M110C Week 6

Goals:

Recap, Warmup.

More Partial Derivatives.

Tangent Planes.

Directional Derivatives.

The Gradient.

Higher Order Derivatives.

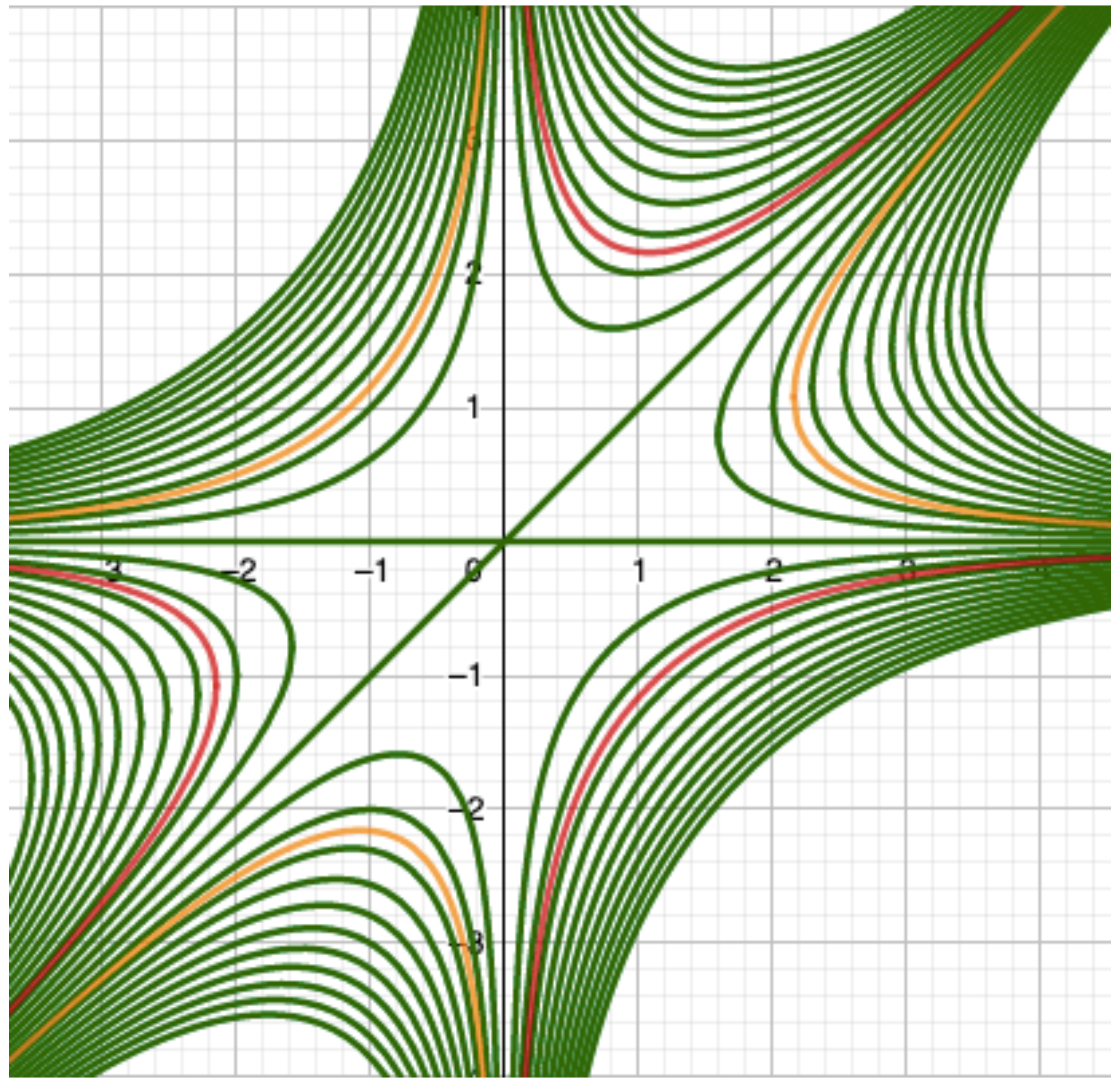
Multivariable Taylor Series.

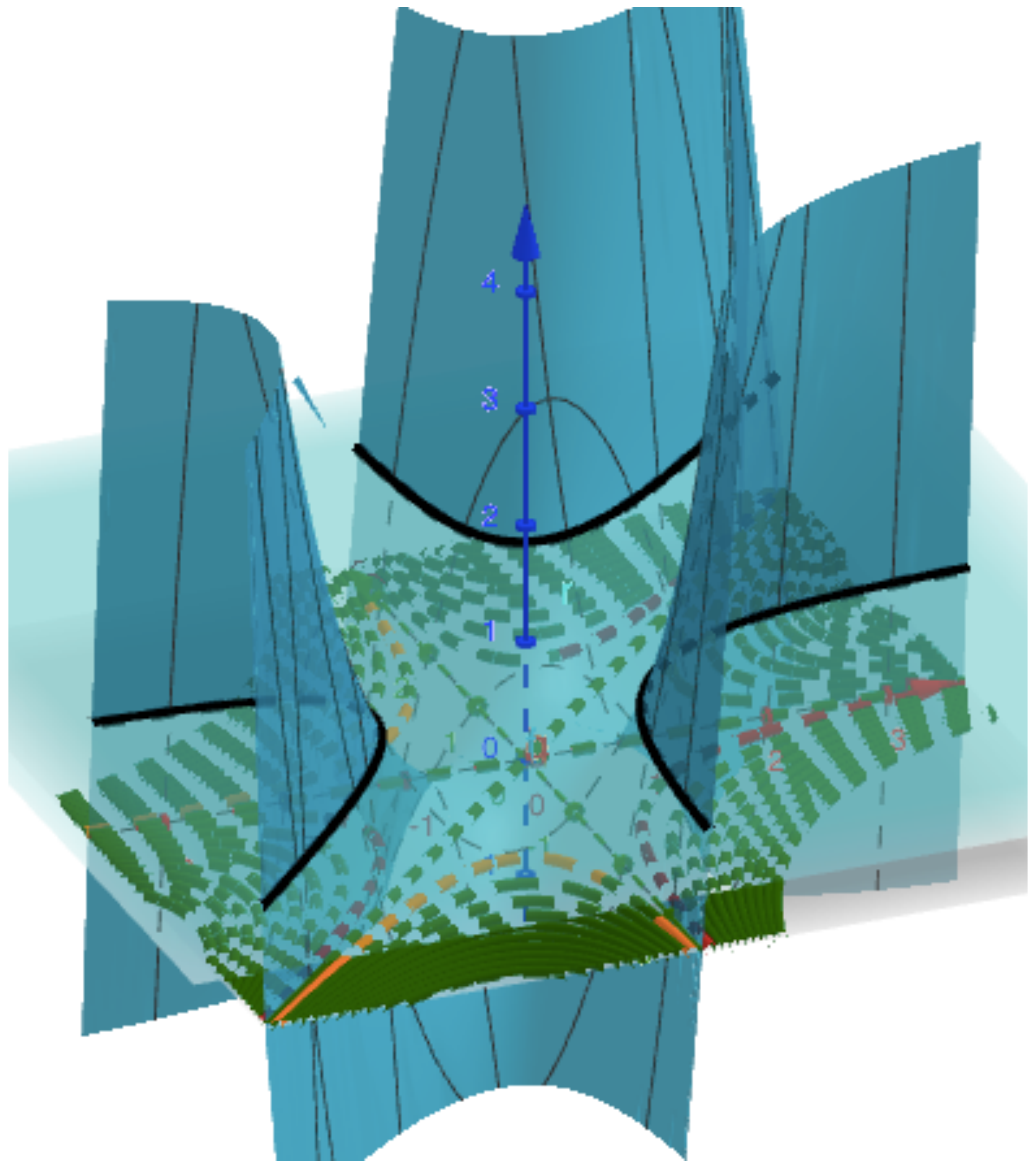
Recap, WarmUp, pg 1.

Last time we saw…

Level Curves and Contour Maps of Multivariable Functions!

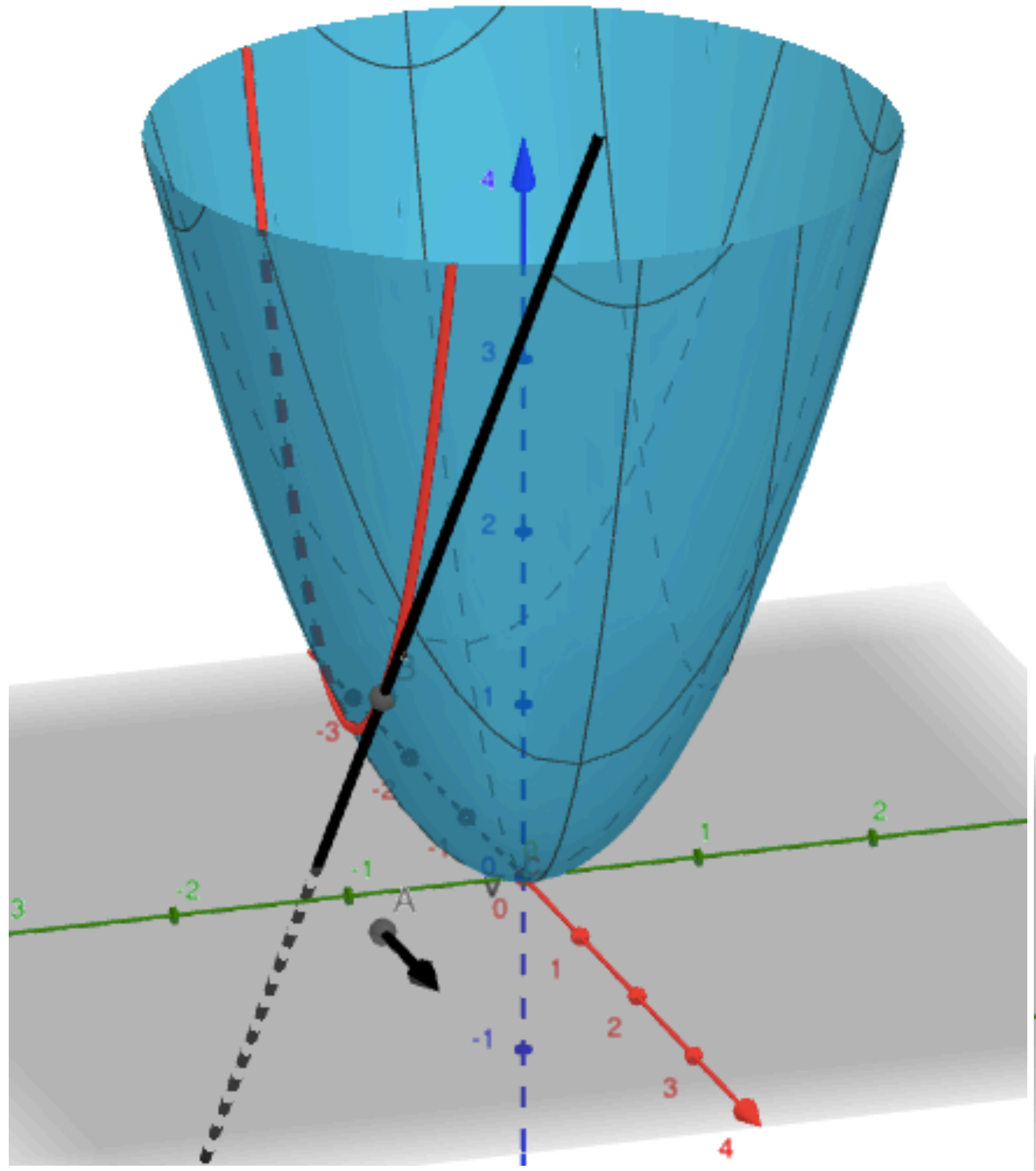
$z = f(x,y) = xy^2 - yx^2$

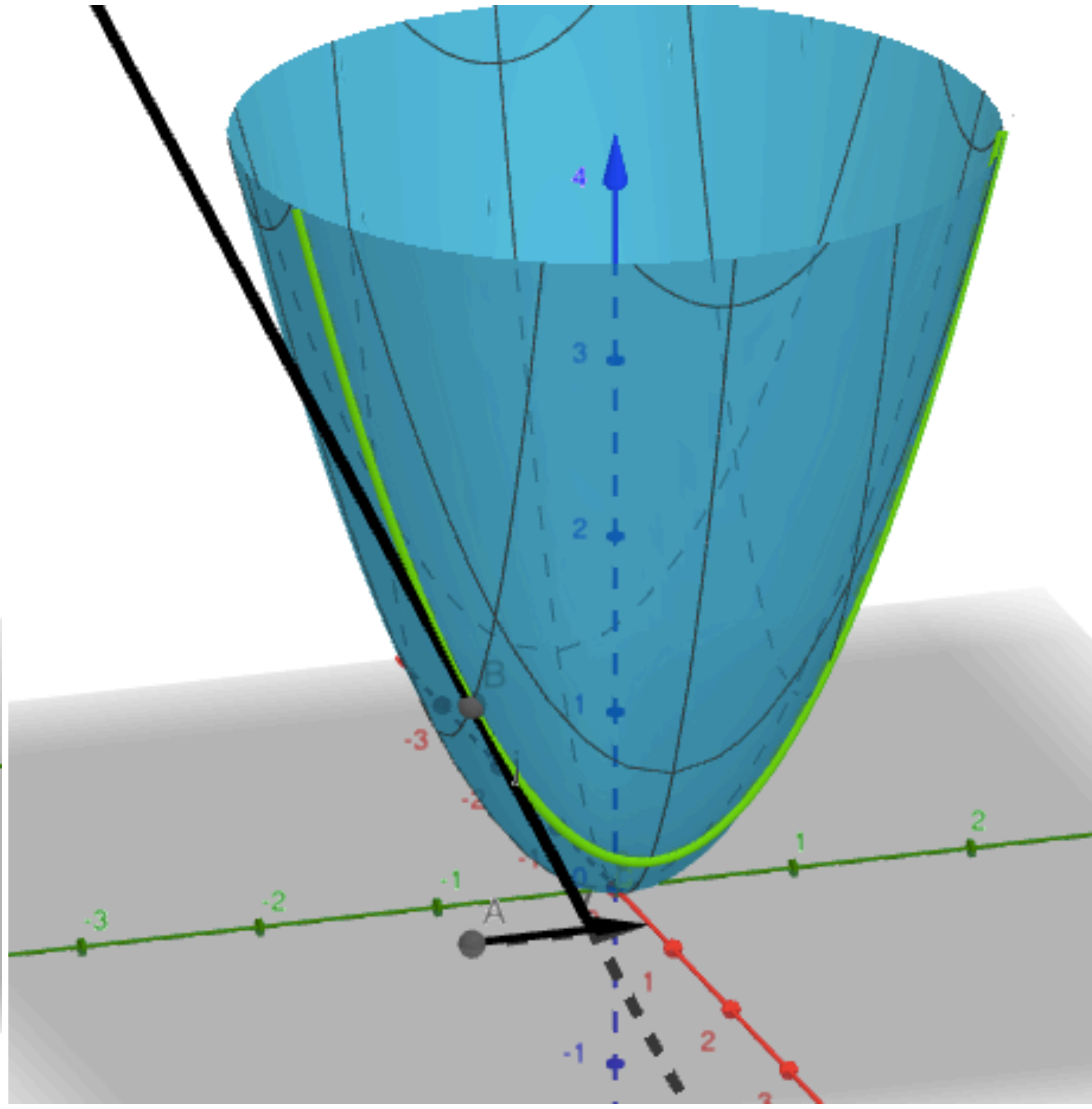




As well as…

Partial Derivatives!





$g(t) = < a+t, b, f(a+t, b) >$

$g(t) = < a, b+t, f(a, b+t) >$

$$\frac{\partial f}{\partial x}(a,b) = f_x(a,b) := g_z'(0) \qquad \frac{\partial f}{\partial y}(a,b) = f_y(a,b) := g_z'(0)$$

We parameterized a curve on the graph of f through a given point, in a given direction.

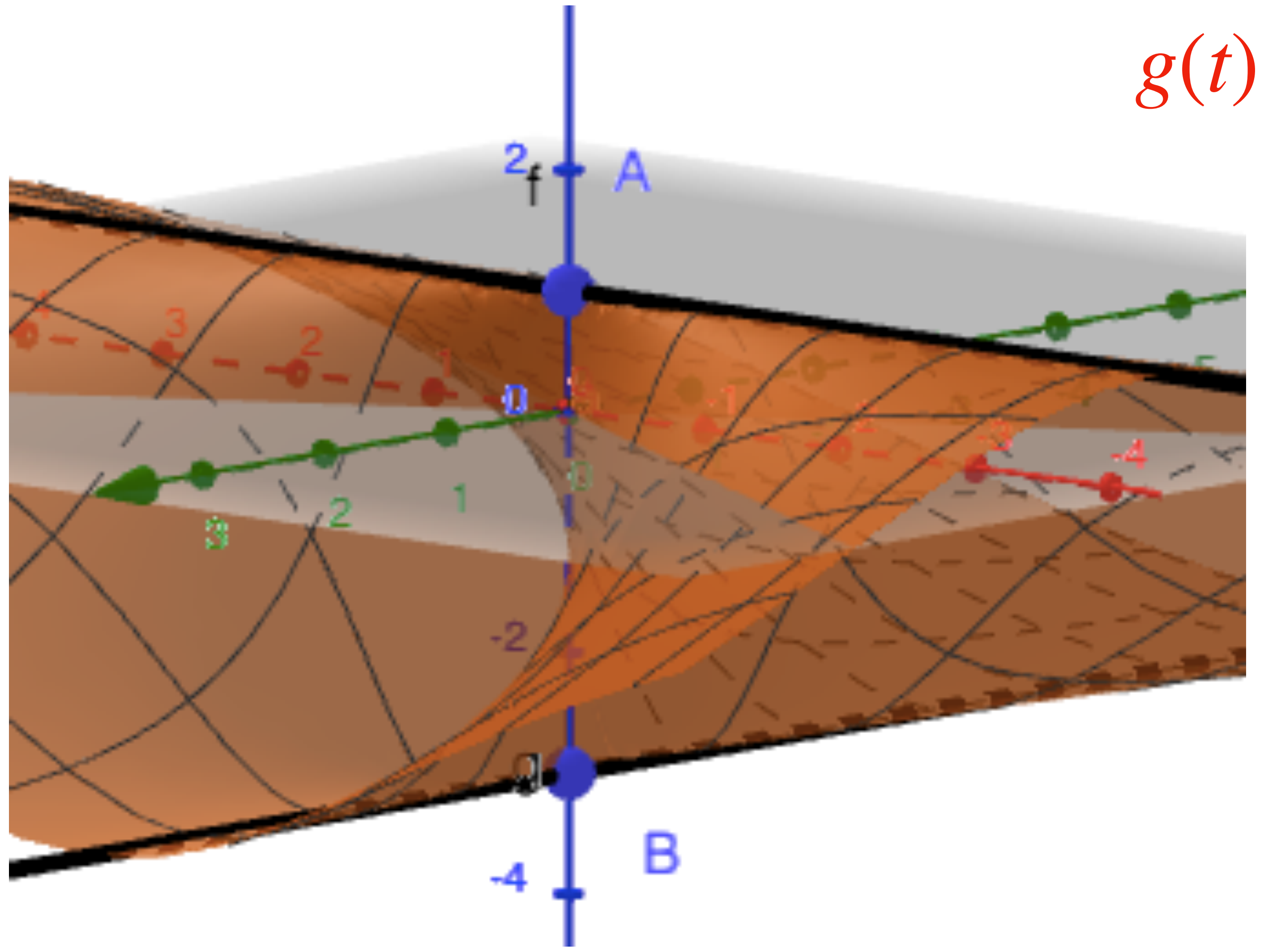
The derivative of the z-component measures the derivative of f at the point, in that direction.

Multivariable limits:

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - 3y^2}{x^2 + y^2} = ?$$

This limit does not exist:

$\dfrac{x^2 - 3y^2}{x^2 + y^2}$ doesn't approach a single number.



Recap, WarmUp, pg 2.

We saw the standard way of computing partial derivatives f_x and f_y.

To compute f_x, treat y as a constant, differentiate in x only.

To compute f_y, treat x as a constant, differentiate in y only.

e.g.

$f(x, y) = \sin(xy)$

Compute $f_y(x, y)$ just like you would compute

$$\frac{d}{dy}\sin(5y) = \cos(5y)\cdot 5 = 5\cos(5y)$$

$$\frac{\partial}{\partial y}\sin(xy) = \cos(xy)\cdot x = x\cos(xy)$$

Examples.

	$f_x(x, y)$	$f_y(x, y)$
$f(x, y) = \sin(xy)$	$y\cos(xy)$	$x\cos(xy)$
$f(x, y) = x^4 + 5xy^3$	$4x^3 + 5y^3$	$15xy^2$
$f(x, y) = x^2 + 3xy - y^2$	$2x + 3y$	$3x - 2y$
$f(x, y) = xy^2 - yx^2$	$y^2 - 2xy$	$2xy - x^2$
$f(x, y) = \cos(xy^3)$	$-y^3\sin(xy^3)$	$-3xy^2\sin(xy^3)$
$f(x, y) = \frac{2x}{3y}$	$\frac{2}{3y}$	$-\frac{2x}{3}y^{-2}$

Partial Derivatives, More Examples.

All of the rules you learned in single-variable calculus still apply in multi-variable calculus:

product rule! quotient rule! chain rule! oh my!

Examples. Compute $\dfrac{\partial}{\partial x}$ and $\dfrac{\partial}{\partial y}$.

1. $p(x, y) = y^2 \sin(xy)$

$$p_x(x, y) = y^2 \cos(xy) \cdot y = y^3 \cos(xy)$$

$$p_y(x, y) = f'(y)g(y) + f(y)g'(y)$$

where $f(y) = y^2, \;\; g(y) = \sin(xy)$

$$p_y(x, y) = 2y \sin(xy) + y^2 \cos(xy) \cdot x$$
$$= 2y \sin(xy) + xy^2 \cos(xy)$$

2. $q(x, y) = \dfrac{\sin(x)e^{2x}}{(x+y)^2}$

$$q_x(x, y) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

where $f(x) = \sin(x)e^{2x}, \;\; g(x) = (x+y)^2$

$$q_x(x, y) = \frac{(\cos(x)e^{2x} + 2\sin(x)e^{2x})(x+y)^2 - \sin(x)e^{2x} \cdot 2(x+y)}{(x+y)^4}$$

$$q_y(x, y) = \frac{\partial}{\partial y}\left(\sin(x)e^{2x}(x+y)^{-2}\right) = \sin(x)e^{2x} \cdot (-2)(x+y)^{-3}$$
$$= \frac{-2\sin(x)e^{2x}}{(x+y)^3}$$

3. $h(x, y) = \sqrt{e^{2x} + x^2y^3\sin(xy)} = \left(e^{2x} + x^2y^3\sin(xy)\right)^{1/2}$

$$h_x(x, y) = \frac{1}{2}\left(e^{2x} + x^2y^3\sin(xy)\right)^{-1/2}(2e^{2x} + 2xy^3\sin(xy) + x^2y^4\cos(xy))$$

$$h_y(x, y) = \frac{1}{2}\left(e^{2x} + x^2y^3\sin(xy)\right)^{-1/2}(3x^2y^2\sin(xy) + x^3y^3\cos(xy))$$

Partial Derivatives, Many many variables. Practice.

Example. Compute all the possible partial derivatives.

1. $f(x, y, z) = z\tan(x^2 + 2yz)$

$$f_x(x, y, z) = z\sec^2(x^2 + 2yz)\cdot 2x$$
$$= 2xz\sec^2(x^2 + 2yz)$$

$$f_y(x, y, z) = z\sec^2(x^2 + 2yz)\cdot 2z$$
$$= 2z^2\sec^2(x^2 + 2yz)$$

$$f_z(x, y, z) = 1\cdot\tan(x^2 + 2yz) + z\sec^2(x^2 + 2yz)\cdot 2y$$
$$= \tan(x^2 + 2yz) + 2yz\sec^2(x^2 + 2yz)$$

2. $h(u, v) = \dfrac{e^v}{u + v^2}$

$$h_u(u, v) = -e^v(u + v^2)^{-2} = \frac{-e^v}{(u + v^2)^2}$$

$$h_v(u, v) = \frac{e^v(u + v^2) - e^v\cdot 2v}{(u + v^2)^2} = \frac{e^v(v^2 - 2v + u)}{(u + v^2)^2}$$

3. $g(x, y, z, t) = \dfrac{xt^2}{y^3z^4}\cos\left(\dfrac{z}{t}\right)$

$$g_x(x, y, z, t) = \frac{t^2}{y^3z^4}\cos\left(\frac{z}{t}\right) \qquad g_y(x, y, z, t) = \frac{-3xt^2}{y^4z^4}\cos\left(\frac{z}{t}\right)$$

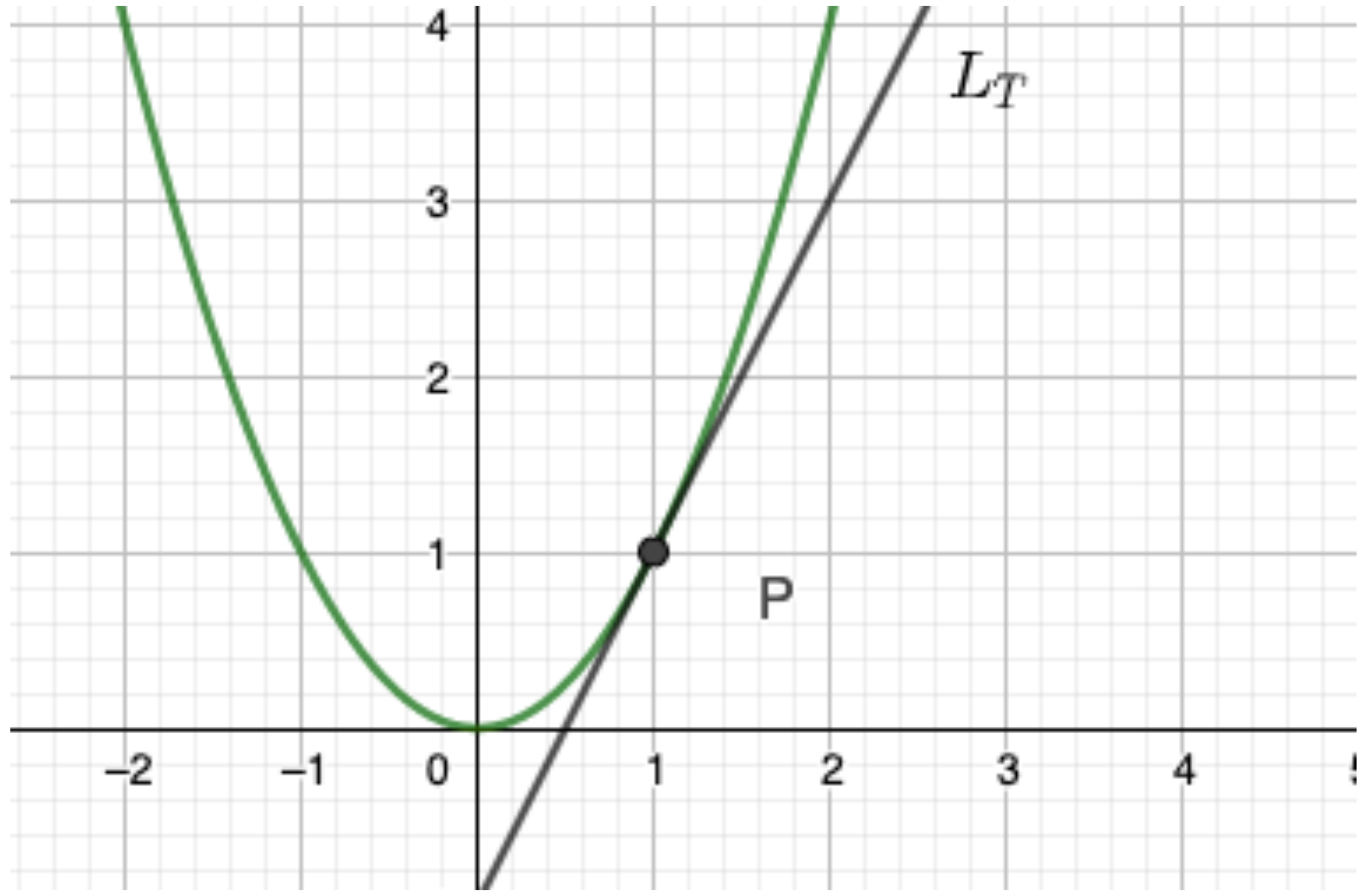
$$g_z(x, y, z, t) = \frac{-4xt^2}{y^3z^5}\cos\left(\frac{z}{t}\right) + \frac{-xt^2}{y^3z^4}\sin\left(\frac{z}{t}\right)\cdot\frac{1}{t}$$
$$= \frac{-4xt^2}{y^3z^5}\cos\left(\frac{z}{t}\right) - \frac{xt}{y^3z^4}\sin\left(\frac{z}{t}\right)$$

$$g_t(x, y, z, t) = \frac{2xt}{y^3z^4}\cos\left(\frac{z}{t}\right) + \frac{-xt^2}{y^3z^4}\sin\left(\frac{z}{t}\right)\cdot\left(-\frac{z}{t^2}\right)$$
$$= \frac{2xt}{y^3z^4}\cos\left(\frac{z}{t}\right) + \frac{x}{y^3z^3}\sin\left(\frac{z}{t}\right)$$

Tangent Planes, pg 1.

A tangent plane to $z = f(x, y)$ at a point $P(a, b, f(a, b))$ is similar.

Remember tangent lines in single-variable calculus.



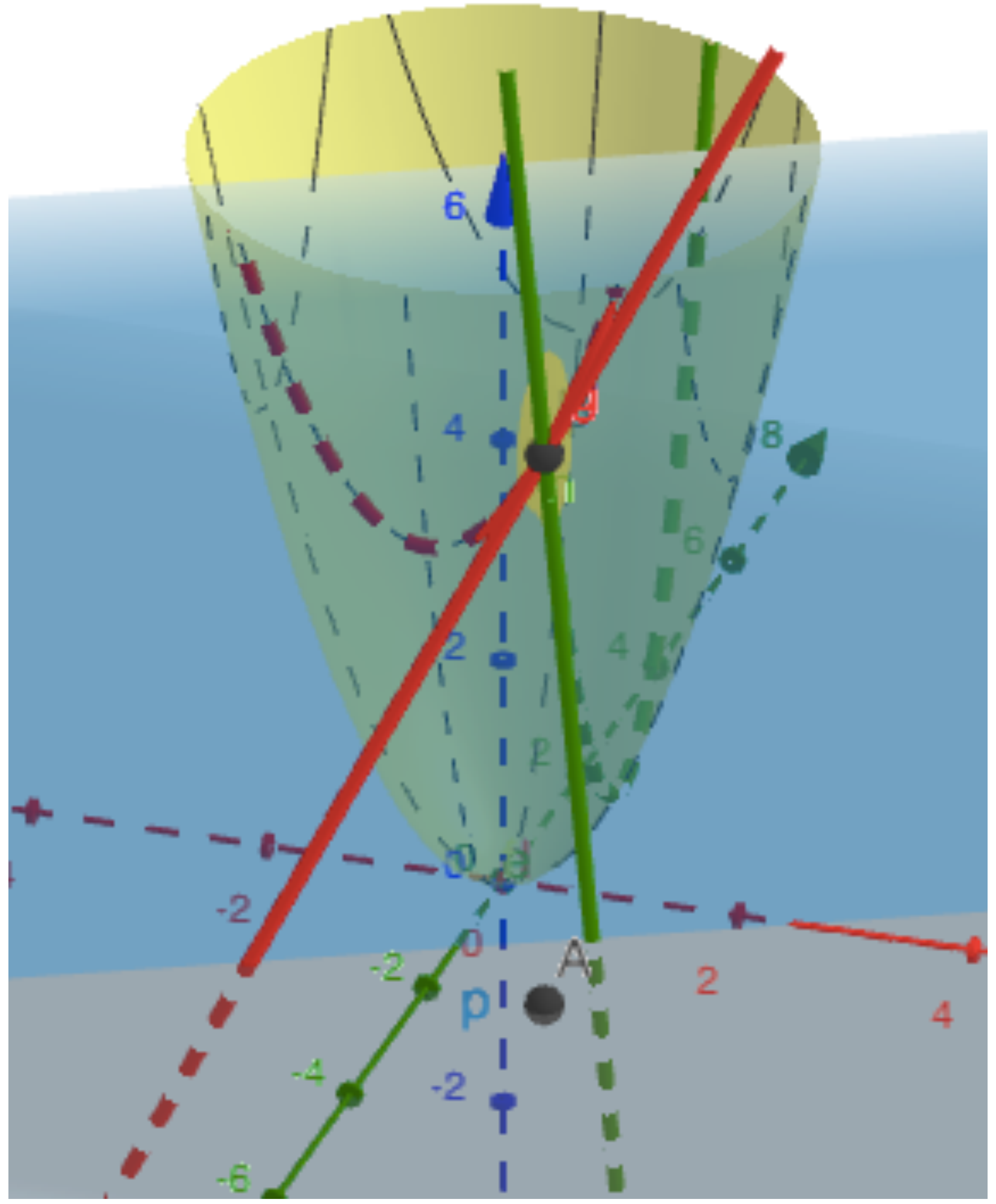
The line tangent to $y = f(x)$ at P goes through the point $P(a, f(a))$ and has slope $f'(a)$

Its equation was

$$y - y_0 = m(x - x_0)$$

$$y - f(a) = f'(a)(x - a)$$

$$y = f(a) + f'(a)(x - a)$$



There's a tangent line in every direction, all of which lie in the tangent plane of f at the point P.

We already know two such lines, namely the one in the x direction (red), and the one in the y direction (green).

The direction vectors of those two lines give us vectors that lie on the plane.

Their cross product is a normal vector i.e. it is perpendicular to the plane.

x-direction (red line): $\mathbf{L}_1(t) = < a, b, f(a, b) > + t \cdot g_1'(0)$

$g_1(t) = < a + t, b, f(a + t, b) >$ $\quad = < a, b, f(a, b) > + t < 1, 0, f_x(a, b) >$

y-direction (green line): $\mathbf{L}_2(t) = < a, b, f(a, b) > + t \cdot g_2'(0)$

$g_2(t) = < a, b + t, f(a, b + t) >$ $\quad = < a, b, f(a, b) > + t < 0, 1, f_y(a, b) >$

$\mathbf{u} = < 1, 0, f_x(a, b) >$

$\mathbf{v} = < 0, 1, f_y(a, b) >$

The normal vector is $\mathbf{w} = \mathbf{u} \times \mathbf{v}$

Tangent Planes, pg 2.

$\mathbf{u} = < 1,0,f_x(a,b) >$

$\mathbf{v} = < 0,1,f_y(a,b) >$

$$\mathbf{w} = \mathbf{u} \times \mathbf{v} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f_x(a,b) \\ 0 & 1 & f_y(a,b) \end{pmatrix}$$

$$= < -f_x(a,b), -f_y(a,b), 1 >$$

The tangent plane at the point $P(a, b, f(a, b))$ is

$$-f_x(a,b)(x-a) - f_y(a,b)(y-b) + 1(z - f(a,b)) = 0$$

Move some stuff to the right side…

$$z - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Compare this with single-variable:

$$y = f(a) + f'(a)(x-a)$$

Examples. Find the tangent plane of the function at the indicated point.

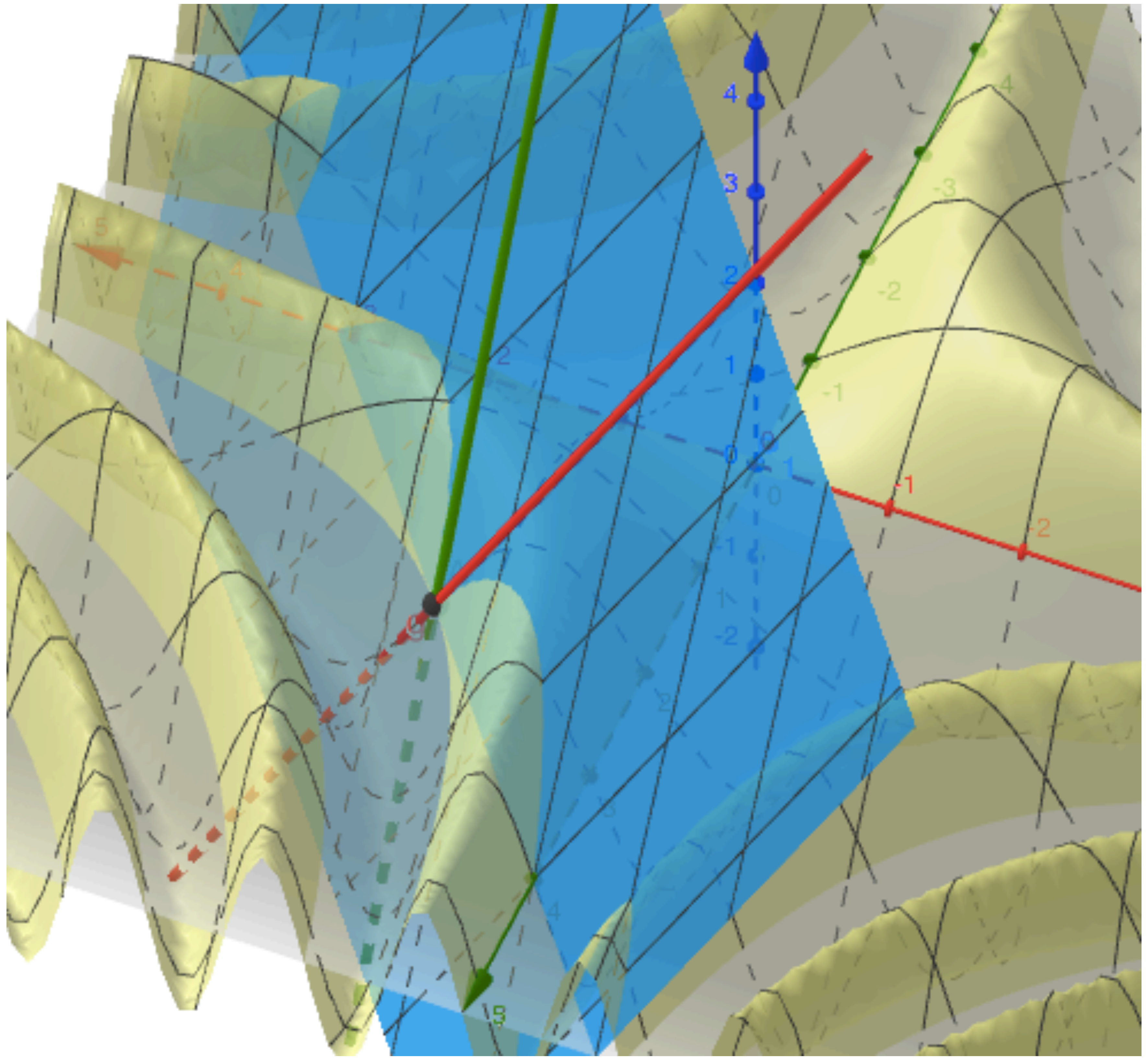
1. $f(x,y) = \sin(xy) \quad P(\pi/2, 2)$

$f(\pi/2, 2) = 0$

$f_x(x,y) = y\cos(xy) \qquad f_x(\pi/2, 2) = -2$

$f_y(x,y) = x\cos(xy) \qquad f_y(\pi/2, 2) = -\pi/2$

$z = -2(x - \pi/2) - \pi/2(y-2)$



Tangent Planes, pg 3, Practice.

$$\mathbf{z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)}$$

Find the tangent plane of the function at the given point.

2. $f(x, y) = \cos(x) - \cos(y)$ $\quad P(\pi/4, 3\pi/4)$ $\quad f(\pi/4, 3\pi/4) = \sqrt{2}$

$$f_x(x, y) = -\sin(x) \qquad f_x(\pi/4, 3\pi/4) = -\sqrt{2}/2$$

$$f_y(x, y) = \sin(y) \qquad f_y(\pi/4, 3\pi/4) = \sqrt{2}/2$$

$$z = \sqrt{2} - \frac{\sqrt{2}}{2}(x - \pi/4) + \frac{\sqrt{2}}{2}(y - 3\pi/4)$$

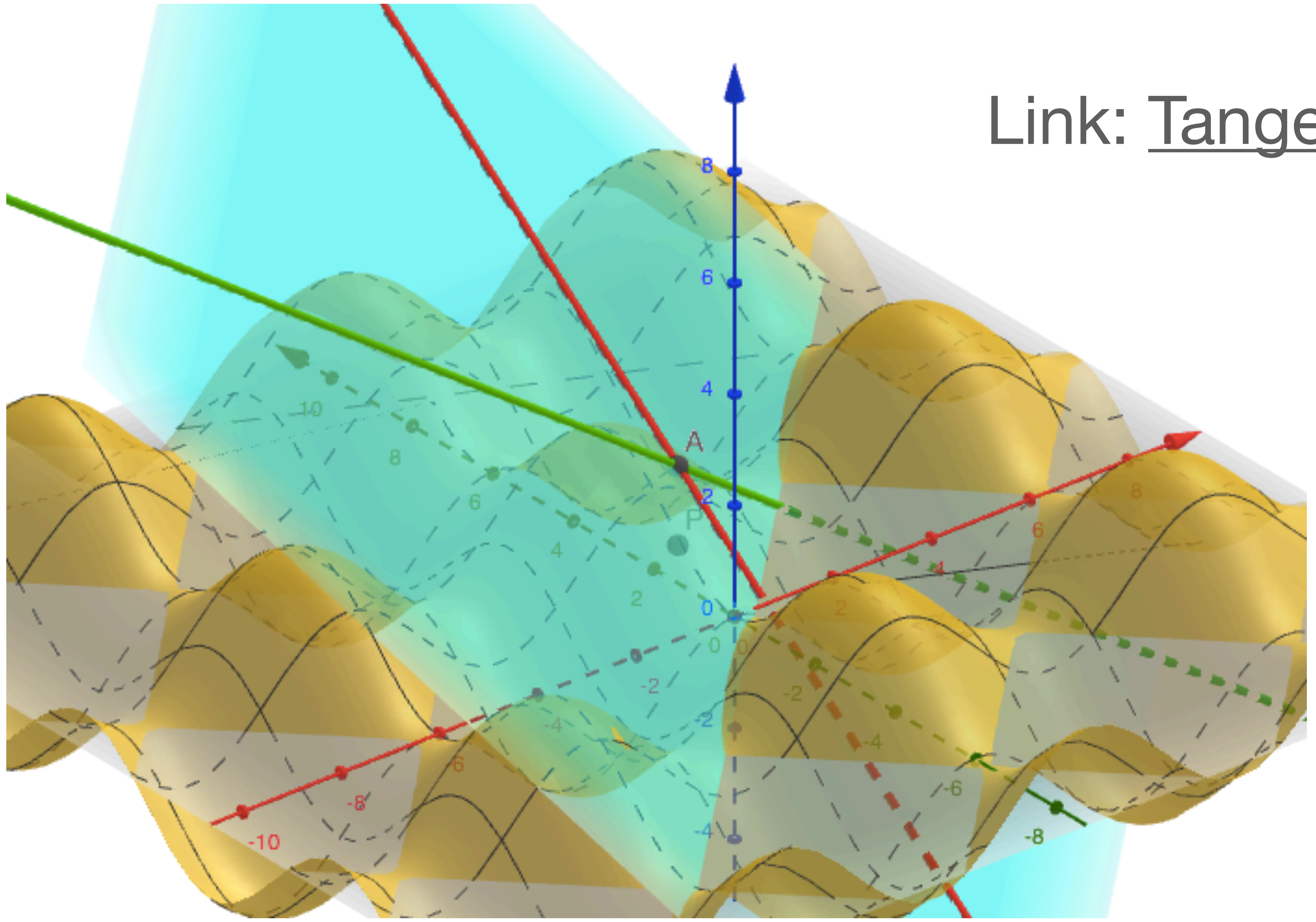
3. $h(x, y) = 1 + x\ln(xy - 5)$ $\quad P(2,3)$ $\quad h(2,3) = 1$

$$h_x(x, y) = 0 + \ln(xy - 5) + x \cdot \frac{1}{xy - 5} \cdot y$$

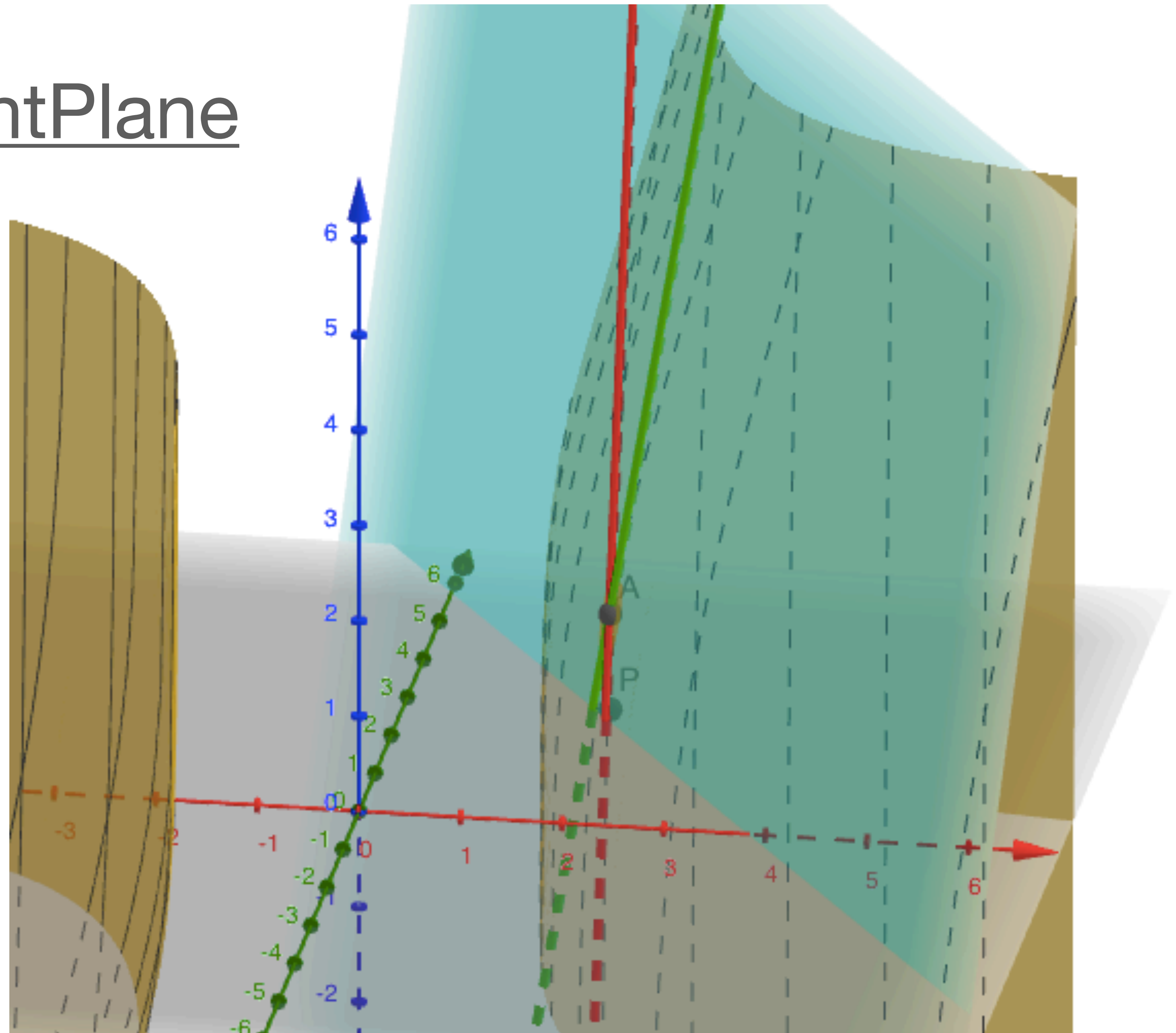
$$= \ln(xy - 5) + \frac{xy}{xy - 5} \qquad h_x(2,3) = 6$$

$$h_y(x, y) = x\frac{1}{xy - 5} \cdot x = \frac{x^2}{xy - 5} \qquad h_y(2,3) = 4$$

$$z = 1 + 6(x - 2) + 4(y - 3)$$



Link: TangentPlane



We can use tangent planes to approximate a function …→

Tangent Planes, pg 4, Approximate a function.

Example1.

Approximate the value of the function $f(x, y) = \sin(xy)$ at the point $(1.6, 2)$

We have the tangent plane to f at the point $(\pi/2, 2) \approx (1.57, 2)$

$z = -2(x - \pi/2) - \pi/2(y - 2)$

Input $(x, y) = (1.6, 2)\ldots$

$z \approx -0.05841$

The actual value is

$f(1.6 \cdot 2) \approx -0.05837$

Example2.

Approximate $h(x, y)$ at (2.2, 3)

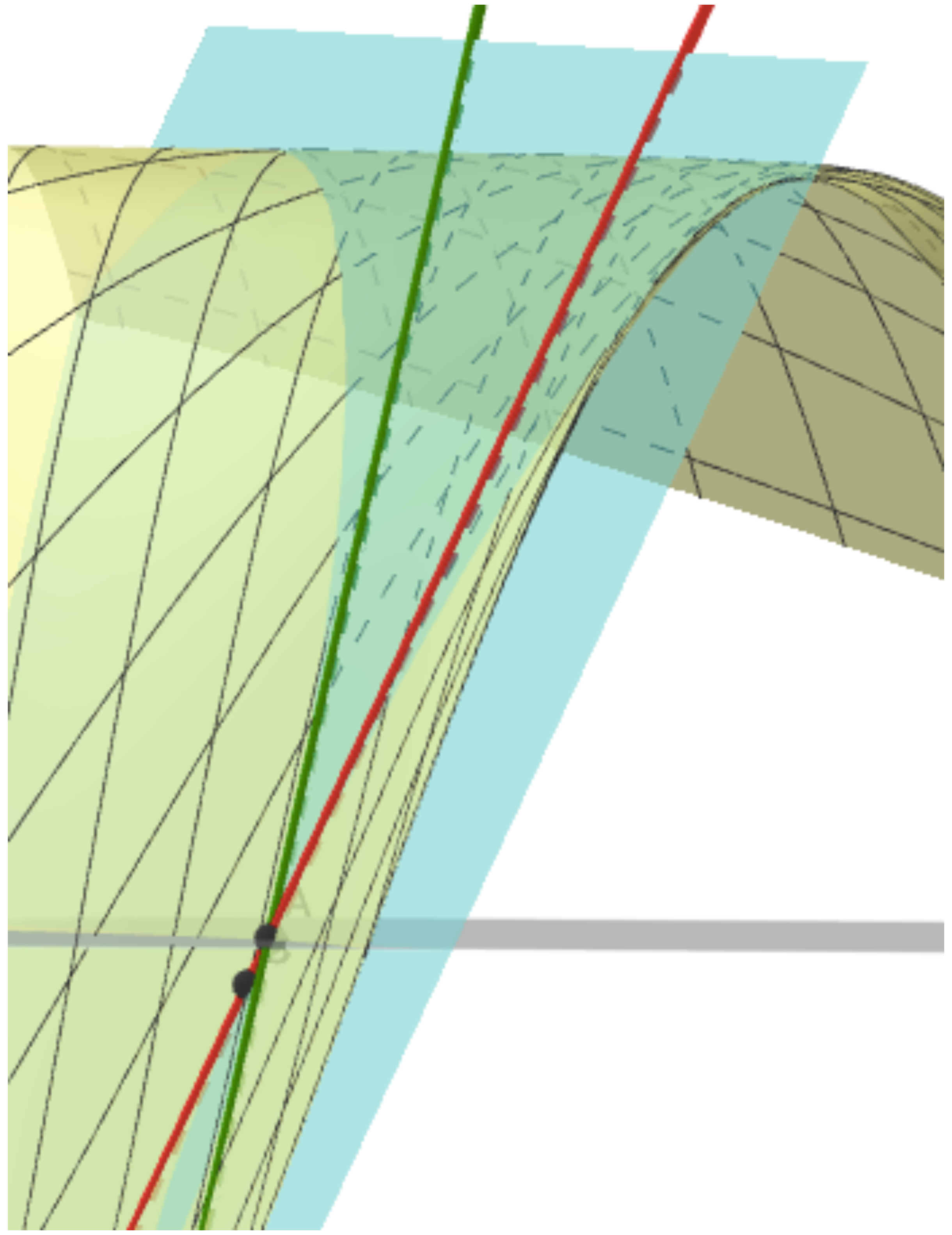
$h(x, y) = 1 + x\ln(xy - 5)$

$z = 1 + 6(x - 2) + 4(y - 3)$

$z(2.2,3) = 2.2 \qquad h(2.2,3) \approx 2.034$

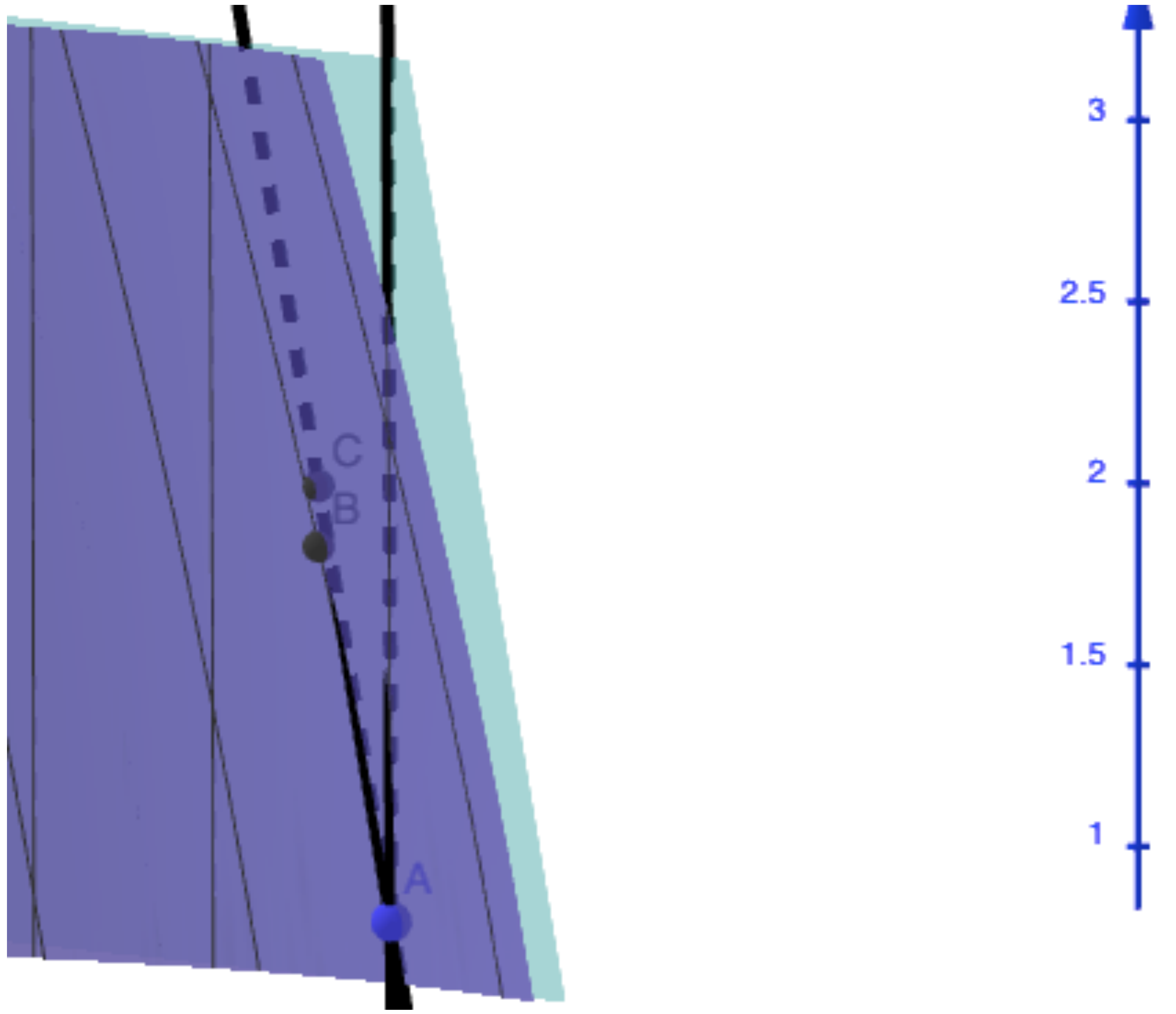
picture of example 1.

B and C are practically indistinguishable.



picture of example 2.

B is on the graph, while C is on the tangent plane



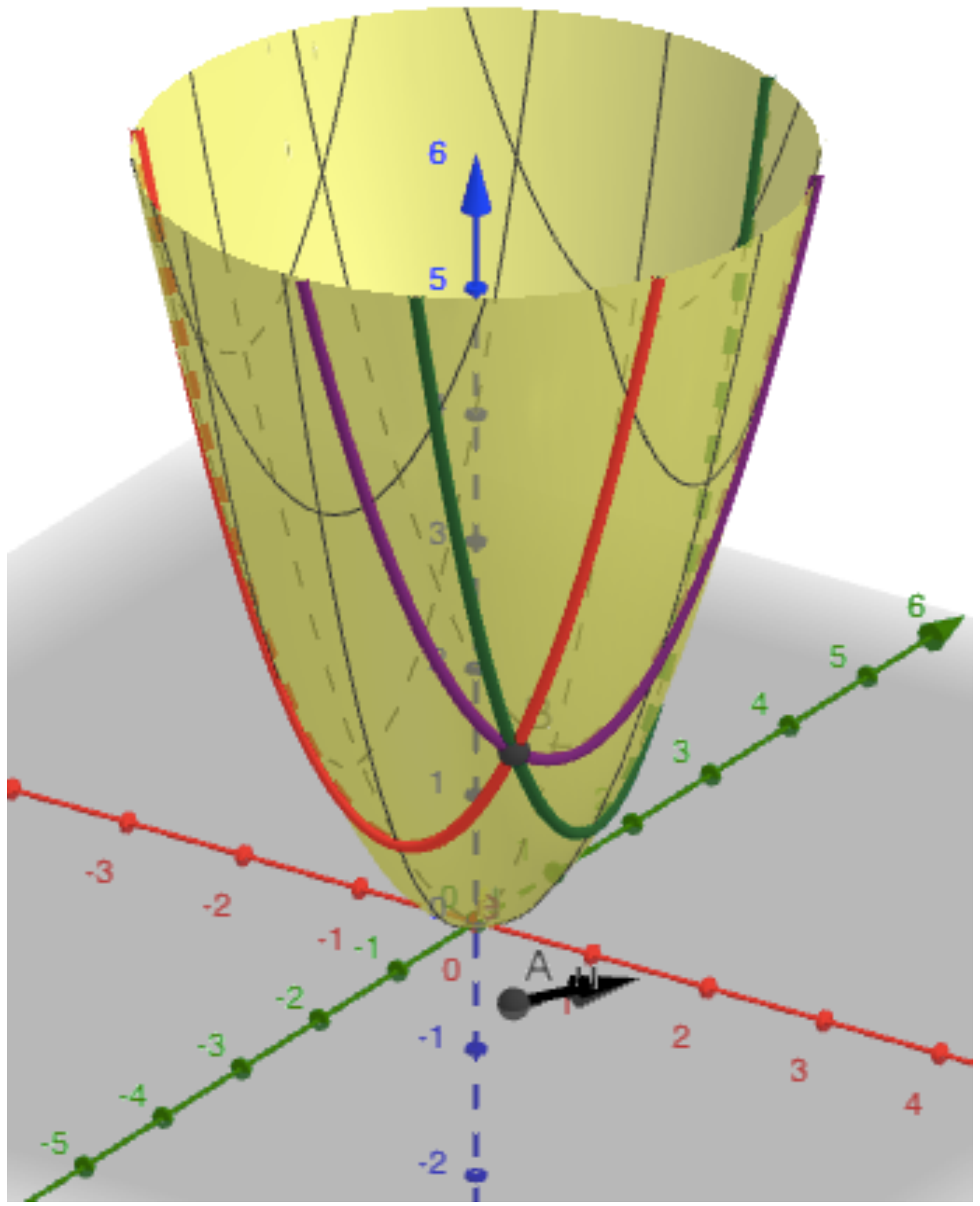
In general, the quantity $z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$ is a first-degree (linear) approximation to $f(x, y)$ at (a, b).

We will see higher-degree Taylor Approximations a bit later.

Directional Derivatives, pg 1.

We have computed derivatives in the x-direction and y-direction.

You can compute a derivative in any direction!



We have seen $f_x(a,b)$ equals the slope of the red curve at the point (a,b,f(a,b)).

$f_y(a,b)$ equals the slope of the green curve at the point (a,b,f(a,b)).

What about the purple curve: the derivative in the direction of a given vector $\mathbf{u}$, $f_u(a,b)$?

The tangent line whose slope represents $f_u(a,b)$ is tangent to another curve on the surface of the graph of f (in purple).

We can parametrize this curve as we did the ones in the x and y directions.

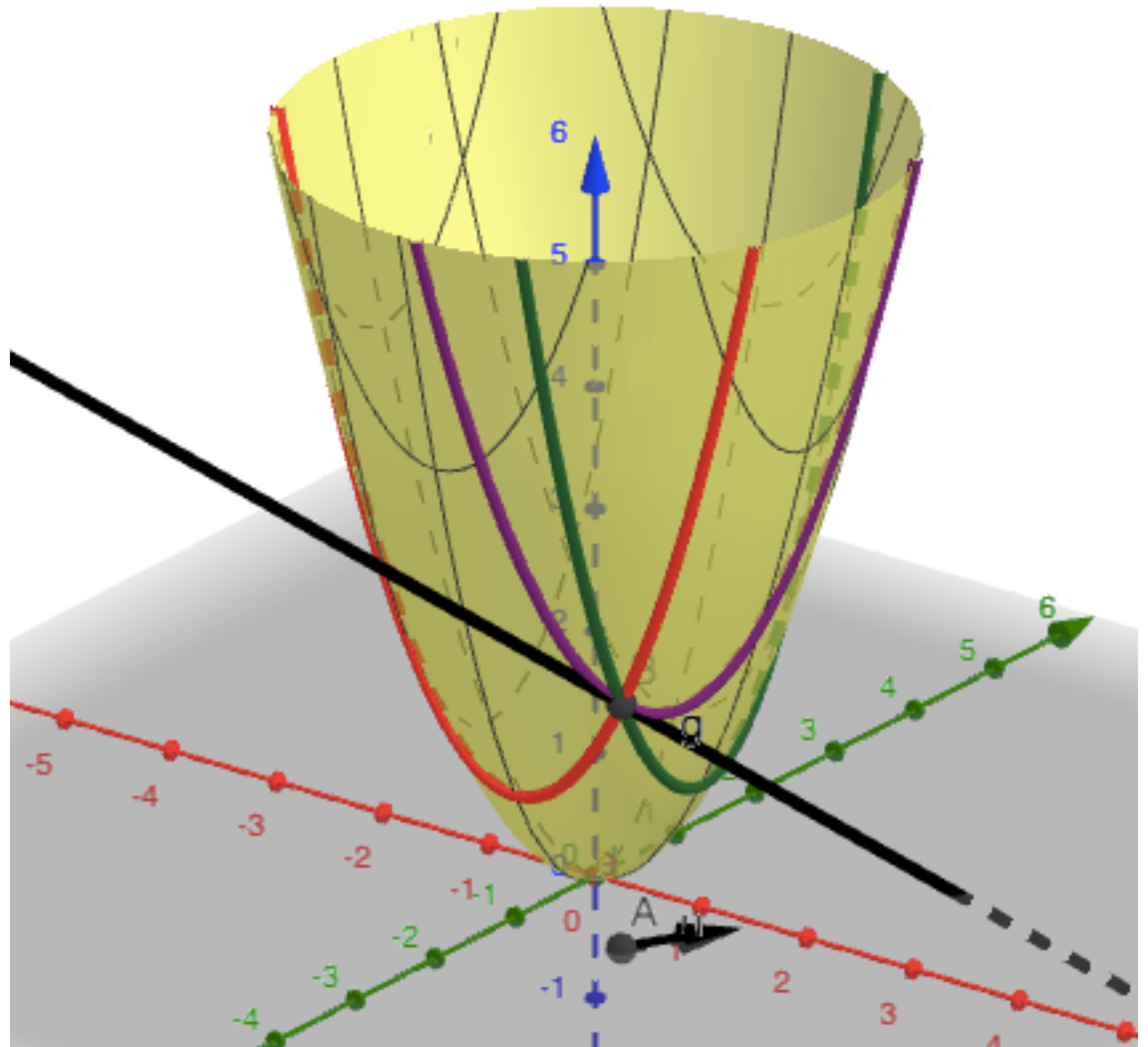
x - direction. $g_1(t) = < a+t, b, f(a+t,b) >$

y - direction. $g_2(t) = < a, b+t, f(a,b+t) >$

in general, in the direction of $\hat{\mathbf{u}} = < \hat{u}_x, \hat{u}_y >$.

$g_u(t) = < a + \hat{u}_x t,\ b + \hat{u}_y t,\ f(a+\hat{u}_x t, b+\hat{u}_y t) >$

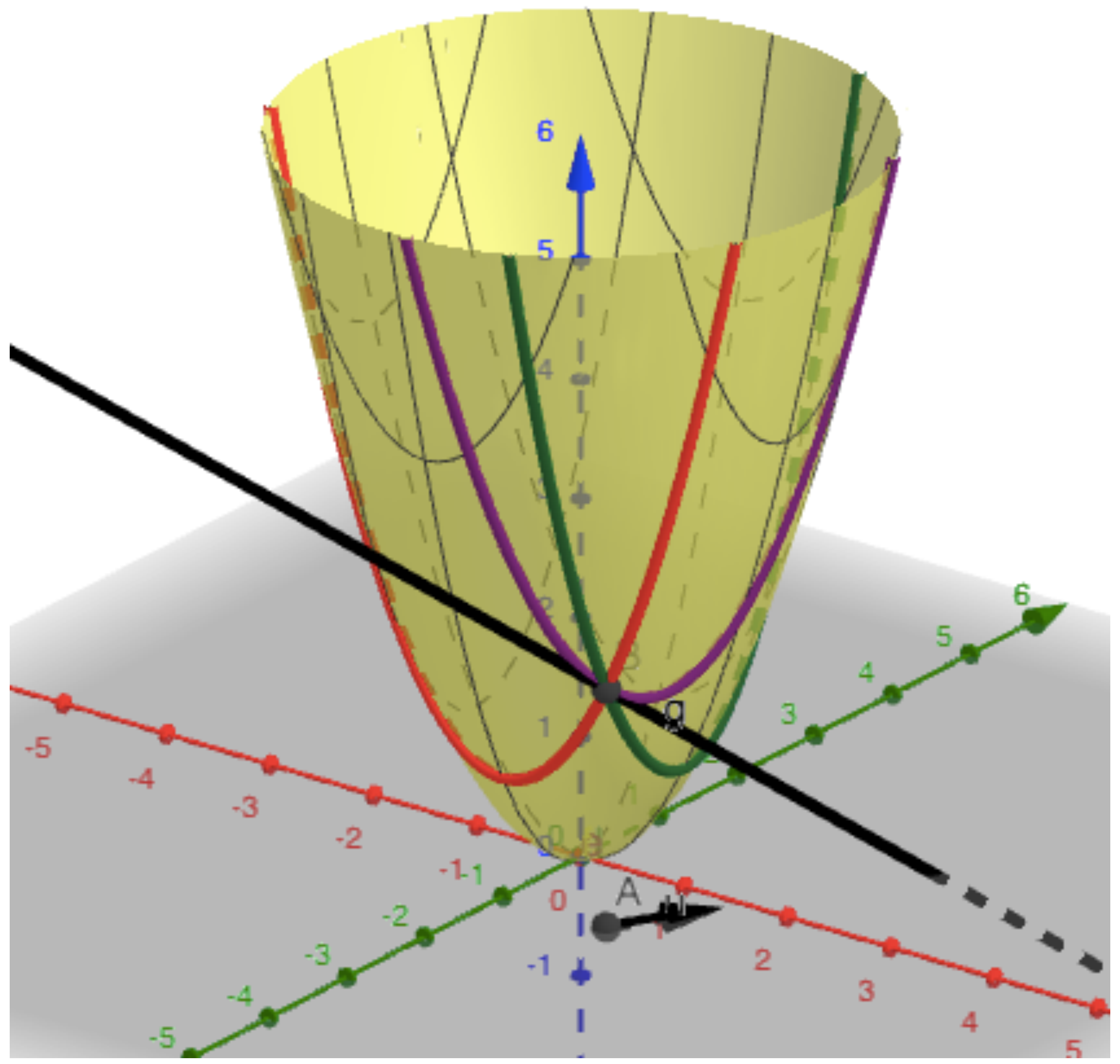
Here $\hat{\mathbf{u}} = < \hat{u}_x, \hat{u}_y >$ is a *unit* vector! $|\hat{\mathbf{u}}| = 1$



As before, the derivative of the z component, $z_u'(0)$, will measure the slope of the tangent line in the desired direction.

Directional Derivatives, pg 2.

Specific Examples.



Example 1.

$f(x, y) = x^2 + y^2$

$A(1, -1)$

$\hat{\mathbf{u}} = < \cos(\pi/3), \sin(\pi/3) >$

$= < 1/2, \sqrt{3}/2 >$

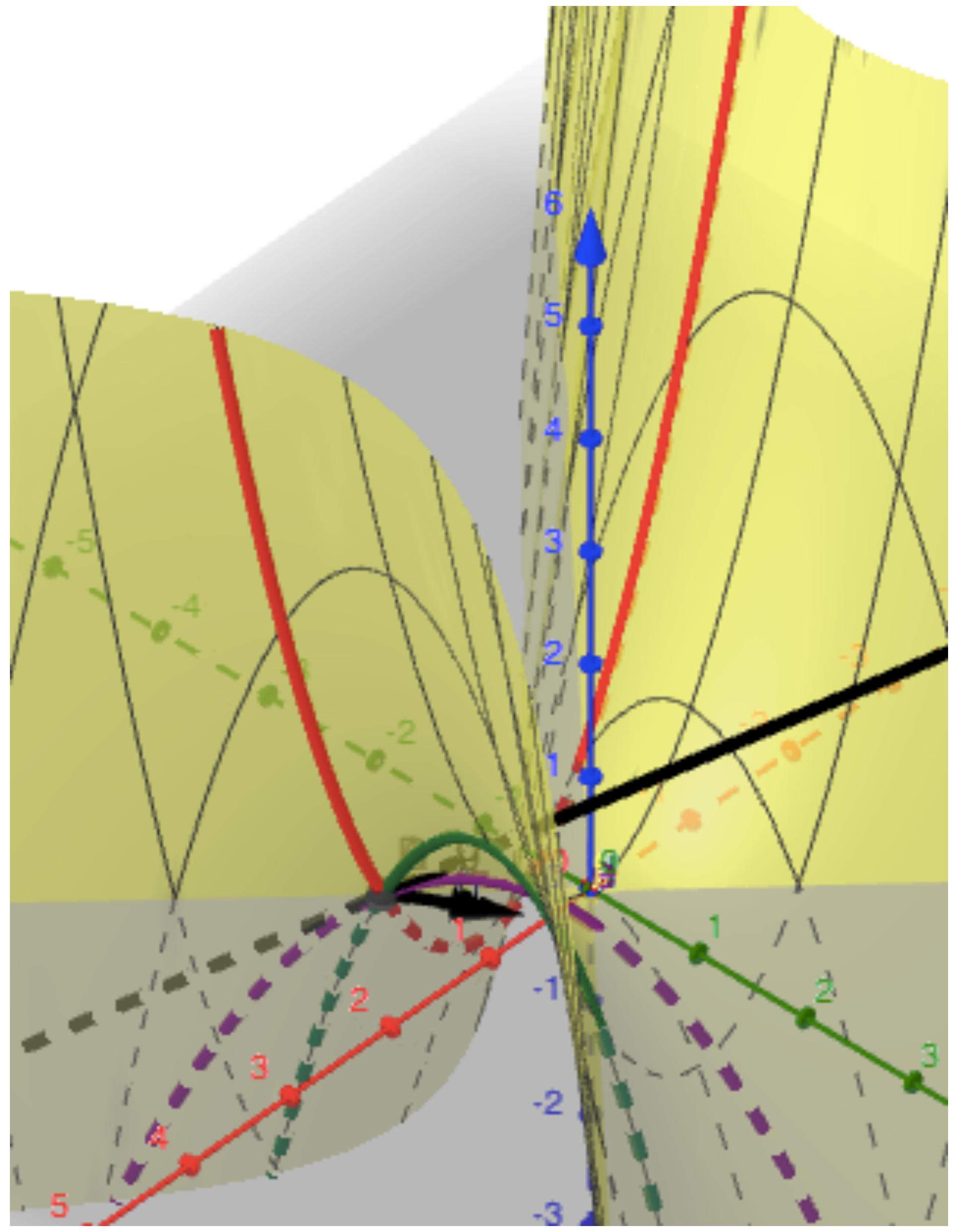
$g_u(t) = < a + \hat{u}_x t, b + \hat{u}_y t, f(a + \hat{u}_x t, b + \hat{u}_y t) >$

$= < 1 + t/2, \ -1 + \sqrt{3}t/2, \ (1 + t/2)^2 + (-1 + \sqrt{3}t/2)^2 >$

$z_u(t) = (1 + t/2)^2 + (-1 + \sqrt{3}t/2)^2$

$z_u'(t) = 2(1 + t/2) \cdot 1/2 + 2\sqrt{3}(-1 + \sqrt{3}t/2) \cdot 1/2$

$z_u'(0) = 1 - \sqrt{3} \approx -0.73$



Example 2.

$f(x, y) = x^2 - y^2$

$A(1, -1)$

$\hat{\mathbf{u}} = < \cos(2\pi/3), \sin(2\pi/3) >$

$= < -1/2, \sqrt{3}/2 >$

$g_u(t) = < 1 - t/2, \ -1 + \sqrt{3}t/2, (1 - t/2)^2 - (-1 + \sqrt{3}t/2)^2 >$

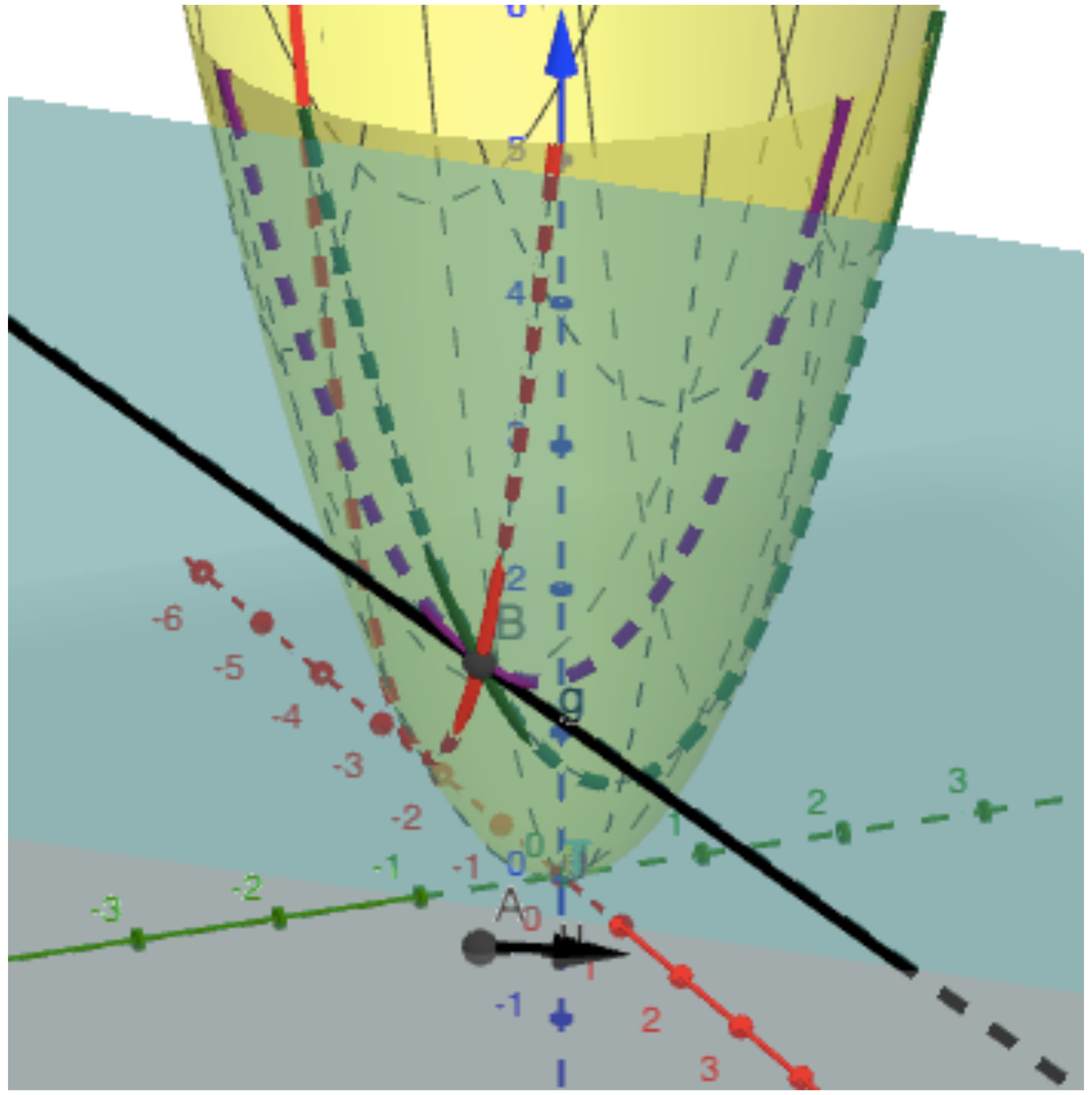
$z_u(t) = (1 - t/2)^2 - (-1 + \sqrt{3}t/2)^2$

$z_u'(t) = 2(1 - t/2) \cdot \frac{-1}{2} - 2\sqrt{3}(-1 + \sqrt{3}t/2) \cdot \frac{1}{2}$

$z_u'(0) = -1 + \sqrt{3} \approx 0.73$

Directional Derivatives, pg 3.

There are other ways to compute $f_u(a,b)$.



We can get the tangent line in the direction of $\mathbf{u}$ from the tangent plane at the point (a,b).

The tangent line has equation:

$$\mathbf{L}(t) = < a + \hat{u}_x t,\ b + \hat{u}_y t,\ z(a + \hat{u}_x t, b + \hat{u}_y t) >$$

where $z(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$$\mathbf{L}(t) = < a + \hat{u}_x t, b + \hat{u}_y t, f(a,b) + f_x(a,b)\hat{u}_x t + f_y(a,b)\hat{u}_y t >$$

The rate of change of the z coordinate is

$$\mathbf{L}_z'(t) = f_x(a,b)\hat{u}_x + f_y(a,b)\hat{u}_y$$

This is the slope that we want, $f_u(a,b)$.

Examples.

$f(x,y) = x^2 + y^2$

$A(1,-1)$

$\hat{\mathbf{u}} = < \cos(\pi/3), \sin(\pi/3) >$

$= < 1/2, \sqrt{3}/2 >$

$f_x(1,-1) = 2$

$f_y(1,-1) = -2$

$f_u(1,-1) = 2\frac{1}{2} - 2\frac{\sqrt{3}}{2}$

$= 1 - \sqrt{3} \approx -0.73$

$f(x,y) = x^2 - y^2$

$A(1,-1)$

$\hat{\mathbf{u}} = < \cos(2\pi/3), \sin(2\pi/3) >$

$= < -1/2, \sqrt{3}/2 >$

$f_x(1,-1) = 2$

$f_y(1,-1) = 2$

$f_u(1,-1) = 2\frac{-1}{2} + 2\frac{\sqrt{3}}{2}$

$= -1 + \sqrt{3}$

≈ 0.73

Directional Derivatives, pg 4, practice.

$$f_u(a,b) = \mathbf{L}_z'(t) = f_x(a,b)\hat{u}_x + f_y(a,b)\hat{u}_y$$

Find the derivative of the given function at the given point, in the given direction.

1. $f(x,y) = \sin(xy) \quad P(\pi/2, 2) \quad \mathbf{u} = <\sqrt{3}/2,\ 1/2>$

$$g_u(t) = < a + \hat{u}_x t,\ b + \hat{u}_y t,\ f(a + \hat{u}_x t, b + \hat{u}_y t) >$$

$$z_u(t) = \sin((a + \hat{u}_x t)(b + \hat{u}_y t)) \qquad (a,b) = (\pi/2,\ 2)$$

$$z_u'(t) = \cos((a + \hat{u}_x t)(b + \hat{u}_y t)) \cdot [\hat{u}_x(b + \hat{u}_y t) + \hat{u}_y(a + \hat{u}_x t)]$$

$$\begin{aligned} z_u'(0) &= \cos(ab) \cdot [\hat{u}_x b + \hat{u}_y a] \\ &= \cos(\pi/2 \cdot 2) \cdot (\sqrt{3}/2 \cdot 2 + 1/2 \cdot \pi/2) \\ &= -(\sqrt{3} + \pi/4) \approx -2.52 \end{aligned}$$

OR

$$f_x(x,y) = y\cos(xy) \qquad f_x(\pi/2, 2) = -2$$

$$f_y(x,y) = x\cos(xy) \qquad f_y(\pi/2, 2) = -\pi/2$$

$$\begin{aligned} f_x(a,b)\hat{u}_x + f_y(a,b)\hat{u}_y &= -2\sqrt{3}/2 - \pi/2 \cdot 1/2 \\ &= -\sqrt{3} - \pi/4 \approx -2.52 \end{aligned}$$

2. $h(x,y) = xy^2 - yx^2 \quad P(2, -1) \quad \mathbf{u} = <\sqrt{2}/2,\ \sqrt{2}/2>$

$$g_u(t) = < a + \hat{u}_x t,\ b + \hat{u}_y t,\ f(a + \hat{u}_x t, b + \hat{u}_y t) >$$

$$z_u(t) = (a + \hat{u}_x t)(b + \hat{u}_y t)^2 - (b + \hat{u}_y t)(a + \hat{u}_x t)^2$$

$$\begin{aligned} z_u'(t) = &\ \hat{u}_x(b + \hat{u}_y t)^2 + 2\hat{u}_y(a + \hat{u}_x t)(b + \hat{u}_y t) \\ &-[\hat{u}_y(a + \hat{u}_x t)^2 + 2\hat{u}_x(b + \hat{u}_y t)(a + \hat{u}_x t)] \end{aligned}$$

$$\begin{aligned} z_u'(0) &= \hat{u}_x b^2 + 2\hat{u}_y ab - \hat{u}_y a^2 - 2\hat{u}_x ab \\ &= \frac{-3\sqrt{2}}{2} \approx -2.12 \end{aligned}$$

OR

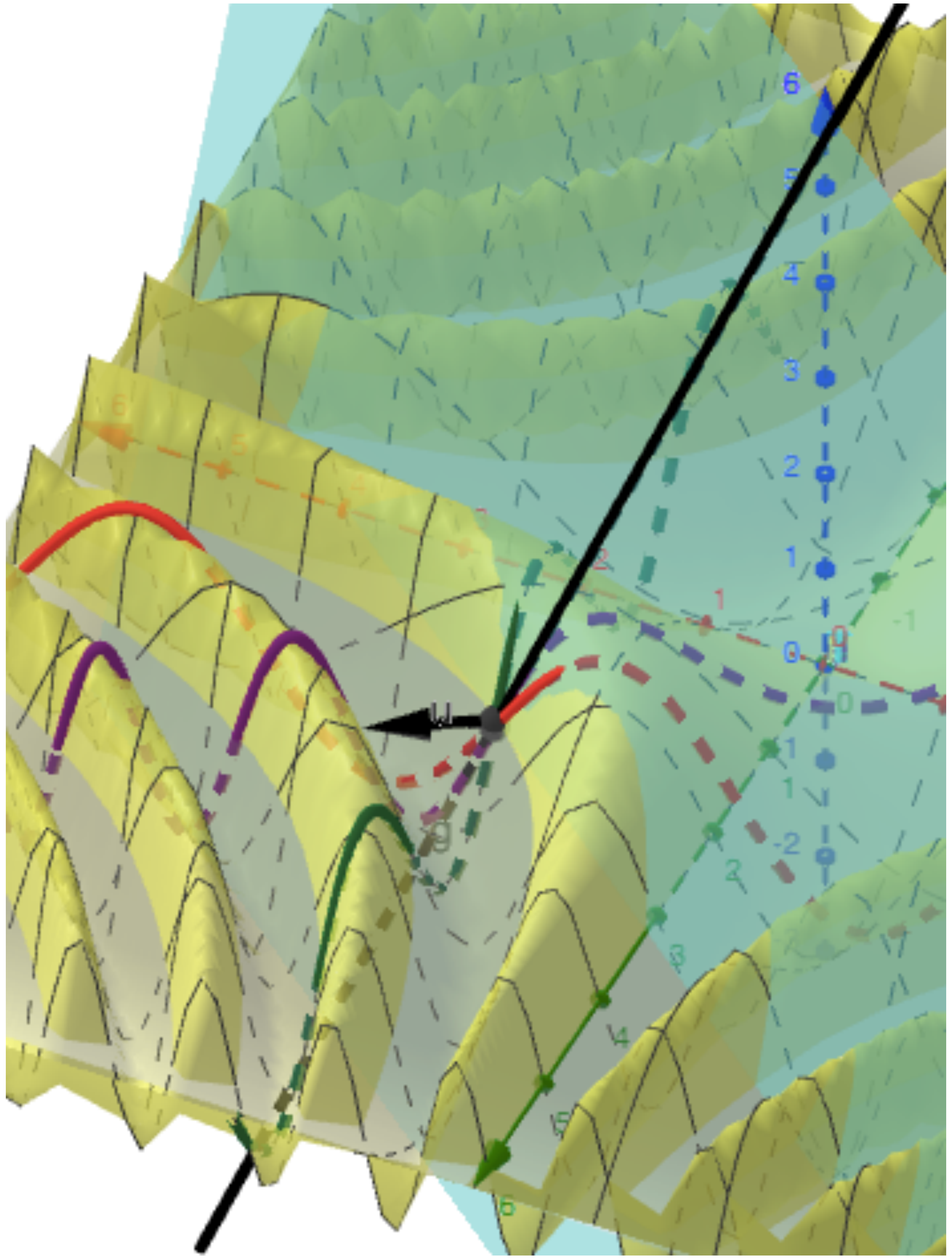
$$h_x(x,y) = y^2 - 2xy \qquad h_x(2, -1) = 5$$

$$h_y(x,y) = 2xy - x^2 \qquad h_y(2, -1) = -8$$

$$\begin{aligned} h_x(a,b)\hat{u}_x + h_y(a,b)\hat{u}_y &= 5\sqrt{2}/2 - 8\sqrt{2}/2 \\ &= -3\sqrt{2}/2 \approx -2.12 \end{aligned}$$

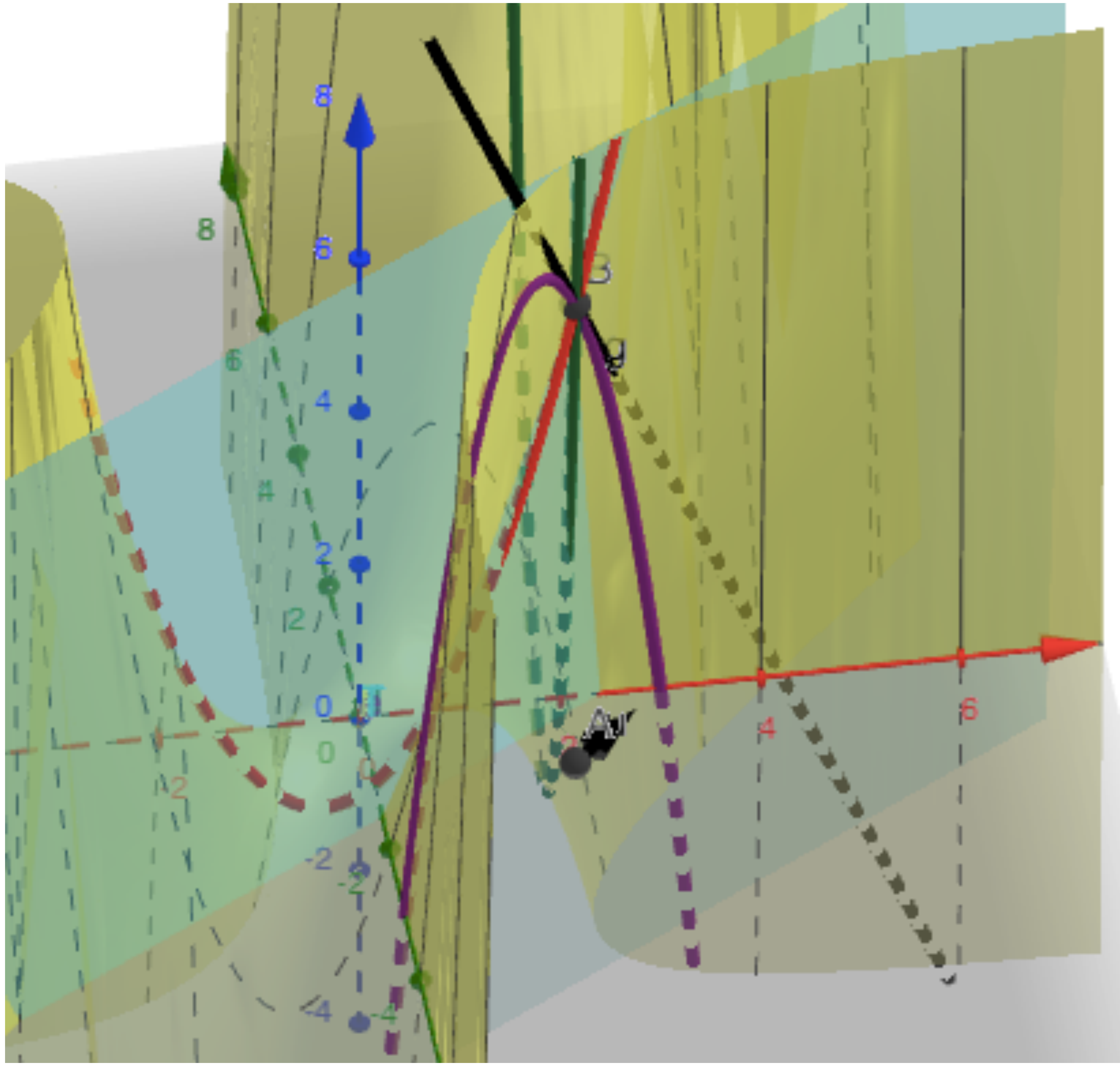
Directional Derivatives, pg 5, Previous Calculations, illustrated.

1. $f(x, y) = \sin(xy)$ $\quad P(\pi/2, 2)$ $\quad \hat{\mathbf{u}} = < \sqrt{3}/2, 1/2 >$



$f_u(\pi/2, 2) \approx -2.52$

2. $h(x, y) = xy^2 - yx^2$ $\quad P(2, -1)$ $\quad \hat{\mathbf{u}} = < \sqrt{2}/2, \sqrt{2}/2 >$



$h_u(2, -1) = -3\sqrt{2}/2 \approx -2.12$

Directional Derivatives, more examples.

1. $f(x,y) = \dfrac{x}{y} \quad P(2,1) \quad \mathbf{u} = <3, 4>$

Make $\mathbf{u}$ a unit vector!

$$\hat{\mathbf{u}} = \frac{\mathbf{u}}{|\mathbf{u}|} = \frac{<3,4>}{\sqrt{3^2+4^2}} = \frac{1}{5} <3, 4>$$

$$z(t) = f(2 + 3t/5,\ 1 + 4t/5)$$

$$= \frac{2+3t/5}{1+4t/5} = \frac{10+3t}{5+4t}$$

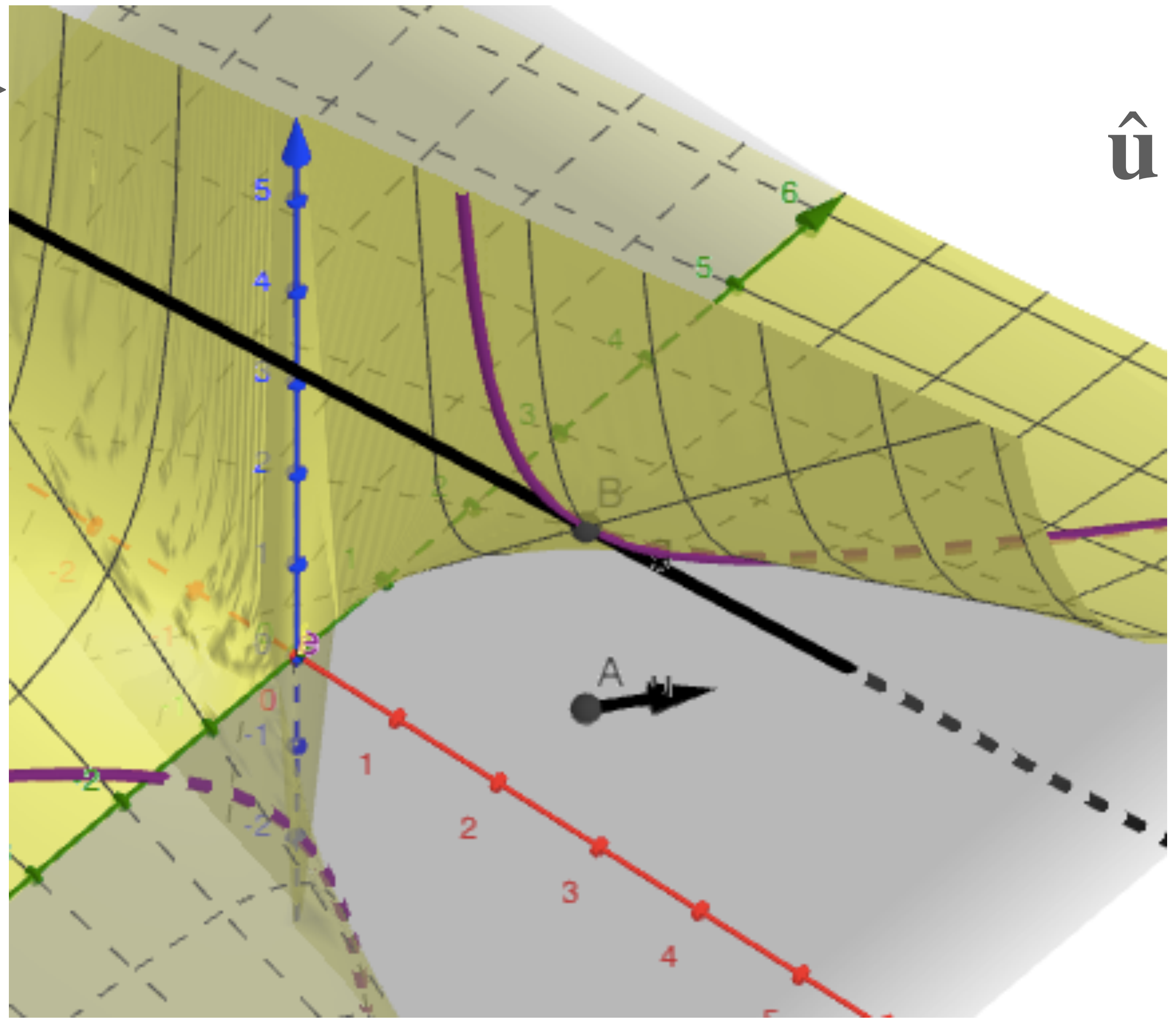
$$z'(t) = \frac{-25}{(5+4t)^2} \quad f_u(2,1) = z'(0) = -1$$

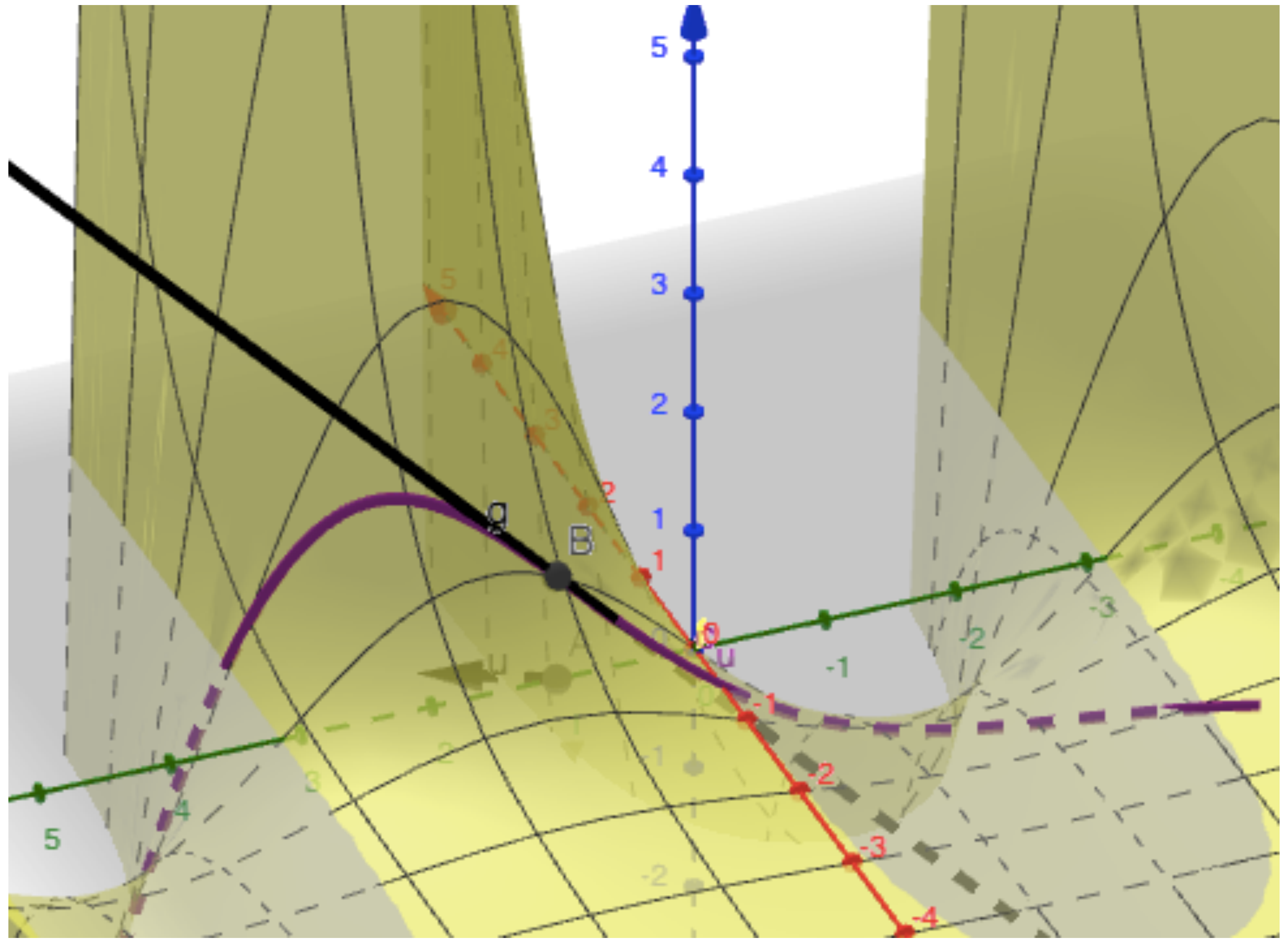
OR $f_u(a,b) = f_x(a,b)\hat{u}_x + f_y(a,b)\hat{u}_y$

$$f_u(2,1) = f_x(2,1)\cdot 3/5 + f_y(2,1)\cdot 4/5$$

$$f_x(x,y) = 1/y \qquad f_y(x,y) = -x/y^2$$

$$f_u(2,1) = 1\cdot 3/5 + -2\cdot 4/5 = -1$$





2. $f(x,y) = e^x \sin(y) \quad (0, \pi/3) \quad \mathbf{u} = <1,2>$

$$\hat{\mathbf{u}} = \frac{1}{\sqrt{5}} <1, 2> \quad z(t) = f\left(0 + \frac{t}{\sqrt{5}}, \frac{\pi}{3} + \frac{2t}{\sqrt{5}}\right)$$

$$= e^{t/\sqrt{5}} \sin(\pi/3 + 2t/\sqrt{5})$$

$$z'(t) = \frac{1}{\sqrt{5}} e^{t/\sqrt{5}} \sin(\pi/3 + 2t/\sqrt{5})$$

$$+ \frac{2}{\sqrt{5}} e^{t/\sqrt{5}} \cos(\pi/3 + 2t/\sqrt{5})$$

$$z'(0) = \frac{\sqrt{3}}{2\sqrt{5}} + \frac{1}{\sqrt{5}} \approx 0.83$$

Using $f_u(a,b) = f_x(a,b)\hat{u}_x + f_y(a,b)\hat{u}_y$

$$f_x(x,y) = e^x \sin(y) \qquad f_y(x,y) = e^x \cos(y)$$

$$f_u(0,\pi/3) = \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{5}} + \frac{1}{2} \cdot \frac{2}{\sqrt{5}} \approx 0.83$$

Directional Derivatives, more examples, pg 2.

Compute some directional derivatives.

3. $f(x,y) = x^2\ln(y)$ $\quad P(3,1)$ $\quad \mathbf{u} = < -5/13,\, 12/13 >$

$$z(t) = (3 + -5t/13)^2 \cdot \ln(1 + 12t/13)$$

$$z'(t) = -\frac{10}{13}(3 - 5t/13)\cdot\ln(1+12t/13)$$

$$+(3-5t/13)^2 \cdot \frac{12}{13+12t}$$

$$z'(0) = \frac{9\cdot 12}{13} = \frac{108}{13} \approx 8.31$$

OR

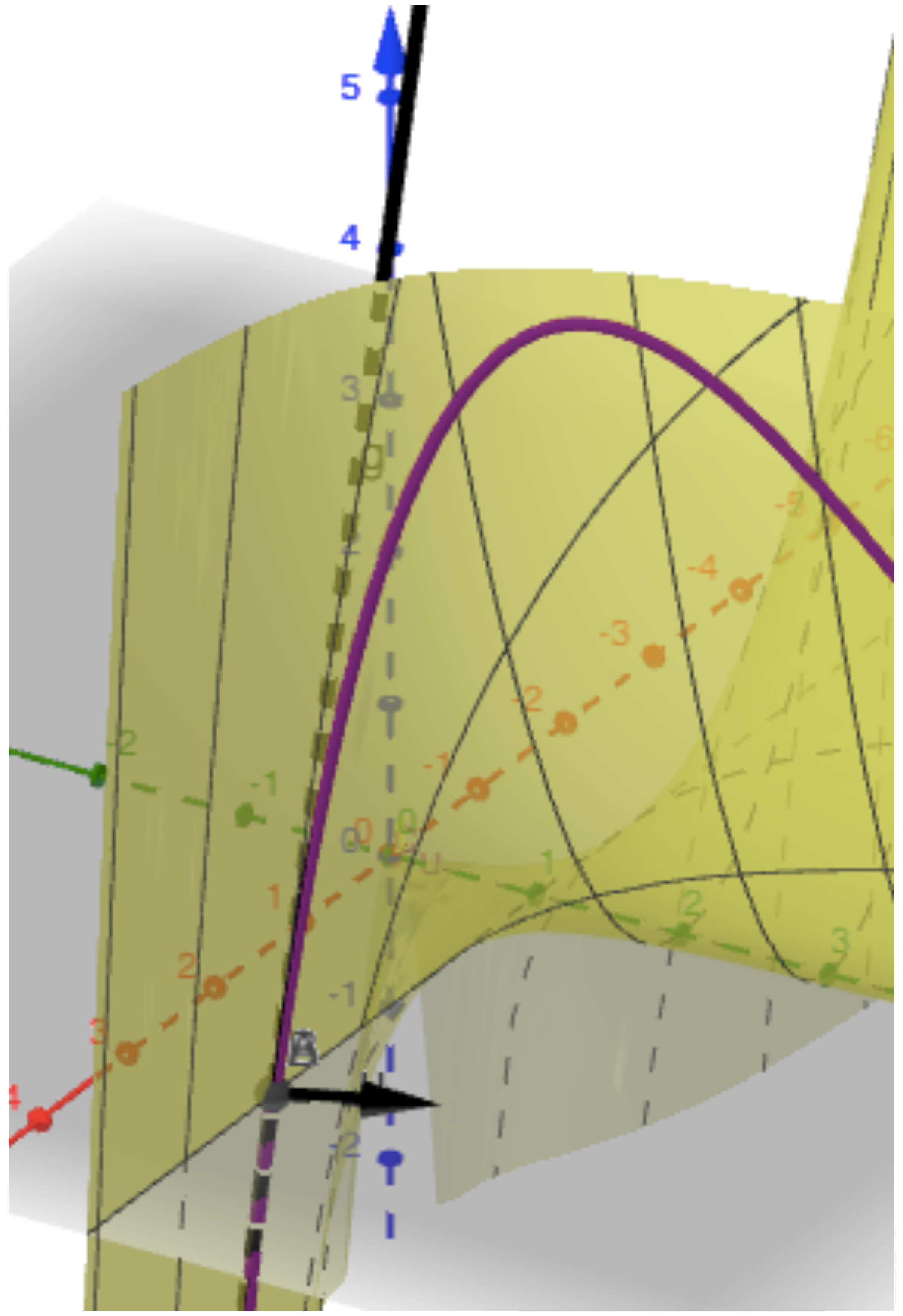
$$f_x(x,y) = 2x\ln(y)$$

$$f_y(x,y) = x^2/y$$

$$f_u(3,1) = 0(-5/13)$$

$$+9(12/13)$$

$$= \frac{108}{13} \approx 8.31$$



4. $f(x,y,z) = xy^2\tan^{-1}z$ $\quad P(2,1,1)$ $\quad \mathbf{u} = < 1,1,1 >$

$$w(t) = (2 + t/\sqrt{3})\cdot(1+t/\sqrt{3})^2\cdot\tan^{-1}(1+t/\sqrt{3}) \qquad \hat{\mathbf{u}} = \frac{1}{\sqrt{3}} < 1,1,1 >$$

$$w'(t) = \frac{1}{\sqrt{3}}\cdot(1+t/\sqrt{3})^2\cdot\tan^{-1}(1+t/\sqrt{3})$$

$$+(2+t/\sqrt{3})\cdot 2(1+t/\sqrt{3})\frac{1}{\sqrt{3}}\cdot\tan^{-1}(1+t/\sqrt{3})$$

$$+(2+t/\sqrt{3})\cdot(1+t/\sqrt{3})^2\cdot\frac{1/\sqrt{3}}{1+(1+t/\sqrt{3})^2}$$

$$w'(0) = \frac{5\pi}{4\sqrt{3}} + \frac{1}{\sqrt{3}} \approx 2.84$$

OR $\quad f_x = y^2\tan^{-1}(z) \quad f_y = 2xy\tan^{-1}(z) \quad f_z = xy^2\cdot\dfrac{1}{1+z^2}$

$$f_x(2,1,1) = \pi/4 \quad f_y(2,1,1) = 4\pi/4 \quad f_z(2,1,1) = 1$$

$$f_u(2,1,1) = \frac{\pi}{4}\cdot\frac{1}{\sqrt{3}} + \frac{4\pi}{4}\cdot\frac{1}{\sqrt{3}} + 1\cdot\frac{1}{\sqrt{3}} = \frac{5\pi}{4\sqrt{3}} + \frac{1}{\sqrt{3}} \approx 2.84$$

Directional Derivatives with some context.

Example. (S14.6 #34)

You're climbing a hill with height

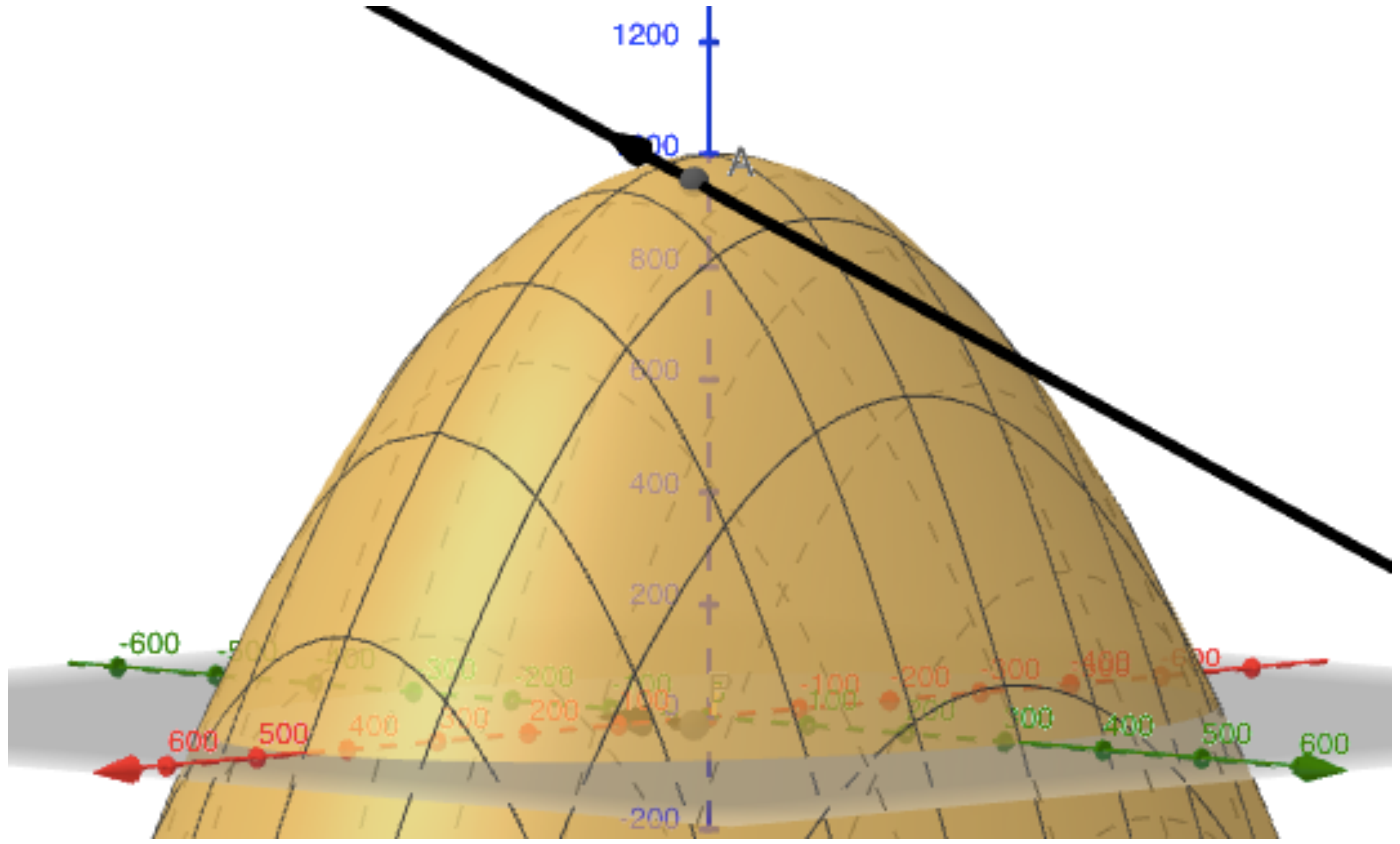
$h(x, y) = 1000 - 0.005x^2 - 0.01y^2$

$x, y,$ and h are all in meters.

You're at the point $P(60,40,966)$

The positive x-axis points east, the positive y-axis points north.

a) If you walk south, will you ascend the hill or descend? How quickly?

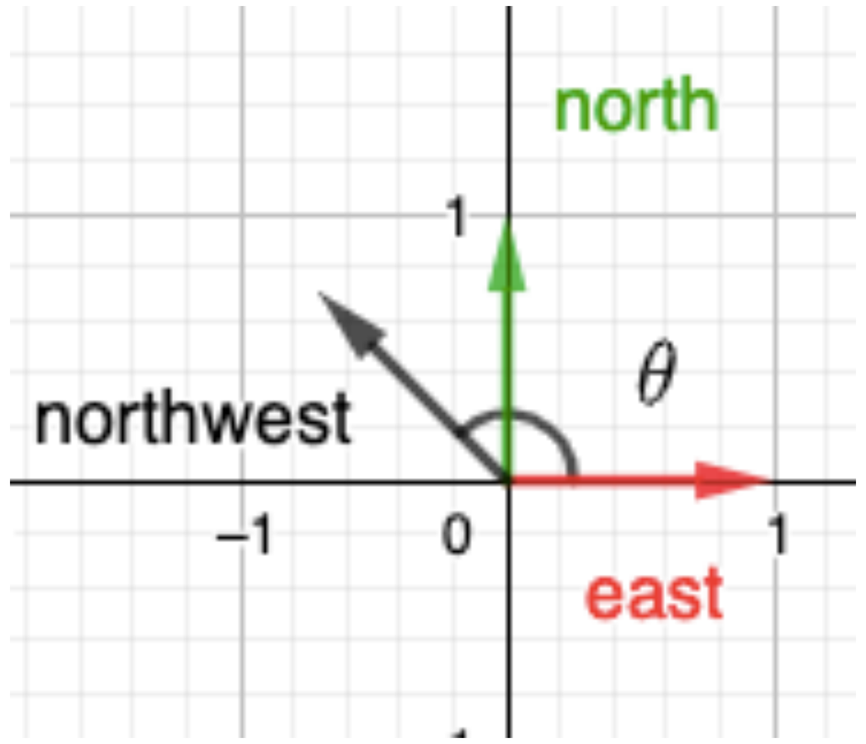


a) i.e. find $h_u(a, b)$, $(a, b) = (60,40)$, $\hat{\mathbf{u}} = < 0, -1 >$

$h_x(x, y) = -0.01x \qquad h_y(x, y) = -0.02y$

$h_x(a, b)\hat{u}_x + h_y(a, b)\hat{u}_y = (-0.6)(0) + (-0.8)(-1) = 0.8$

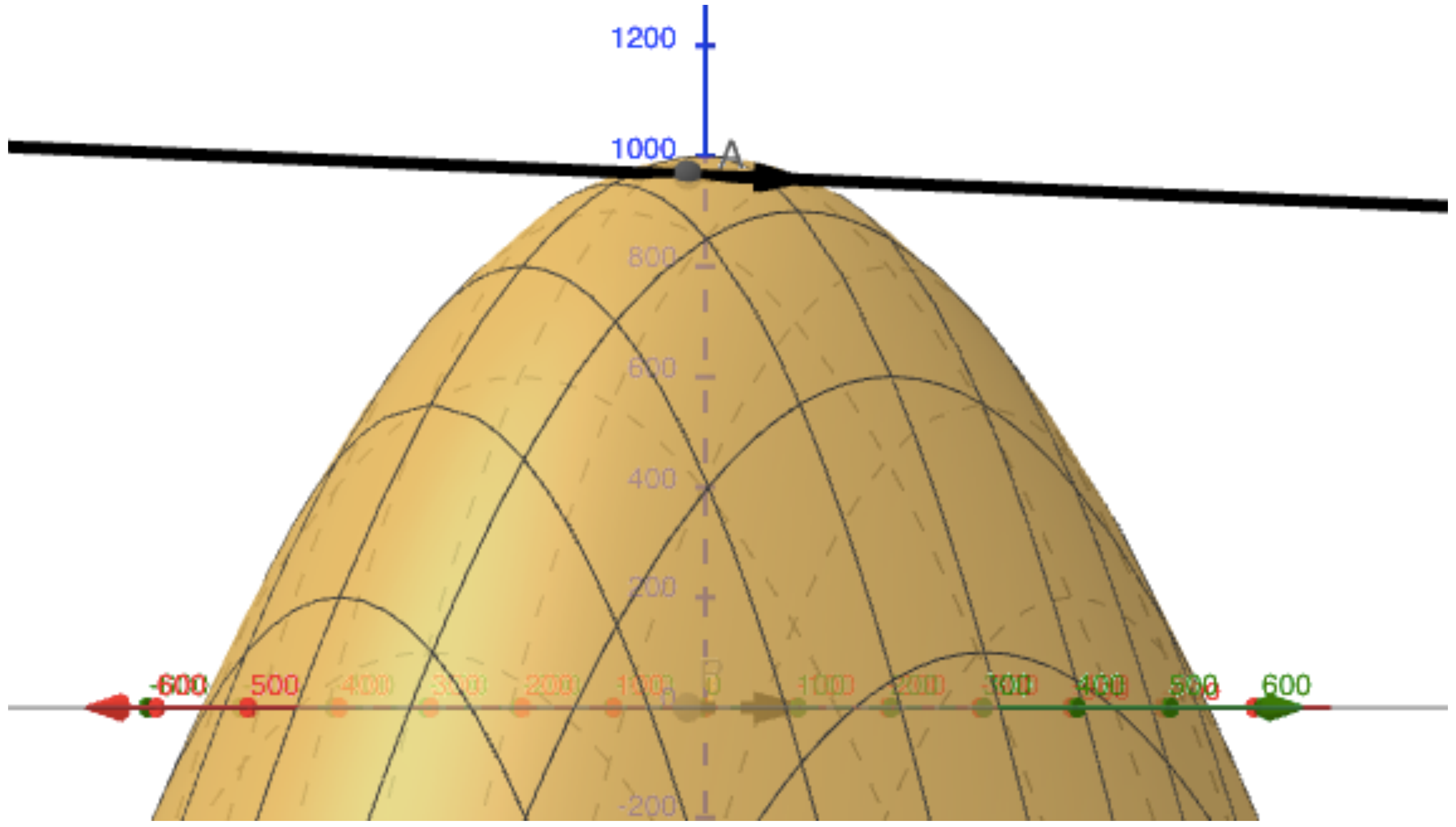
We will ascend the hill at a rate of 0.8 meters up per meter south.



b) If you walk northwest, will you ascend the hill or descend? How quickly?

i.e. find $h_u(a, b)$, $(a, b) = (60,40)$, $\hat{\mathbf{u}} = < \cos(3\pi/4), \sin(3\pi/4) >$

$= < -1/\sqrt{2}, 1/\sqrt{2} >$



Link: MountainAscent

$h_x(a, b)\hat{u}_x + h_y(a, b)\hat{u}_y$

$= (-0.6)(-1/\sqrt{2}) + (-0.8)(1/\sqrt{2})$

$= -0.2/\sqrt{2} \approx -0.14$

We will descend the hill at a rate of 0.14 meters down per meters northwest.

Directional Derivatives with context, The gradient, pg 1.

$h(x, y) = 1000 - 0.005x^2 - 0.01y^2$

c) Which direction has the steepest ascent??

Note that the directional derivative

$h_x(a, b)\hat{u}_x + h_y(a, b)\hat{u}_y$

can be expressed as a dot product.

$< h_x(a, b), h_y(a, b) > \cdot < \hat{u}_x, \hat{u}_y >$

Which vector $\hat{\mathbf{u}}$ makes this dot product the largest?

$$| < h_x(a, b), h_y(a, b) > \cdot < \hat{u}_x, \hat{u}_y > |$$

$$= | < h_x(a, b), h_y(a, b) > | \cdot | < \hat{u}_x, \hat{u}_y > | \cdot |\cos(\alpha)|$$

$$= | < h_x(a, b), h_y(a, b) > | \cdot |\cos(\alpha)|$$

Where α is the angle between the two vectors in the dot product.

For a given function (in this case h), the direction which results in the largest derivative is the one for which $\cos(\alpha) = 1$, i.e. $\theta = 0$.

thus $\hat{\mathbf{u}}$ should be the unit vector of $< h_x(a, b), h_y(a, b) >$

In our example,

$h(x, y) = 1000 - 0.005x^2 - 0.01y^2, \quad (a, b) = (60,40)$

$< h_x(x, y), h_y(x, y) > = < -0.01x, -0.02y >$

$< h_x(60,40), h_y(60,40) > = < -0.6, -0.8 >$

$$\hat{\mathbf{u}} = \frac{< -0.6, -0.8 >}{\sqrt{0.36 + 0.64}} = < -0.6, -0.8 >$$

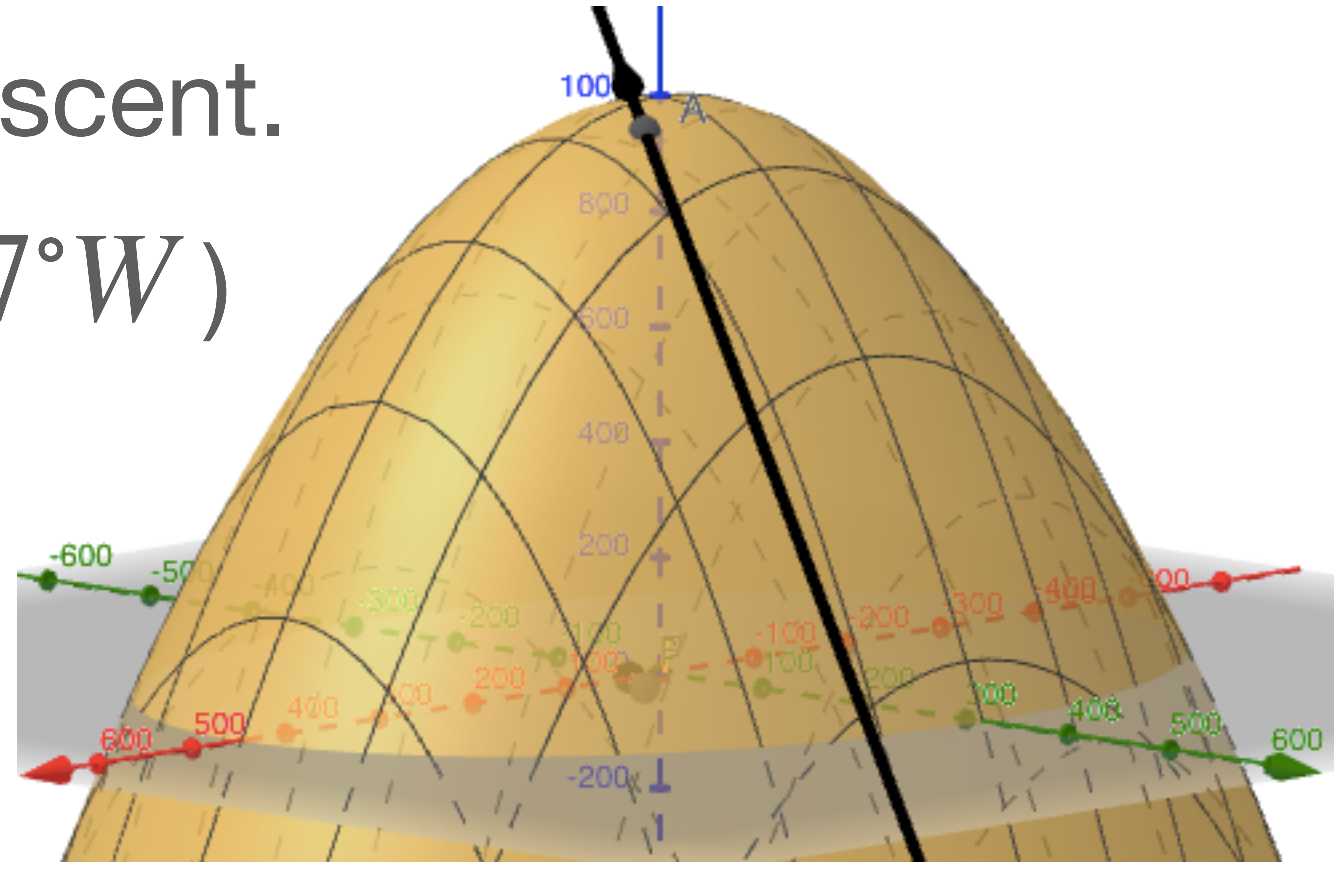
$< h_x, h_y >$ is already a unit vector at (60,40).

This is the direction of steepest ascent.

(Some describe this as $\approx S36.87°W$)

The rate of steepest ascent is

$| < h_x(60,40), h_y(60,40) > | = 1$



The gradient, pg 2.

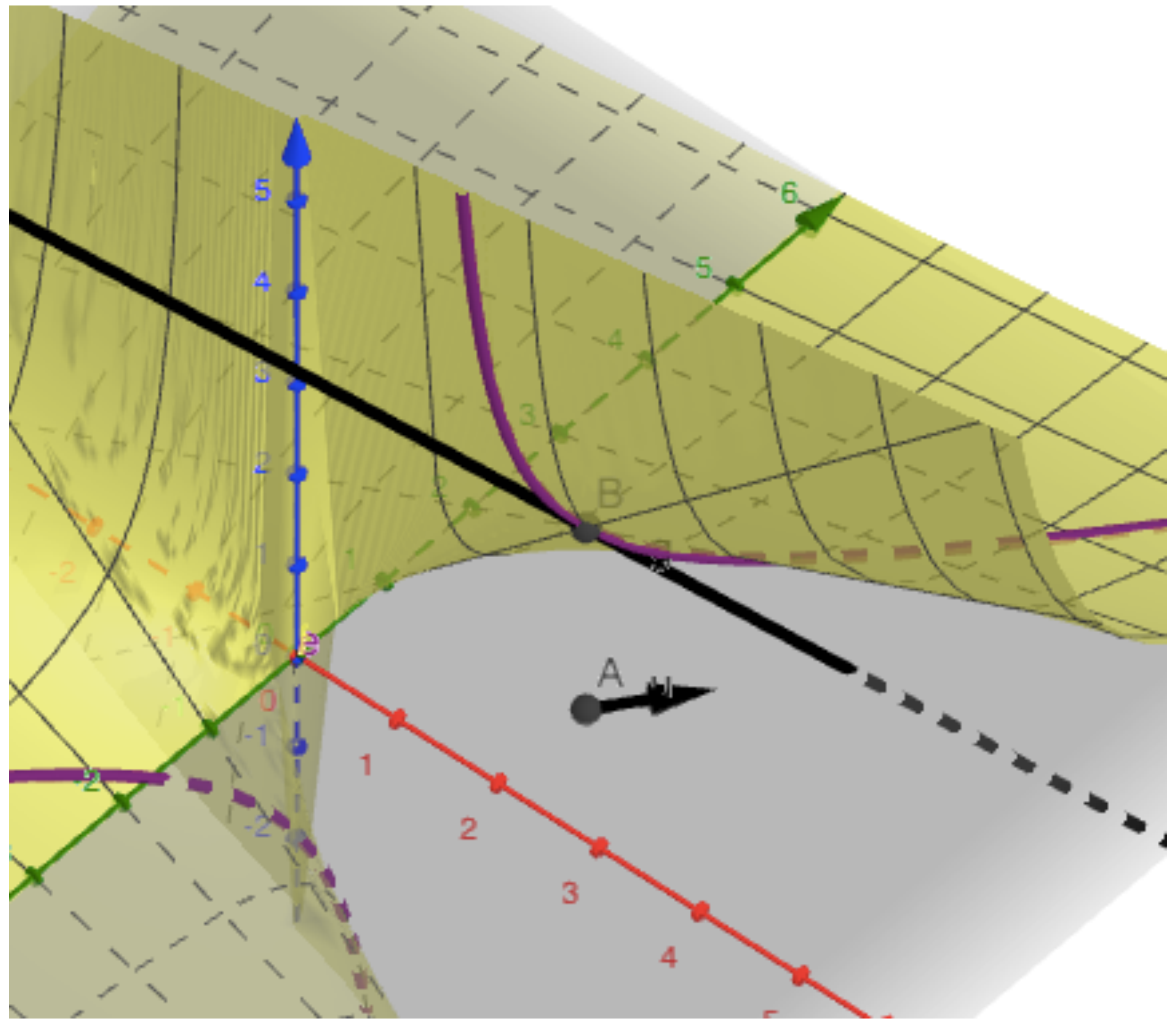
In general, the *gradient* of a function $f(x, y)$ at a point (a, b) in the domain is the vector

$$\nabla f(a, b) = \langle f_x(a, b), f_y(a, b)\rangle$$

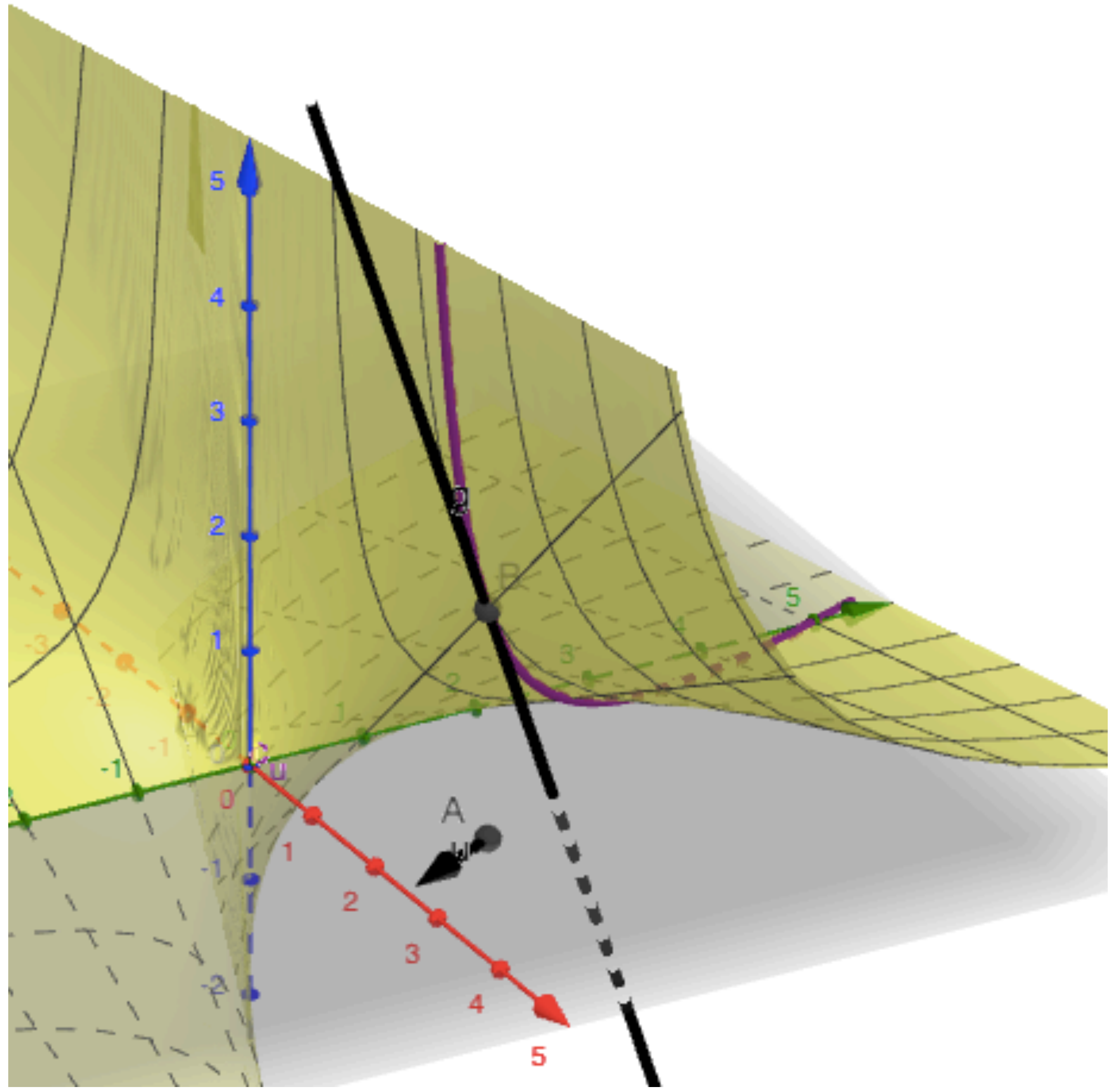
The gradient's direction is the direction of the largest directional derivative; its magnitude is the largest rate of change.

Example. $f(x, y) = \dfrac{x}{y}$ $\quad P(2,1) \quad \mathbf{u} = < 3,4 >$

We computed $f_u(a, b) = -1$



$\hat{\mathbf{u}} = < 3/5, 4/5 >$



$\hat{\mathbf{u}} = < 1/\sqrt{5}, -2/\sqrt{5} >$

At $P(2,1)$, which direction has the steepest climb?

$$\nabla f = < 1/y, -x/y^2 >$$

$$\nabla f(2,1) = < 1, -2 >$$

The direction of steepest ascent is given by $\hat{\mathbf{u}} = \dfrac{1}{\sqrt{5}} < 1, -2 >$

The magnitude of steepest ascent is $|\nabla f(2,1)| = |< 1, -2 >| = \sqrt{5}$

Example. (many variables, same idea).

$$f(x, y, z) = xy^2 \tan^{-1} z \quad P(2,1,1)$$

$$\nabla f = < y^2 \tan^{-1}(z), 2xy \tan^{-1}(z), \frac{xy^2}{1 + z^2} >$$

$$\nabla f(2,1,1) = < \pi/4, \pi, 1 >$$

Steepest ascent: $|\nabla f(2,1,1)| = \dfrac{\sqrt{17\pi^2 + 16}}{4} \approx 3.39$

Direction of steepest ascent: $\hat{\mathbf{u}} = \dfrac{4 < \pi/4, \pi, 1 >}{\sqrt{17\pi^2 + 16}}$

The gradient, pg 3, practice.

S14.6 #30 Near a buoy (at the origin), the depth of a lake is $d(x, y) = 200 + 0.02x^2 - 0.001y^3$

A boat starts at the point (80m, 60m).

a) How does the depth of the lake change when the boat moves towards the buoy?

b) The boat wants deep water for good fishing. Which direction should the boat go in?

S14.6 #31 The temperature in a solid metal ball is inversely proportional to the distance to the center of the ball (the origin). The temperature at the point (1m, 2m, 2m) is 120°

a) how quickly does temperature change as one moves from (1, 2, 2) towards (2, 1, 3)?

b) what is the largest rate of temperature change from the point (1, 2, 2) ?

$$a)\, \nabla d = < 0.04x, -0.003y^2 >$$

$$\nabla d(80{,}60) = < 3.2, -10.8 >$$

$$\mathbf{u} = < -80, -60 >$$

$$\hat{\mathbf{u}} = < -0.8, -0.6 >$$

$$d_u(80{,}60) = 3.2(-0.8) - 10.8(-0.6) = 3.92$$

$$b)\, |\nabla d(80{,}60)| = \sqrt{3.2^2 + 10.8^2} = \sqrt{126.88}$$

$$\hat{\mathbf{u}} = \frac{1}{\sqrt{126.88}} < 3.2, -10.8 >$$

$$a) T(x, y, z) = \frac{k}{\sqrt{x^2 + y^2 + z^2}} \qquad 120 = T(1{,}2{,}2) = \frac{k}{3}, \quad k = 360$$

$$T_x = -1/2(360)(x^2 + y^2 + z^2)^{-3/2} \cdot 2x = -360x(x^2 + y^2 + z^2)^{-3/2}$$
$$T_y = -1/2(360)(x^2 + y^2 + z^2)^{-3/2} \cdot 2y = -360y(x^2 + y^2 + z^2)^{-3/2}$$
$$T_z = -1/2(360)(x^2 + y^2 + z^2)^{-3/2} \cdot 2z = -360z(x^2 + y^2 + z^2)^{-3/2}$$

$$\nabla T = \frac{-360 < x, y, z >}{(x^2 + y^2 + z^2)^{3/2}} \qquad \nabla T(1{,}2{,}2) = -\frac{40}{3} < 1{,}2{,}2 >$$

$$\mathbf{u} = < 1, -1{,}1 >, \quad \hat{\mathbf{u}} = \frac{1}{\sqrt{3}} < 1, -1{,}1 >$$

$$T_u(1{,}2{,}2) = \nabla T(1{,}2{,}2) \cdot \hat{\mathbf{u}} = -40/(3\sqrt{3}) \qquad b)\, |\nabla T(1{,}2{,}2)| = 40$$

Higher order derivatives.

Example. $f(x,y) = \sin(xy^2)$

$f_x(x,y) = y^2\cos(xy^2)$

$f_y(x,y) = 2xy\cos(xy^2)$

$f_{xx}(x,y) = \dfrac{\partial}{\partial x}(f_x) = -\,y^4\sin(xy^2)$

$f_{yy}(x,y) = \dfrac{\partial}{\partial y}(f_y) = 2x\cos(xy^2) - 4x^2y^2\sin(xy^2)$

$f_{xy}(x,y) = \dfrac{\partial}{\partial y}(f_x) = 2y\cos(xy^2) - 2xy^3\sin(xy^2)$

$f_{yx}(x,y) = \dfrac{\partial}{\partial x}(f_y) = 2y\cos(xy^2) - 2xy^3\sin(xy^2)$

$f_{xy} = f_{yx}$

Coincidence?!

Nope!

Example. $f(x,y) = x^4y - 2x^3y^2$

$f_x = 4x^3y - 6x^2y^2$

$f_y = x^4 - 4x^3y$

$f_{xx} = 12x^2y - 12xy^2$

$f_{yy} = -4x^3$

$f_{xy} = 4x^3 - 12x^2y$

$f_{yx} = 4x^3 - 12x^2y$

Note: The mixed partials f_{xy} and f_{yx} will always be equal whenever they are continuous. This is *Clairaut's Theorem*.

Consequently, the mixed partials of a nice 'smooth' (i.e. infinitely differentiable) function, f_I and f_J, will be equal.

Here I and J are lists of x's and y's with an equal number of each. e.g. $f_{xxyyx} = f_{xyxyx}$

$f_{xxyxyyy} = f_{yyxxxyy}$

Proof of Clairaut's Theorem: pg A48, Stewart's textbook.

Taylor Series, pg 1.

Remember single-variable Taylor Series Polynomial approximations.

The tangent line to $f(x)$ at $x = a$ is

$$T_1(x) = f(a) + f'(a)(x - a)$$

The tangent parabola to $f(x)$ at $x = a$ is

$$T_2(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2$$

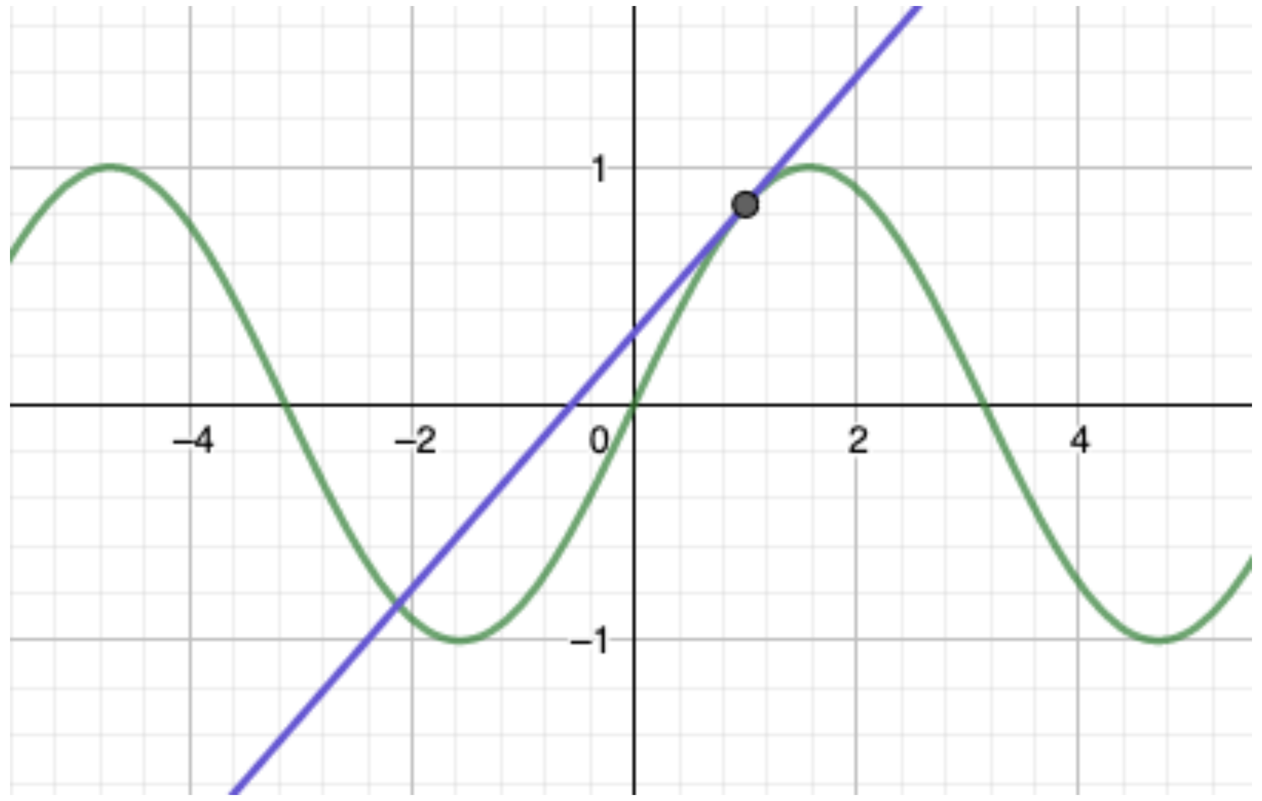
The tangent cubic to $f(x)$ at $x = a$ is

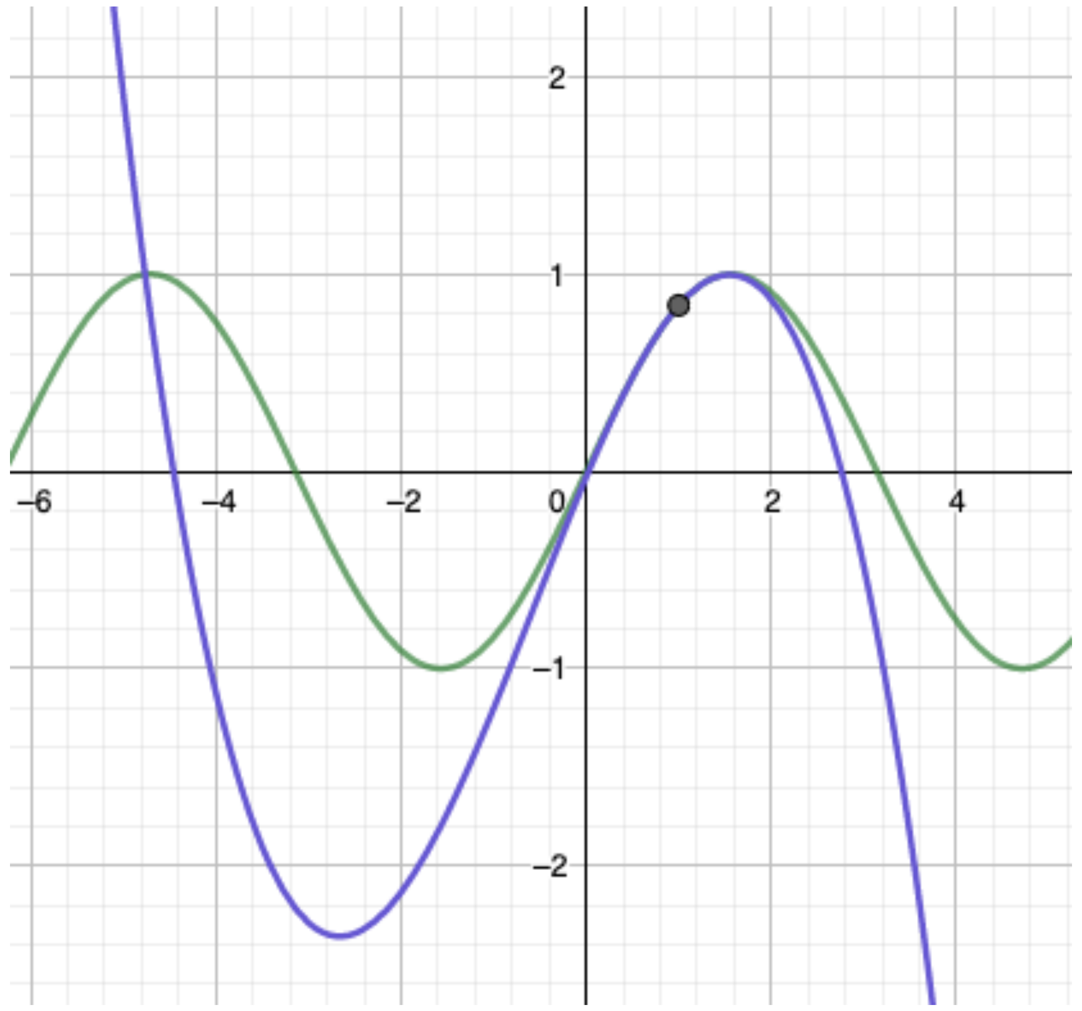
$$T_3(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3$$

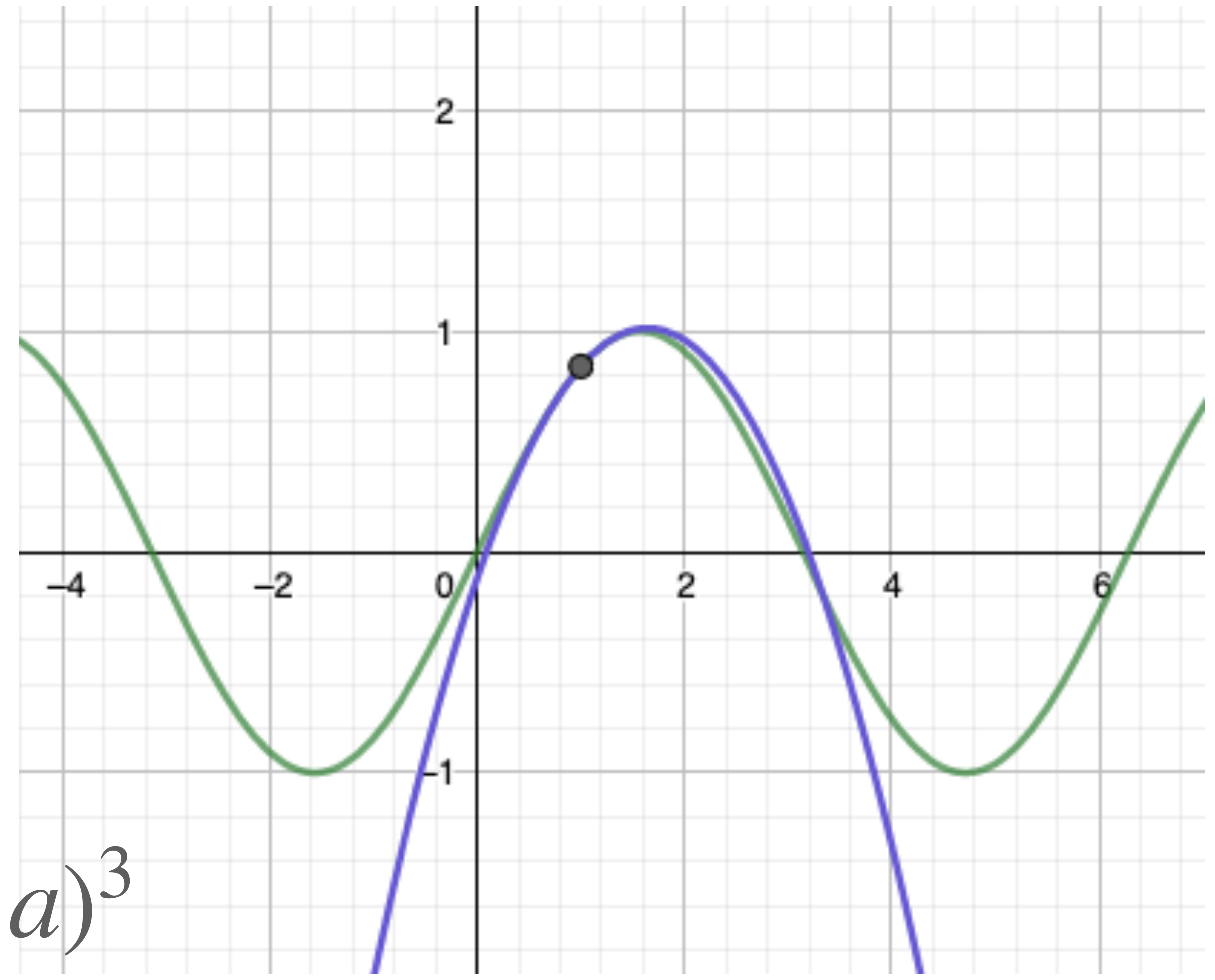
In general, if f has enough derivatives, then you can find a tangent polynomial function of degree n.

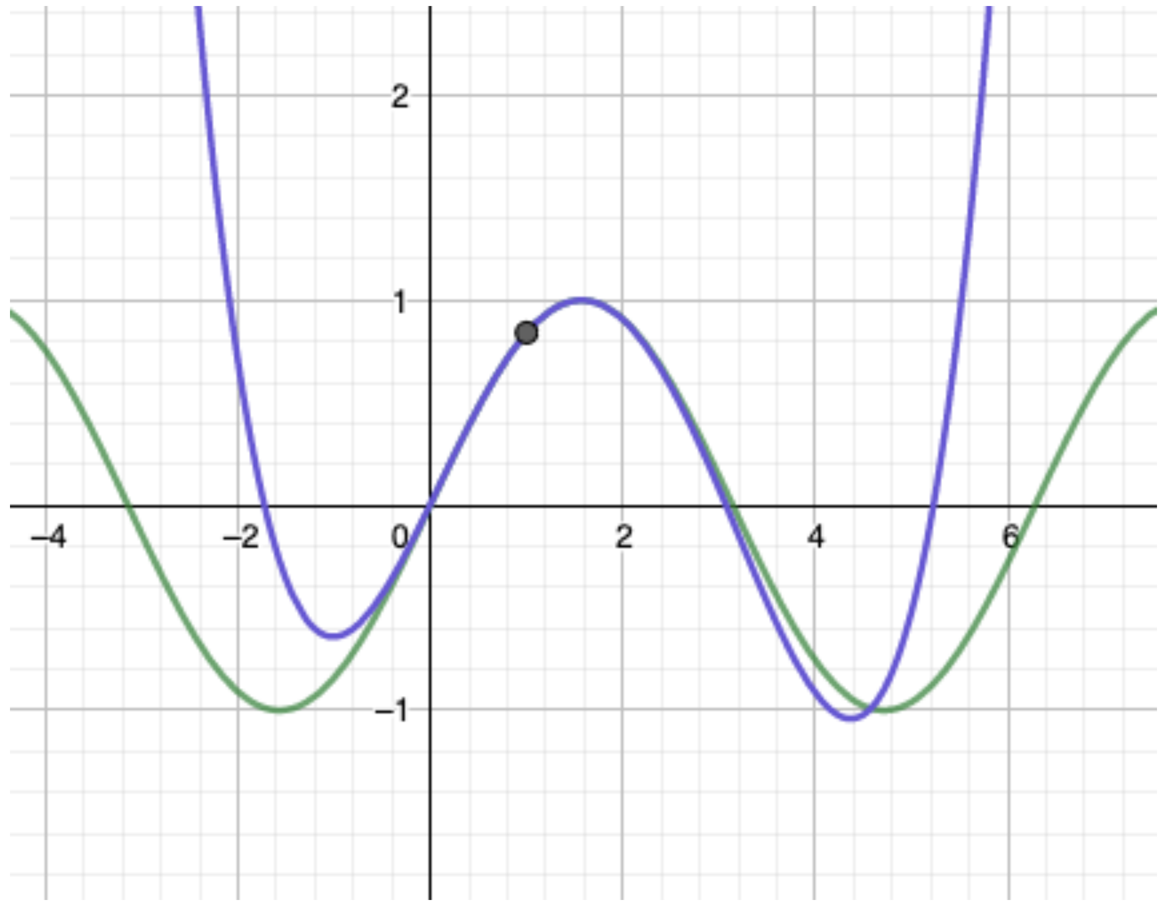
$$T_n(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \ldots \frac{f^{(n)}(a)}{n!}(x - a)^n$$

$$= \sum_{j=0}^{j=n} \frac{f^{(j)}(a)}{j!}(x - a)^j$$









Link: TaylorSeries

The nth degree Taylor polynomial has the exact same first n derivatives as f does at x=a:

$$\begin{cases} T_n(a) = f(a) \\ T_n'(a) = f'(a) \\ T_n''(a) = f''(a) \\ \vdots \\ T_n^{(n)}(a) = f^{(n)}(a) \end{cases}$$

Taylor Series, pg 2.

We can use the same idea with multivariable functions:

build some (multivariable) polynomial functions that have the same partial derivatives as some given differentiable function, $f(x, y)$.

The nth degree Taylor polynomial has all partial derivatives up to the nth degree equal to those of the given function at the given point (a,b).

$T_0(x, y) = f(a, b)$

(0th degree: constant function)

$T_1(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$

(1st degree: linear function (the tangent plane))

$T_2(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$

$+\frac{1}{2!}(f_{xx}(a, b)(x - a)^2 + f_{xy}(a, b)(x - a)(y - b) + f_{yx}(a, b)(y - b)(x - a) + f_{yy}(a, b)(y - b)^2)$

$= f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) + \frac{1}{2!}(f_{xx}(a, b)(x - a)^2 + 2f_{xy}(a, b)(x - a)(y - b) + f_{yy}(a, b)(y - b)^2)$

e.g. $\frac{\partial}{\partial x}T_2(a, b) = f_x(a, b)$ $\quad \frac{\partial}{\partial y}T_2(a, b) = f_y(a, b)$

$\frac{\partial^2}{\partial x^2}T_2(a, b) = f_{xx}(a, b)$ $\quad \frac{\partial^2}{\partial y^2}T_2(a, b) = f_{yy}(a, b)$

$\frac{\partial^2}{\partial x \partial y}T_2(a, b) = f_{xy}(a, b)$ $\quad \frac{\partial^2}{\partial y \partial x}T_2(a, b) = f_{yx}(a, b)$

(2nd degree: quadric surface (the tangent quadric!))

$T_3(x, y) = T_2(x, y) + \frac{1}{3!}(f_{xxx}(a, b)(x - a)^3 + 3f_{xxy}(a, b)(x - a)^2(y - b) + 3f_{xyy}(a, b)(x - a)(y - b)^2 + f_{yyy}(a, b)(y - b)^3)$

(3rd degree!)

Taylor Series, pg 3.

Example.

$f(x, y) = \sin(xy), \quad (a, b) = (\pi/2, 2)$

$f_x(x, y) = y\cos(xy)$

$f_y(x, y) = x\cos(xy)$

$f_x(a, b) = -2$

$f_y(a, b) = -\pi/2$

$T_1(x, y) = 0 - 2(x - \pi/2) - \pi/2(y - 2)$

$f_{xx}(x, y) = -y^2\sin(xy)$

$f_{xy}(x, y) = \cos(xy) - xy\sin(xy)$

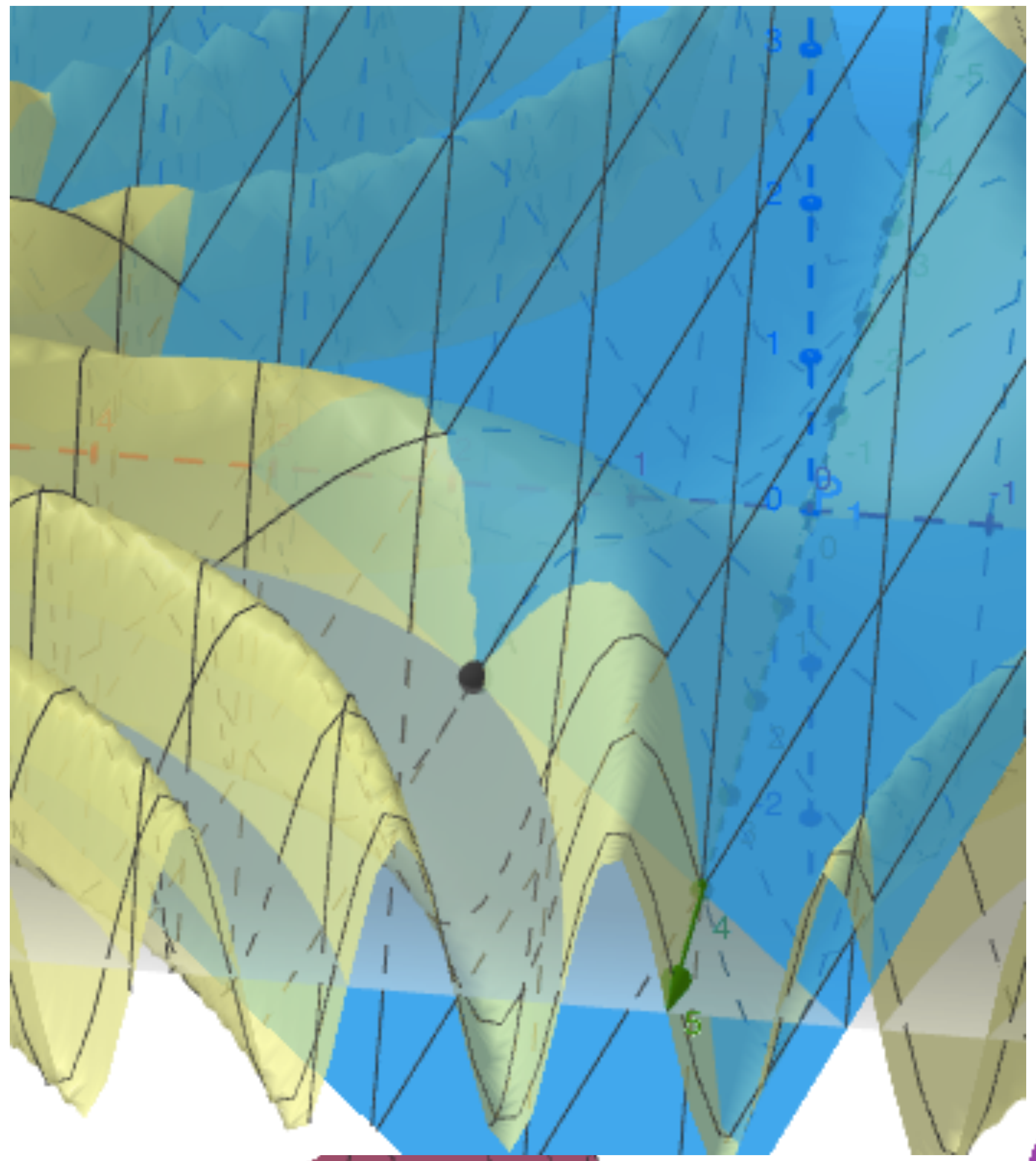
$f_{yy}(x, y) = -x^2\sin(xy)$

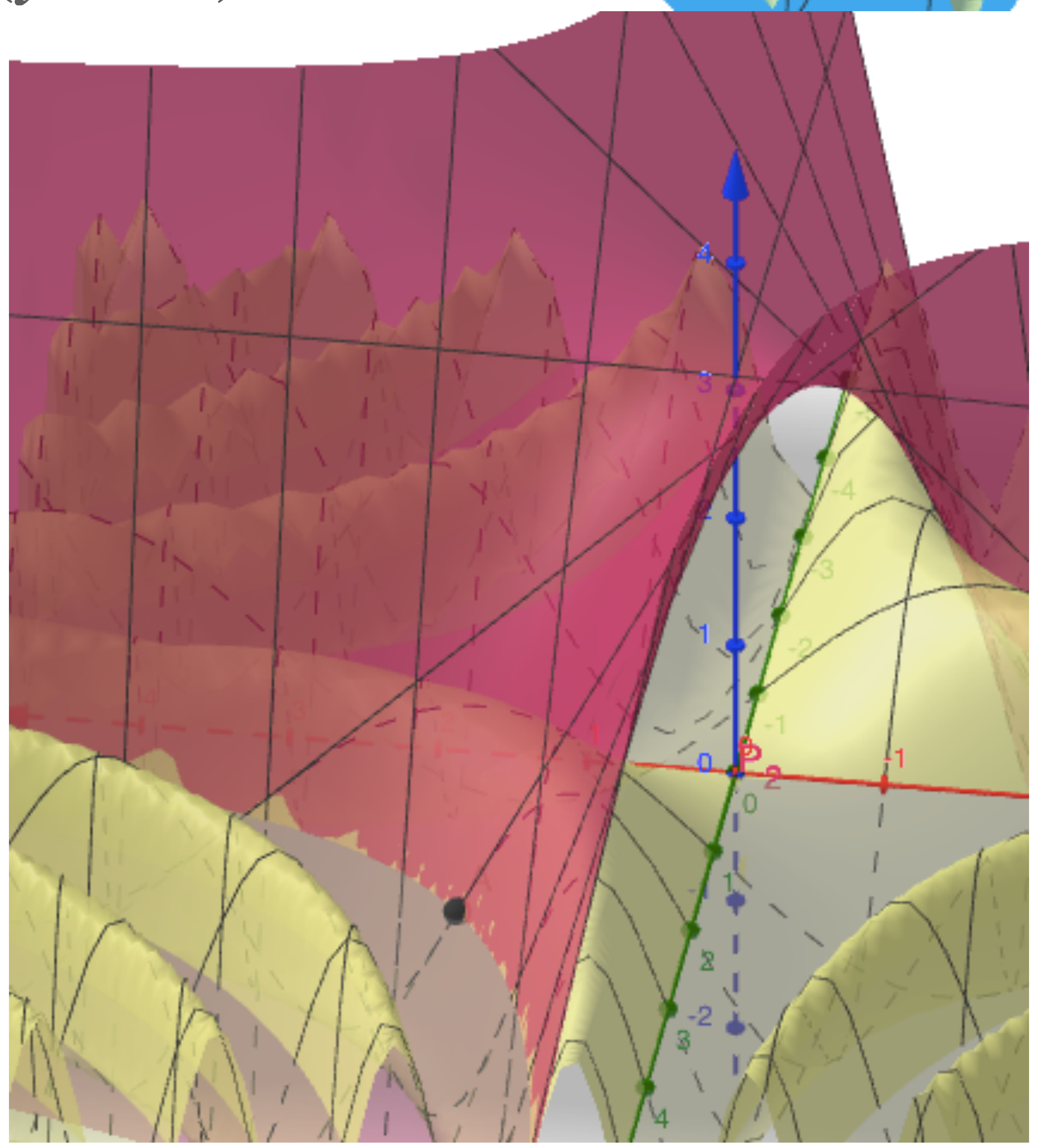
$f_{xx}(a, b) = 0$

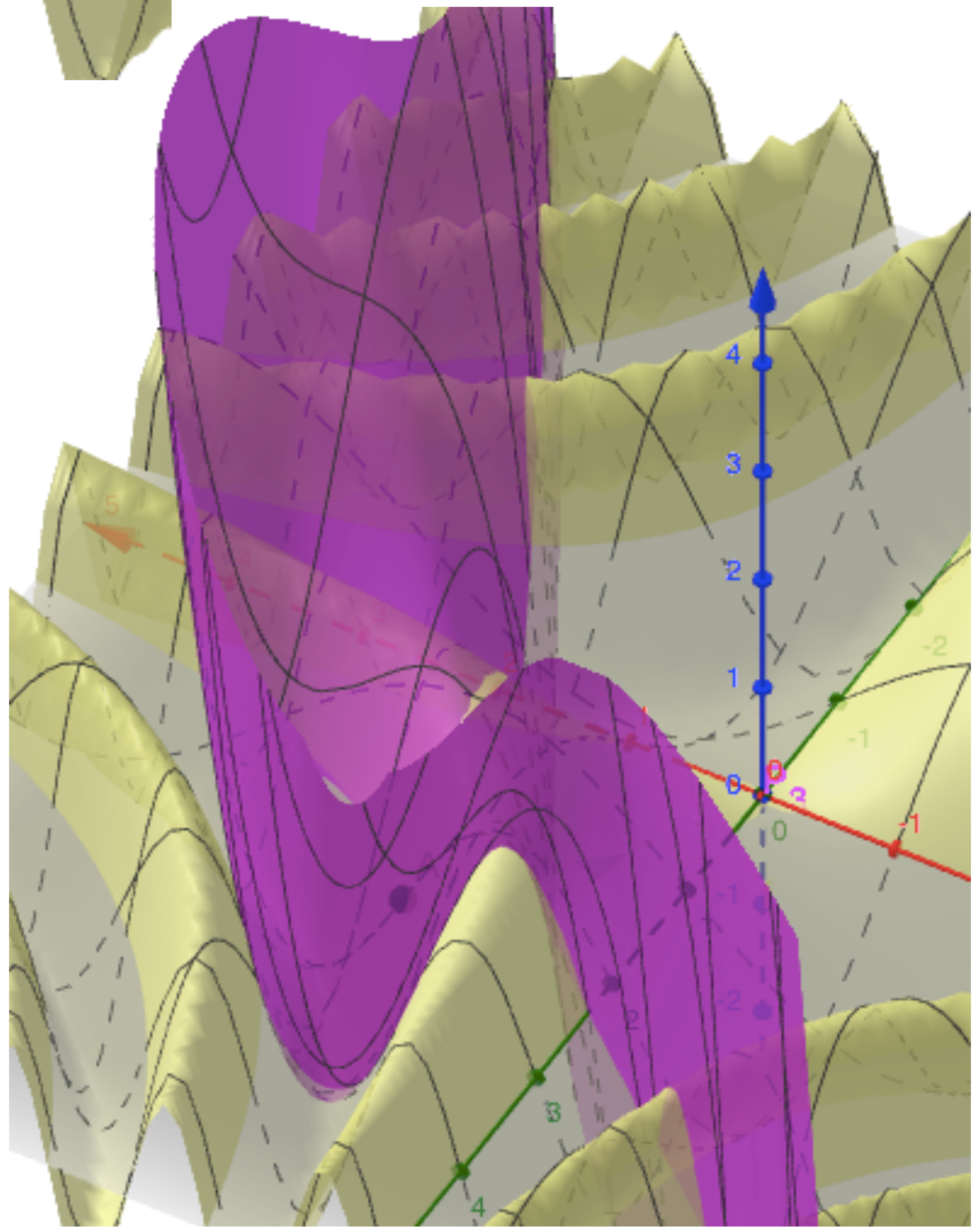
$f_{xy}(a, b) = -1$

$f_{yy}(a, b) = 0$

$T_2(x, y) = T_1(x, y) + \frac{1}{2!}(0(x - \pi/2)^2 - 2(x - \pi/2)(y - 2) + 0(y - 2)^2)$







$f_{xxx}(x, y) = -y^3\cos(xy)$

$f_{xxy}(x, y) = -2y\sin(xy) - xy^2\cos(xy)$

$f_{xyy}(x, y) = -2x\sin(xy) - x^2y\cos(xy)$

$f_{yyy}(x, y) = -x^3\cos(xy)$

$f_{xxx}(a, b) = 8$

$f_{xxy}(a, b) = 2\pi$

$f_{xyy}(a, b) = \pi^2/2$

$f_{yyy}(a, b) = \pi^3/8$

$$T_3(x, y) = T_2(x, y) + \frac{1}{3!}\Big(+8(x - \pi/2)^3 +6\pi(x - \pi/2)^2(y - 2) +\frac{3\pi^2}{2}(x - \pi/2)(y - 2)^2 +\pi^3/8(y - 2)^3 \Big)$$

Taylor Series, pg 4.

Link: MultiD TaylorSeries

$f(x, y) = \cos(x) - \cos(y), \quad (a, b) = (0,0)$

Find the 4th degree Taylor Polynomial.

$f_x(x, y) = -\sin(x)$

$f_y(x, y) = \sin(y)$

$f_{xx}(x, y) = -\cos(x)$

$f_{xy}(x, y) = 0$

$f_{yy}(x, y) = \cos(y)$

$f_{xxx}(x, y) = \sin(x)$

$f_{yyy}(x, y) = -\sin(y)$

$f_{xxxx}(x, y) = \cos(x)$

$f_{yyyy}(x, y) = -\cos(y)$

(All mixed partials are 0.)

	at (a,b) = (0,0)
f_x	0
f_y	0
f_xx	-1
f_yy	1
f_xxx	0
f_yyy	0
f_xxxx	1
f_yyyy	-1

$T_1(x, y) = 0$

$T_2(x, y) = \frac{1}{2!}(-x^2 + y^2)$

$T_3(x, y) = T_2(x, y)$

$T_4(x, y) = \frac{1}{2!}(-x^2 + y^2) + \frac{1}{4!}(x^4 - y^4)$