# Lecture 2 DAGs

**Def:** $G = ([m], E)$ a DAG

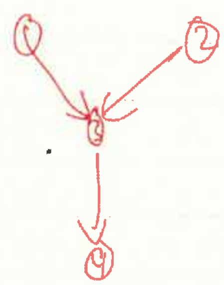
A path $\langle i_0, i_1, \dots, i_\ell \rangle$ in $G$ is d-connecting given $C \subseteq [m]$, if

① $\forall\, i_k$ s.t. $i_{k-1} \rightarrow i_k \leftarrow i_{k+1}$ (collider node)

we have $i_k \in C$ or $i_k \in An(C)$
i.e. $i_k \rightarrow \cdots \rightarrow x \in C$

② if $\langle i_{k-1}, i_k, i_{k+1} \rangle$ is not a collider then $i_k \notin C$.

$A, B \subseteq [m]$ are d-seperated given $C$ if there is no d-connecting path from $a \in A$ to $b \in B$.



① & ② d-connected given ③
① & 2 d-connected given 4

1 & 2 d-separated given $\emptyset$
2 & 4 d-connected given $\emptyset$
2 & 4 d-separated given 3

Def: $G = ([m], E)$ a DAG
A distribution $\mathbb{P}$ over $X_1, \ldots, X_m$ is said to satisfy

(DL) Directed Local Markov Property w.r.t $G$ if $\forall i \in [m]$

$$X_i \overset{\mathbb{P}}{\perp\!\!\!\perp} X_{nd_G(i) \setminus pa_G(i)} \mid pa_G(i)$$

(DG) Directed Global Markov Property if $X_A \overset{\mathbb{P}}{\perp\!\!\!\perp} X_B \mid X_C$ holds in $\mathbb{P}$ whenever $A, B$ d-seperated given $C$.

A distribution $\mathbb{P}$ over $X_1, \ldots, X_m$ with density $f$ factorizes according to $G$ if

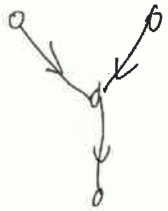
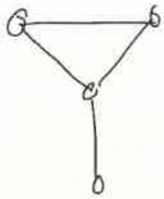
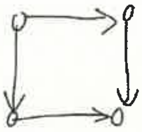
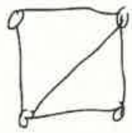
(DF) $$f(x) = \prod_{i \in [m]} f(x_i \mid x_{pa_G(i)})$$

Will prove that these are equivalent. → That is, the graphs are good models of the distributions

Def: $G = ([m], E)$ a DAG. The moral graph of $G$, denoted $G^m$ is the undirected graph $G^m = ([m], E')$

$$E' = \{ ij \mid i \to j \in E \text{ or } ch_G(i) \cap ch_G(j) \neq \emptyset \}$$

We will use the results from the first lecture about undirected graphs.

$G =$  $G^{\hat{m}}$  ; $G =$  , $G^{\hat{m}} =$ 

Prop 1: If $\mathbb{P}$ factorize according to a DAG $G$ (DF)
Then it obeys $(G)$ w.r.t $G^{\hat{m}}$

Proof: $\mathbb{P}$ has density $f$ with

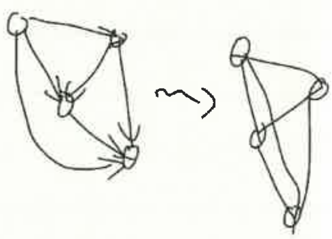
$$f(x) = \prod_{i \in [m]} f(x_i \mid x_{pa_G(i)})$$

the set of parents of any $i \in [m]$ will form a clique in $G^{\hat{m}}$ we may take functions.

$$\psi_{pa_G(i) \cup \{i\}}(x) = f(x_i \mid x_{pa_G(i)})$$

$G$ DAG so all maxl clique are of the form $\{i\} \cup pa_G(i)$ for some $i$.

Hence $\mathbb{P}$ factorizes over $G^{\hat{m}}$
Then we use $(F) \Rightarrow (G)$



(DF) for $G$ => (P) for $G^m$ => (G) for $G^m$

**Prop 2:** If $\mathbb{P}$ factorizes according to a DAG $G = ([n], E)$ and $A \subseteq [n]$ which is ancestrally closed ($An(A) = A$), then the marginal distribution $\mathbb{P}_A$ factorizes over $G|_A$

**Proof:** Exercise just a restriction

(DM) Moralization Markov Property

$X_A \perp\!\!\!\perp X_B \mid X_C$ holds if $A, B$ separated by $C$ in $\left(G|_{An(A \cup B \cup C)}\right)^m$

(DF) => (DM)

**Prop 3:** $\mathbb{P}$ factorizes according to $G = ([n], E)$. Then (DM) holds.

**Proof:**
- $\mathbb{P}_{An(A \cup B \cup C)}$ factorizes according to $G_{An(A \cup B \cup C)}$ by Prop 2. ~~just a~~ restriction
- Prop 1 => $\mathbb{P}_{An(A \cup B \cup C)}$ obeys (G) w.r.t $\left(G_{An(A \cup B \cup C)}\right)^m$
- Hence $A \perp\!\!\!\perp B \mid C$ in $\mathbb{P}_{An(A \cup B \cup C)}$ hence in $\mathbb{P}$. $\square$

from the Lecture 1


**Prop 4** (DM) is equiv. to (DG)

**Proof:** Must prove

(1) $A, B$ d-separated given $C$ in $G$

$\Downarrow$ Exercise

$\Updownarrow$

(2) $A, B$ separated in $(G_{An(A \cup B \cup C)})^m$ given $C$.

**Exercise:** $\Uparrow$ suppose (1) false $\Rightarrow \exists$ d-connecting path $a = i_0, \ldots, i_m = b$ given $C$ in $G$
$\Rightarrow \{i_1, \ldots, i_m\} \subseteq An(A \cup B \cup C) \Rightarrow \exists$ path circumventing $C$.

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**Theorem** $G = ([m], E)$ a DAG, $\mathbb{P}$ distr. $X_1, \ldots, X_m$

$$(DF) \Leftrightarrow (DG) \Leftrightarrow (DL)$$

**Proof:**

(DF) $\Rightarrow$ (DG) prop 3 & Prop 4

(DL) $\Rightarrow$ (DF) By induction over $m$.

Base case: $m = 1, 2$

I.H. Assume holds for $G$ with less than $m$ vertices.

$G = ([m], E)$ DAG, let $i$ be a sink in $G$.

$$\frac{f(x)}{f(x_{[m]\setminus i})} = f(x_i \mid x_{[m]\setminus i}) = f(x_i \mid x_{pa(i)}, x_{[m]\setminus pa(i) \cup \{i\}})$$

$x_{nd(i)\setminus pa(i)}$

$$\overset{(DL)}{=} f(x_i \mid x_{pa(i)})$$

(DG) $\Rightarrow$ (DL)

See Lauritzen

Let $i \in [m]$ need to show $X_i \perp\!\!\!\perp X_{nd(i)\setminus pa(i)} \mid X_{pa(i)}$

(DG) $\Rightarrow$ suffices to show $i$ & $nd(i)\setminus pa(i)$ d-sep given $pa(i)$.

for contradiction.

S'pose $\exists$ d-connecting path $\pi = \langle i_0 = i, i_1, \dots, i_m = j \rangle$ given $pa(i)$, $j \in nd(i)$

Note $i_0 \leftarrow i_1$ not possible since then $i_1 \in pa(i)$

If $\pi$ contains no collider $i \to \circ - \cdots \to j$ contradicting $j \in nd(i)$

If $\pi$ contains collider $i_{k-1} \to i_k \leftarrow i_{k+1}$ $\Rightarrow$ $i_k \in An(pa(i))$

$\in pa(i)$ $\quad$ $\exists$ cycle! ↯

$\blacksquare$