M110C Week4

Goals:

- Recap, Warmup.
- Parametrization.
- Arc-Length.
- Curvature.
- Components of Acceleration.
- Orthonormal Sets of Vectors.
- Torsion.
- Optional: Find the curve, given curvature and torsion.

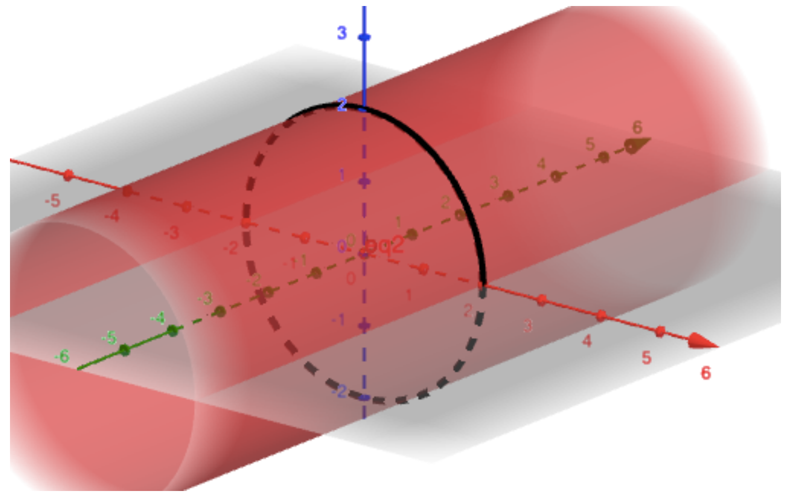
Recap, pg 1.

What did we see last week?

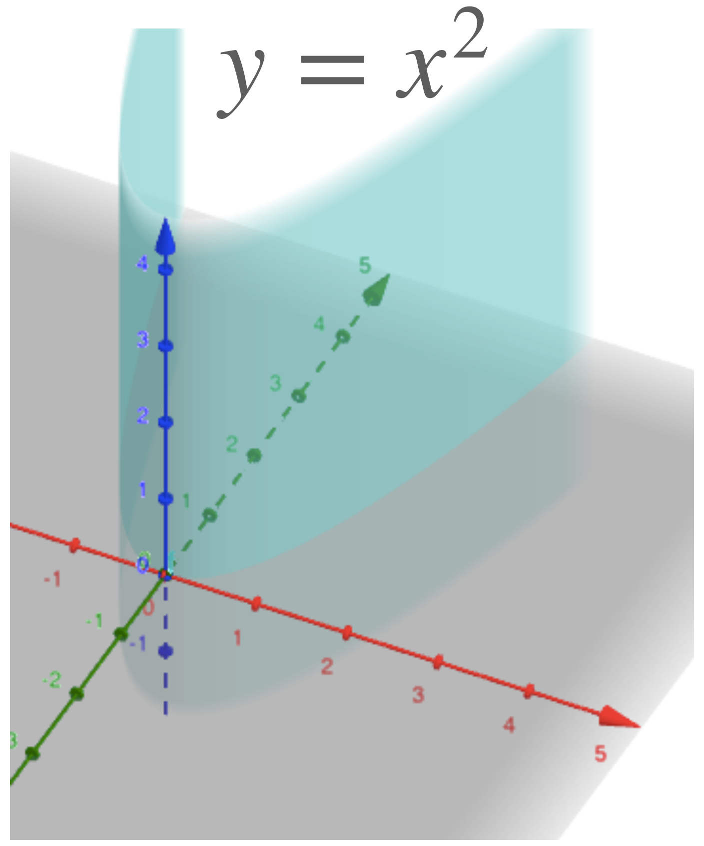
- Some 3D surfaces, including…

… Cylinders!

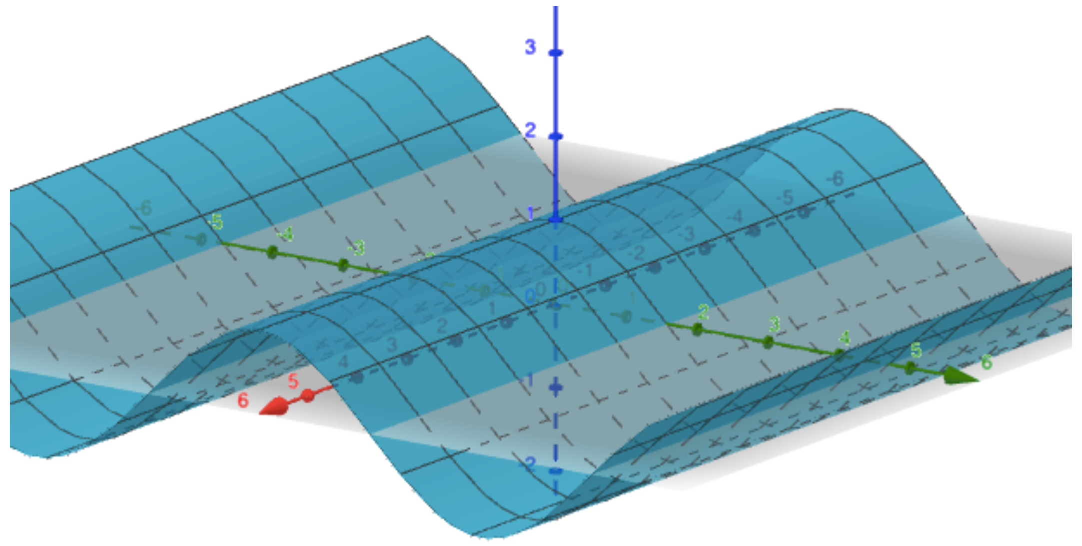
$x^2 + z^2 = 4$



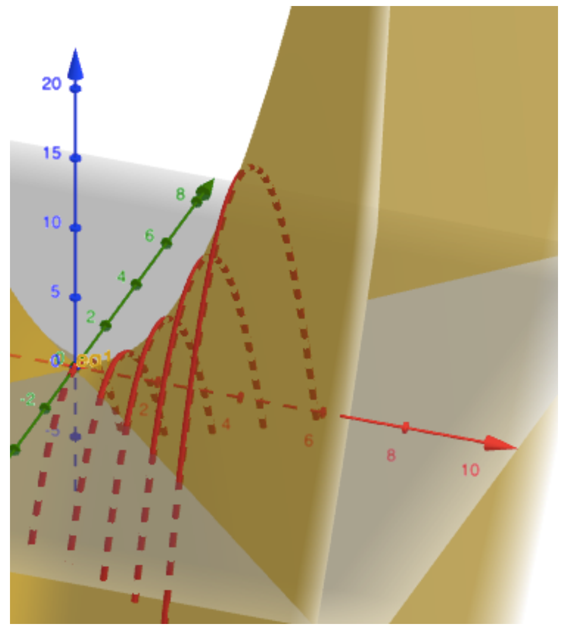
$y = x^2$

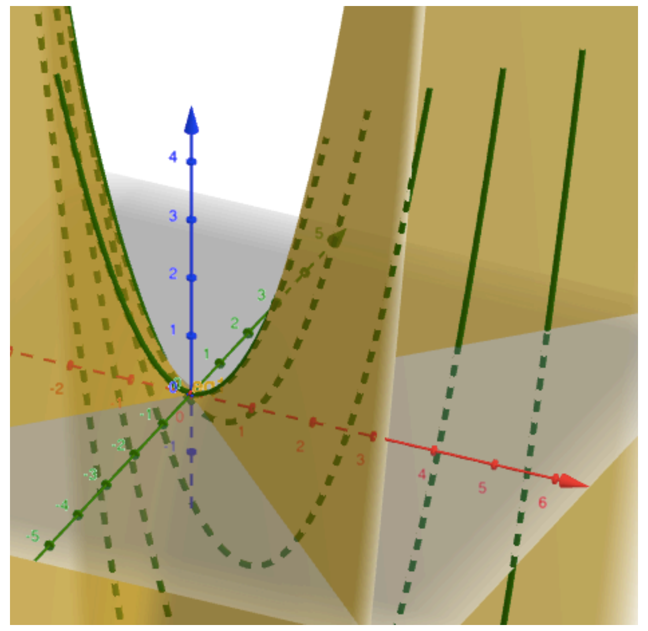


$z = \cos(y)$

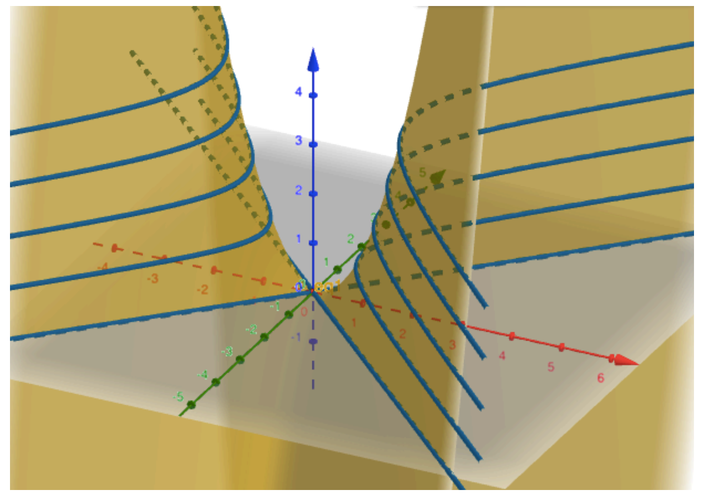


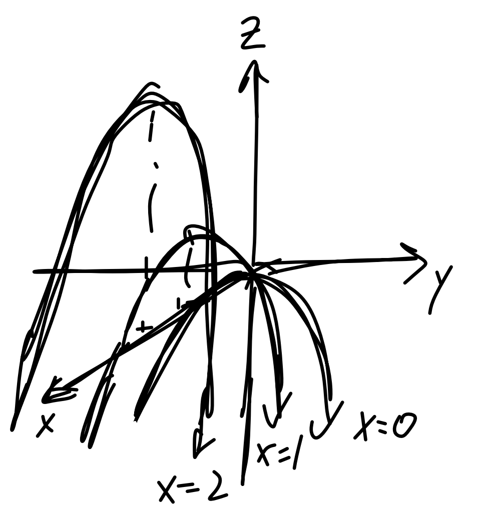
… Quadric Surfaces.

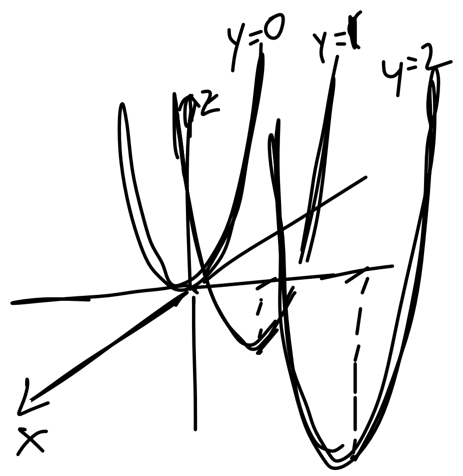


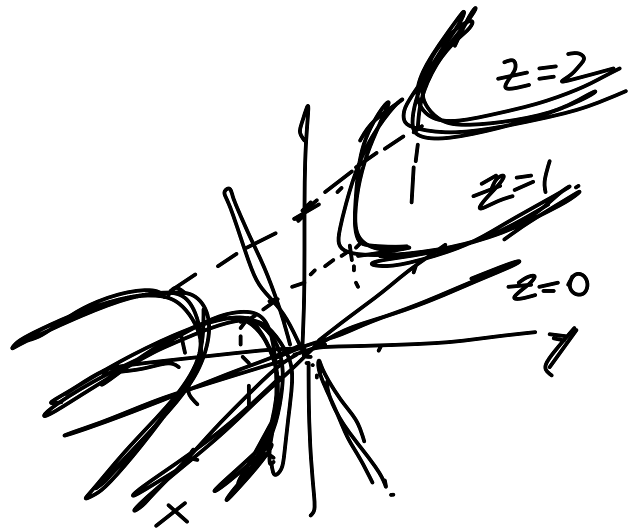


$z = x^2 - y^2$







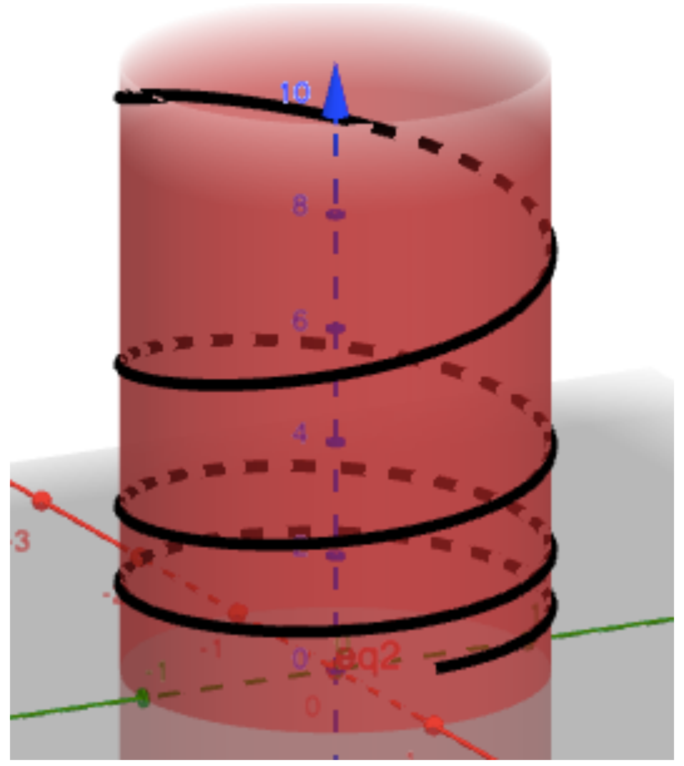


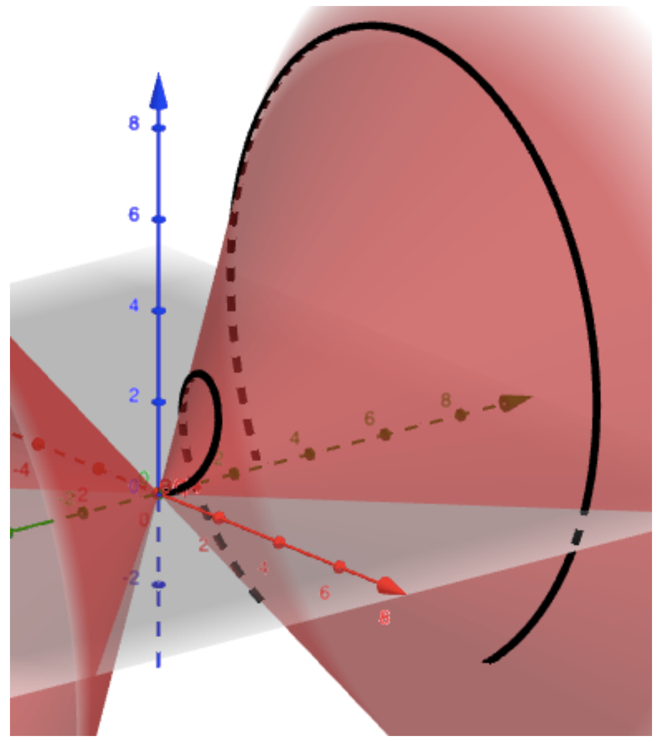
- Vector-Valued Functions and their graphs - curves in 3 dimensions.

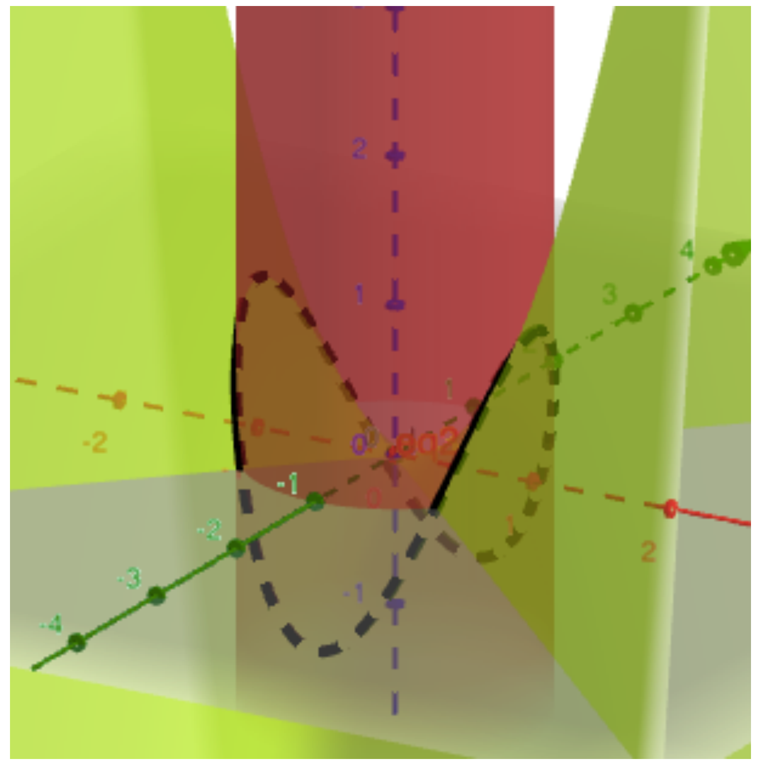
$\mathbf{r}_1(t) = < t\cos(t), t, t\sin(t) >\ ,\quad t \geq 0$

$\mathbf{r}_4(t) = < \cos(t), \sin(t), \cos(2t) >$

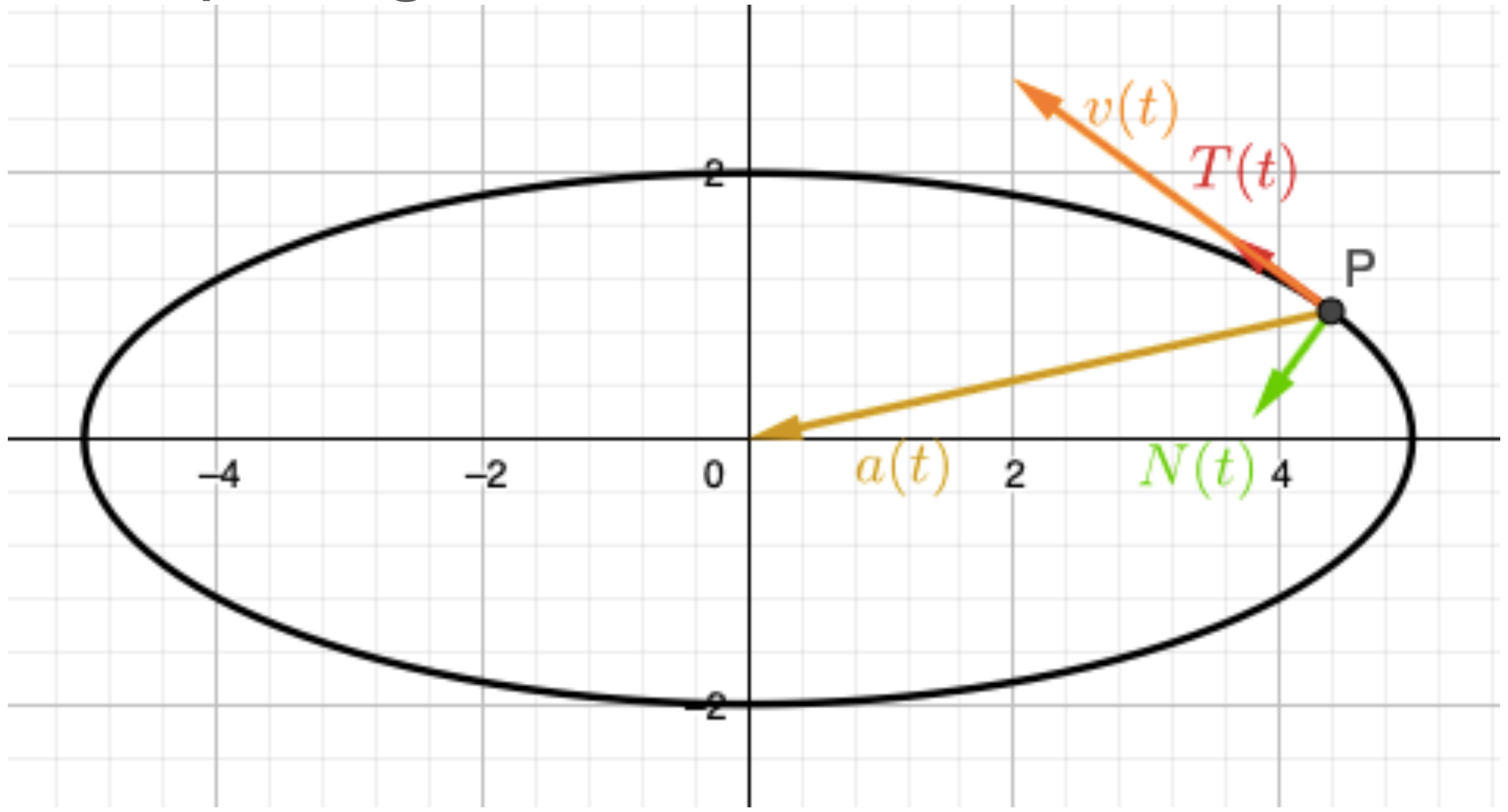
$\mathbf{r}_5(t) = < \cos(8t), \sin(8t), e^{0.8t} >\ ,\ t \geq 0$







- Calculus of Vector-Valued Functions: computing $\mathbf{v}$, $\mathbf{a}$ and $\mathbf{T}$, $\mathbf{N}$



$\mathbf{v}$ and $\mathbf{a}$ are velocity and acceleration.

$\mathbf{T}(t)$ is the unit tangent vector at the point $\mathbf{r}(t)$

$\mathbf{N}(t)$ is the unit normal vector at the point $\mathbf{r}(t)$

Warm-Up, pg 1.

Given $\mathbf{a}$, and some initial values, compute $\mathbf{r}$.

Example.

$$\mathbf{a}(t) = < 2t, \sin(t), \cos(t) >$$

$$\mathbf{v}(0) = < 1,0,0 > , \quad \mathbf{r}(0) = < 0,1,0 >$$

$$\mathbf{r}(t) = ???$$

$$\mathbf{v}(t) = \int \mathbf{a}(t)\, dt = < t^2 + c_1, -\cos(t) + c_2, \sin(t) + c_3 >$$

$$= < t^2, -\cos(t), \sin(t) > + \mathbf{c} \quad \text{where } \mathbf{c} = < c_1, c_2, c_3 >$$

$$\mathbf{v}(0) = < 0 + c_1, -1 + c_2, 0 + c_3 > = < 1,0,0 >$$

$$< c_1, c_2, c_3 > = < 1,1,0 >$$

$$\mathbf{v}(t) = < t^2 + 1, -\cos(t) + 1, \sin(t) >$$

$$\mathbf{r}(t) = \int \mathbf{v}(t)\, dt = < 1/3t^3 + t + k_1, -\sin(t) + t + k_2, -\cos(t) + k_3 >$$

$$\mathbf{r}(0) = < k_1, k_2, -1 + k_3 > = < 0,1,0 > , \ < k_1, k_2, k_3 > = < 0,1,1 >$$

$$\mathbf{r}(t) = < 1/3t^3 + t, -\sin(t) + t + 1, -\cos(t) + 1 >$$

Example. (S13.4 #18)

$$\mathbf{a}(t) = < t, e^t, e^{-t} >$$

$$\mathbf{v}(0) = \mathbf{k}, \quad \mathbf{r}(0) = \mathbf{j} + \mathbf{k}$$

$$\mathbf{r}(t) = ???$$

$$\mathbf{v}(t) = < 1/2t^2, e^t, -e^{-t} > + \mathbf{c}$$

$$\mathbf{v}(0) = < 0,1,-1 > + \mathbf{c} = \mathbf{k} = < 0,0,1 >$$

$$\mathbf{c} = < 0,-1,2 >$$

$$\mathbf{v}(t) = < 1/2t^2, e^t - 1, -e^{-t} + 2 >$$

$$\mathbf{r}(t) = < 1/6t^3, e^t - t, e^{-t} + 2t > + \mathbf{c}$$

$$\mathbf{r}(0) = < 0,1,1 > + \mathbf{c} = \mathbf{j} + \mathbf{k} = < 0,1,1 >$$

$$\mathbf{c} = < 0,0,0 >$$

$$\mathbf{r}(t) = < 1/6t^3, e^t - t, e^{-t} + 2t >$$

Recap, pg 2.

$\mathbf{v}, \mathbf{a}, \mathbf{T}, \mathbf{N}, \mathbf{B}$

Given $\mathbf{r}(t)\ldots$

$$\mathbf{v}(t) = \mathbf{r}'(t)$$

$$\mathbf{a}(t) = \mathbf{v}'(t)$$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$$

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

We did these calculations:

2. $\mathbf{r}(t) = < \sqrt{2}t, e^t, e^{-t} >$

$$\mathbf{v}(t) = < \sqrt{2}, e^t, -e^{-t} >$$

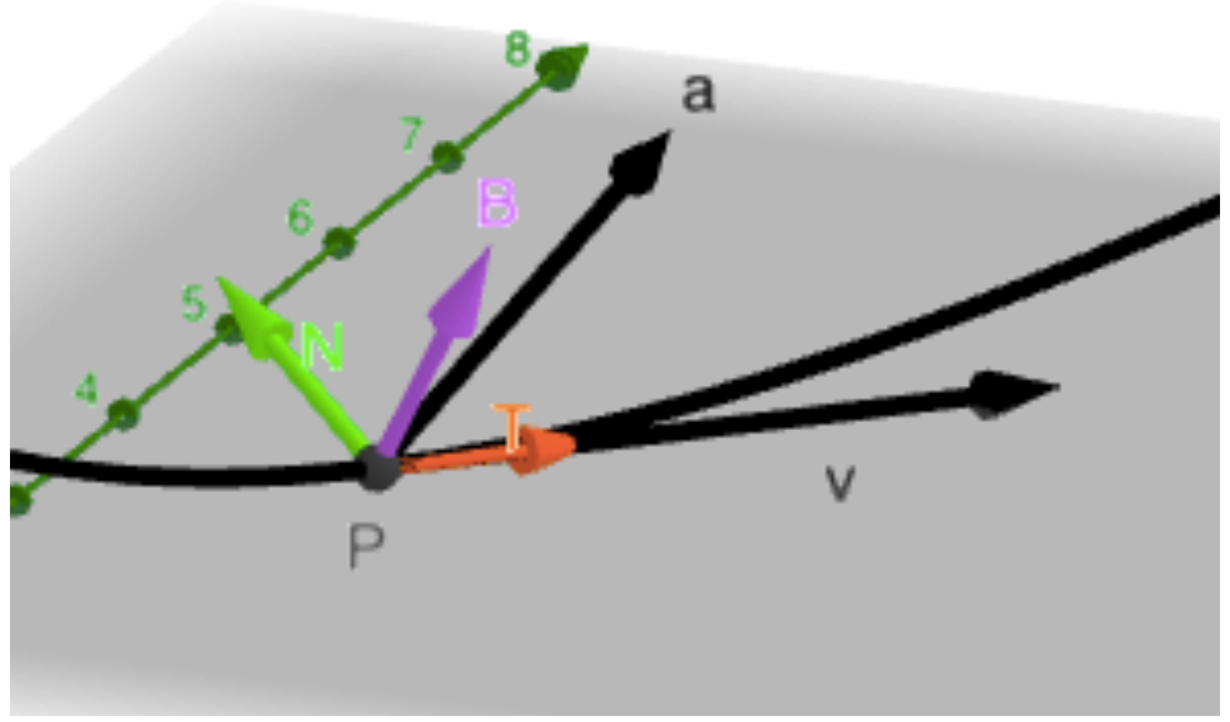
$$\mathbf{a}(t) = < 0, e^t, e^{-t} >$$

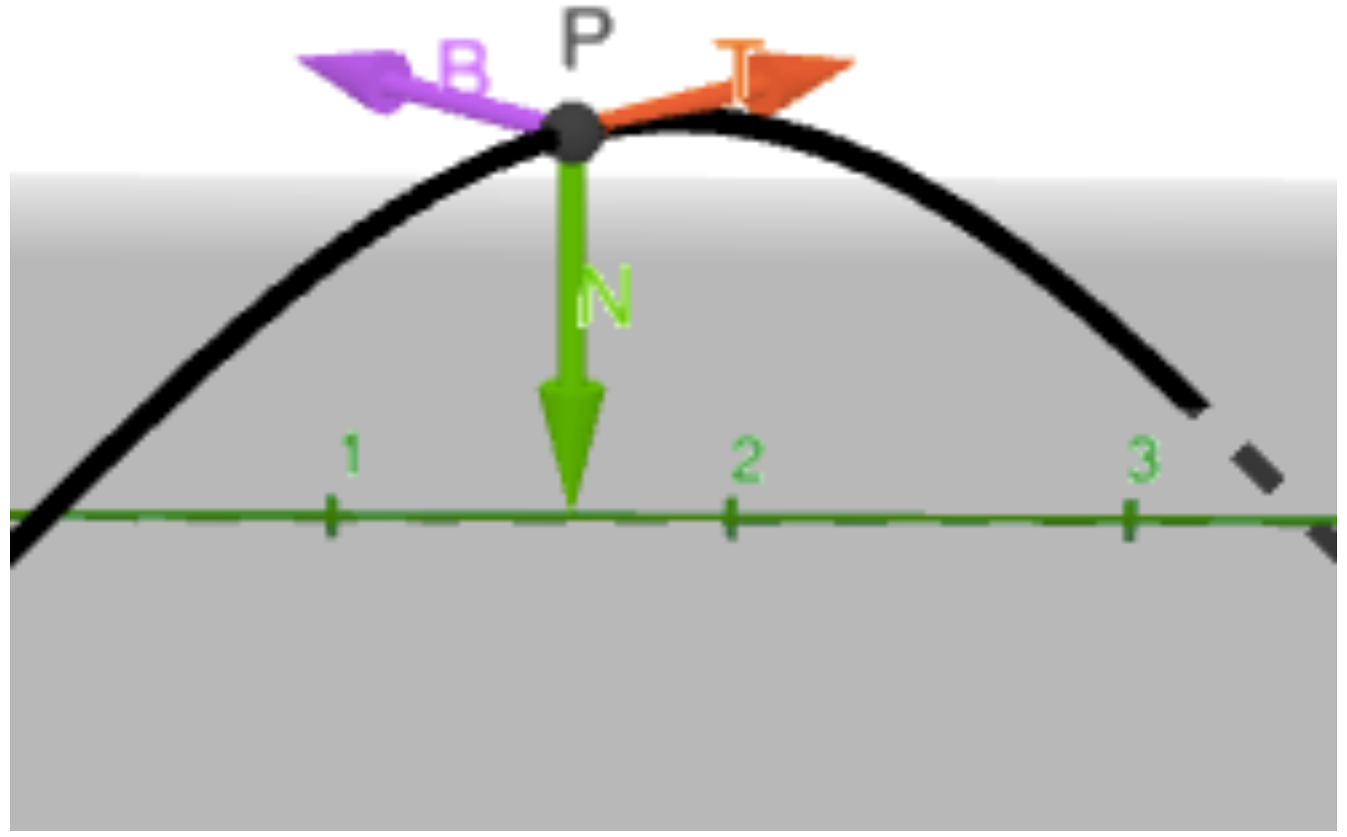
$$\mathbf{T}(t) = (e^t + e^{-t})^{-1} < \sqrt{2}, e^t, -e^{-t} >$$

$$\mathbf{N}(t) = (e^t + e^{-t})^{-1} < e^{-t} - e^t, \sqrt{2}, \sqrt{2} >$$

$$\mathbf{B}(t) = (e^t + e^{-t})^{-2} < \sqrt{2}(e^t + e^{-t}), -1 - e^{-2t}, 1 + e^{2t} >$$

$$= (e^t + e^{-t})^{-1} < \sqrt{2}, -e^{-t}, e^t >$$





3. $\mathbf{r}(t) = < \cos(t), t, \sin(t) >$

$$\mathbf{v}(t) = < -\sin(t), 1, \cos(t) >$$

$$\mathbf{a}(t) = < -\cos(t), 0, -\sin(t) >$$

$$\mathbf{T}(t) = \frac{1}{\sqrt{2}} < -\sin(t), 1, \cos(t) >$$

$$\mathbf{N}(t) = < -\cos(t), 0, -\sin(t) >$$

$$\mathbf{B}(t) = \frac{1}{\sqrt{2}} < -\sin(t), -1, \cos(t) >$$

Examples.

1. $\mathbf{r}(t) = < t, \sin(t), 0 >$

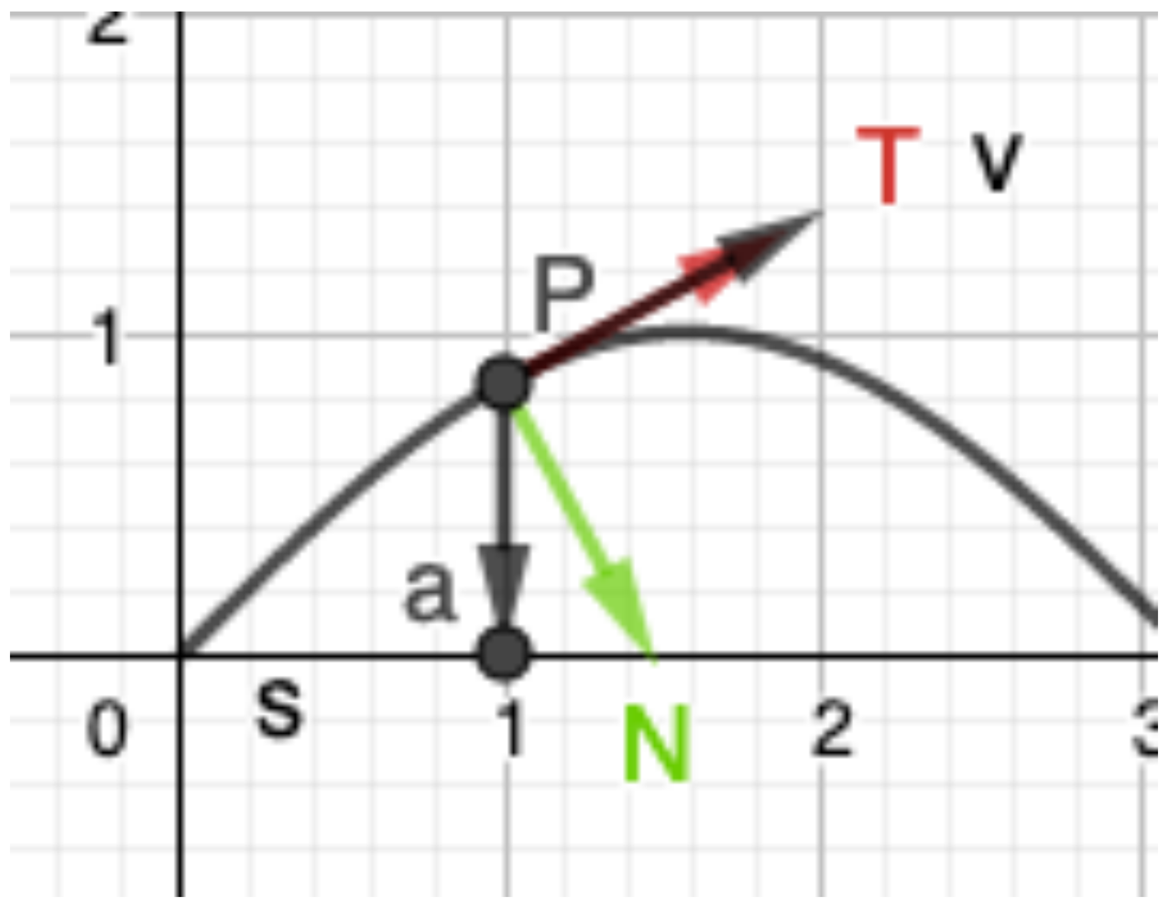
$$\mathbf{v}(t) = < 1, \cos(t), 0 >$$

$$\mathbf{a}(t) = < 0, -\sin(t), 0 >$$

$$\mathbf{T}(t) = (1 + \cos^2(t))^{-1/2} < 1, \cos(t), 0 >$$

$$\mathbf{N}(t) = (1 + \cos^2(t))^{-1/2} < \cos(t), -1, 0 >$$

$$t \in (0, \pi)$$



Warm-Up, pg 2.

Find $\mathbf{v}, \mathbf{a}, \mathbf{T}, \mathbf{N}, \mathbf{B}$ for the following:

1. $\mathbf{r}(u) = < e^u, e^{2u}, 0 > \; u \in \mathbf{R}$
2. $\mathbf{r}(t) = < t, t^2, t^3 > \; t \in \mathbf{R}$ (next slide)

1. $\mathbf{r}(u) = < e^u, e^{2u}, 0 > \; u \in \mathbf{R}$

$$\mathbf{v}(u) = < e^u, 2e^{2u}, 0 >$$

$$\mathbf{a}(u) = < e^u, 4e^{2u}, 0 >$$

$$|\mathbf{v}(u)| = (e^{2u} + 4e^{4u})^{1/2}$$
$$= e^u(1 + 4e^{2u})^{1/2}$$

$$\mathbf{T}(u) = (1 + 4e^{2u})^{-1/2} < 1, 2e^u, 0 >$$

$$\mathbf{T}'(u) = -\frac{1}{2}(1 + 4e^{2u})^{-3/2} 8e^{2u} < 1, 2e^u, 0 > + (1 + 4e^{2u})^{-1/2} < 0, 2e^u, 0 >$$

$$= (1 + 4e^{2u})^{-3/2}\left(-4e^{2u} < 1, 2e^u, 0 > + (1 + 4e^{2u}) < 0, 2e^u, 0 > \right)$$

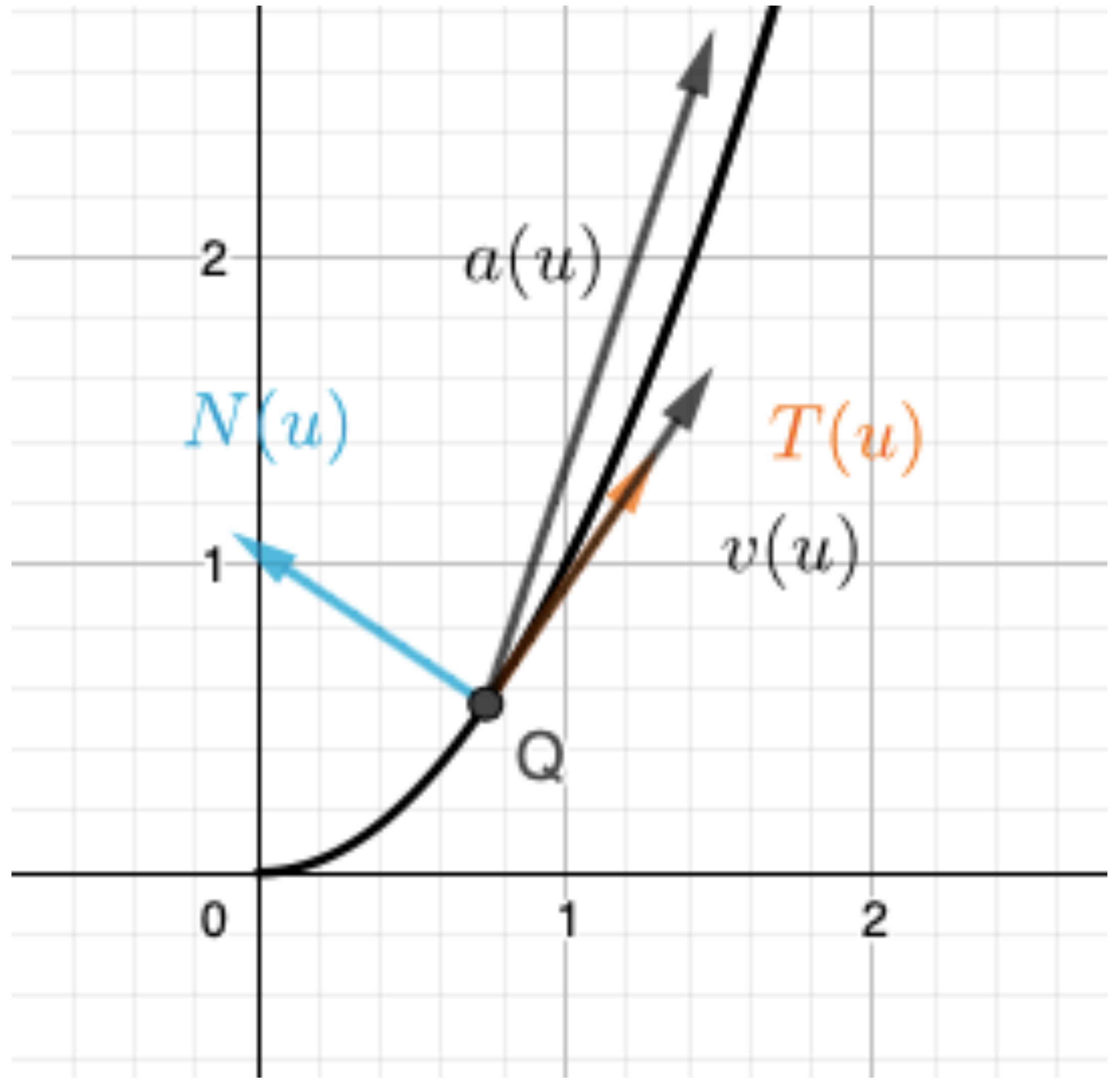
$$= (1 + 4e^{2u})^{-3/2} < -4e^{2u}, 2e^u, 0 >$$

$$= 2e^u(1 + 4e^{2u})^{-3/2} < -2e^u, 1, 0 >$$

$$|\mathbf{T}'(u)| = 2e^u(1 + 4e^{2u})^{-3/2} | < -2e^u, 1, 0 > |$$
$$= 2e^u(1 + 4e^{2u})^{-1}$$

$$\mathbf{N}(u) = (1 + 4e^{2u})^{-1/2} < -2e^u, 1, 0 >$$

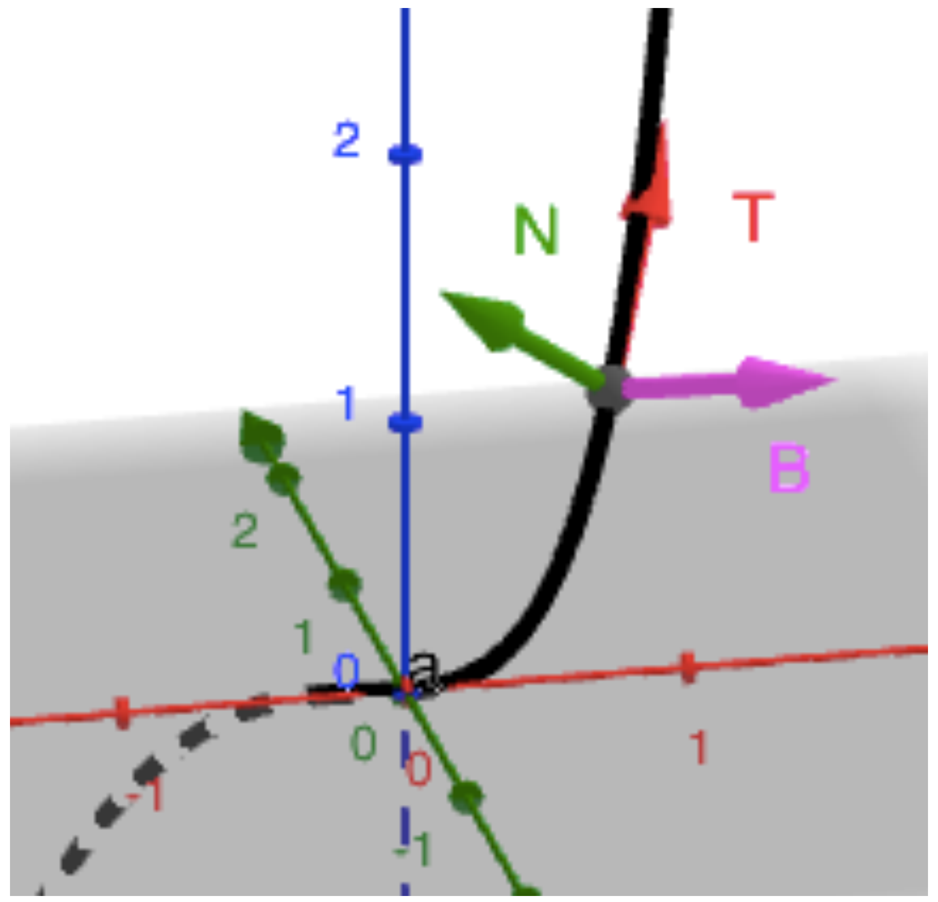
$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$
$$= (1 + 4e^{2u})^{-1} \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2e^u & 0 \\ -2e^u & 1 & 0 \end{pmatrix}$$
$$= < 0, 0, 1 >$$



Link: MotionOnParabola

Warm-Up, pg 3.

Link: TNB QuadraticCubic



2. $\mathbf{r}(t) = < t, t^2, t^3 > \quad t \in \mathbf{R}$

$$\mathbf{v}(t) = < 1,2t,3t^2 >$$

$$\mathbf{a}(t) = < 0,2,6t >$$

$$\mathbf{T}(t) = (1+4t^2+9t^4)^{-1/2} < 1,2t,3t^2 >$$

$$\mathbf{T}'(t) = -\frac{1}{2}(1+4t^2+9t^4)^{-3/2}(8t+36t^3) < 1,2t,3t^2 >$$
$$+(1+4t^2+9t^4)^{-1/2} < 0,2,6t >$$

$$= (1+4t^2+9t^4)^{-3/2}\Big[(-4t-18t^3) < 1,2t,3t^2 >$$
$$+(1+4t^2+9t^4) < 0,2,6t > \Big]$$

$$= \frac{< -4t-18t^3,\ 2-18t^4,\ 6t+12t^3 >}{(1+4t^2+9t^4)^{3/2}}$$

$$= \frac{2 < -2t-9t^3,\ 1-9t^4,\ 3t+6t^3 >}{(1+4t^2+9t^4)^{3/2}}$$

$$|\mathbf{T}'| = \frac{2\Big((-2t-9t^3)^2+(1-9t^4)^2+(3t+6t^3)^2\Big)^{1/2}}{(1+4t^2+9t^4)^{3/2}} = \ldots$$

$$\ldots = \frac{2(1+13t^2+54t^4+117t^6+81t^8)^{1/2}}{(1+4t^2+9t^4)^{3/2}}$$

$$= \frac{2(1+4t^2+9t^4)^{1/2}(1+9t^2+9t^4)^{1/2}}{(1+4t^2+9t^4)^{3/2}}$$

$$= \frac{2(1+9t^2+9t^4)^{1/2}}{(1+4t^2+9t^4)}$$

$$\mathbf{N}(t) = \frac{< -2t-9t^3,\ 1-9t^4,\ 3t+6t^3 >}{(1+4t^2+9t^4)^{1/2}(1+9t^2+9t^4)^{1/2}}$$

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

$$\mathbf{B}(t) = \frac{\det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2t & 3t^2 \\ -2t-9t^3 & 1-9t^4 & 3t+6t^3 \end{pmatrix}}{(1+4t^2+9t^4)(1+9t^2+9t^4)^{1/2}}$$

$$= \frac{< 3t^2,\ -3t,1 >}{(1+9t^2+9t^4)^{1/2}}$$

Warm-Up, extension, pg 4.

The calculations can get onerous.

There may be other ways to do them!

Note that both $\mathbf{v}(t)$ and $\mathbf{a}(t)$ both lie in the same plane as $\mathbf{T}(t)$ and $\mathbf{N}(t)$.

$$\begin{aligned}
\mathbf{v}(t) &= v(t)\mathbf{T}(t) \\
\mathbf{a}(t) &= \mathbf{v}'(t) \\
&= v'(t)\mathbf{T}(t) + v(t)\mathbf{T}'(t) \\
&= v'(t)\mathbf{T}(t) + v(t)\,|\,\mathbf{T}'(t)\,|\,\mathbf{N}(t)
\end{aligned}$$

A result of this is that $\mathbf{v}(t) \times \mathbf{a}(t)$ is in the same direction as $\mathbf{B}(t)$.

$$\begin{aligned}
&\mathbf{v}(t) \times \mathbf{a}(t) \\
&= v(t)\mathbf{T}(t) \times \left(v'(t)\mathbf{T}(t) + v(t)\,|\,\mathbf{T}'(t)\,|\,\mathbf{N}(t)\right) \\
&= v(t)v'(t)\mathbf{T}(t) \times \mathbf{T}(t) + v^2(t)\,|\,\mathbf{T}'(t)\,|\,\mathbf{T}(t) \times \mathbf{N}(t) \\
&= 0 + v^2(t)\,|\,\mathbf{T}'(t)\,|\,\mathbf{B}(t)
\end{aligned}$$

Consequently, we can get to $\mathbf{B}(t)$ by looking at the unit vector in the direction of $\mathbf{v}(t) \times \mathbf{a}(t)$

$$\mathbf{B}(t) = \frac{\mathbf{v}(t) \times \mathbf{a}(t)}{|\,\mathbf{v}(t) \times \mathbf{a}(t)\,|}$$

Furthermore, with $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$, you can show that $\mathbf{N}(t) = \mathbf{B}(t) \times \mathbf{T}(t)$. So it's possible to get $\mathbf{N}$ from $\mathbf{B}$ and $\mathbf{T}$.

Example.

2. $\mathbf{r}(t) = \,<\sqrt{2}t, e^t, e^{-t}>$

$\mathbf{v}(t) = \,<\sqrt{2}, e^t, -e^{-t}>$

$\mathbf{a}(t) = \,<0, e^t, e^{-t}>$

$$\mathbf{v} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sqrt{2} & e^t & -e^{-t} \\ 0 & e^t & e^{-t} \end{vmatrix}$$

$$= \,<2, -\sqrt{2}e^{-t}, \sqrt{2}e^t>$$

$$\begin{aligned}
|\,\mathbf{v} \times \mathbf{a}\,| &= \sqrt{2}(2 + e^{-2t} + e^{2t})^{1/2} \\
&= \sqrt{2}(e^t + e^{-t})
\end{aligned}$$

$$\begin{aligned}
\frac{\mathbf{v} \times \mathbf{a}}{|\,\mathbf{v} \times \mathbf{a}\,|} &= (e^t + e^{-t})^{-1} <\sqrt{2}, -e^{-t}, e^t> \\
&= \mathbf{B}(t) \quad ✓
\end{aligned}$$

Warm-Up, extension, pg 5.

Example.

2. $\mathbf{r}(t) = <\sqrt{2}t, e^t, e^{-t}>$

$\mathbf{v}(t) = <\sqrt{2}, e^t, -e^{-t}>$

$\mathbf{a}(t) = <0, e^t, e^{-t}>$

$\mathbf{T}(t) = (e^t + e^{-t})^{-1} <\sqrt{2}, e^t, -e^{-t}>$

$\mathbf{B}(t) = (e^t + e^{-t})^{-1} <\sqrt{2}, -e^{-t}, e^t>$

We can get $\mathbf{N}$ from $\mathbf{B}$ and $\mathbf{T}$

$$\mathbf{N}(t) = \mathbf{B}(t) \times \mathbf{T}(t)$$

$$= (e^t + e^{-t})^{-2} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sqrt{2} & -e^{-t} & e^t \\ \sqrt{2} & e^t & -e^{-t} \end{vmatrix}$$

$$= (e^t + e^{-t})^{-2} <e^{-2t} - e^{2t}, \sqrt{2}(e^{-t} + e^t), \sqrt{2}(e^t + e^{-t})>$$

$$= (e^t + e^{-t})^{-1} <e^{-t} - e^t, \sqrt{2}, \sqrt{2}> \text{ as before} ✓$$

Try these if you dare!

$a)$ $\mathbf{r}(t) = <\cos(t), t, \sin(t)>$

$\mathbf{v}(t) = <-\sin(t), 1, \cos(t)>$

$\mathbf{a}(t) = <-\cos(t), 0, -\sin(t)>$

$$\mathbf{T}(t) = \frac{1}{\sqrt{2}} <-\sin(t), 1, \cos(t)>$$

$b)$ $\mathbf{r}(t) = <t, t^2, t^3>$ $t \in \mathbf{R}$

$\mathbf{v}(t) = <1, 2t, 3t^2>$

$\mathbf{a}(t) = <0, 2, 6t>$

$\mathbf{T}(t) = (1 + 4t^2 + 9t^4)^{-1/2} <1, 2t, 3t^2>$

$a)$ $\mathbf{v} \times \mathbf{a} = <-\sin(t), -1, \cos(t)>$ $\quad |\mathbf{v} \times \mathbf{a}| = \sqrt{2}$

$$\frac{\mathbf{v} \times \mathbf{a}}{|\mathbf{v} \times \mathbf{a}|} = \frac{1}{\sqrt{2}} <-\sin(t), -1, \cos(t)> = \mathbf{B}(t) ✓$$

$$a) \ \mathbf{N}(t) = \mathbf{B}(t) \times \mathbf{T}(t) = \frac{1}{2} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin(t) & -1 & \cos(t) \\ -\sin(t) & 1 & \cos(t) \end{vmatrix}$$

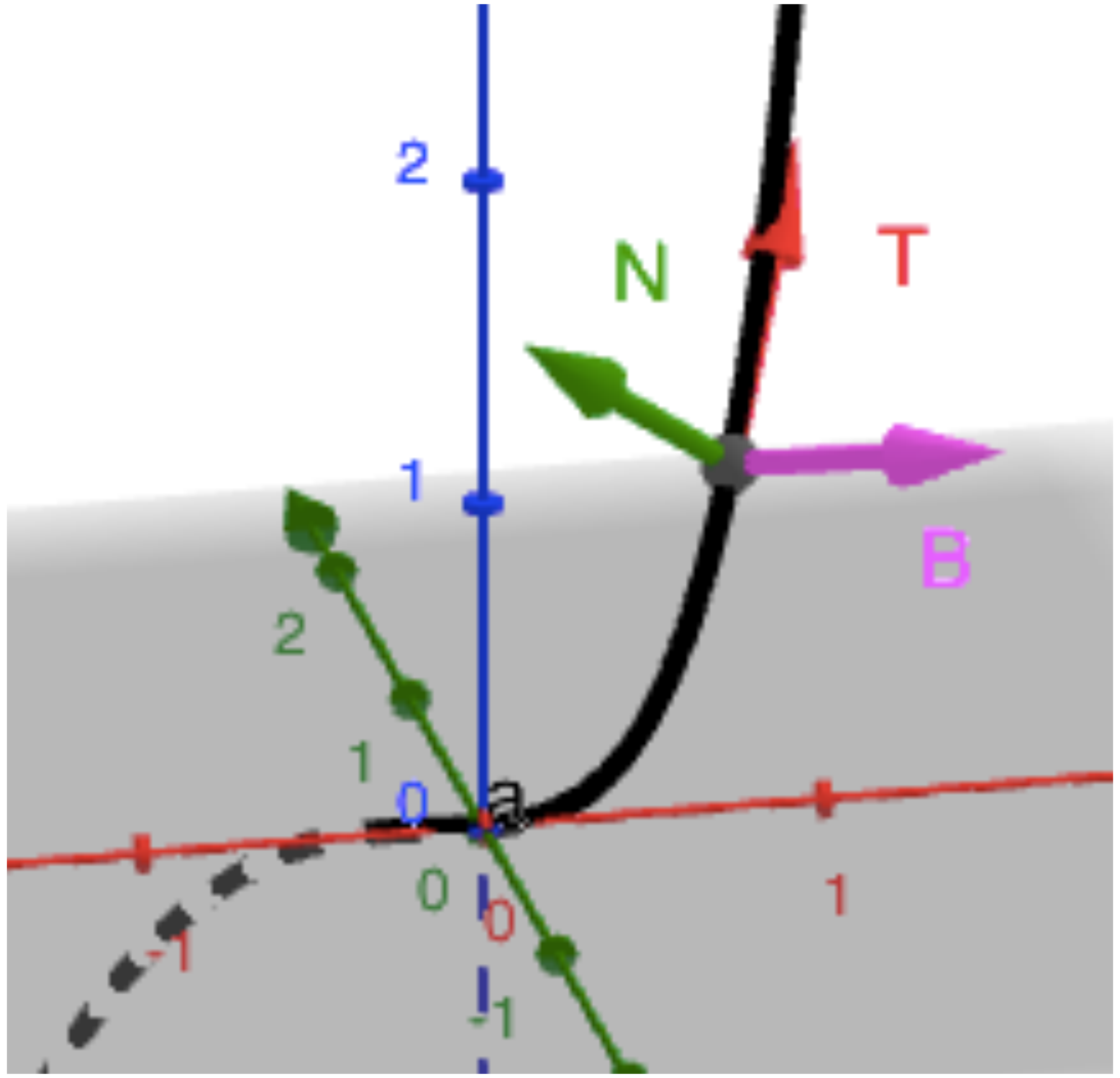
$$= \frac{1}{2} <-2\cos(t), 0, -2\sin(t)>$$

$$= <-\cos(t), 0, -\sin(t)> ✓$$

(compare with page 4)

Warm-Up, extension, pg 6.

Link: TNB QuadraticCubic



$b)\quad \mathbf{r}(t) = < t, t^2, t^3 > \quad t \in \mathbf{R}$

$$\mathbf{v}(t) = < 1, 2t, 3t^2 >$$

$$\mathbf{a}(t) = < 0, 2, 6t >$$

$$\mathbf{T}(t) = (1 + 4t^2 + 9t^4)^{-1/2} < 1, 2t, 3t^2 >$$

$$\mathbf{v} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = < 6t^2, -6t, 2 >$$

$$|\mathbf{v} \times \mathbf{a}| = 2(9t^4 + 9t^2 + 1)^{1/2}$$

$$\mathbf{B}(t) = \frac{\mathbf{v} \times \mathbf{a}}{|\mathbf{v} \times \mathbf{a}|}$$

$$= (9t^2 + 9t^2 + 1)^{-1/2} < 3t^2, -3t, 1 > \checkmark$$

$$\mathbf{N}(t) = \mathbf{B}(t) \times \mathbf{T}(t)$$

$$= (1 + 4t^2 + 9t^4)^{-1/2}(1 + 9t^2 + 9t^4)^{-1/2} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3t^2 & -3t & 1 \\ 1 & 2t & 3t^2 \end{vmatrix} = \frac{< -9t^3 - 2t, -9t^4 + 1, 6t^3 + 3t >}{(1 + 4t^2 + 9t^4)^{-1/2}(1 + 9t^2 + 9t^4)^{-1/2}} \checkmark$$

Changing Parameters.

We know that the same curve can be described algebraically, or *parametrized*, in many different ways.

Velocity and acceleration depend on the parametrization. $\mathbf{T}, \mathbf{N}, \mathbf{B}$ do not.

Velocity and acceleration are properties of the particle, while $\mathbf{T}, \mathbf{N}, \mathbf{B}$ are properties of the curve.

Example.

$\mathbf{r}(t) = < t, t^2, 0 > \quad t \in (0, \infty)$

$\mathbf{v}(t) = < 1, 2t, 0 >$

$\mathbf{a}(t) = < 0, 2, 0 >$

$\mathbf{T}(t) = (1 + 4t^2)^{-1/2} < 1, 2t, 0 >$

$\mathbf{N}(t) = (1 + 4t^2)^{-1/2} < -2t, 1, 0 >$

New Parametrization:

$\mathbf{r}(u) = < e^u, e^{2u}, 0 > \quad u \in (-\infty, \infty)$

$\mathbf{v}(u) = < e^u, 2e^{2u}, 0 >$

$\mathbf{a}(u) = < e^u, 4e^{2u}, 0 >$

$\mathbf{T}(u) = (1 + 4e^{2u})^{-1/2} < 1, 2e^u, 0 >$

$\mathbf{N}(u) = (1 + 4e^{2u})^{-1/2} < -2e^u, 1, 0 >$

Let's compare $\mathbf{v}, \mathbf{a}, \mathbf{T}, \mathbf{N}, \mathbf{B}$ at some point on the curve.

e.g $(3, 9, 0)$ is on the curve.

$t = 3$ or $u = \ln(3)$

Using the first parametrization when $t = 3$ we have

$\mathbf{r}(3) = < 3, 9, 0 >$

$\mathbf{v}(3) = < 1, 6, 0 >$

$\mathbf{a}(3) = < 0, 2, 0 >$

$\mathbf{T}(3) = 37^{-1/2} < 1, 6, 0 >$

$\mathbf{N}(3) = 37^{-1/2} < -6, 1, 0 >$

Using the other parametrization when $u = \ln(3)$ we have

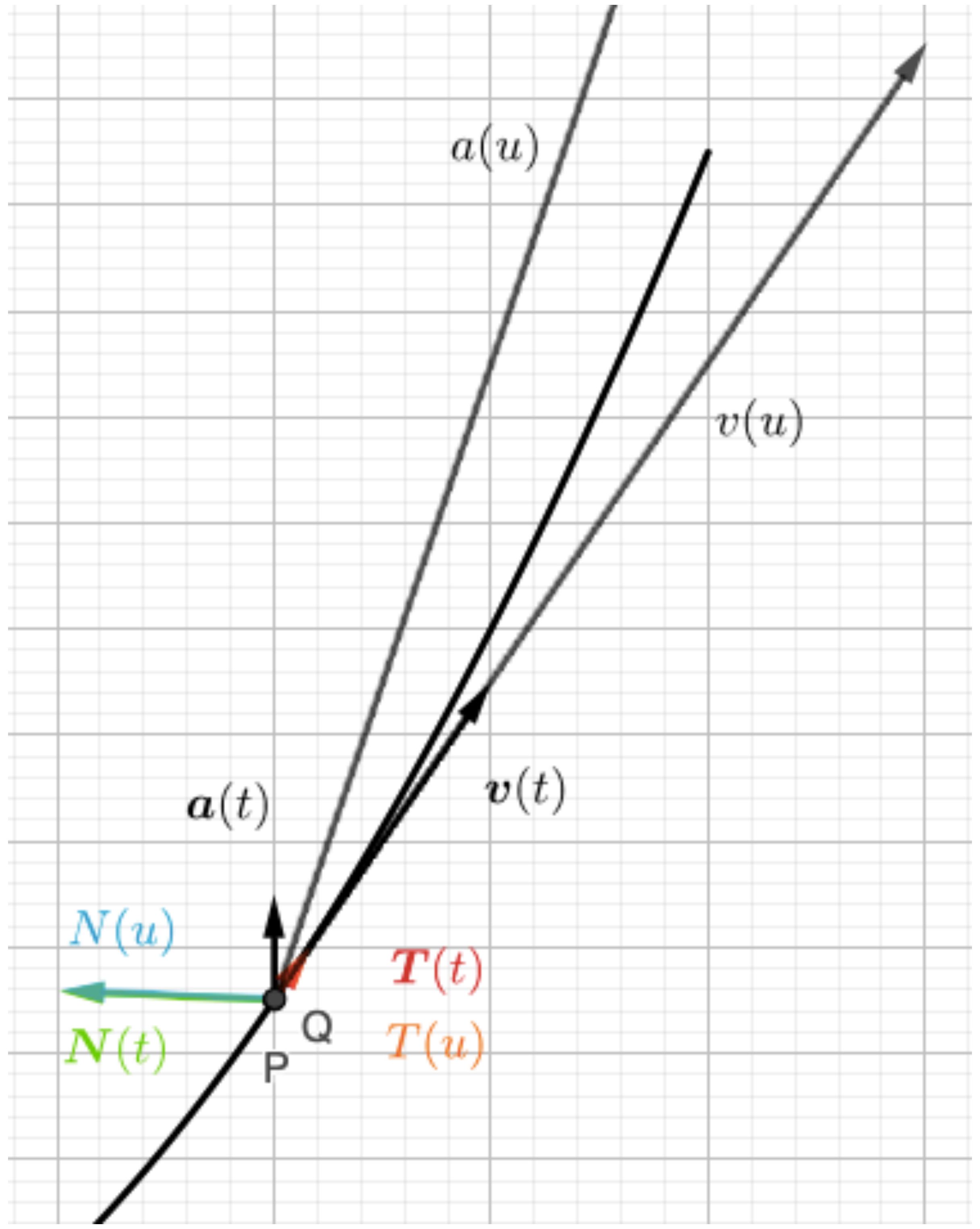
$\mathbf{r}(\ln(3)) = < 3, 9, 0 >$

$\mathbf{v}(\ln(3)) = < 3, 18, 0 >$

$\mathbf{a}(\ln(3)) = < 3, 36, 0 >$

$\mathbf{T}(\ln(3)) = 37^{-1/2} < 1, 6, 0 >$

$\mathbf{N}(\ln(3)) = 37^{-1/2} < -6, 1, 0 >$



Link: MotionOnParabola

Changing Parameters, pg 2.

$P\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

Compute $\mathbf{v}, \mathbf{a}, \mathbf{T}, \mathbf{N}$ at the point P.

1. $\mathbf{r}(t) = < t, (1-t^2)^{1/2}, 0 > \; t \in (-1,1)$
2. $\mathbf{r}(\theta) = < -\cos(\theta), \sin(\theta), 0 > \; \theta \in [0,\pi]$

$$\mathbf{v}(t) = < 1, -t(1-t^2)^{-1/2}, 0 >$$

$$\mathbf{a}(t) = < 0, -(1-t^2)^{-3/2}, 0 >$$

$$|\mathbf{v}(t)| = (1 + t^2(1-t^2)^{-1})^{1/2}$$

$$= (1-t^2)^{-1/2}$$

$$\mathbf{T}(t) = (1-t^2)^{1/2} < 1, -t(1-t^2)^{-1/2}, 0 >$$

$$\mathbf{T}'(t) = (1-t^2)^{-1/2} < -t, -(1-t^2)^{1/2}, 0 >$$

$$|\mathbf{T}'(t)| = (1-t^2)^{-1/2}$$

$$\mathbf{N}(t) = < -t, -(1-t^2)^{1/2}, 0 >$$

$$\mathbf{v}(\theta) = < \sin(\theta), \cos(\theta), 0 >$$

$$\mathbf{a}(\theta) = < \cos(\theta), -\sin(\theta), 0 >$$

$$|\mathbf{v}(\theta)| = 1$$

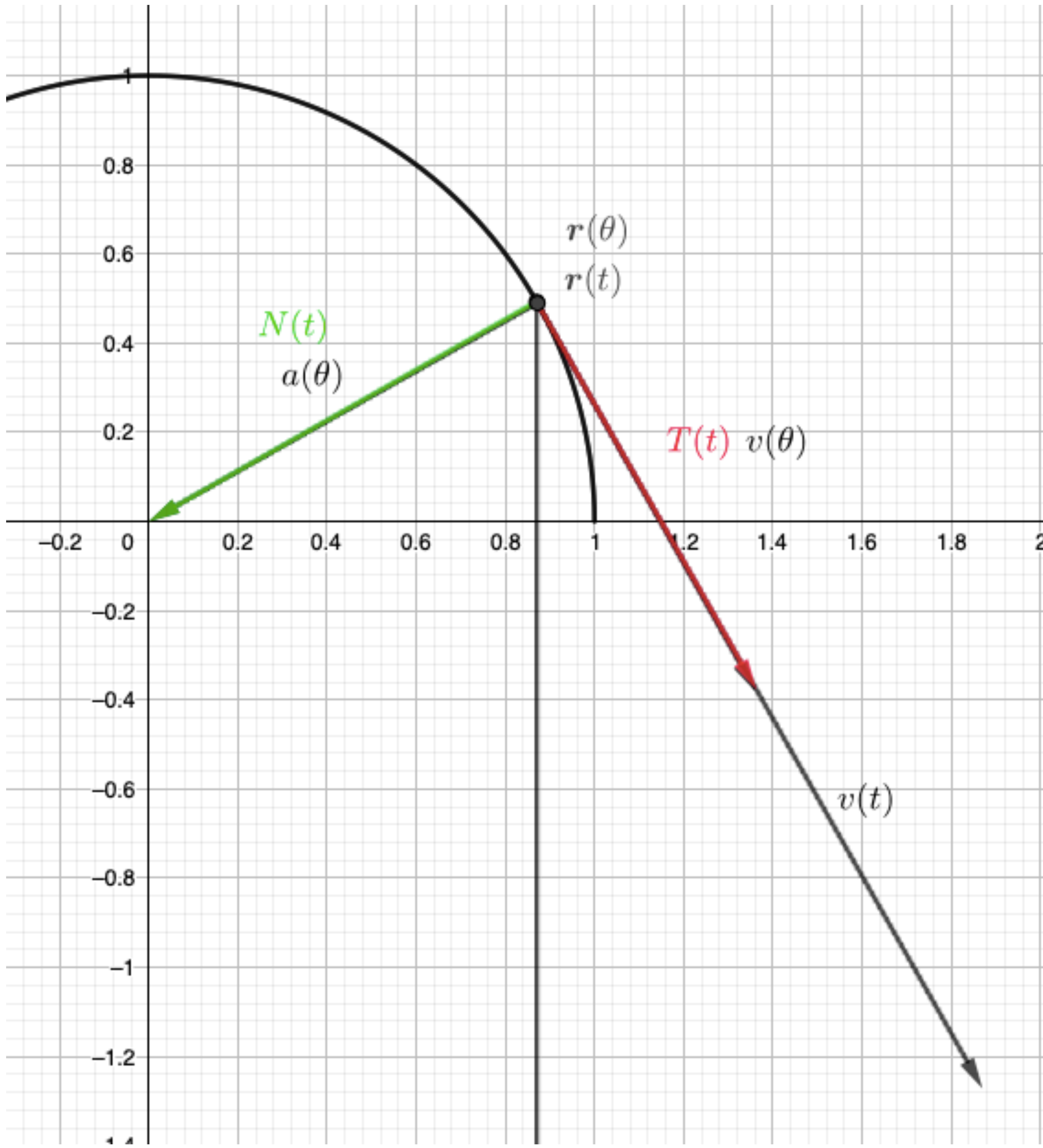
$$\mathbf{T}(\theta) = < \sin(\theta), \cos(\theta), 0 >$$

$$\mathbf{T}'(\theta) = < \cos(\theta), -\sin(\theta), 0 >$$

$$|\mathbf{T}'(\theta)| = 1$$

$$\mathbf{N}(\theta) = < \cos(\theta), -\sin(\theta), 0 >$$

Link: MotionOnAHalfCircle



At P

$$t = \frac{\sqrt{3}}{2} \qquad \theta = \frac{5\pi}{6}$$

$$\mathbf{v}(t) = < 1, -\sqrt{3}, 0 >$$

$$\mathbf{a}(t) = < 0, -8, 0 >$$

$$\mathbf{T}(t) = < 1/2, -\sqrt{3}/2, 0 >$$

$$\mathbf{N}(t) = < -\sqrt{3}/2, -1/2, 0 >$$

$$\mathbf{v}(\theta) = < 1/2, -\sqrt{3}/2, 0 >$$

$$\mathbf{a}(\theta) = < \sqrt{3}/2, -1/2, 0 >$$

$$\mathbf{T}(\theta) = < 1/2, -\sqrt{3}/2, 0 >$$

$$\mathbf{N}(\theta) = < \sqrt{3}/2, -1/2, 0 >$$

Reparameterization.

There are infinitely many ways to parameterize a curve.

Example: Parabola (right half).

$\mathbf{r}(t) = < t, t^2, 0 > \; t \in (0, \infty)$

$\mathbf{r}(u) = < e^u, e^{2u}, 0 > \quad f(u) = e^u,\; u \in (-\infty, \infty)$

$\mathbf{r}(w) = < w^3, w^6, 0 > \quad f(w) = w^3,\; w \in (0, \infty)$

$\mathbf{r}(v) = < \frac{1}{v}, \frac{1}{v^2}, 0 > \quad f(v) = \frac{1}{v},\; v \in (0, \infty)$

Example: Helix ($y > 0$ half).

$\mathbf{r}(t) = < \cos(t), t, \sin(t) > \; t \in (0, \infty)$

$\mathbf{r}(u) = < \cos(2u), 2u, \sin(2u) > \quad f(u) = 2u$

$\mathbf{r}(v) = < \cos(v^2), v^2, \sin(v^2) > \quad f(v) = v^2$

$\mathbf{r}(s) = \left\langle \cos\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}}, \sin\left(\frac{s}{\sqrt{2}}\right) \right\rangle \; f(s) = \frac{s}{\sqrt{2}}$

In general, given $\mathbf{r}(t) = < x(t), y(t), z(t) > \; t \in (a, b)$ and a one-to-one function $f : (c, d) \to (a, b),\; f(u) = t,$

then you can reparameterize $\mathbf{r}$ using composition:

$\mathbf{r}(u) = < x \circ f(u), y \circ f(u), z \circ f(u) > \quad u \in (c, d)$

You might think of parameterizations like fractions.

Many different fractions represent the same number.

$$\ldots \frac{7}{14} = \frac{6}{12} = \frac{5}{10} = \frac{4}{8} = \frac{3}{6} = \frac{2}{4} = \frac{1}{2} = \frac{-2}{-4} = \ldots$$

But there's one special fraction representation, the *reduced* fraction, where nothing can be canceled out.

If you have a curve, is there one special parameterization of it? Yes!

parameterization with respect to arc length!

Arc Length, pg 1.

$\mathbf{r}(t) = < x(t), y(t), z(t) >$

A particle's speed is

$$v(t) = |\mathbf{v}(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$$

The distance that a particle travels in the amount of time Δt is about

$$v(t_*)\Delta t = \sqrt{x'(t_*)^2 + y'(t_*)^2 + z'(t_*)^2}\ \Delta t$$

(here t_* is a time between t and $t + \Delta t$.)

The total distance a particle travels from time t_0 to time t_1 is

$$s = \int_{t_0}^{t_1} |\mathbf{v}(t)|\ dt = \int_{t_0}^{t_1} \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} dt$$

This is the length of the curve, or the *Arc Length,* between points $\mathbf{r}(t_0)$ and $\mathbf{r}(t_1)$.

We can describe arc length as a function of t

$$s(t) = \int_0^t \sqrt{x'(u)^2 + y'(u)^2 + z'(u)^2} du$$

$s(t)$ expresses the length of the curve from some initial point $\mathbf{r}(0)$ to an arbitrary point $\mathbf{r}(t)$.

(The ‘initial’ point could happen at any moment. You could have some specific t_0 in place of 0.)

Examples.

Find the length of the given curve.

S13.3 #6 $\quad \mathbf{r}(t) = t^2\mathbf{i} + 9t\mathbf{j} + 4t^{3/2}\mathbf{k} \quad t \in [1,4]$

$\mathbf{r}(t) = < t^2, 9t, 4t^{3/2} >$

$\mathbf{v}(t) = < 2t, 9, 6t^{1/2} >$

$v(t) = (4t^2 + 81 + 36t)^{1/2} = |2t + 9|$

$$s = \int_1^4 |2t + 9|\ dt = \int_1^4 2t + 9\ dt = t^2 + 9t\Big|_1^4 = 42$$

Arc Length, pg 2.

Wanna try? Find the arc length.

Example2.

S13.3#4 $\mathbf{r}(t) = < \cos(t), \sin(t), \ln(\cos(t)) > \; t \in [0,\pi/4]$

$$\mathbf{v}(t) = \left\langle -\sin(t), \cos(t), \frac{-\sin(t)}{\cos(t)} \right\rangle$$

$$v(t) = (1 + \tan^2(t))^{1/2}$$

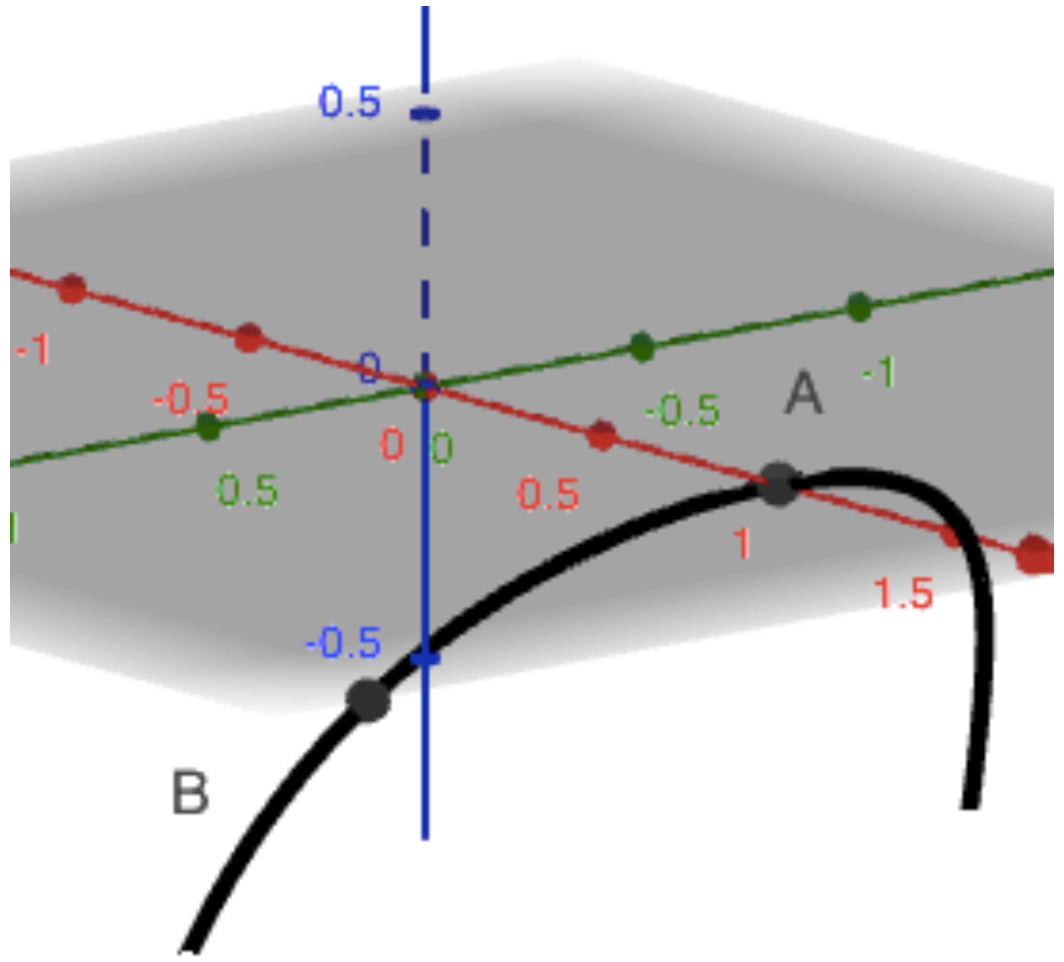
$$= |\sec(t)|$$

$$s = \int_0^{\pi/4} |\sec(t)| \; dt$$

$$= \int_0^{\pi/4} \sec(t) \; dt$$

$$= \ln|\sec(t) + \tan(t)| \Big|_0^{\pi/4}$$

$$= \ln(1 + \sqrt{2}) \approx 0.88$$

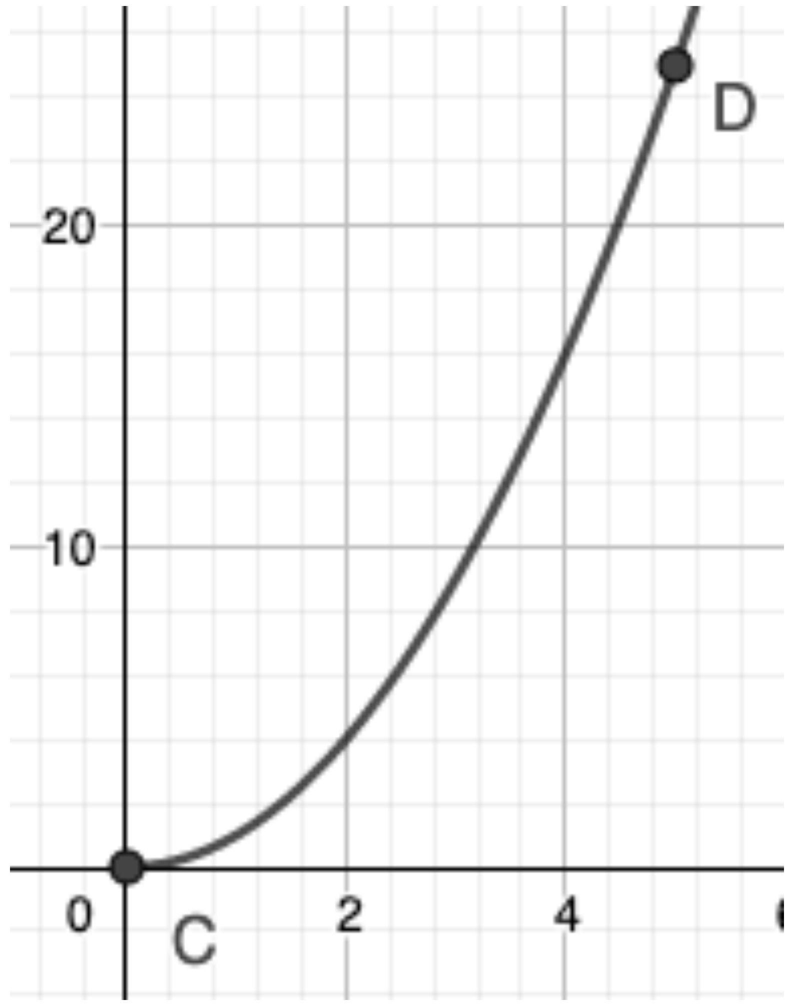


Example3. $\mathbf{r}(t) = < t, t^2, 0 > \; t \in [0,5]$

$$\mathbf{v}(t) = < 1, 2t, 0 >$$

$$v(t) = (1 + 4t^2)^{1/2}$$

$$s = \int_0^5 \sqrt{1 + 4t^2} \; dt$$



Try a trig substitution: $t = \tan(\theta)/2$

by parts!

$$s = \ldots = \frac{1}{2} \int_{\tan^{-1}(0)}^{\tan^{-1}(10)} \sec^3(\theta) \; d\theta \qquad u = \sec(\theta) \qquad du = \sec(\theta)\tan(\theta)d\theta$$

$$dv = \sec^2(\theta)d\theta \qquad v = \tan(\theta)$$

$$\int \sec^3(\theta) \; d\theta = \sec(\theta)\tan(\theta) - \int \left(\sec^3(\theta) - \sec(\theta)\right) \; d\theta$$

$$s = \frac{1}{2} \int_{\tan^{-1}(0)}^{\tan^{-1}(10)} \sec^3(\theta) \; d\theta$$

$$= \frac{1}{4} \left(\sec(\theta)\tan(\theta) + \ln(\sec(\theta) + \tan(\theta))\right) \Big|_0^{\tan^{-1}(10)} \approx 25.87$$

Arc Length, pg 3. Does parameterization matter?

Does the computation of arc length depend on how the curve is parameterized?

(Is it a property of the curve itself, or of a particle that traverses the curve?)

arc-length is a property of the curve. It shouldn't depend on parameterization!

Example. We just computed an arc length

$\mathbf{r}(t) = < \cos(t), \sin(t), \ln(\cos(t)) > \ t \in [0,\pi/4]$

What if we had used this function instead?

$\mathbf{r}(u) = < \cos(u^2), \sin(u^2), \ln(\cos(u^2)) > \ u \in [0,\sqrt{\pi/4}]$

$$\mathbf{v}(u) = \left\langle -2u\sin(u^2),\ 2u\cos(u^2), \frac{-2u\sin(u^2)}{\cos(u^2)} \right\rangle$$

$$s = \int_0^{\sqrt{\pi/4}} \sqrt{4u^2\sin^2(u^2) + 4u^2\cos^2(u^2) + \frac{4u^2\sin^2(u^2)}{\cos^2(u^2)}}\ du$$

$$= \int_0^{\sqrt{\pi/4}} 2u\sqrt{1+\tan^2(u^2)}\ du = \int_0^{\sqrt{\pi/4}} 2u\sec(u^2)\ du = \ldots$$

substitution:

$$t = u^2$$
$$dt = 2udu$$

$$\ldots = \int_0^{\pi/4} \sec(t)\ dt$$

← We had this before

In general, if you have reparameterized curve

$$\mathbf{r}(t) = < x(t), y(t), z(t) > , \quad t \in (a,b)$$

$$\mathbf{r}(u) = < x(f(u)), y(f(u)), z(f(u)) > , \quad u \in (c,d)$$

Arc length of the reparameterized curve equals the arc length of the original curve:

$$s = \int_c^d |\mathbf{v}(u)|\ du$$

$$= \int_c^d \left(x'(f(u))^2 f'(u)^2 + y'(f(u))^2 f'(u)^2 + z'(f(u))^2 f'(u)^2\right)^{1/2} du$$

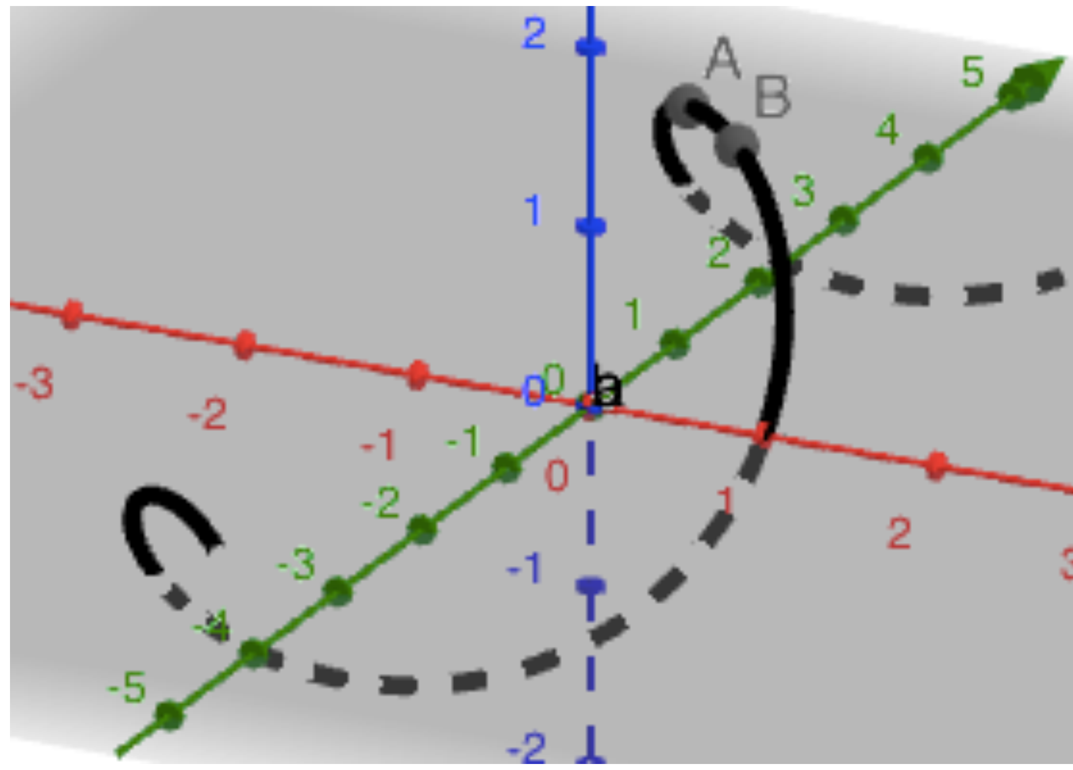
$$= \int_c^d \left(x'(f(u))^2 + y'(f(u))^2 + z'(f(u))^2\right)^{1/2} f'(u)\ du$$

substitution:

$$t = f(u)$$
$$dt = f'(u)du$$

$$= \int_a^b \left(x'(t)^2 + y'(t)^2 + z'(t)^2\right)\ dt = \int_a^b |\mathbf{v}(t)|\ dt$$

Arc Length, pg 3. Parametrization via arc-length.



Sometimes we can use arc length function to give a curve a new parameter.

Example.

$$\mathbf{r}(t) = < \cos(t), t, \sin(t) >$$

$$\mathbf{v}(t) = < -\sin(t), 1, \cos(t) >$$

$$v(t) = \sqrt{2}$$

$$s(t) = \int_0^t v(u)\, du \; = \int_0^t \sqrt{2}\, du$$

$$= \sqrt{2}u \Big|_0^t = \sqrt{2}t$$

$$s(t) = \sqrt{2}t$$

Solve $s = \sqrt{2}t$ for t to get a new parameter.

$$t = \frac{s}{\sqrt{2}} \qquad \mathbf{r}(s) = \left\langle \cos\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}}, \sin\left(\frac{s}{\sqrt{2}}\right) \right\rangle$$

A: $t = 2$

B: $s = 2$

B is the point on the curve that is 2 units away from the initial point at $(1,0)$.

In this parameterization, velocity has magnitude 1.

$$\mathbf{v}(s) = \left\langle -\frac{1}{\sqrt{2}} \sin\left(\frac{s}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \cos\left(\frac{s}{\sqrt{2}}\right) \right\rangle$$

$$= \frac{1}{\sqrt{2}} \left\langle -\sin\left(\frac{s}{\sqrt{2}}\right), 1, \cos\left(\frac{s}{\sqrt{2}}\right) \right\rangle$$

$$|\mathbf{v}(s)| = \frac{1}{\sqrt{2}} \sqrt{\sin^2\left(\frac{1}{\sqrt{2}}\right) + 1 + \cos^2\left(\frac{1}{\sqrt{2}}\right)} = \frac{1}{\sqrt{2}} \cdot \sqrt{1+1} = 1$$

If we measure arc-length with this parameter, we get $\displaystyle\int_0^s 1\, du = s$

This is the special parameterization that we were looking for.

We call this parameterization *with respect to arc-length.*

Parametrization with respect to arc length.

$\mathbf{r}(t) = < x(t), y(t), z(t) >$

In general, to parameterize w.r.t. arc length, start with the arc-length function:

$$s(t) = \int_a^t |\mathbf{v}(u)|\, du \quad t \in (a, b)$$

$s : (a, b) \to (0,L)$ is increasing, so it's one-to-one. It has an inverse:

$$F : (0,L) \to (a, b), \quad F(s) = t$$

We get a parametrization

$$\mathbf{r}(s) = < x \circ F(s), y \circ F(s), z \circ F(s) >$$

This is the parametrization *with respect to arc length*.

It results in a *unit-speed* curve, because…

$$\mathbf{r}'(s) = < x'(F(s))F'(s), y'(F(s))F'(s), z'(F(s))F'(s) >$$

$$= F'(s) < x'(t), y'(t), z'(t) >$$

Note: $\dfrac{ds}{dt} = |\mathbf{v}(t)|$ Also $F(s(t)) = t$

$$\frac{dF}{ds}\frac{ds}{dt} = 1 \qquad \frac{dF}{ds} = \frac{1}{ds/dt} = \frac{1}{|\mathbf{v}(t)|}$$

So $\mathbf{r}'(s) = \dfrac{1}{|\mathbf{v}(t)|} < x'(t), y'(t), z'(t) >$

$$|\mathbf{r}'(s)| = \frac{1}{|\mathbf{v}(t)|} | < x'(t), y'(t), z'(t) > | = 1$$

We can parameterize a curve with respect to arc length *in theory*, but in practice, finding F might be really really hard. Nevertheless, most curves have this theoretical parametrization. It is used quite frequently in the theory of curves.

Parametrization with respect to arc length, pg 2, Practice.

previous example:

$\mathbf{r}(t) = <\cos(t), t, \sin(t)>$

$\mathbf{v}(t) = <-\sin(t), 1, \cos(t)>$

$v(t) = \sqrt{2}$

$$s(t) = \int_0^t v(u)\,du = \int_0^t \sqrt{2}\,du$$

$$= \sqrt{2}u\Big|_0^t = \sqrt{2}t$$

$s(t) = \sqrt{2}t$

Solve $s = \sqrt{2}t$ for t to get a new parameter. $t = \dfrac{s}{\sqrt{2}}$

$$\mathbf{r}(s) = \left\langle \cos\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}}, \sin\left(\frac{s}{\sqrt{2}}\right) \right\rangle$$

Try this.

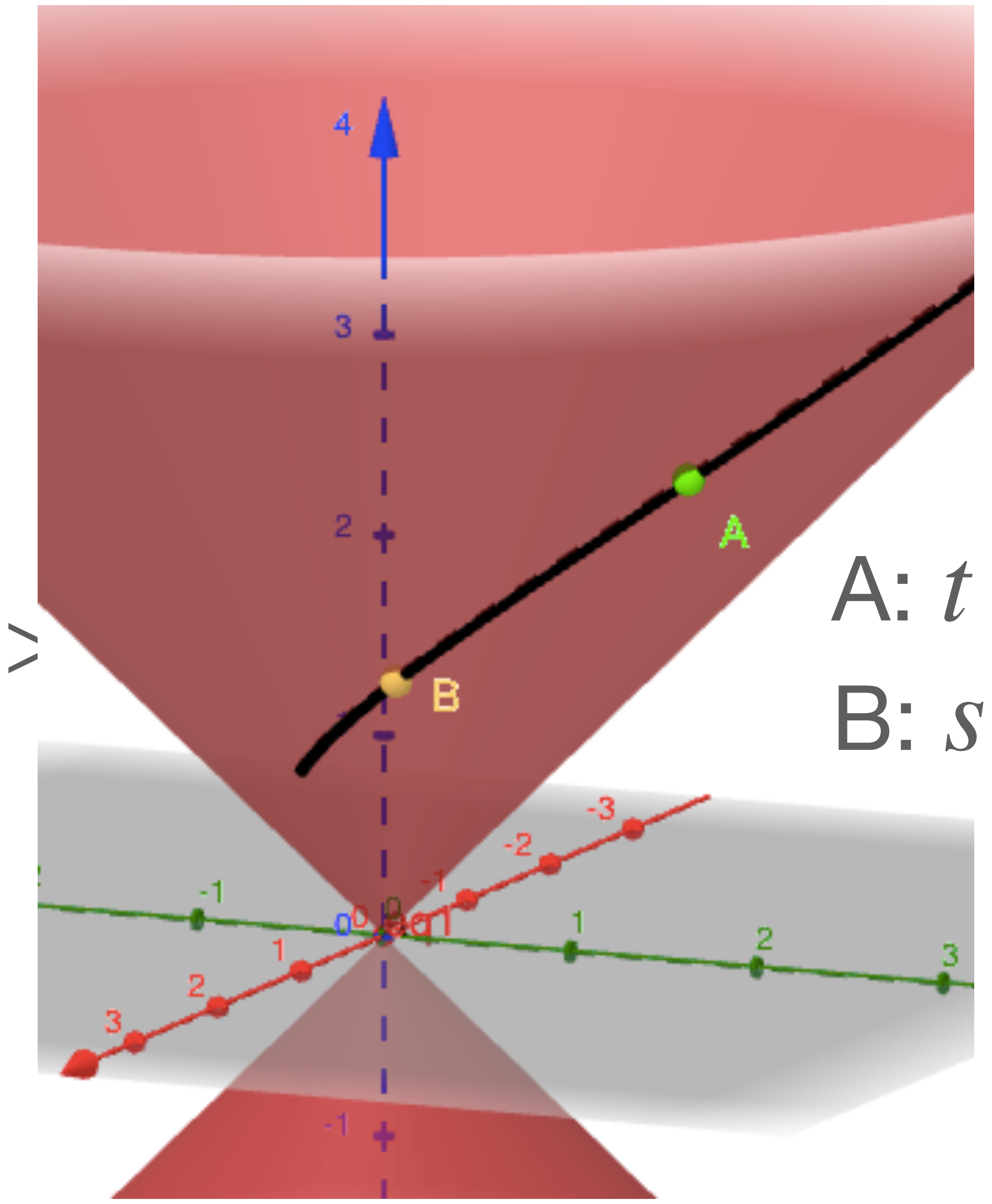
$\mathbf{r}(t) = e^t <\cos(t), \sin(t), 1>$

$$\begin{aligned}\mathbf{v}(t) &= e^t <\cos(t), \sin(t), 1> \\ &\quad + e^t <-\sin(t), \cos(t), 0> \\ &= e^t <\cos(t) - \sin(t),\ \cos(t) + \sin(t),\ 1>\end{aligned}$$

$v(t) = |\mathbf{v}(t)| = e^t\sqrt{3}$

$$s(t) = \int_0^t e^u\sqrt{3}\,du = \sqrt{3}(e^t - 1)$$

$$s = \sqrt{3}(e^t - 1) \qquad t = \ln\left(\frac{s}{\sqrt{3}} + 1\right)$$



A: $t = 1$

B: $s = 1$

Link: ArcLengthParam

$$\mathbf{r}_{arc}(s) = \left(\frac{s}{\sqrt{3}} + 1\right)\left\langle \cos\left(\ln\left(\frac{s}{\sqrt{3}} + 1\right)\right), \sin\left(\ln\left(\frac{s}{\sqrt{3}} + 1\right)\right), 1 \right\rangle$$

In the picture, when t=1, using the original parameterization gives $\mathbf{r}(1) = A$. Using the parameterization w.r.t arc length gives $\mathbf{r}_{arc}(1) = B$. B is 1 unit away from $\mathbf{r}_{arc}(0)$ along the curve.

Curvature, pg 1.

Definition: $\kappa(s) := |\mathbf{T}'(s)|$

The *curvature* of a curve is the magnitude of the tangent vector's derivative when parameterized by arc length.

Example.

1. $\mathbf{r}(t) = < \cos(t), t, \sin(t) >$

$$\mathbf{v}(t) = < -\sin(t), 1, \cos(t) >$$

$$v(t) = \sqrt{2}$$

$$s(t) = \int_0^t v(u)\,du$$

$$= \int_0^t \sqrt{2}\,du$$

$$s = \sqrt{2}t$$

$$t = \frac{s}{\sqrt{2}} = F(s)$$

$$\mathbf{r}(s) = \left\langle \cos\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}}, \sin\left(\frac{s}{\sqrt{2}}\right) \right\rangle$$

$$\mathbf{v}(s) = \frac{1}{\sqrt{2}}\left\langle -\sin\left(\frac{s}{\sqrt{2}}\right), 1, \cos\left(\frac{s}{\sqrt{2}}\right) \right\rangle$$

$$|\mathbf{v}(s)| = 1$$

$$\mathbf{T}(s) = \mathbf{v}(s)$$

$$\mathbf{T}'(s) = \frac{1}{2}\left\langle -\cos\left(\frac{s}{\sqrt{2}}\right), 0, -\sin\left(\frac{s}{\sqrt{2}}\right) \right\rangle$$

$$\kappa(s) = |\mathbf{T}'(s)| = \frac{1}{2}$$

A helix has constant curvature!

Example.

$$\mathbf{r}(t) = < \sqrt{2}t, e^t, e^{-t} >$$

$$\mathbf{v}(t) = < \sqrt{2}, e^t, -e^{-t} >$$

$$|\mathbf{v}(t)| = (2 + e^{2t} + e^{-2t})^{1/2}$$

$$= e^t + e^{-t}$$

$$s(t) = \int_0^t e^u + e^{-u}\,du$$

$$= e^t - e^{-t}$$

$$s = e^t - e^{-t}$$

$$t = \ldots$$

$$\ldots = \ln\left(\frac{s + \sqrt{s^2+4}}{2}\right) = F(s)$$

$$\mathbf{r}(s) = \left\langle \sqrt{2}\ln\left(\frac{s + \sqrt{s^2+4}}{2}\right), \frac{s + \sqrt{s^2+4}}{2}, \frac{2}{s + \sqrt{s^2+4}} \right\rangle$$

😩😩😩 Isn't there a better way?!

Curvature, pg 2. Another formula.

Say we have some other parameter t.

We get from arc-length via $s(t) = \int_a^t |\mathbf{v}(u)|\, du$

$$\mathbf{T}(t) = \mathbf{T}(s(t))$$

$$\mathbf{T}'(t) = \frac{d}{ds}\big(\mathbf{T}(s)\big) \cdot \frac{ds}{dt}$$

$$\mathbf{T}'(s) = \frac{\mathbf{T}'(t)}{s'(t)} = \frac{\mathbf{T}'(t)}{|\mathbf{v}(t)|}$$

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{v}(t)|}$$

Example.

$$\mathbf{r}(t) = < \cos(t), t, \sin(t) >$$

$$\mathbf{v}(t) = < -\sin(t), 1, \cos(t) >$$

$$v(t) = |\mathbf{v}(t)| = \sqrt{2}$$

$$\mathbf{T}(t) = \frac{1}{\sqrt{2}} < -\sin(t), 1, \cos(t) >$$

$$\mathbf{T}'(t) = \frac{1}{\sqrt{2}} < -\cos(t), 0, -\sin(t) >$$

$$|\mathbf{T}'(t)| = \frac{1}{\sqrt{2}} \qquad \kappa(t) = \frac{\frac{1}{\sqrt{2}}}{\sqrt{2}} = \frac{1}{2}$$

← this equals our calculation using arc-length.

Example.

$$\mathbf{r}(t) = < \sqrt{2}t, e^t, e^{-t} >$$

$$\mathbf{v}(t) = < \sqrt{2}, e^t, -e^{-t} >$$

$$|\mathbf{v}(t)| = e^t + e^{-t}$$

$$\mathbf{T}(t) = (e^t + e^{-t})^{-1} < \sqrt{2}, e^t, -e^{-t} >$$

$$\mathbf{T}'(t) = (e^t + e^{-t})^{-2} < \sqrt{2}(e^{-t} - e^t), 2, 2 >$$

$$|\mathbf{T}'(t)| = \sqrt{2}(e^t + e^{-t})^{-1}$$

$$\kappa(t) = \sqrt{2}(e^t + e^{-t})^{-2}$$

Curvature, pg 3. Yet another formula.

There are other formulas for curvature.

$\mathbf{v}(t) = v(t) \cdot \mathbf{T}(t)$ where $v(t) = |\mathbf{v}(t)|$

$\mathbf{v}'(t) = v(t)' \cdot \mathbf{T}(t) + v(t) \cdot \mathbf{T}'(t)$

$$\mathbf{v}(t) \times \mathbf{v}'(t) = v(t) \cdot \mathbf{T}(t) \times \left(v(t)' \cdot \mathbf{T}(t) + v(t) \cdot \mathbf{T}'(t)\right)$$

$$= v(t)^2 \left(\mathbf{T}(t) \times \mathbf{T}'(t)\right)$$

$$|\mathbf{v}(t) \times \mathbf{v}'(t)| = v(t)^2 |\mathbf{T}(t)| |\mathbf{T}'(t)| \sin(\theta)$$

$$= v(t)^2 \cdot 1 \cdot |\mathbf{T}'(t)| \cdot 1$$

$$|\mathbf{T}'(t)| = \frac{|\mathbf{v}(t) \times \mathbf{v}'(t)|}{v(t)^2}$$

$$\kappa(t) = \frac{|\mathbf{v}(t) \times \mathbf{v}'(t)|}{v(t)^3}$$

$$= \frac{|\mathbf{v}(t) \times \mathbf{a}(t)|}{v(t)^3}$$

Example.

sec13.3#25 $\mathbf{r}(t) = < t, t^2, t^3 >$

Find the curvature at the point $P(1,1,1)$

$\mathbf{v}(t) = < 1,2t,3t^2 >, \quad \mathbf{a}(t) = < 0,2,6t >$

$$|\mathbf{v}(t)| = (1 + 4t^2 + 9t^4)^{1/2}$$

$$\mathbf{T}(t) = (1 + 4t^2 + 9t^4)^{-1/2} < 1,2t,3t^2 >$$

$$\mathbf{T}'(t) = 2(1 + 4t^2 + 9t^4)^{-3/2} < -2t - 9t^3, 1 - 9t^4, 3t + 6t^3 >$$

$$|\mathbf{T}'(t)| = 2(1 + 4t^2 + 9t^4)^{-1}(1 + 9t^2 + 9t^4)^{1/2}$$

$$\kappa(t) = \frac{2(1 + 9t^2 + 9t^4)^{1/2}}{(1 + 4t^2 + 9t^4)^{3/2}}$$

or $\quad \mathbf{v} \times \mathbf{a} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{pmatrix} = < 6t^2, -6t, 2 >$

$$|\mathbf{v} \times \mathbf{a}| = 2(9t^4 + 9t^2 + 1)^{1/2} \qquad \kappa(t) = \frac{2(1 + 9t^2 + 9t^4)^{1/2}}{(1 + 4t^2 + 9t^4)^{3/2}}$$

Curvature, pg 4. Practice. Find the curvature.

S13.3#20 $\mathbf{r}(t) = < t, \frac{1}{2}t^2, t^2 >$

$\mathbf{v}(t) = < 1,t,2t >$

$\mathbf{a}(t) = < 0,1,2 >$

$v(t) = (1+5t^2)^{1/2}$

$\mathbf{T}(t) = (1+5t^2)^{-1/2} < 1,t,2t >$

$\mathbf{T}'(t) = (1+5t^2)^{-3/2} < -5t,1,2 >$

$|\mathbf{T}'(t)| = \dfrac{\sqrt{5}}{1+5t^2} \qquad \kappa(t) = \dfrac{\sqrt{5}}{(1+5t^2)^{3/2}}$

or $\quad \mathbf{v}\times\mathbf{a} = \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & t & 2t \\ 0 & 1 & 2 \end{pmatrix} = < 0, -2,1 >$

$$\frac{|\mathbf{v}\times\mathbf{a}|}{v^3} = \frac{\sqrt{5}}{(1+5t^2)^{3/2}}$$

S13.3#18 $\mathbf{r}(t) = < t^2, \sin(t) - t\cos(t), \cos(t) + t\sin(t) > \quad t > 0$

$\mathbf{v}(t) = t < 2, \sin(t), \cos(t) >$

$\mathbf{a}(t) = < 2, \sin(t) + t\cos(t), \cos(t) - t\sin(t) >$

$v(t) = |t|\sqrt{5} = \sqrt{5}t$

$\mathbf{T}(t) = 5^{-1/2} < 2, \sin(t), \cos(t) >$

$\mathbf{T}'(t) = 5^{-1/2} < 0, \cos(t), -\sin(t) >$

$|\mathbf{T}'(t)| = 5^{-1/2} \qquad \kappa(t) = \dfrac{1}{5t} \quad$ or $\quad \mathbf{v}\times\mathbf{a} = \ldots$

$$\ldots = t\cdot\det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & \sin(t) & \cos(t) \\ 2 & \sin(t)+t\cos(t) & \cos(t)-t\sin(t) \end{pmatrix}$$

$$= t < -t, -2t\sin(t), 2t\cos(t) > = t^2 < -1, -2\sin(t), 2\cos(t) >$$

$$\frac{|\mathbf{v}\times\mathbf{a}|}{v^3} = \frac{\sqrt{5}t^2}{(\sqrt{5}t)^3} = \frac{1}{5t}$$

Curvature, pg 5. Real-Valued functions' curvature.

We can get the curvature of the graph of a real-valued function.

$$\mathbf{r}(t) = < t, f(t), 0 >$$

$$\mathbf{v}(t) = < 1, f'(t), 0 >$$

$$\mathbf{a}(t) = < 1, f''(t), 0 >$$

$$\mathbf{v} \times \mathbf{a} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & f'(t) & 0 \\ 0 & f''(t) & 0 \end{pmatrix}$$

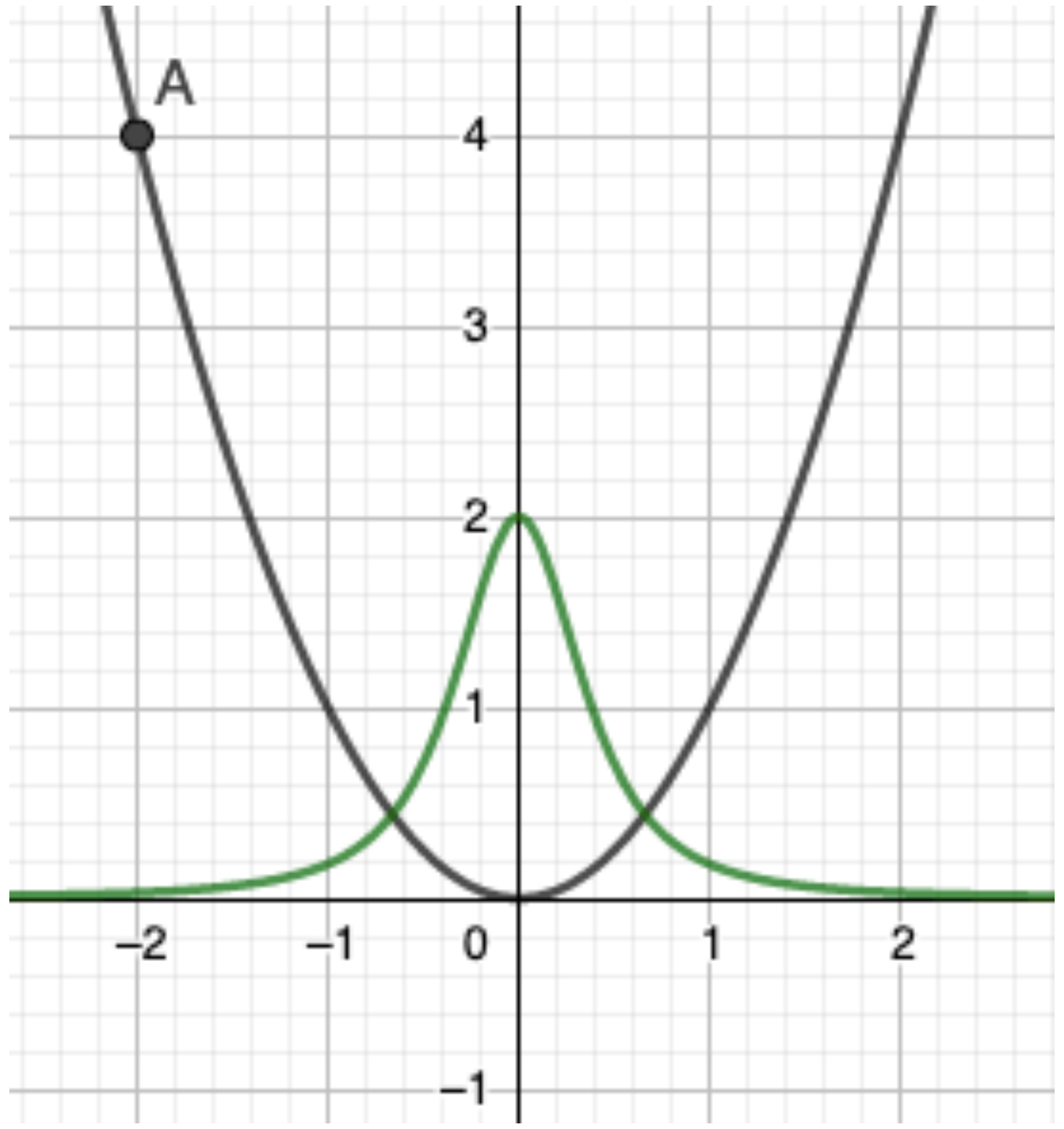
$$= < 0, 0, f''(t) >$$

$$\kappa(t) = \frac{|f''(t)|}{(1 + f'(t)^2)^{\frac{3}{2}}}$$

Examples.

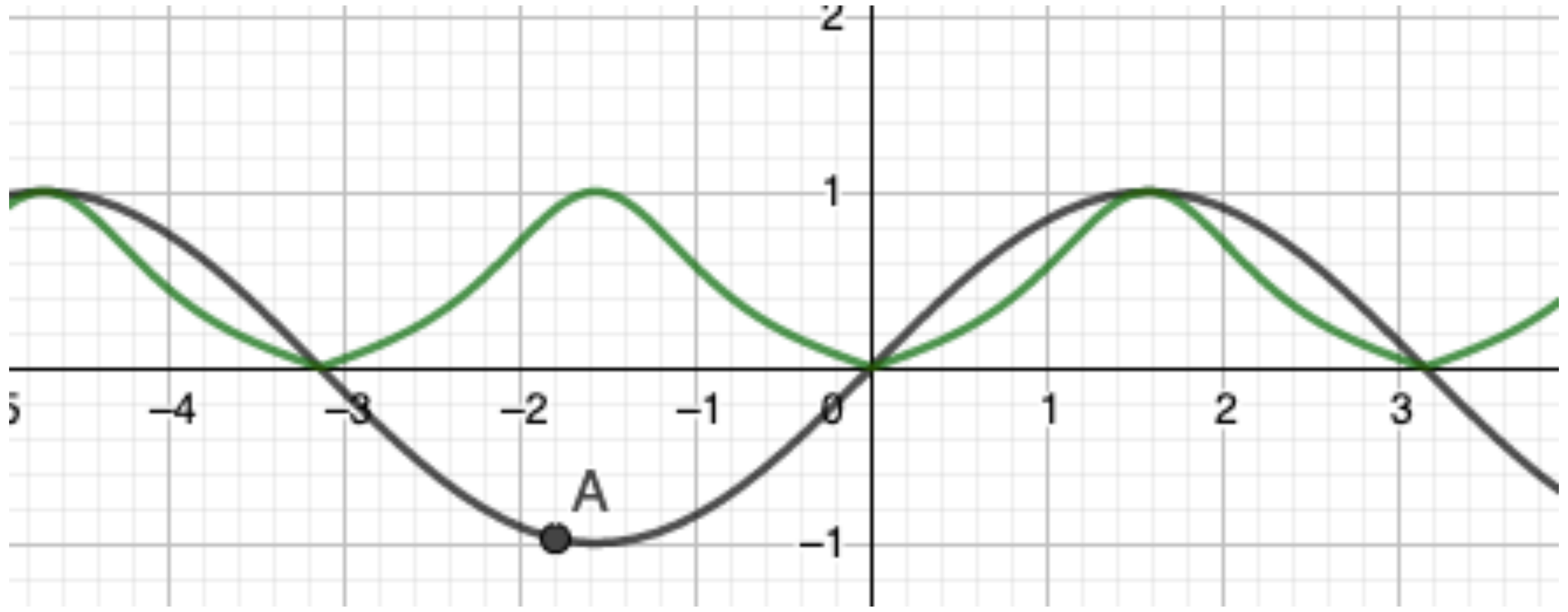
1. $\mathbf{r}(t) = < t, t^2, 0 >$

$$\kappa(t) = \frac{2}{(1+4t^2)^{3/2}}$$



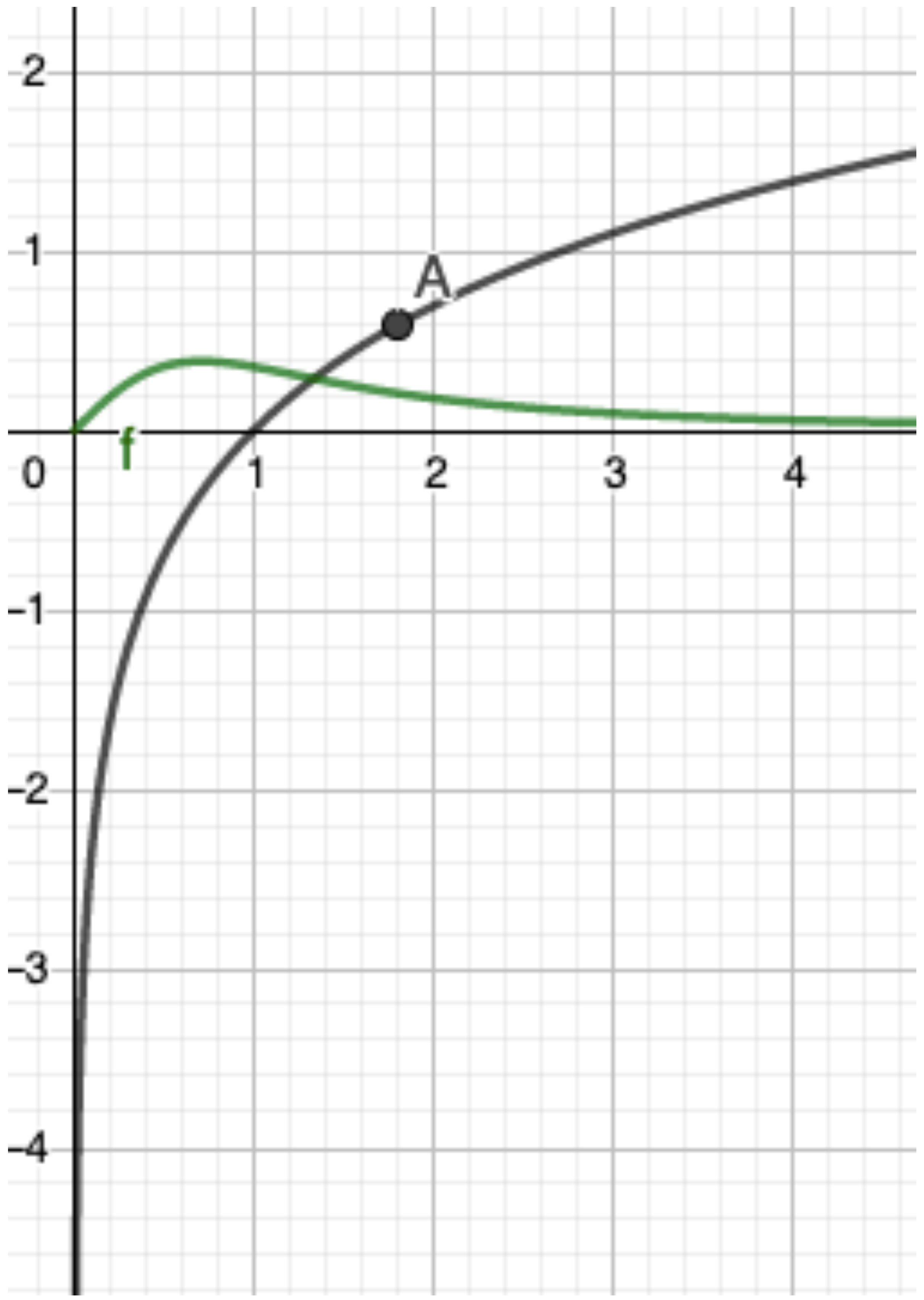
2. $\mathbf{r}(t) = < t, \sin(t), 0 >$

$$\kappa(t) = \frac{|\sin(t)|}{(1+\cos^2(t))^{3/2}}$$



3. $\mathbf{r}(t) = < t, \ln(t), 0 >$

$$\kappa(t) = \frac{t^{-2}}{(1+t^{-2})^{3/2}} = \frac{1}{t^2(1+t^{-2})^{3/2}}$$



Decomposing $\overrightarrow{a}$ using $\overrightarrow{T}$ and $\overrightarrow{N}$.

Say $|\mathbf{v}(t)| = v(t)$

Then $\mathbf{v}(t) = v(t)\mathbf{T}(t)$

$$\mathbf{a}(t) = \mathbf{v}'(t)$$
$$= v'(t)\mathbf{T}(t) + v(t)\mathbf{T}'(t)$$

Recall $\mathbf{T}'(t) = |\mathbf{T}'(t)|\mathbf{N}(t)$

$$= \kappa(t)v(t)\mathbf{N}(t)$$

So $\mathbf{a}(t) = v'(t)\mathbf{T}(t) + \kappa(t)v^2(t)\mathbf{N}(t)$

$$= \mathbf{a}_T\mathbf{T} + \mathbf{a}_N\mathbf{N}$$

Note this description of $\mathbf{a}$ depends on the parameter, (as does $\mathbf{a}$ itself).

If we use arc length, then the speed is constant 1, so

$$\mathbf{a}(s) = 0\mathbf{T}(\mathbf{s}) + \kappa(s)1^2\mathbf{N}(s) = \kappa(s)\mathbf{N}(s)$$

Note2: regarding the components' meaning.

Part of acceleration comes from changing the speed, namely $v'(t)\mathbf{T}(t)$.

Another part comes from changing the direction, namely $\kappa(t)v^2(t)\mathbf{N}(t)$.

Note3: We can describe the components in terms of $\mathbf{a}$ and $\mathbf{v}$.

$$v(t) = |\mathbf{v}(t)| = (\mathbf{v}(t)\cdot\mathbf{v}(t))^{1/2}$$

$$a_{\mathbf{T}} = v'(t) = \frac{1}{2}(\mathbf{v}\cdot\mathbf{v})^{-1/2}(\mathbf{v}'\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v}') = \frac{2\mathbf{v}\cdot\mathbf{v}'}{2(\mathbf{v}\cdot\mathbf{v})^{1/2}} = \frac{\mathbf{v}\cdot\mathbf{a}}{v}$$

$$a_{\mathbf{N}} = \kappa(t)v^2(t) = \frac{|\mathbf{v}\times\mathbf{a}|v^2}{v^3} = \frac{|\mathbf{v}\times\mathbf{a}|}{v}$$

Examples.

$$\mathbf{r}(t) = < t, 2e^t, e^{2t} >$$
$$\mathbf{v}(t) = < 1, 2e^t, 2e^{2t} >$$
$$\mathbf{a}(t) = < 0, 2e^t, 4e^{2t} >$$

$$a_{\mathbf{T}} = \frac{\mathbf{v}\cdot\mathbf{a}}{v} = \frac{4e^{2t} + 8e^{4t}}{(1 + 4e^{2t} + 4e^{4t})^{1/2}} = 4e^{2t}$$

$$a_{\mathbf{N}} = \frac{|\mathbf{v}\times\mathbf{a}|}{v} = \frac{|< 4e^{3t}, -4e^{2t}, 2e^t >|}{1 + 2e^{2t}} = 2e^t$$

Decomposing $\vec{a}$ using $\vec{T}$ and $\vec{N}$, pg 2. Practice.

$$a_{\mathrm{T}} = \frac{\mathbf{v} \cdot \mathbf{a}}{v} \qquad a_{\mathrm{N}} = \frac{|\mathbf{v} \times \mathbf{a}|}{v}$$

1. $\mathbf{r}(t) = < t^2 + 1, t^3, 0 >$

$\mathbf{v}(t) = < 2t, 3t^2, 0 >$

$\mathbf{a}(t) = < 2, 6t, 0 >$

$$a_{\mathrm{T}} = \frac{4t + 18t^3}{(4t^2 + 9t^4)^{1/2}} = \frac{4 + 18t^2}{(4 + 9t^2)^{1/2}}$$

$$a_{\mathrm{N}} = \frac{| < 0, 0, 6t^2 > |}{(4t^2 + 9t^4)^{1/2}} = \frac{6t}{(4 + 9t^2)^{1/2}}$$

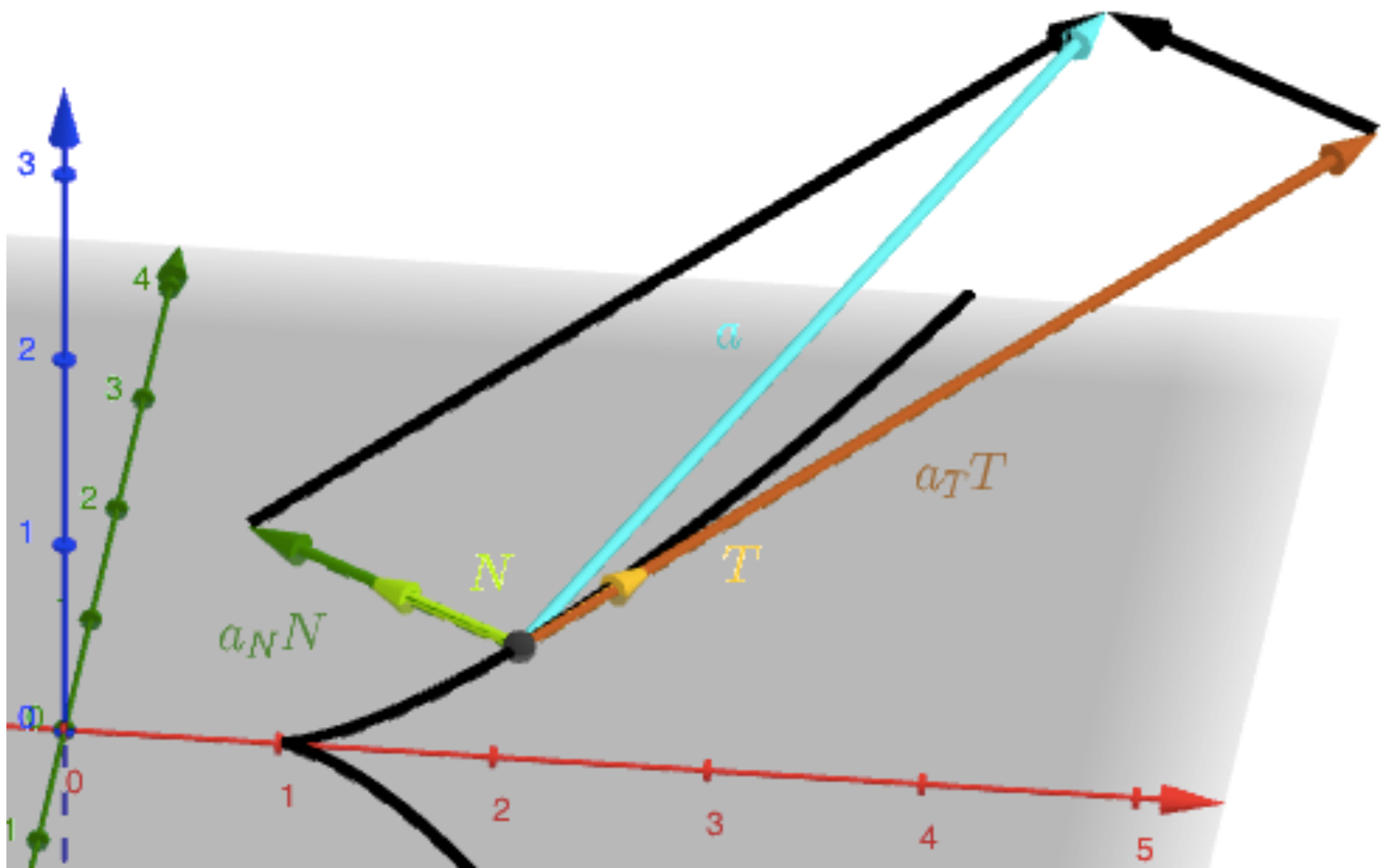
2. $\mathbf{r}(t) = < \cos(t), \sin(t), t >$

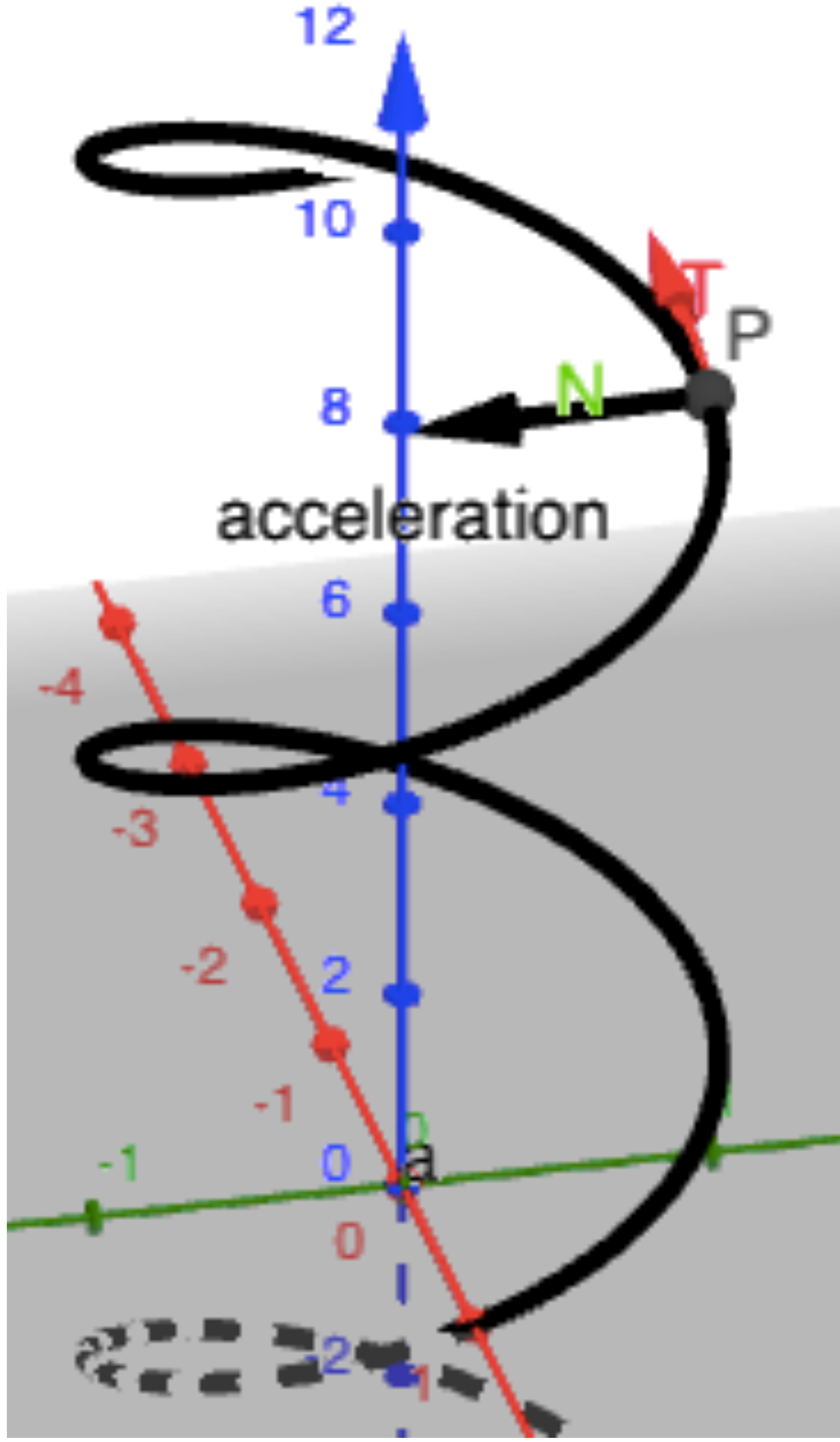
$\mathbf{v}(t) = < -\sin(t), \cos(t), 1 >$

$\mathbf{a}(t) = < -\cos(t), -\sin(t), 0 >$

$$a_{\mathrm{T}} = \frac{0}{\sqrt{2}} = 0$$

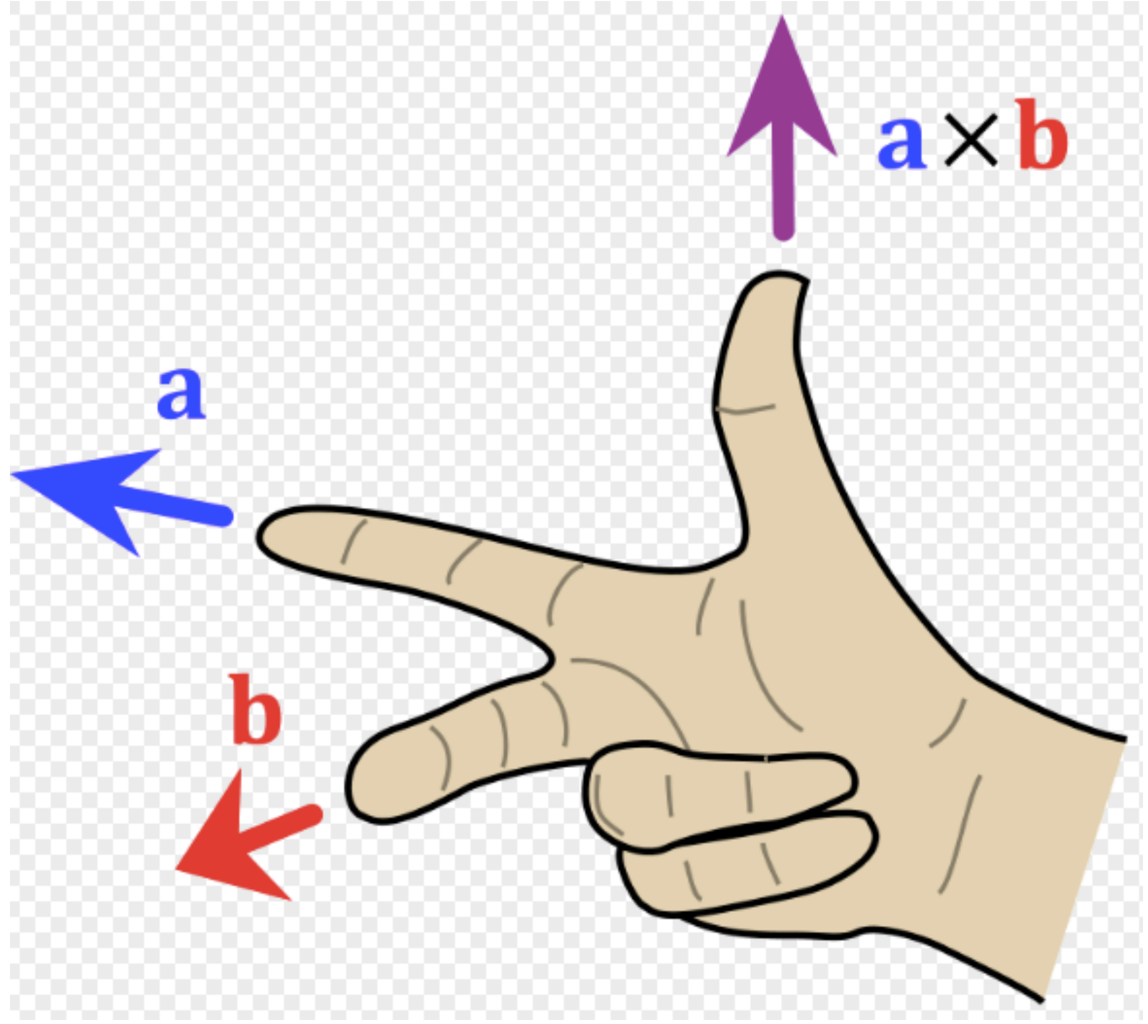
$$a_{\mathrm{N}} = \frac{| < \sin(t), -\cos(t), 1 > |}{\sqrt{2}} = 1$$





Orthonormal sets of vectors.

$$\mathbf{i}\times\mathbf{j}=\mathbf{k} \quad \mathbf{j}\times\mathbf{i}=-\mathbf{k}$$
$$\mathbf{j}\times\mathbf{k}=\mathbf{i} \quad \mathbf{k}\times\mathbf{j}=-\mathbf{i}$$
$$\mathbf{k}\times\mathbf{i}=\mathbf{j} \quad \mathbf{i}\times\mathbf{k}=-\mathbf{j}$$



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Recall $\mathbf{i} = < 1,0,0 >$, $\mathbf{j} = < 0,1,0 >$, $\mathbf{k} = < 0,0,1 >$

They are subject to the right-hand rule.

They are all related to one another via the cross product.

We have other sets of vectors that behave a similar way…

$\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$!!!

At every location on a curve, at every instant t, the vectors $\mathbf{T}(t)$, $\mathbf{N}(t)$, $\mathbf{B}(t)$ are of unit length, and mutually perpendicular.

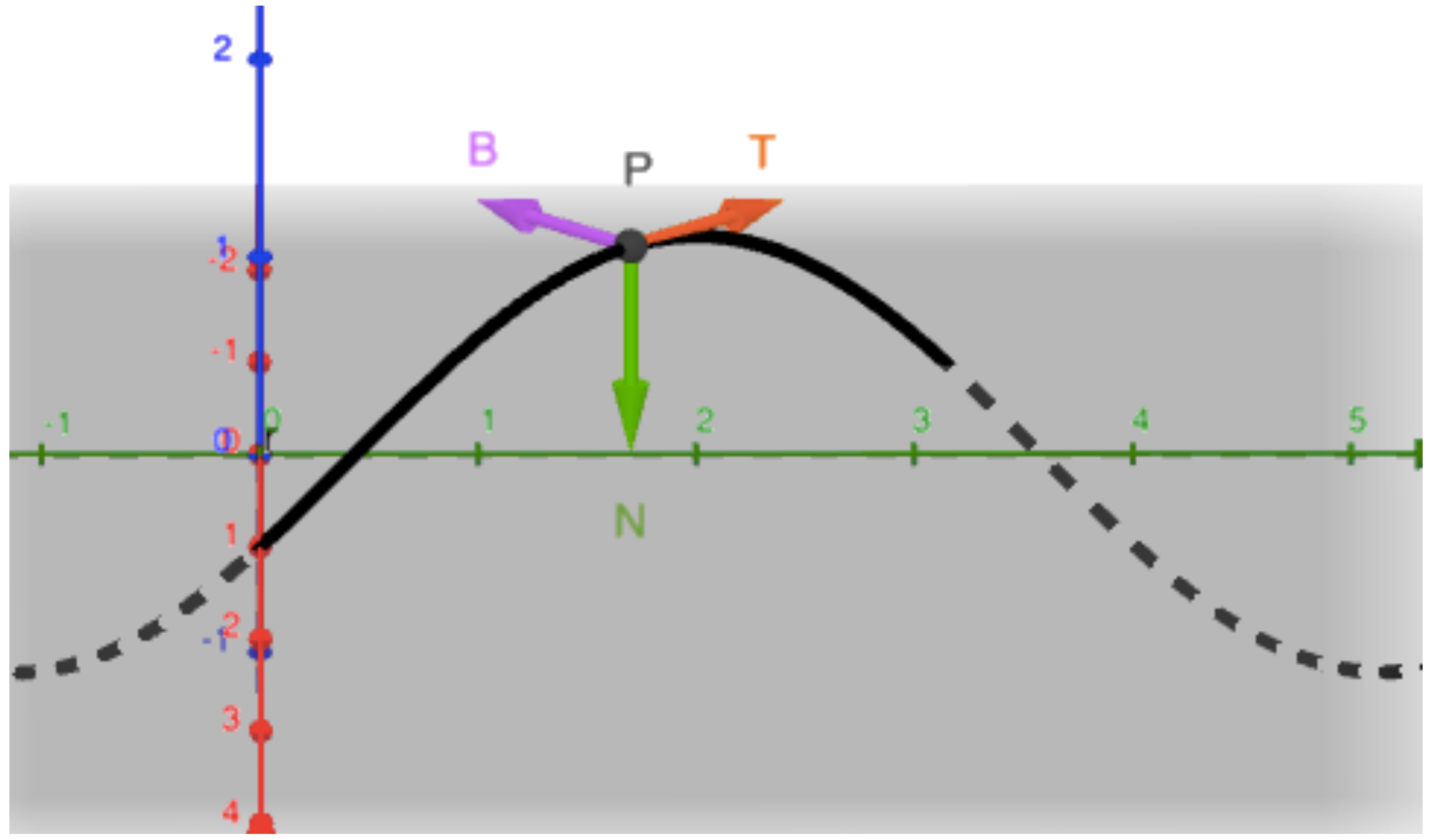
$\mathbf{T}\times\mathbf{N}=\mathbf{B}$ (by definition)

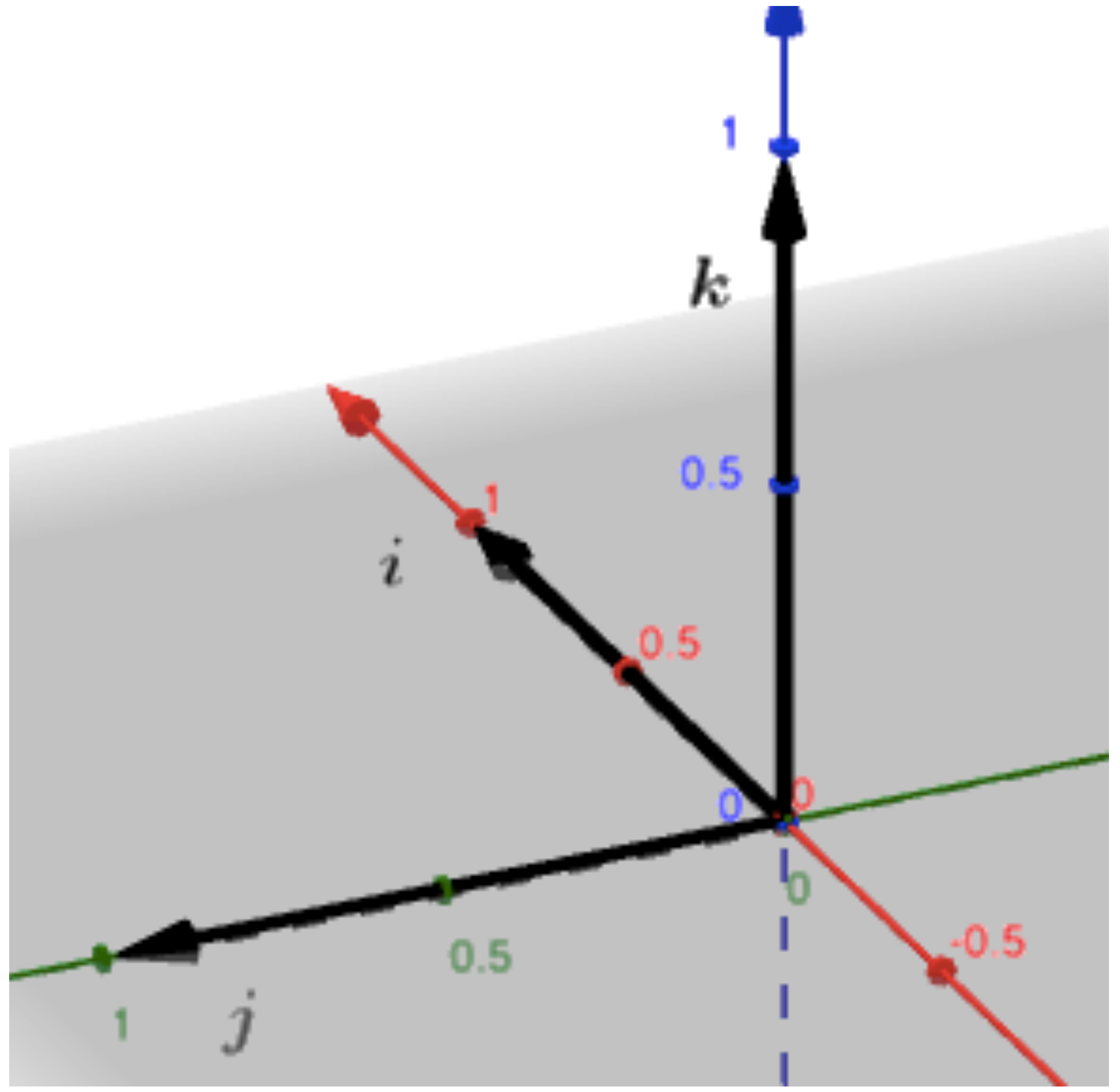
$\mathbf{N}\times\mathbf{T}=-\mathbf{B}$

but also…

$\mathbf{B}\times\mathbf{T}=\mathbf{N}$ $\quad$ $\mathbf{T}\times\mathbf{B}=-\mathbf{N}$

$\mathbf{N}\times\mathbf{B}=\mathbf{T}$ $\quad$ $\mathbf{B}\times\mathbf{N}=-\mathbf{T}$





Sets that behave like $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ and $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ are called *orthonormal bases* of 3 dimensional space.

(You can learn more about orthonormal bases at CCSF in M120, and M130 !!!)

Torsion, pg 1.

Suppose we have a unit-speed curve (i.e. parametrized by arc length), $\mathbf{r}(s)$.

We have seen that curvature is measured by the magnitude $|\mathbf{T}'(s)|$.

i.e. $\mathbf{T}'(s) = \kappa(s)\mathbf{N}(s)$

What about $\mathbf{N}'(s)$ and $\mathbf{B}'(s)$? Start with $\mathbf{B}'(s)$.

$|\mathbf{B}(s)| = 1$ for all s. Thus $\mathbf{B}'(s)$ is orthogonal (i.e. perpendicular) to $\mathbf{B}(s)$.

(This perpendicularity has the same reasons as that $\mathbf{T}'$ is perpendicular to $\mathbf{T}$.)

Furthermore, $\mathbf{B}'(s)$ is also perpendicular to $\mathbf{T}(s)$!

This comes from the definition of $\mathbf{B}$, and the product rule for differentiation applied to cross products…

The definition was $\mathbf{B} = \mathbf{T} \times \mathbf{N}$

The derivative is $\mathbf{B}' = \mathbf{T}' \times \mathbf{N} + \mathbf{T} \times \mathbf{N}'$

The first term, $\mathbf{T}' \times \mathbf{N}$, is 0 because $\mathbf{T}'$ and $\mathbf{N}$ are parallel.

So $\mathbf{B}'(s) = \mathbf{T}(s) \times \mathbf{N}'(s)$.

This is perpendicular to, in particular, $\mathbf{T}(s)$.

Thus $\mathbf{B}'(s)$ is perpendicular to both $\mathbf{B}(s)$ and $\mathbf{T}(s)$.

$\mathbf{B}'(s)$ is parallel to $\mathbf{B} \times \mathbf{T} = \mathbf{N}$.

$\mathbf{B}'(s) = -\tau(s) \cdot \mathbf{N}(s)$

This is the definition of the *torsion,* $\tau(s)$, of a curve. (The negative sign is a convention.)

Analogous to curvature, we have $|\tau(s)| = |\mathbf{B}'(s)|$

But unlike curvature, torsion can be positive or negative. $\mathbf{B}$ may change toward or away from $\mathbf{N}$.

Torsion, pg 2. Why is it important?

Using curvature and torsion together, we get a system of differential equations in $\mathbf{T}, \mathbf{N}, \mathbf{B}$

We already know $\mathbf{T}'(s) = \kappa(s)\mathbf{N}(s)$ and $\mathbf{B}'(s) = -\tau(s)\mathbf{N}(s)$

What about $\mathbf{N}'(s)$?

$$\mathbf{N}(s) = \mathbf{B}(s) \times \mathbf{T}(s)$$

$$\begin{aligned}
\mathbf{N}'(s) &= \mathbf{B}'(s) \times \mathbf{T}(s) + \mathbf{B}(s) \times \mathbf{T}'(s) \\
&= -\tau(s)\mathbf{N}(s) \times \mathbf{T}(s) + \kappa(s)\mathbf{B}(s) \times \mathbf{N}(s) \\
&= \tau(s)\mathbf{B}(s) - \kappa(s)\mathbf{T}(s) \\
&= -\kappa(s)\mathbf{T}(s) + \tau(s)\mathbf{B}(s)
\end{aligned}$$

All together…

$$\begin{aligned}
\mathbf{T}'(s) &= 0\mathbf{T}(s) + \kappa(s)\mathbf{N}(s) + 0\mathbf{B}(s) \\
\mathbf{N}'(s) &= -\kappa(s)\mathbf{T}(s) + 0\mathbf{N}(s) + \tau(s)\mathbf{B}(s) \\
\mathbf{B}'(s) &= 0\mathbf{T}(s) - \tau(s)\mathbf{N}(s) + 0\mathbf{B}(s)
\end{aligned}$$

You may see this system written like this:

$$\begin{pmatrix} \mathbf{T}'(s) \\ \mathbf{N}'(s) \\ \mathbf{B}'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}(s) \end{pmatrix}$$

If the curve is not parameterized by arc length, but in some parameter t…

$$\begin{pmatrix} \mathbf{T}'(t) \\ \mathbf{N}'(t) \\ \mathbf{B}'(t) \end{pmatrix} = \begin{pmatrix} \frac{d\mathbf{T}}{ds} \cdot \frac{ds}{dt} \\ \frac{d\mathbf{N}}{ds} \cdot \frac{ds}{dt} \\ \frac{d\mathbf{B}}{ds} \cdot \frac{ds}{dt} \end{pmatrix} = \begin{pmatrix} \frac{d\mathbf{T}}{ds} \cdot v(t) \\ \frac{d\mathbf{N}}{ds} \cdot v(t) \\ \frac{d\mathbf{B}}{ds} \cdot v(t) \end{pmatrix} = \begin{pmatrix} \mathbf{T}'(s) \\ \mathbf{N}'(s) \\ \mathbf{B}'(s) \end{pmatrix} \cdot v(t)$$

where $\dfrac{ds}{dt} = |\mathbf{v}(t)| = v(t)$

The differential equation becomes…

$$\begin{pmatrix} \mathbf{T}'(t) \\ \mathbf{N}'(t) \\ \mathbf{B}'(t) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(t)v(t) & 0 \\ -\kappa(t)v(t) & 0 & \tau(t)v(t) \\ 0 & -\tau(t)v(t) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T}(t) \\ \mathbf{N}(t) \\ \mathbf{B}(t) \end{pmatrix}$$

Torsion, pg 3, Computing Torsion.

You can use the definition $\mathbf{B}'(s) = -\tau(s)\mathbf{N}(s)$

If you don't want to (or can't) parametrize in arc length, then $\mathbf{B}'(t) = -\tau(t)v(t)\mathbf{N}(t)$

Example1.

$\mathbf{r}(t) = <\sqrt{2}t, e^t, e^{-t}>$

We have already computed $\mathbf{T}, \mathbf{N}, \mathbf{B}$ (see warm-up, slide 4)

$$v(t) = |\mathbf{v}(t)| = e^t + e^{-t}$$

$$\mathbf{T}(t) = (e^t + e^{-t})^{-1} < \sqrt{2}, e^t, -e^{-t} >$$

$$\mathbf{N}(t) = (e^t + e^{-t})^{-1} < e^{-t} - e^t, \sqrt{2}, \sqrt{2} >$$

$$\mathbf{B}(t) = (e^t + e^{-t})^{-1} < \sqrt{2}, -e^{-t}, e^t >$$

$$\mathbf{B}'(t) = -(e^t + e^{-t})^{-2}(e^t - e^{-t}) < \sqrt{2}, -e^{-t}, e^t >$$

$$+(e^t + e^{-t})^{-1} < 0, e^{-t}, e^t >$$

$$= \ldots = (e^t + e^{-t})^{-2} < \sqrt{2}(e^{-t} - e^t), 2, 2 > = \frac{\sqrt{2}}{e^t + e^{-t}}\mathbf{N}(t)$$

$$\mathbf{B}'(t) = -\tau(t)v(t)\mathbf{N}(t)$$

$$\tau(t) = -\frac{1}{v(t)} \cdot \frac{\sqrt{2}}{e^t + e^{-t}} = \frac{-\sqrt{2}}{(e^t + e^{-t})^2}$$

Example2.

$\mathbf{r}(t) = < \cos(t), t, \sin(t) >$ $\qquad v(t) = \sqrt{2}$

$$\mathbf{T}(t) = \frac{1}{\sqrt{2}} < -\sin(t), 1, \cos(t) >$$

$$\mathbf{N}(t) = < -\cos(t), 0, -\sin(t) >$$

$$\mathbf{B}(t) = \frac{1}{\sqrt{2}} < -\sin(t), -1, \cos(t) >$$

$$\mathbf{B}'(t) = \frac{1}{\sqrt{2}} < -\cos(t), 0, -\sin(t) > = \frac{1}{\sqrt{2}}\mathbf{N}(t)$$

$$\mathbf{B}'(t) = -\tau(t)v(t)\mathbf{N}(t)$$

$$\tau(t) = -\frac{1}{v(t)} \cdot \frac{1}{\sqrt{2}} = -\frac{1}{2}$$

Torsion, pg 4: Deriving a Formula for Torsion.

We expressed curvature in terms of $\mathbf{v}$ and $\mathbf{a}$.

$$\kappa(t) = \frac{|\mathbf{v}(t) \times \mathbf{a}(t)|}{v(t)^3}$$

We can express torsion in terms of velocity $\mathbf{v}$, acceleration $\mathbf{a}$, and $\mathbf{a}'$.

Recently we saw how to express $\mathbf{a}$ in terms of $\mathbf{T}$ and $\mathbf{N}$

$$\mathbf{a}(t) = v'(t)\mathbf{T}(t) + \kappa(t)v^2(t)\mathbf{N}(t)$$

Its derivative is..

$$\mathbf{a}' = v''\,\mathbf{T} + v'\,\mathbf{T}'$$

$$+(\kappa' v^2 + 2\kappa v v')\mathbf{N} + \kappa v^2 \mathbf{N}'$$

Using $\mathbf{T}' = \kappa v \mathbf{N}$ and $\mathbf{N}' = -\kappa v \mathbf{T} + \tau v \mathbf{B}$

$$\begin{aligned}\mathbf{a}' &= v''\,\mathbf{T} + v'\kappa v\,\mathbf{N} \\ &\quad +(\kappa' v^2 + 2\kappa v v')\mathbf{N} + \kappa v^2(-\kappa v \mathbf{T} + \tau v \mathbf{B}) \\ &= (v'' - \kappa^2 v^3)\mathbf{T} + (\kappa' v^2 + 3\kappa v v')\mathbf{N} + \kappa v^3 \tau \mathbf{B}\end{aligned}$$

Also, using $\mathbf{a}(t) = v'(t)\mathbf{T} + \kappa(t)v^2(t)\mathbf{N}(t)$ and $\mathbf{v}(t) = v(t)\mathbf{T}$, we can express $\mathbf{v} \times \mathbf{a}$.

$$\begin{aligned}\mathbf{v} \times \mathbf{a} &= v\mathbf{T} \times (v'\mathbf{T} + \kappa v^2 \mathbf{N}) \\ &= vv'\mathbf{T} \times \mathbf{T} + \kappa v^3 \mathbf{T} \times \mathbf{N} \\ &= vv'\mathbf{0} + \kappa v^3 \mathbf{B} = \kappa v^3 \mathbf{B}\end{aligned}$$

The dot product $(\mathbf{v} \times \mathbf{a}) \cdot \mathbf{a}'$ leave us with some τ.

$$\begin{aligned}(\mathbf{v} \times \mathbf{a}) \cdot \mathbf{a}' &= \kappa v^3 \mathbf{B} \cdot \left((v'' - \kappa^2 v^3)\mathbf{T} + (\kappa' v^2 + 3\kappa v v')\mathbf{N} + \kappa v^3 \tau \mathbf{B}\right) \\ &= 0 + 0 + \kappa^2 v^6 \tau \mathbf{B} \cdot \mathbf{B} = \kappa^2 v^6 \tau\end{aligned}$$

Because $\mathbf{B} \cdot \mathbf{T} = \mathbf{B} \cdot \mathbf{N} = 0$ and $\mathbf{B} \cdot \mathbf{B} = |\mathbf{B}|^2 = 1$

Also note that $|\mathbf{v} \times \mathbf{a}| = \kappa v^3$

Finally then, $\tau = \dfrac{(\mathbf{v} \times \mathbf{a}) \cdot \mathbf{a}'}{|\mathbf{v} \times \mathbf{a}|^2}$

Torsion, pg 5. More computations of τ.

$$\tau = \frac{(\mathbf{v} \times \mathbf{a}) \cdot \mathbf{a}'}{|\mathbf{v} \times \mathbf{a}|^2}$$

Example.

$$\mathbf{r}(t) = < \sqrt{2}t, e^t, e^{-t} >$$

$$\mathbf{v}(t) = < \sqrt{2}, e^t, -e^{-t} >$$

$$\mathbf{a}(t) = < 0, e^t, e^{-t} >$$

$$\mathbf{a}'(t) = < 0, e^t, -e^{-t} >$$

$$\mathbf{v} \times \mathbf{a} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sqrt{2} & e^t & -e^{-t} \\ 0 & e^t & e^{-t} \end{pmatrix}$$

$$= < 2, -\sqrt{2}e^{-t}, \sqrt{2}e^t >$$

$$= \sqrt{2} < \sqrt{2}, -e^{-t}, e^t >$$

$$|\mathbf{v} \times \mathbf{a}|^2 = 2(e^t + e^{-t})^2$$

$$(\mathbf{v} \times \mathbf{a}) \cdot \mathbf{a}' = -2\sqrt{2}$$

$$\tau = \frac{(\mathbf{v} \times \mathbf{a}) \cdot \mathbf{a}'}{|\mathbf{v} \times \mathbf{a}|^2} = \frac{-\sqrt{2}}{(e^t + e^{-t})^2}$$

Example.

$$\mathbf{r}(t) = < \cos(t), t, \sin(t) >$$

$$\mathbf{v}(t) = < -\sin(t), 1, \cos(t) >$$

$$\mathbf{a}(t) = < -\cos(t), 0, -\sin(t) >$$

$$\mathbf{a}'(t) = < \sin(t), 0, -\cos(t) >$$

$$\mathbf{v} \times \mathbf{a} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin(t) & 1 & \cos(t) \\ -\cos(t) & 0 & -\sin(t) \end{pmatrix}$$

$$= < -\sin(t), -1, \cos(t) >$$

$$|\mathbf{v} \times \mathbf{a}|^2 = 2$$

$$(\mathbf{v} \times \mathbf{a}) \cdot \mathbf{a}' = -1$$

$$\tau = \frac{(\mathbf{v} \times \mathbf{a}) \cdot \mathbf{a}'}{|\mathbf{v} \times \mathbf{a}|^2} = \frac{-1}{2}$$

Example.

$$\mathbf{r}(t) = < t, t^2, t^3 >$$

$$\mathbf{v}(t) = < 1, 2t, 3t^2 >$$

$$\mathbf{a}(t) = < 0, 2, 6t >$$

$$\mathbf{a}'(t) = < 0, 0, 6 >$$

$$\mathbf{v} \times \mathbf{a} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{pmatrix}$$

$$= < 6t^2, -6t, 2 >$$

$$|\mathbf{v} \times \mathbf{a}|^2 = 36t^4 + 36t^2 + 4$$

$$(\mathbf{v} \times \mathbf{a}) \cdot \mathbf{a}' = 12$$

$$\tau = \frac{(\mathbf{v} \times \mathbf{a}) \cdot \mathbf{a}'}{|\mathbf{v} \times \mathbf{a}|^2} = \frac{12}{36t^4 + 36t^2 + 4}$$

$$= \frac{3}{9t^4 + 9t^2 + 1}$$

Visualizing Torsion, pg1. The osculating plane.

The *osculating plane* of a curve $\mathbf{r}(t)$ at a point on the curve is the plane that contains the vectors $\mathbf{T}(t)$ and $\mathbf{N}(t)$.

This is the plane that is perpendicular to $\mathbf{B}(t)$.

Example. Find the osculating plane of the given curve at the given point.

$$\mathbf{r}(t) = <\sqrt{2}t, e^t, e^{-t}> , \qquad P(0,1,1)$$

The vector perpendicular to this plane is what we computed earlier.

$$\mathbf{B}(t) = (e^t + e^{-t})^{-1} < \sqrt{2}, -e^{-t}, e^t >$$

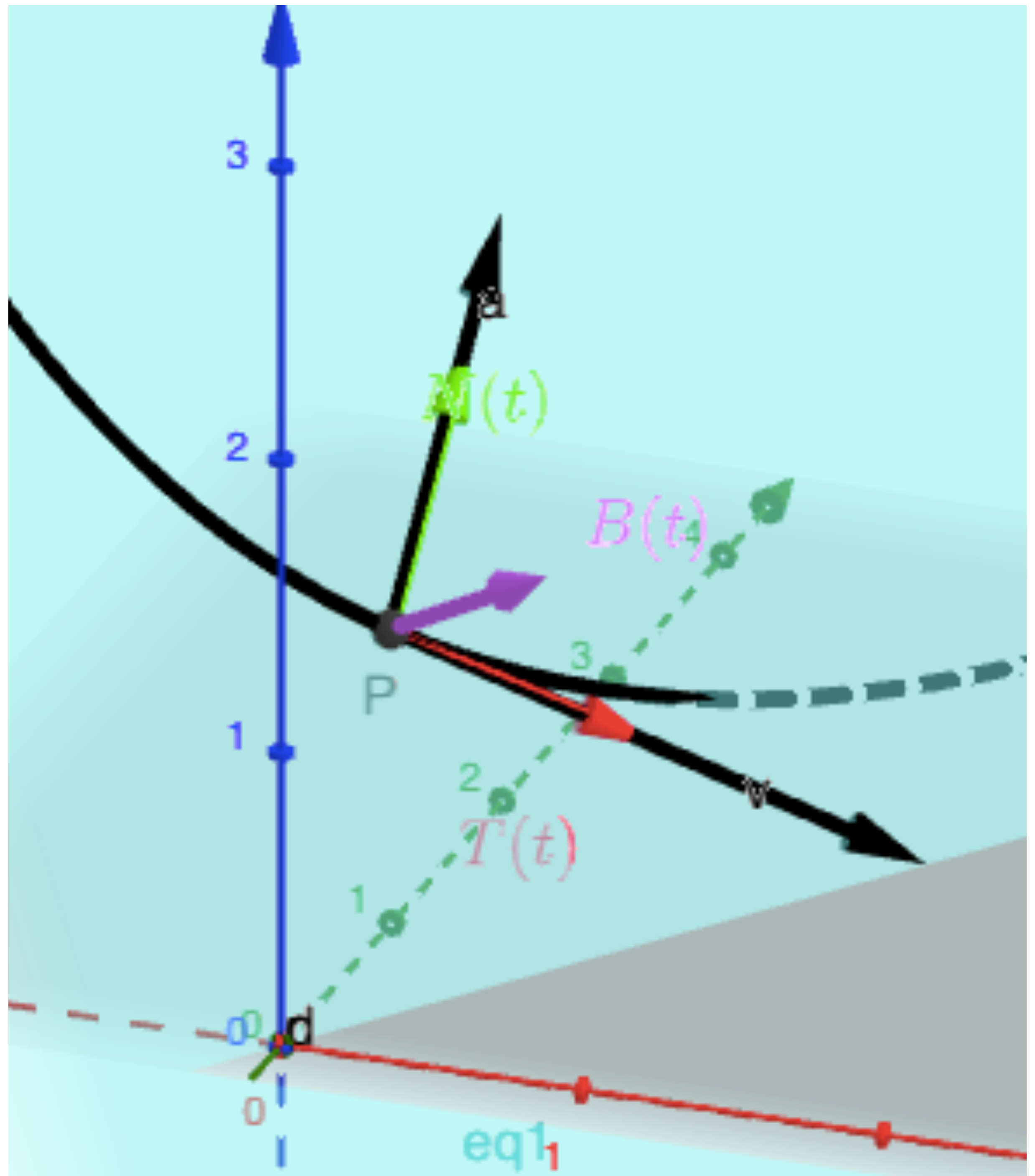
At time t the osculating plane is

$$\sqrt{2}(x - \sqrt{2}t) - e^{-t}(y - e^t) + e^t(z - e^{-t}) = 0$$

When $t = 0$ this plane is

$$\sqrt{2}(x - 0) - (y - 1) + (z - 1) = 0$$

or $\sqrt{2}x - y + z = 0$



Example.

$$\mathbf{r}(t) = < \cos(t), t, \sin(t) > , \quad P(0,\pi/2,1)$$

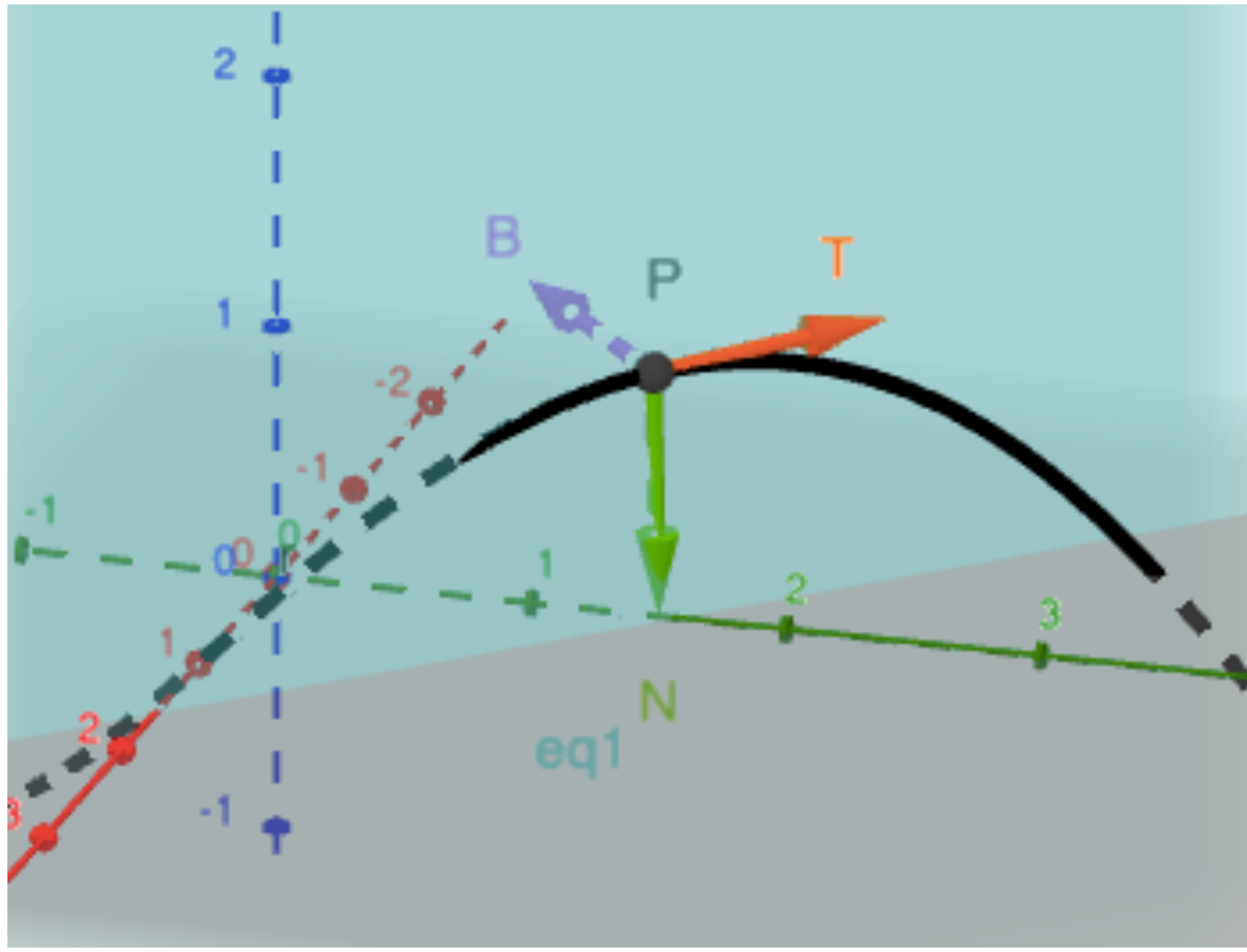
$$\mathbf{B}(t) = \frac{1}{\sqrt{2}} < -\sin(t), -1, \cos(t) >$$

At time t the osculating plane is

$$-\sin(t)(x - \cos(t)) - (y - t) + \cos(t)(z - \sin(t)) = 0$$

At time $t = \pi/2$ this is

$$-(x - 0) - (y - \pi/2) = 0 \quad \text{or } x + y = \pi/2$$



Visualizing Torsion, pg 2. The osculating plane.

For planar curves like $\mathbf{r}(t) = < x(t), y(t), 0 >$, torsion is 0 because the osculating plane is constant.

This is so because for curves that live in the xy plane, we always have $\mathbf{B}(t) = \pm < 0,0,1 >$.

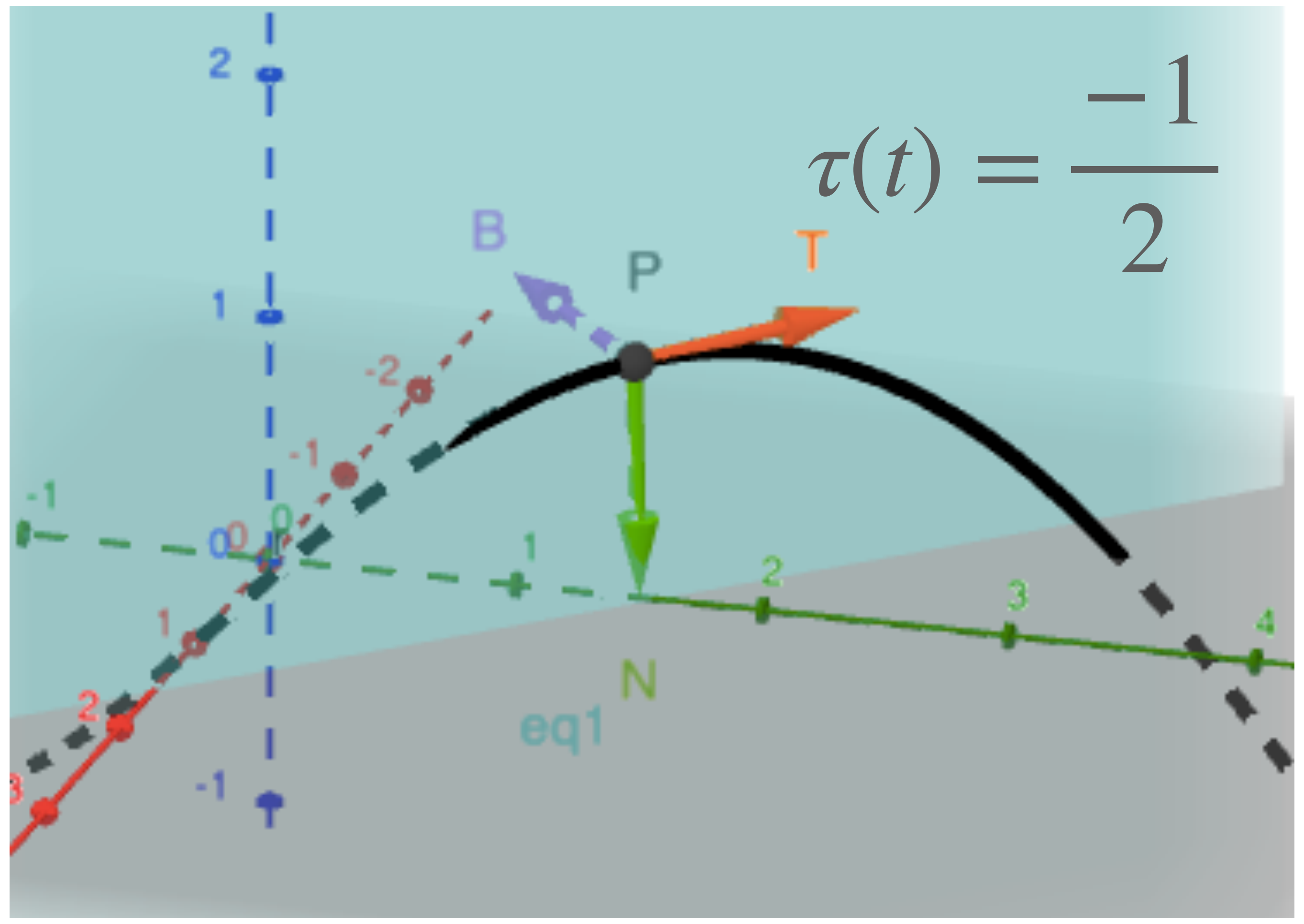
For curves in space, however, the osculating plane changes with time.

The rate at which it twists is measured by $|\mathbf{B}'(t)| = |\tau(t)| v(t)$.

Constant Torsion:

Link: MotionOnAHelix

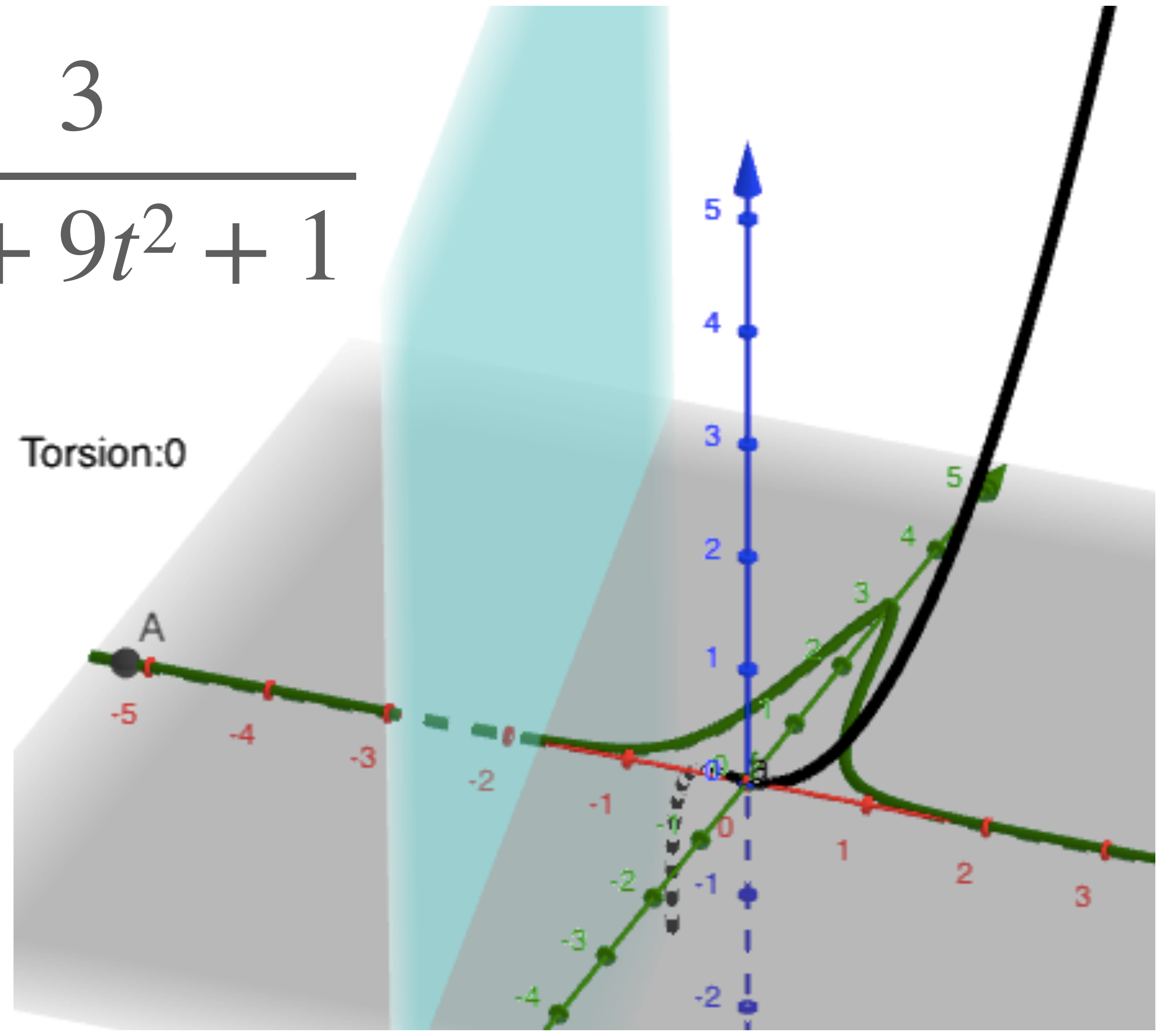
$$\tau(t) = \frac{-1}{2}$$



Almost all 0 torsion:

Link: MotionAlong<t,t^2,t^3>

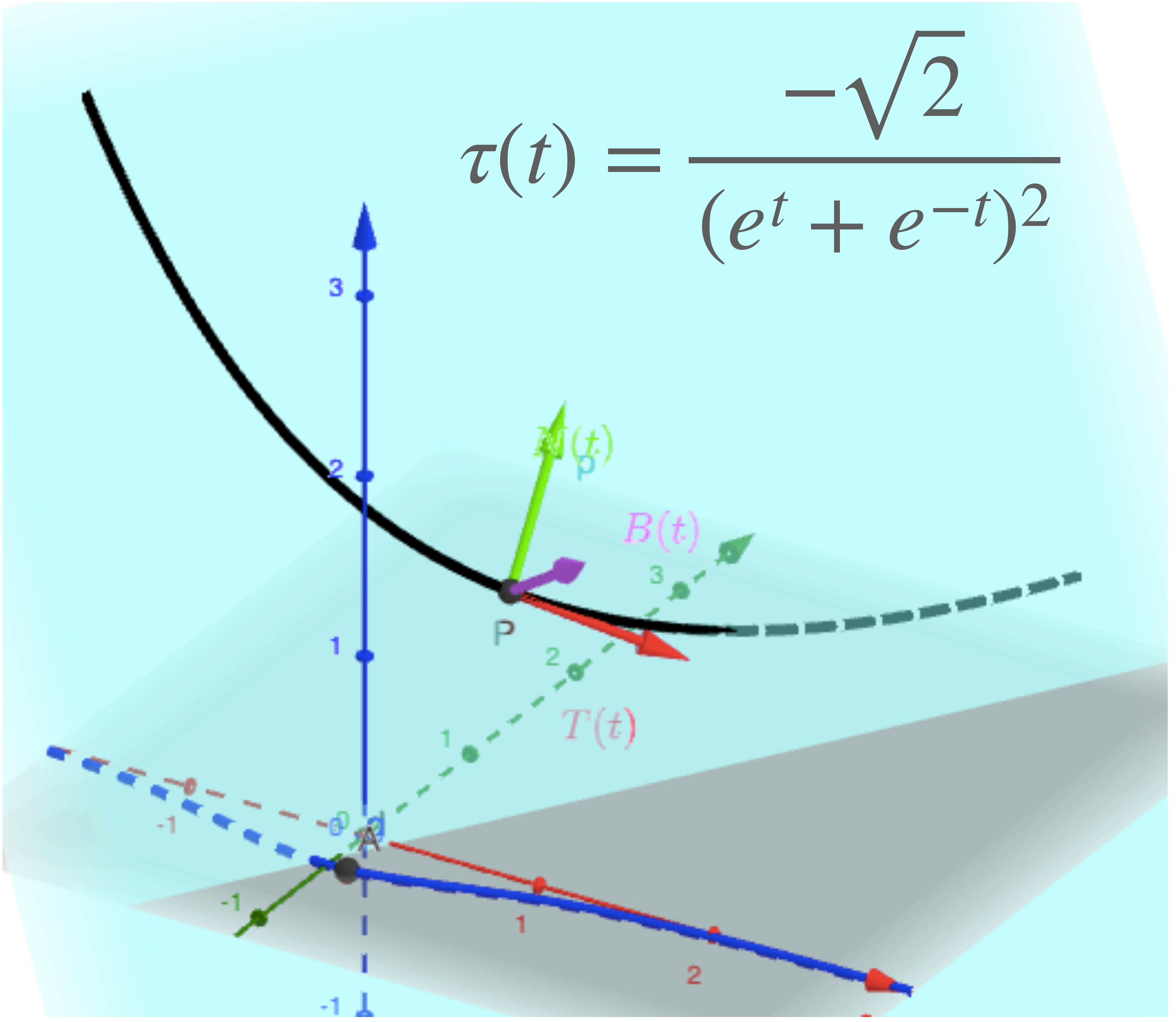
$$\tau(t) = \frac{3}{9t^4 + 9t^2 + 1}$$



Negative Torsion.

Link: HyperbolicCrash

$$\tau(t) = \frac{-\sqrt{2}}{(e^t + e^{-t})^2}$$



Optional: Solving the Matrix Differential Equation, pg 1.

If we are given the curvature and torsion of the unit-speed parameterization of a curve, then we can determine the curve by solving a differential equation.

First a small lemma.

We know how derivatives of $\sin$ and $\cos$ behave.

$$\frac{d}{dt}\Big(A\sin(\omega t)\Big) = A\omega\cos(\omega t)$$

$$\frac{d^2}{dt^2}\Big(A\sin(\omega t)\Big) = -A\omega^2\sin(\omega t)$$

$$\frac{d}{dt}\Big(A\cos(\omega t)\Big) = -A\omega\sin(\omega t)$$

$$\frac{d^2}{dt^2}\Big(A\cos(\omega t)\Big) = -A\omega^2\cos(\omega t)$$

An interesting fact is that there is a converse to this behavior:

Lemma: If you have a function whose second derivative is a negative multiple of the function itself,

then this function must be some combination of sine and cosine functions.

If $f''(t) = -c \cdot f(t)$, then $f(t) = A\cos(\omega t) + B\sin(\omega t)$

where $c > 0$, $\omega = \sqrt{c}$. (If you're interested in some details, you can learn about this more at CCSF in M125 and/or M130.)

If you know some initial conditions, such as $f(0)$ and $f'(0)$, then you can use these to determine the values of coefficients A and B.

Furthermore, the combination of sines and cosines can be simplified into a single function:

$$A\cos(\omega t) + B\sin(\omega t) = H\cos(\omega t - \theta) = H\sin(\omega t + \alpha)$$

where $H = \sqrt{A^2 + B^2}$, $\quad \theta = \arctan\left(\dfrac{B}{A}\right)$

Optional: Solving the Matrix Differential Equation, pg 2.

Suppose we're given a unit-speed curve with constant curvature and zero torsion.

What kind of curve might this be?!

It must be a circle. Why?

Say the curvature is $\kappa(s) = k$

$$\begin{aligned}
\mathbf{T}'(s) &= 0\mathbf{T}(s) + \kappa(s)\mathbf{N}(s) + 0\mathbf{B}(s) \\
\mathbf{N}'(s) &= -\kappa(s)\mathbf{T}(s) + 0\mathbf{N}(s) + \tau(s)\mathbf{B}(s) \\
\mathbf{B}'(s) &= 0\mathbf{T}(s) - \tau\mathbf{N}(s) + 0\mathbf{B}(s)
\end{aligned}$$

$$\begin{aligned}
\mathbf{T}'(s) &= k\mathbf{N}(s) \\
\mathbf{N}'(s) &= -k\mathbf{T}(s) \\
\mathbf{B}'(s) &= \overrightarrow{\mathbf{0}}
\end{aligned}$$

Torsion being zero implies $\mathbf{B}(s)$ is a constant vector. This means that the curve remains in a single plane, the plane whose normal vector is $\mathbf{B}$.

If we choose the x,y and z axes appropriately, then we can have this curve contained in the xy plane, so its z component is 0, and we're only concerned with $\mathbf{T}(s)$ and $\mathbf{N}(s)$.

We get a second-order differential equation in $\mathbf{T}(s)$

$$\mathbf{T}''(s) = (k\mathbf{N}(s))' = k\mathbf{N}'(s) = k(-k\mathbf{T}(s)) = -k^2\mathbf{T}(s)$$

Now using components of $\mathbf{T}$: $\mathbf{T}(s) = < x(s), y(s), 0 >$

$$\mathbf{T}''(s) = < x''(s), y''(s), 0 >$$

$$-k^2\mathbf{T}(s) = < -k^2x(s), -k^2y(s), 0 >$$

Thus $x''(s) = -k^2x(s), \quad y''(s) = -k^2y(s)$

Applying some differential equation theory (the lemma from the previous slide),each of the components of $\mathbf{T}$ must be a combination of sines and cosines.

$$x(s) = A\cos(ks) + B\sin(ks) \qquad y(s) = C\cos(ks) + D\sin(ks)$$

Optional: Solving the Matrix Differential Equation, pg 3.

$$x(s) = A\cos(ks) + B\sin(ks) \qquad y(s) = C\cos(ks) + D\sin(ks)$$

$$\mathbf{T}(s) = < A\cos(ks) + B\sin(ks), C\cos(ks) + D\sin(ks) > = \mathbf{v}(s)$$

Since this is the arc-length parameterization ↑

$$\mathbf{r}(s) = \int \mathbf{v}(s)\, ds = \int \mathbf{T}(s)\, ds$$

$$= \frac{1}{k} < A\sin(ks) - B\cos(ks), C\sin(ks) - D\cos(ks) > + \mathbf{c}$$

We can argue that $\mathbf{c}$ is the center of a circle of the curve.

Try to show $|\mathbf{r}(s) - \mathbf{c}| =$ constant

$$|\mathbf{r}(s) - \mathbf{c}|^2$$

$$= \frac{1}{k^2}\big(A^2\sin^2(ks) - 2AB\sin(ks)\cos(ks) + B^2\cos^2(ks)$$

$$+C^2\sin^2(ks) - 2CD\sin(ks)\cos(ks) + D^2\cos^2(ks)\big)$$

$$= \frac{1}{k^2} | < -A\sin(ks) + B\cos(ks), -C\sin(ks) + D\cos(ks) > |^2 = \dots$$

$$\dots = \frac{1}{k^2}\left(\frac{|\mathbf{T}'(s)|}{k}\right)^2$$

$$= \frac{1}{k^2}\cdot 1 = \frac{1}{k^2}$$

$\mathbf{r}(s)$ is the arc-length parameterization of a circle centered at $\mathbf{c}$, with radius $1/k$, the reciprocal of the given curvature.

Extra Credit:

a) Can you find the unit-speed curve whose curvature and torsion are both 1?

b) What kind of unit-speed curves have constant curvature and torsion?