M110C Week 3

Goals:

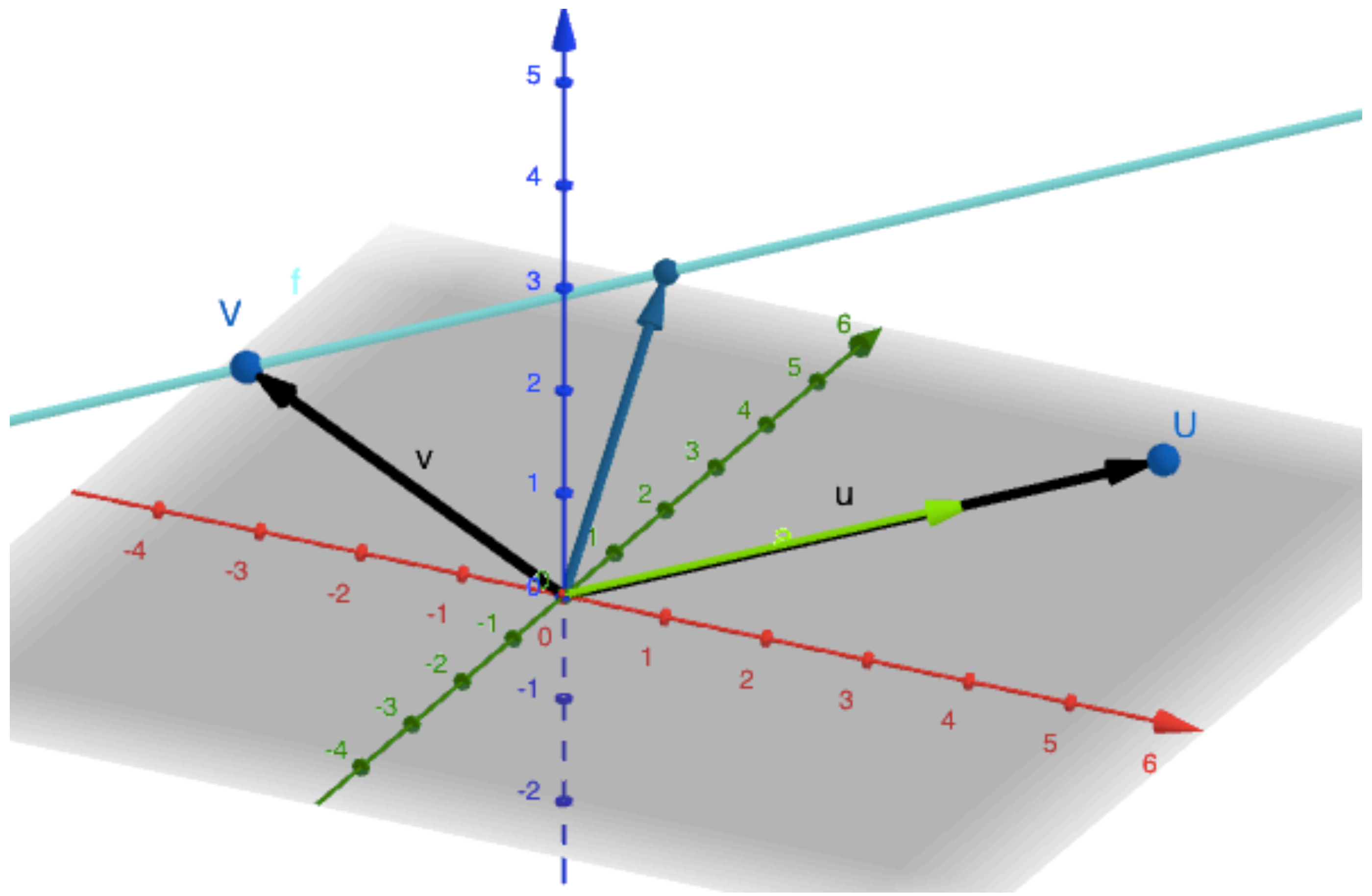
- -Recap.
- -3D Graphs.
  - Cylinders
  - Quadric Surfaces
  - Traces
- -Curves in Space.
  - Vector-Valued Functions
  - Parameters and Intersection
  - Calculus: Derivatives, Integrals
  - Plane Curves (2D)
  - Tangent and Normal Vectors
  - Space Curves (3D)
  - Binormal Vector

Recap.

What did we see last week?

- Equations of Lines.

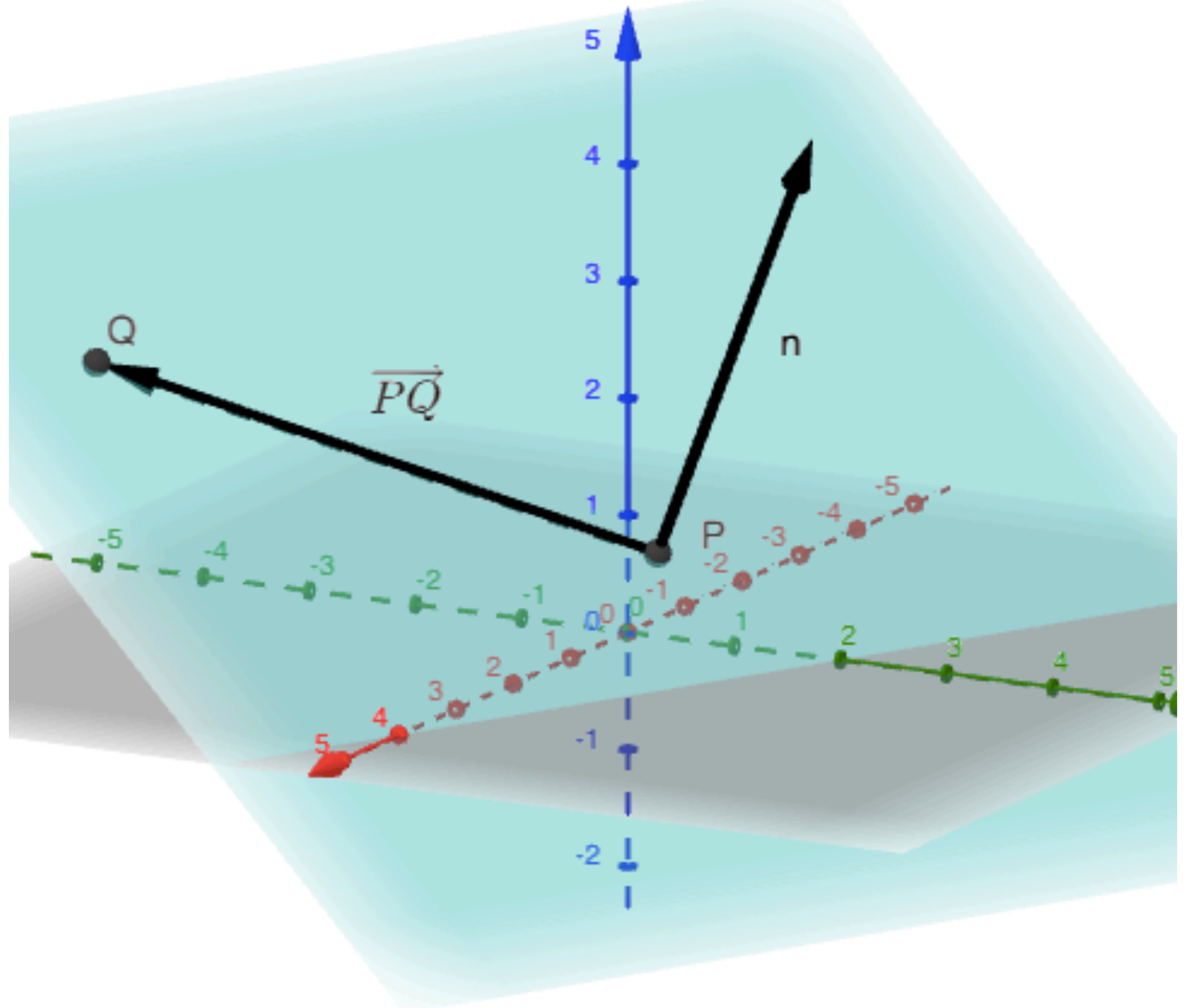
$$\mathbf{L}(t) = \mathbf{v} + t \cdot \mathbf{u}$$
$$= < v_x + tu_x, v_y + tu_y, v_z + tu_z >$$



- Equations of Planes.

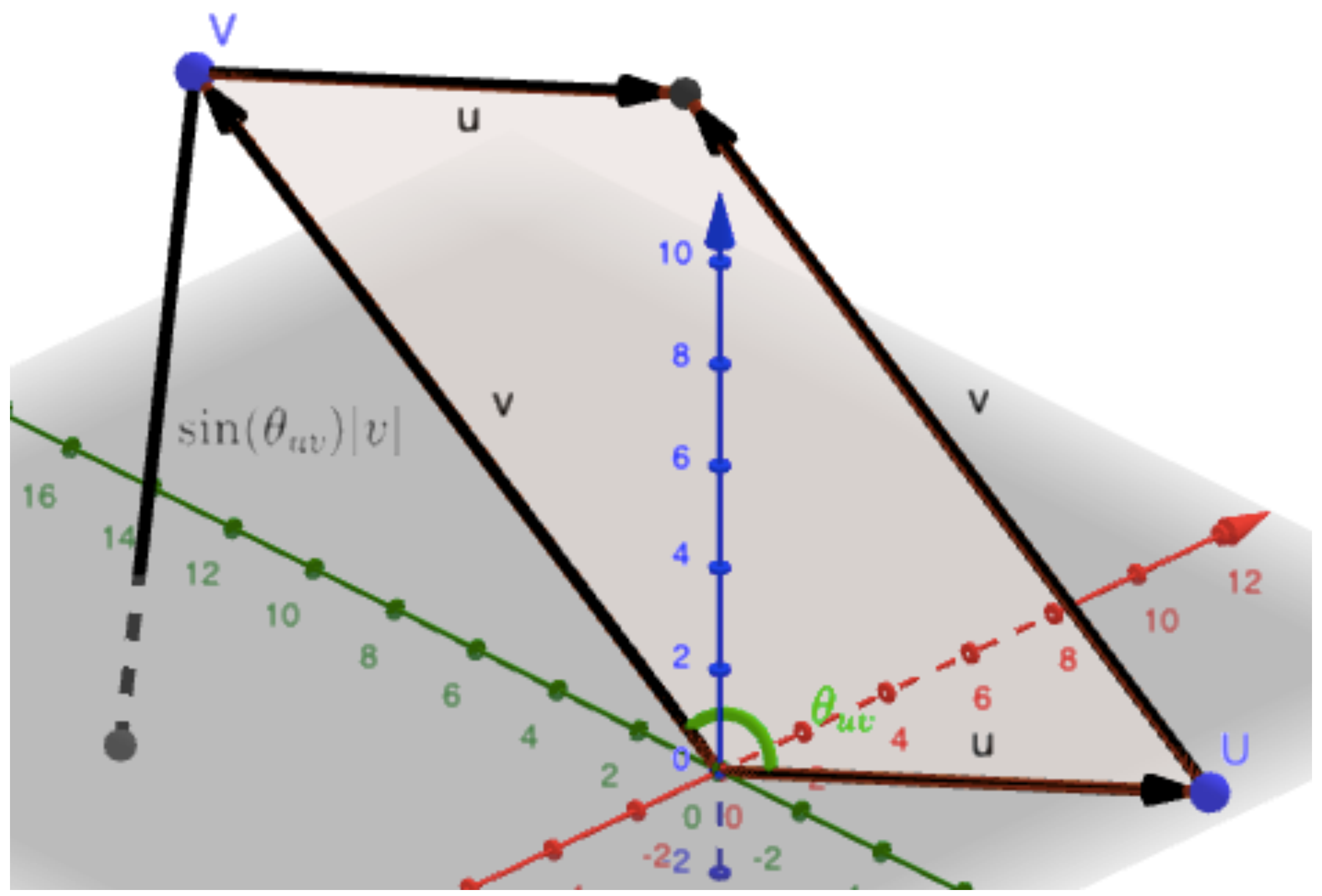
$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

where $\mathbf{n} = < a, b, c >$ is a vector perpendicular to the plane, and $P(x_0, y_0, z_0)$ is a point on the plane.



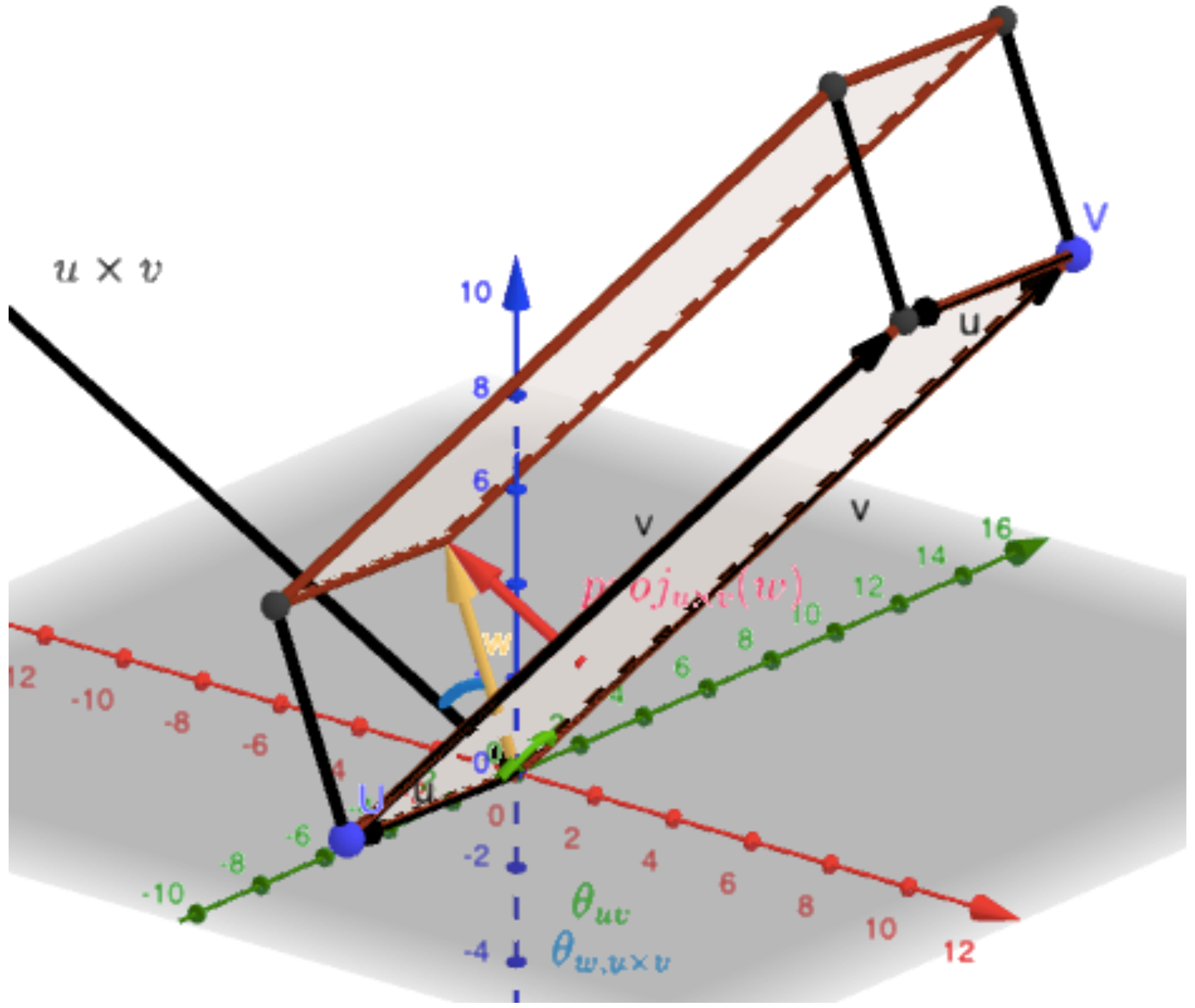
- Properties of Cross Product.

$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin(\theta)$ is the area of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$

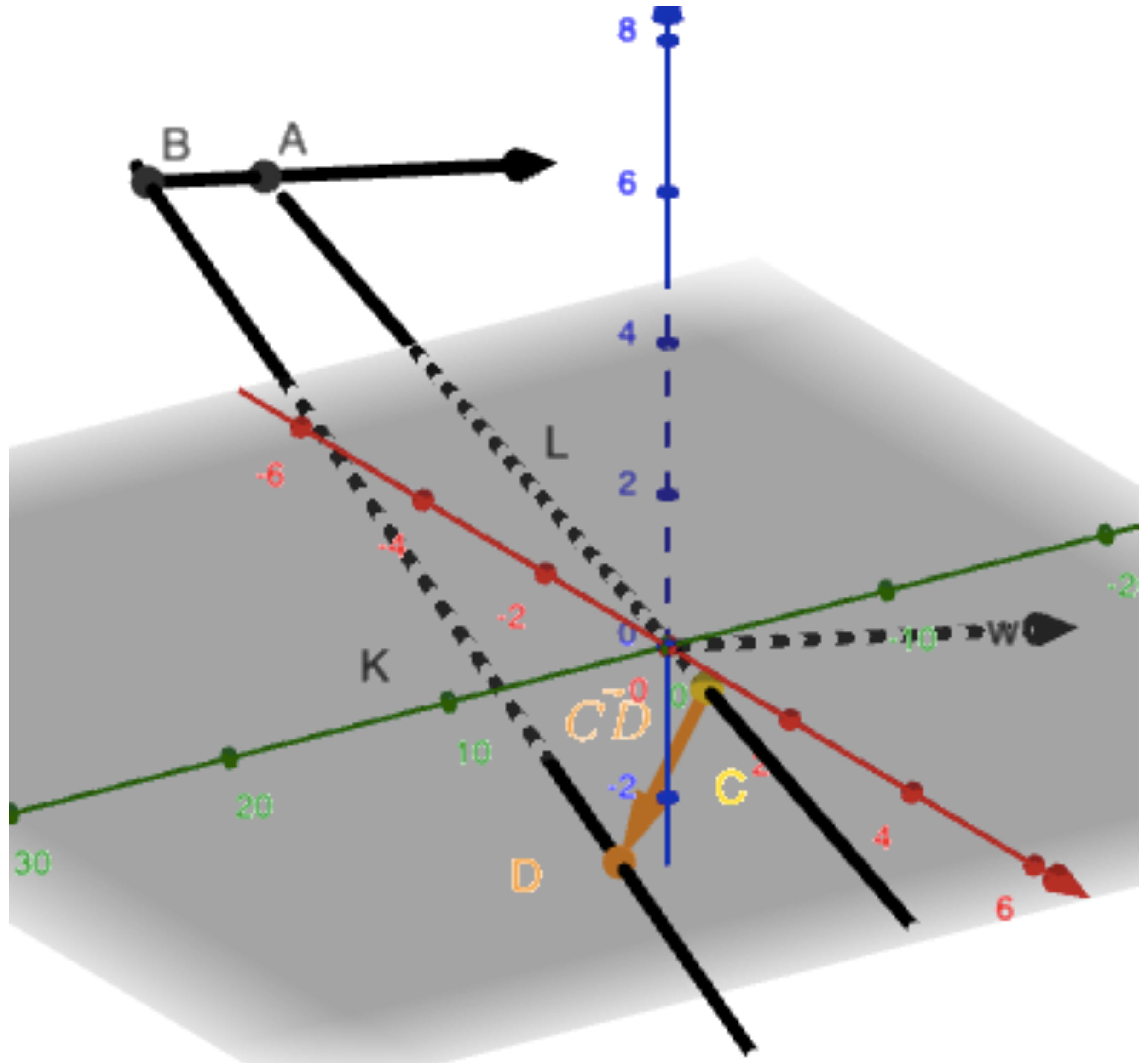


$|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = |\mathbf{u} \times \mathbf{v}||\mathbf{w}||\cos(\theta)|$

is the volume of this parallelepiped.



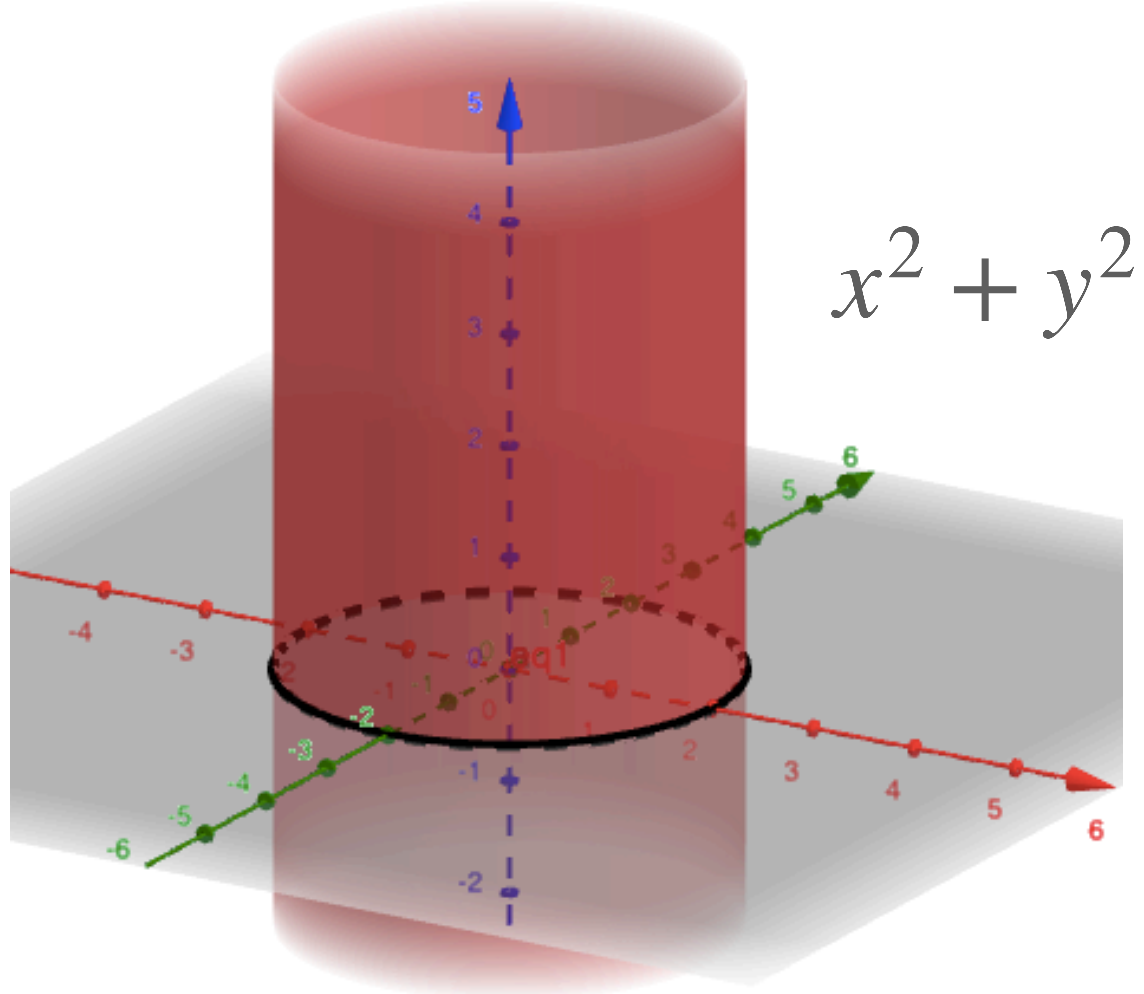
- Distances between objects in 3D.



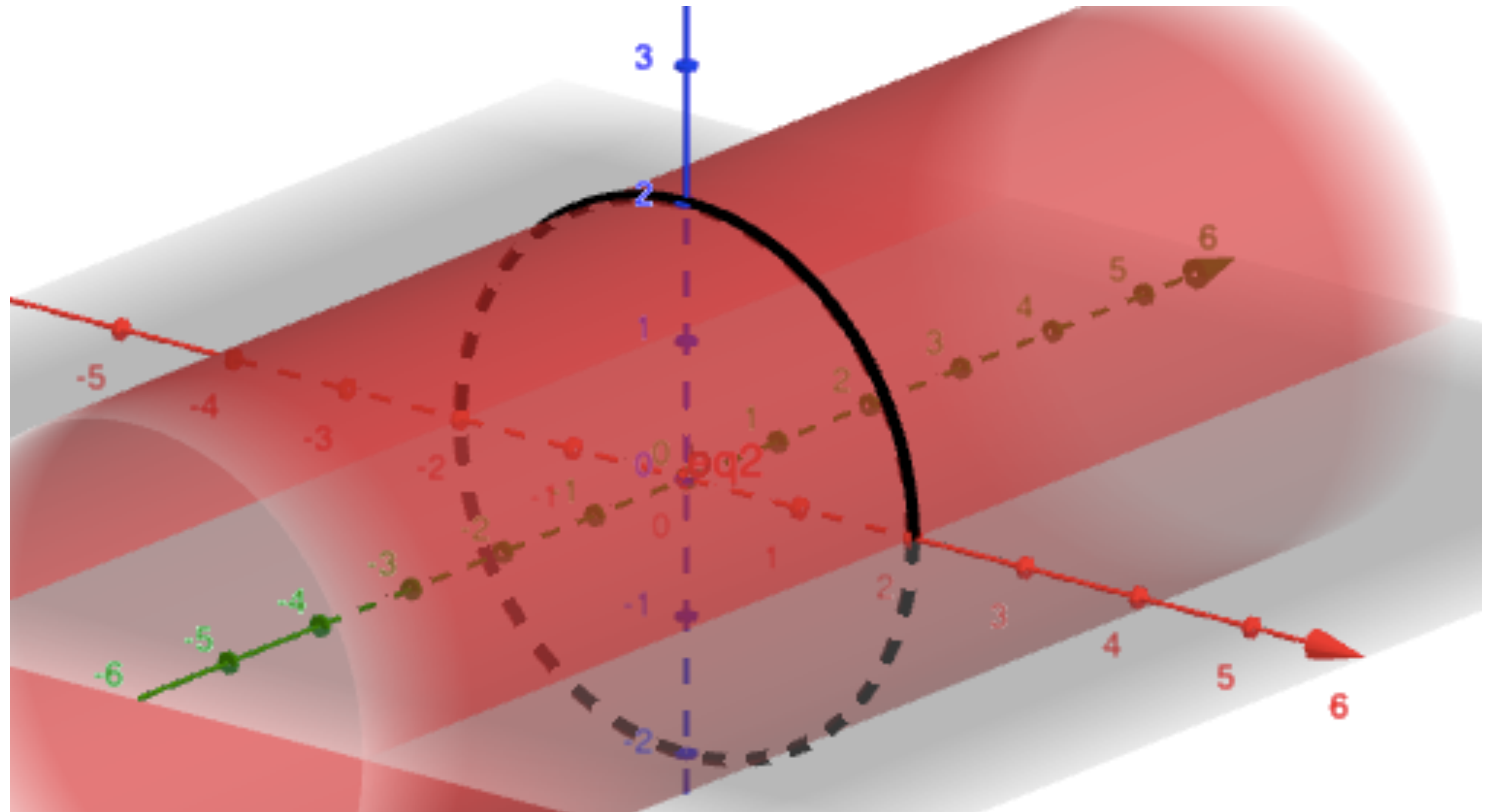
Cylinders, pg 1.

The graph of a *cylinder* consists of lines through a curve in a plane that are all parallel to a given line.

We have seen a few….



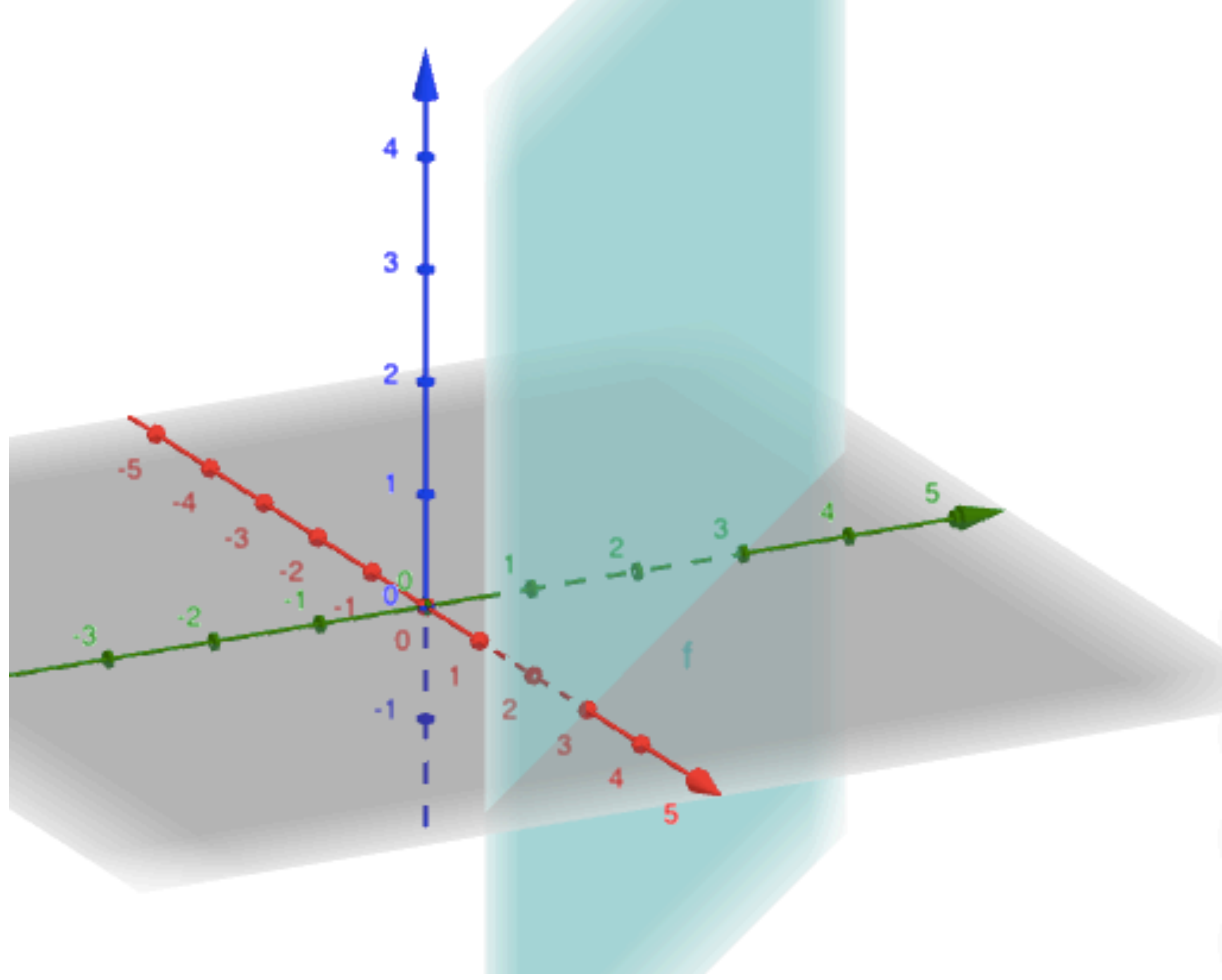
$x^2 + y^2 = 4$



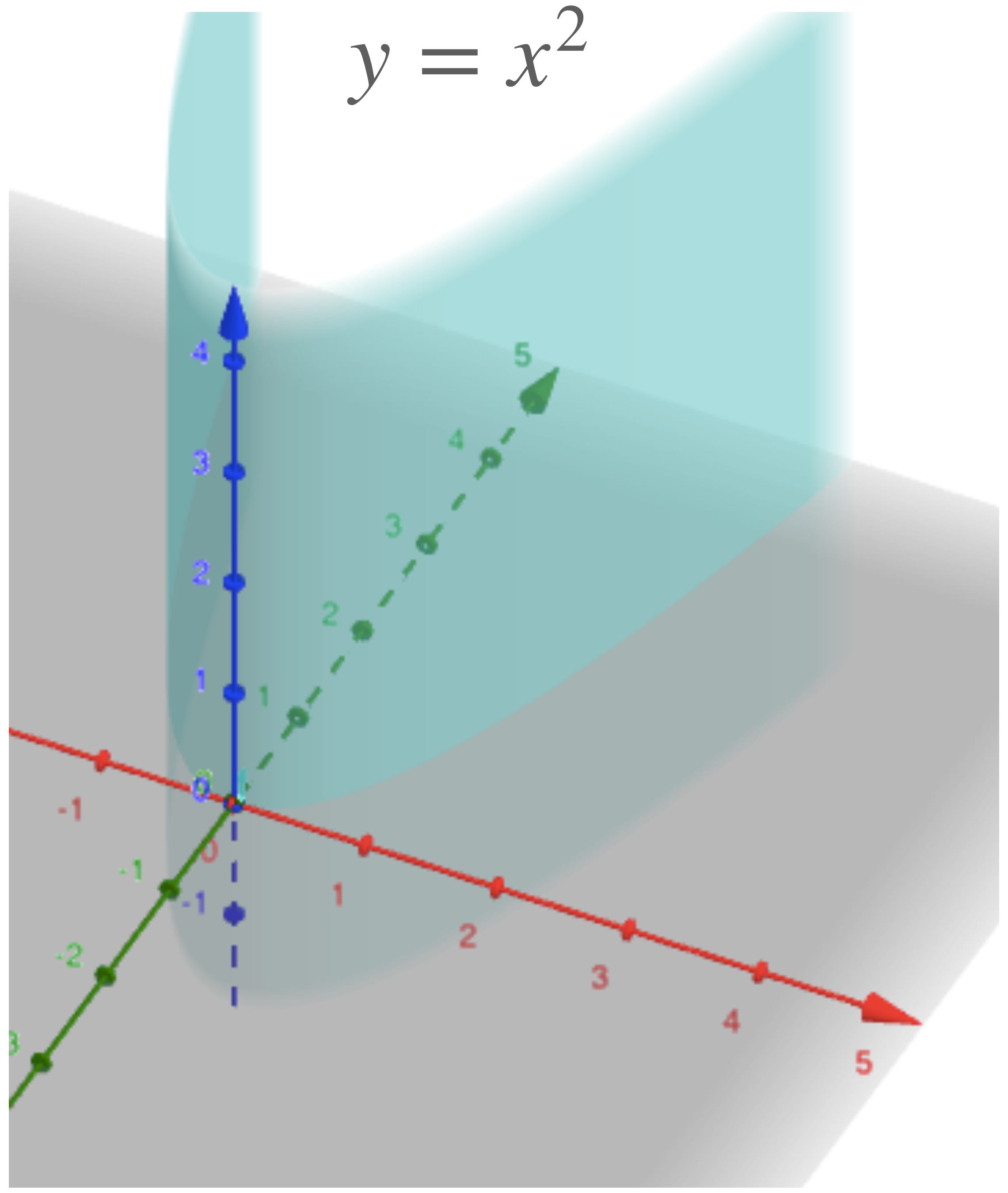
$x^2 + z^2 = 4$

Here are a few more…

$x + y = 3$

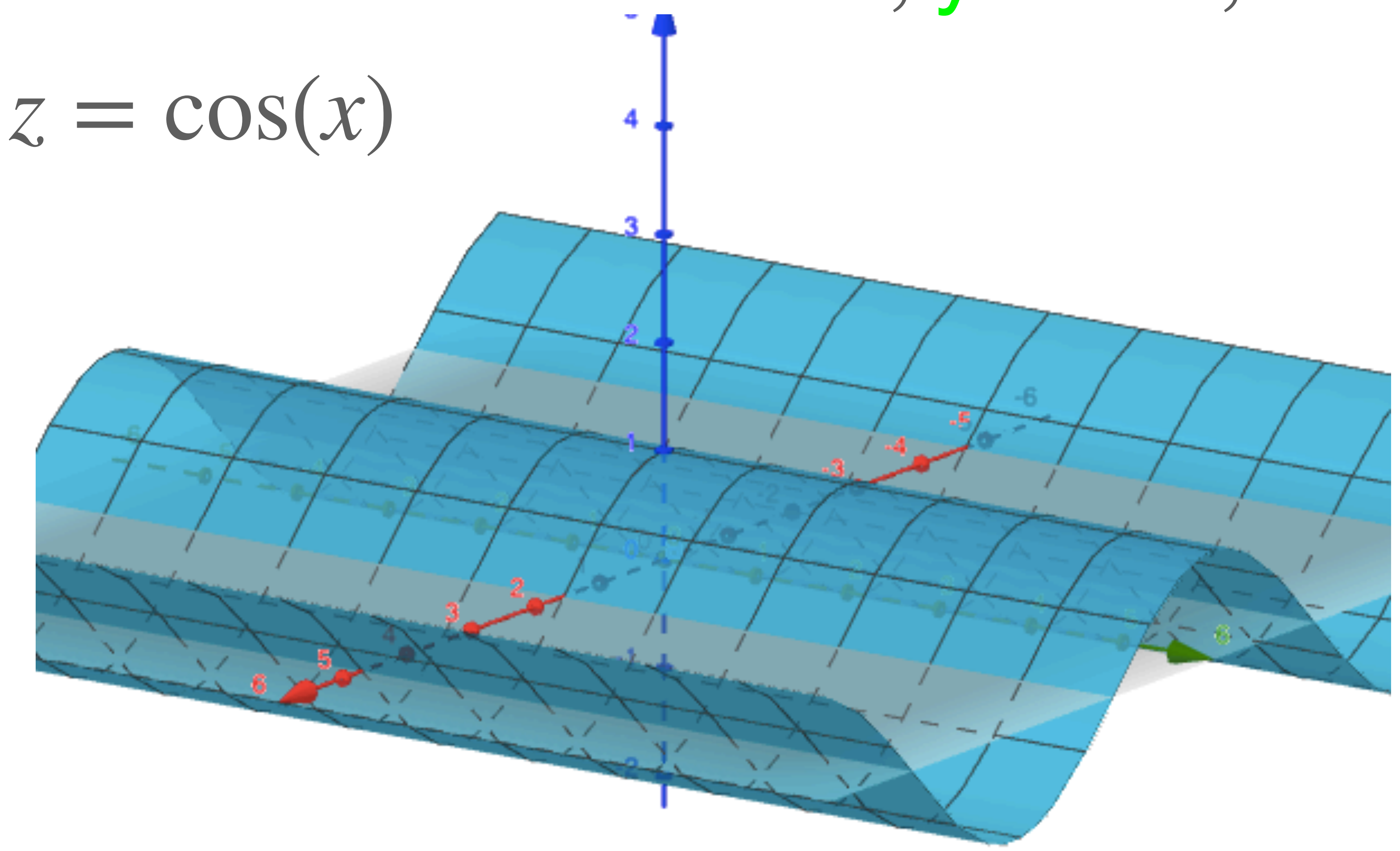


$y = x^2$

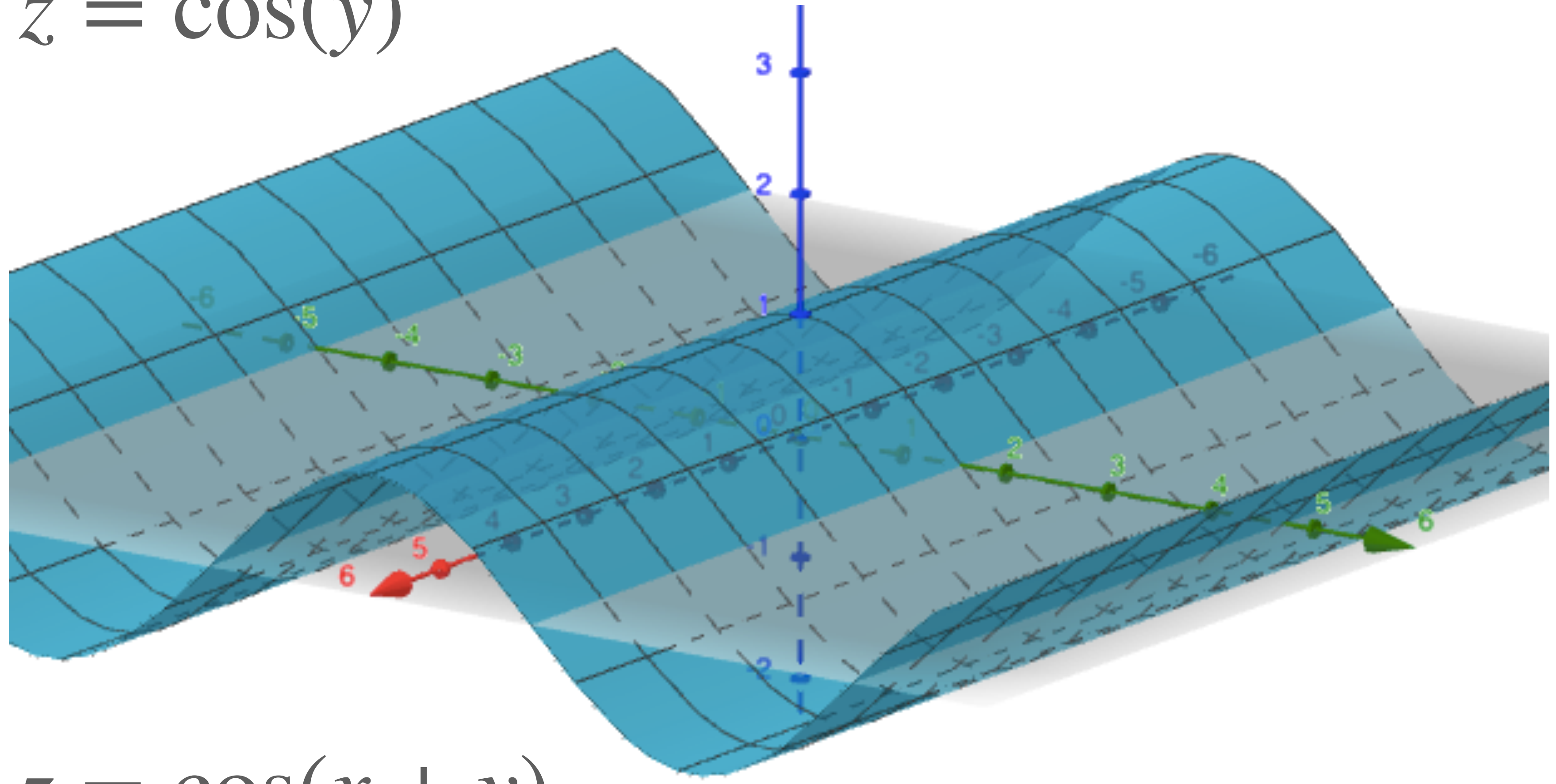


Illustrations have an x-axis ; y-axis ; z-axis

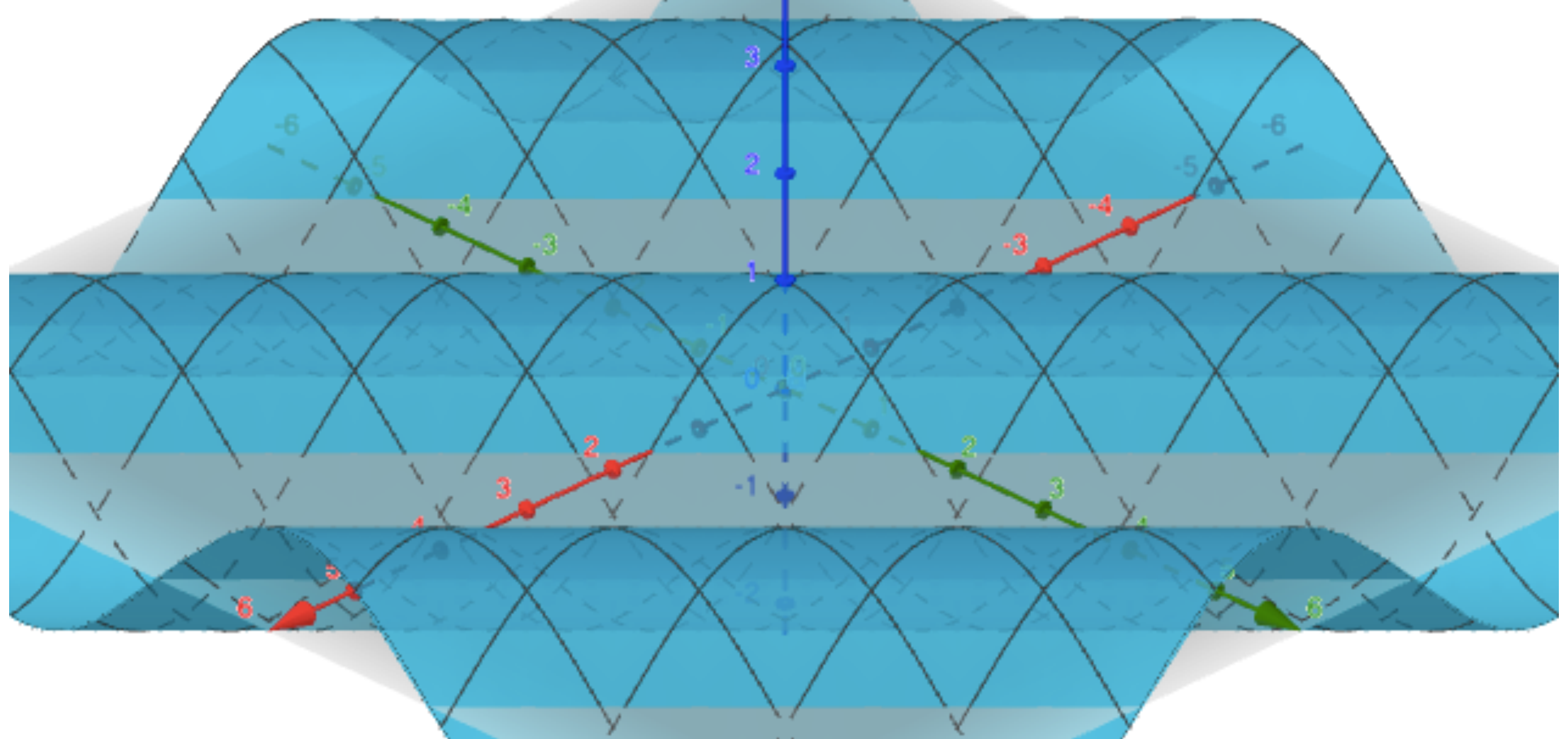
$z = \cos(x)$



$z = \cos(y)$



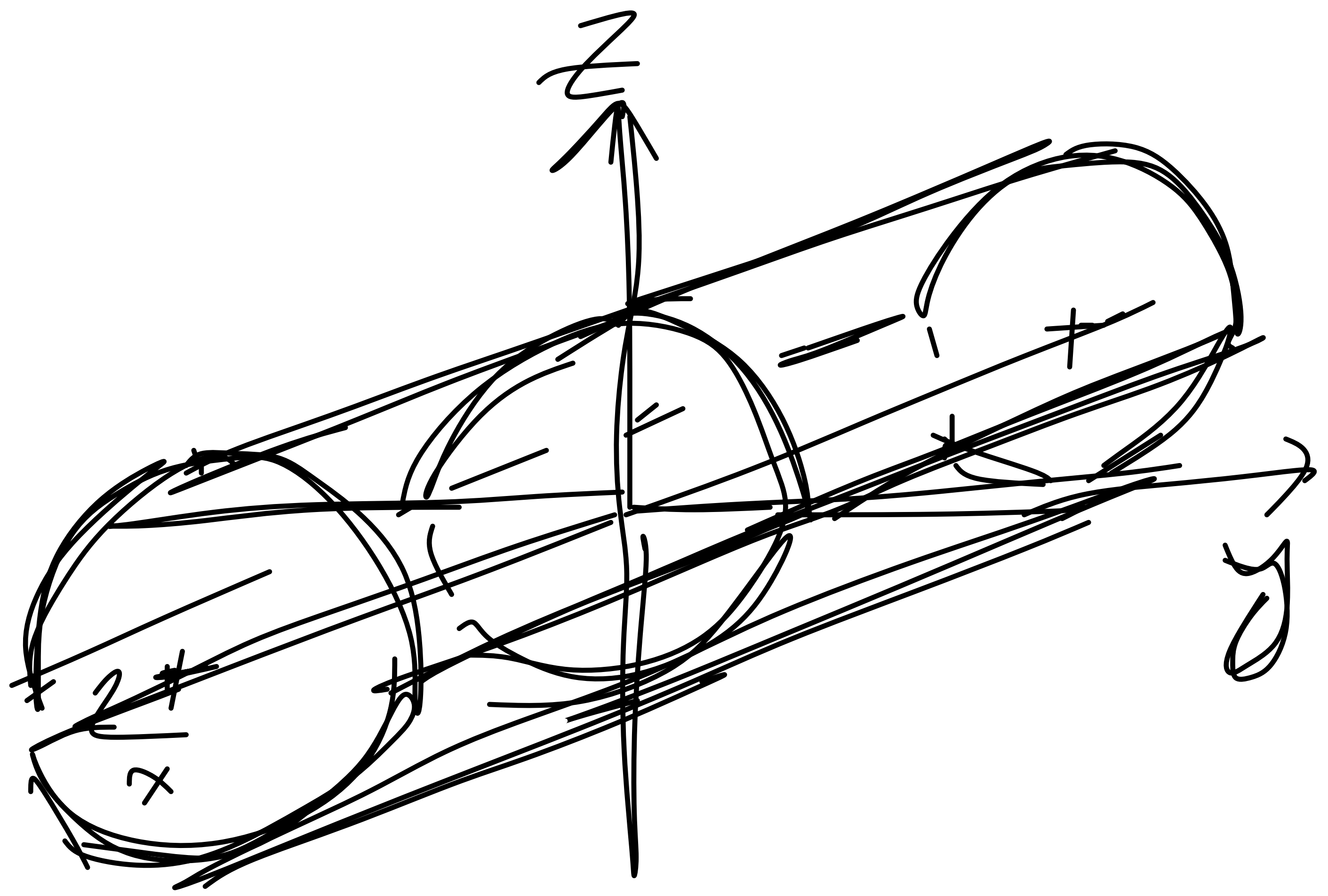
$z = \cos(x + y)$



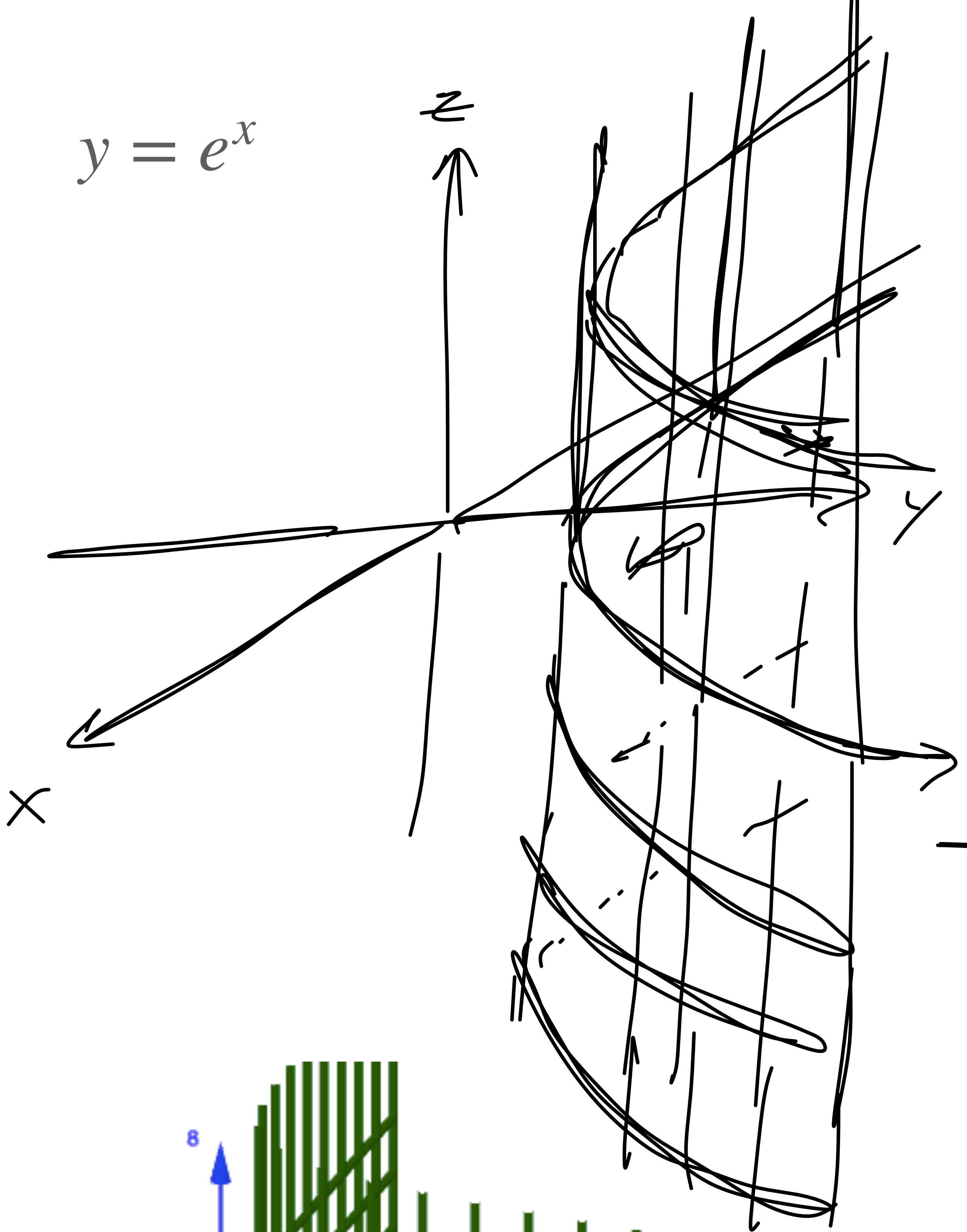
Cylinders, pg 2.

More Examples.

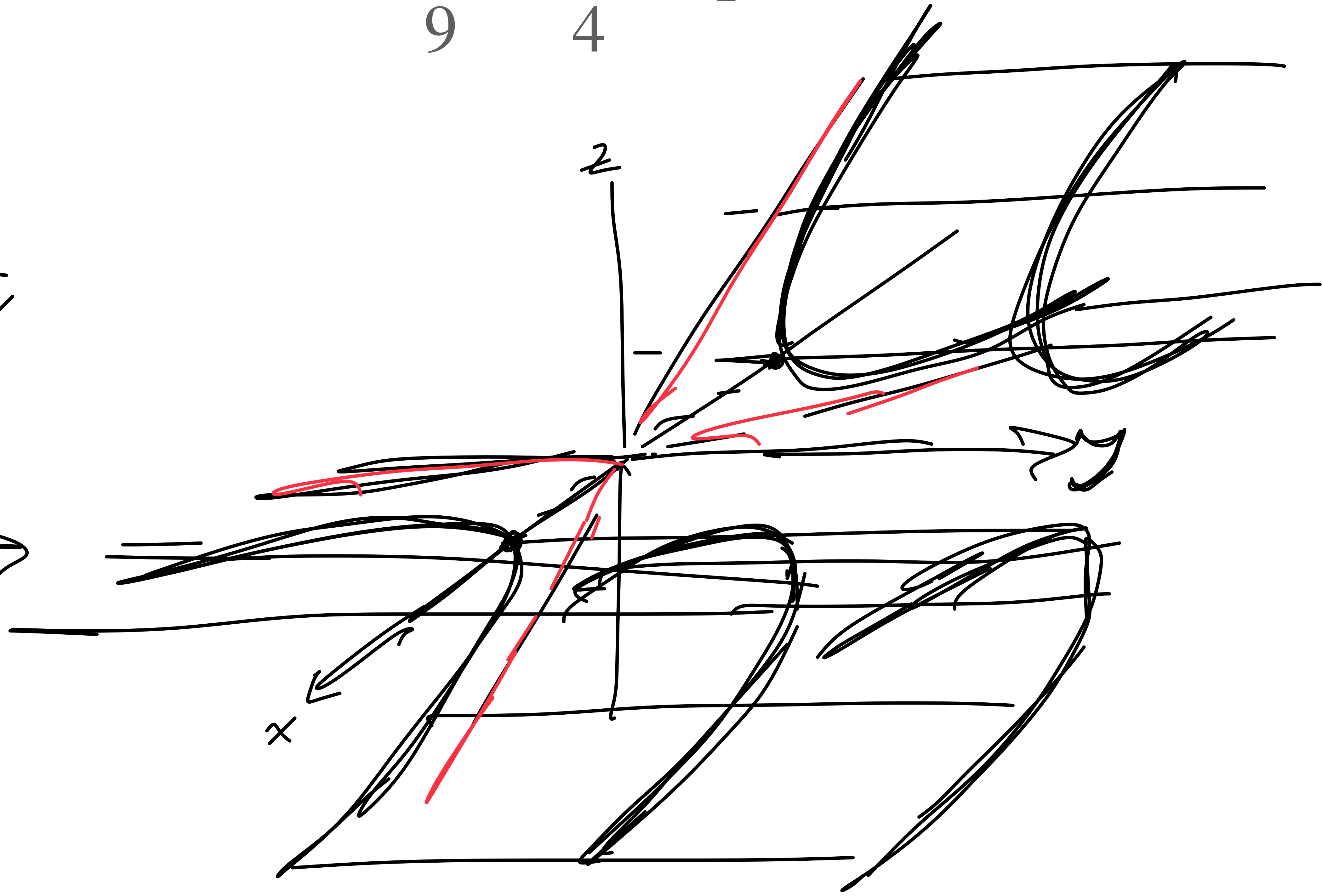
$y^2 + z^2 = 4$

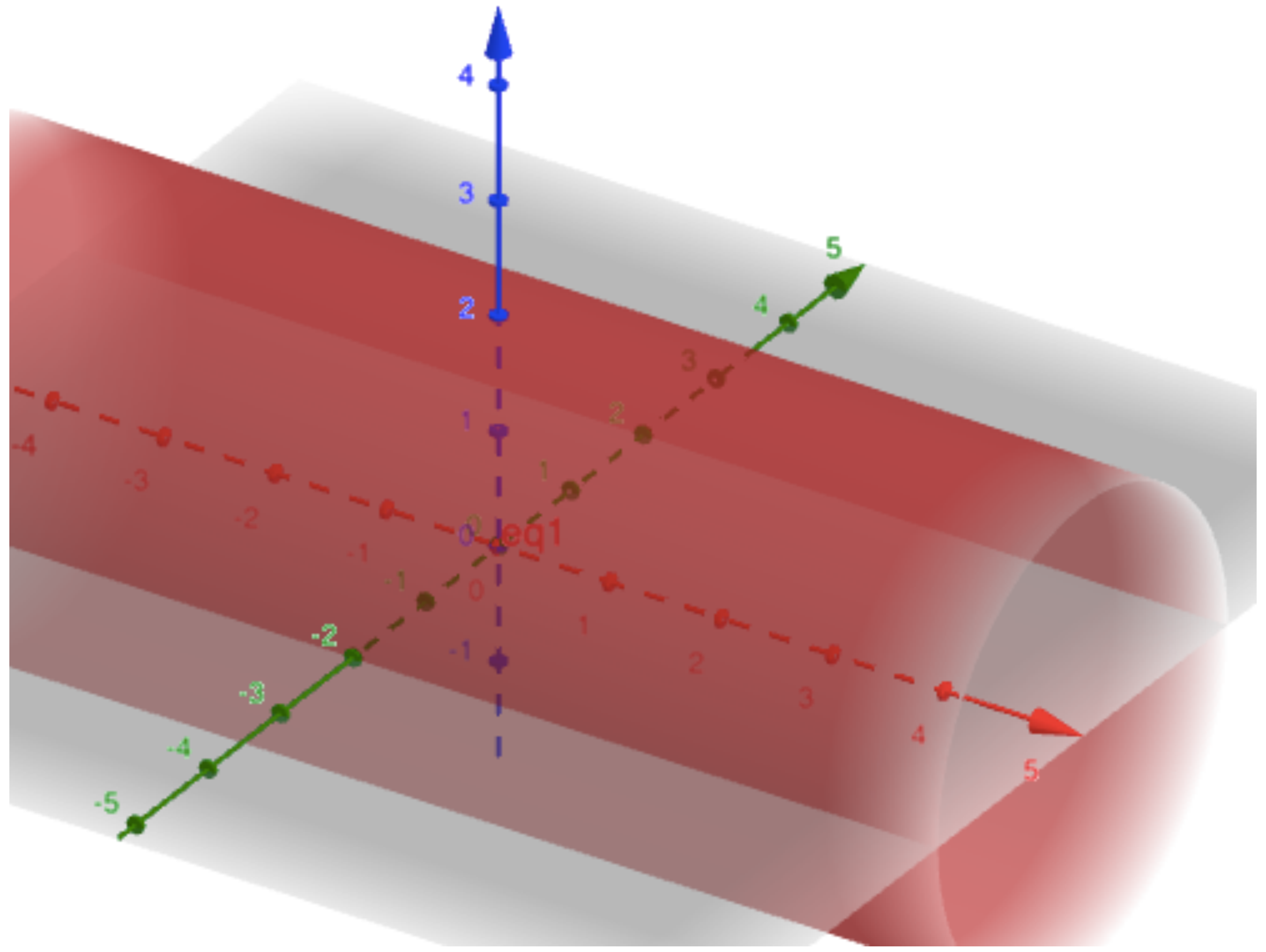


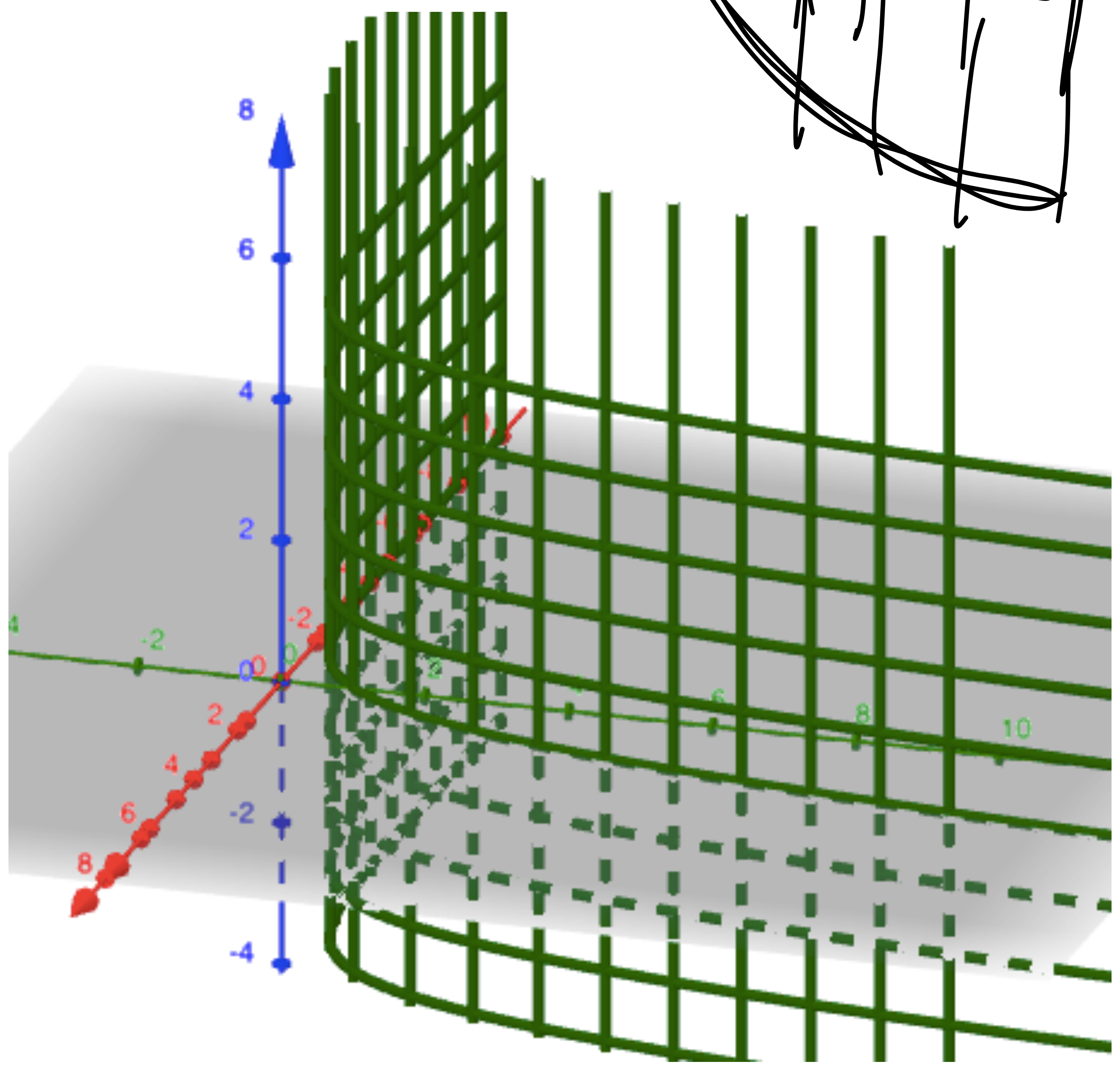
$y = e^x$

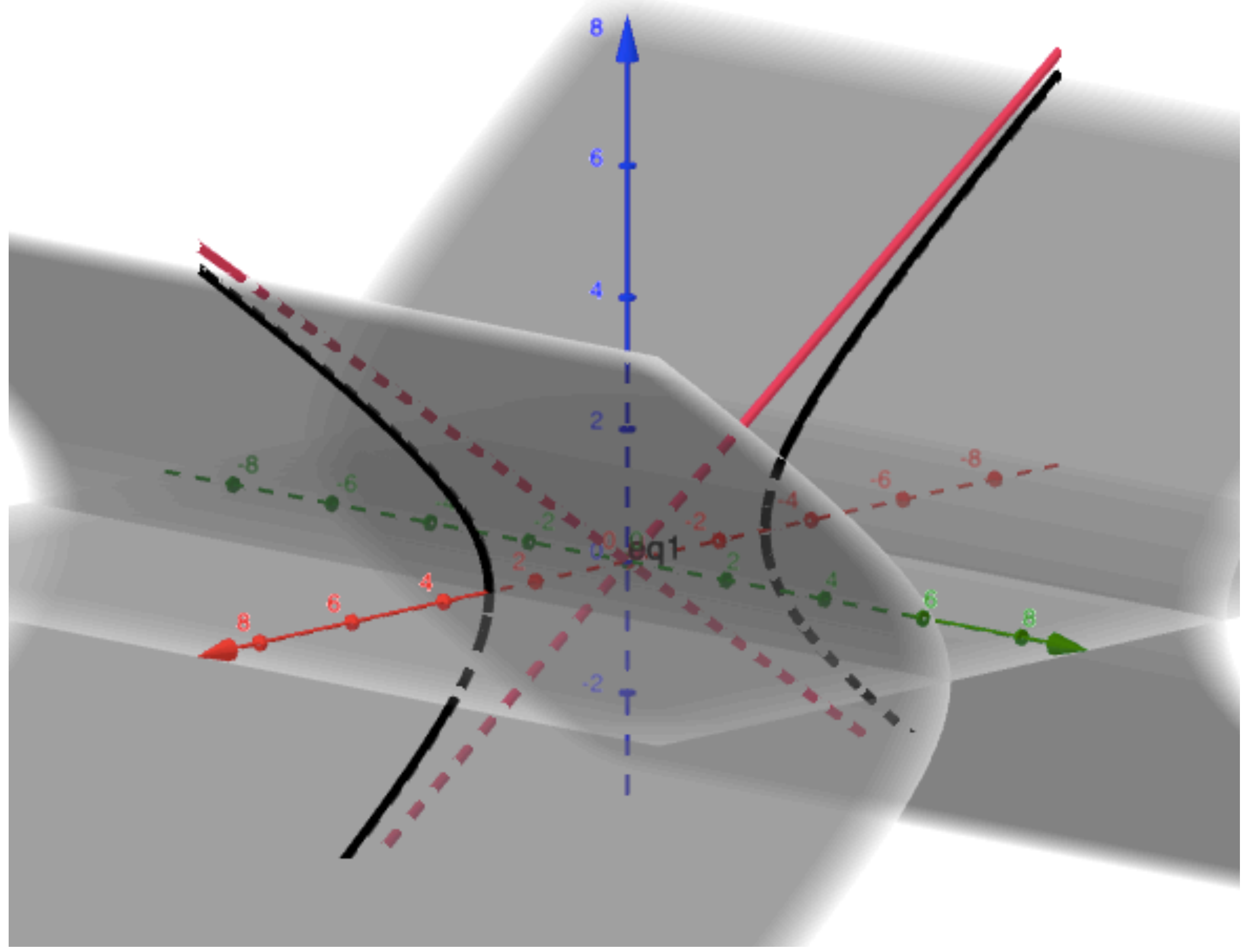


$$\frac{x^2}{9} - \frac{z^2}{4} = 1$$



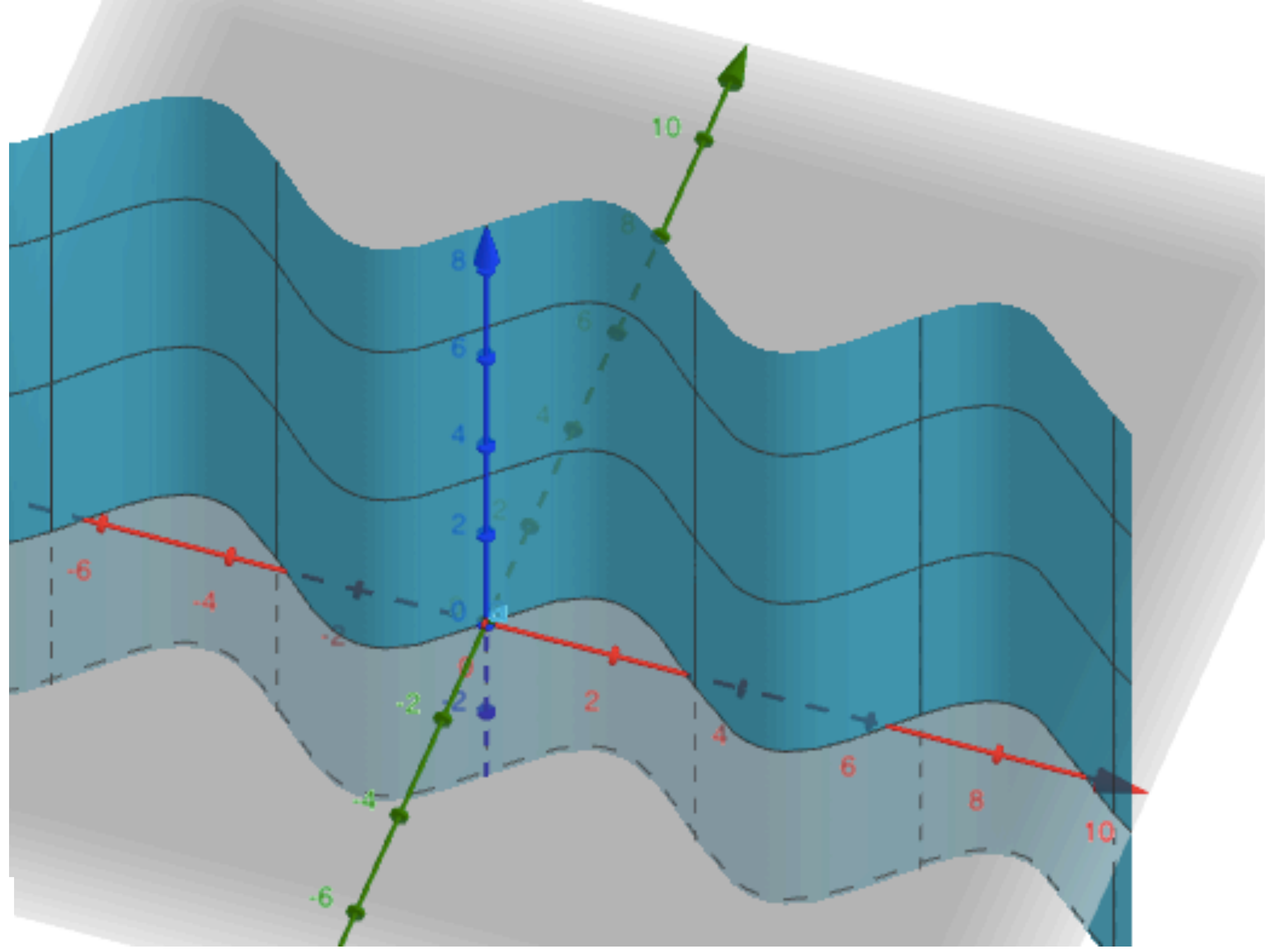






Cylinders, pg 3.

1. $y = \sin(x)$
2. $x + z = 4$
3. $xy = 1$
4. $y = 1 - z^2$



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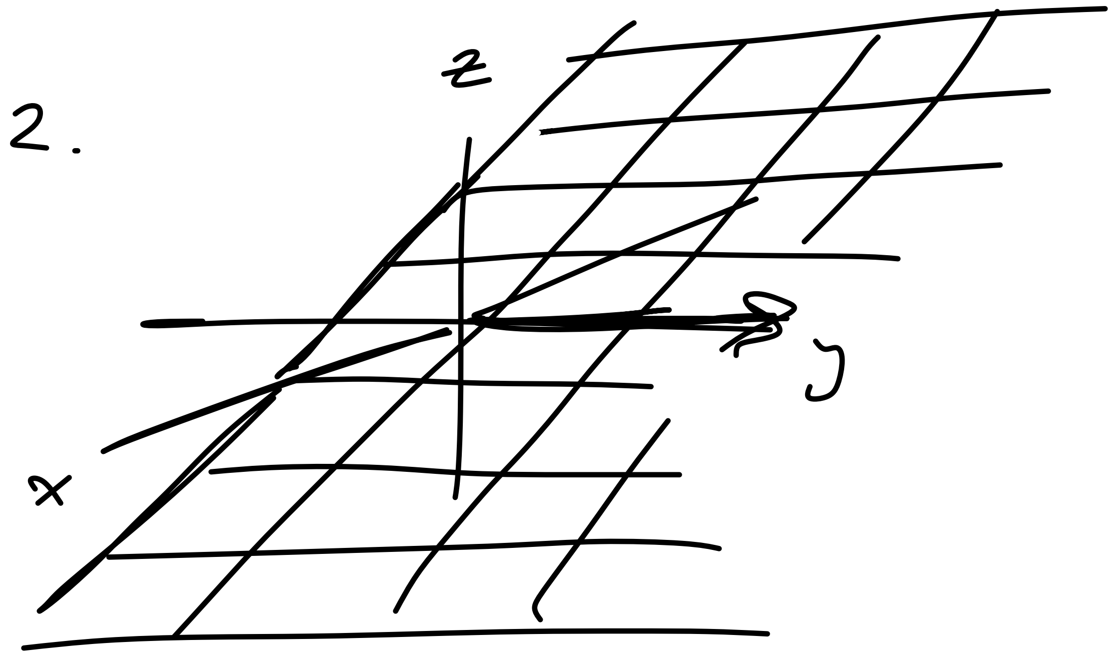

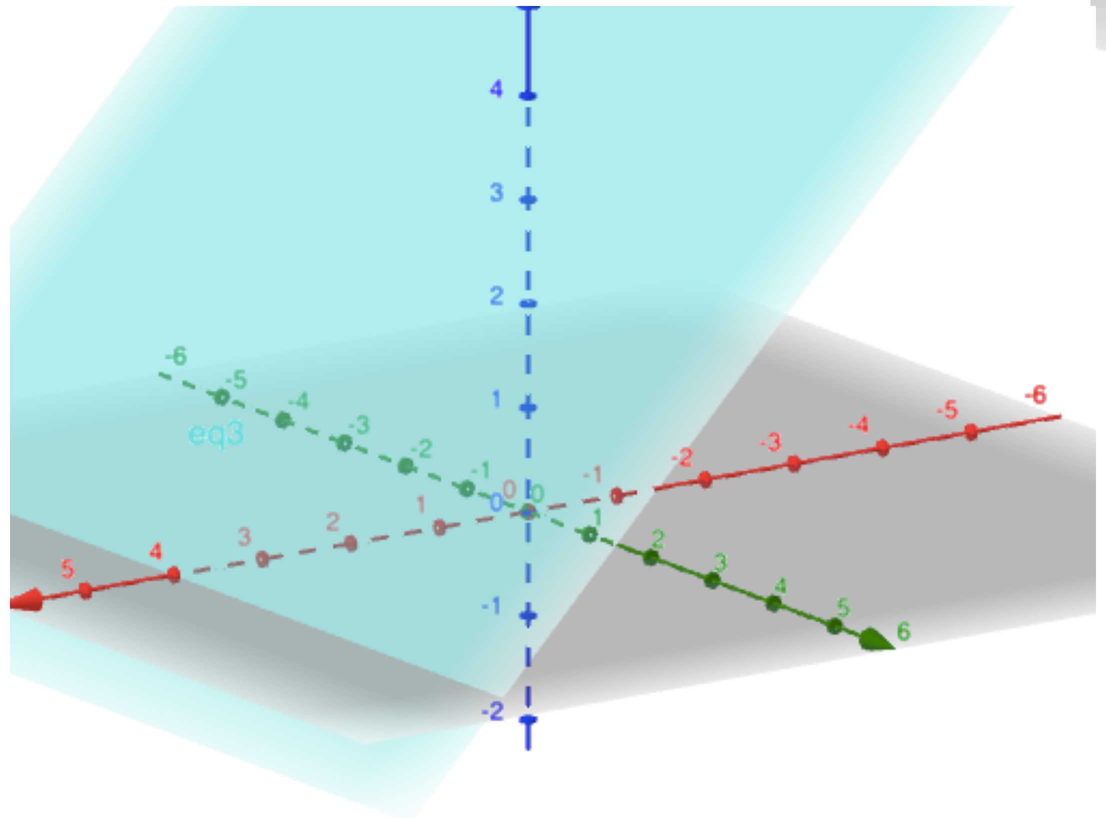

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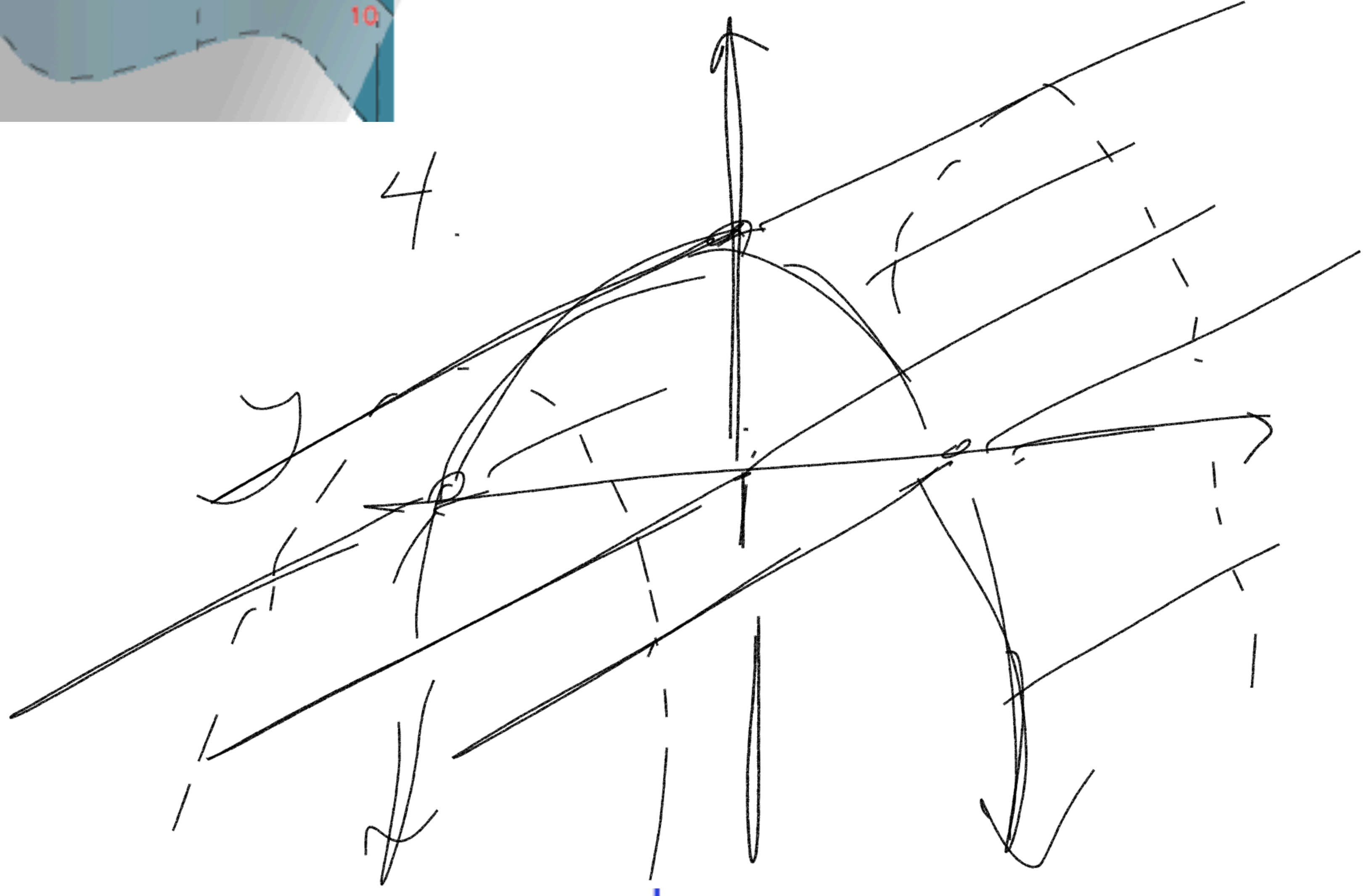
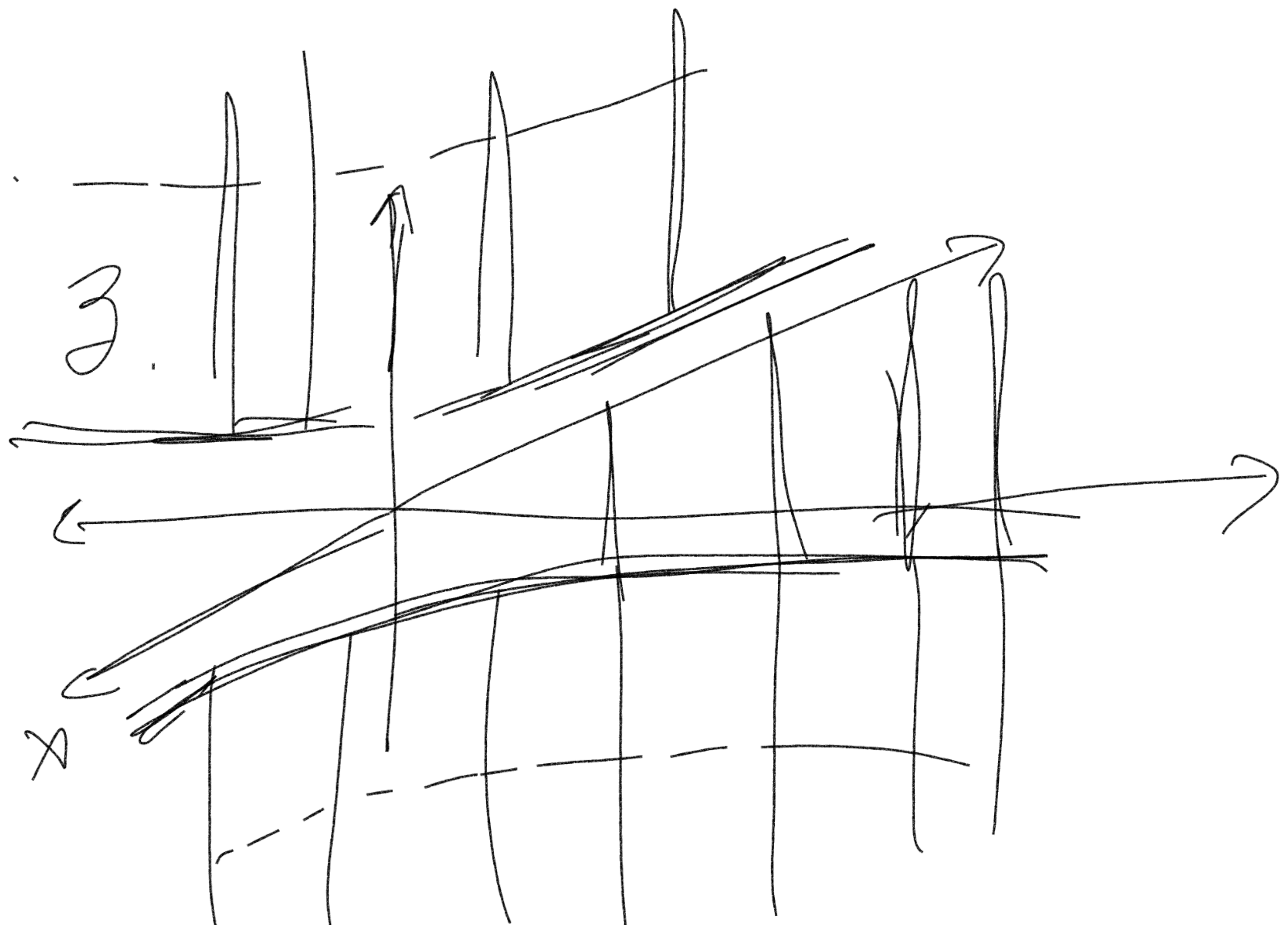
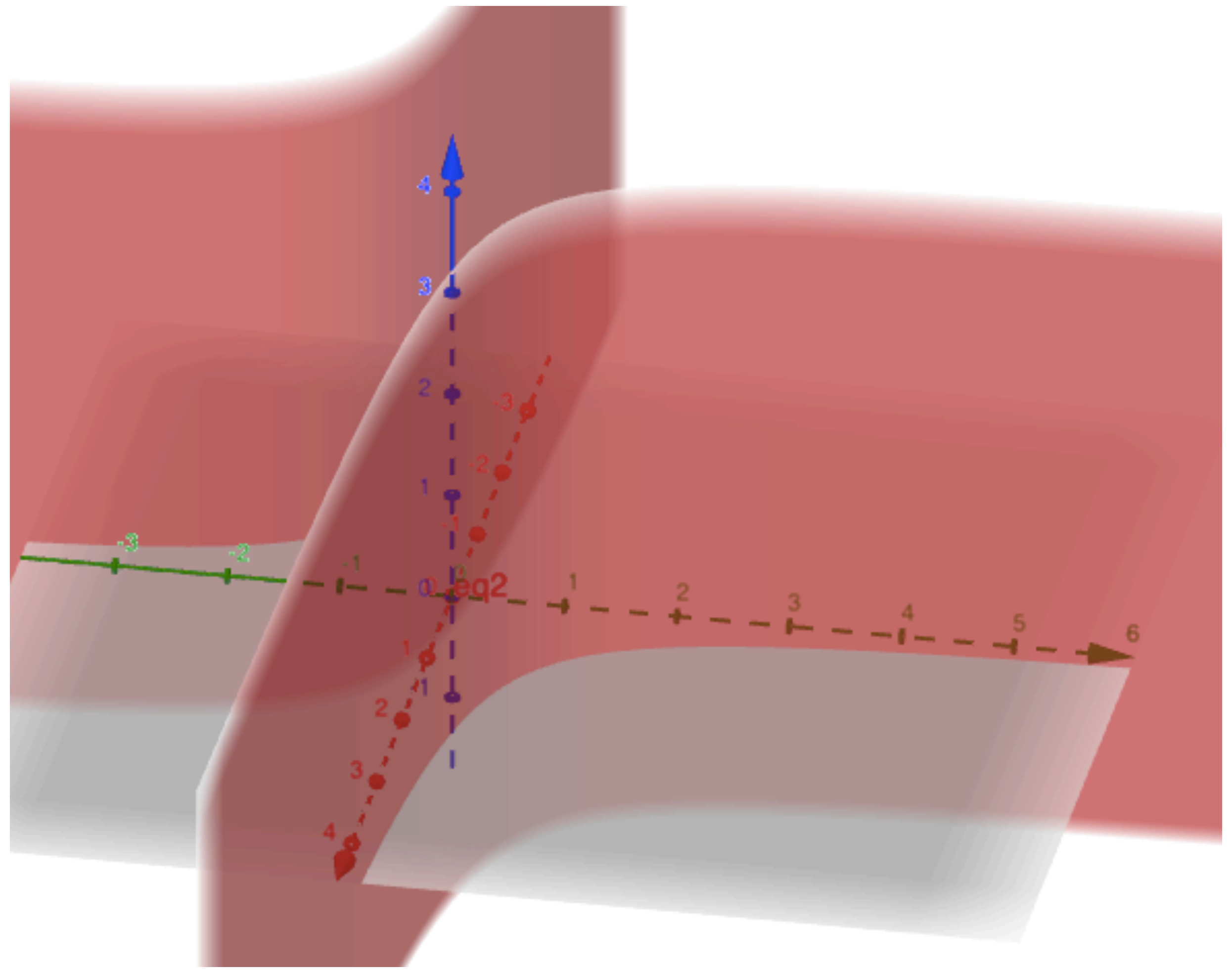
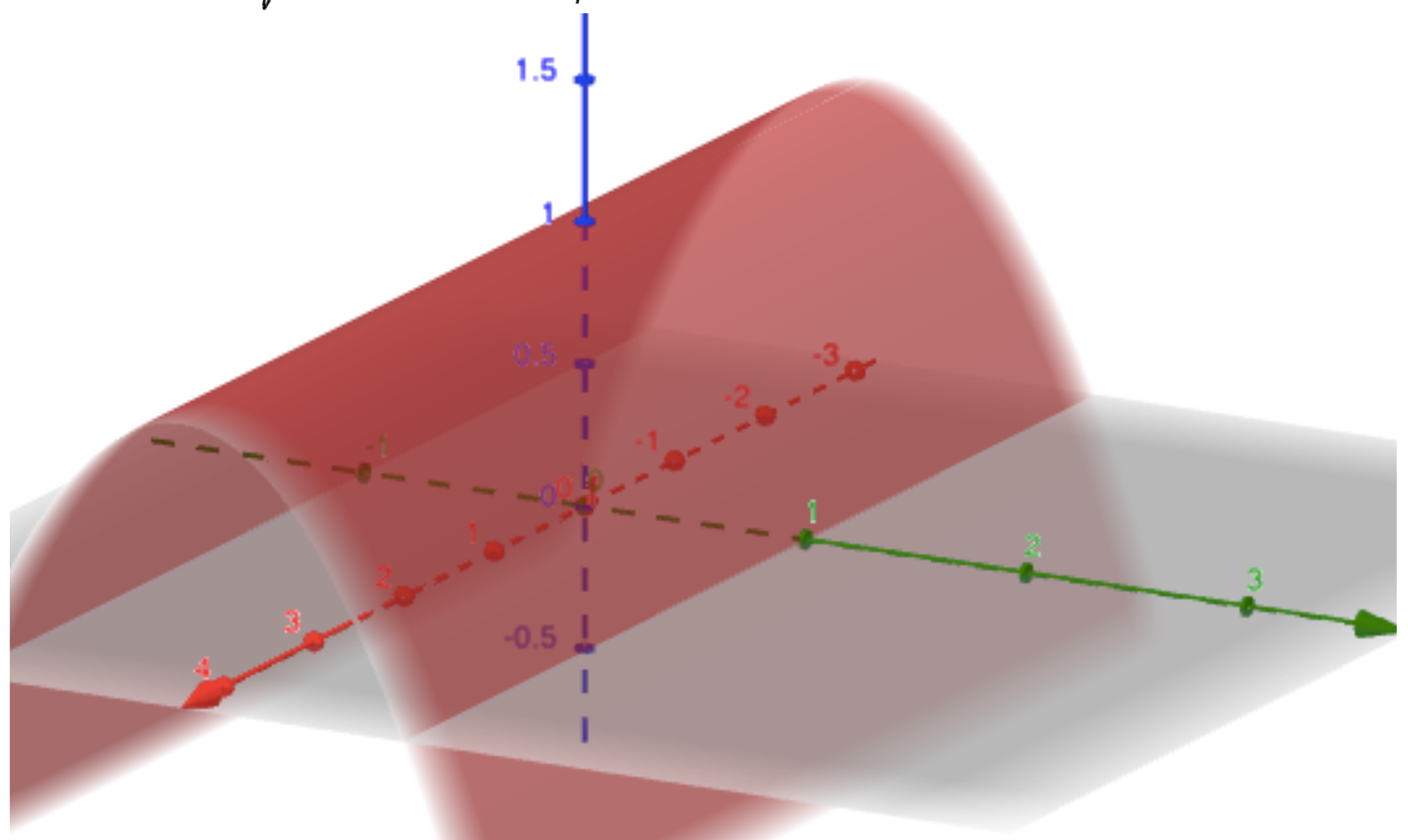


3.







Quadric Surfaces, pg 1. Traces.

A quadric surface is given by an equation of the form:

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

This is a conic section of a higher dimension!

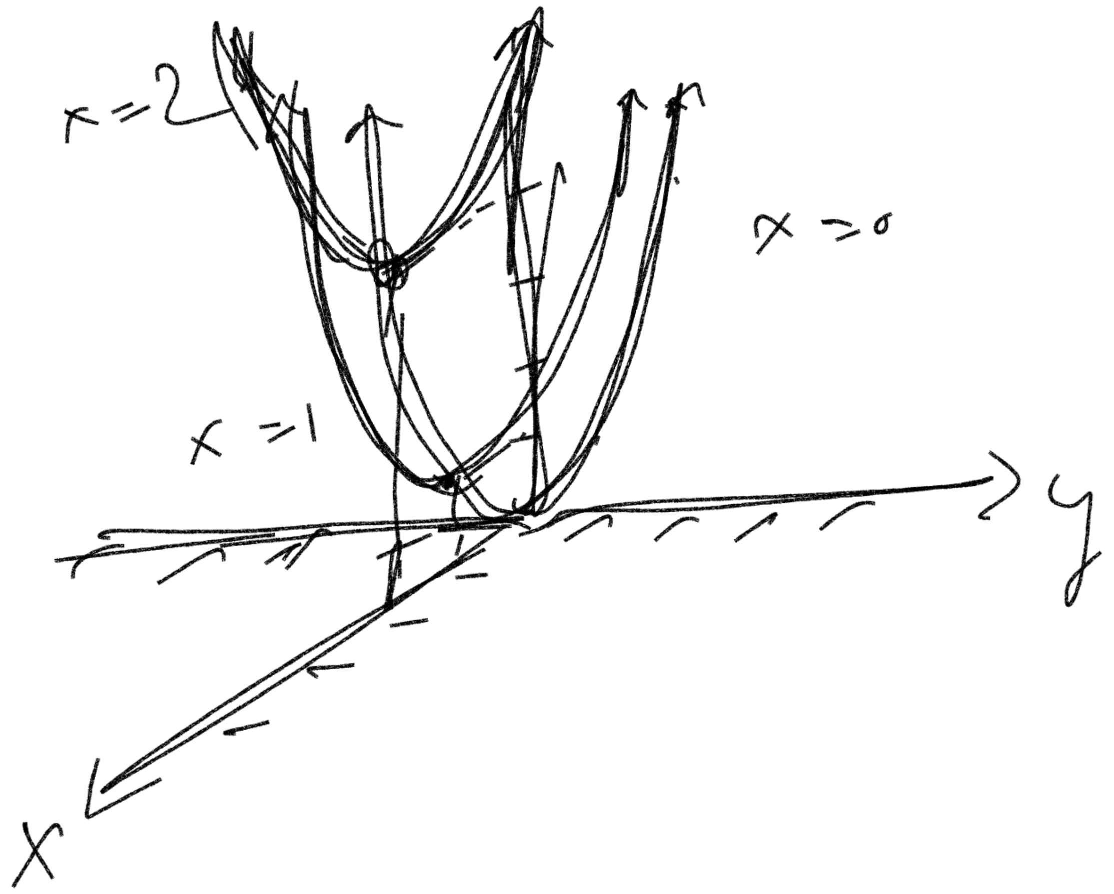
As with conic sections, you can change coordinates (via translations and rotations) to get a simpler quadric surface of the form

$Ax^2 + By^2 + Cz^2 + J = 0$, or $Ax^2 + By^2 + Iz = 0$

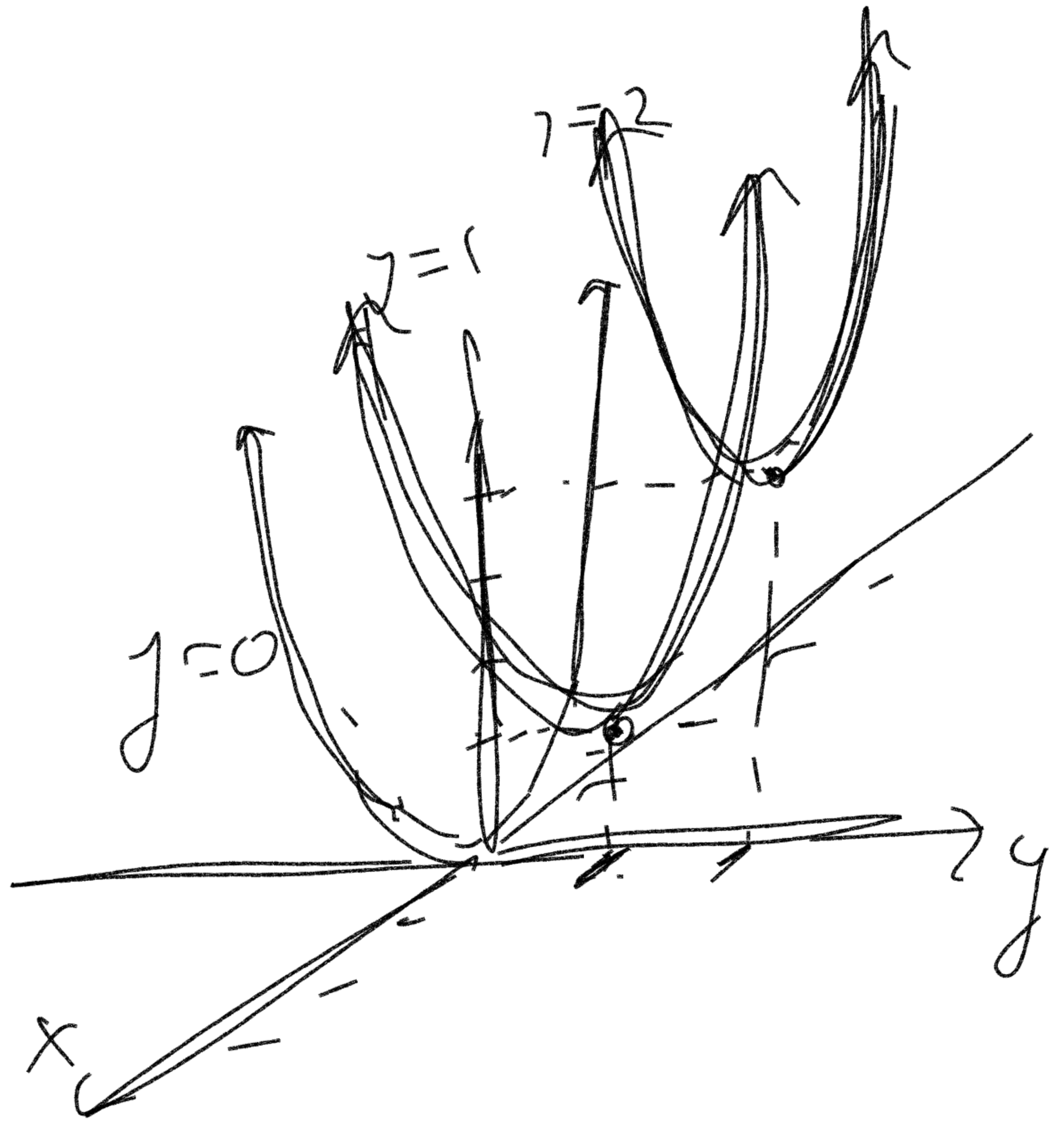
Examples. 1. $z = x^2 + y^2$

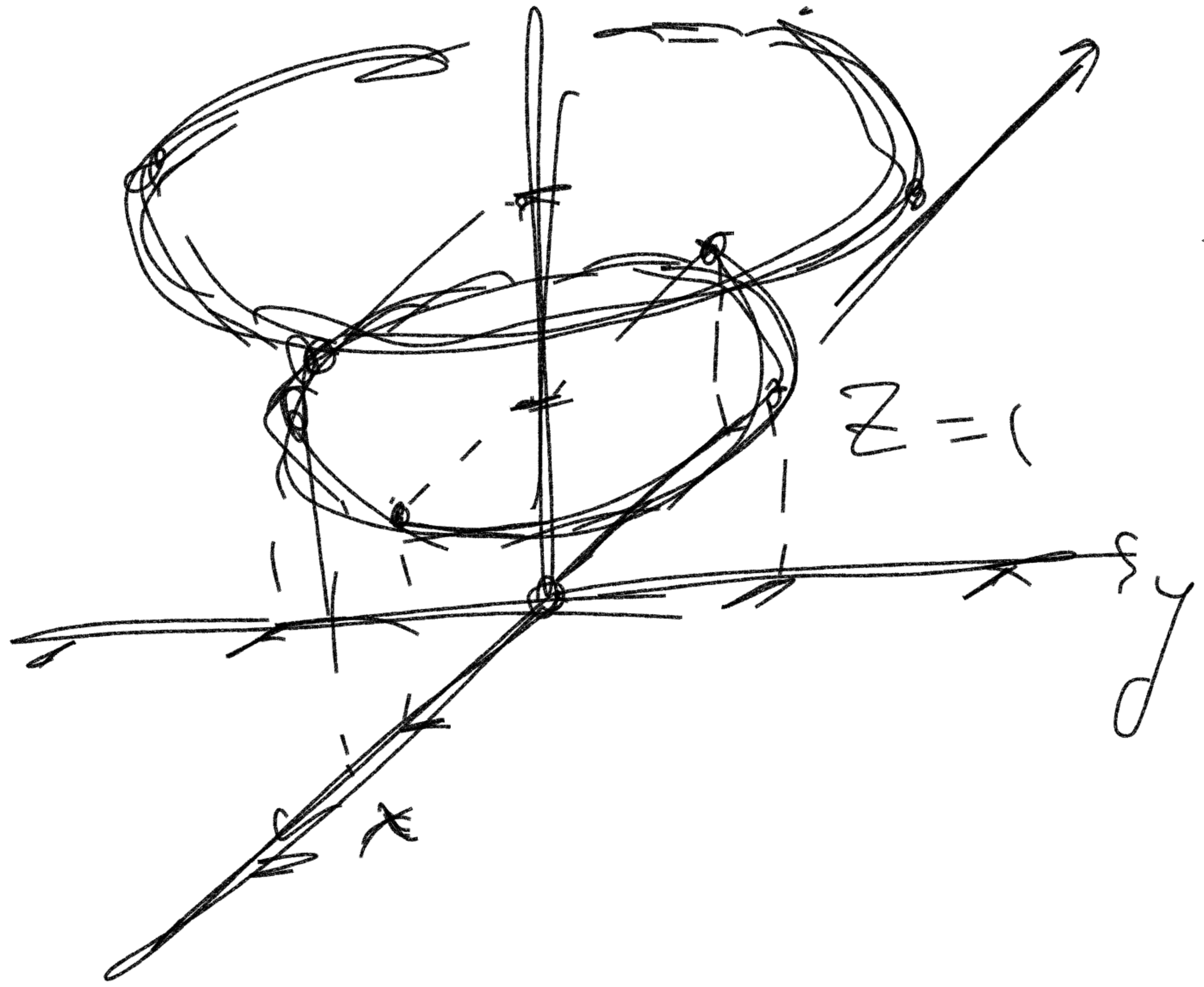
To draw a graph, look at **traces** parallel to each coordinate plane (the xy; xz; yz planes).

$$\begin{array}{ll} x = 0 & z = y^2 \\ x = 1 & z = 1 + y^2 \\ x = 2 & z = 4 + y^2 \\ \vdots & \vdots \end{array}$$



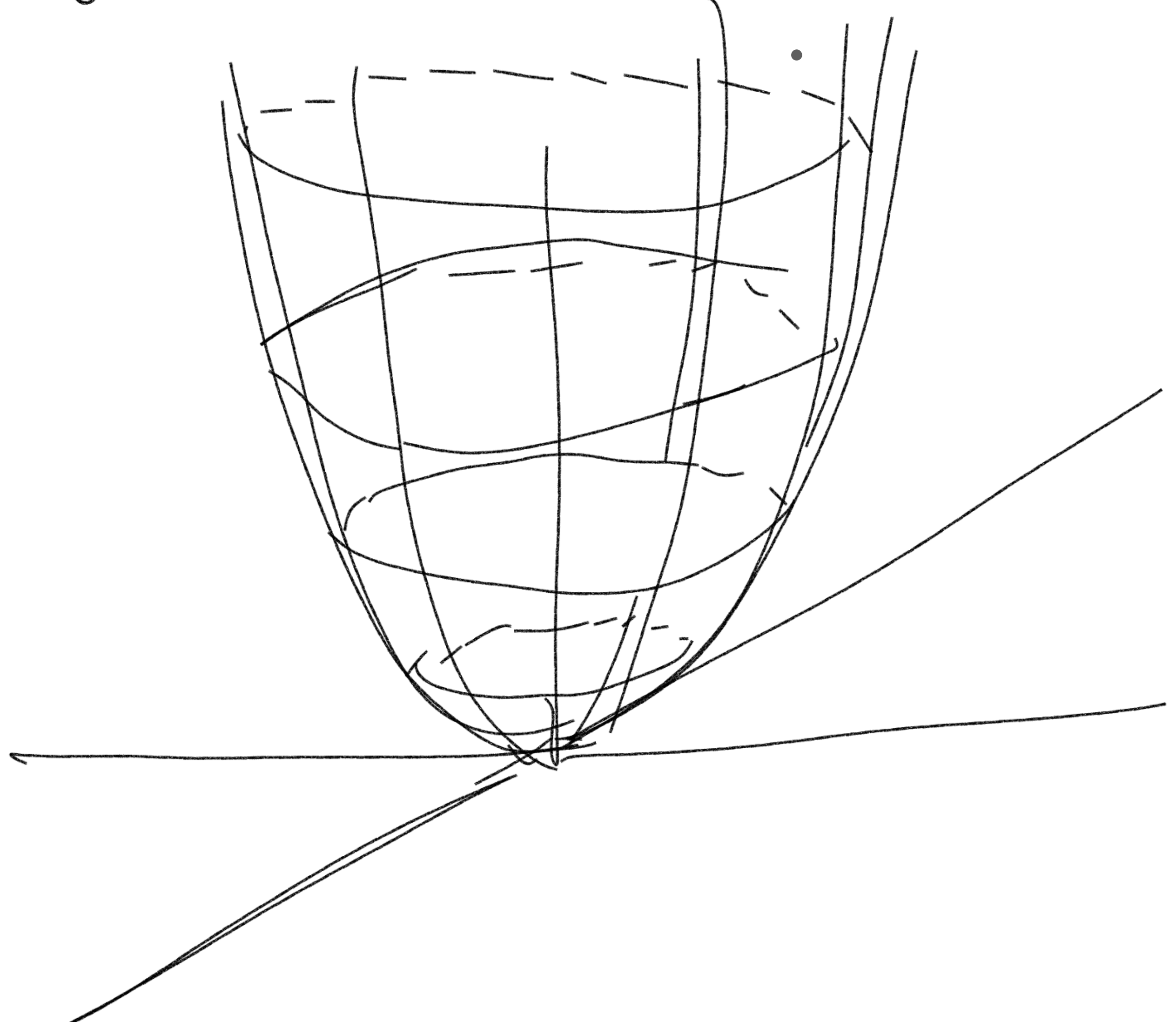
$$\begin{array}{ll} y = 0 & z = x^2 \rightarrow \\ y = 1 & z = x^2 + 1 \\ y = 2 & z = x^2 + 4 \\ \vdots & \vdots \end{array}$$





$$\begin{array}{ll} \leftarrow z = 0 & 0 = x^2 + y^2 \\ z = 1 & 1 = x^2 + y^2 \\ z = 2 & 2 = x^2 + y^2 \\ \vdots & \vdots \end{array}$$

Putting all the traces together:



Quadric Surfaces, pg 2, geogebra.

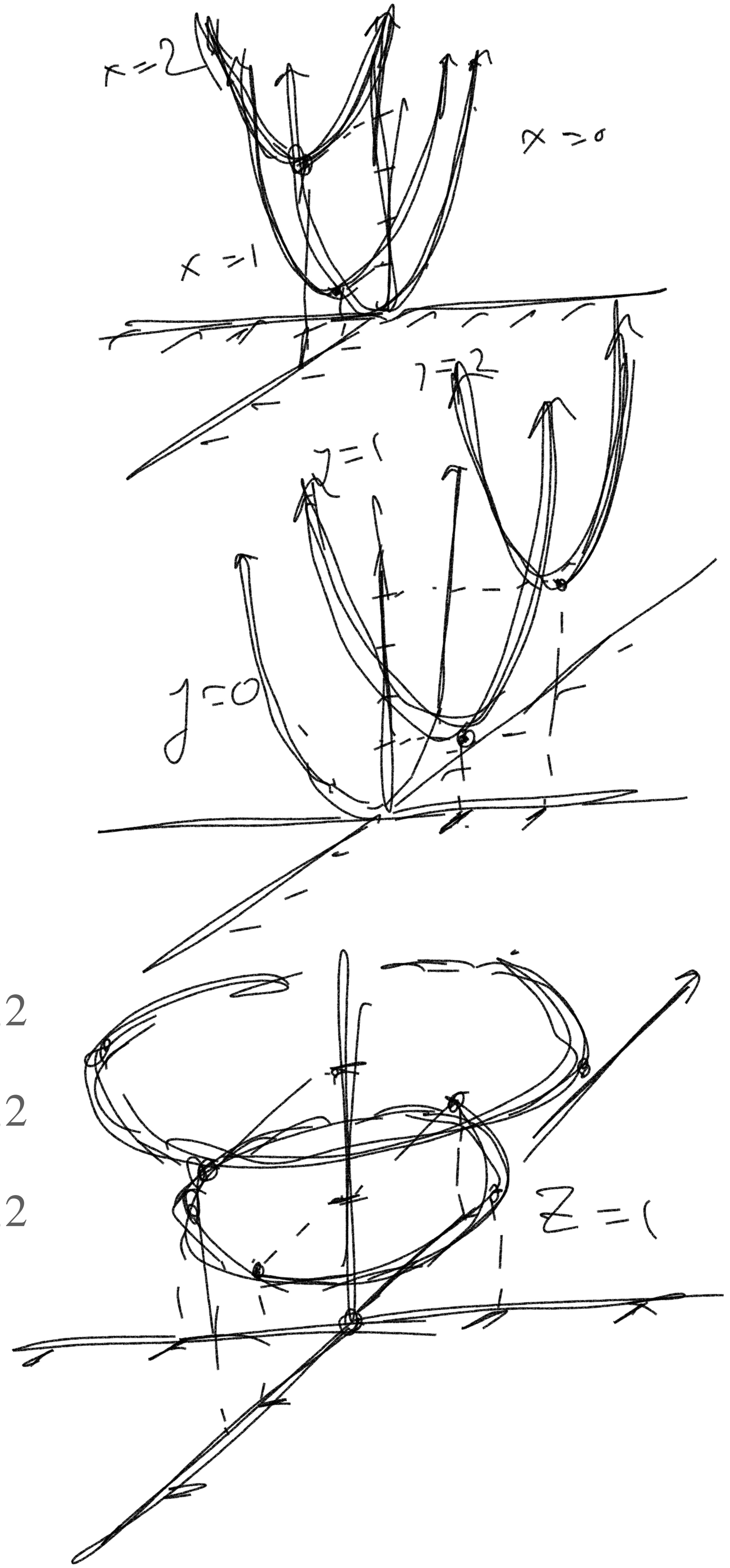
Geogebra is very helpful.

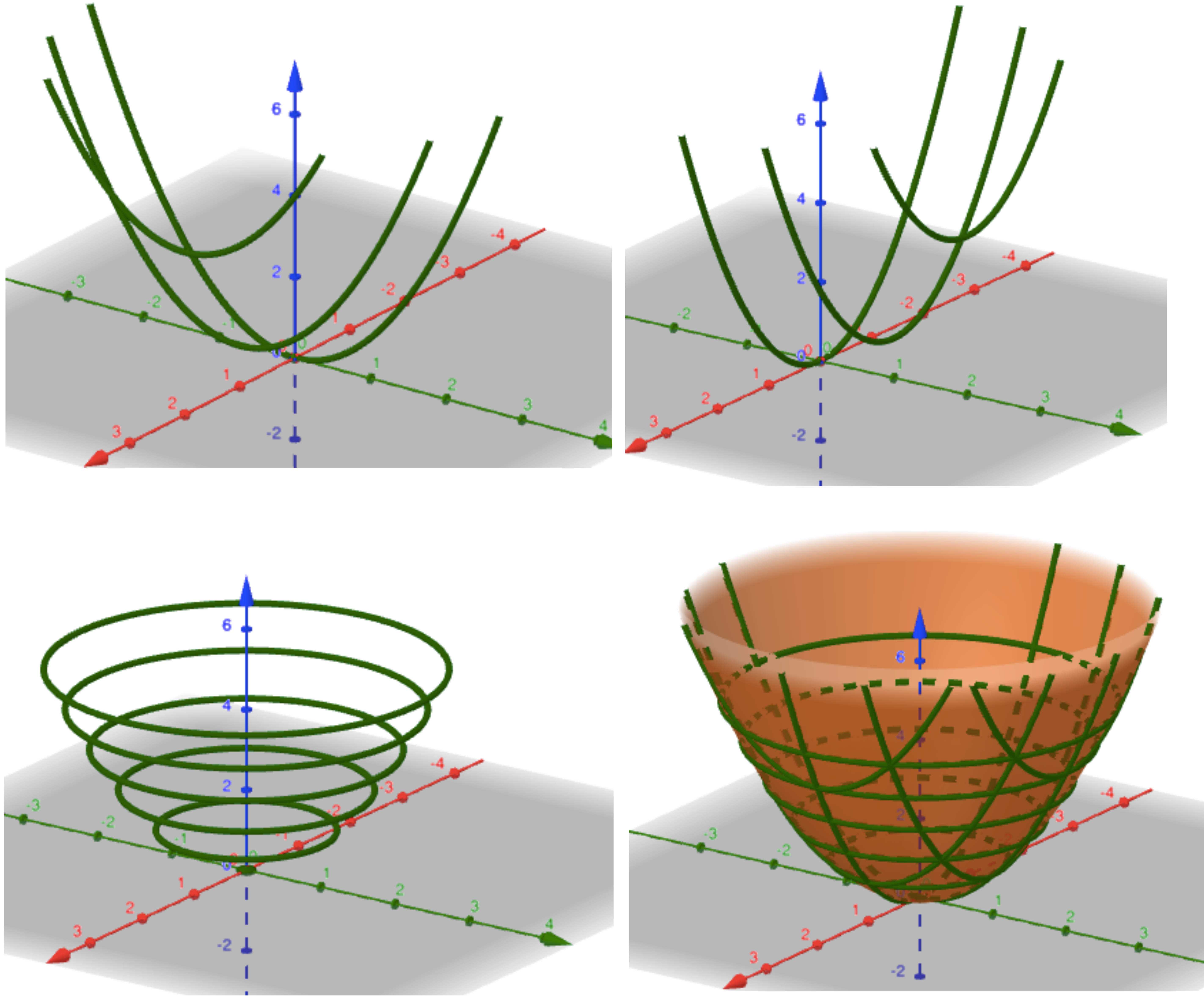
1. $z = x^2 + y^2$

$$\begin{array}{ll} x = 0 & z = y^2 \\ x = 1 & z = 1 + y^2 \\ x = 2 & z = 4 + y^2 \\ \vdots & \vdots \end{array}$$

$$\begin{array}{ll} y = 0 & z = x^2 \\ y = 1 & z = x^2 + 1 \\ y = 2 & z = x^2 + 4 \\ \vdots & \vdots \end{array}$$

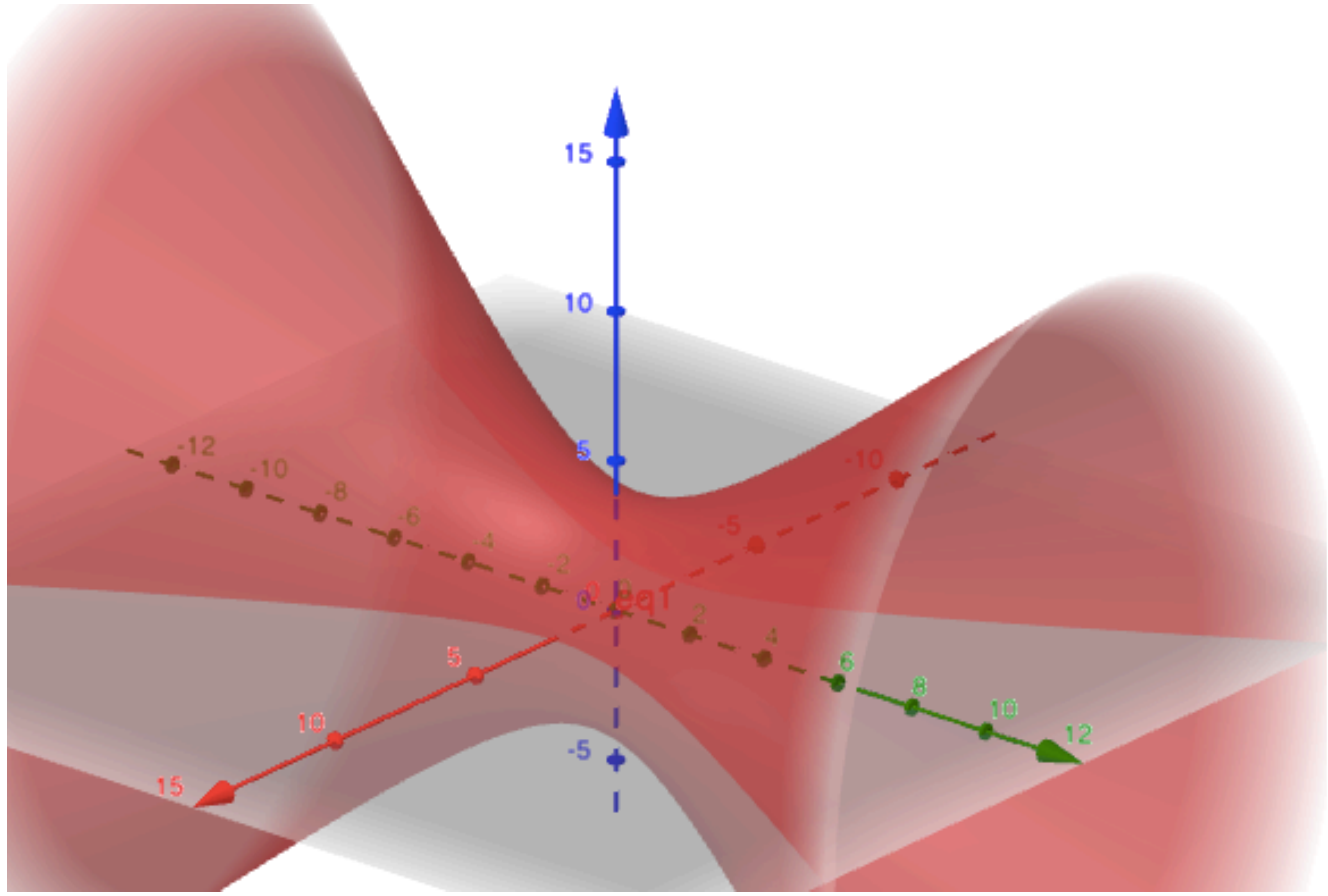
$$\begin{array}{ll} z = 0 & 0 = x^2 + y^2 \\ z = 1 & 1 = x^2 + y^2 \\ z = 2 & 2 = x^2 + y^2 \\ \vdots & \vdots \end{array}$$





Quadric Surfaces, pg 3, example2.

Example: $x^2 - \frac{y^2}{2} + \frac{z^2}{3} = 5$

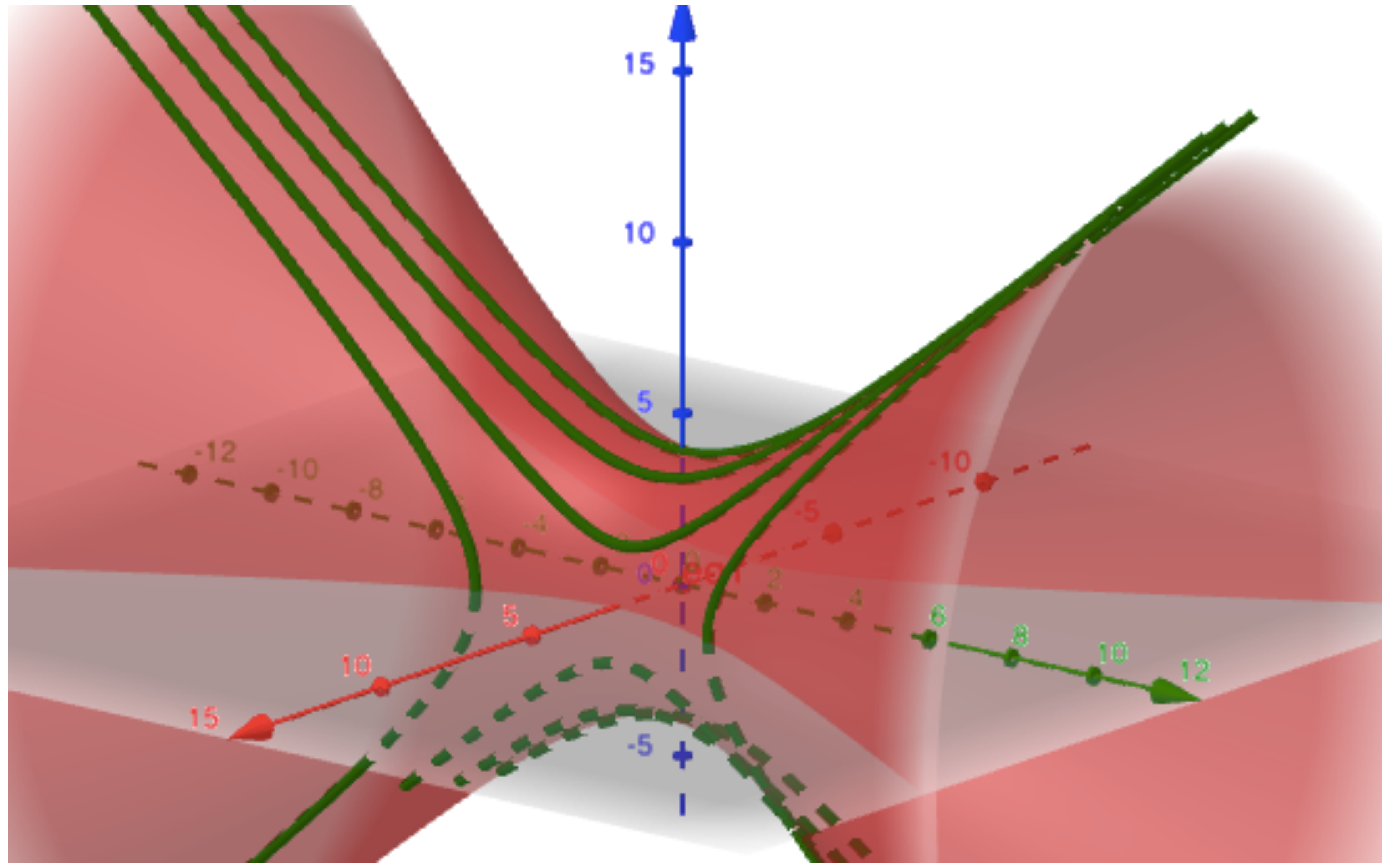


$$x = 0: \quad -\frac{y^2}{2} + \frac{z^2}{3} = 5$$

$$x = 1: \quad -\frac{y^2}{2} + \frac{z^2}{3} = 4$$

$$x = 2: \quad -\frac{y^2}{2} + \frac{z^2}{3} = 1$$

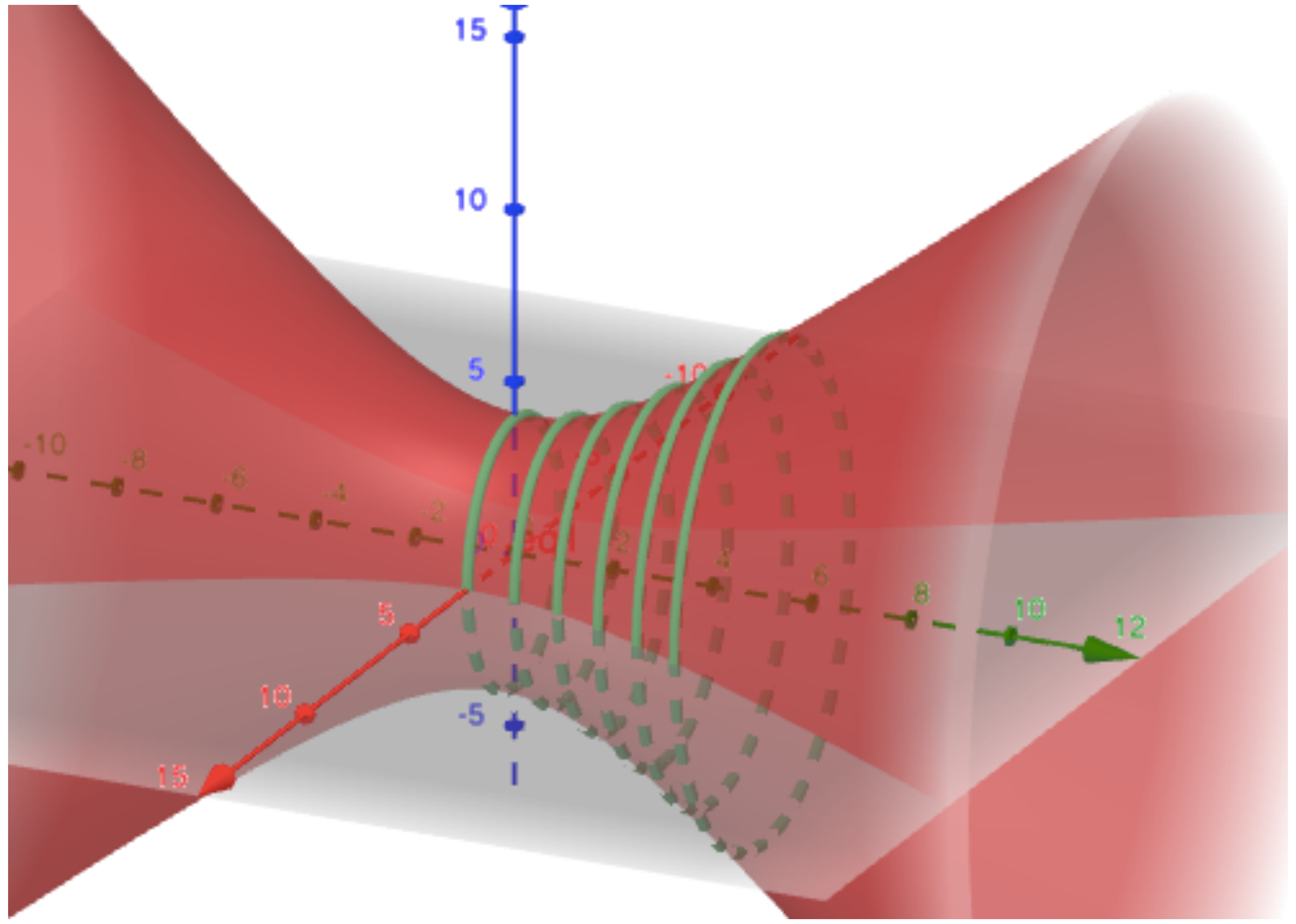
$$x = 3: \quad -\frac{y^2}{2} + \frac{z^2}{3} = -4$$



$$y = 0: \quad x^2 + \frac{z^2}{3} = 5$$

$$y = 1: \quad x^2 + \frac{z^2}{3} = 5.5$$

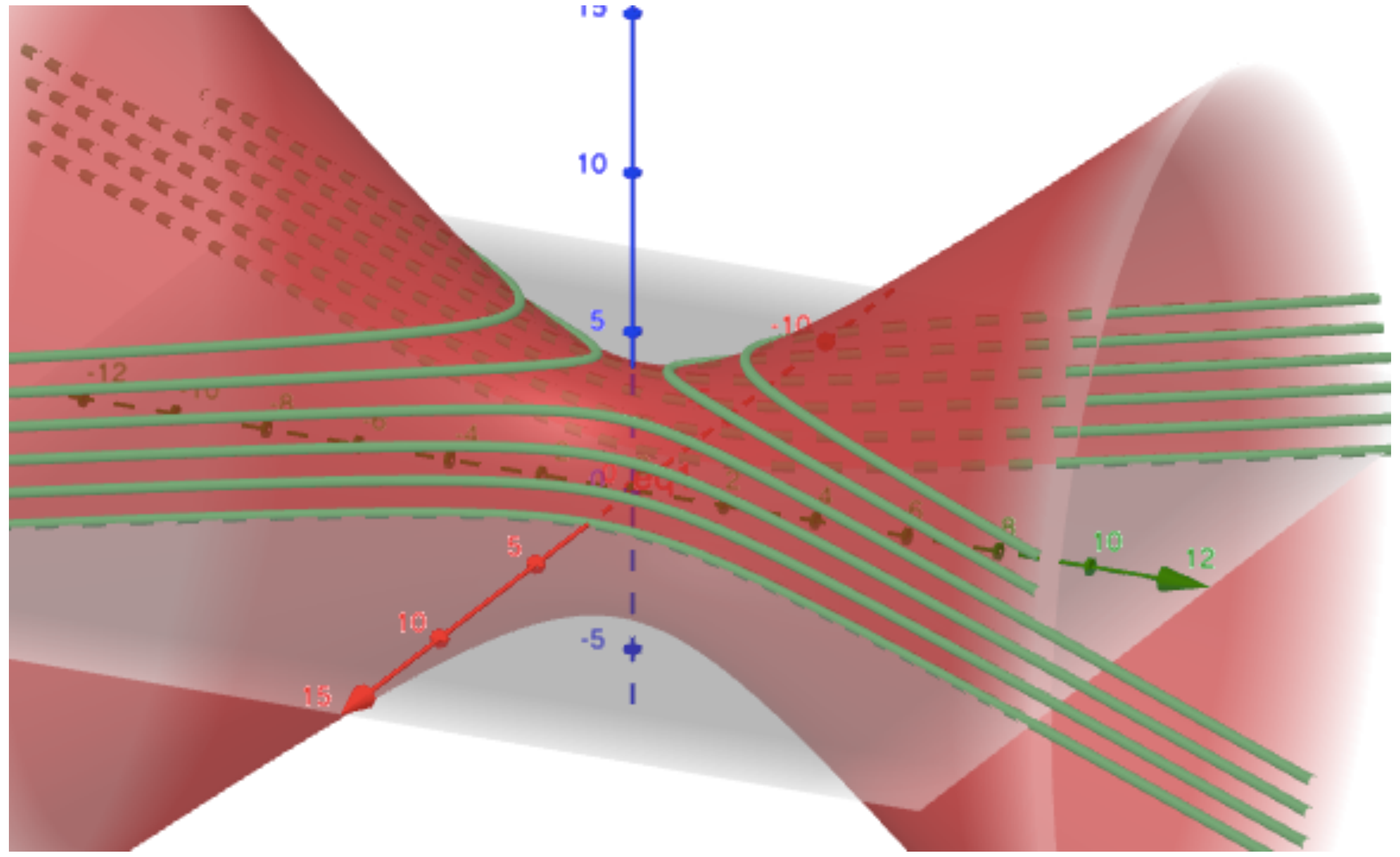
$$y = 2: \quad x^2 + \frac{z^2}{3} = 7$$



$$z = 0: \quad x^2 - \frac{y^2}{2} = 5$$

$$z = 1: \quad x^2 - \frac{y^2}{2} = \frac{14}{3}$$

$$z = 2: \quad x^2 - \frac{y^2}{2} = \frac{11}{3}$$



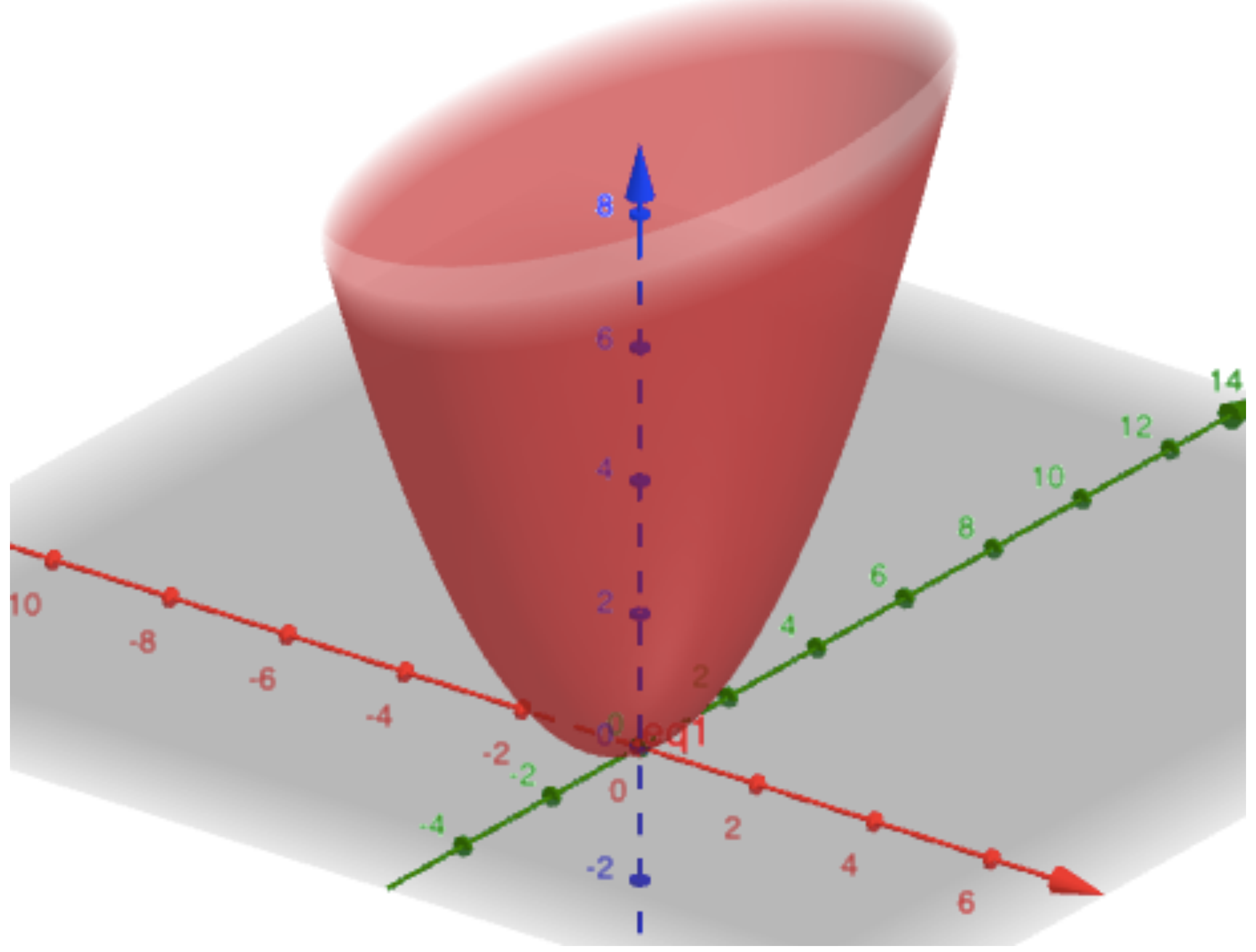
Quadric Surfaces, pg 4.
Standard examples.

Parabaloid:

$$\frac{x}{c} = \frac{y^2}{a^2} + \frac{z^2}{b^2}$$

$$\frac{y}{c} = \frac{x^2}{a^2} + \frac{z^2}{b^2}$$

$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

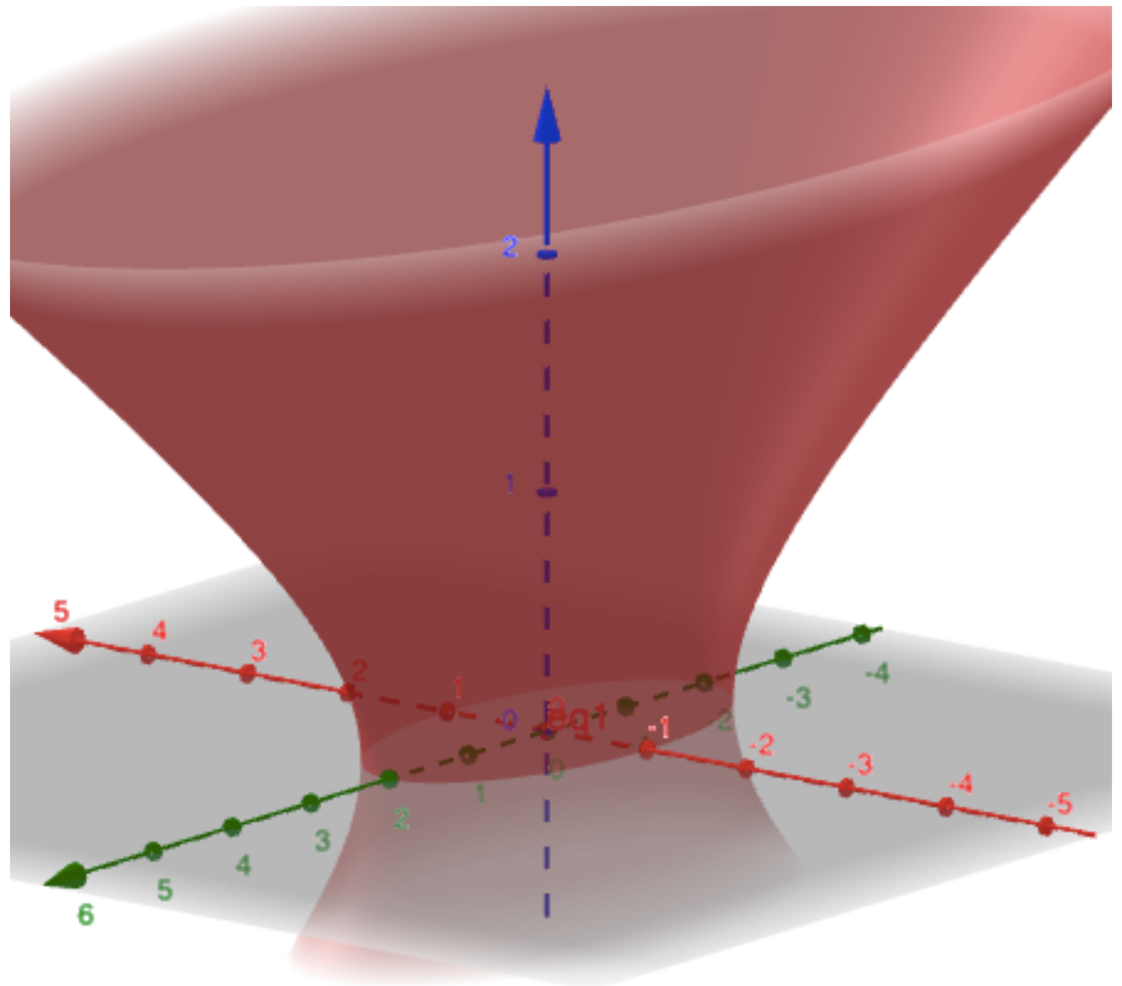


Hyperboloid (one-sheet)

$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

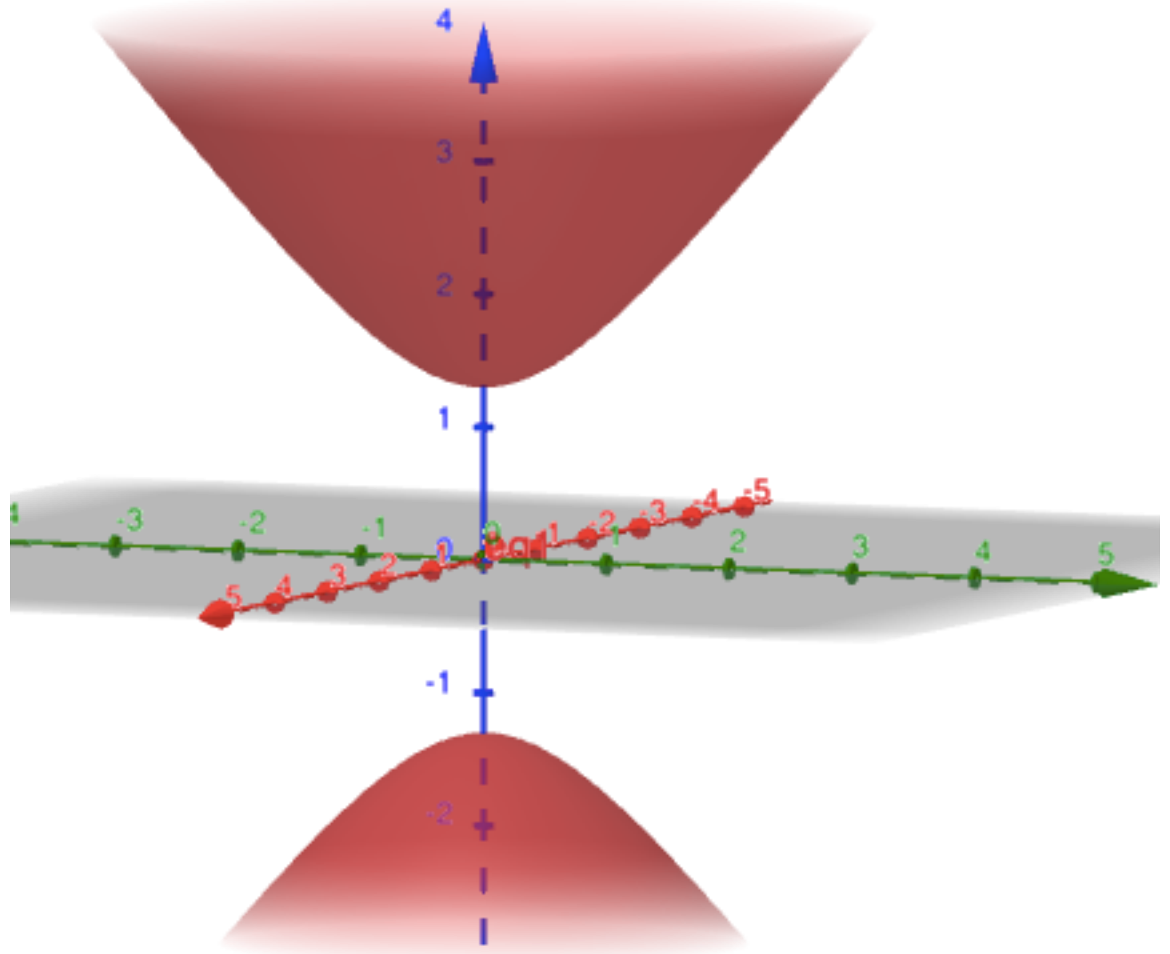


Hyperboloid (two-sheets)

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

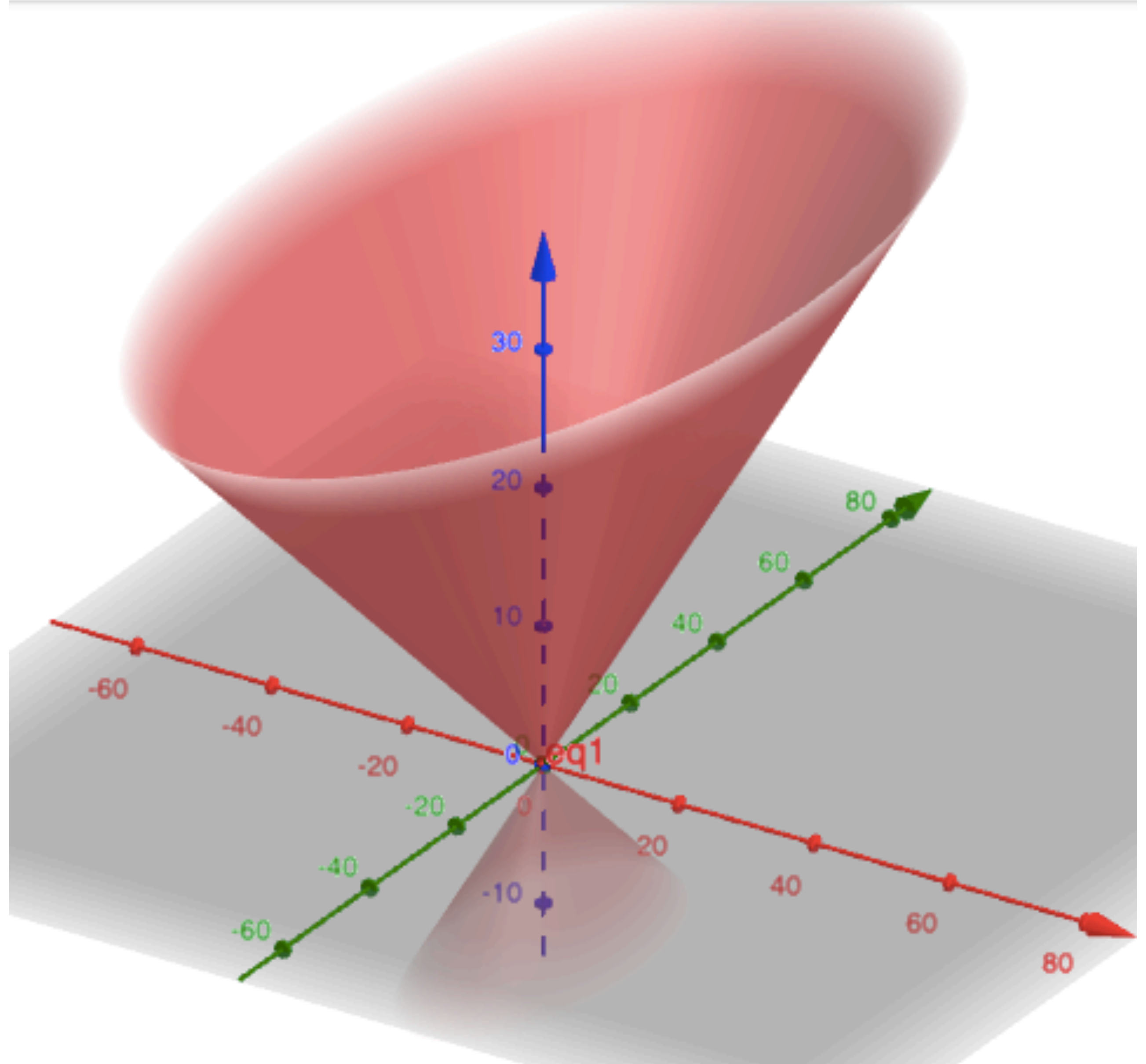


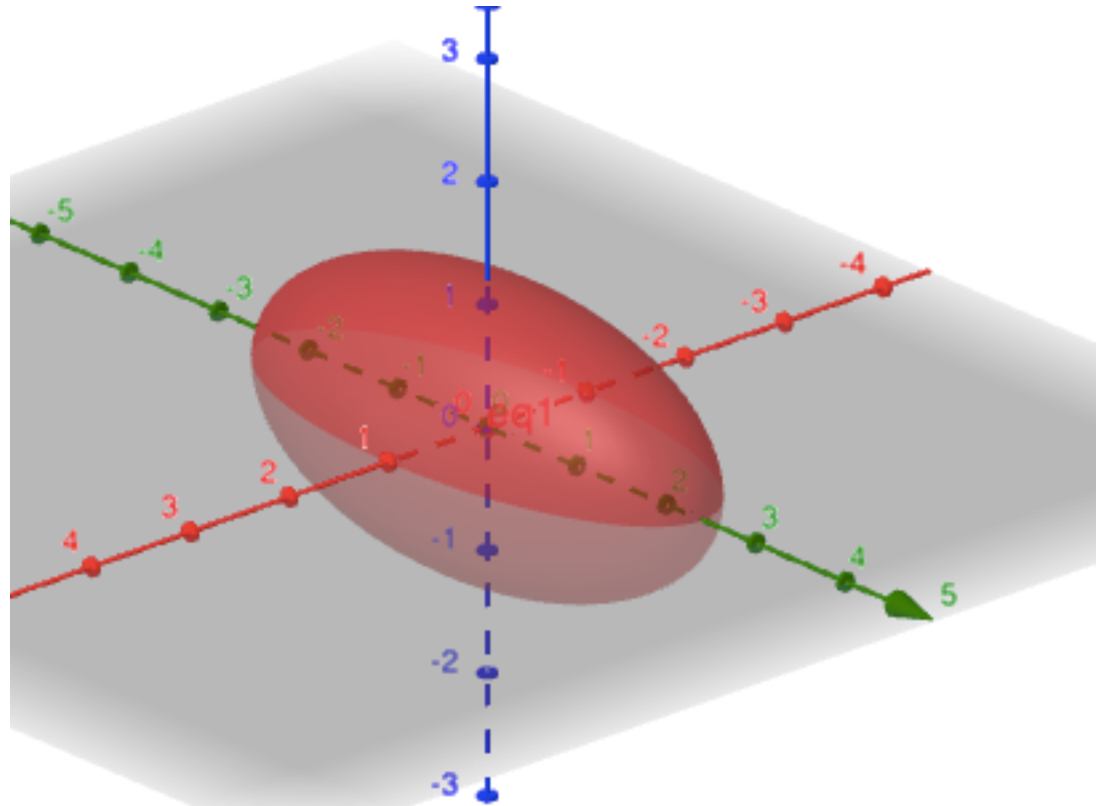
Cone:

$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$





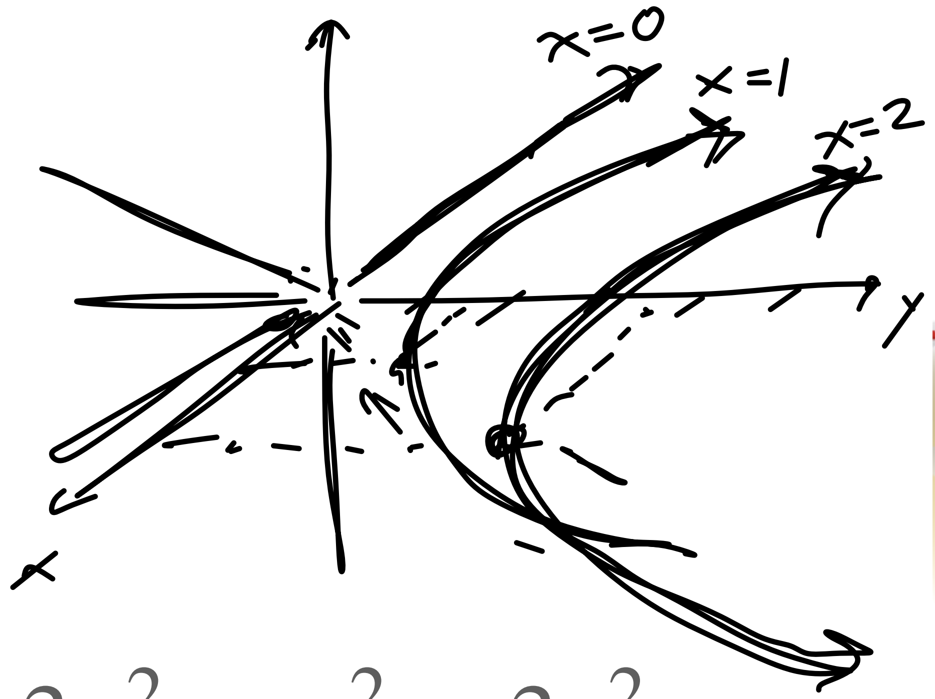
Ellipsoid

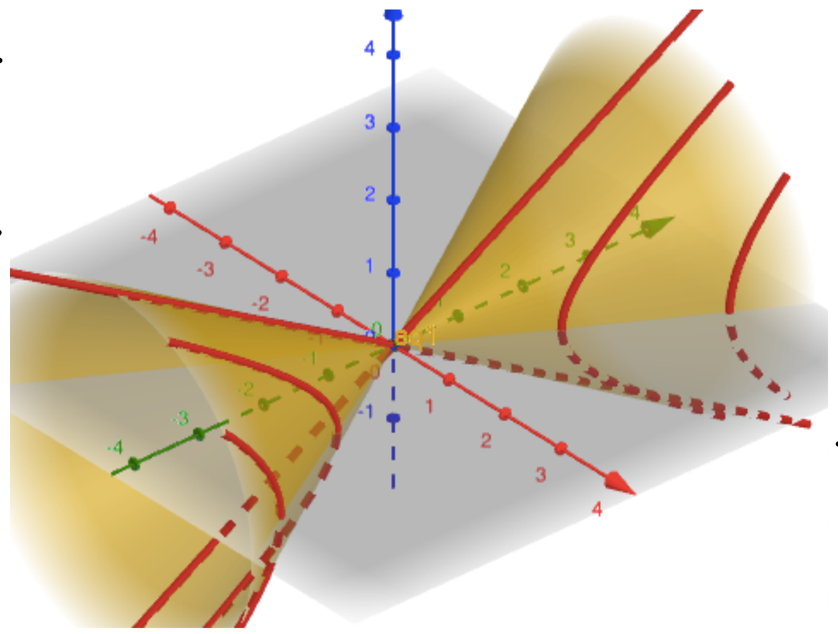
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Quadric Surfaces, pg 5, more examples.

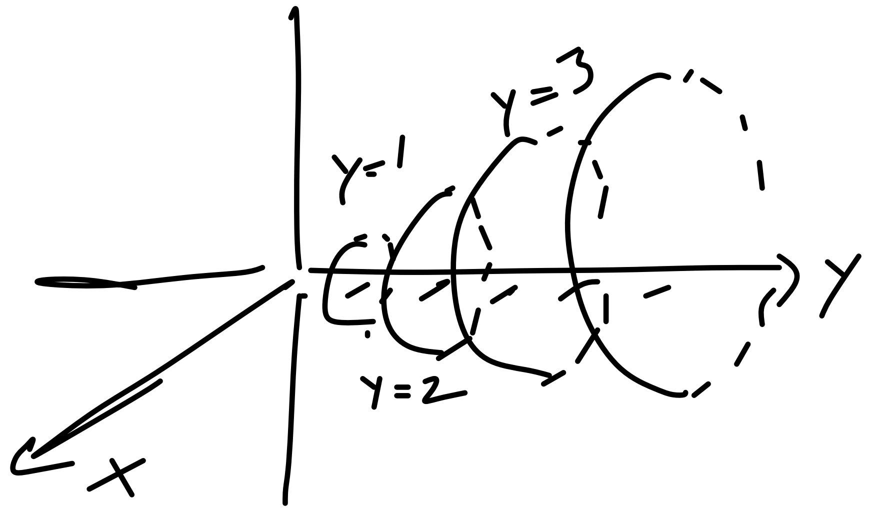
Draw some traces along the x,y and z-axes of the following.

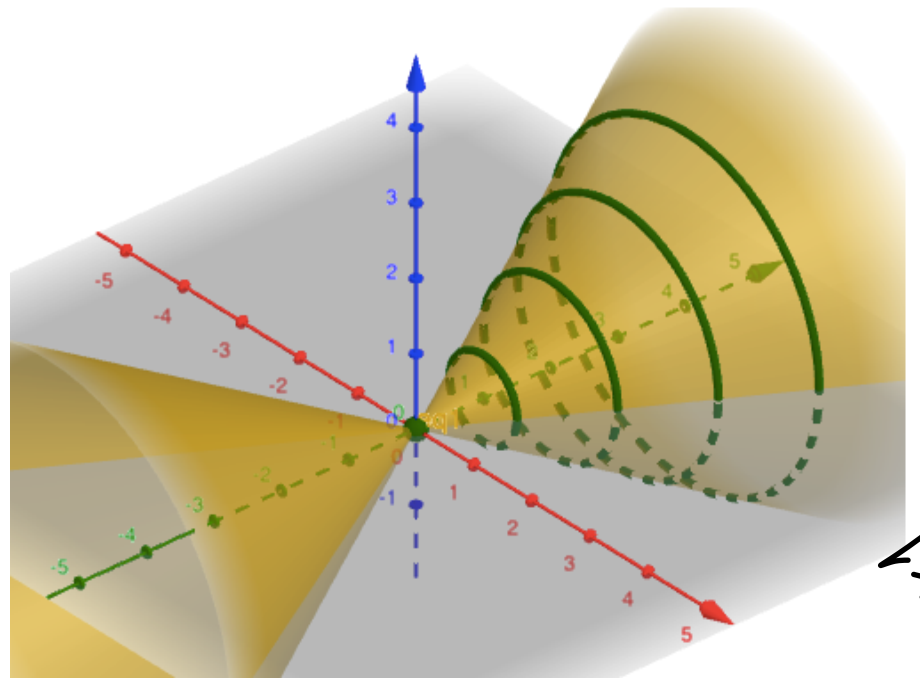
1. $3x^2 - y^2 + 3z^2 = 0$



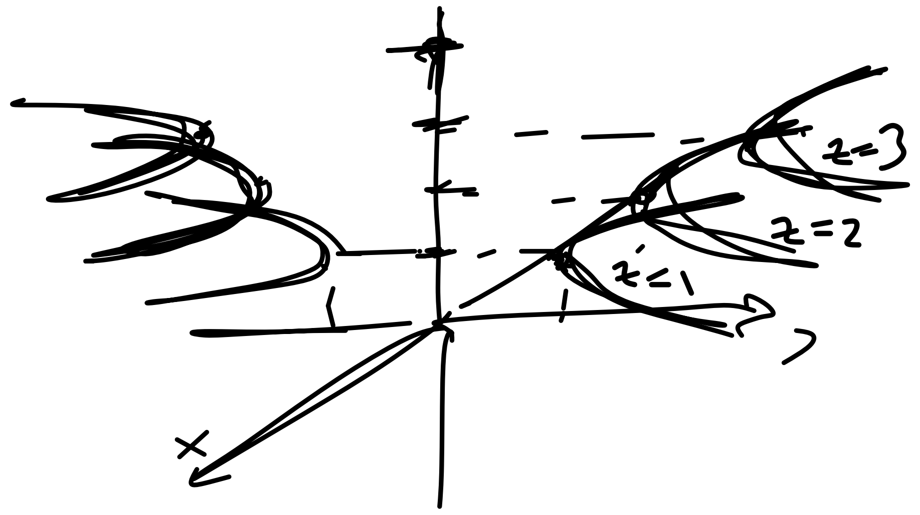


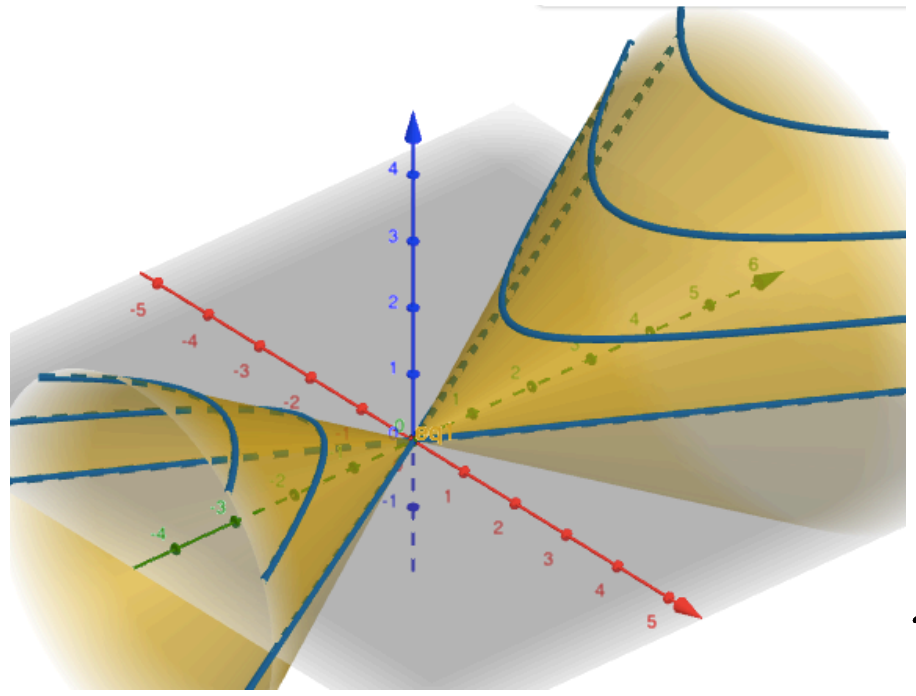
$3x^2 = y^2 - 3z^2$





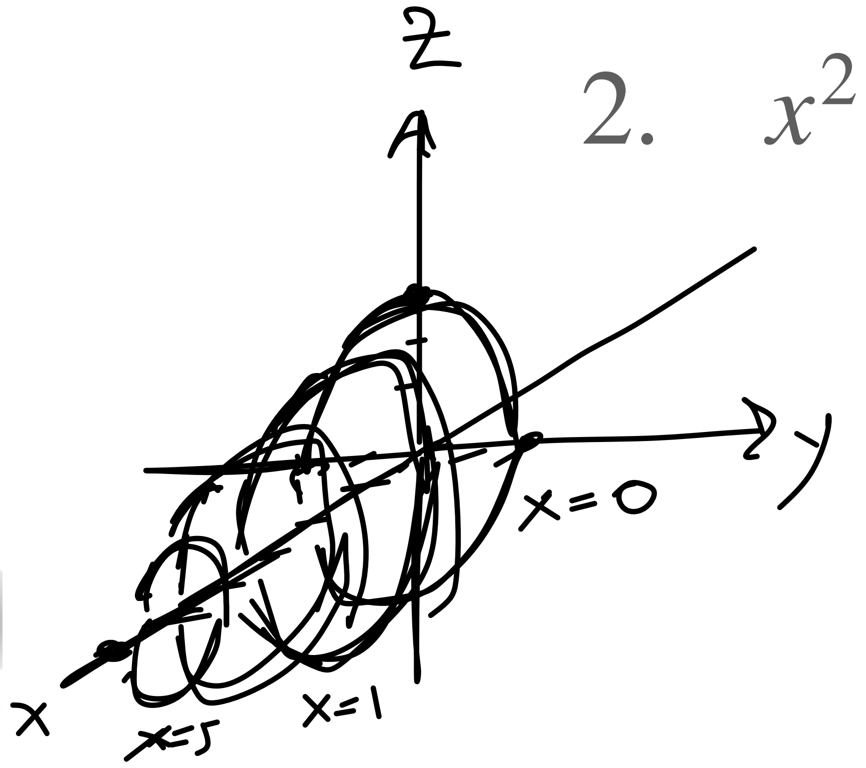
$y^2 = 3x^2 + 3z^2$

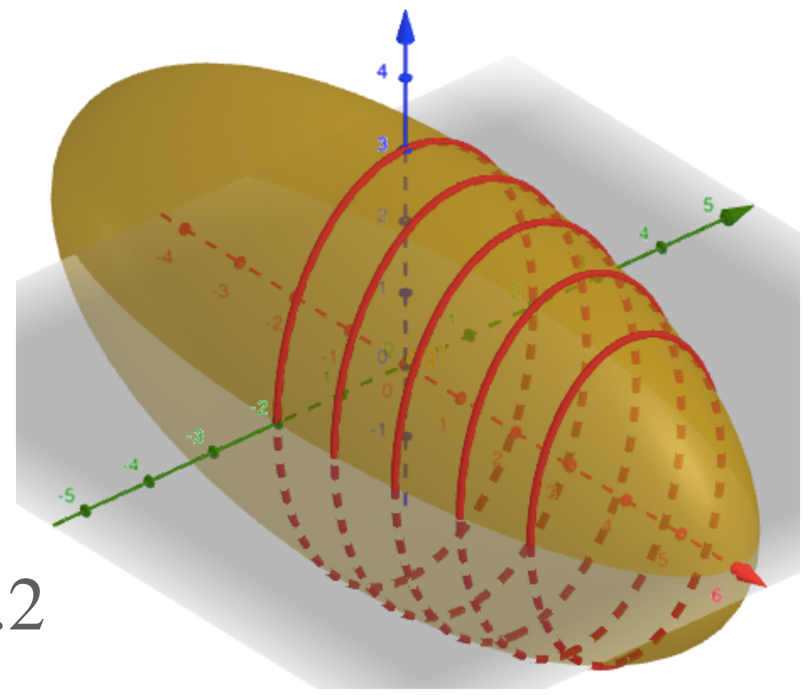




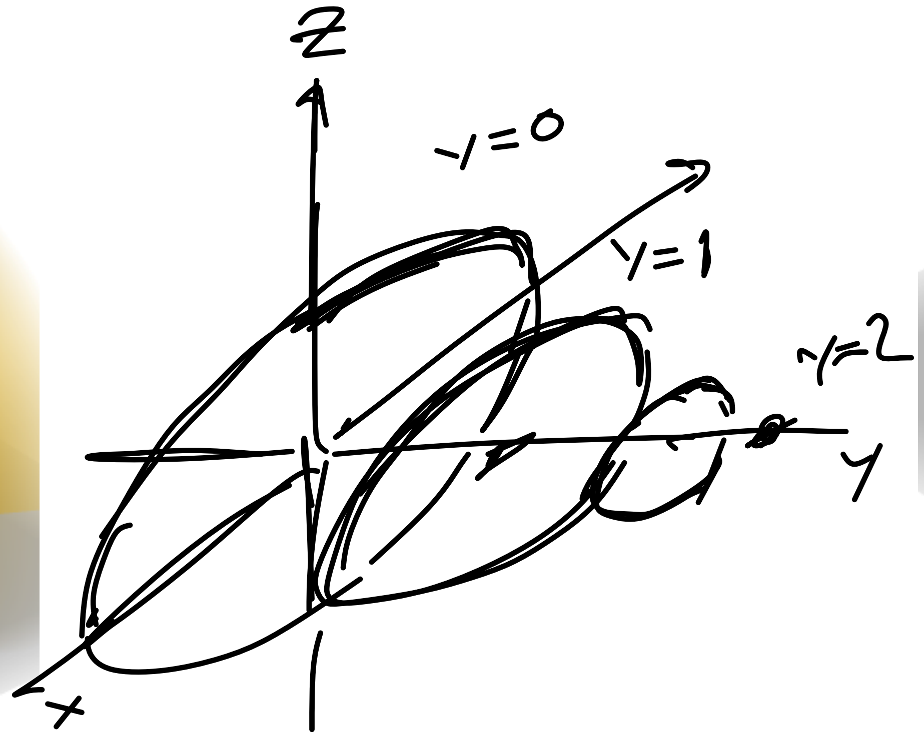
$3z^2 = -3x^2 + y^2$

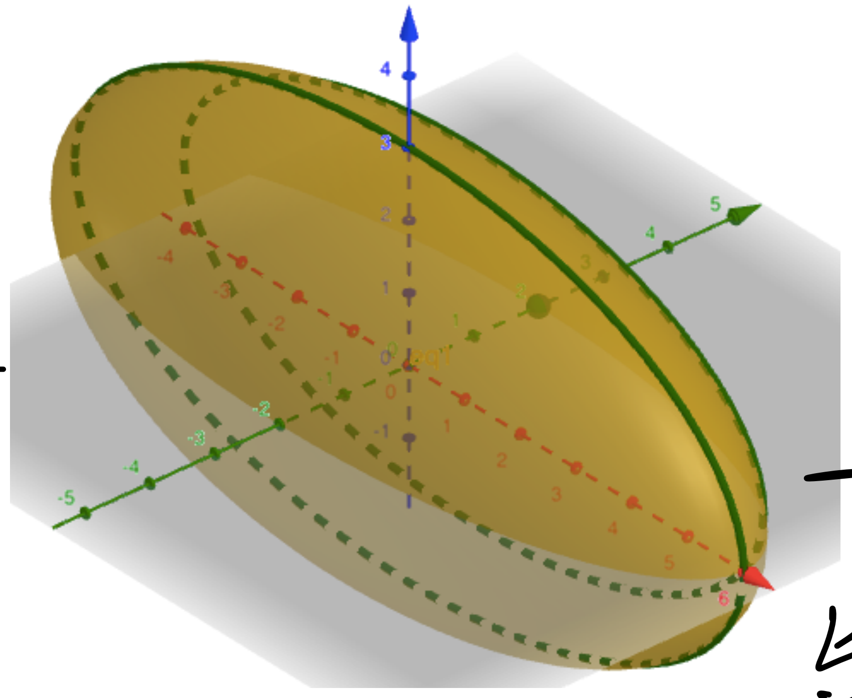
2. $x^2 + 9y^2 + 4z^2 = 36$



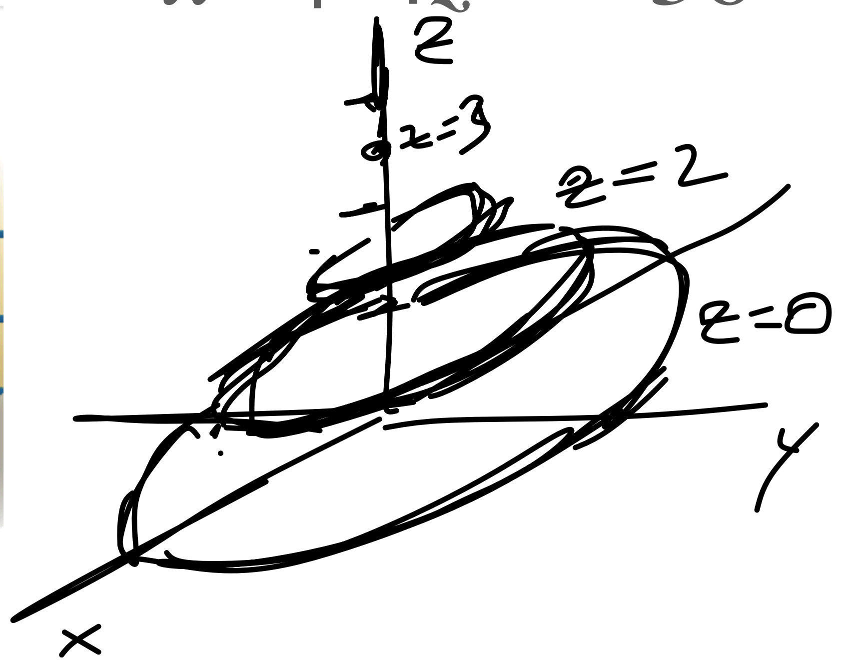


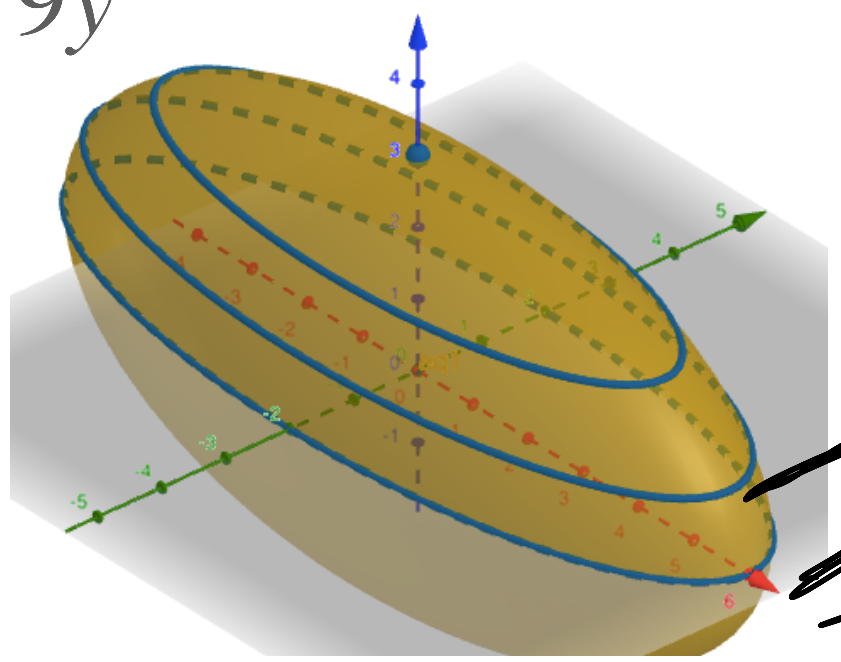
$9y^2 + 4z^2 = 36 - x^2$





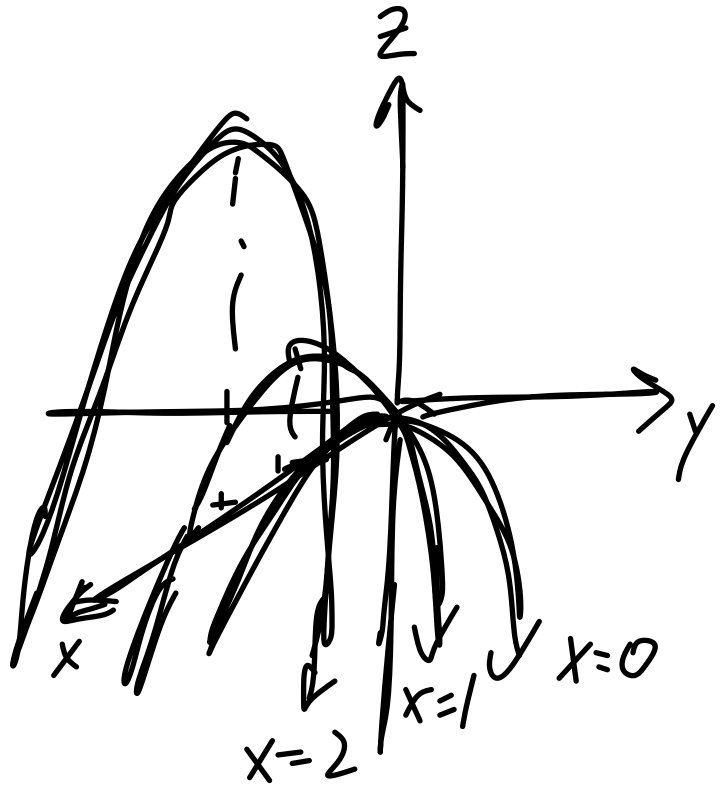
$x^2 + 4z^2 = 36 - 9y^2$

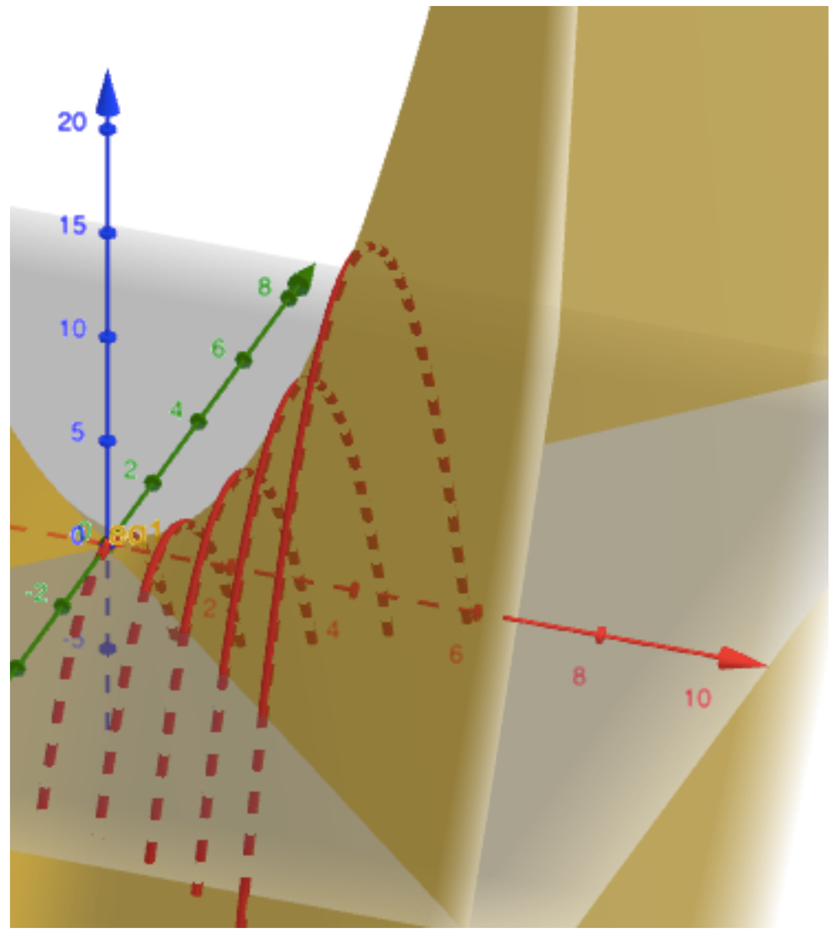


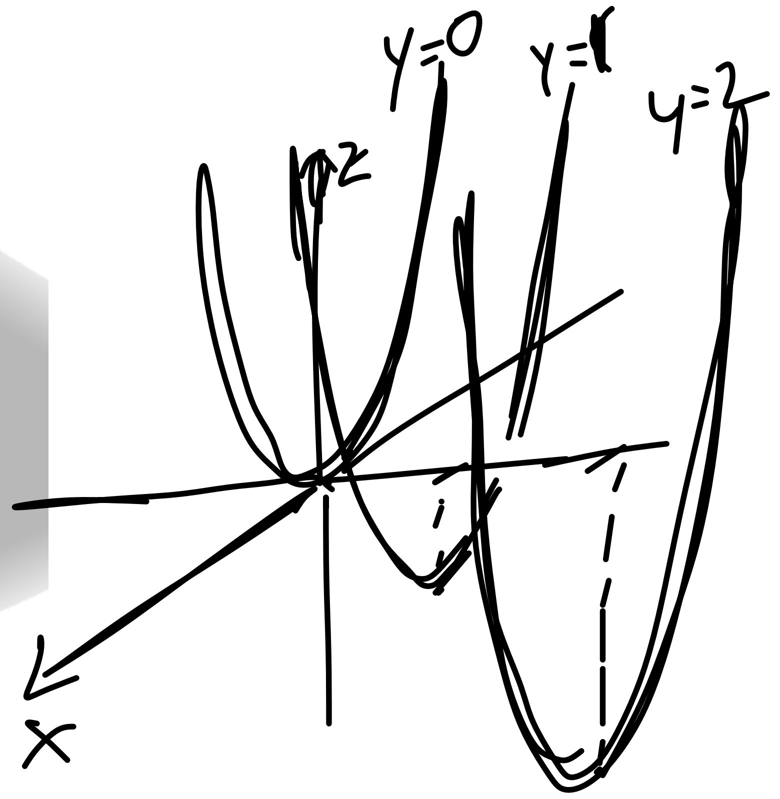


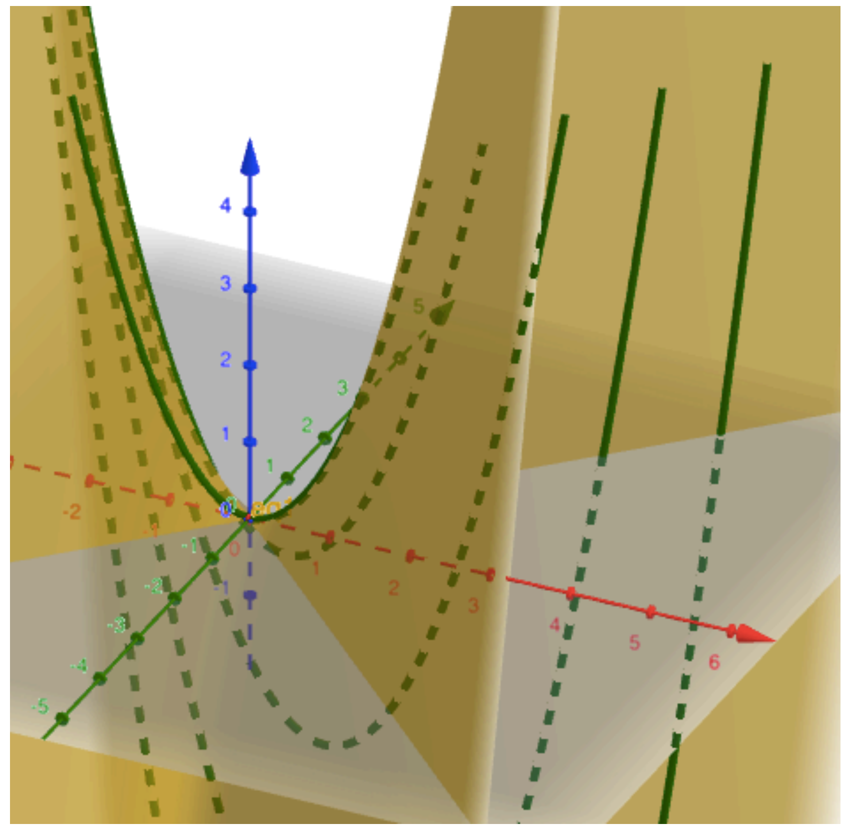
$x^2 + 9y^2 = 36 - 4z^2$

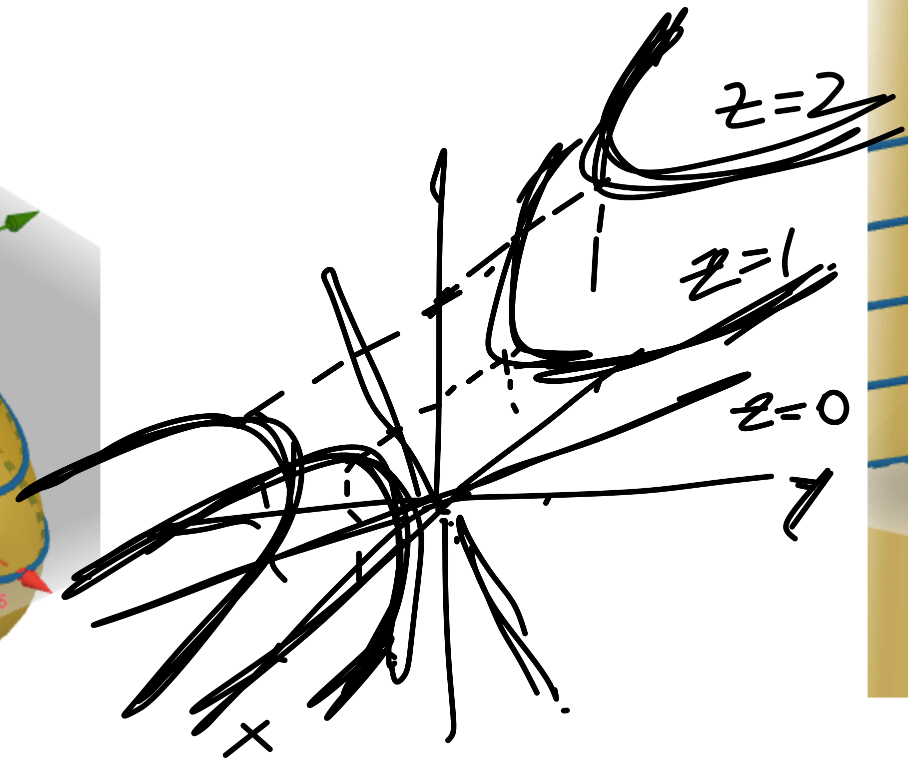
3. $z = x^2 - y^2$

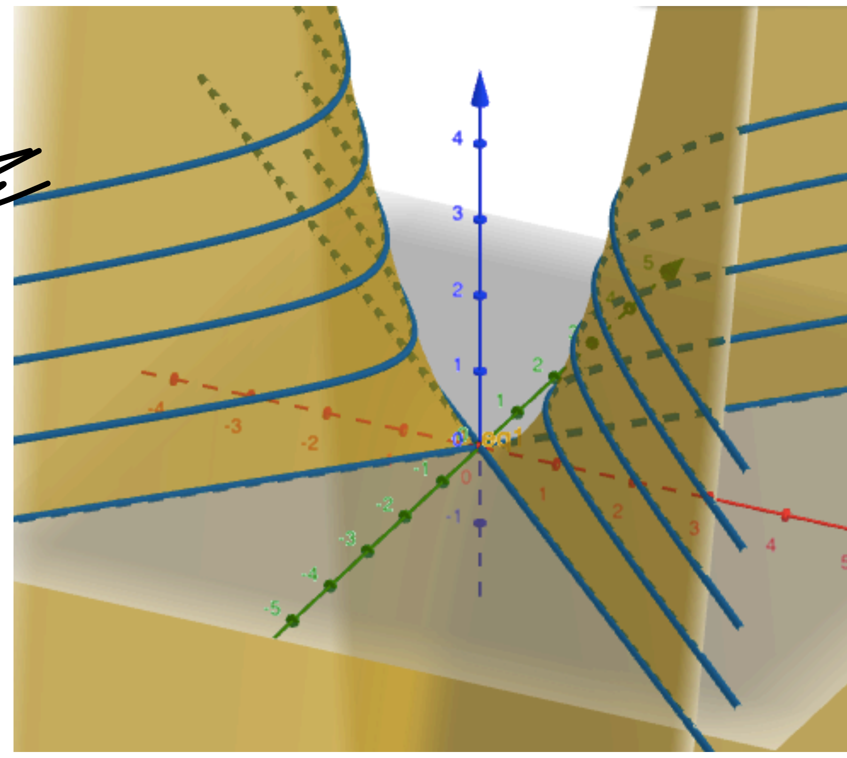












Curves in Space. Vector-Valued Functions

You have seen lots of *real-valued* functions (aka *scalar* functions): functions that output real numbers.

e.g. e^x, $\sin(\theta)$, $\ln(x)$, $x^3, \ldots$

In our class we'll talk more about *vector-valued* functions: functions that output vectors.

We already know one such kind of function. $\mathbf{L}(t) = \mathbf{v} + t\mathbf{u} = < v_x + t \cdot u_x, v_y + t \cdot u_y, v_z + t \cdot v_z >$

More generally, vector-valued functions can have any kind of scalar component functions:

$\mathbf{r}(t) = < x(t), y(t), z(t) >$

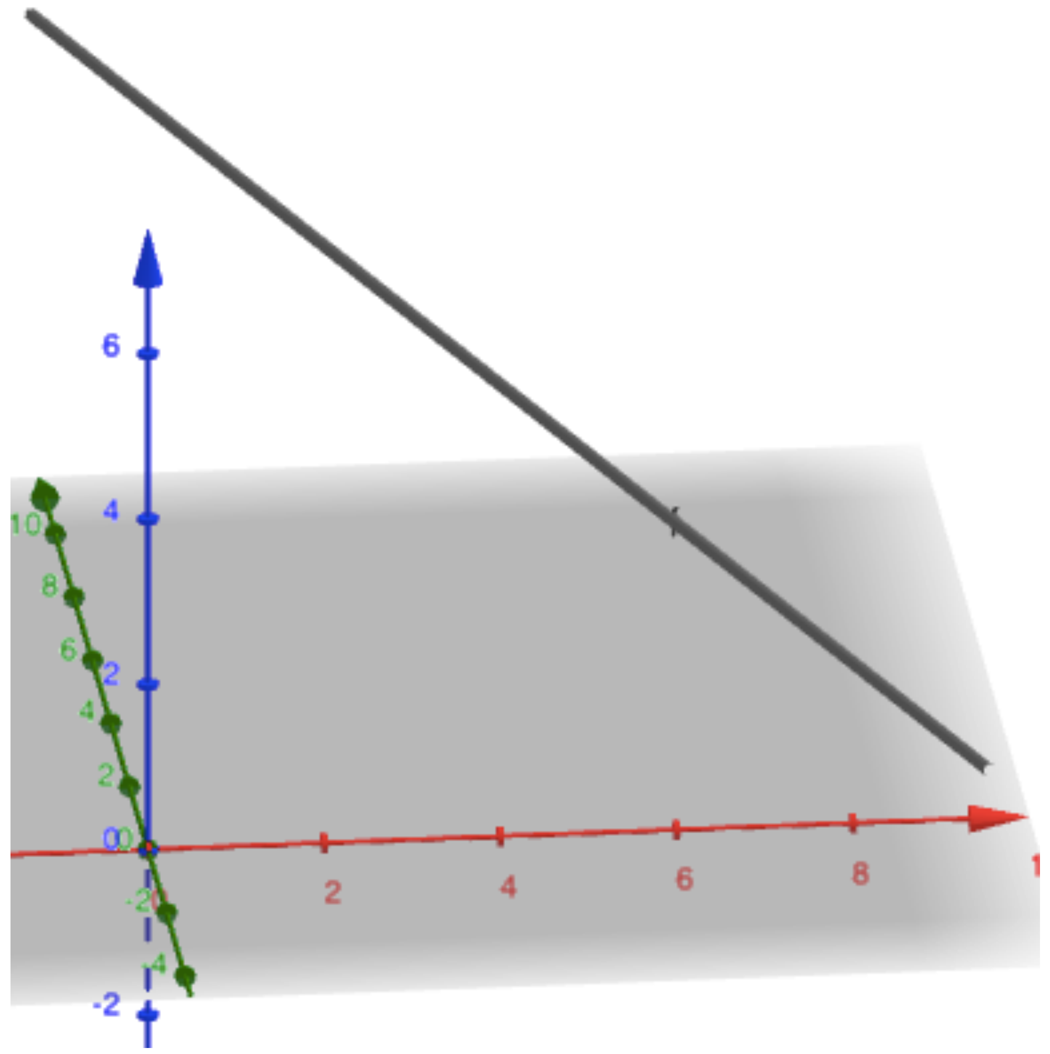
The scalar functions $x(t), y(t), z(t)$ may be any of the above real-valued functions.

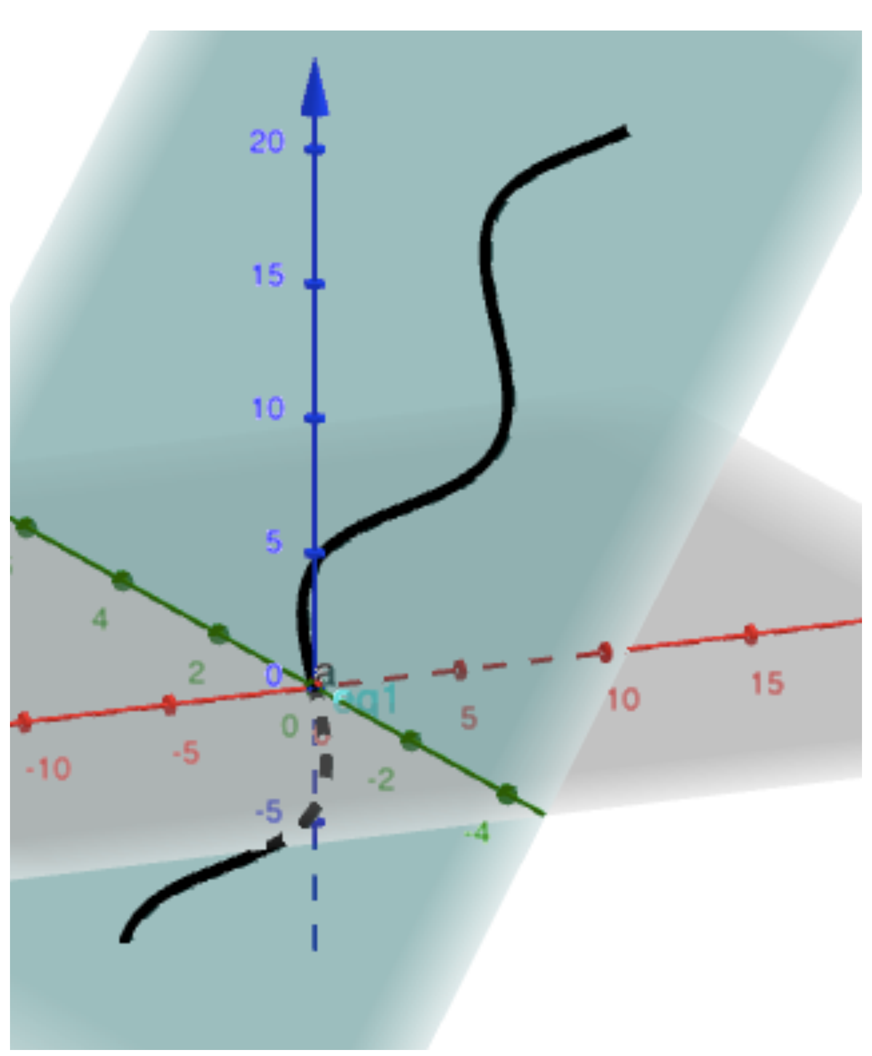
Examples.

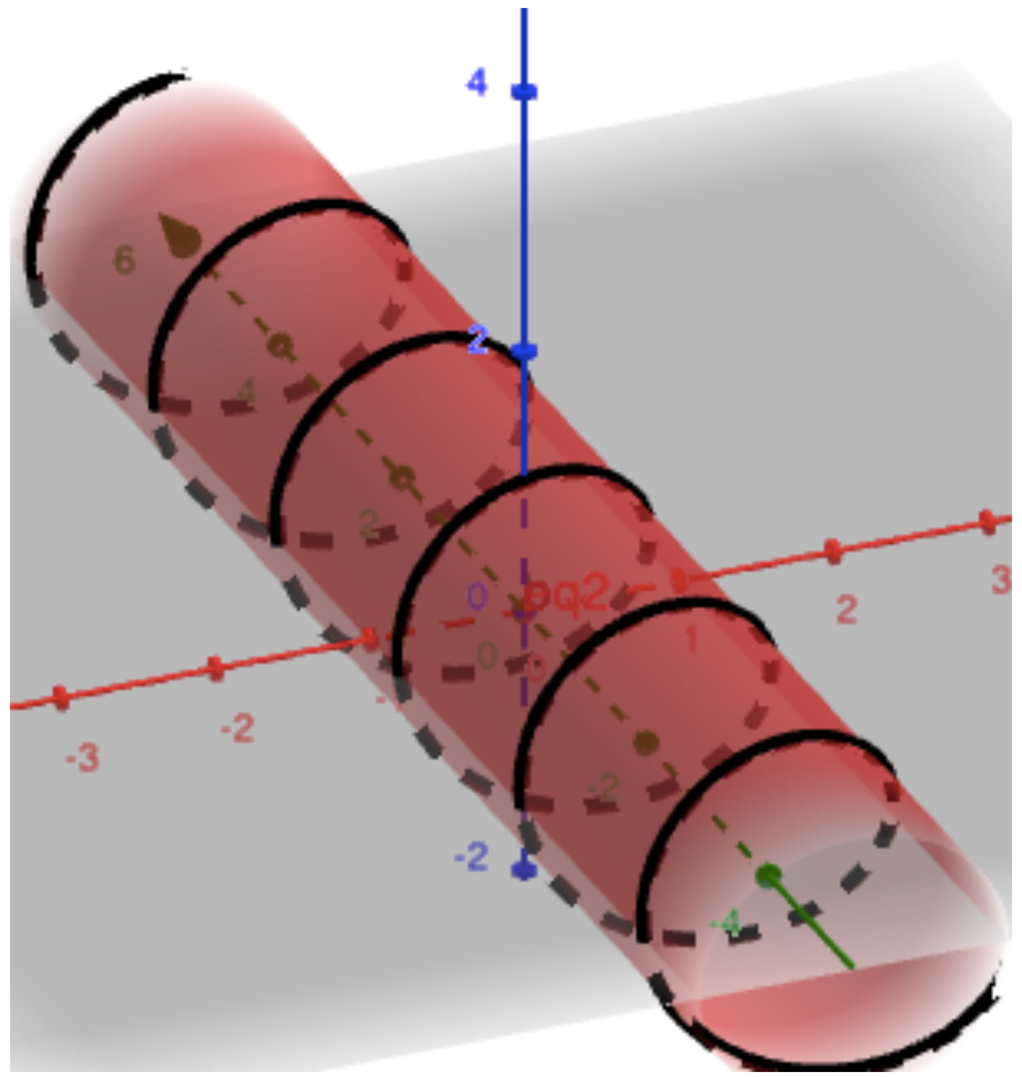
1. $\mathbf{r}(t) = < 5 - t, 2 + 3t, 4 >$

2. $\mathbf{r}(t) = < t, \sin(t), 2t >$

3. $\mathbf{r}(t) = < \sin(\pi t), t, \cos(\pi t) >$







Curves in Space, Parameters.

$x(t), y(t), z(t)$ are also called the *parametric equations* of the curve.

The independent variable, often t or s, is called a parameter.

Many different vector-valued functions can describe the same curves in space!

This can come from different *parameterizations* of the same cuvre.

These curves are graphs of the vector-values function $t \mapsto \mathbf{r}(t) = < x(t), y(t), z(t) >$

The input variable, the parameter t, is hard to see.

Think of t as a particular time when a particle is located at some point on the curve.

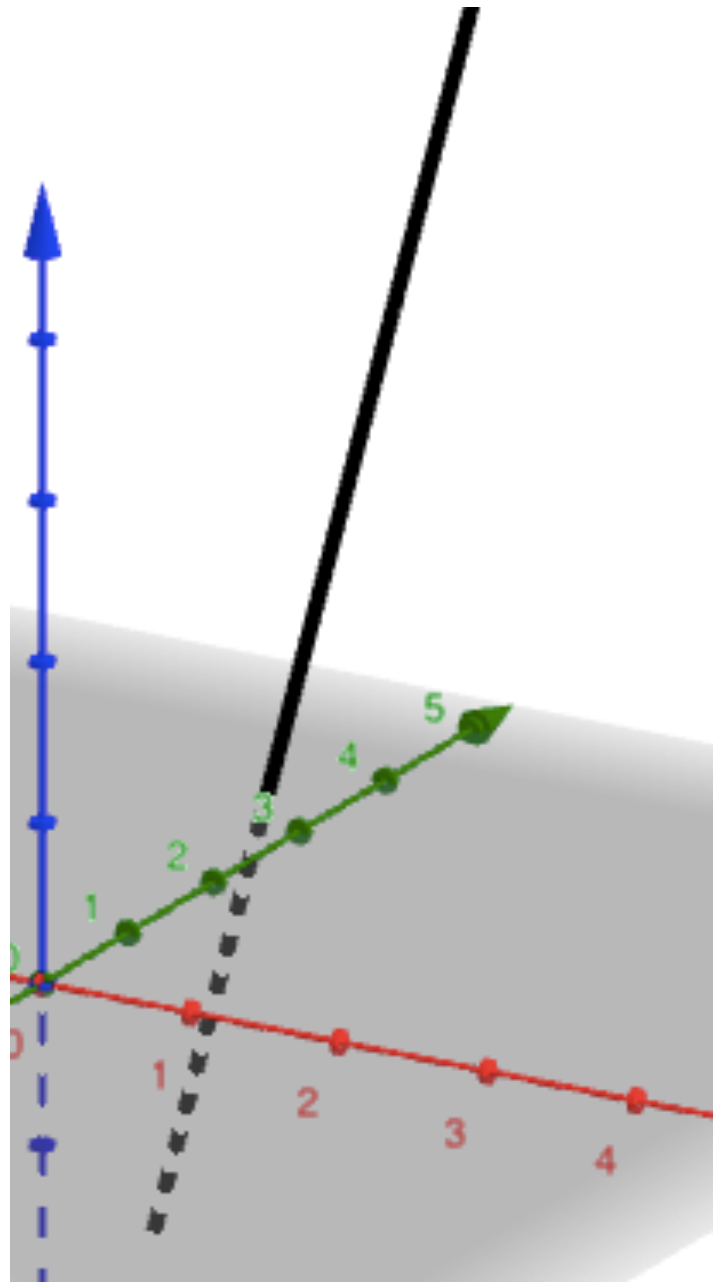
The compilation of all the particle's locations over time constitutes the curve that you see.

Different parameterizations of the same curve feature particles traversing the curve at different rates, or different directions.

Examples.

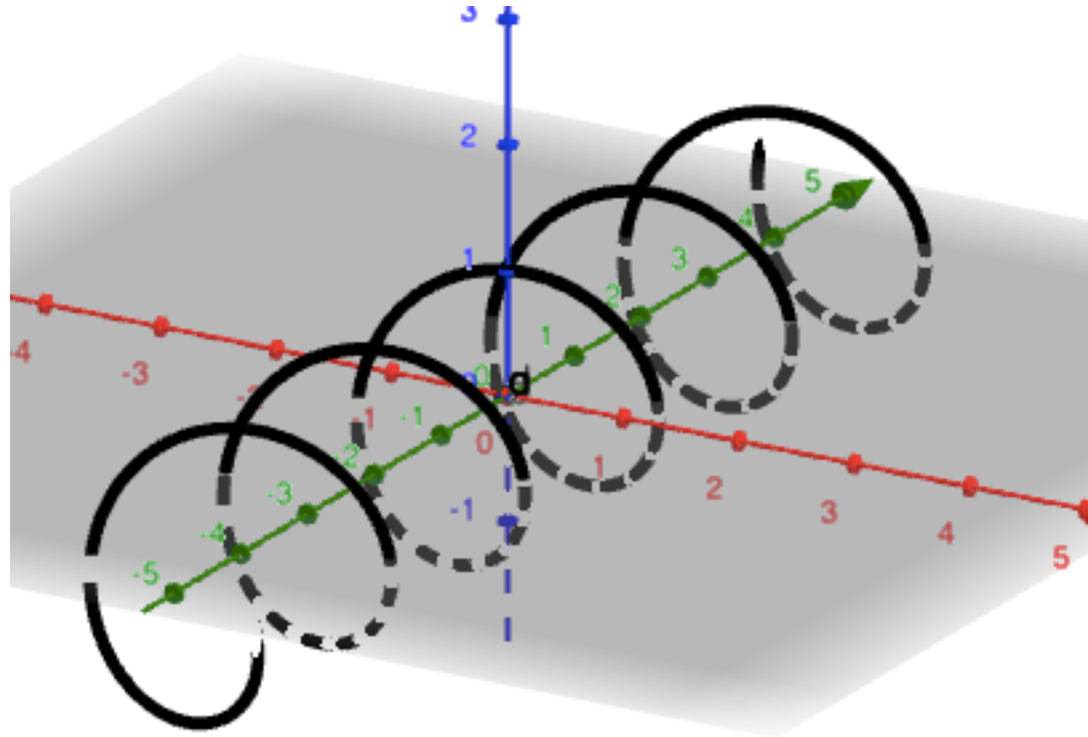
1.

$$\mathbf{r}_1(t) = < 1+t, 2-t, 3+2t >$$
$$\mathbf{r}_2(t) = < 1+2t, 2-2t, 3+4t >$$
$$\mathbf{r}_3(t) = < 2+t, 1-t, 5+2t >$$



2.

$$\mathbf{r}_1(t) = < \sin(\pi t), t, \cos(\pi t) >$$
$$\mathbf{r}_2(t) = < \sin(t), t/\pi, \cos(t) >$$
$$\mathbf{r}_3(t) = < \sin(\pi t^2), t^2, \cos(\pi t^2) >$$



See some particle's race, link: [Racing Particles.](#) →

Curves in Space, Projections.

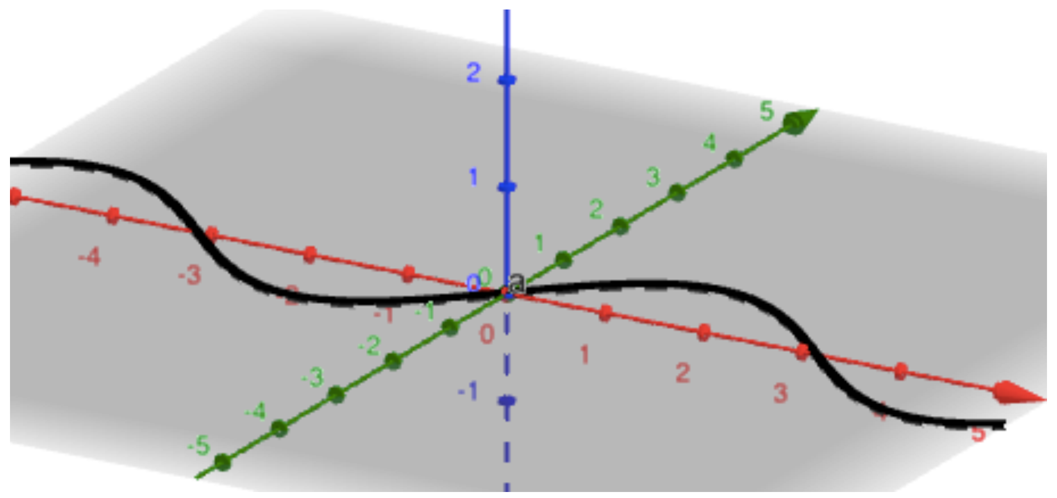
By ignoring a component function, we can project a curve onto the xy, xz, or yz planes.

Example1. $\mathbf{r}(t) = <t,\ \sin(t), 2\cos(t)>$

$$= t\mathbf{i} + \sin(t)\mathbf{j} + 2\cos(t)\mathbf{k}$$

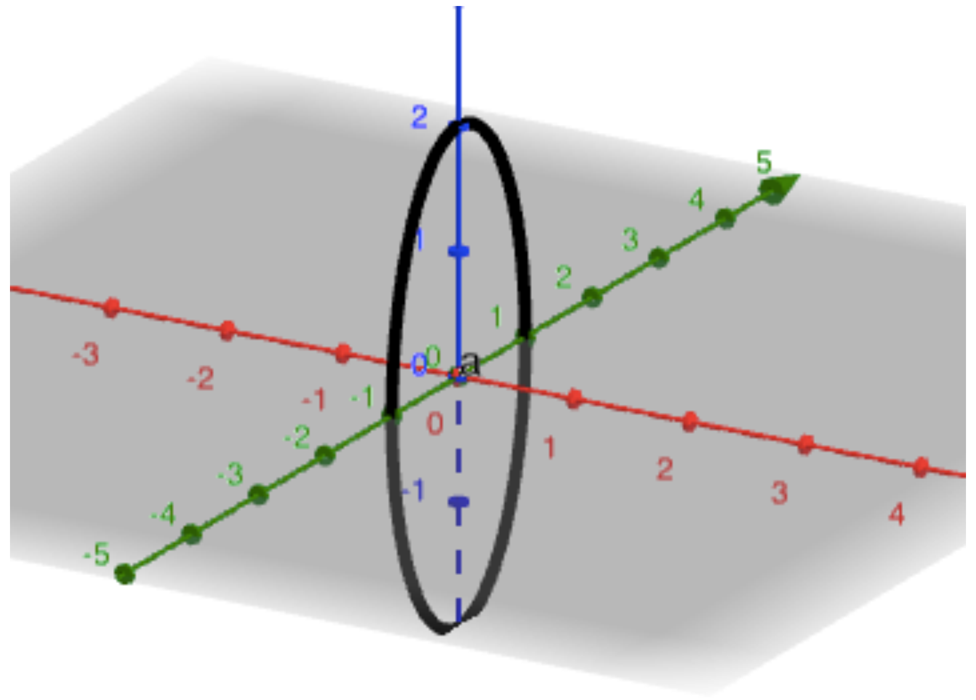
xy plane:

$\mathbf{r}(t) = <t, \sin(t), 0>$



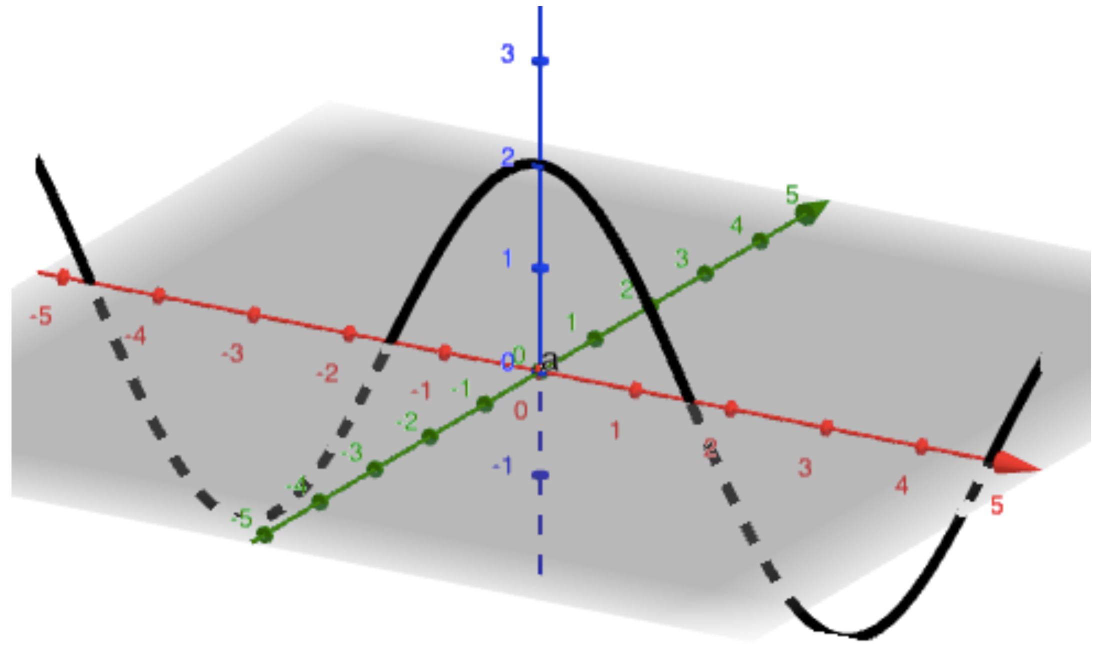
yz plane:

$\mathbf{r}(t) = <0, \sin(t), 2\cos(t)>$

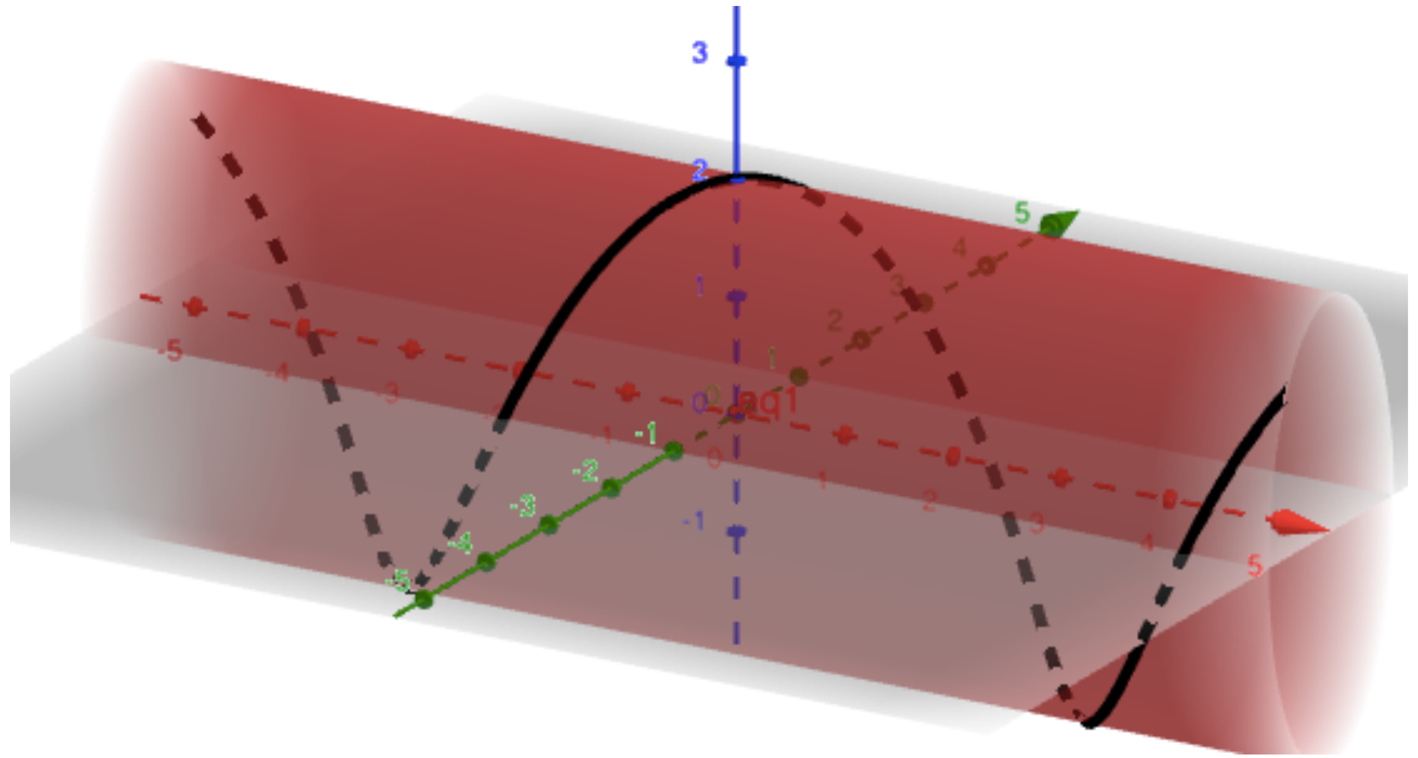


xz plane:

$\mathbf{r}(t) = <t, 0, 2\cos(t)>$



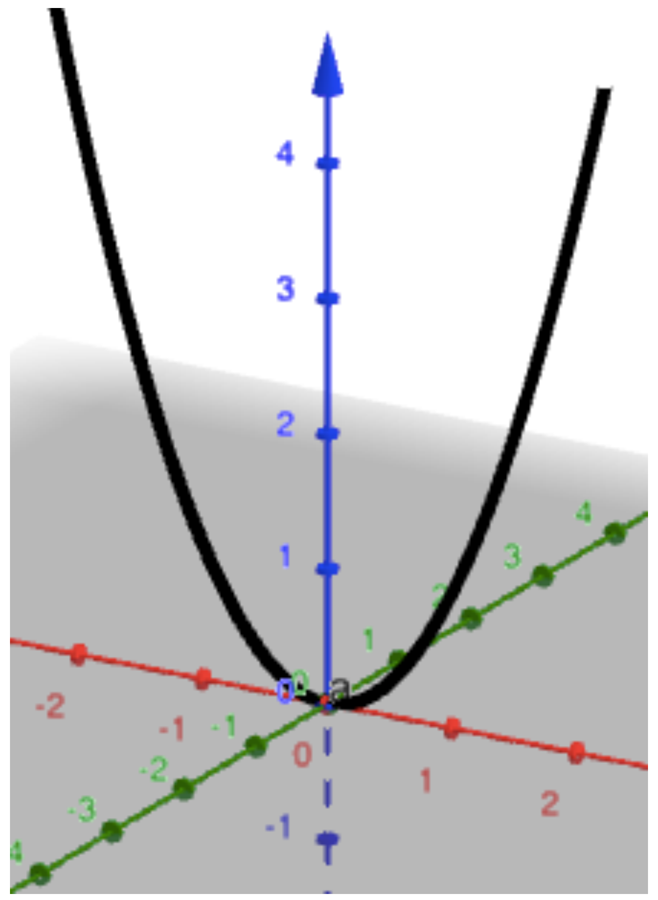
All together:



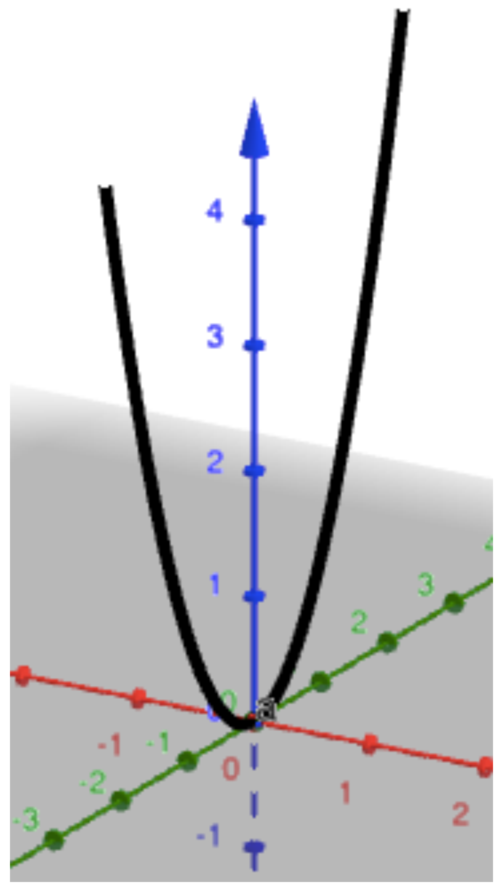
Re Example1: Note that the curve lies on the surface of $z^2 + 4y^2 = 1$.

Example2. $\mathbf{r}(t) = <t, t, t^2>$

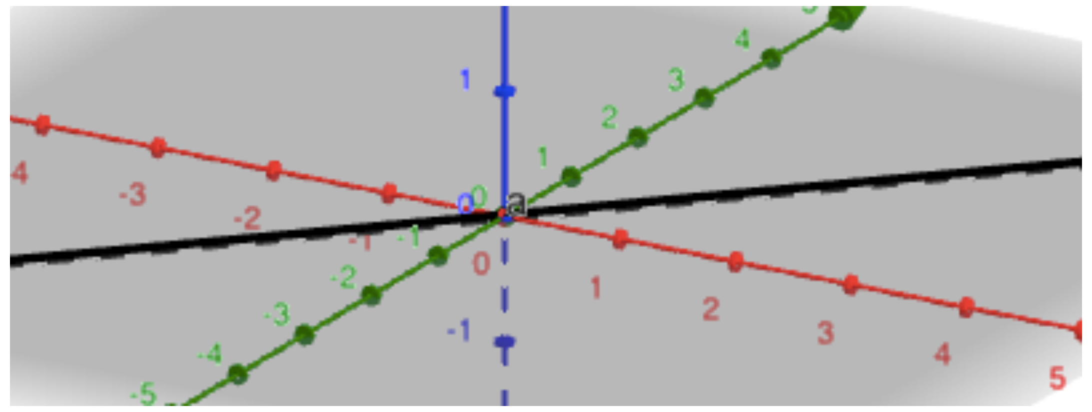
xz plane:



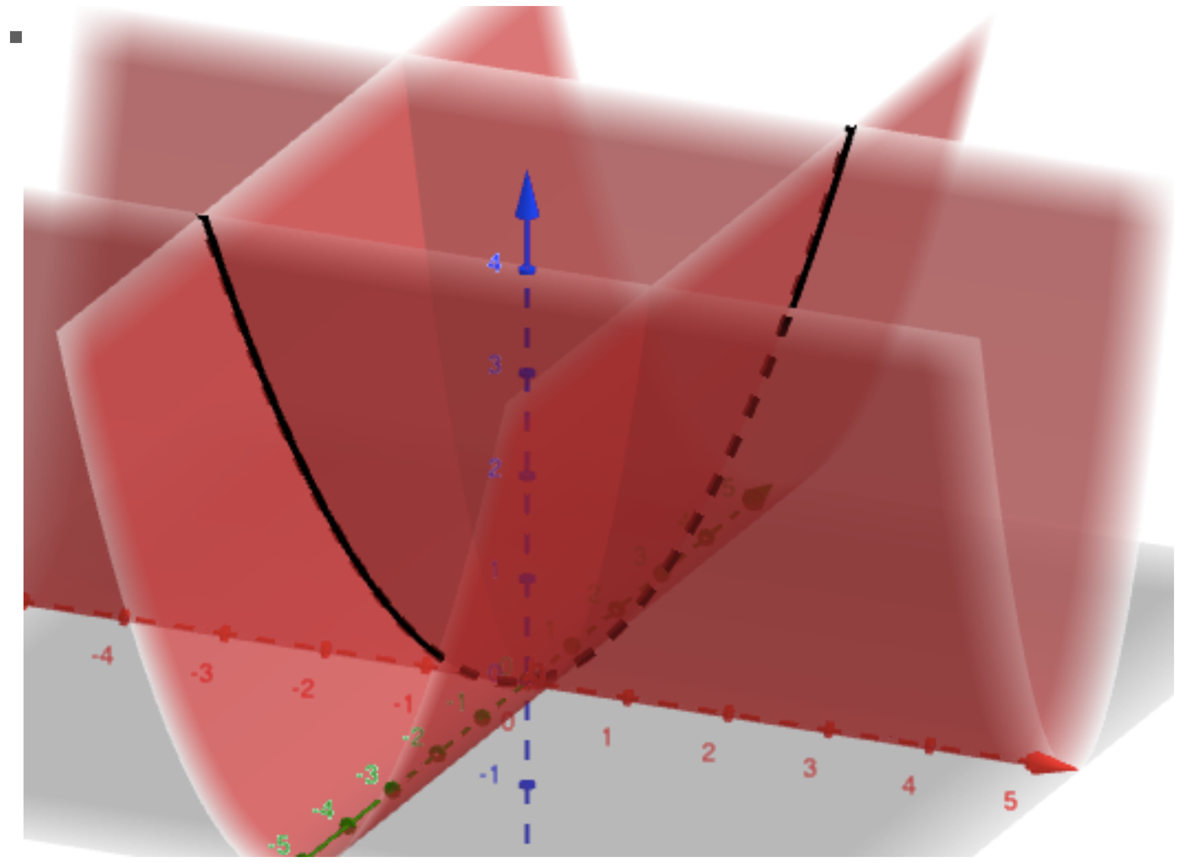
yz plane (further right):



xy plane:



All together↘:



Re Example2: This curves components satisfy both $z = x^2$and $z = y^2$. So the curve lies on both of the surfaces given by those equations.

Curves in Space, Practice.

(S13.1 #21-26).

Try matching these vector-valued functions to their graphs, the curves in space.

$$\mathbf{r}_1(t) = < t\cos(t), t, t\sin(t) > \ , \quad t \geq 0$$

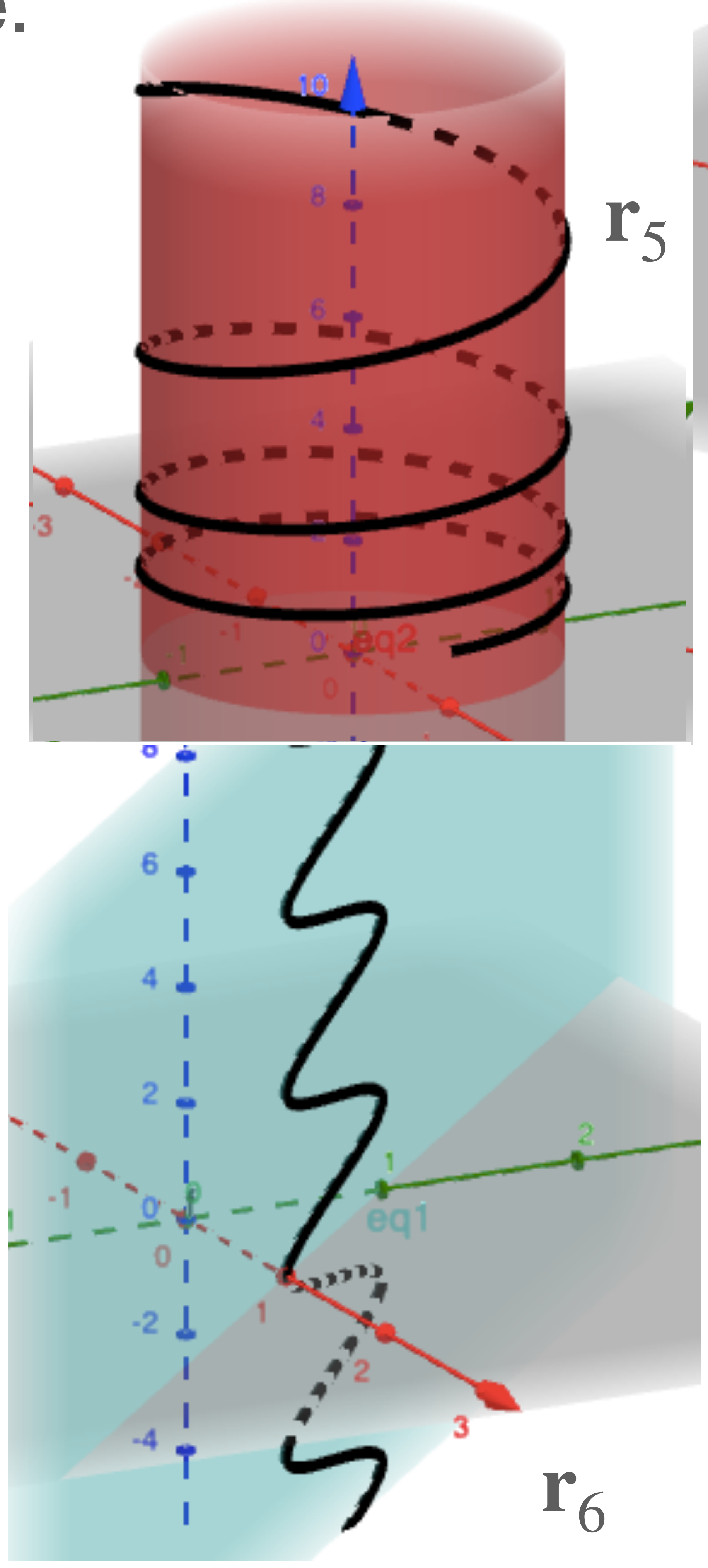
$$\mathbf{r}_2(t) = \left\langle \cos(t), \sin(t), \frac{1}{1+t^2} \right\rangle$$

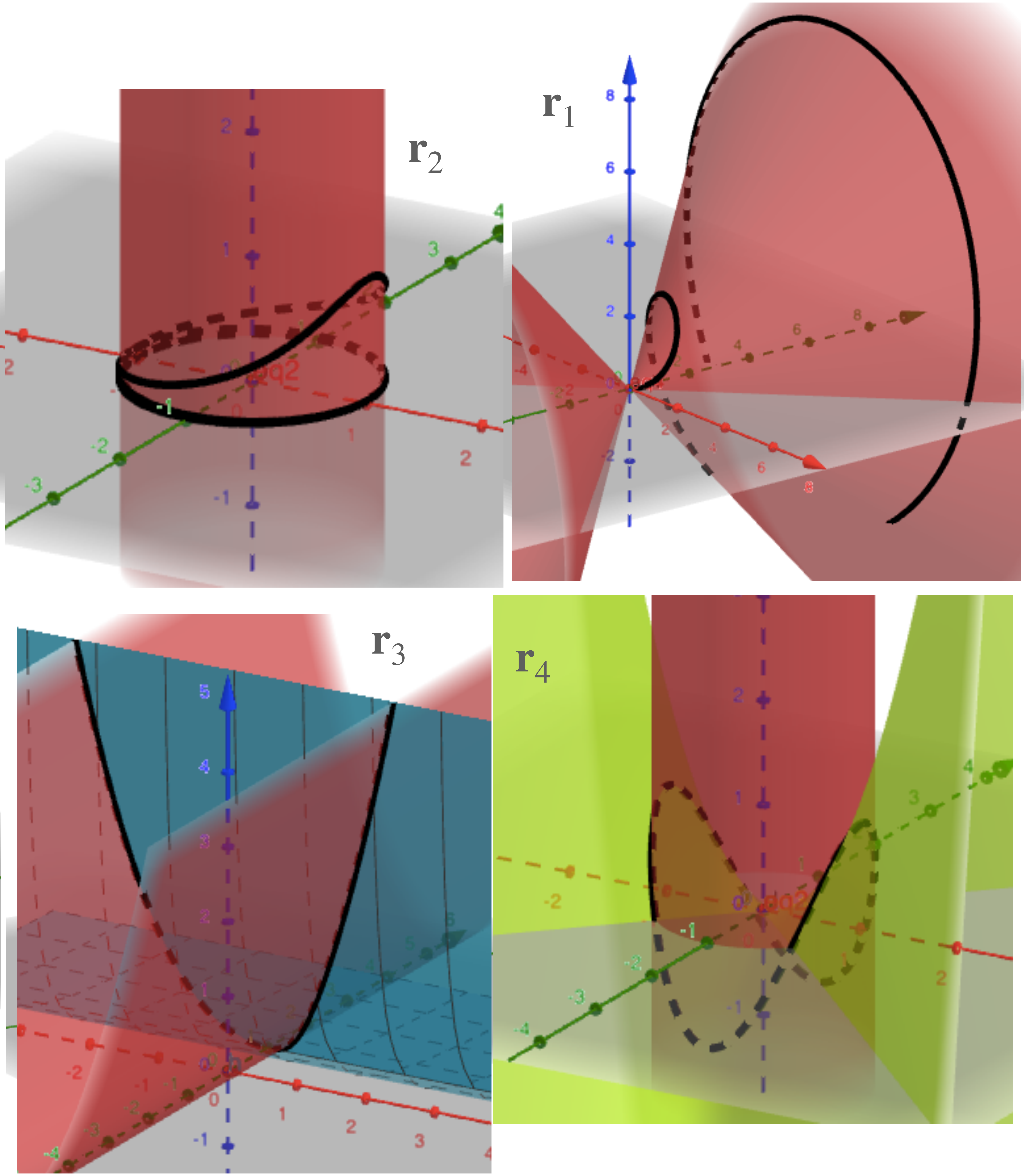
$$\mathbf{r}_3(t) = \left\langle t, \frac{1}{1+t^2}, t^2 \right\rangle$$

$$\mathbf{r}_4(t) = < \cos(t), \sin(t), \cos(2t) >$$

$$\mathbf{r}_5(t) = < \cos(8t), \sin(8t), e^{0.8t} > \ , \ t \geq 0$$

$$\mathbf{r}_6(t) = < \cos^2(t), \sin^2(t), t >$$





Curves in Space, Intersections, pg 1.

S13.1 #50

Particles travel along trajectories given by $\mathbf{r}_1(t) = < t, t^2, t^3 >$ and $\mathbf{r}_2(t) = < 1+2t, 1+6t, 1+14t >$

a) do the particles collide?

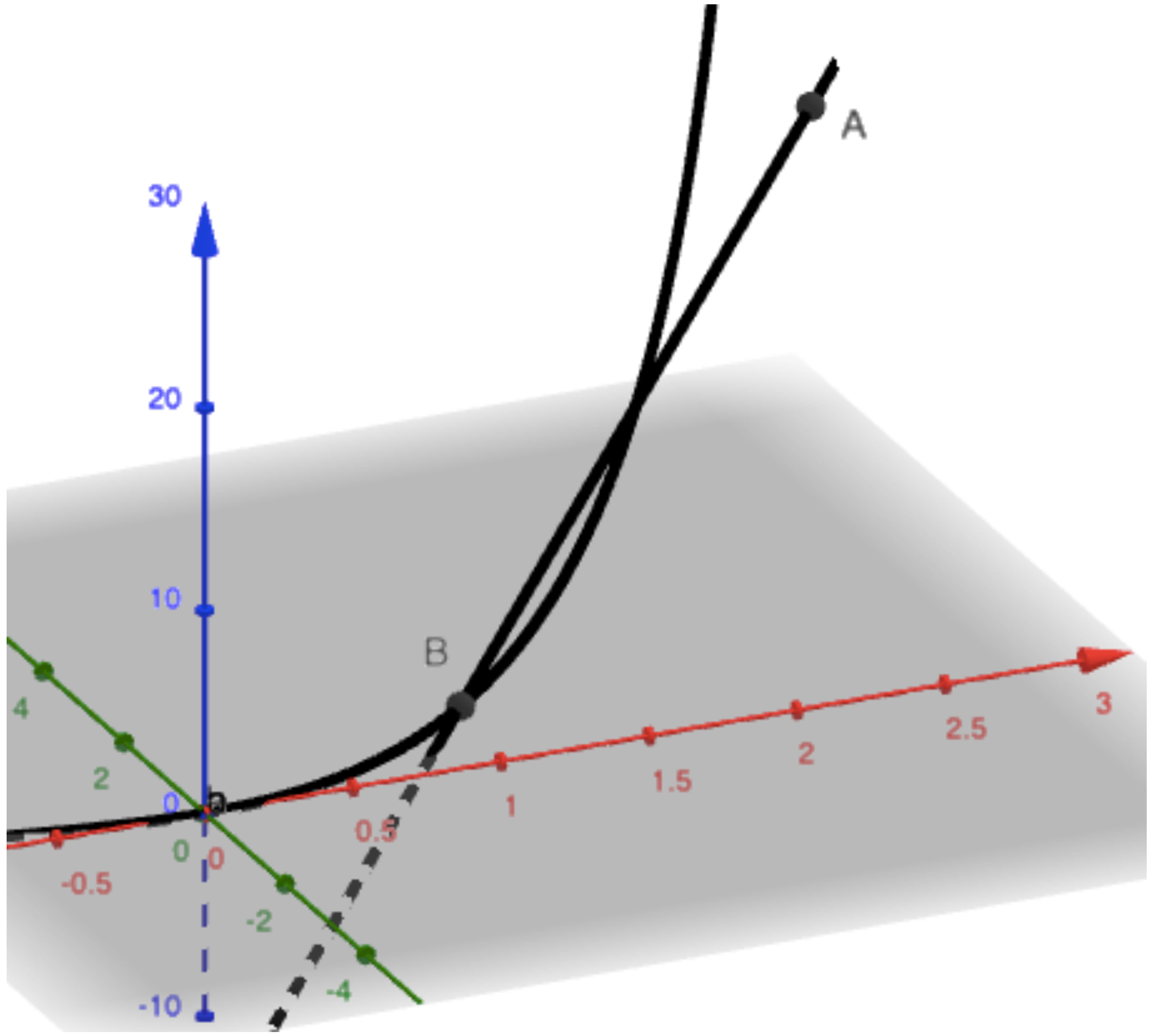
b) do their paths cross?

$$\begin{cases} (1+2s)^2 = 1+6s \\ (1+2s)^3 = 1+14s \end{cases} \quad \ldots \quad s = 0; \quad s = \frac{1}{2}$$

a) they don't collide. The particles are never at the same location at the same time, because this system of equations…

$$\begin{cases} t = 1+2t \\ t^2 = 1+6t \\ t^3 = 1+14t \end{cases}$$ … has no solution.

Both particles visit the points $(1,1,1)$ and $(2,4,8)$, at different times.



b) their paths do collide.

Say $\mathbf{r}_2(s) = < 1+2s, 1+6s, 1+14s >$

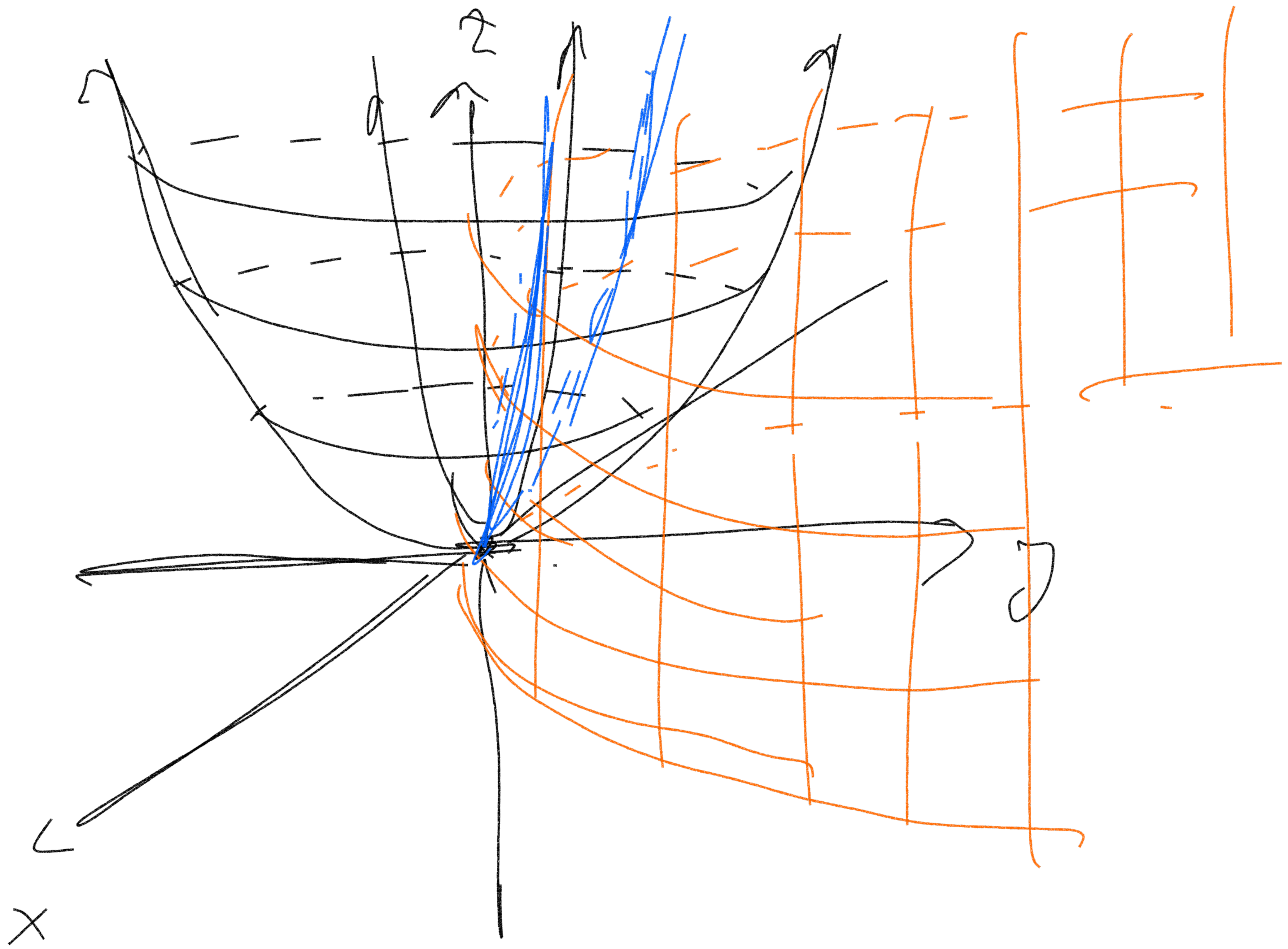
Is their ever a 'time' s when…

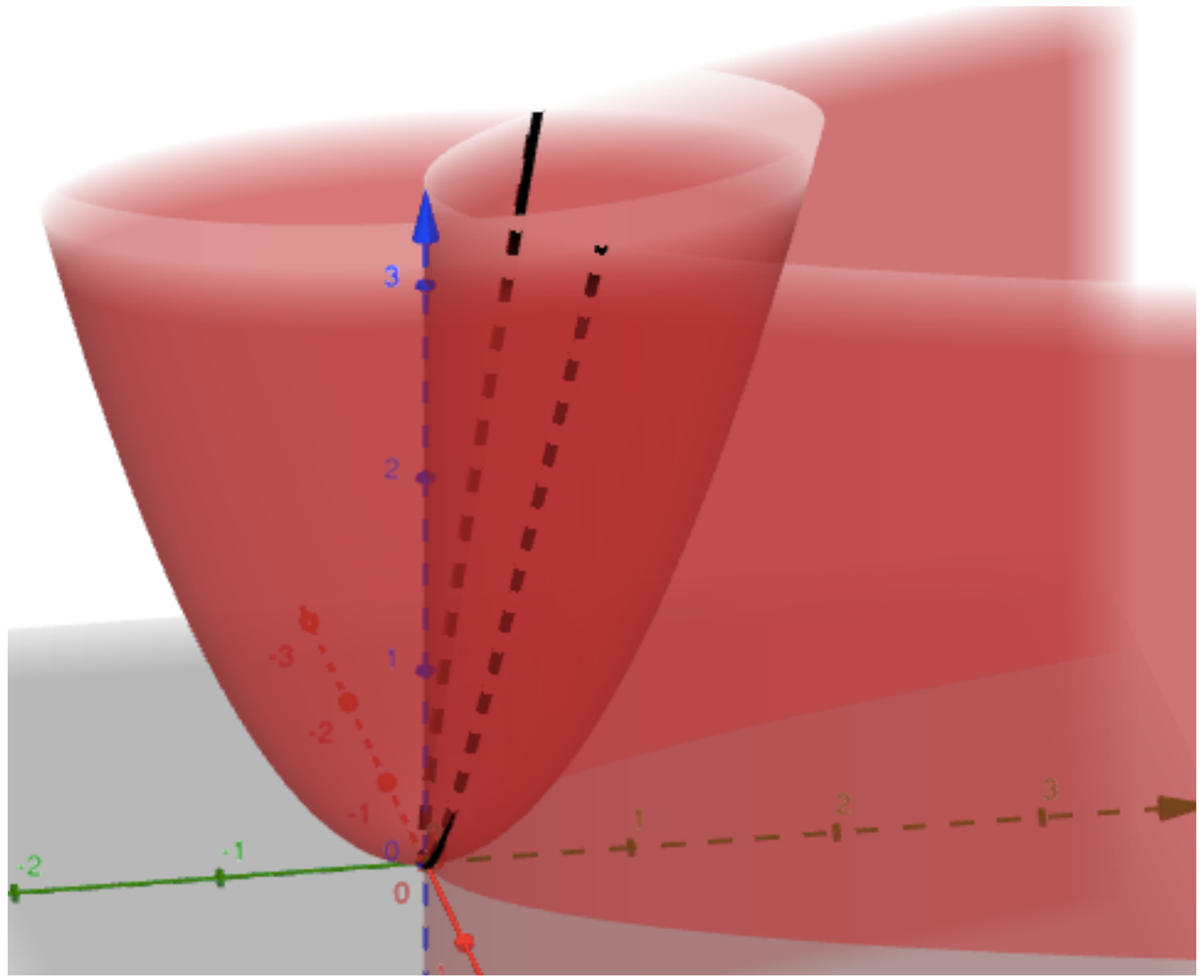
$$\begin{cases} t = 1+2s \\ t^2 = 1+6s \\ t^3 = 1+14s \end{cases}$$

Curves in Space, Intersections, pg 2.

Find a vector function that represents the curve of intersection of the two given surfaces.

S13.1 #44 The paraboloid $z = 4x^2 + y^2$ and the parabolic cylinder $y = x^2$.





First y, and then z can be defined in terms of x: $y = x^2$; $\quad z = 4x^2 + x^4$

Make x the parameter of the curve, i.e. $x = t$

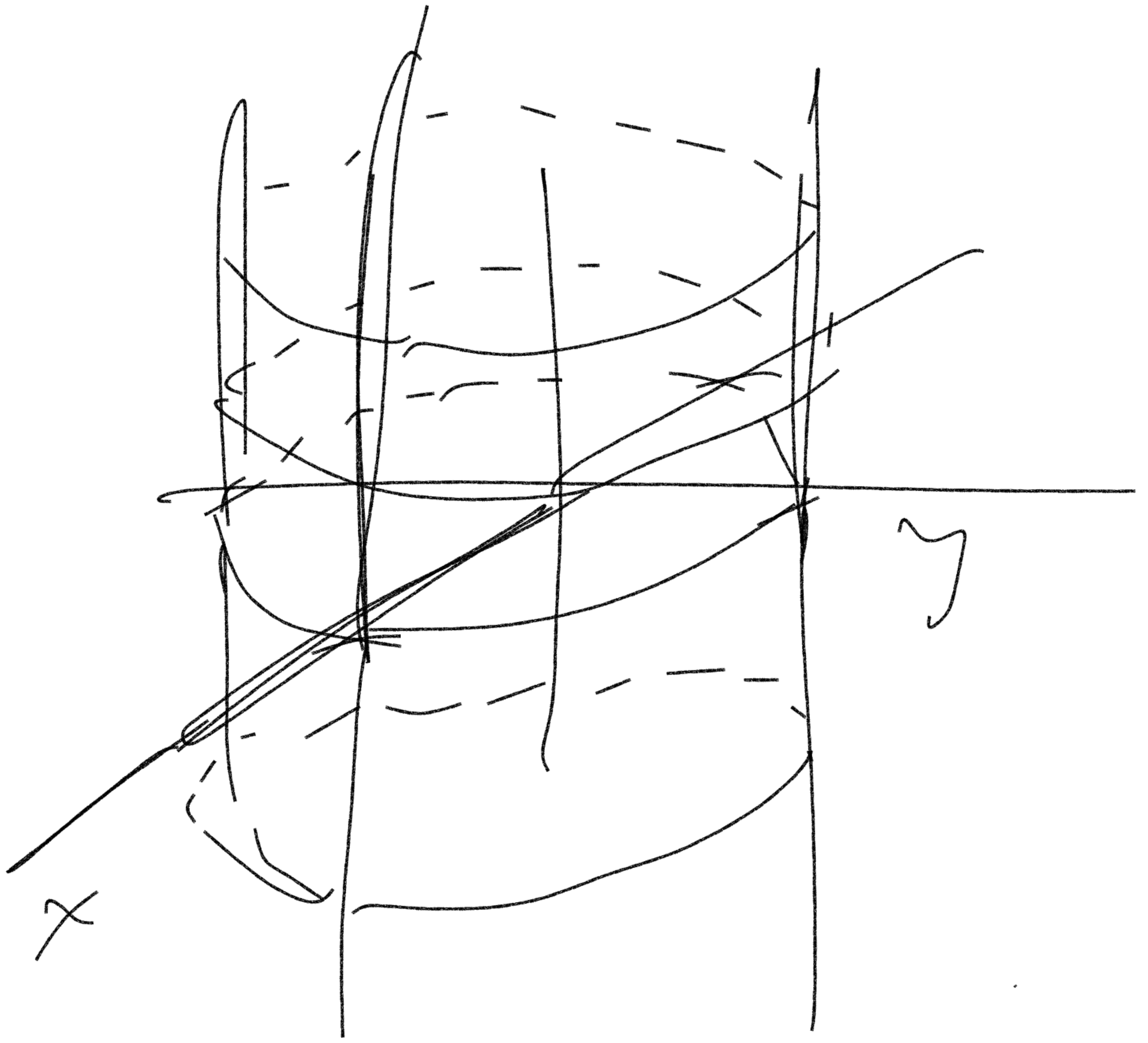
Then $y = t^2$, $z = 4t^2 + t^4$

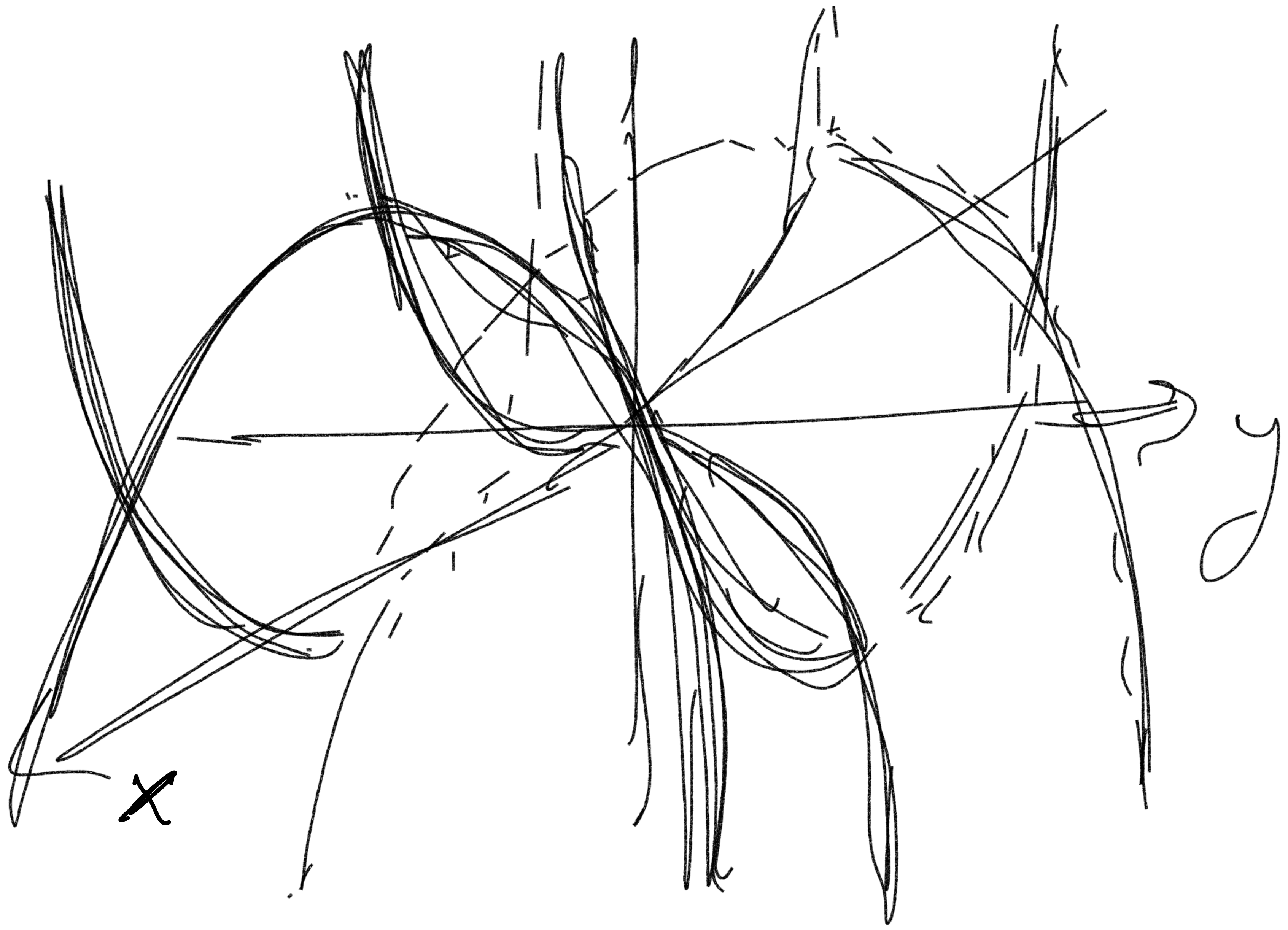
$\mathbf{r}(t) = < t, t^2, 4t^2 + t^4 >$ domain: $t \in (-\infty, \infty)$

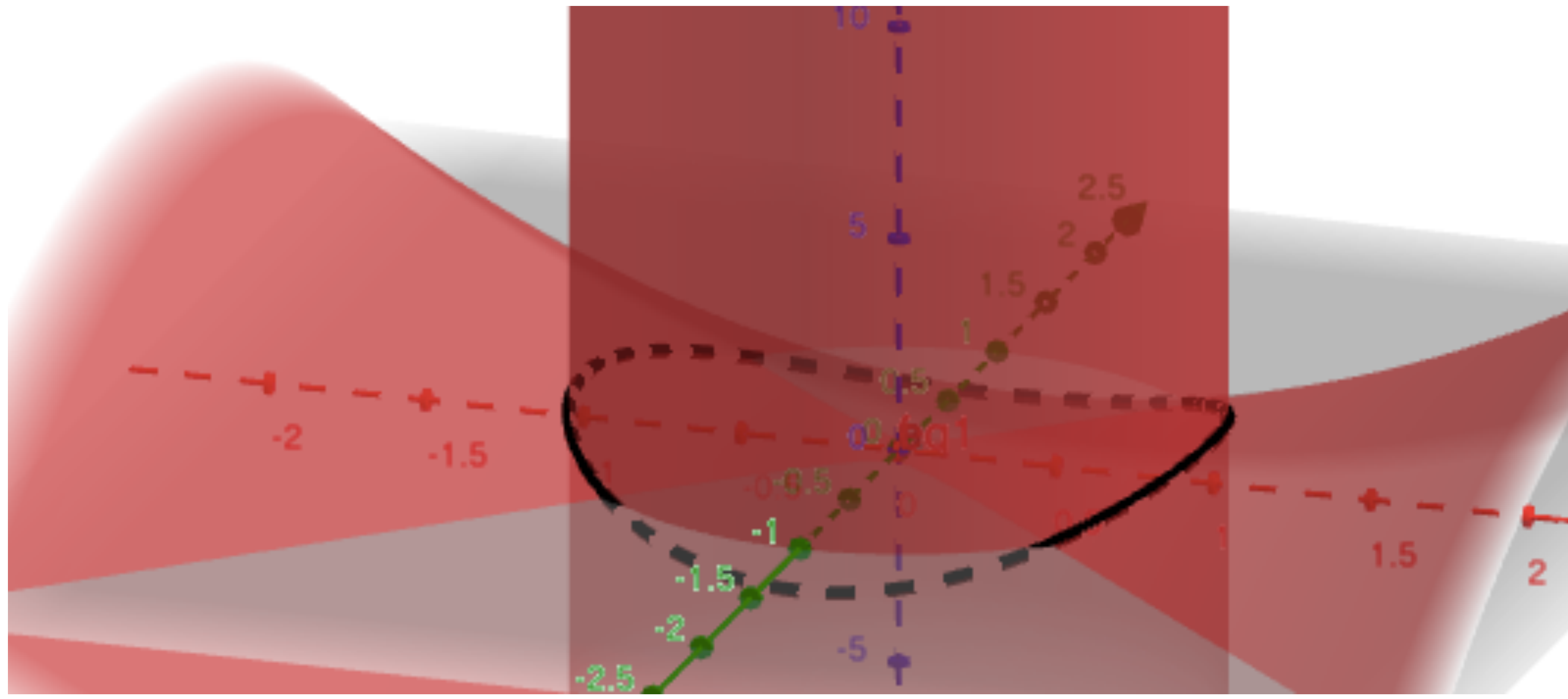
Curves in Space, Intersections, pg 3.

Find a vector function that represents the curve of intersection of the two given surfaces.

S13.1 #45 The saddle $z = x^2 - y^2$ and the cylinder $x^2 + y^2 = 1$.







We could make one of our variables a parameter t

e.g. $y = t \quad x = \pm\sqrt{1 - t^2} \quad z = (1 - t^2) - t^2 = 1 - 2t^2$

We get two curves:

$$\mathbf{r}_1(t) = < -\sqrt{1 - t^2}, t, 1 - 2t^2 > , \qquad t \in [-1,1]$$

$$\mathbf{r}_2(t) = < +\sqrt{1 - t^2}, t, 1 - 2t^2 > , \qquad t \in [-1,1]$$

OR, we could parameterize using trig functions (very useful with circles, ellipses, cylinders etc.)

$$x = \cos(t), \quad y = \sin(t), \qquad z = \cos^2(t) - \sin^2(t)$$
$$= \cos(2t)$$

$$\mathbf{r}(t) = < \cos(t), \sin(t), \cos(2t) > , \quad t \in [0,2\pi)$$

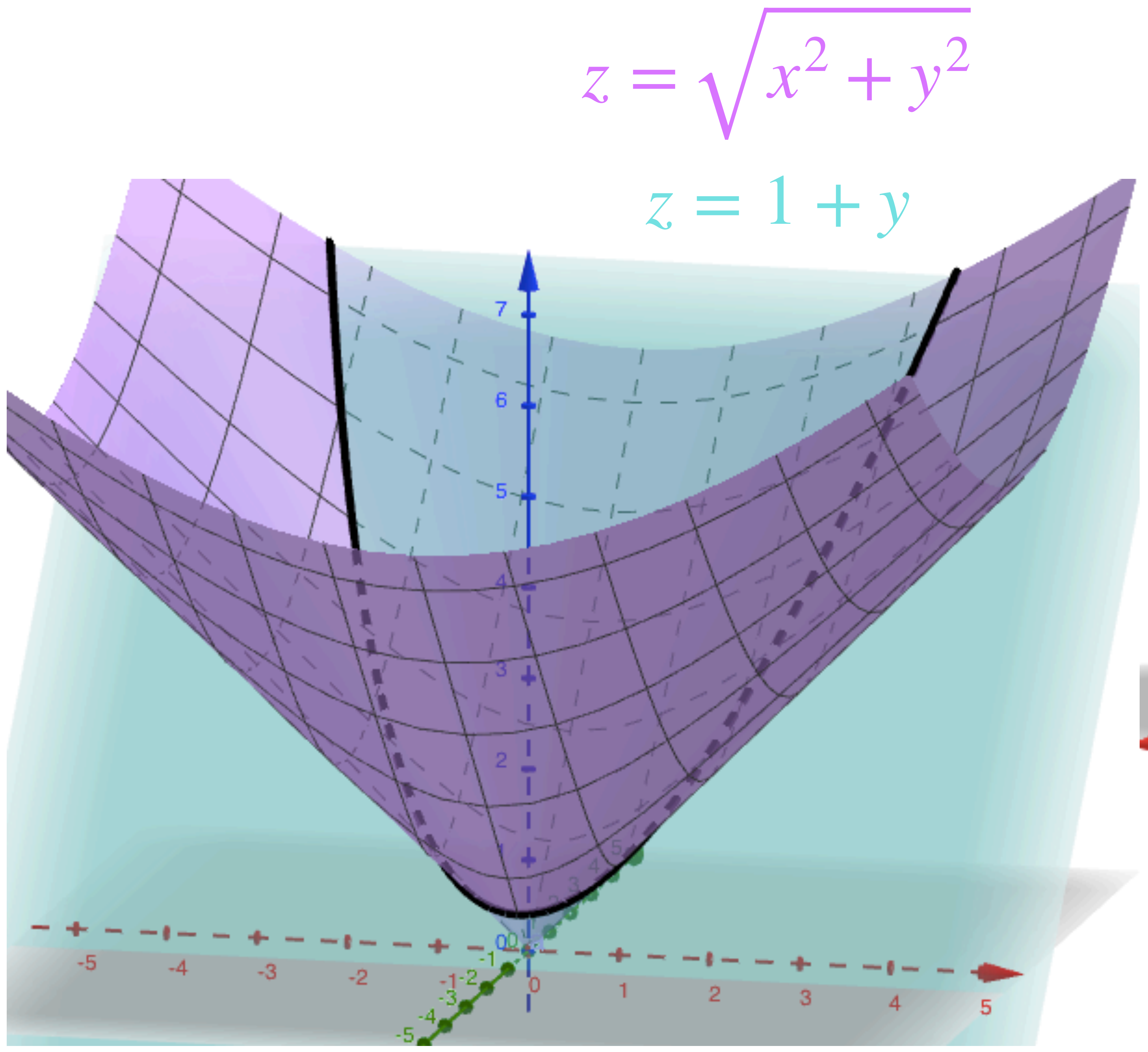
Curves in Space, Intersections. Practice.

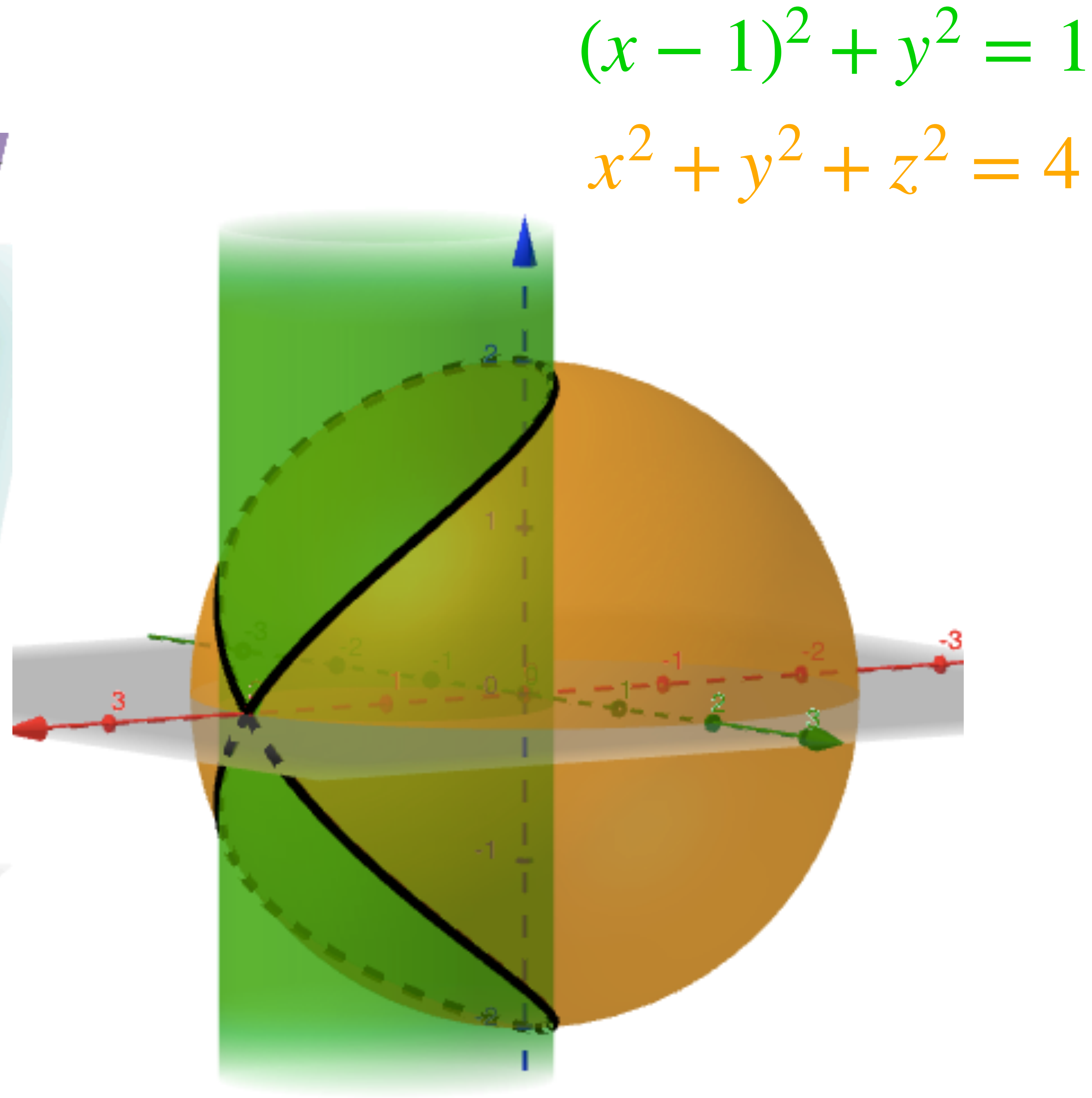
Parameterize the curve(s) where the two surfaces intersect.

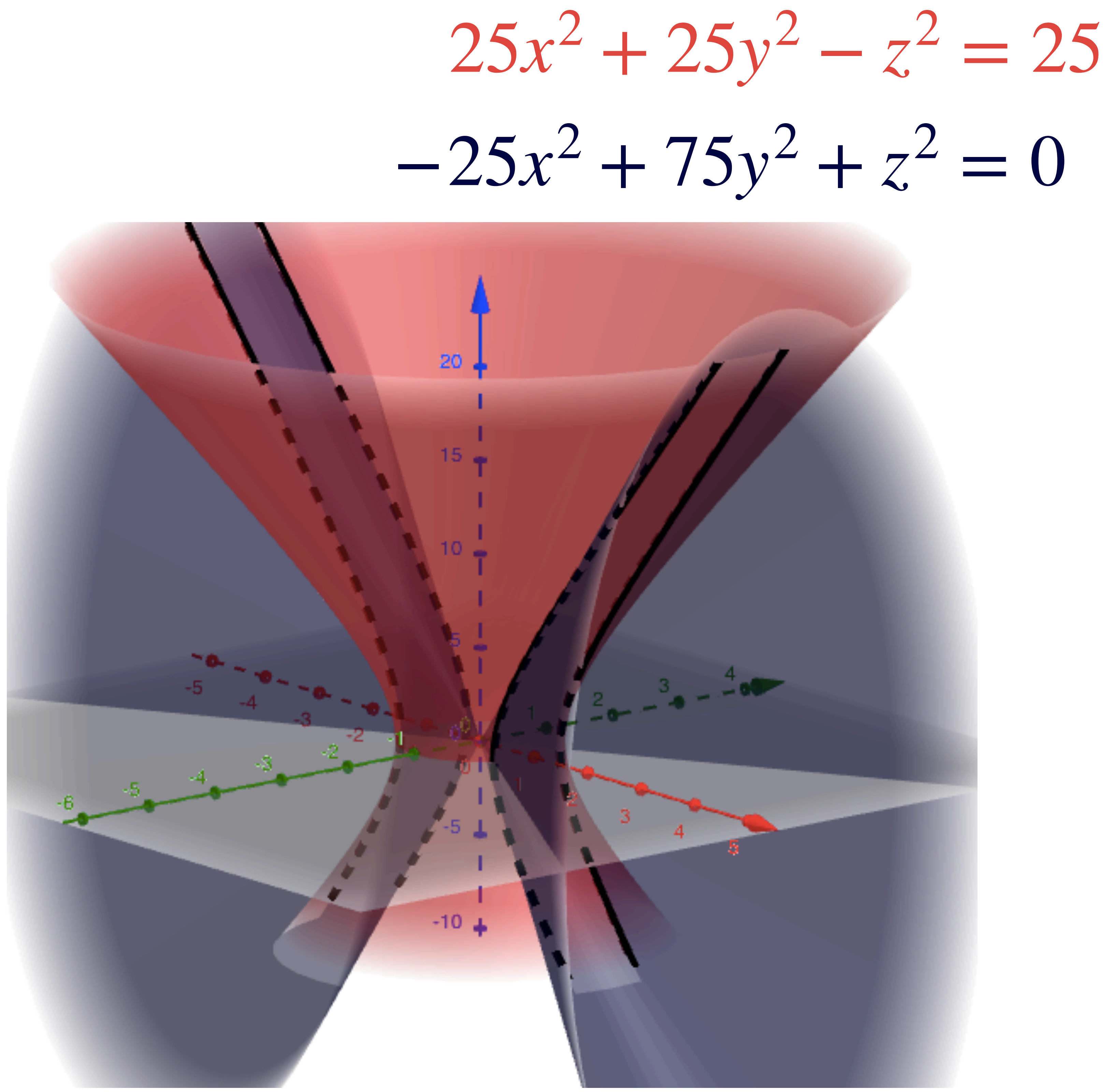
1. (S13.1 #43) The cone $z = \sqrt{x^2 + y^2}$ and the surface $z = 1 + y$.

2. (OX2.6 #355) The cylinder $(x-1)^2 + y^2 = 1$ and the sphere $x^2 + y^2 + z^2 = 4$.

3. (OX2.6 #356) The hyperboloid of one sheet given by $25x^2 + 25y^2 - z^2 = 25$ and the elliptic cone given by $-25x^2 + 75y^2 + z^2 = 0$







Curves in Space, Intersections. Practice, pg 2.

Parameterize the curve(s) where the two surfaces intersect.

1. (S13.1 #43) The cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 1 + y$.

We can describe both x and z in terms of y:

$$z = y + 1 \quad x = \pm\sqrt{z^2 - y^2} = \pm\sqrt{(y+1)^2 - y^2} = \pm\sqrt{2y+1}$$

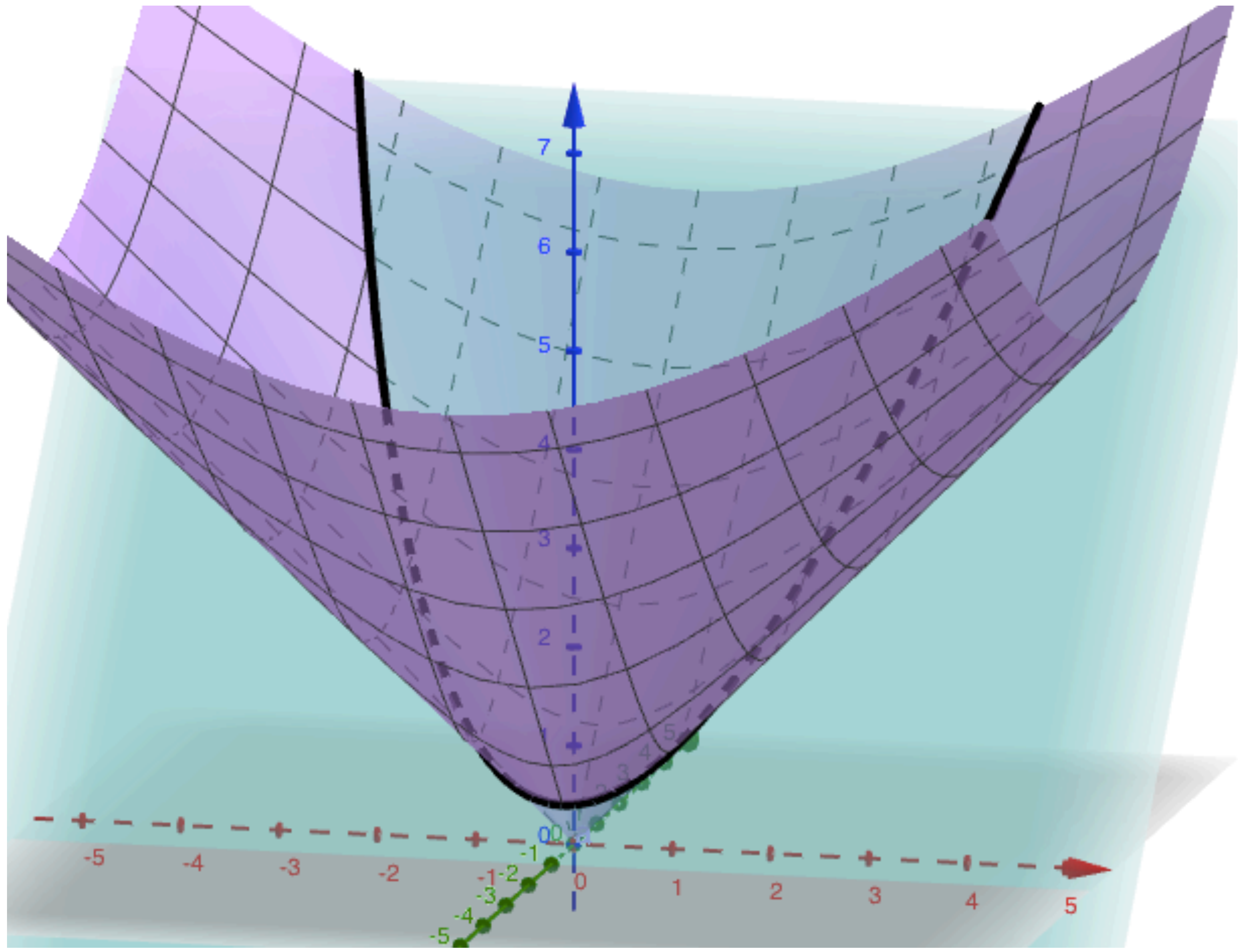
Make $y = t$

$$\mathbf{r}_1(t) = < \sqrt{2t+1}, t, t+1 > , \quad t \in [-0.5, \infty)$$

$$\mathbf{r}_2(t) = < -\sqrt{2t+1}, t, t+1 > , \quad t \in [-0.5, \infty)$$

$z = \sqrt{x^2 + y^2}$

$z = 1 + y$



2. (OX2.6 #355) The cylinder $(x-1)^2 + y^2 = 1$ and the sphere $x^2 + y^2 + z^2 = 4$.

When you see x's and y's squared, try some sines and cosines: $x = \cos(t) + 1, \quad y = \sin(t)$

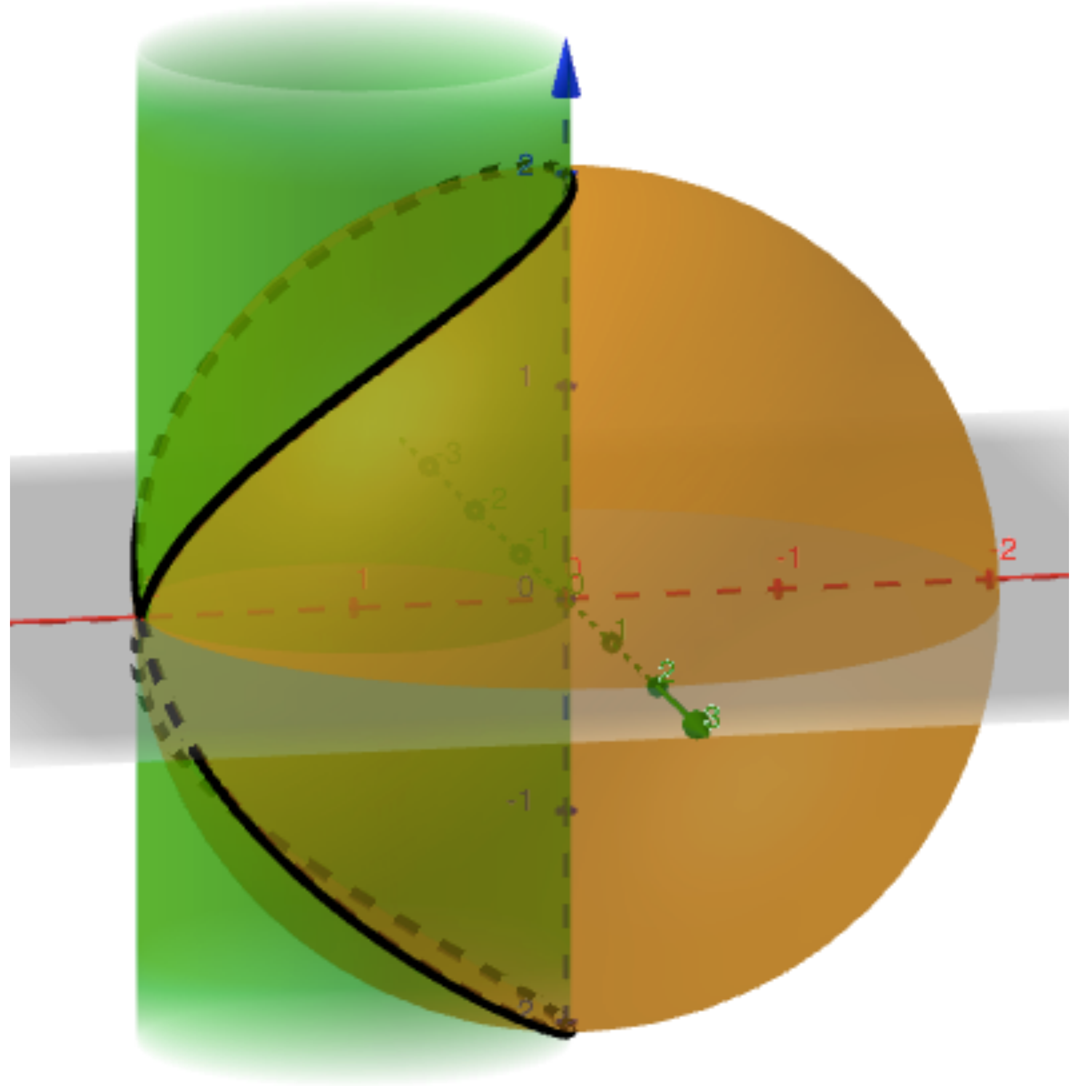
$$z^2 = 4 - (x^2 + y^2) = 4 - (2 + 2\cos(t)) \qquad z = \pm\sqrt{2 - 2\cos(t)}$$

$$\mathbf{r}_1(t) = < \cos(t) + 1,\ \sin(t),\ \sqrt{2 - 2\cos(t)} > , \quad t \in [0, 2\pi]$$

$$\mathbf{r}_1(t) = < \cos(t) + 1,\ \sin(t),\ -\sqrt{2 - 2\cos(t)} > , \quad t \in [0, 2\pi]$$

$(x-1)^2 + y^2 = 1$

$x^2 + y^2 + z^2 = 4$



Curves in Space, Intersections. Practice, pg 3.

3. (OX2.6 #356) The hyperboloid of one sheet given by $25x^2 + 25y^2 - z^2 = 25$ and the elliptic cone given by $-25x^2 + 75y^2 + z^2 = 0$

If both these equations are satisfied, then y is constant.

Adding the two equations together yields $100y^2 = 25$

Putting this into either equation results in a hyperbola 1/2 unit away from the xz plane. $y = \pm 1/2$

$$-25x^2 + z^2 = -\frac{75}{4}, \quad \frac{4}{3}x^2 - \frac{4}{75}z^2 = 1$$

You can parameterize this using some hyperbolic trig functions, known as the 'cosh' and 'sinch.'

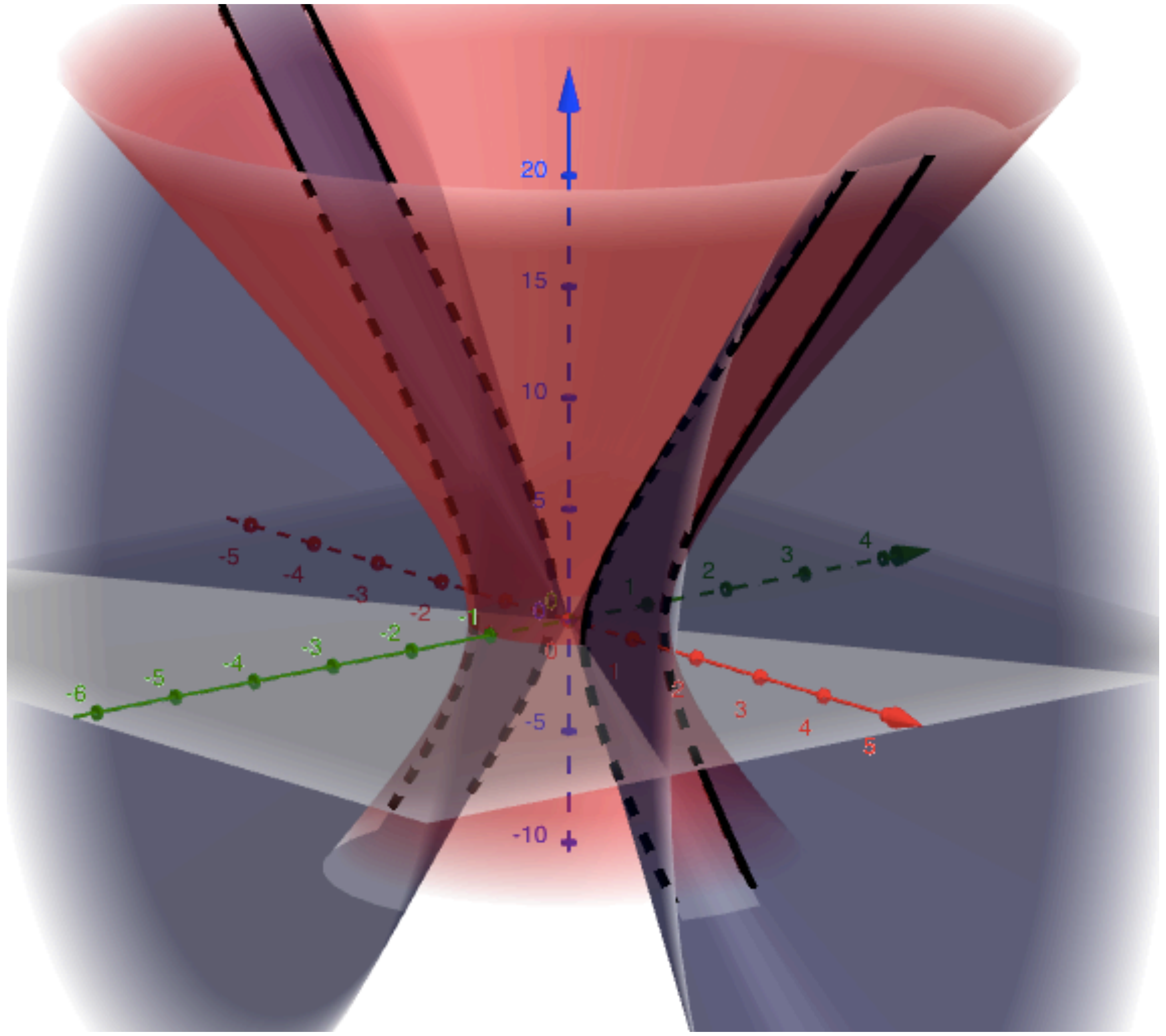
$$\cosh(t) = \frac{e^t + e^{-t}}{2}, \quad \sinh(t) = \frac{e^t - e^{-t}}{2}$$

Note: $\cosh^2(t) - \sinh^2(t) = 1$

$$\frac{4}{3}x^2 - \frac{4}{75}z^2 = 1$$

$$x = \frac{\sqrt{3}}{2}\cosh(t), \quad z = \frac{5\sqrt{3}}{2}\sinh(t)$$

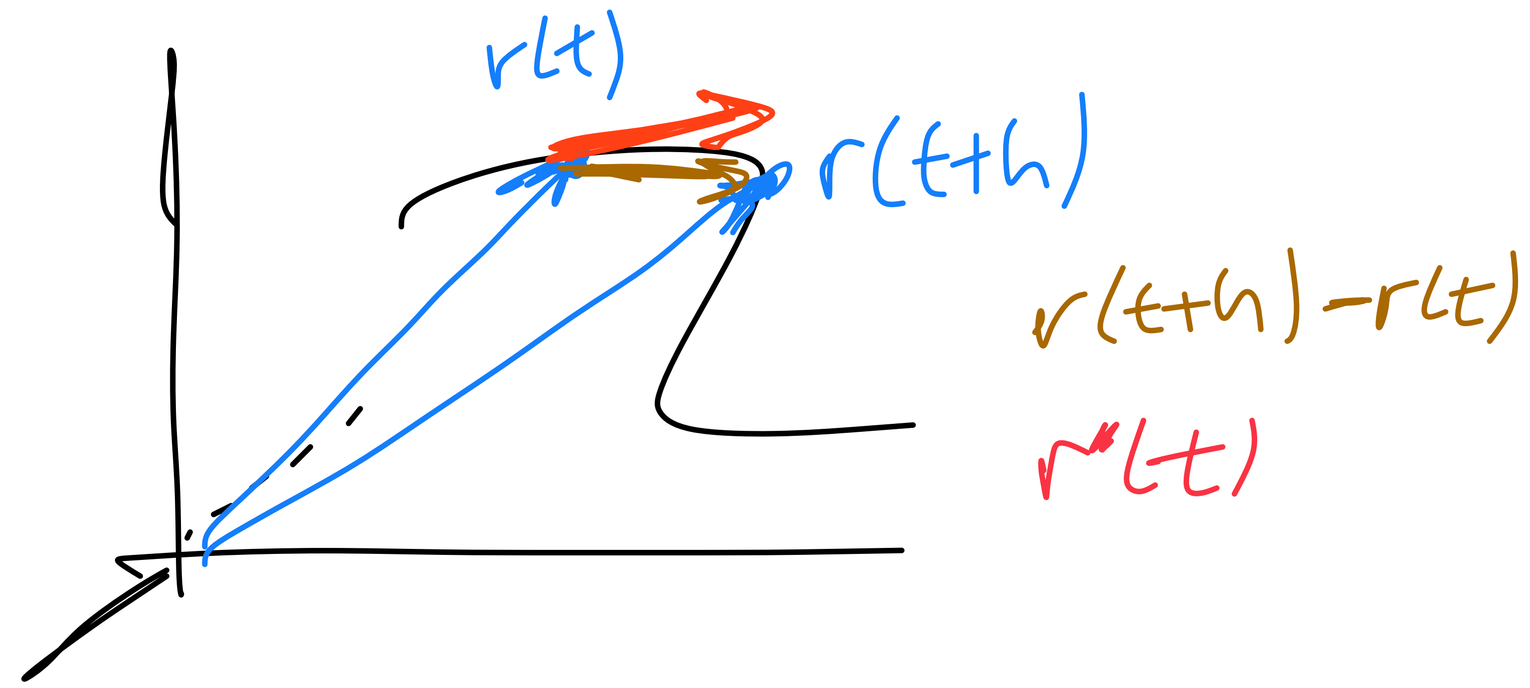
$$\mathbf{r}(t) = < \pm\frac{\sqrt{3}}{2}\cosh(t), \ \pm\frac{1}{2}, \ \frac{5\sqrt{3}}{2}\sinh(t) >$$



$$25x^2 + 25y^2 - z^2 = 25$$

$$-25x^2 + 75y^2 + z^2 = 0$$

Some Calculus of Vector-Valued Functions, pg 1.



Limits, Continuity, Derivatives and Integrals all work component-wise.

$$\lim_{t\to a} \mathbf{r}(t) = \lim_{t\to a} < x(t), y(t), z(t) > := < \lim_{t\to a} x(t), \lim_{t\to a} y(t), \lim_{t\to a} z(t) >$$

$$\frac{d}{dt}\mathbf{r}(t) = \mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} = \left\langle \lim_{h\to 0} \frac{x(t+h) - x(t)}{h}, \lim_{h\to 0} \frac{y(t+h) - y(t)}{h}, \lim_{h\to 0} \frac{z(t+h) - z(t)}{h} \right\rangle = < x'(t), y'(t), z'(t) >$$

$$\int_a^b \mathbf{r}(t)\ dt = \lim_{n\to\infty} \sum_{j=0}^{n} \mathbf{r}(t_j)\Delta t_j = \left\langle \lim_{n\to\infty} \sum_{j=0}^{n} x(t_j)\Delta t_j, \lim_{n\to\infty} \sum_{j=0}^{n} y(t_j)\Delta t_j, \lim_{n\to\infty} \sum_{j=0}^{n} z(t_j)\Delta t_j \right\rangle = \left\langle \int_a^b x(t)\ dt, \int_a^b y(t)\ dt, \int_a^b z(t)\ dt \right\rangle$$

Example.

$$\mathbf{r}(t) = < \sin(t), t, \cos(t) >$$

$$\mathbf{r}'(t) = < \cos(t), 1, -\sin(t) >$$

$$\int \mathbf{r}(t)\ dt = < -\cos(t) + c_1, 1/2t^2 + c_2, \sin(t) + c_3 > = < -\cos(t), 1/2t^2, \sin(t) > + \mathbf{c}$$

where $\mathbf{c} = < c_1, c_2, c_3 >$ is a constant vector of integration.

Some Calculus of Vector-Valued Functions, pg 2.

The same rules that you have seen for scalar functions still apply.

1. $\frac{d}{dt}(\mathbf{r}_1(t) \pm \mathbf{r}_2(t)) = \mathbf{r}_1'(t) \pm \mathbf{r}_2'(t)$

2. $\frac{d}{dt}(c \cdot \mathbf{r}(t)) = c\mathbf{r}'(t)$

3. product rules:

3.1 $\frac{d}{dt}(f(t) \cdot \mathbf{r}(t)) = f'(t)\mathbf{r}(t) + f(t)\mathbf{r}'(t)$

3.2 $\frac{d}{dt}(\mathbf{r}_1(t) \cdot \mathbf{r}_2(t)) = \mathbf{r}_1'(t) \cdot \mathbf{r}_2(t) + \mathbf{r}_1(t) \cdot \mathbf{r}_2'(t)$

3.3 $\frac{d}{dt}(\mathbf{r}_1(t) \times \mathbf{r}_2(t)) = \mathbf{r}_1'(t) \times \mathbf{r}_2(t) + \mathbf{r}_1(t) \times \mathbf{r}_2'(t)$

4.chain rule: $\frac{d}{dt}\mathbf{r}(f(t)) = \mathbf{r}'(f(t))f'(t)$

Proof of 3.1:

$$\frac{d}{dt}(f(t) \cdot \mathbf{r}(t)) = \frac{d}{dt} < f(t)x(t), f(t)y(t), f(t)z(t) > = \ldots$$

$$\ldots = < \frac{d}{dt}f(t)x(t), \frac{d}{dt}f(t)y(t), \frac{d}{dt}f(t)z(t) >$$

$$= < f'(t)x(t) + f(t)x'(t), f'(t)y(t) + f(t)y'(t), f'(t)z(t) + f(t)z'(t) >$$

$$= f'(t) < x(t), y(t), z(t) > + f(t) < x'(t), y'(t), z'(t) >$$

$$= f'(t)\mathbf{r}(t) + f(t)\mathbf{r}'(t)$$

Proof of 3.2:

Say $\mathbf{r}_1(t) = < u_1(t), u_2(t), u_3(t) >$, $\quad \mathbf{r}_2(t) = < v_1(t), v_2(t), v_3(t) >$

$$\begin{aligned}
\mathbf{r}_1(t) \cdot \mathbf{r}_2(t) &= u_1(t)v_1(t) + u_2(t)v_2(t) + u_3(t)v_3(t) \\
&= u_1v_1 + u_2v_2 + u_3v_3 \\
(\mathbf{r}_1 \cdot \mathbf{r}_2)' &= (u_1v_1 + u_2v_2 + u_3v_3)' \\
&= u_1'v_1 + u_1v_1' + u_2'v_2 + u_2v_2' + u_3'v_3 + u_3v_3' \\
&= < u_1', u_2', u_3' > \cdot < v_1, v_2, v_3 > \\
&\quad + < u_1, u_2, u_3 > \cdot < v_1', v_2', v_3' > = \mathbf{r}_1' \cdot \mathbf{r}_2 + \mathbf{r}_1 \cdot \mathbf{r}_2'
\end{aligned}$$

Practice.

$\mathbf{r}(t) = < 1, t^2, t^3 >$

$\mathbf{p}(t) = < \cos(t), \sin(t), t >$

$f(t) = e^t$

Compute some derivatives.

1. $\mathbf{r}'(t)$
2. $\mathbf{p}'(t)$
3. $\int \mathbf{p}(t)\, dt$
4. $\frac{d}{dt}\left(f(t)\mathbf{r}(t)\right)$
5. $\frac{d}{dt}\left(\mathbf{r}(t) \cdot \mathbf{p}(t)\right)$

Answers:

1. $\mathbf{r}'(t) = < 0, 2t, 3t^2 >$
2. $\mathbf{p}'(t) = < -\sin(t), \cos(t), 1 >$
3. $\int \mathbf{p}(t)\, dt = < \sin(t), -\cos(t), t^2/2 > + \mathbf{C}$

4.directly

$$\frac{d}{dt} < e^t, t^2 e^t, t^3 e^t >$$
$$= < e^t, e^t(t^2 + 2t), e^t(t^3 + 3t^2) >$$
$$= e^t < 1, t^2 + 2t, t^3 + 3t^2 >$$

4.with a product rule

$$\frac{d}{dt}\left(e^t < 1, t^2, t^3 > \right)$$
$$= e^t < 1, t^2, t^3 > + e^t < 0, 2t, 3t^2 >$$
$$= e^t < 1, t^2 + 2t, t^3 + 3t^2 >$$

5.directly

$$\frac{d}{dt}\left(\cos(t) + t^2 \sin(t) + t^4\right)$$
$$= -\sin(t) + 2t\sin(t) + t^2\cos(t) + 4t^3$$

5.with a product rule

$$\frac{d}{dt}\left(\mathbf{r}(t) \cdot \mathbf{p}(t)\right) = \mathbf{r}'(t) \cdot \mathbf{p}(t) + \mathbf{r}(t) \cdot \mathbf{p}'(t)$$
$$= < 0, 2t, 3t^2 > \cdot < \cos(t), \sin(t), t >$$
$$+ < 1, t^2, t^3 > \cdot < -\sin(t), \cos(t), 1 >$$
$$= -\sin(t) + 2t\sin(t) + t^2\cos(t) + 4t^3$$

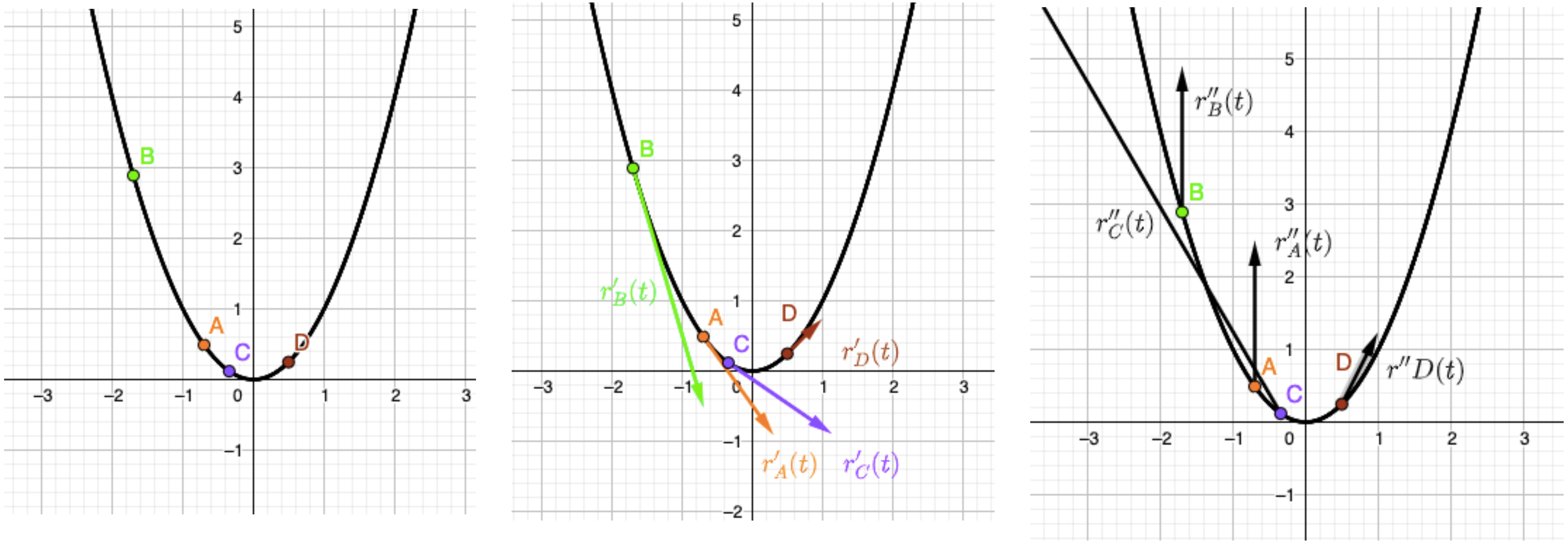
Some Significance of Derivatives and Integrals.

If a particle traverses a curve according to a parameterization $\mathbf{r}(t) = < x(t), y(t), z(t) >$

Then the particle's velocity is $\mathbf{v}(t) = \mathbf{r}'(t) = < x'(t), y'(t), z'(t) >$

Then the particle's acceleration is $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) = < x''(t), y''(t), z''(t) >$

Example.



$\mathbf{r}_A(t) = < t, t^2, 0 >$ | $\mathbf{v}_A(t) = < 1, 2t, 0 >$ | $\mathbf{a}_A(t) = < 0, 2, 0 >$

$\mathbf{r}_B(t) = < t - 1, (t - 1)^2, 0 >$ | $\mathbf{v}_B(t) = < 1, 2(t - 1), 0 >$ | $\mathbf{a}_B(t) = < 0, 2, 0 >$

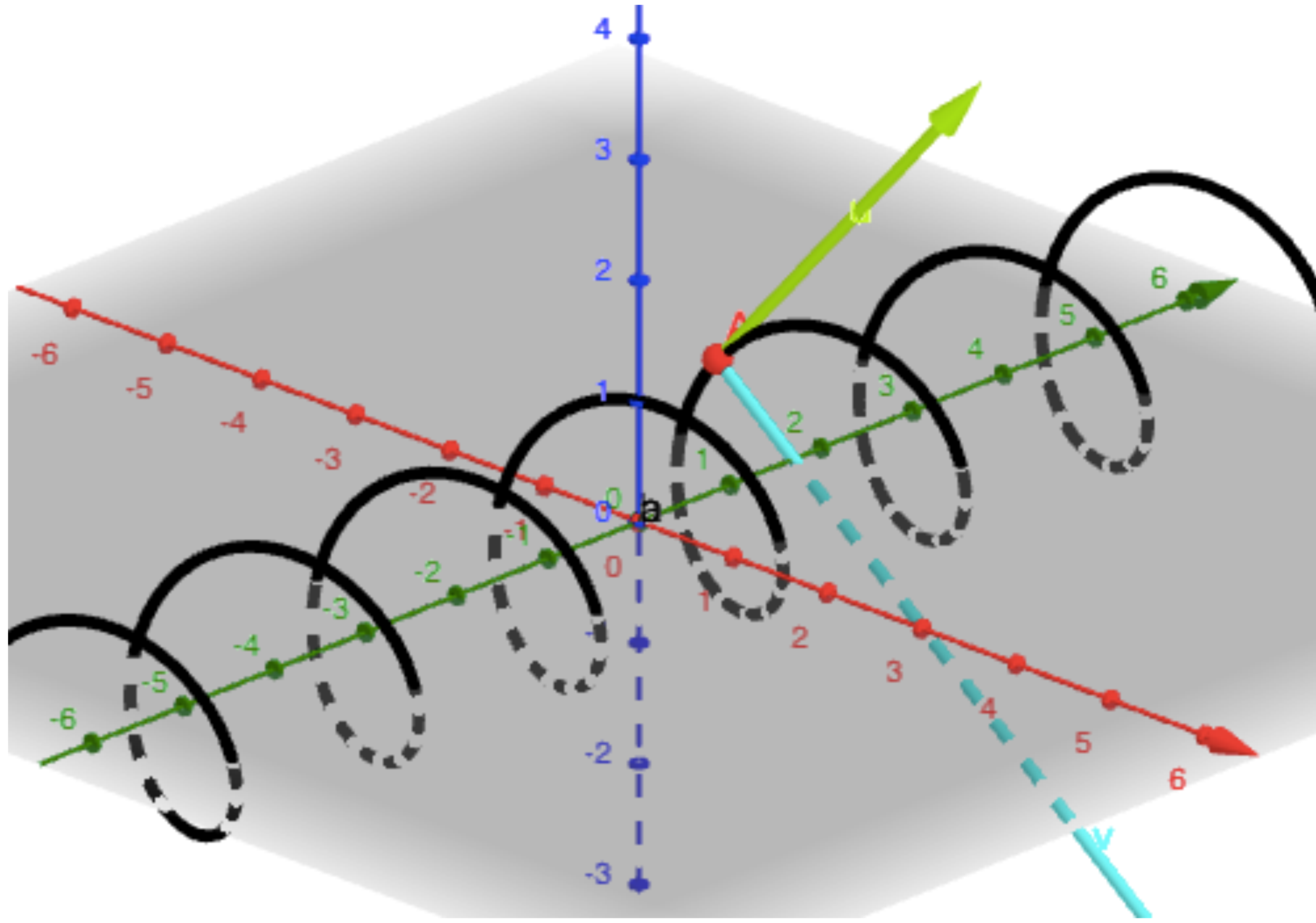
$\mathbf{r}_C(t) = < t^3, t^6, 0 >$ | $\mathbf{v}_C(t) = < 3t^2, 6t^5, 0 >$ | $\mathbf{a}_C(t) = < 6t, 30t^4, 0 >$

$\mathbf{r}_D(t) = < e^t, e^{2t}, 0 >$ | $\mathbf{v}_D(t) = < e^t, 2e^{2t}, 0 >$ | $\mathbf{a}_D(t) = < e^t, 4e^{2t}, 0 >$

Link: ParticleMotion(2D)

Some Significance of Derivatives and Integrals.

Link: MotiononHelix



Given position (curve parameterization), find velocity and acceleration.

$$\mathbf{r}(t) = < \sin(\pi t), t, \cos(\pi t) >$$

$$\mathbf{v}(t) = \mathbf{r}'(t) = < \cos(\pi t) \cdot \pi, 1, -\sin(\pi t) \cdot \pi > = < \pi \cos(\pi t), 1, -\pi \sin(\pi t) >$$

$$\mathbf{a}(t) = \mathbf{v}'(t) = < -\pi^2 \sin(\pi t), 0, -\pi^2 \cos(\pi t) >$$

Given acceleration, initial velocity and position, find velocity and position.

$$\mathbf{a}(t) = < -\sin(t), 3t^2, e^t >, \quad \mathbf{v}(0) = < 2, 0, -1 >, \quad \mathbf{r}(0) = < 0,0,0 >$$

$$\mathbf{v}(t) = \int \mathbf{a}(t)\, dt = < \cos(t) + c_1, t^3 + c_2, e^t + c_3 > = < \cos(t), t^3, e^t > + \mathbf{c}$$

we can specify $\mathbf{c}$ by using initial conditions.

$$< 2, 0, -1 > = \mathbf{v}(0) = < 1,0,1 > + \mathbf{c} \qquad \mathbf{c} = < 2,0,-1 > - < 1,0,1 > = < 1,0,-2 >$$

$$\mathbf{v}(t) = < \cos(t), t^3, e^t > + < 1, 0, -2 > = < \cos(t) + 1, t^3, e^t - 2 >$$

$$\mathbf{r}(t) = \int \mathbf{v}(t)\, dt = < \sin(t) + t, \frac{1}{4}t^4, e^t - 2t > + \mathbf{c}$$

$$< 0,0,0 > = \mathbf{r}(0) = < 0,0,1 > + \mathbf{c} \qquad \mathbf{c} = < 0,0,-1 >$$

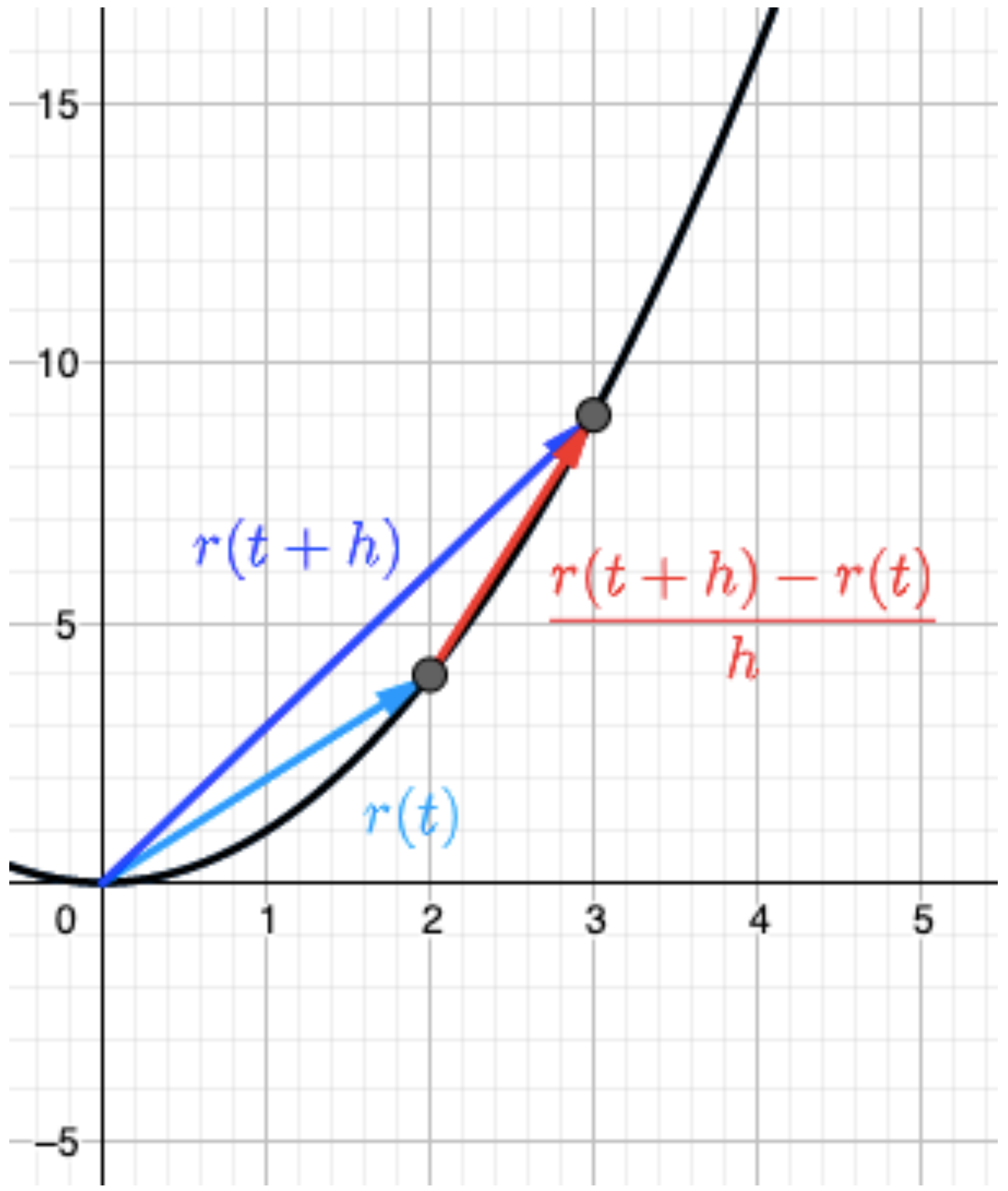
$$\mathbf{r}(t) = < \sin(t) + t, \frac{1}{4}t^4, e^t - 2t > + < 0,0,-1 > = < \sin(t) + t, \frac{1}{4}t^4, e^t - 2t - 1 >$$

Plane Curves, pg 1. Tangent vectors, lines.

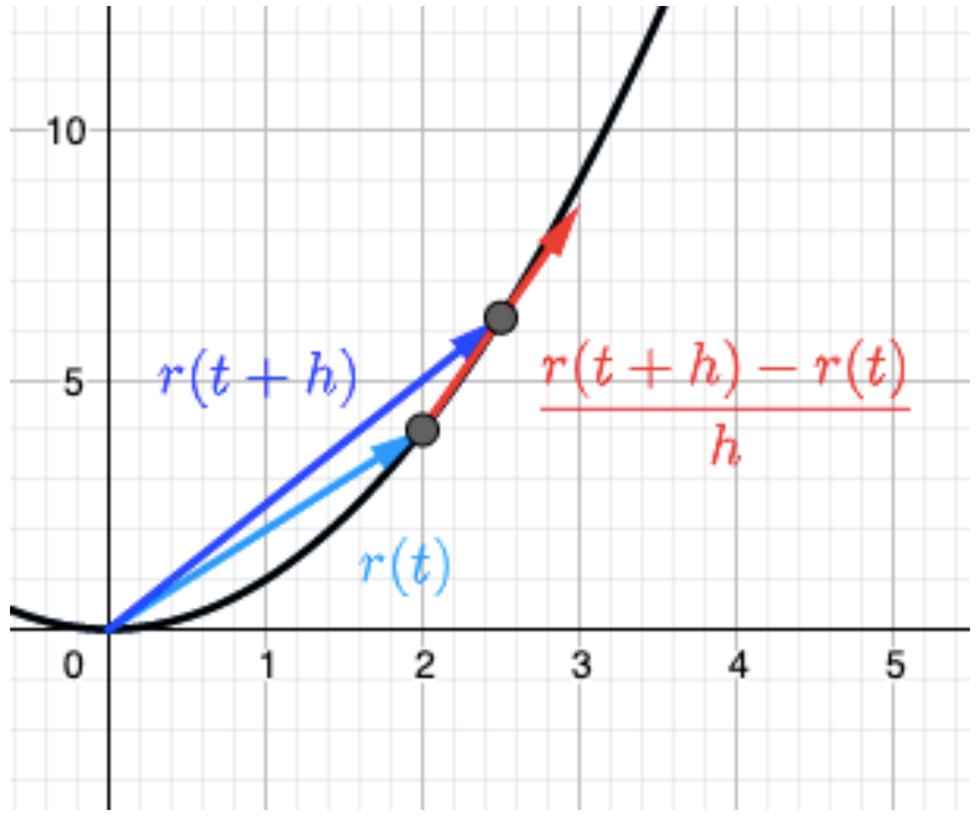
We'll start by working in the xy plane.
So our functions will look like

$$\mathbf{r}(t) = < x(t), y(t), 0 >$$

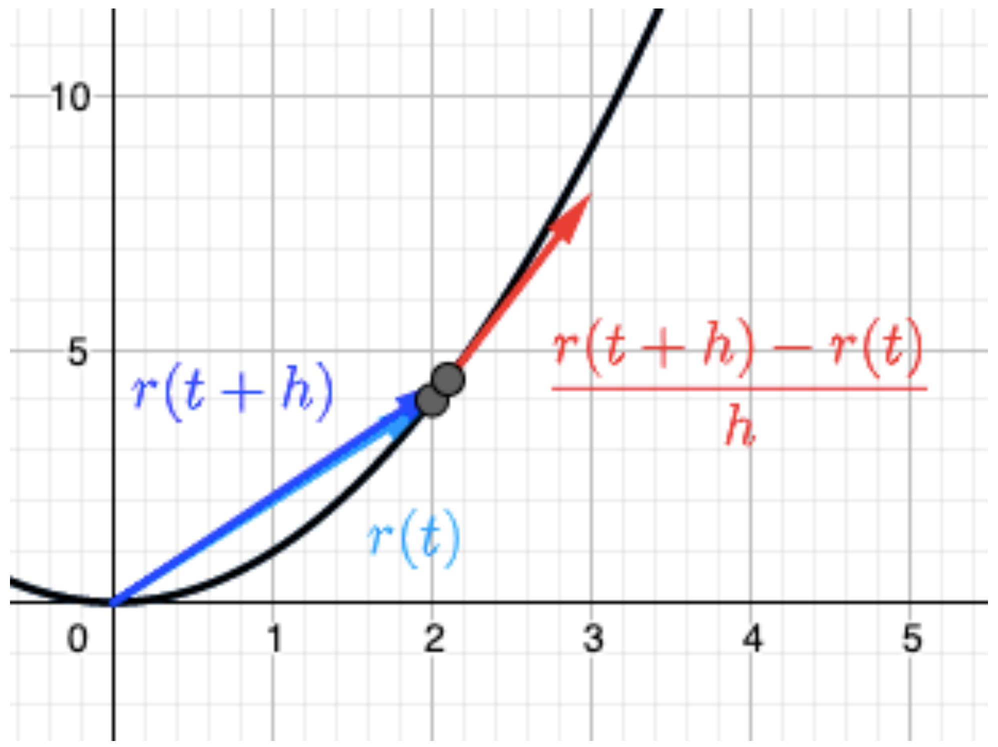
The velocity, in particular, is a tangent vector.



$\mathbf{r}(t+1) - \mathbf{r}(t)$



$\mathbf{r}(t+0.5) - \mathbf{r}(t)$



$\mathbf{r}(t+0.1) - \mathbf{r}(t)$

$$\mathbf{v}(t) := \mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} = < x'(t), y'(t), 0 >$$

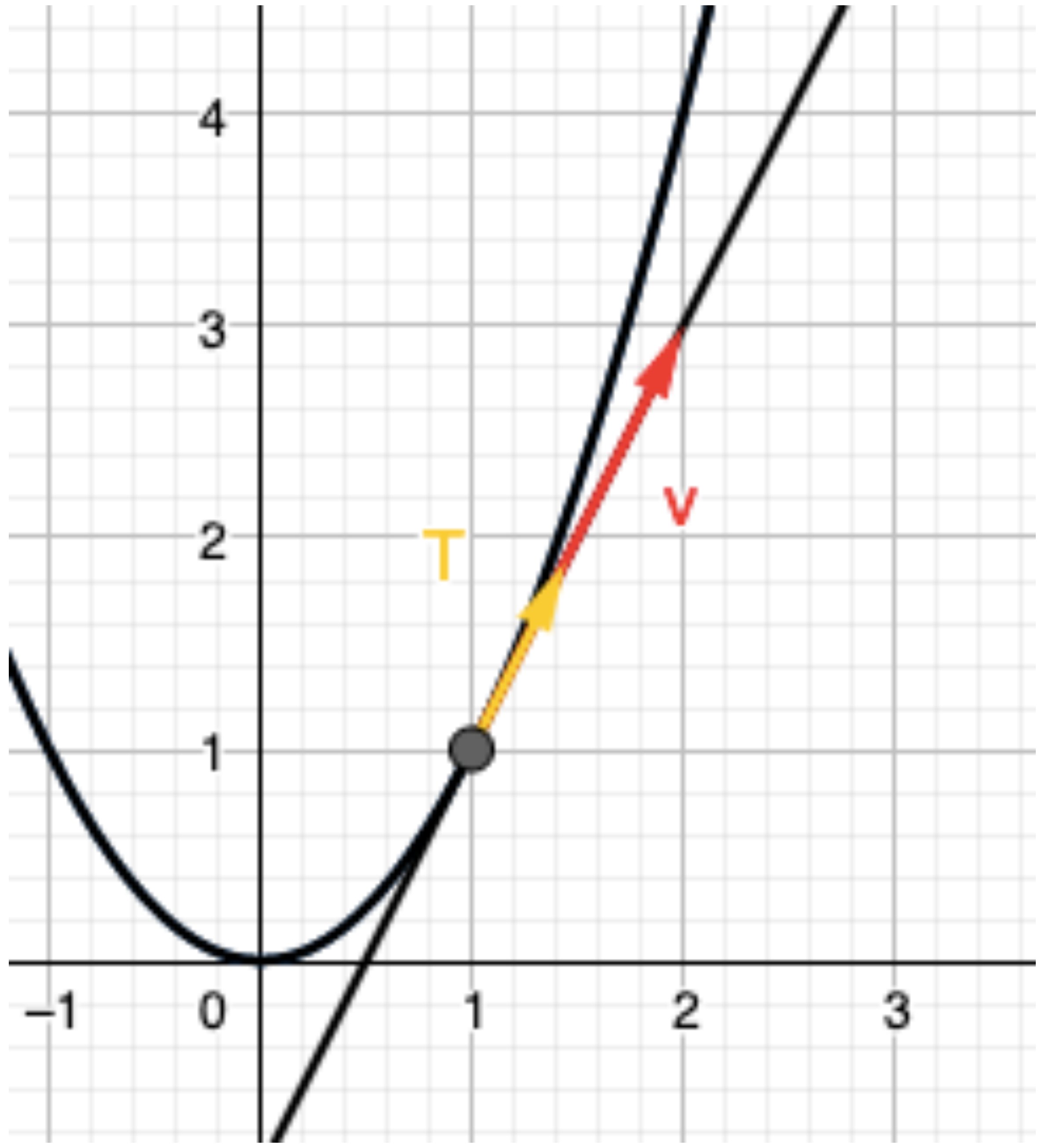
The derivative tells us the instantaneous velocity, $\mathbf{r}'(t) = \mathbf{v}(t)$, of a particle on the curve. It tells us the direction in which a particle is moving, and how quickly it's moving in that direction, the particle's speed.

Speed is the magnitude of velocity, $|\mathbf{v}(t)| = v(t)$.

To isolate a particle's direction, we can compute the unit tangent vector.

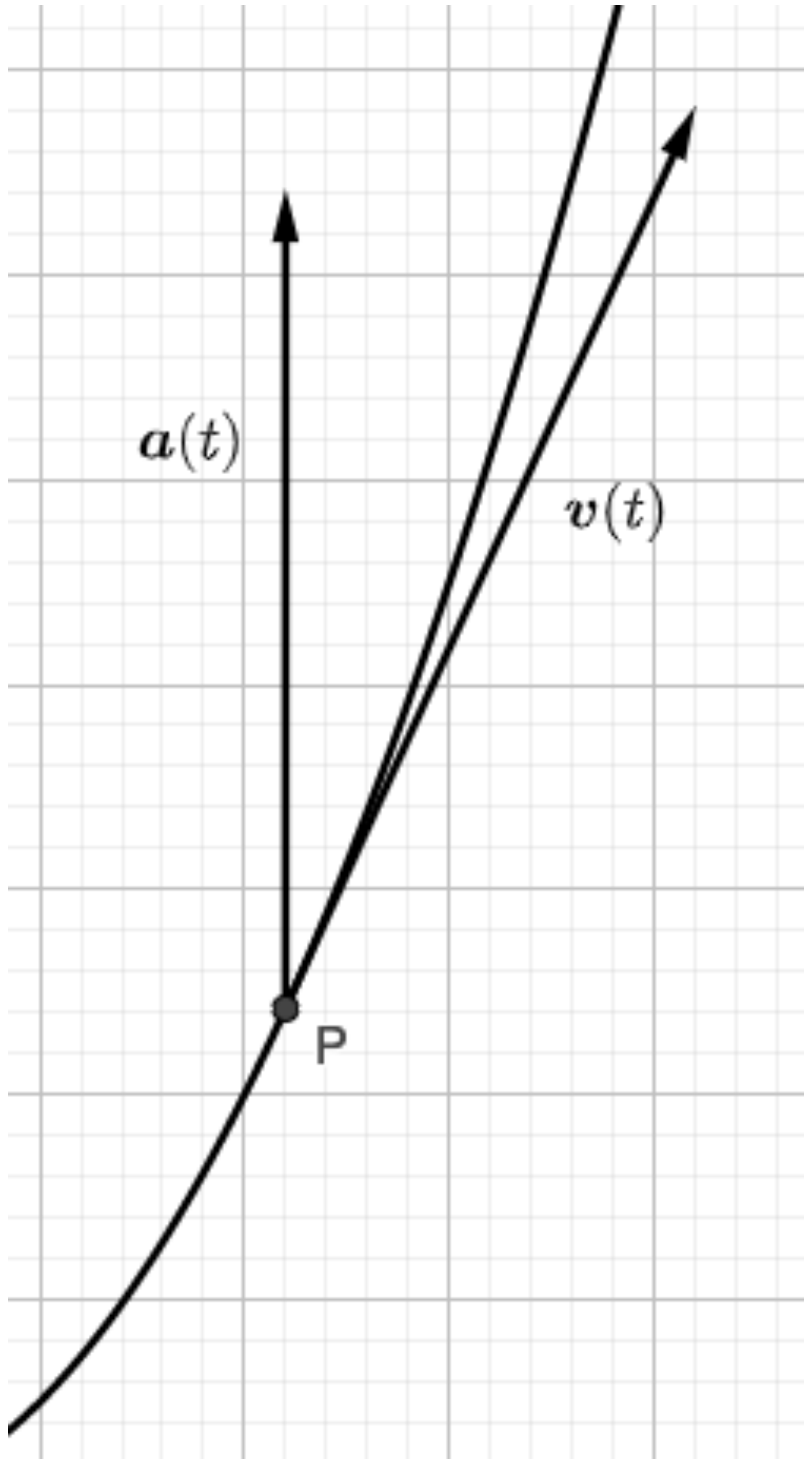
$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\mathbf{v}(t)}{v(t)}$$

We can use either $\mathbf{T}(t)$ or $\mathbf{v}(t) = v(t)\mathbf{T}(t)$ to describe the tangent line at any location along the curve.



Plane Curves, pg 2. Normal vectors.

We have seen acceleration is the derivative of velocity, $\mathbf{v}'(t) = \mathbf{a}(t)$.

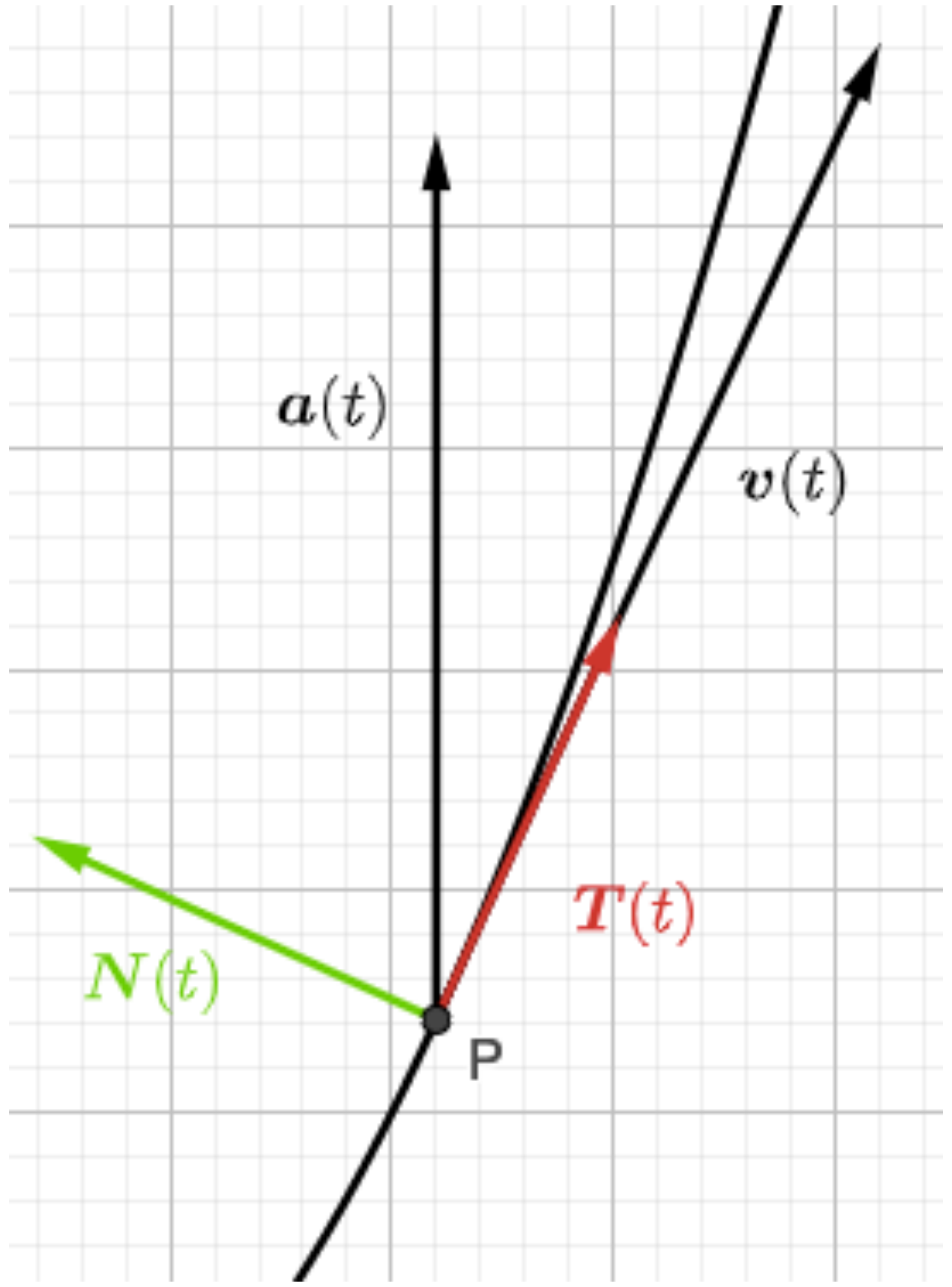


Acceleration indicates how the velocity is changing.

The derivative of $\mathbf{T}(t)$, meanwhile, is perpendicular to the curve.

The unit vector in this direction is

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$$



$\mathbf{N}$ tells us the direction in which the particle is curving.

Why must $\mathbf{T}'(\mathbf{t})$ be perpendicular to $\mathbf{T}(t)$?

Instance of this theorem:
If a vector-valued function $\mathbf{r}(t)$ has a constant magnitude over time, then $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$

Proof:

Suppose $|\mathbf{r}(t)| = k$ for all t in the domain of $\mathbf{r}$. Then…

$$k^2 = |\mathbf{r}(t)|^2 = \mathbf{r}(t) \cdot \mathbf{r}(t)$$

Take the derivative of both sides. use the product rule on the right.

$$0 = \mathbf{r}'(t) \cdot \mathbf{r}(t) + \mathbf{r}(t) \cdot \mathbf{r}'(t)$$

$$0 = 2\mathbf{r}(t) \cdot \mathbf{r}'(t)$$

$$0 = \mathbf{r}(t) \cdot \mathbf{r}'(t)$$

Plane Curves, pg 3. Some practice.

Given $\mathbf{r}(t)$, compute $\mathbf{v}(t), \mathbf{a}(t), \mathbf{T}(t), \mathbf{N}(t)$

Example1.

$$\mathbf{r}(t) = < t, t^2, 0 >$$

$$\mathbf{v}(t) = < 1, 2t, 0 >$$

$$\mathbf{a}(t) = < 0, 2, 0 >$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}$$

$$|\mathbf{v}(t)| = \sqrt{1+4t^2}$$

$$\mathbf{T}(t) = (1+4t^2)^{-1/2} < 1, 2t, 0 >$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$$

$$\mathbf{T}'(t) = -\frac{1}{2}(1+4t^2)^{-3/2} \cdot 8t < 1, 2t, 0 >$$

$$+(1+4t^2)^{-1/2} < 0, 2, 0 > = \dots$$

$$\dots = (1+4t^2)^{-3/2}\Big(-4t < 1, 2t, 0 > + (1+4t^2) < 0, 2, 0 > \Big)$$

$$= (1+4t^2)^{-3/2} < -4t, 2, 0 >$$

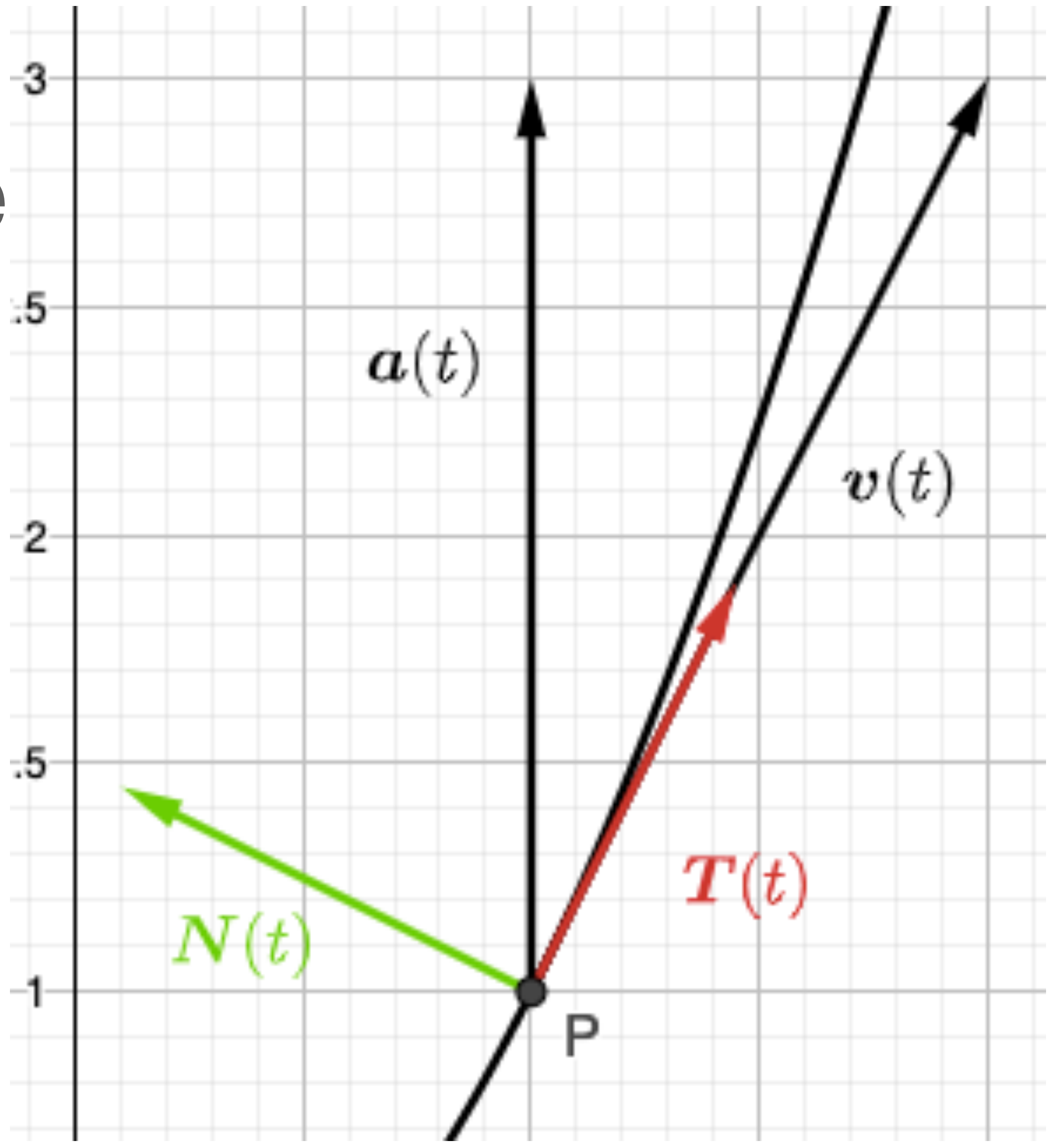
$$|\mathbf{T}'(t)| = (1+4t^2)^{-3/2} | < -4t, 2, 0 > |$$

$$= (1+4t^2)^{-3/2}(16t^2+4)^{1/2}$$

$$= 2(1+4t^2)^{-1}$$

$$\mathbf{N}(t) = (1+4t^2)^{-1/2} < -2t, 1, 0 >$$

picture when $t = 1$ $\rightarrow$

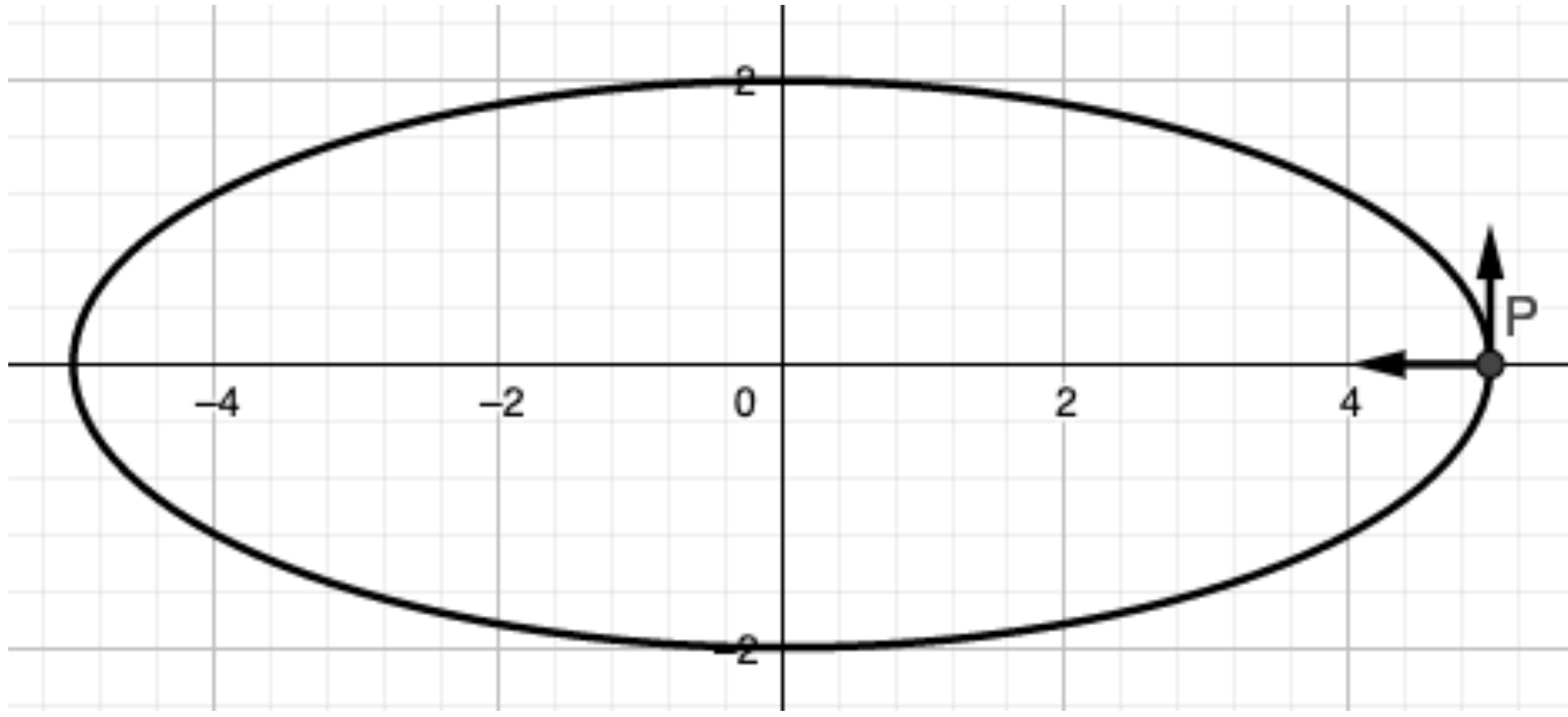


Link: MotionOnParabola

Example2.

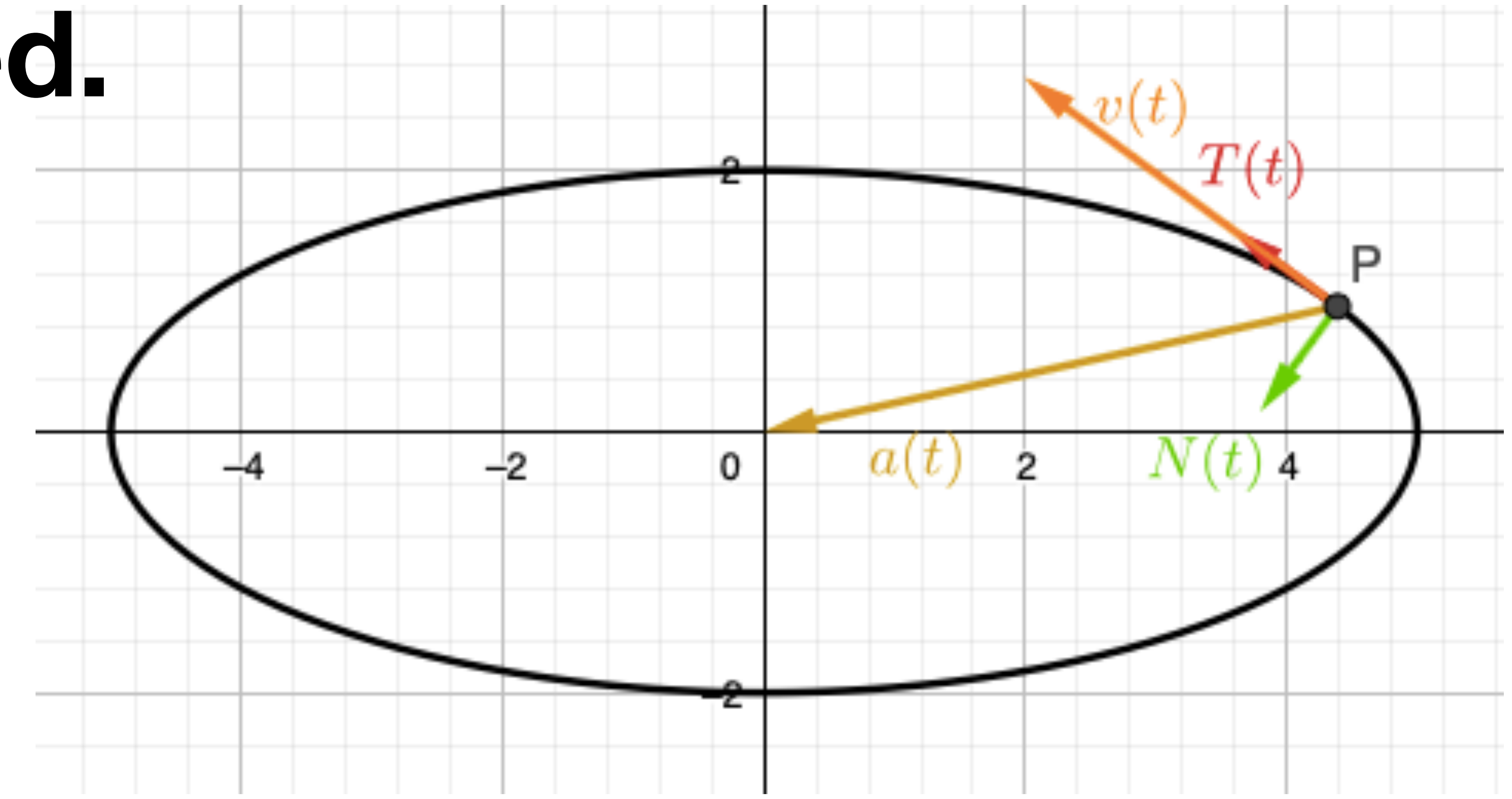
$$\mathbf{r}(t) = < a\cos(t), b\sin(t), 0 >$$

$\mathbf{r}(t)$ traverses an ellipse on the xy plane.



$\mathbf{T}(t) = ?$ $\mathbf{N}(t) = ?$

Plane Curves, pg 4. Some practice, continued.



$$\mathbf{r}(t) = < a\cos(t), b\sin(t), 0 >$$

$$\mathbf{v}(t) = < -a\sin(t), b\cos(t), 0 >$$

$$\mathbf{a}(t) = < -a\cos(t), -b\sin(t), 0 >$$

$$|\mathbf{v}(t)| = (a^2\sin^2(t) + b^2\cos^2(t))^{1/2}$$

$$\mathbf{T}(t) = (a^2\sin^2(t) + b^2\cos^2(t))^{-1/2} < -a\sin(t), b\cos(t), 0 >$$

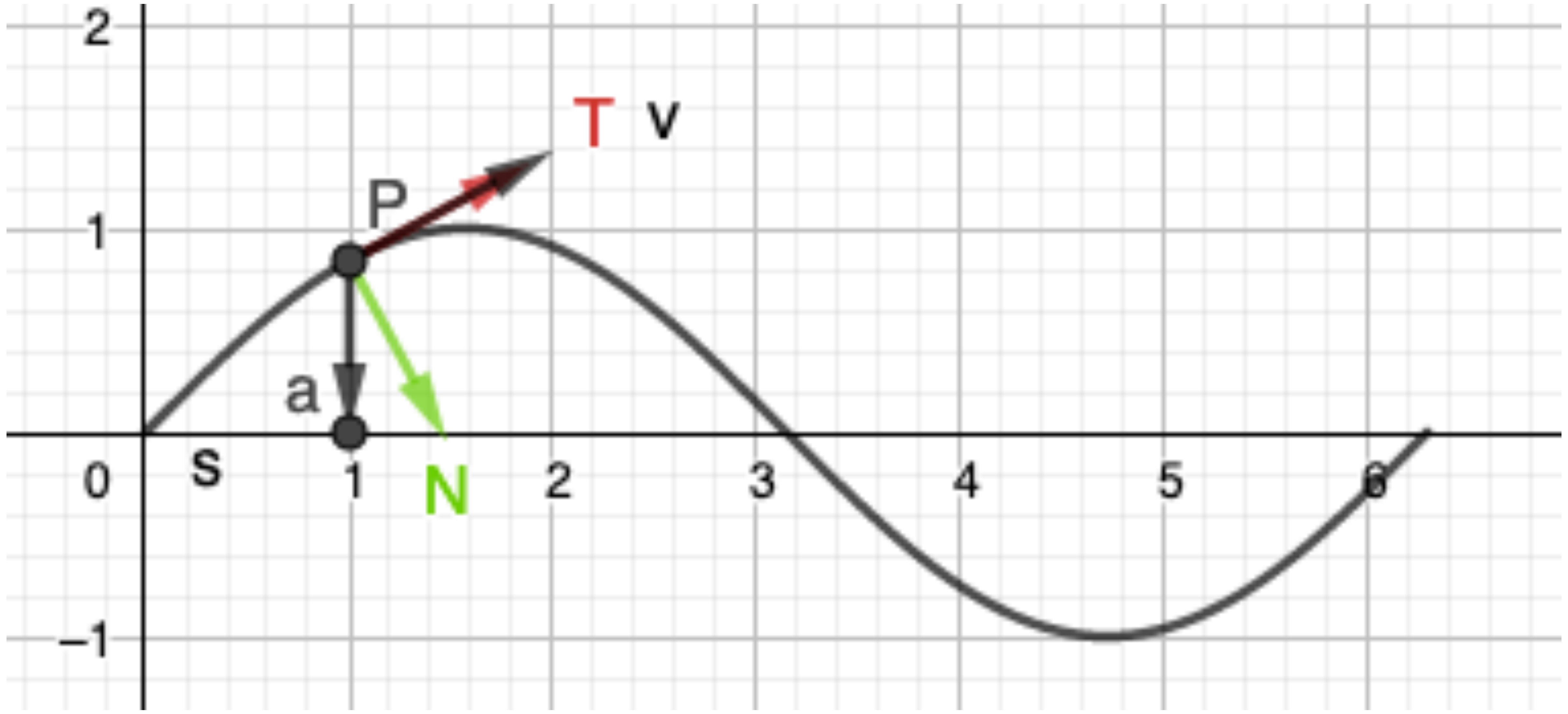
$$\mathbf{T}'(t) = -\frac{1}{2}(a^2\sin^2(t) + b^2\cos^2(t))^{-3/2} \cdot (2a^2\sin(t)\cos(t) - 2b^2\cos(t)\sin(t)) < -a\sin(t), b\cos(t), 0 >$$
$$+(a^2\sin^2(t) + b^2\cos^2(t))^{-1/2} < -a\cos(t), -b\sin(t), 0 >$$
$$= (a^2\sin^2(t) + b^2\cos^2(t))^{-3/2}\big[(-a^2\sin(t)\cos(t) + b^2\cos(t)\sin(t)) < -a\sin(t), b\cos(t), 0 >$$
$$+(a^2\sin^2(t) + b^2\cos^2(t)) < -a\cos(t), -b\sin(t), 0 > \big]$$
$$= (a^2\sin^2(t) + b^2\cos^2(t))^{-3/2}\big[< a^3\sin^2(t)\cos(t) - ab^2\sin^2(t)\cos(t), -a^2b\sin(t)\cos^2(t) + b^3\cos^2(t)\sin(t), 0 >$$
$$+ < -a^3\sin^2(t)\cos(t) - ab^2\cos^3(t), -a^2b\sin^3(t) - b^3\cos^2(t)\sin(t), 0 > \big]$$
$$= (a^2\sin^2(t) + b^2\cos^2(t))^{-3/2} < -ab^2\cos(t), -ba^2\sin(t), 0 >$$

$$|\mathbf{T}'(t)| = (a^2\sin^2(t) + b^2\cos^2(t))^{-3/2}(a^2b^4\cos^2(t) + b^2a^4\sin^2(t))^{1/2}$$
$$= ab(a^2\sin^2(t) + b^2\cos^2(t))^{-1}$$

$$\mathbf{N}(t) = (a^2\sin^2(t) + b^2\cos^2(t))^{-1/2} < -b\cos(t), -a\sin(t), 0 >$$

Planetary motion, link: <u>Orbit</u>

Plane Curves, pg 5. You try.



Given $\mathbf{r}(t)$, compute $\mathbf{v}(t), \mathbf{a}(t), \mathbf{T}(t), \mathbf{N}(t)$

$$\mathbf{r}(t) = < t, \sin(t), 0 >$$

$$\mathbf{v}(t) = < 1, \cos(t), 0 >$$

$$\mathbf{a}(t) = < 0, -\sin(t), 0 >$$

$$|\mathbf{v}(t)| = (1+\cos^2(t))^{1/2}$$

$$\mathbf{T}(t) = (1+\cos^2(t))^{-1/2} < 1, \cos(t), 0 >$$

$$\mathbf{T}'(t) = -\frac{1}{2}(1+\cos^2(t))^{-3/2}(-2\cos(t)\sin(t)) < 1, \cos(t), 0 > + (1+\cos^2(t))^{-1/2} < 0, -\sin(t), 0 > = \ldots$$

$$\mathbf{T}'(t) = \ldots = (1+\cos^2(t))^{-3/2}\Big[\cos(t)\sin(t) < 1, \cos(t), 0 > + (1+\cos^2(t)) < 0, -\sin(t), 0 > \Big]$$

$$= (1+\cos^2(t))^{-3/2} < \cos(t)\sin(t), -\sin(t), 0 >$$

$$= \sin(t)(1+\cos^2(t))^{-3/2} < \cos(t), -1, 0 >$$

$$|\mathbf{T}'(t)| = |\sin(t)(1+\cos^2(t))^{-3/2}| \cdot | < \cos(t), -1, 0 > |$$

$$= |\sin(t)| \cdot (1+\cos^2(t))^{-3/2}(\cos^2(t)+1)^{1/2}$$

$$= |\sin(t)|(1+\cos^2(t))^{-1}$$

$$\mathbf{N}(t) = \frac{\sin(t)}{|\sin(t)|}(1+\cos^2(t))^{-1/2} < \cos(t), -1, 0 >$$

$$= \begin{cases} (1+\cos^2(t))^{-1/2} < \cos(t), -1, 0 > & t \in (0, \pi) \\ (1+\cos^2(t))^{-1/2} < -\cos(t), 1, 0 > & t \in (\pi, 2\pi) \end{cases}$$

Space Curves, pg 1.

The concepts of $\mathbf{v}$, $\mathbf{a}$, $\mathbf{T}$, $\mathbf{N}$ extend to curves in 3D.

There's also the *binormal vector,*

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

Example.

$$\mathbf{r}(t) = <\sqrt{2}t, e^t, e^{-t}>$$

$$\mathbf{v}(t) = <\sqrt{2}, e^t, -e^{-t}>$$

$$\mathbf{a}(t) = <0, e^t, e^{-t}>$$

$$|\mathbf{v}(t)| = (2 + e^{2t} + e^{-2t})^{1/2}$$
$$= e^t + e^{-t}$$

$$\mathbf{T}(t) = (e^t + e^{-t})^{-1} <\sqrt{2}, e^t, -e^{-t}>$$

$$\mathbf{T}'(t) = -(e^t + e^{-t})^{-2}(e^t - e^{-t}) <\sqrt{2}, e^t, -e^{-t}>$$
$$+(e^t + e^{-t})^{-1} <0, e^t, e^{-t}>$$

$$= (e^t + e^{-t})^{-2}\left[(e^{-t} - e^t) <\sqrt{2}, e^t, -e^{-t}> + (e^t + e^{-t}) <0, e^t, e^{-t}>\right] = \dots$$

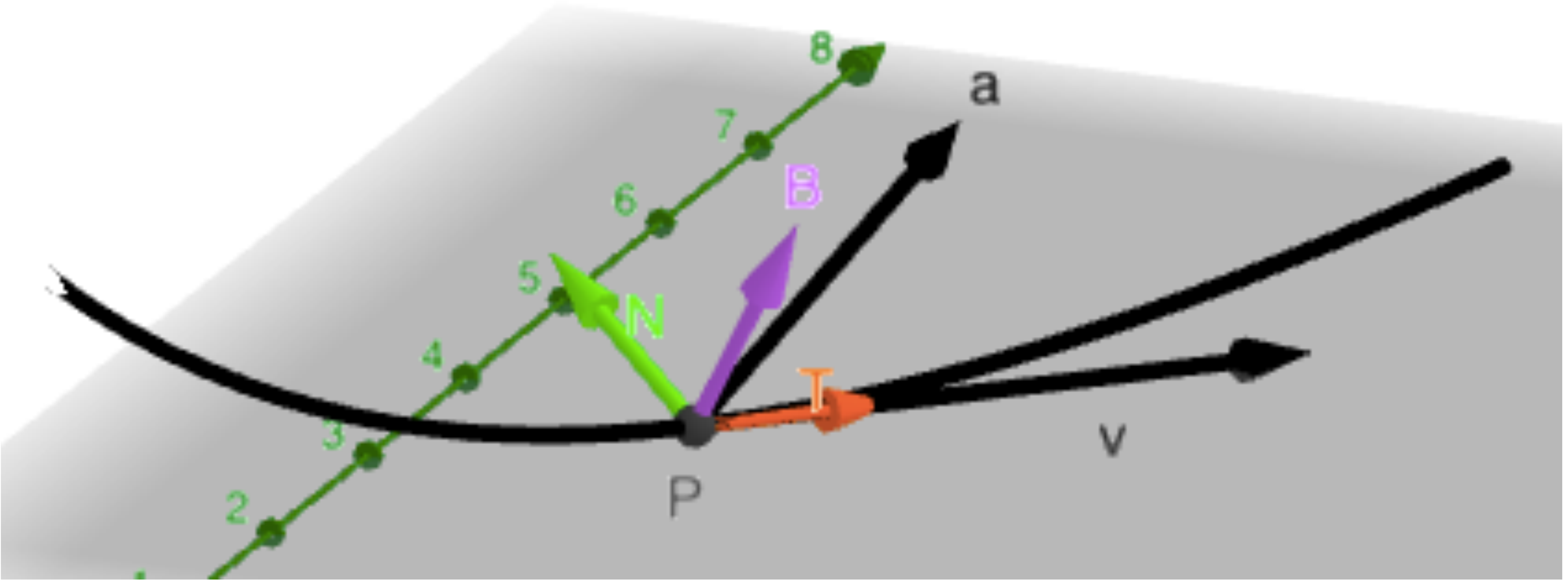
$$\mathbf{T}'(t) = \dots = (e^t + e^{-t})^{-2} <\sqrt{2}(e^{-t} - e^t), 2, 2>$$

$$|\mathbf{T}'(t)| = \sqrt{2}(e^t + e^{-t})^{-2}((e^{-t} - e^t)^2 + 2 + 2)^{1/2} = \sqrt{2}(e^t + e^{-t})^{-1}$$

$$\mathbf{N}(t) = (e^t + e^{-t})^{-1} <e^{-t} - e^t, \sqrt{2}, \sqrt{2}>$$

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) \quad \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sqrt{2} & e^t & -e^{-t} \\ e^{-t} - e^t & \sqrt{2} & \sqrt{2} \end{pmatrix}$$
$$= (e^t + e^{-t})^{-2} \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sqrt{2} & e^t & -e^{-t} \\ e^{-t} - e^t & \sqrt{2} & \sqrt{2} \end{pmatrix}$$

$$= (e^t + e^{-t})^{-2} <\sqrt{2}(e^t + e^{-t}), -1 - e^{-2t}, 1 + e^{2t}>$$



Link: HyperbolicRollerCoaster

Space Curves, pg 2.

Given $\mathbf{r}(t)$, compute $\mathbf{v}(t)$, $\mathbf{a}(t)$, $\mathbf{T}(t)$, $\mathbf{N}(t)$ and $\mathbf{B}(t)$

$$\mathbf{r}(t) = < \cos(t), t, \sin(t) >$$

$$\mathbf{v}(t) = < -\sin(t), 1, \cos(t) >$$

$$\mathbf{a}(t) = < -\cos(t), 0, -\sin(t) >$$

$$|\mathbf{v}(t)| = \sqrt{2}$$

$$\mathbf{T}(t) = \frac{1}{\sqrt{2}} < -\sin(t), 1, \cos(t) >$$

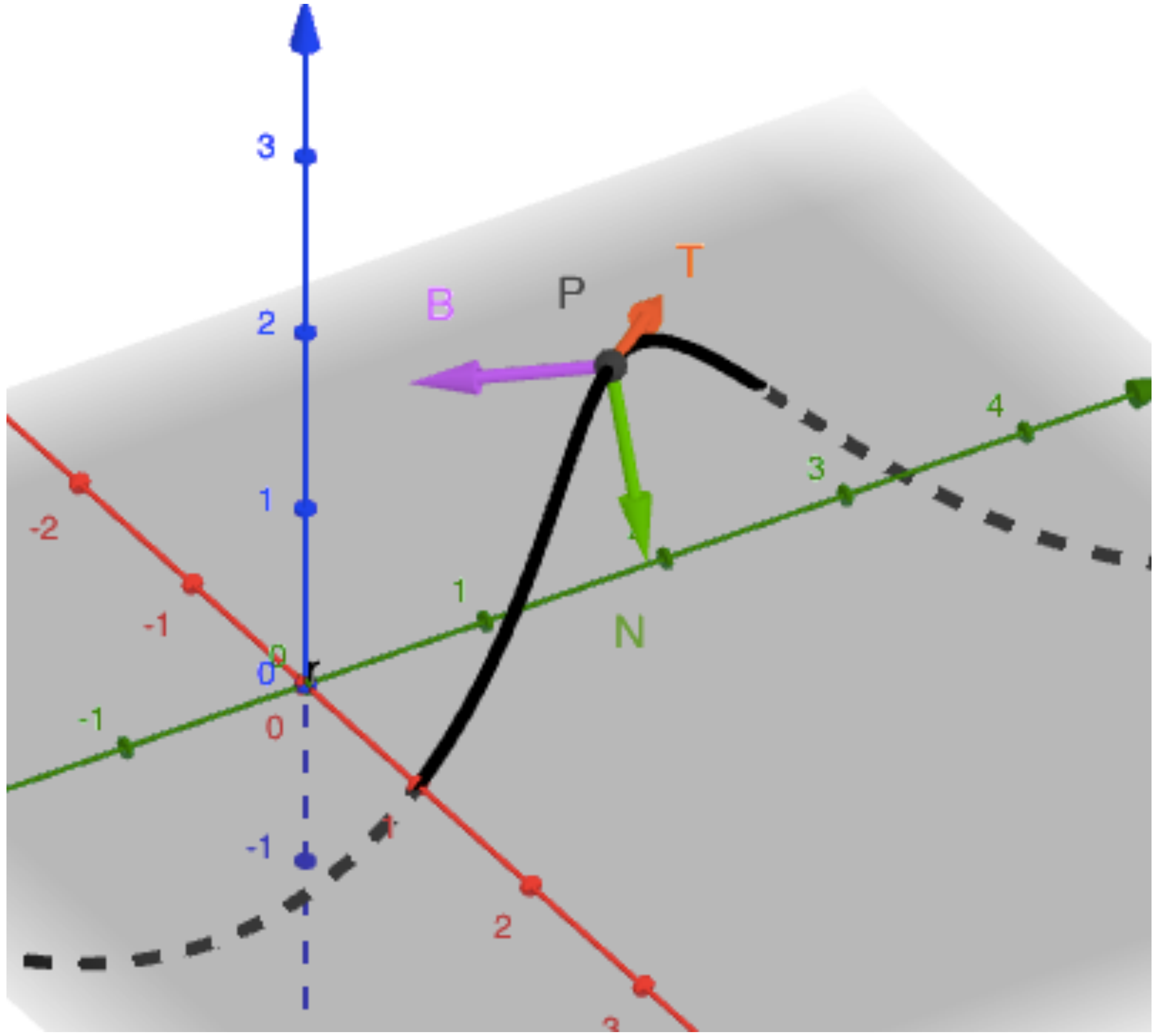
$$\mathbf{T}'(t) = \frac{1}{\sqrt{2}} < -\cos(t), 0, -\sin(t) >$$

$$|\mathbf{T}'(t)| = \frac{1}{\sqrt{2}}$$

$$\mathbf{N}(t) = < -\cos(t), 0, -\sin(t) >$$

$$\mathbf{B}(t) = \frac{1}{\sqrt{2}} \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin(t) & 1 & \cos(t) \\ -\cos(t) & 0 & -\sin(t) \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} < -\sin(t), -1, \cos(t) >$$



Link: RideTheHelix

Link (wikipedia): Frenet_Serret_Formulas