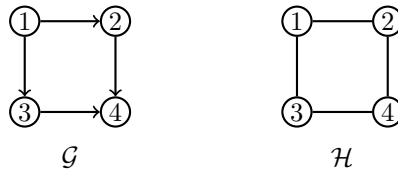


Probabilistic Graphical Models: Problem Set 5

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1. What is the treewidth of each of the following graphs?



2. Let $\mathcal{G} = ([m], E)$ be a DAG and let $\Phi = \{f_{X_i}(X_i \mid X_{\text{pa}_{\mathcal{G}}(i)})\}_{i=1}^m$ be its set of factors. What is the relationship between \mathcal{G} and \mathcal{H}_{Φ} ?
3. Show that every induced graph $\mathcal{G}_{\Phi, \prec}$ is chordal.
4. Let \mathcal{G} be a connected, undirected graph. Show that \mathcal{G} is a tree if and only if its treewidth is $\omega_{\mathcal{G}} = 1$.
5. (Medical diagnosis again.) Suppose in our medical diagnosis example we assume that the distribution $\mathbf{X} = [Con, M, F, H, CC, C19, S]^T$ is Markov to the DAG presented in the lecture. We assume \mathbf{X} abides by the following hierarchy:

$S \sim \text{Uniform}(\{0, 1, 2, 3\})$	$C19 \sim \text{Ber}(\theta_9)$	
$H S = 0 \sim \text{Ber}(\theta_1)$	$Con H = 0, CC = 0, C19 = 0 \sim \text{Ber}(\theta_{10})$	$M CC = 0, C19 = 0 \sim \text{Ber}(\theta_{18})$
$H S = 1 \sim \text{Ber}(\theta_2)$	$Con H = 0, CC = 0, C19 = 1 \sim \text{Ber}(\theta_{11})$	$M CC = 0, C19 = 1 \sim \text{Ber}(\theta_{19})$
$H S = 2 \sim \text{Ber}(\theta_3)$	$Con H = 0, CC = 1, C19 = 0 \sim \text{Ber}(\theta_{12})$	$M CC = 1, C19 = 0 \sim \text{Ber}(\theta_{20})$
$H S = 3 \sim \text{Ber}(\theta_4)$	$Con H = 1, CC = 0, C19 = 0 \sim \text{Ber}(\theta_{13})$	$M CC = 1, C19 = 1 \sim \text{Ber}(\theta_{21})$
$CC S = 0 \sim \text{Ber}(\theta_5)$	$Con H = 0, CC = 1, C19 = 1 \sim \text{Ber}(\theta_{14})$	$F CC = 0, C19 = 0 \sim \text{Ber}(\theta_{22})$
$CC S = 1 \sim \text{Ber}(\theta_6)$	$Con H = 1, CC = 0, C19 = 1 \sim \text{Ber}(\theta_{15})$	$F CC = 0, C19 = 1 \sim \text{Ber}(\theta_{23})$
$CC S = 2 \sim \text{Ber}(\theta_7)$	$Con H = 1, CC = 1, C19 = 0 \sim \text{Ber}(\theta_{16})$	$F CC = 1, C19 = 0 \sim \text{Ber}(\theta_{24})$
$CC S = 3 \sim \text{Ber}(\theta_8)$	$Con H = 1, CC = 1, C19 = 1 \sim \text{Ber}(\theta_{17})$	$F CC = 1, C19 = 1 \sim \text{Ber}(\theta_{25})$

We assume that, based on historical data, we have fit the model parameters as

$$[\theta_1, \dots, \theta_{25}]^T = [0.03, 0.70, 0.48, 0.65, 0.73, \\ 0.16, 0.22, 0.96, 0.81, 0.57, \\ 0.56, 0.36, 0.01, 0.42, 0.32, \\ 0.64, 0.12, 0.53, 0.57, 0.52, \\ 0.11, 0.53, 0.59, 0.54, 0.06]^T.$$

Let $\mathbf{Y} = [S, Con, M, F]^T$. We observe for a new patient with the data $\mathbf{y} = [0, 1, 0, 1]^T$. Use variable elimination to compute the posterior expectations

$$\mathbb{E}[H|\mathbf{Y} = \mathbf{y}], \quad \mathbb{E}[CC|\mathbf{Y} = \mathbf{y}], \quad \text{and} \quad \mathbb{E}[C19|\mathbf{Y} = \mathbf{y}].$$