WASP Basic Graph Theory


"Luke, you must learn the ways of the force"
"I'm ready, Obi Wan."
"Oooookay. Let's see here. After you've logged in, you're gonna want to go to the student portal and click Jedi...."

Graph Theory

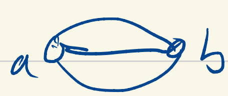
A graph $G = (V, E)$

vertex set (for V), edge set (for E)

$a - b$	$a \to b$	$a \leftarrow b$	$a \leftrightarrow b$
ab			
$\{a, b\}$	(a, b)	(b, a)	

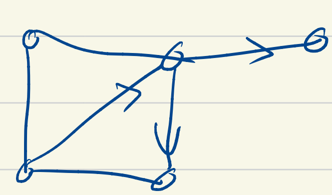
a & b are adjacent / neighbors $\quad$ a, b endpoints of the edge.

No loops $\quad$ 

Most often no multiple edges $\quad$ 

$\Big\}$ simple graphs.

Loopless mixed graphs LMG



$a - b$ undirected graph

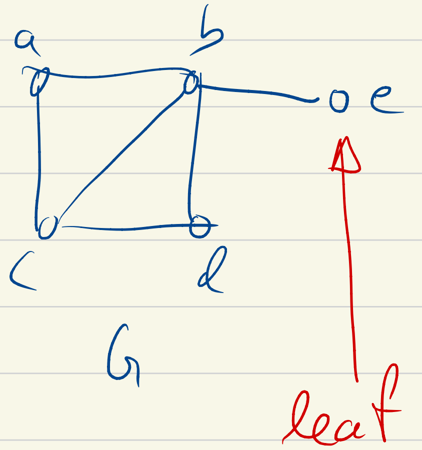
$\left.\begin{array}{l} a \to b \\ a \leftarrow b \end{array}\right\}$ directed graph.

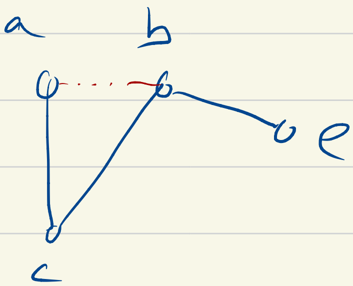
The skeleton of G is obtained by replacing each $a \to b$, $a \leftrightarrow b$ with $a - b$

subgraph – take away some edges & vertices

induced subgraph – take away some vertices (& incident edges)

Example





subgraph, <u>not</u> induced.

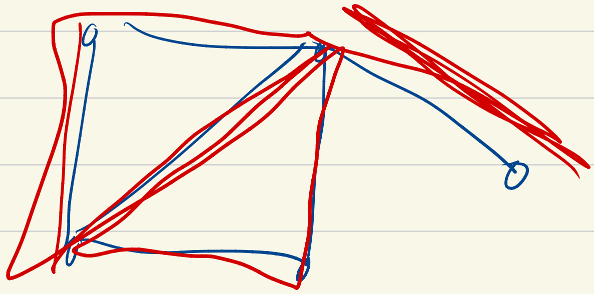
$N_G(a) = \{b, c\}$ neighborhood

$N_G[a] = \{a, b, c\}$ closed neighborhood.

$G = (V, E)$, $A \subseteq V$

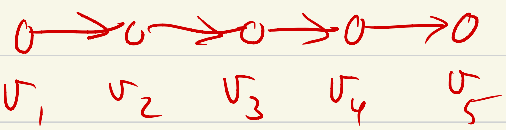
$G|_A$ the induced subgraph on A, i.e.
all vertices in $V \setminus A$ removed.

A walk



A path must not reuse a vertex

directed path

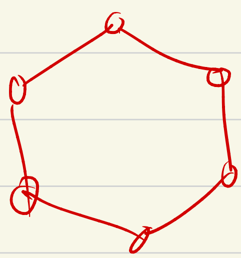


Also as subgraphs.

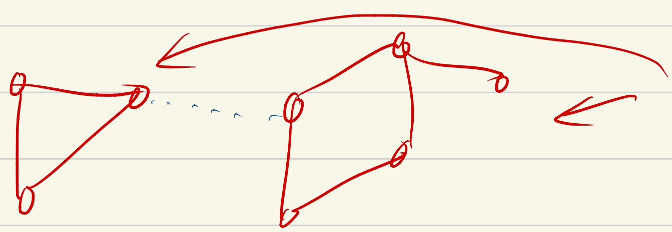
$\langle v_1, v_2, v_3, v_4, v_5 \rangle$

subpath a subgraph that is also a path.

cycle



G is connected if $\exists$ paths between every pair of nodes.

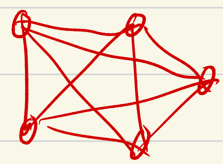


components

not connected

A <u>tree</u> is a connected graph with no cycle.

A <u>complete</u> graph has all possible (undirected) edges.
<u>clique</u>



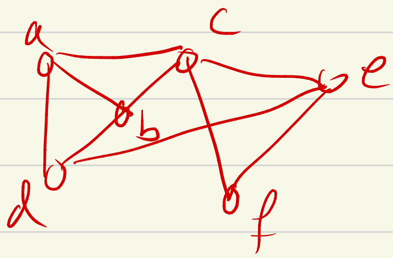
Undirected graphs

$G=(V,E)$ $\quad A,B,C \subseteq V$

A, B <u>connected</u> given C if $\exists$ path from $a \in A$ to $b \in B$ in $G \setminus C$

otherwise
A, B are <u>separated</u> given C, denoted $A \overset{G}{\perp} B \mid C$

<u>Example</u>



$\{a,b\} \overset{G}{\perp} \{e,f\} \mid \{c,d\}$

$\{a,b\} \overset{G}{\not\perp} \{e,f\} \mid \{c\}$

<u>Note</u> if $x \in A \cap B$ but $x \notin C$ then

A, B connected given C.

2. $A \overset{G}{\perp} (B \cup D) \mid C \Rightarrow A \overset{G}{\perp} B \mid C$

We call C an AB-separator if

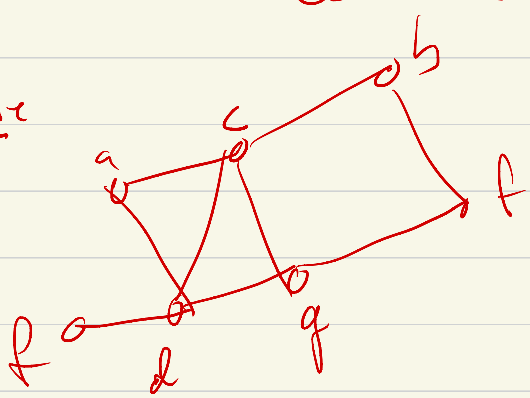
$$A \overset{G}{\perp} B \mid C \quad \text{but} \quad A \overset{G}{\not\perp} B \mid \emptyset$$

In particular $A = \{a\}$, $B = \{b\}$, C is an ab-separator.

i.e. a & b in same component of G $\quad a, b \notin C$
but not in $G \setminus C$

C is a <u>minimal</u> ab-separator if no proper subset of C is also an ab-separator

<u>Example</u>



$\{c, d, g\}$ is an ab-separator

$\{c, d\}$ is a minimal ab-separator

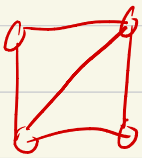
$C \subseteq V$ is a <u>separator</u> if ab-separator for some $a, b \in V$.

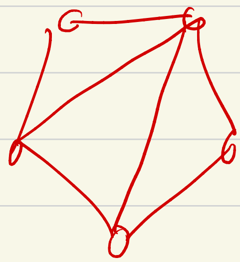
<u>Note:</u> $\{c, d\}$ not a minimal separator since $\{d\}$ is also a separator.

Chordal graphs

Def: $G=(V,E)$ is chordal if $\nexists A \subseteq V$ s.t.

$G|_A$ is a cycle on 4 or more vertices.





Theorem TFAE

a) G is chordal

b) Every minimal vertex separator of G is complete.

c) G is weakly decomposable

d) G has a perfect elimination order

e) $\exists$ a clique tree T_π for G.

Def: $G=(V,E)$ $A,B,C \subseteq V$ disjoint sets

(A,B,C) is a proper weak decomposition if

0) $A,B \neq \emptyset$

1) $V = A \cup B \cup C$

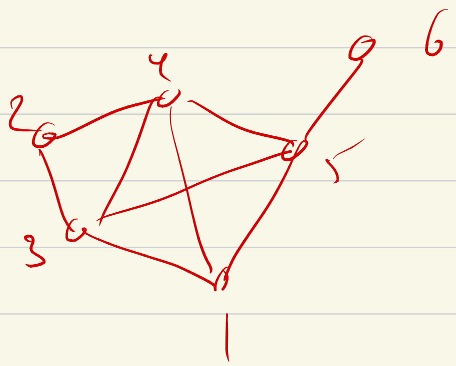
2) $A \perp_G B \mid C$

3) $G|_C$ is complete

G <u>weakly decomposable</u> if G is complete or has a proper weak decomposition s.t.

$G|_{A \cup C}$ & $G|_{B \cup C}$ are weakly decomposable.

<u>Example:</u>



$(\{2\}, \{6\}, \{1,3,4,5\})$

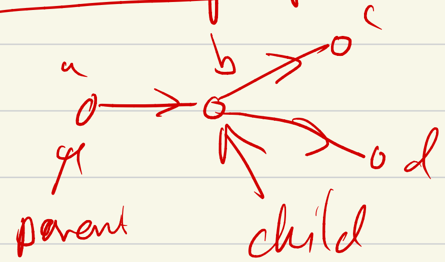
proper weak decomp.

$G = (V, E)$ $\quad V = \{v_1, \dots, v_m\}$ $\quad \mathcal{L}_i = \{v_{i+1}, \dots, v_m\}$

$v_1, \dots, v_m$ is a PEO if

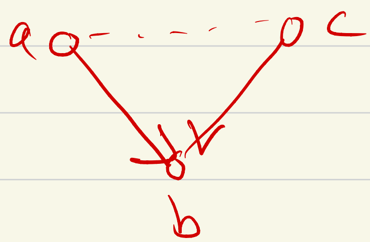
$N(v_i) \cap \mathcal{L}_i$ is a clique $\forall 1 \leq i \leq m-1$

Directed graphs

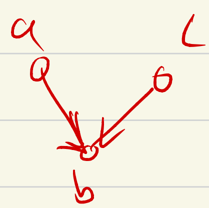


$an(d) = \{a, b, d\}$

DAG: Directed graph with no directed cycle



a collider

a v-structure: collider 

with no edge a to c.

unshielded collider

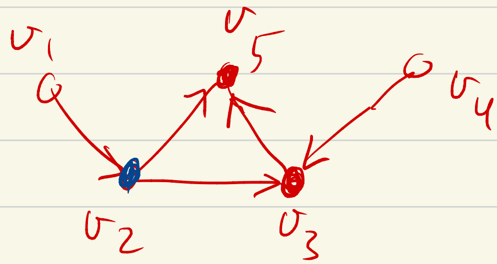
Separation in DAG's

$G = (V, E)$, $C \subseteq V$

A path S is d-connecting (disconnecting) given C if all of its colliders are contained in $an_G(C) := \bigcup_{u \in C} an_G(u)$

and no non-collider nodes of S are in C.

Example



$S = \langle v_1, v_2, v_3, v_4 \rangle$

collider

S d-connecting given $\{v_3\}$

S d-connecting given $\{v_5\}$

S d-separating given $\{v_2\}$