2, Y_3, Y_4, Y_5]^T$:

   $$\begin{aligned} \mathbf{y}_1 &= [2, 4, 4, 2, 3]^T, \\ \mathbf{y}_2 &= [1, 0, 1, 3, 3]^T, \\ \mathbf{y}_3 &= [2, 2, 3, 4, 2]^T, \\ \mathbf{y}_4 &= [1, 3, 2, 1, 3]^T. \end{aligned}$$

   With the help of a clique tree, compute the marginal posteriors

   $$f_{X_1|\mathbf{Z}}(x_1|\mathbf{z}), \quad f_{X_2|\mathbf{Z}}(x_2|\mathbf{z}), \quad f_{X_3|\mathbf{Z}}(x_3|\mathbf{z}), \quad f_{X_4|\mathbf{Z}}(x_4|\mathbf{z}), \quad \text{and } f_{X_5|\mathbf{Z}}(x_5|\mathbf{z}).$$