M110C Week1

Goals:

2D Vectors.
- definition
- arithmetic
- properties

Dot product.
- angle between vectors
- projections of vectors

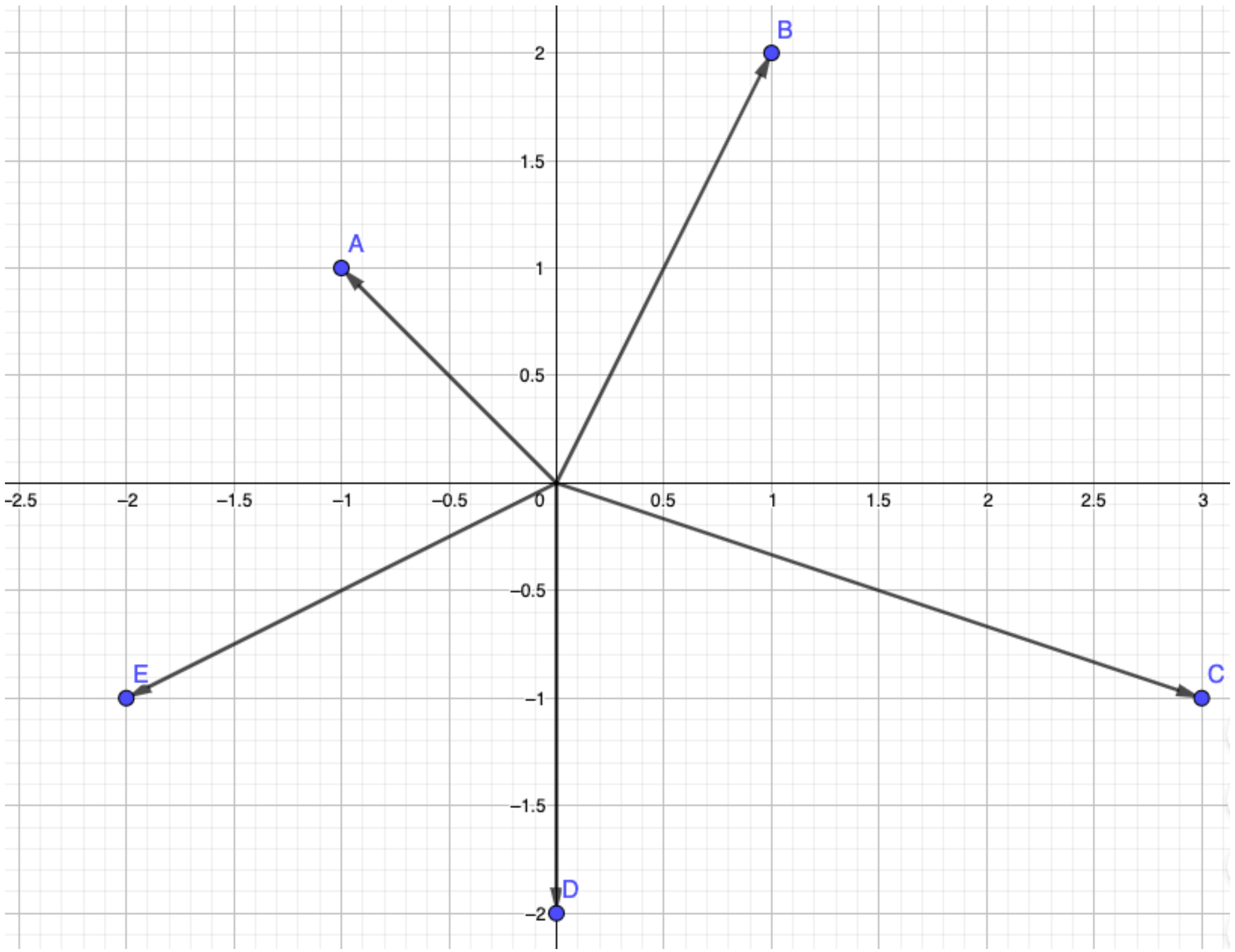
3 Dimensional Space.
- plotting points
- sketching some surfaces
- 3D vectors, angles, projections

Cross Product.
- derivation
- examples

Vectors: Definition.

A **vector** in math is an object that has **magnitude** and **direction**.

Here is a picture of some vectors.



Aside from a picture, how do we represent vectors?

Vectors can be described more specifically using x- and y-components.

Examples:

$\mathbf{a} = < -1,1 >$

$\mathbf{b} = < 1,2 >$

$\mathbf{c} = < 3, -1 >$

$\mathbf{d} = < 0, -2 >$

$\mathbf{e} = < -2, -1 >$

You may see many different notations for vectors, such as…

$$\mathbf{a} = < -1,1 > = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

or even

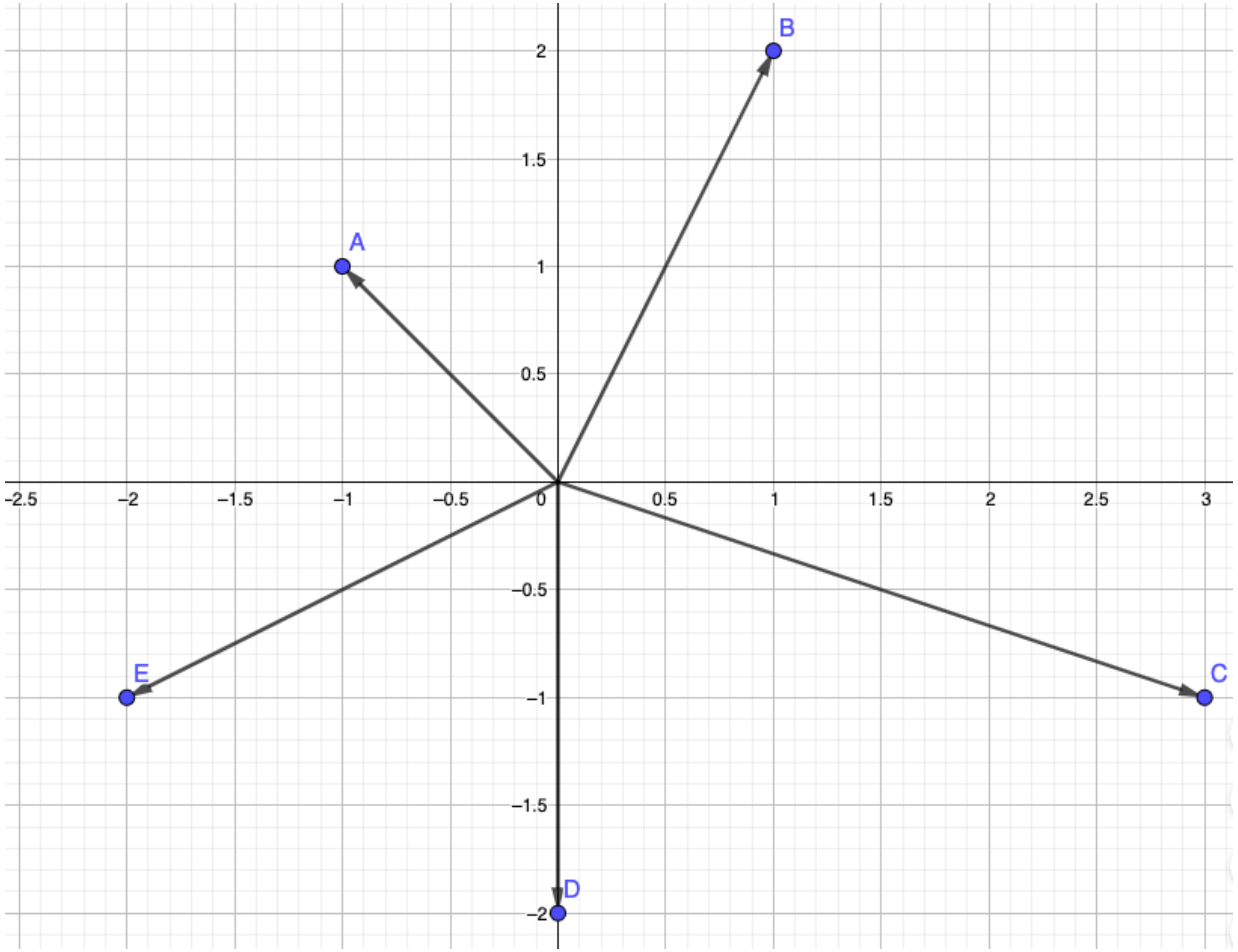
$$\mathbf{a} = (-1,1)$$

You may also see an arrow instead of bold notation:

$$\vec{a} = < -1,1 >$$

Vectors: Magnitude.

In the context of vectors, **magnitude** means length.



Examples.

$\mathbf{a} = < -1,1 >$

The magnitude of $\vec{a}$ is

$|\mathbf{a}| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$

Similarly

$|\mathbf{b}| = \sqrt{1^2 + 2^2} = \sqrt{5}$

Try to compute $|\mathbf{c}|$, $|\mathbf{d}|$, and $|\mathbf{e}|$.

$|\mathbf{c}| = \sqrt{10}$

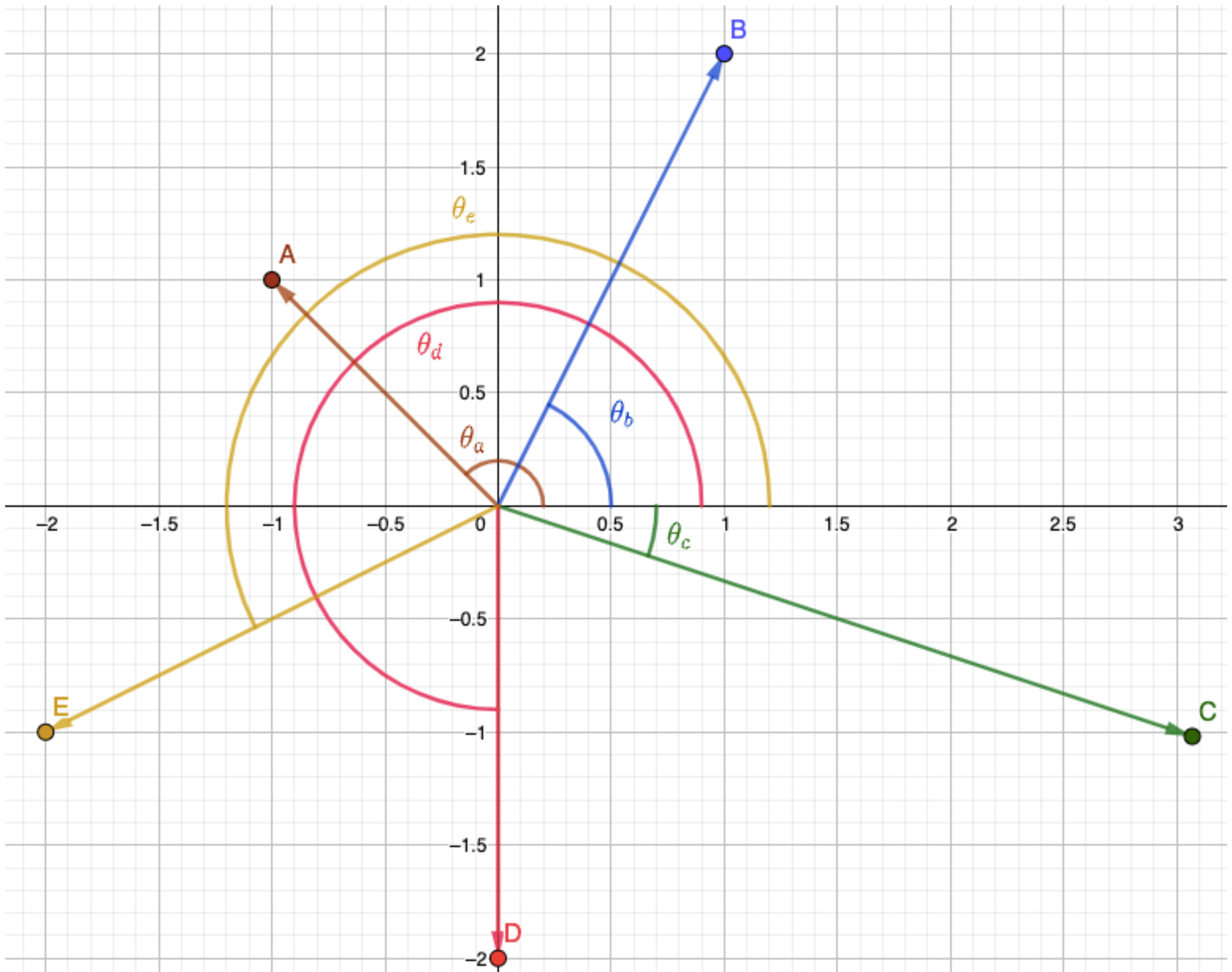
$|\mathbf{d}| = \sqrt{4} = 2$

$|\mathbf{e}| = \sqrt{5}$

2D Vectors: Direction.

There are many ways to define direction of a vector.

For 2D vectors, one common measure of direction is the angle off of the positive x axis.



Examples

$\mathbf{a} = < -1,1 >$ $\quad \tan(\theta_a) = \dfrac{1}{-1} = -1$

$$\theta_a = \tan^{-1}(-1) \in \left\{ -\frac{\pi}{4} \pm n\pi \,\middle|\, n \in \mathbf{N} \right\}$$

$$\theta_a = -\frac{\pi}{4} + 1\pi = \frac{3\pi}{4}$$

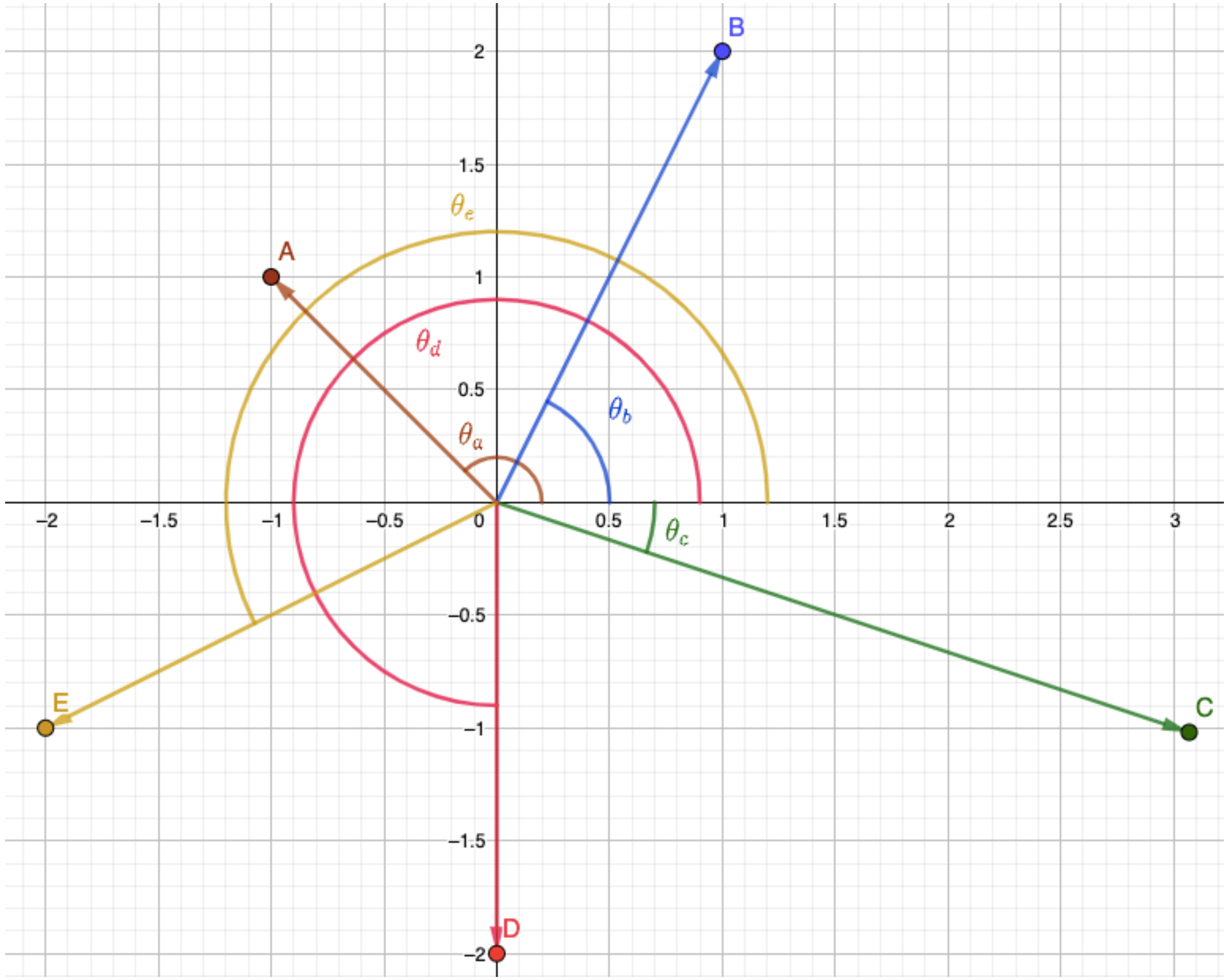
$\mathbf{b} = < 1,2 >$ $\quad \tan(\theta_b) = \dfrac{2}{1} = 2$

$$\theta_b = \tan^{-1}(2) \in \{1.107 \pm n\pi \,|\, n \in \mathbf{N}\}$$

$\theta_b \approx 1.107$ rad $\approx 63.43°$

Try to find $\theta_c, \theta_d, \theta_e$

Vectors: Direction, pg 2.



(continued from previous slide…)

$\mathbf{c} = < 3, -1 >$

$\theta_c = \tan^{-1}\left(-\frac{1}{3}\right) \approx -0.322$ rad $\approx -18.43°$

$\mathbf{d} = < 0, -2 >$

$\theta_d = \ldots = \frac{3\pi}{2}$ $\longleftarrow$ A calculator didn't help here. We need to recognize angles on the y axis.

$\mathbf{e} = < -2, -1 >$

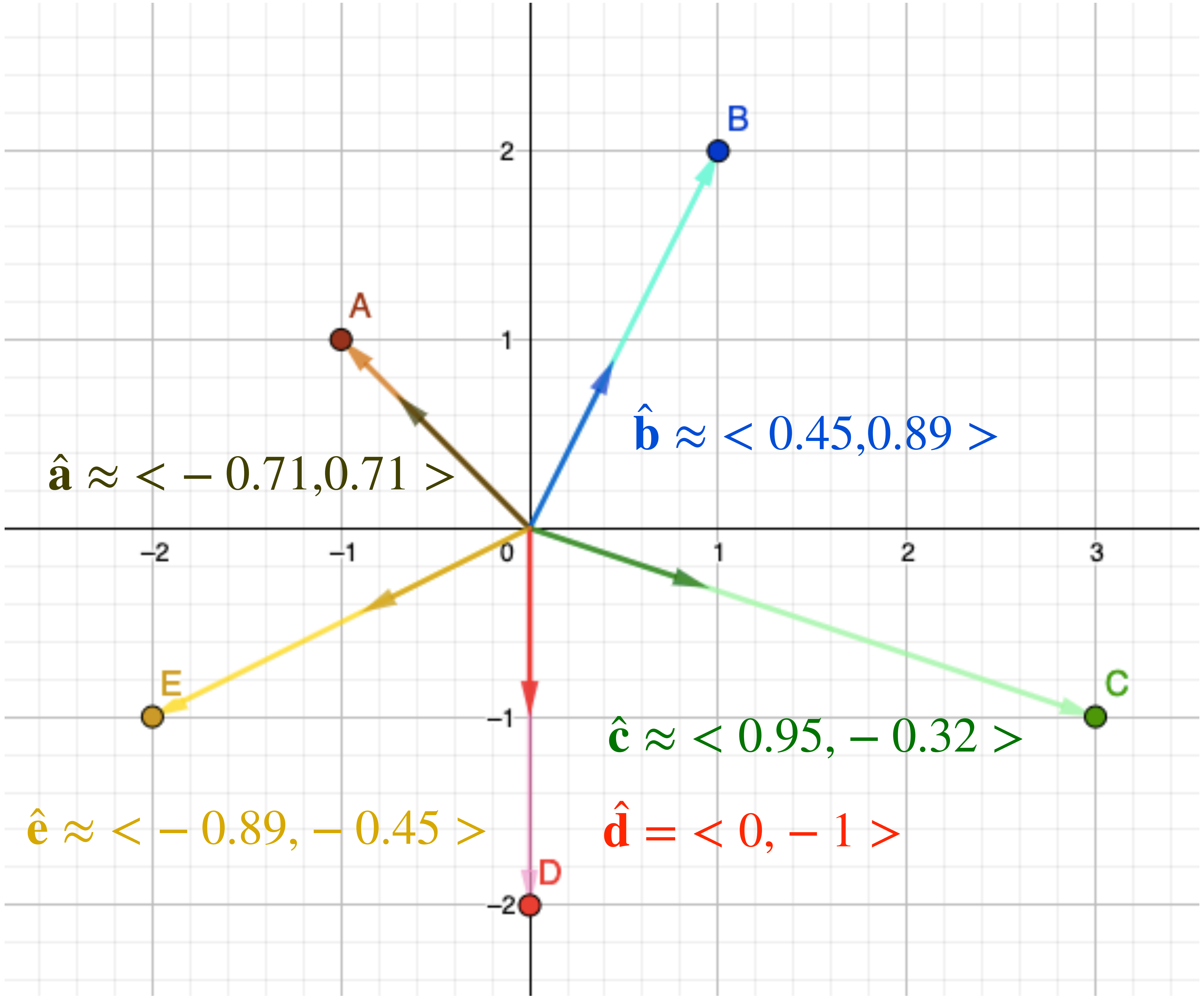
$\theta_e = \tan^{-1}\left(\frac{-1}{-2}\right) \approx 0.464 + \pi$ rad $\approx 206.565°$

NOTE: There are many angles that can describe a single vector!

e.g. θ_d can be any of these: $\left\{ \ldots -\frac{9\pi}{2}, -\frac{5\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}, \ldots \right\} = \left\{ -\frac{\pi}{2} \pm 2\pi n \,\middle|\, n \in \mathbf{N} \right\}$

2D Vectors: Direction, pg 3.

You can also measure direction by finding the *unit vector* of a given vector.



To get the unit vector of a given $\mathbf{v}$, you divide $\mathbf{v}$ by its magnitude.

$$\hat{\mathbf{v}} := \frac{\mathbf{v}}{|\mathbf{v}|}$$

Examples:

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{< -1, 1 >}{\sqrt{2}} = \left\langle \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle \approx < -0.71, 0.71 >$$

$$\hat{\mathbf{b}} = \frac{\mathbf{b}}{|\mathbf{b}|} = \frac{< 1, 2 >}{\sqrt{5}} = \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle \approx < 0.45, 0.89 >$$

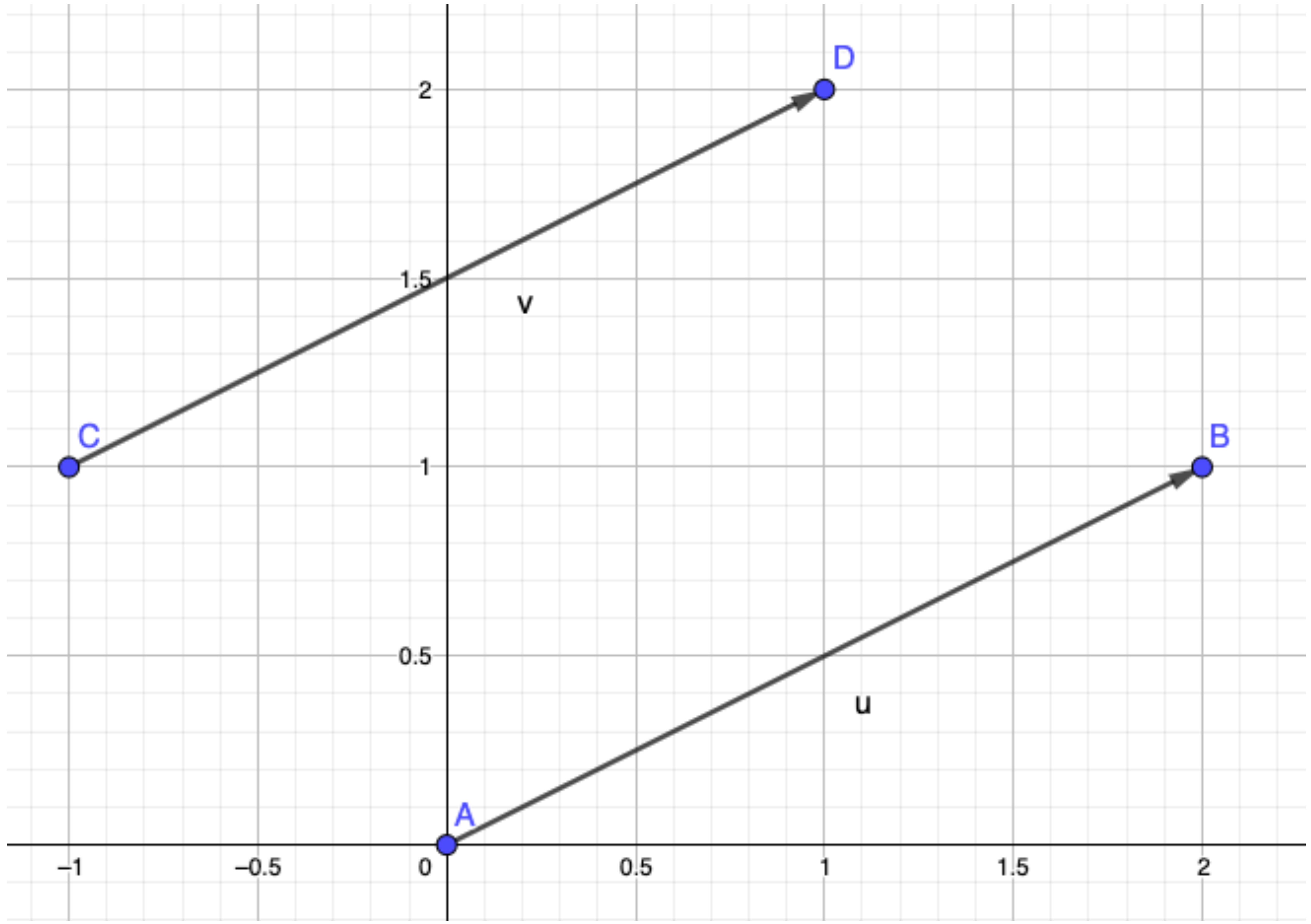
$$\hat{\mathbf{c}} = \frac{\mathbf{c}}{|\mathbf{c}|} = \frac{< 3, -1 >}{\sqrt{10}} = \left\langle \frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}} \right\rangle \approx < 0.95, -0.32 >$$

$$\hat{\mathbf{d}} = \frac{\mathbf{d}}{|\mathbf{d}|} = \frac{< 0, -2 >}{2} = \left\langle \frac{0}{2}, \frac{-2}{2} \right\rangle = < 0, -1 >$$

$$\hat{\mathbf{e}} = \frac{\mathbf{e}}{|\mathbf{e}|} = \frac{< -2, -1 >}{\sqrt{5}} = \left\langle \frac{-2}{\sqrt{5}}, \frac{-1}{\sqrt{5}} \right\rangle \approx < -0.89, -0.45 >$$

Vectors: Initial and Terminal Points.

Vectors don't have to start at the origin. They can start anywhere!



The *initial point* of $\mathbf{u}$ is $A(0,0)$, its *terminal point* is $B(2,1)$.

It's the vector $\mathbf{u} = < 2,1 >$

The initial point of $\mathbf{v}$ is $C(-1,1)$, and its terminal point is $D(1,2)$.

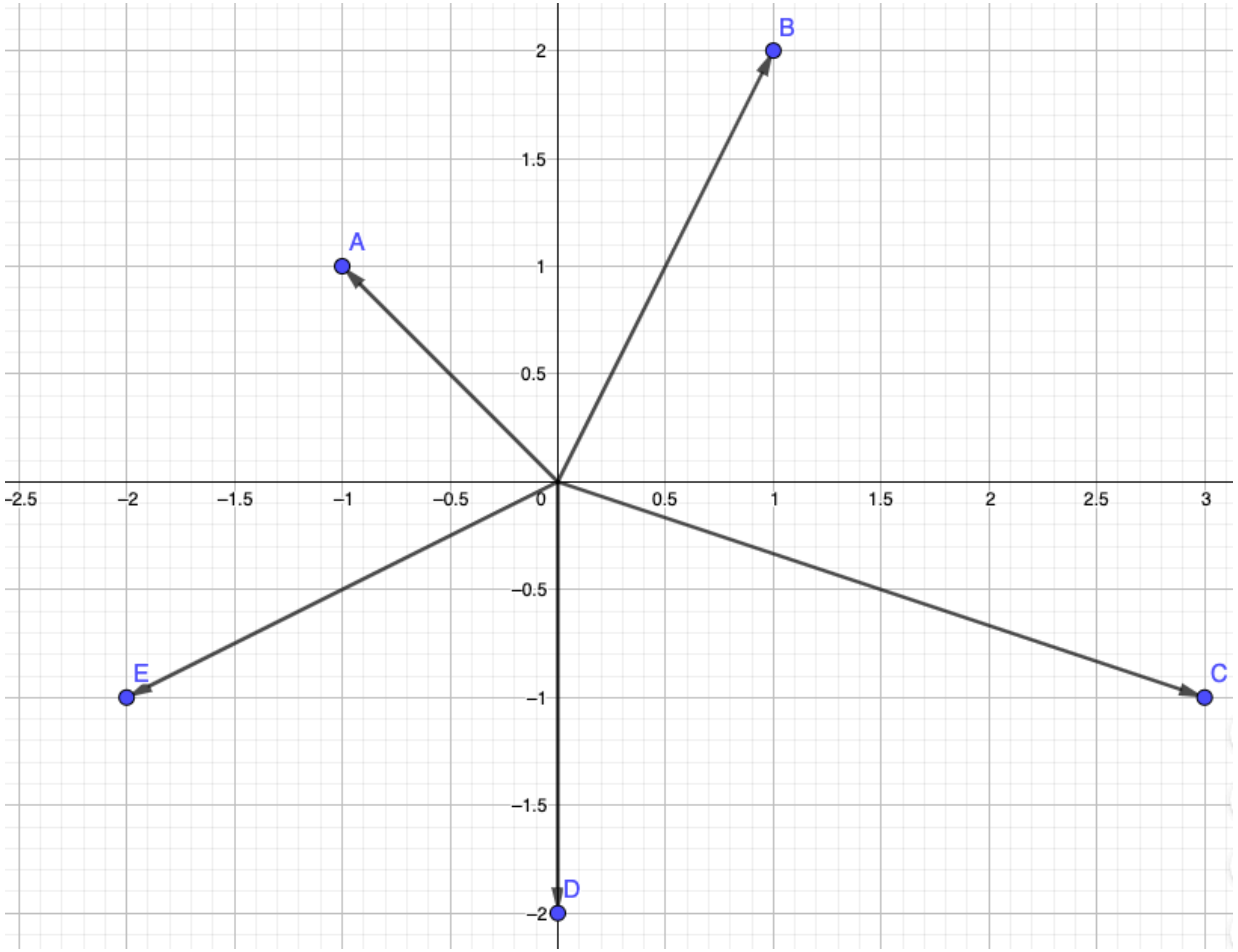
It's the vector $\mathbf{v} = < 2,1 >$

$\mathbf{u}$ and $\mathbf{v}$ are considered equal vectors!

They both have the same *displacement* from their initial to their terminal points, namely, 2 units in the positive x direction, and 1 unit in the positive y direction.

Vector Arithmetic: Algebraic Perspective.

Examples.



$\mathbf{a} = < -1,1 >$

$\mathbf{b} = < 1,2 >$

$\mathbf{c} = < 3, -1 >$

$\mathbf{d} = < 0, -2 >$

$\mathbf{e} = < -2, -1 >$

Vector Addition Example.

$$\mathbf{a} + \mathbf{b} = < -1,1 > + < 1,2 >$$
$$= < -1+1,\ 1+2 > = < 0,3 >$$

Vector Subtraction Example.

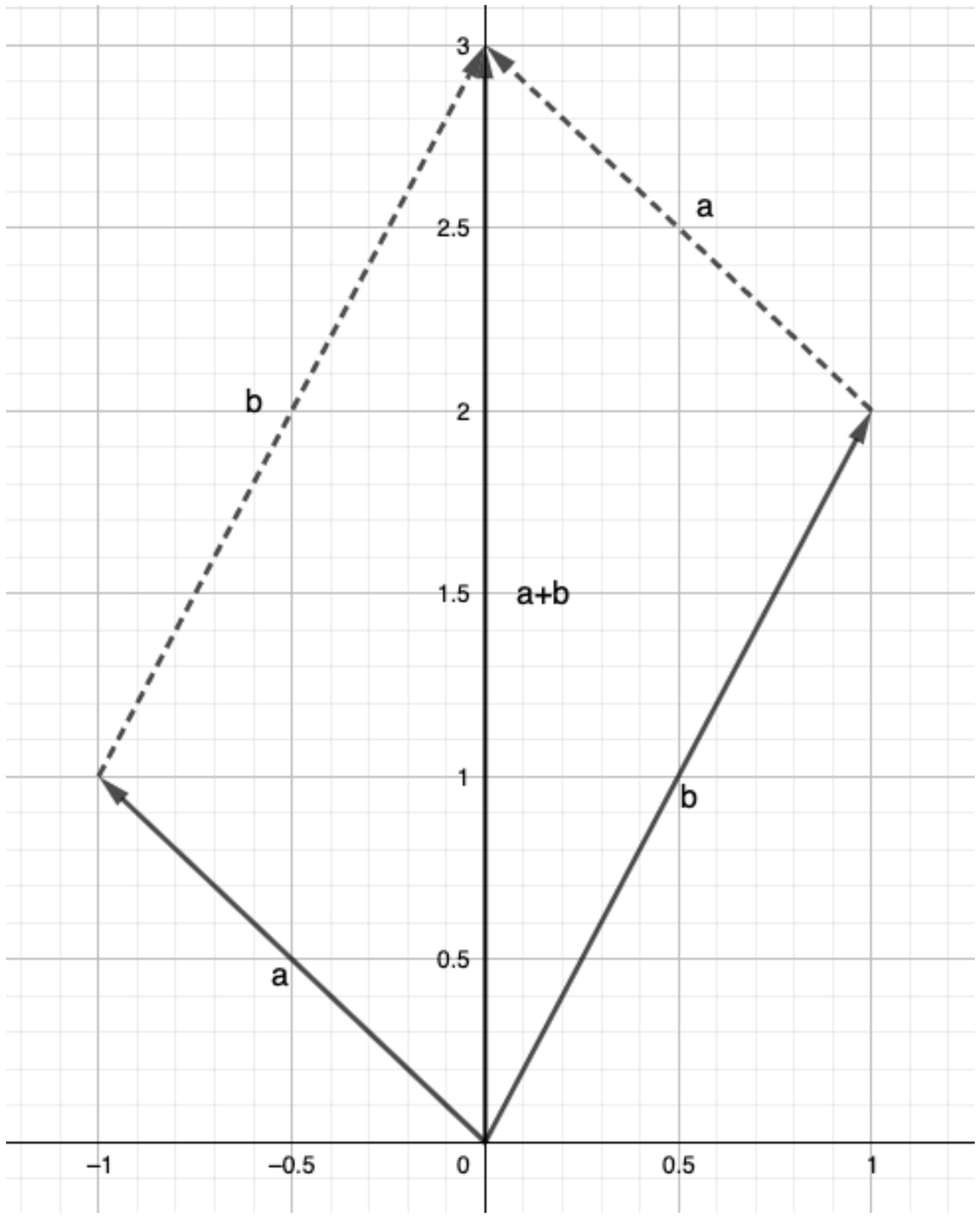
$$\mathbf{a} - \mathbf{b} = < -1,1 > - < 1,2 >$$
$$= < -1-1, 1-2 > = < -2, -1 >$$

Scalar Multiplication Example.

$$5 \cdot \mathbf{a} = 5 \cdot < -1,1 >$$
$$= < 5 \cdot (-1), 5 \cdot 1 > = < -5,5 >$$

Vector Arithmetic, Geometric Perspective : Addition.

$\mathbf{a} + \mathbf{b} = < -1{,}1 > + < 1{,}2 > = < 0{,}3 >$



You can compute $\mathbf{a} + \mathbf{b}$ by placing a copy of $\mathbf{b}$ at the terminal point of $\mathbf{a}$.

$\mathbf{a} + \mathbf{b}$ is then the vector that starts at the initial point of $\mathbf{a}$, and ends at the terminal point of the copy of $\mathbf{b}$.

Also,…

$\mathbf{a} + \mathbf{b}$ is a vector along one of the diagonals of the parallelogram formed by two vectors $\mathbf{a}$ and two vectors $\mathbf{b}$.

Vector Addition. You try!

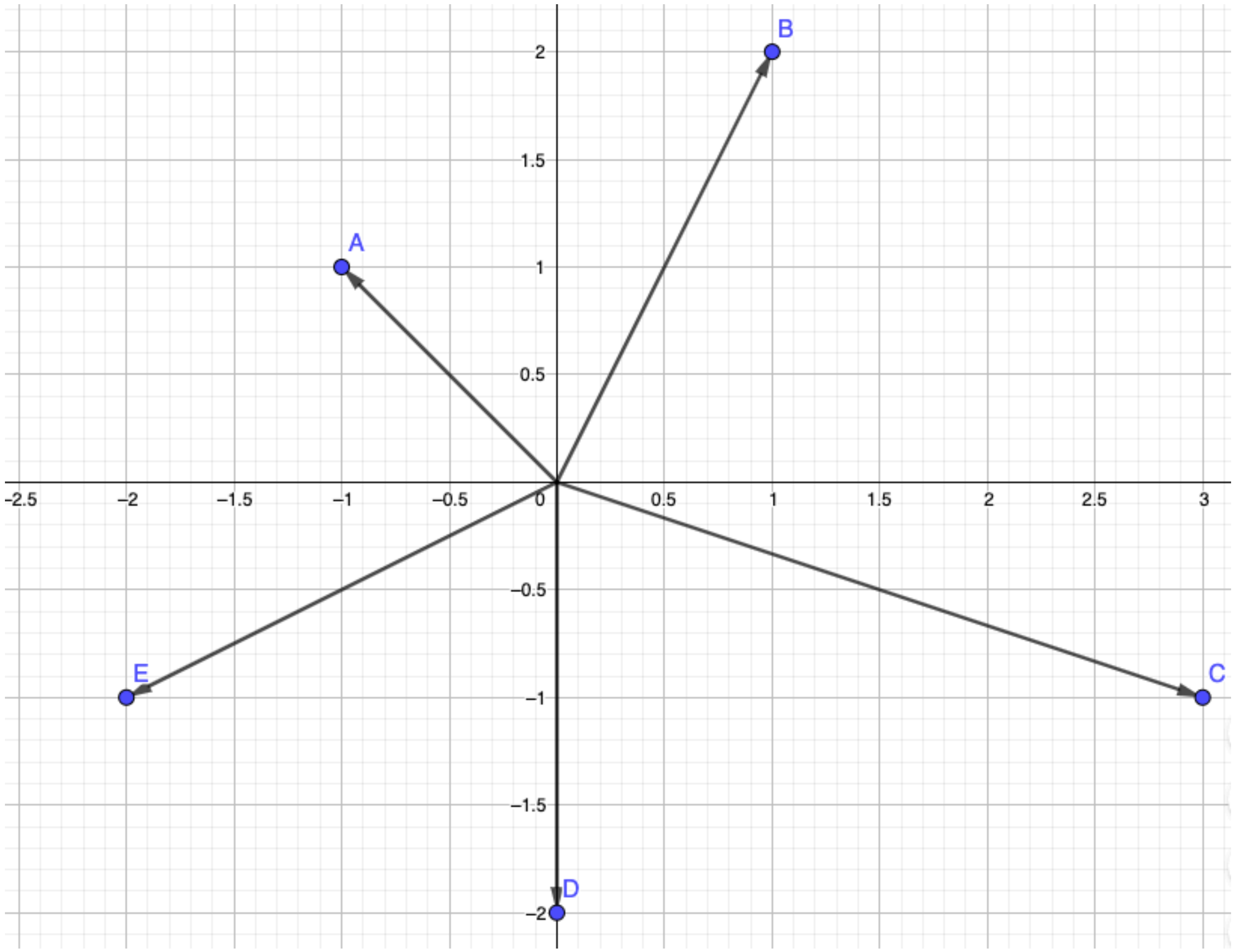
Try computing the following vector sums.
Do both the algebraic computation,
and draw a picture!

$\mathbf{a} = < -1,1 >$ $\mathbf{c} = < 3, -1 >$

$\mathbf{b} = < 1,2 >$ $\mathbf{d} = < 0, -2 >$

$\mathbf{e} = < -2, -1 >$

1. $\mathbf{a} + \mathbf{d}$ 2. $\mathbf{b} + \mathbf{c}$ 3. $\mathbf{e} + \mathbf{c}$



Solutions:

$$\mathbf{a} + \mathbf{d} = < -1 + 0, 1 + -2 > = < -1, -1 >$$

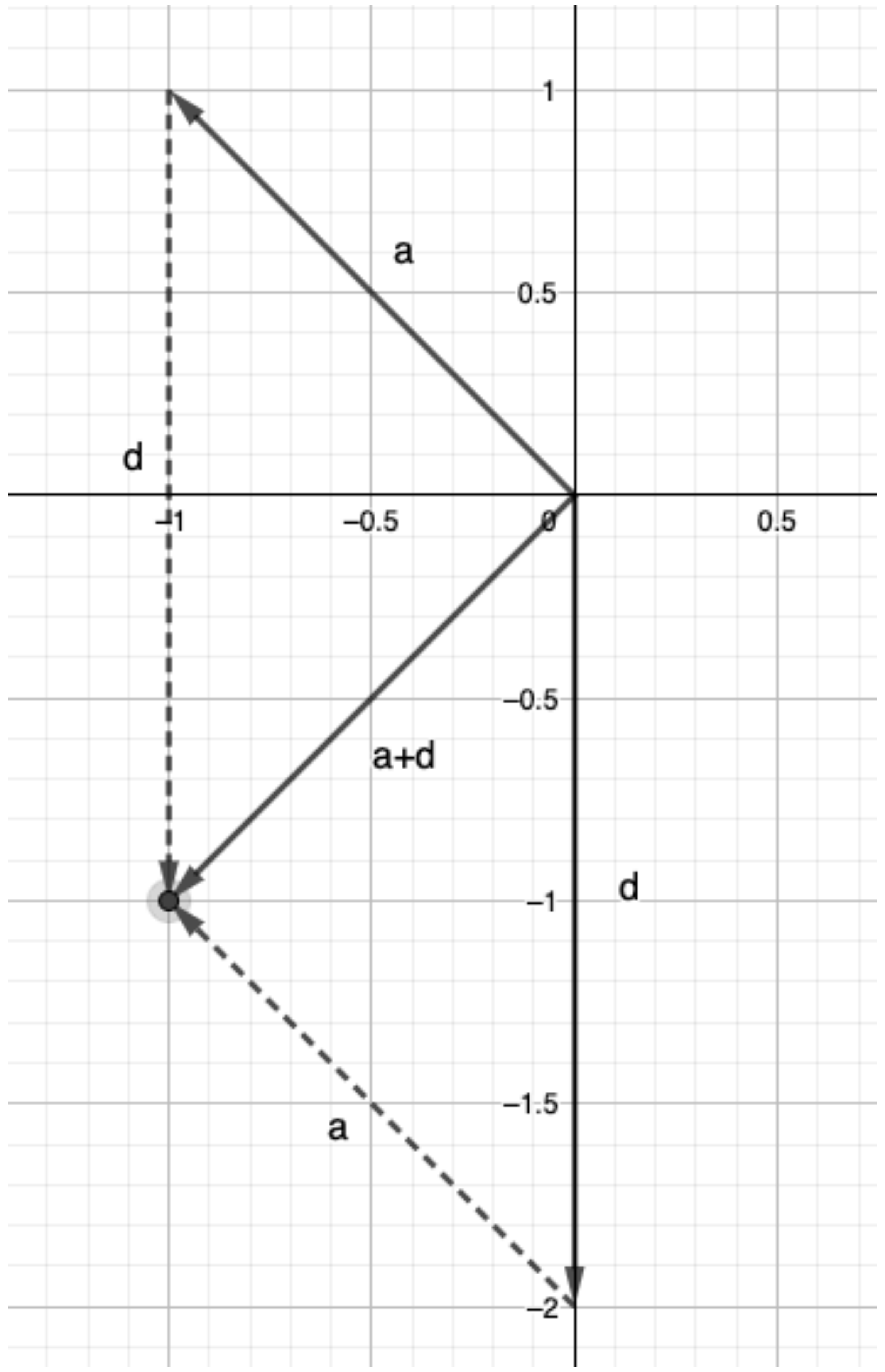
$$\mathbf{b} + \mathbf{c} = < 1 + 3, 2 + -1 > = < 4, 1 >$$

$$\mathbf{e} + \mathbf{c} = < -2 + 3, -1 + -1 > = < 1, -2 >$$

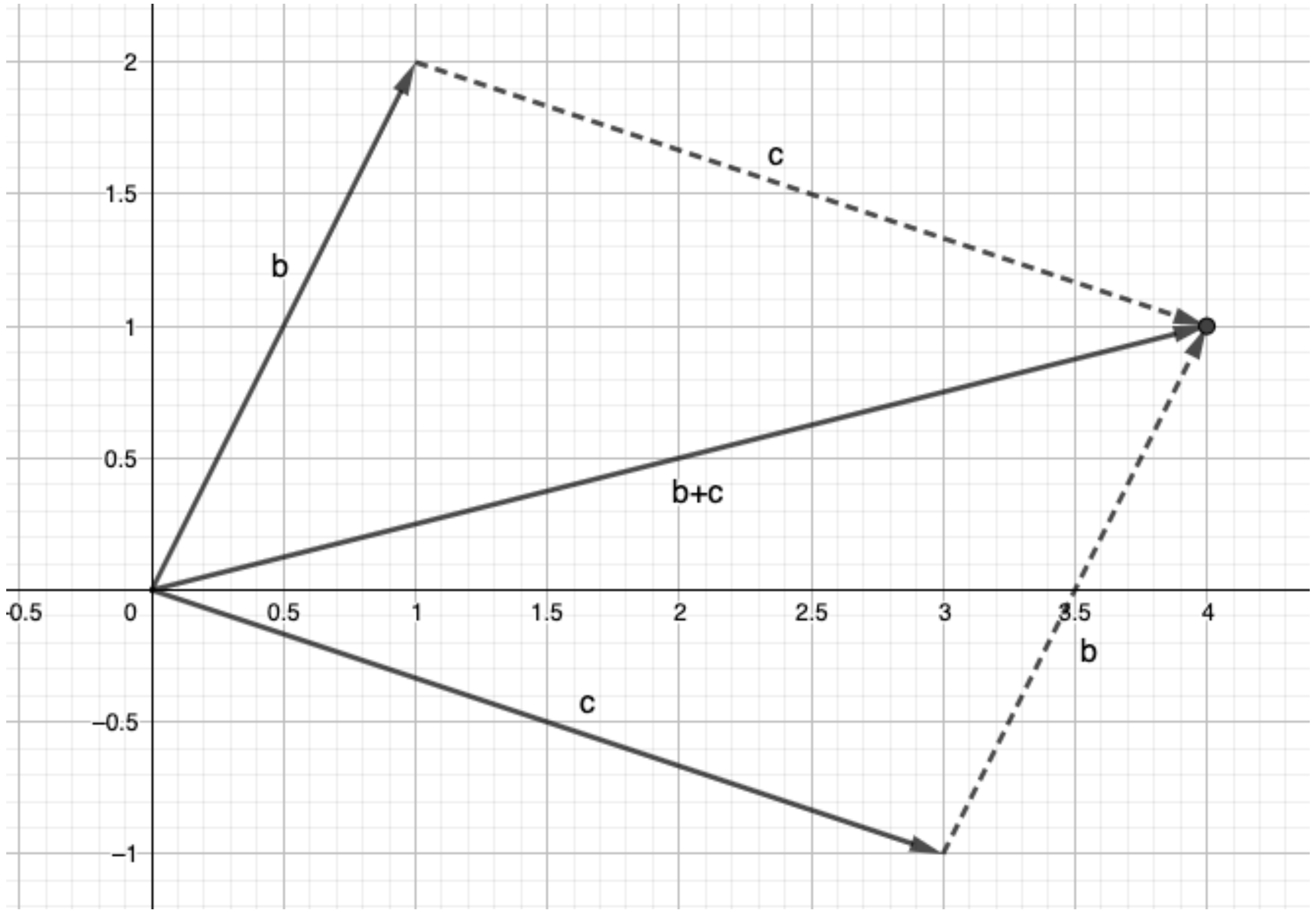
(Pictures are on the next page.)

Vector Addition: previous examples' pictures

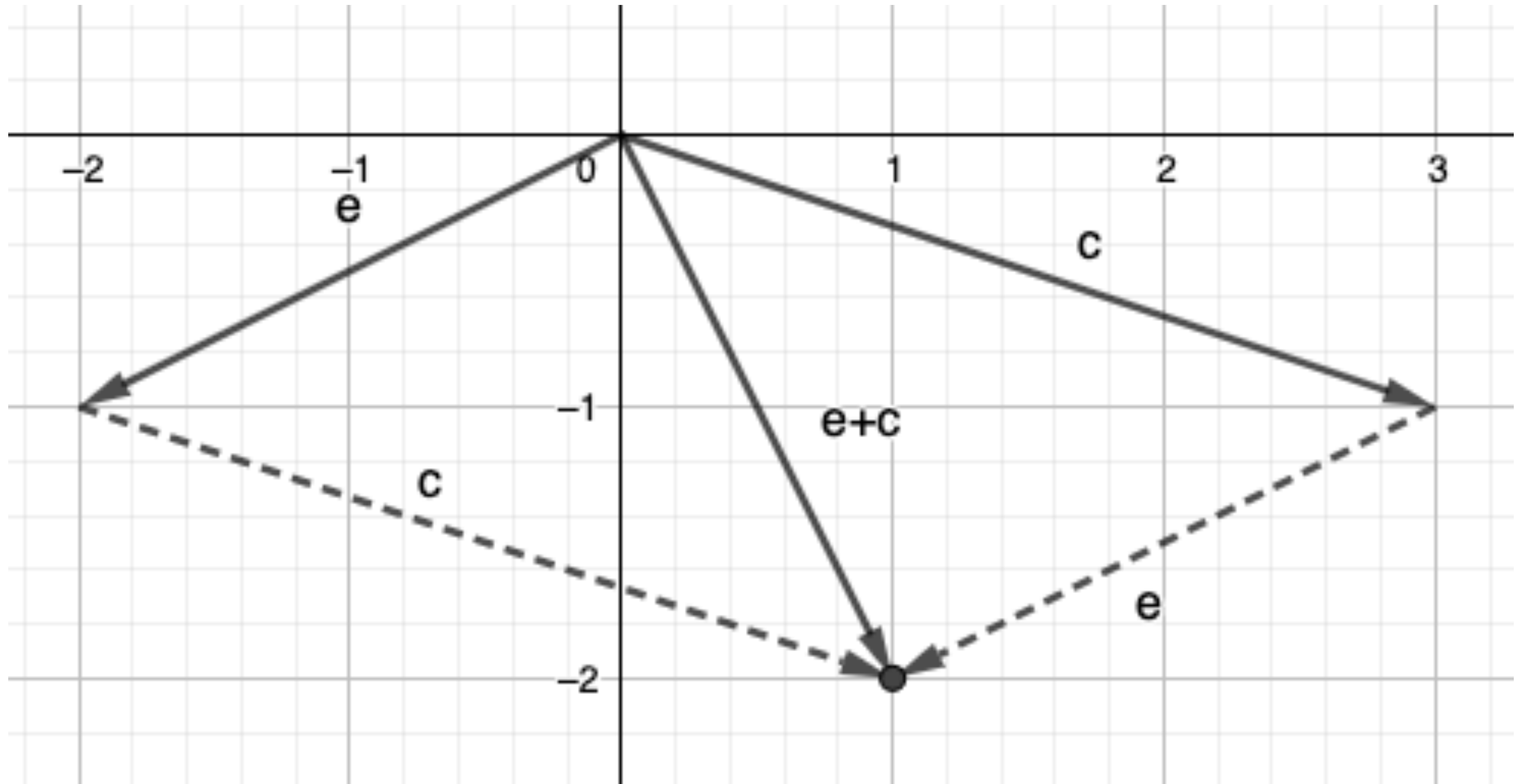
$$\mathbf{a} + \mathbf{d} = < -1,1 > + < 0,-2 > = < -1,-1 >$$



$$\mathbf{b} + \mathbf{c} = < 1,2 > + < 3,-1 > = < 4,1 >$$

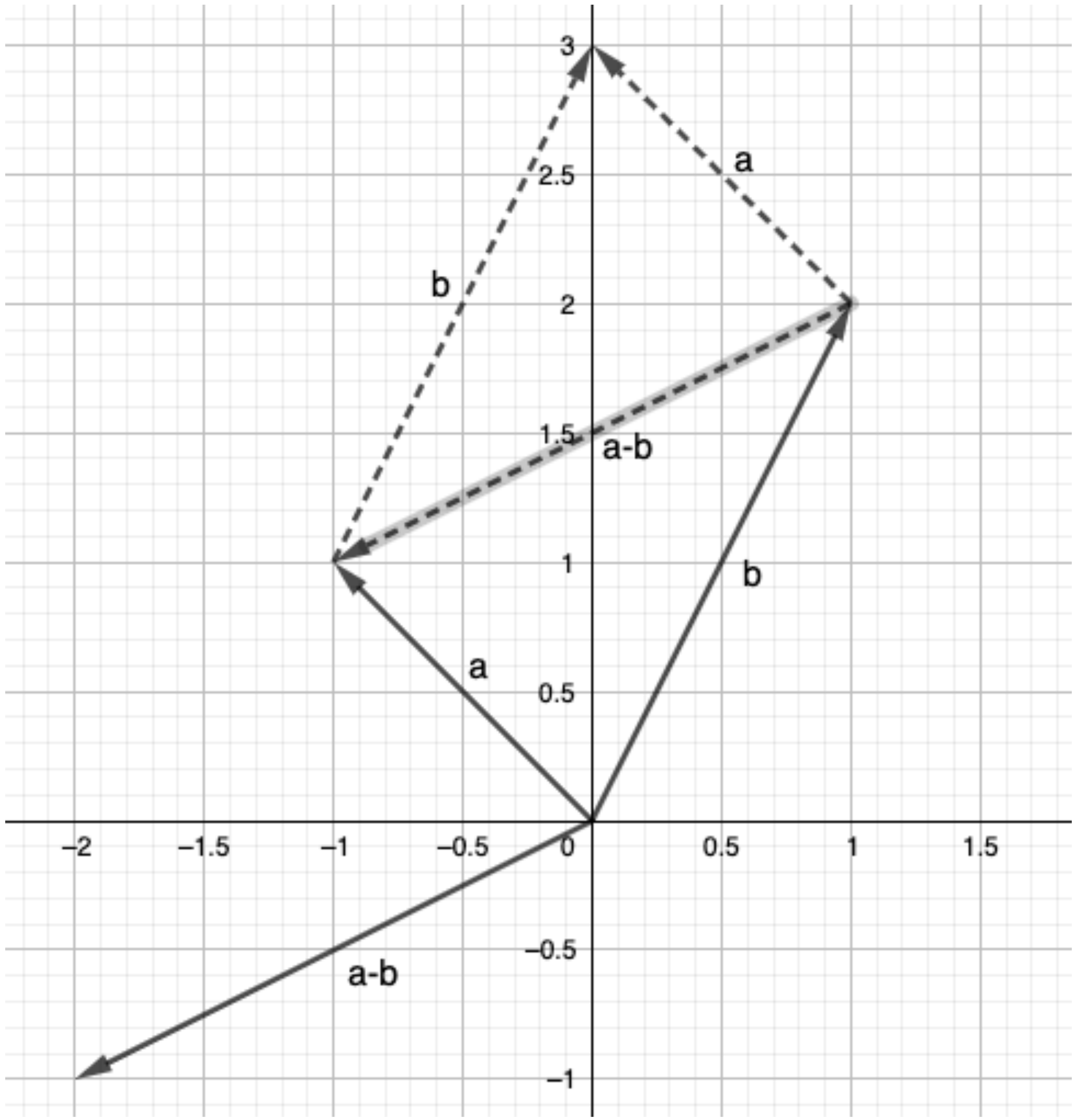


$$\mathbf{e} + \mathbf{c} = < -2,-1 > + < 3,-1 > = < 1,-2 >$$



Vector Arithmetic, Geometric Perspective : Subtraction.

$\mathbf{a} - \mathbf{b} = < -1,1 > - < 1,2 > = < -2, -1 >$



Earlier we saw that $\mathbf{a} + \mathbf{b}$ formed a diagonal of the parallelogram formed by two $\mathbf{a}$'s and two $\mathbf{b}$'s.

$\mathbf{a} - \mathbf{b}$ is equal to the vector that forms the other diagonal of the same parallelogram!

Also, …

$\mathbf{a} - \mathbf{b}$ completes a triangle whose other two sides are formed by $\mathbf{a}$ and $\mathbf{b}$:

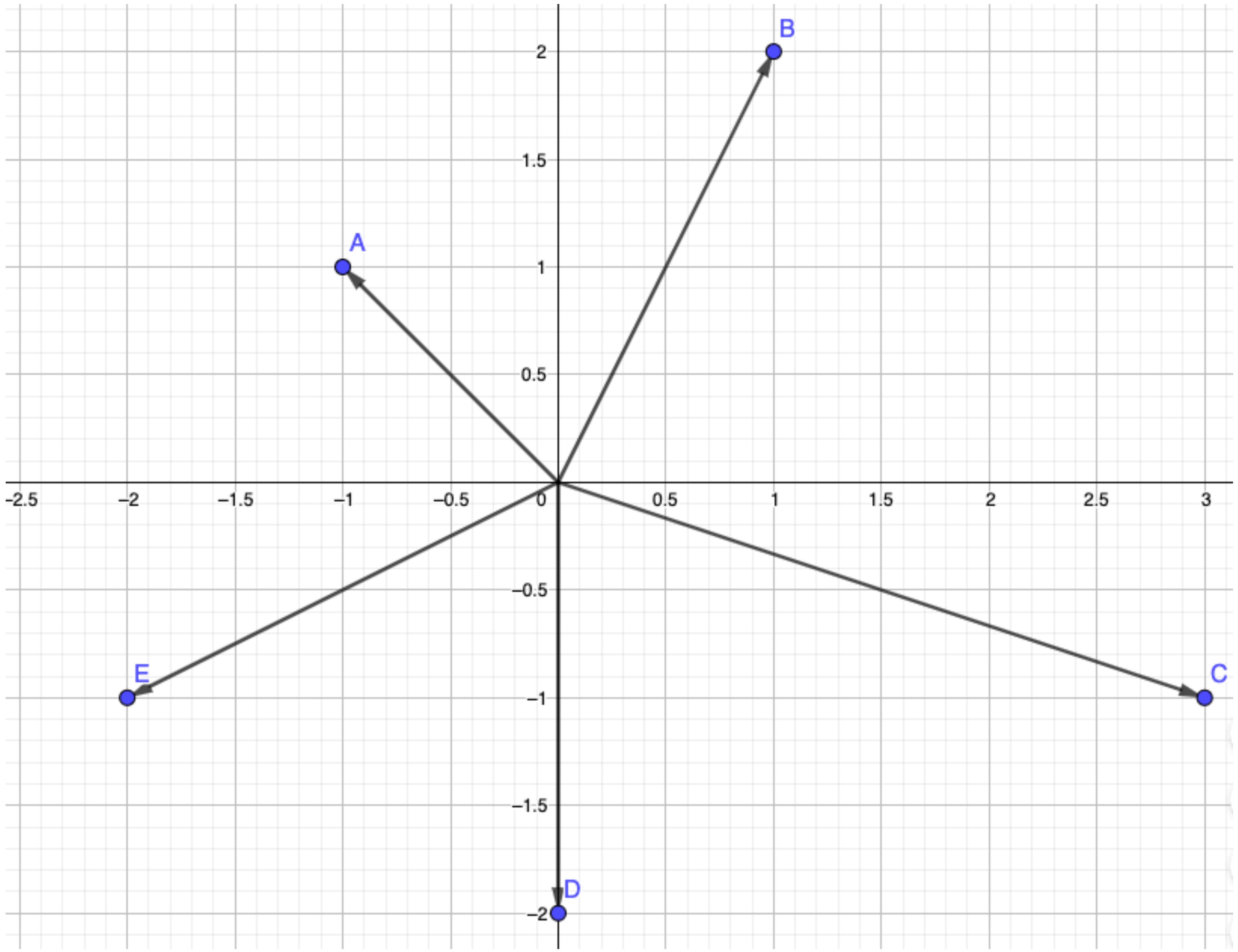
The initial point of $\mathbf{a} - \mathbf{b}$ is the terminal point of $\mathbf{b}$
The terminal point of $\mathbf{a} - \mathbf{b}$ is the terminal point of $\mathbf{a}$.

Note that subtracting in the other order does this:

$\mathbf{b} - \mathbf{a} = < 1,2 > - < -1,1 > = < 2,1 > = -(\mathbf{a} - \mathbf{b})$

In the picture $\mathbf{b} - \mathbf{a}$ has the same magnitude as $\mathbf{a} - \mathbf{b}$, but the opposite direction.

Vector Subtraction: You Try!



$\mathbf{a} = < -1,1 >$

$\mathbf{b} = < 1,2 >$

$\mathbf{c} = < 3, -1 >$

$\mathbf{d} = < 0, -2 >$

$\mathbf{e} = < -2, -1 >$

Try to compute the following.
Do the computation algebraically,
and draw a nice picture!

1. $\mathbf{b} - \mathbf{c}$

2. $\mathbf{e} - \mathbf{a}$

Solutions.

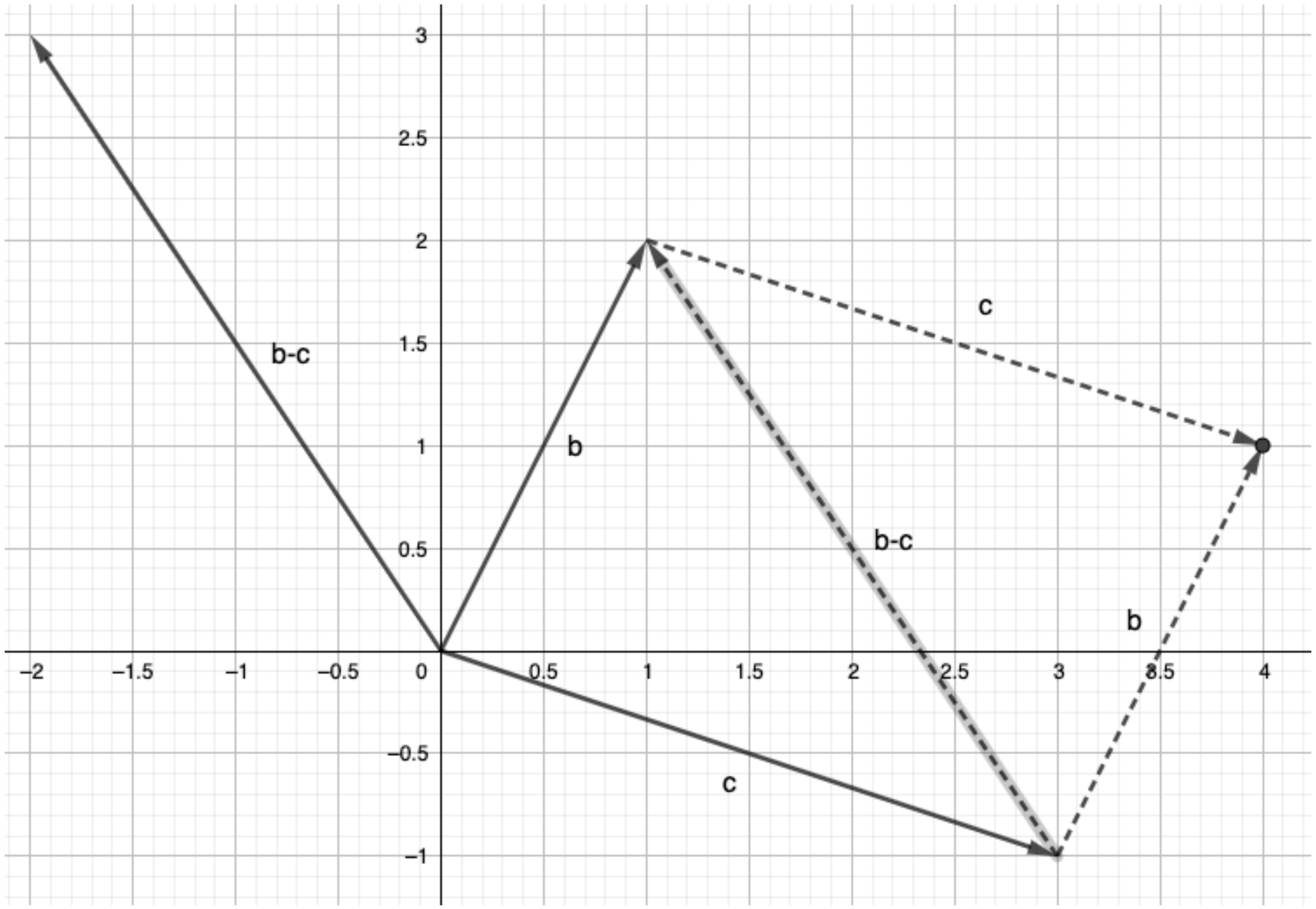
$\mathbf{b} - \mathbf{c} = < 1,2 > - < 3, -1 > = < -2,3 >$

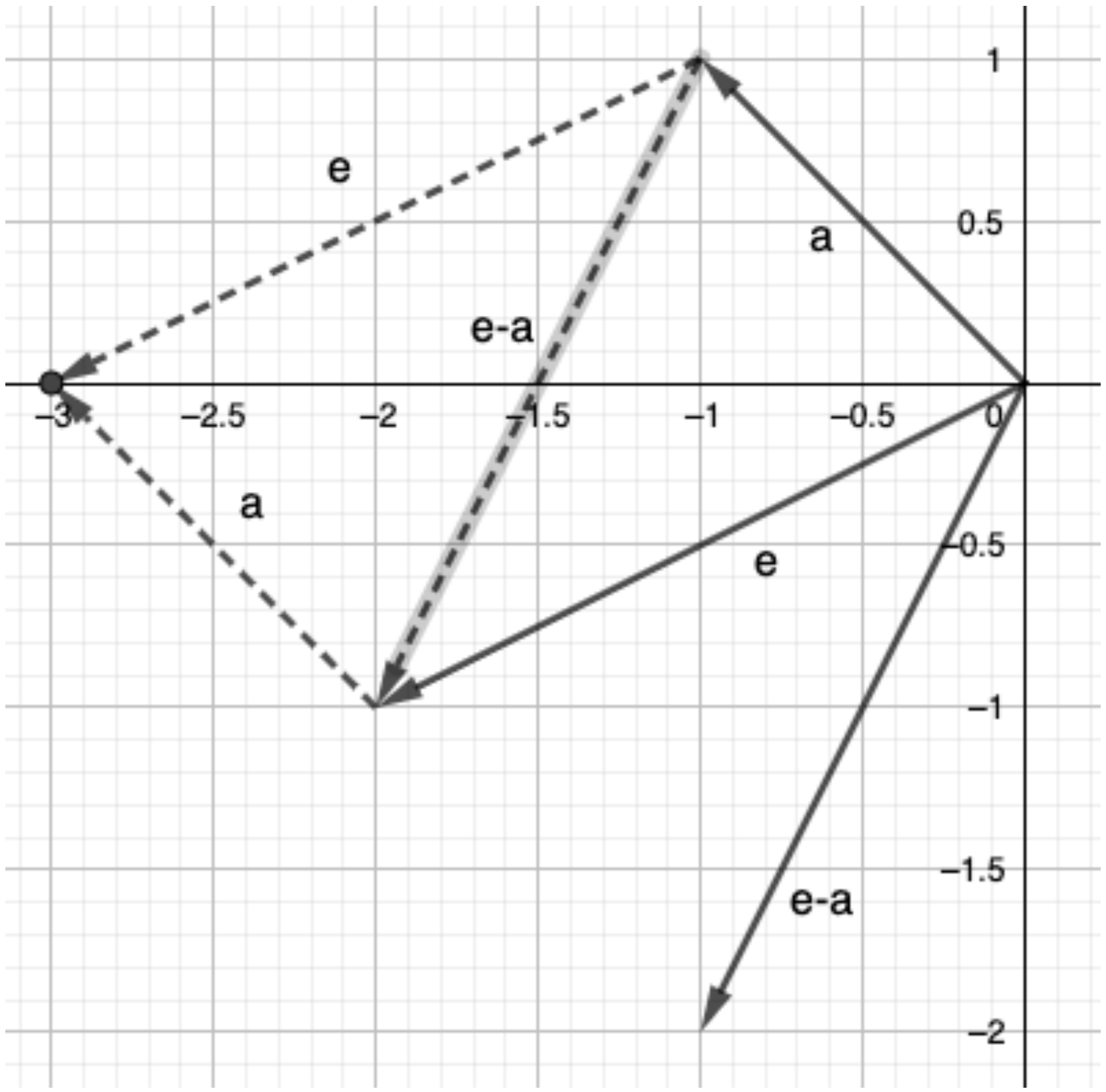
$\mathbf{e} - \mathbf{a} = < -2, -1 > - < -1,1 > = < -1, -2 >$

Vector Subtraction, Illustrations of previous.

$\mathbf{b} - \mathbf{c} = < 1,2 > - < 3, -1 > = < -2,3 >$

$\mathbf{e} - \mathbf{a} = < -2, -1 > - < -1,1 > = < -1, -2 >$

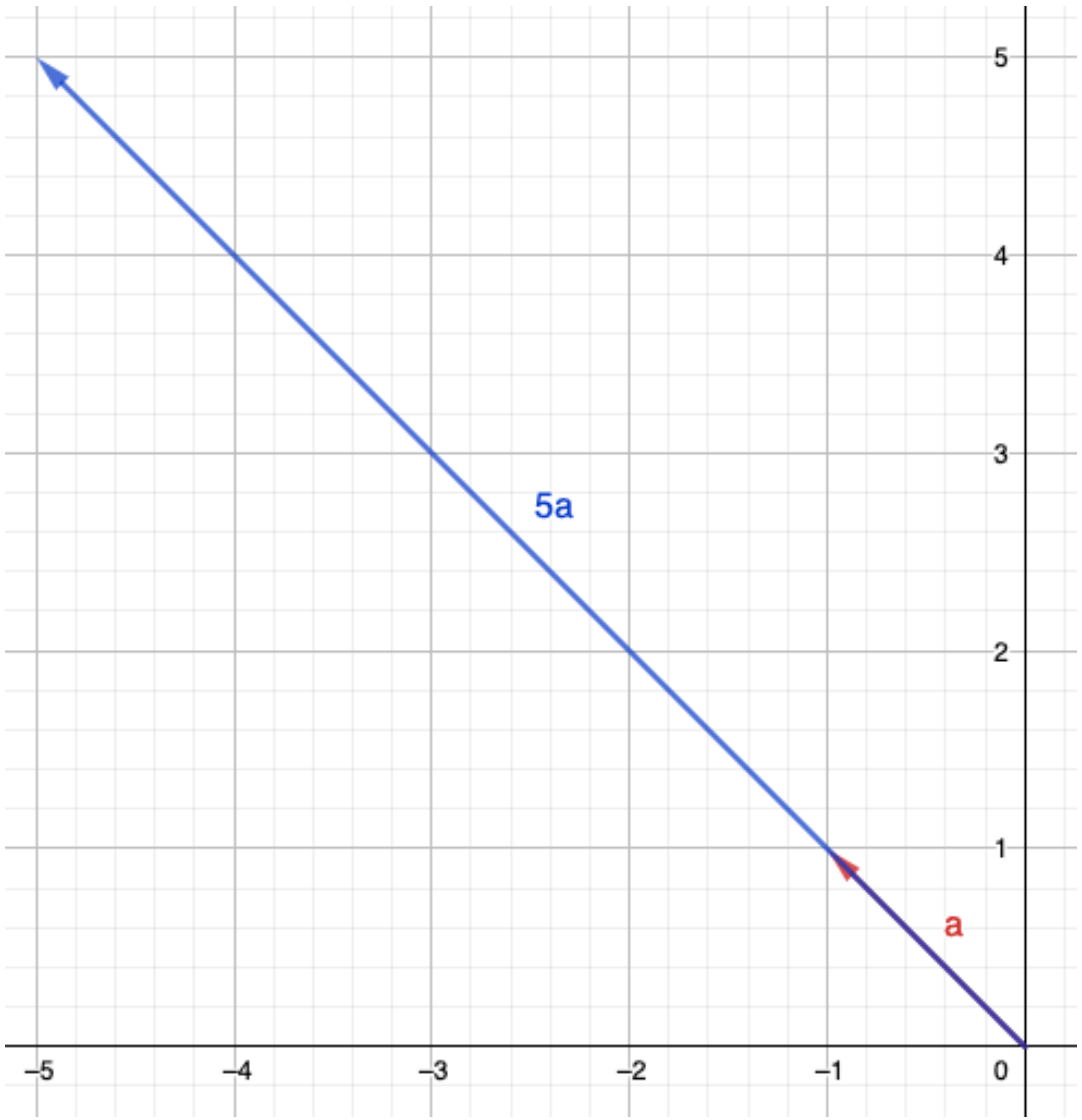




Vector Arithmetic, Geometric Perspective : Scalar Multiplication.

Previously we saw this calculation… $\mathbf{a} = < -1,1 >$

$$5 \cdot \mathbf{a} = 5 \cdot < -1,1 > = < 5 \cdot (-1), 5 \cdot 1 > = < -5,5 >$$



Multiplying $\mathbf{a}$ by 5 has stretched $\mathbf{a}$ so that it's 5 times larger.

$5 \cdot \mathbf{a}$ has the same direction as $\mathbf{a}$, but its magnitude has increased.

Here are some for you to try:

$\mathbf{b} = < 1,2 >$

Compute the following. Include a picture!

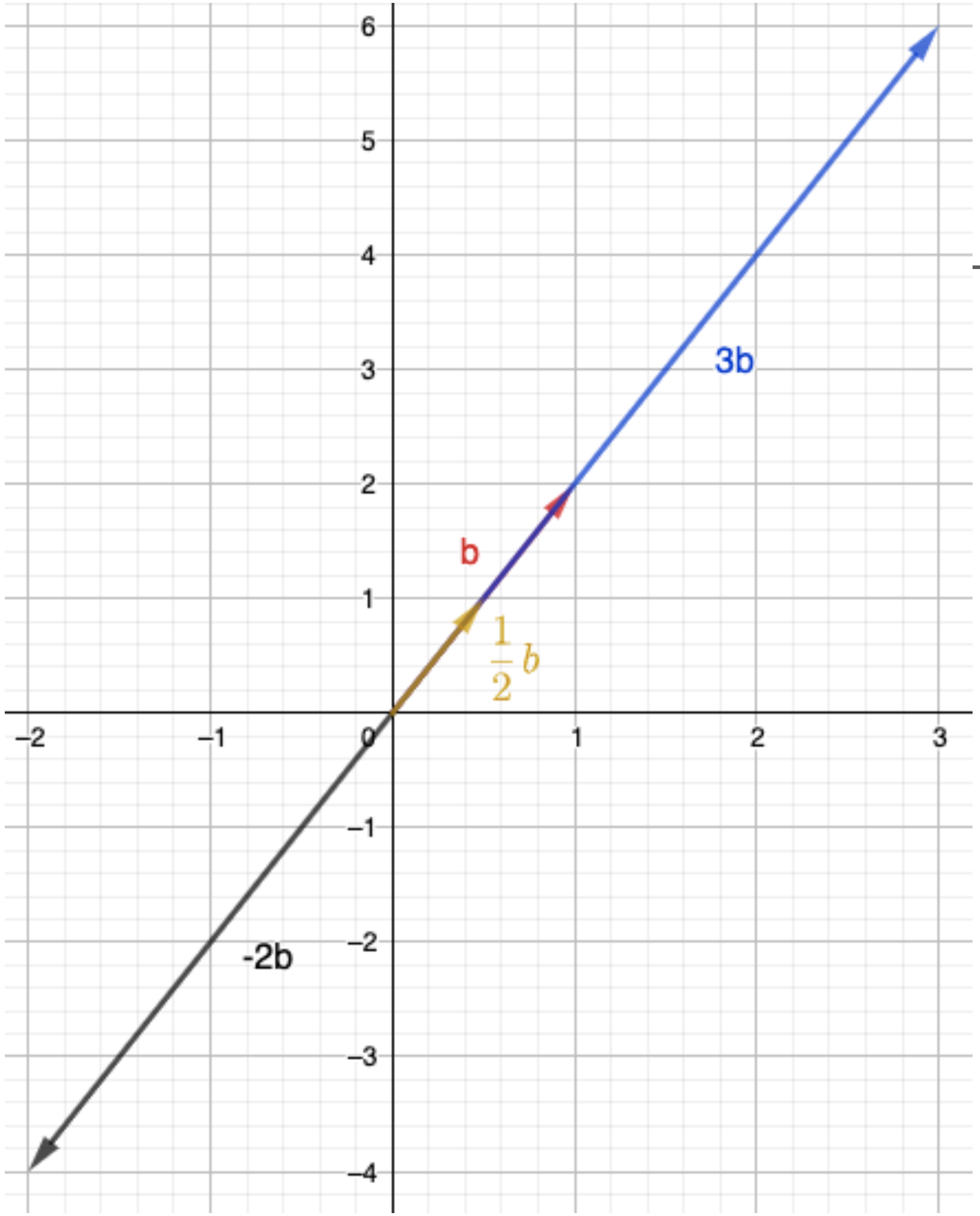
$3 \cdot \mathbf{b} = < 3,6 >$

$-2 \cdot \mathbf{b} = < -2, -4 >$

$$\frac{1}{2} \cdot \mathbf{b} = \left\langle \frac{1}{2}, 1 \right\rangle$$

Pictures on the next slide…

Scalar Multiplication of Vectors: Illustrations.



$\mathbf{b} = < 1{,}2 >$

$3 \cdot \mathbf{b} = < 3{,}6 >$ (**b** is stretched by a factor of 3.)

$-2 \cdot \mathbf{b} = < -2, -4 >$ (**b** is stretched by a factor of 2, and reflected about the origin.)

$\frac{1}{2} \cdot \mathbf{b} = \left\langle \frac{1}{2}, 1 \right\rangle$ (**b** is scaled by a factor of 1/2; you might say it's 'squished' by a factor of 2)

Note 1: scaling a vector by a negative number reflects the vector about the origin.

Note 2: scaling a vector by a scalar k with $|k| < 1$ results in a shorter vector (i.e. one with smaller magnitude).

In fact $|c \cdot \mathbf{v}| = |c| \cdot |\mathbf{v}|$ for any vector $\mathbf{v}$ and any scalar $c \in \mathbf{R}$

Here are some more properties of vectors…

Properties of Vectors.

Suppose $\mathbf{u}, \mathbf{v},$ and $\mathbf{w}$ are vectors, and that $c, d \in \mathbf{R}$ are scalars.

THEN the following properties hold:

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. $\mathbf{0} + \mathbf{u} = \mathbf{u}$ Where $\mathbf{0} = < 0{,}0 >$
4. $-\mathbf{u} + \mathbf{u} = \mathbf{0}$
5. $c \cdot (\mathbf{u} + \mathbf{v}) = c \cdot \mathbf{u} + c \cdot \mathbf{v}$
6. $(c + d) \cdot \mathbf{u} = c \cdot \mathbf{u} + d \cdot \mathbf{u}$
7. $(cd) \cdot \mathbf{u} = c \cdot (d \cdot \mathbf{u})$
8. $1 \cdot \mathbf{u} = \mathbf{u}$

Theoretical Note: These properties are upheld by many mathematical objects, not just the vectors that you have seen.

What other things satisfy these properties?!?!

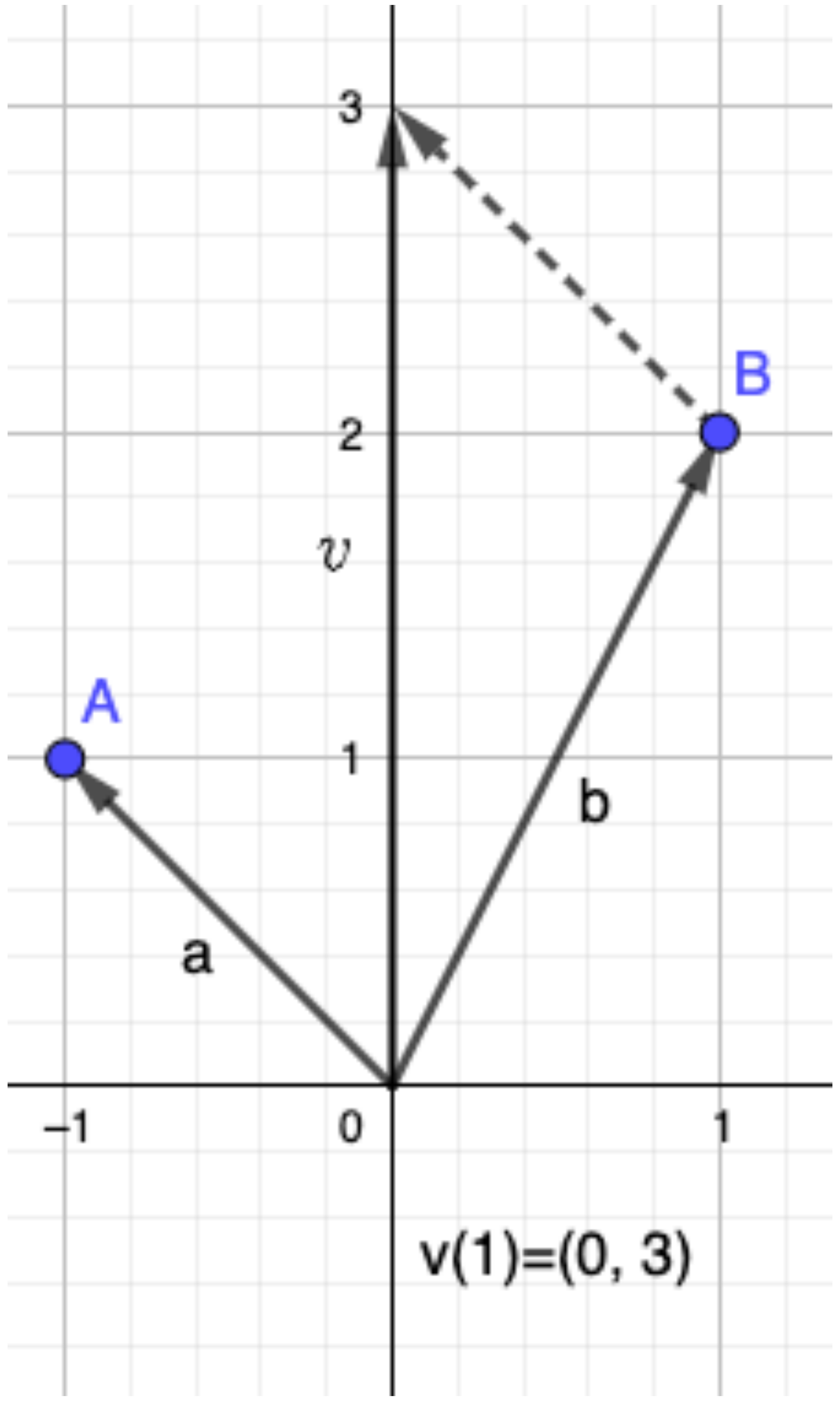
- Real Numbers !
- Complex Numbers!
- Functions!
- also…
- Matrices!!
- Solutions to certain differential equations!!

(Advertisement: You can study Matrices and/or Differential Equations here at CCSF in M120, M125, M130.)

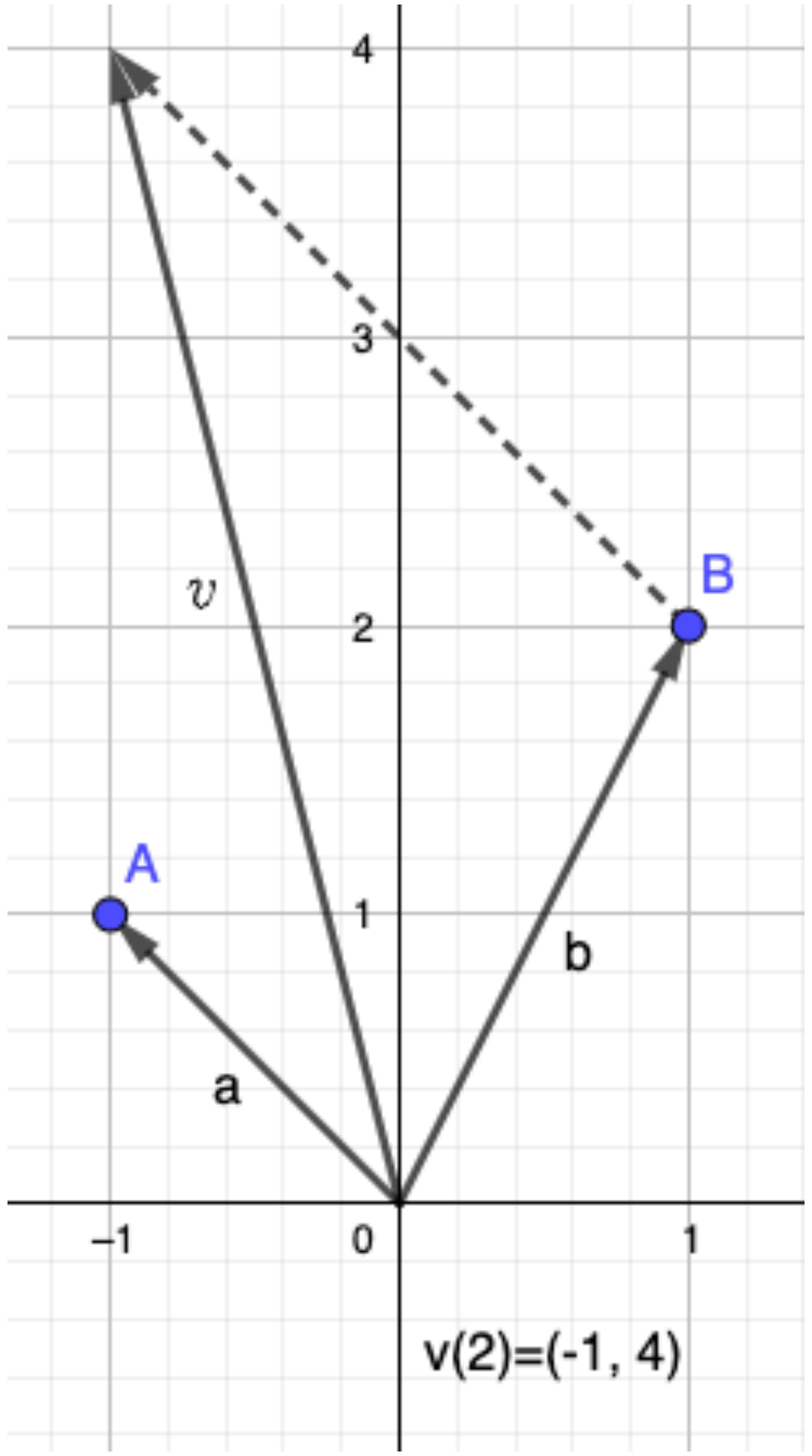
but for us in M110C, why are vectors important…?

One Reason Why Vectors are Cool.

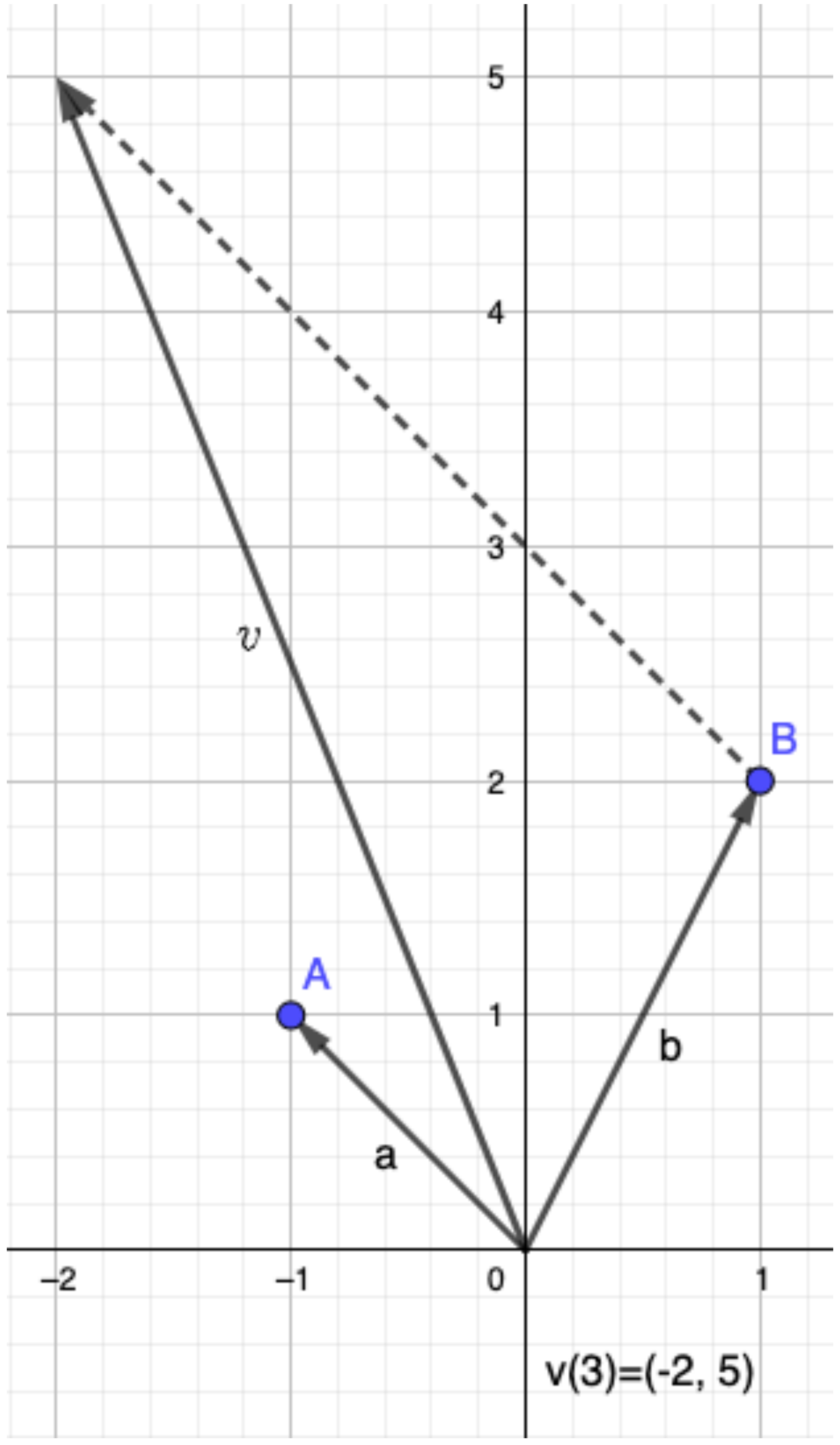
Suppose $\mathbf{a} = < -1{,}1 >$ and $\mathbf{b} = < 1{,}2 >$



$\mathbf{v}(1) = \mathbf{b} + \mathbf{a}$

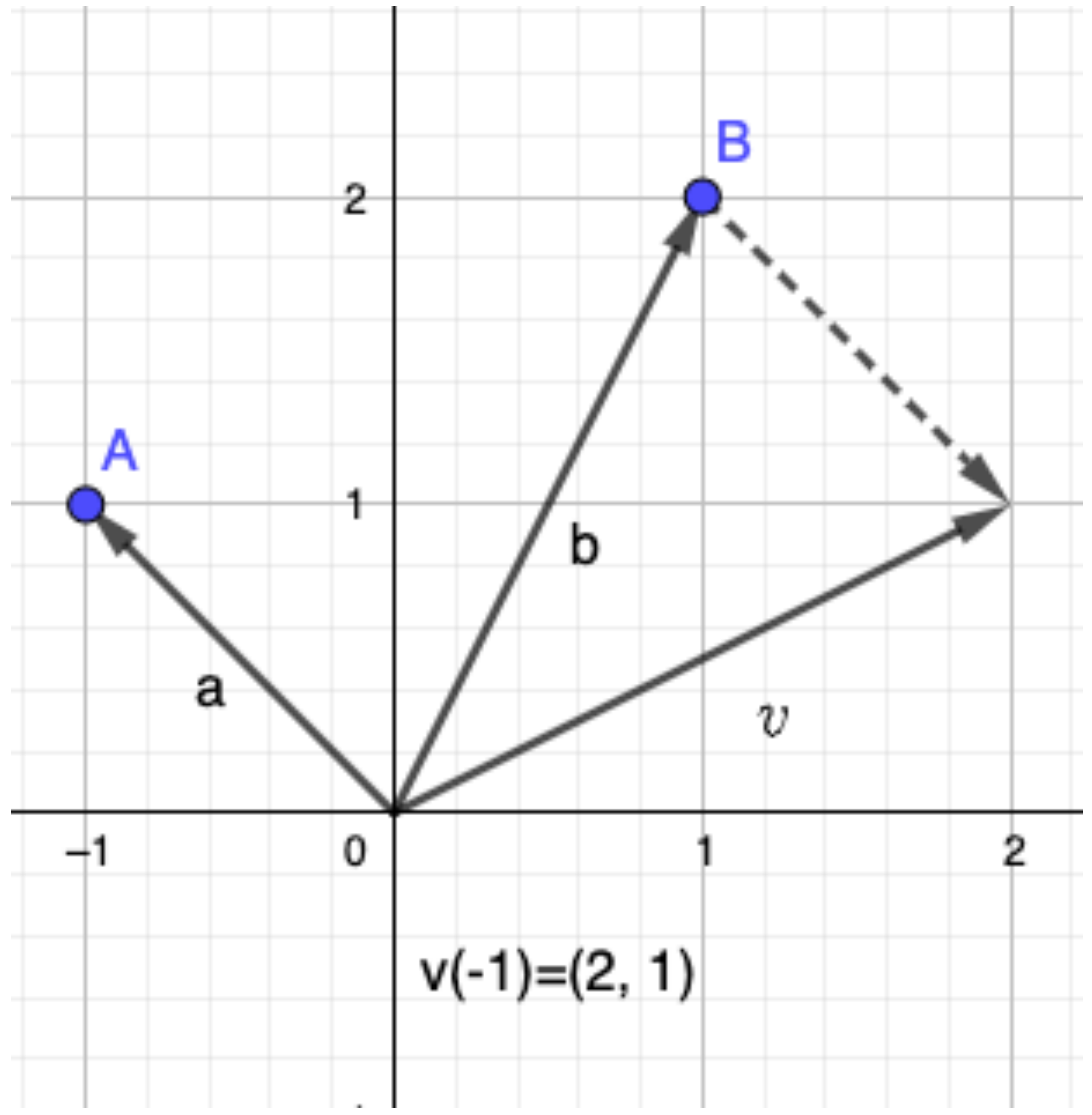


$\mathbf{v}(2) = \mathbf{b} + 2 \cdot \mathbf{a}$



$\mathbf{v}(3) = \mathbf{b} + 3 \cdot \mathbf{a}$

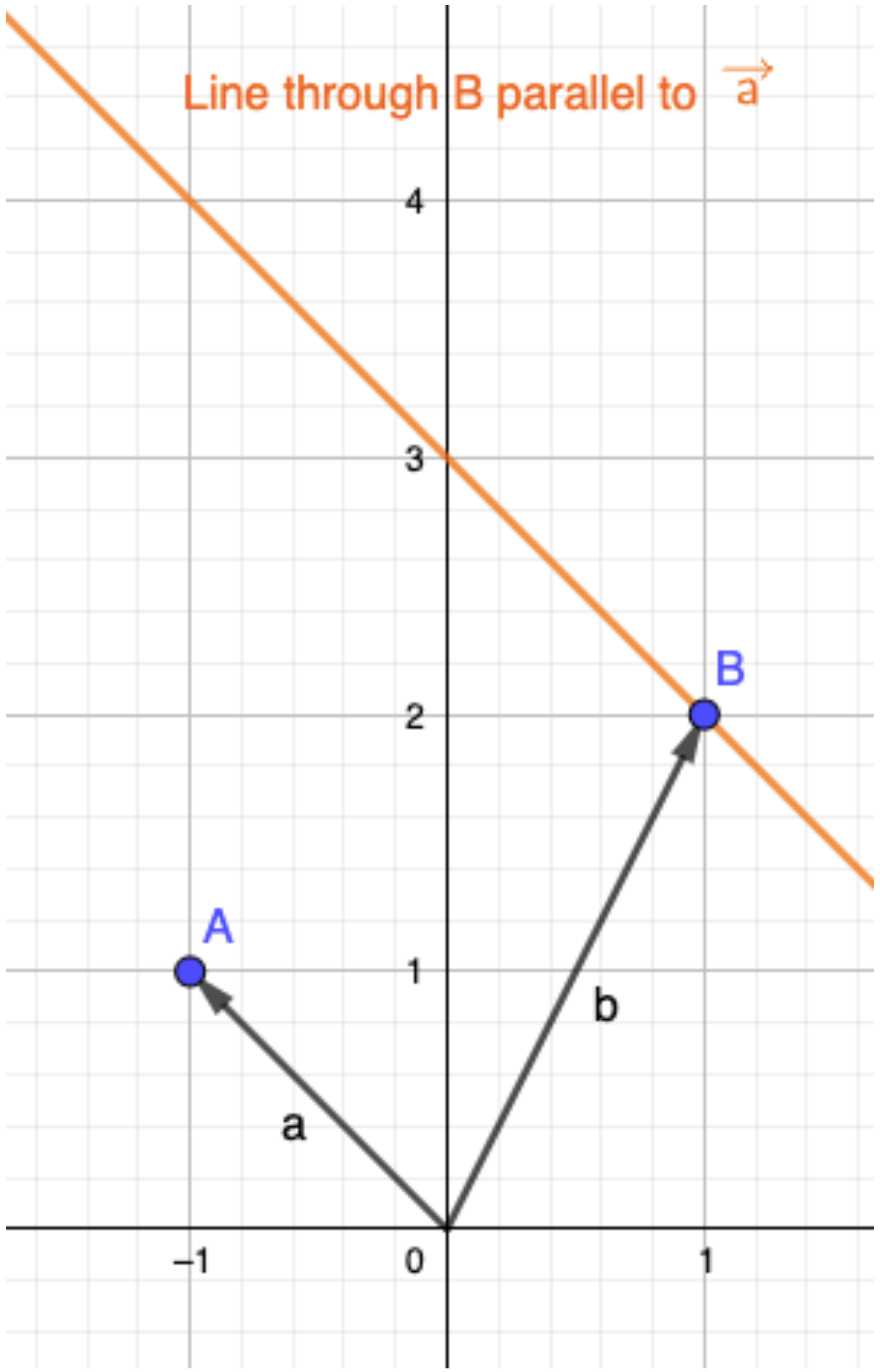
What if we subtract multiples of $\mathbf{a}$ from $\mathbf{b}$?



$\mathbf{v}(-1) = \mathbf{b} - \mathbf{a}$

All told, what do we get when we add different multiples of $\mathbf{a}$ to $\mathbf{b}$??

One Reason Why Vectors are Cool, pg 2.

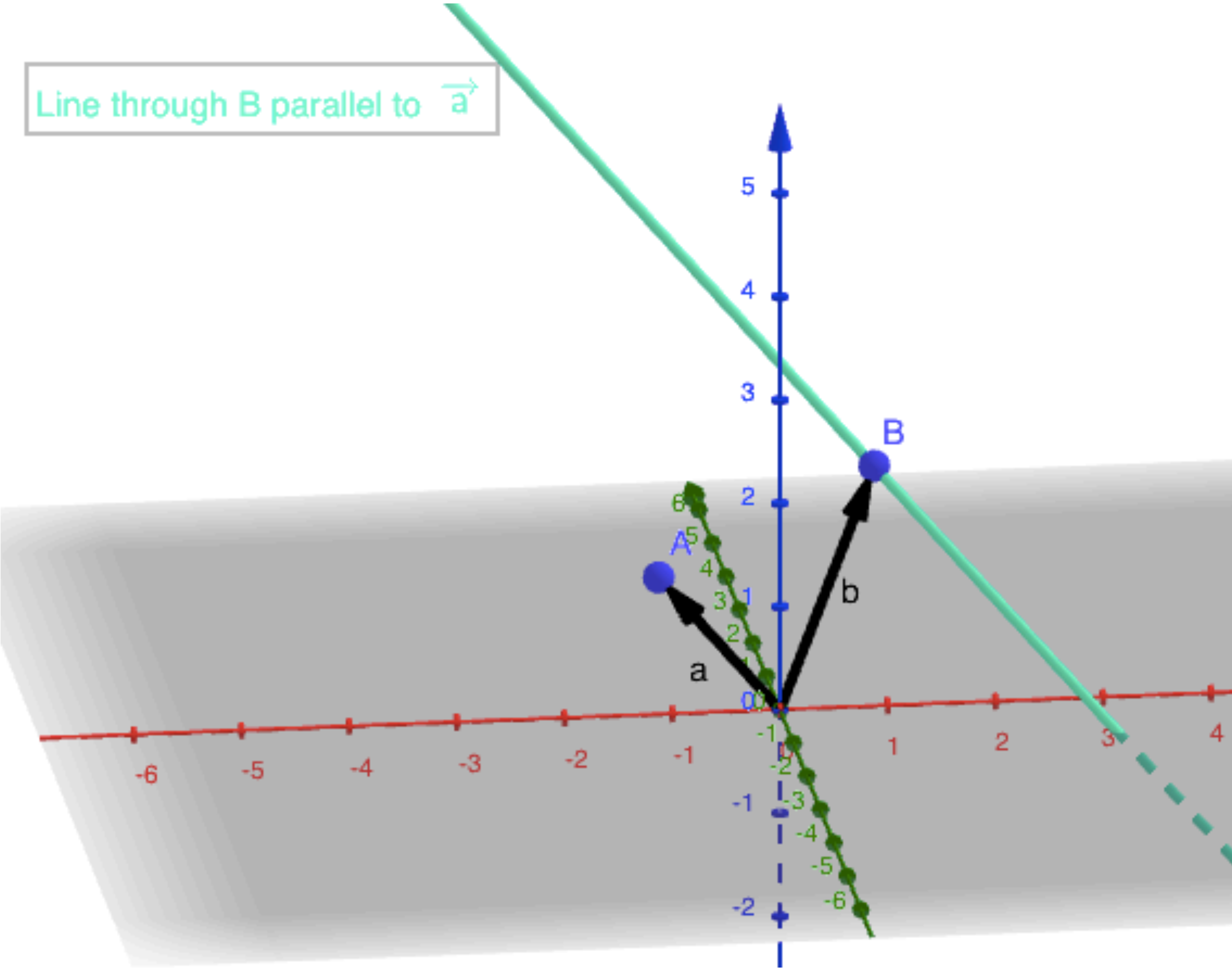


By adding multiples of $\mathbf{a}$ to $\mathbf{b}$, we can describe all the points on the line which goes through the terminal point of $\mathbf{b}$ parallel to the vector $\mathbf{a}$.

Why describe lines using these vectors when we already know how to describe lines using slopes and intercepts?

Link: [2DLineWithVectors]()

This vector approach to describing lines is used for lines in three dimensions!!!



We will see more in the coming lessons.

Link: [3DLinesWithVectors]()

The dot product (2D).

Suppose $\mathbf{a} = < a_1, a_2 >$ and $\mathbf{b} = < b_1, b_2 >$

Then the *dot product* or *inner product* of $\mathbf{a}$ and $\mathbf{b}$ is the scalar (number) $\mathbf{a} \cdot \mathbf{b} := a_1 \cdot b_1 + a_2 \cdot b_2$

Examples. $\mathbf{a} = < 1,3 >$, $\quad \mathbf{b} = < -2,4 >$, $\quad \mathbf{c} = < 2,1 >$

$\mathbf{a} \cdot \mathbf{b} = 1 \cdot -2 + 3 \cdot 4 = -2 + 12 = 10$

$\mathbf{a} \cdot \mathbf{c} = 5$

You try:

$\mathbf{b} \cdot \mathbf{c} = -2 \cdot 2 + 4 \cdot 1 = 0$

$\mathbf{a} \cdot \mathbf{a} = 1 \cdot 1 + 3 \cdot 3 = 10$

$(2 \cdot \mathbf{a}) \cdot \mathbf{b} = < 2,6 > \cdot < -2,4 > = 2 \cdot -2 + 6 \cdot 4 = 20$

$(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = (< 1,3 > + < -2,4 >) \cdot < 2,1 >$

$= < -1,7 > \cdot < 2,1 > = 5$

Observations: The following are true for any vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and scalar k:

1. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
2. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$, or $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$
3. $(k \cdot \mathbf{a}) \cdot \mathbf{b} = k \cdot (\mathbf{a} \cdot \mathbf{b})$
4. $(\mathbf{a} \pm \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} \pm \mathbf{b} \cdot \mathbf{c}$
5. $\mathbf{a} \cdot \mathbf{b} = 0$ exactly when…

…$\mathbf{a}$ and $\mathbf{b}$ are perpendicular.

Dot Product: The Angle between two vectors.

$\mathbf{a} = < 1{,}3 >$, $\mathbf{b} = < -2{,}4 >$

What's the angle, θ, between $\mathbf{a}$ and $\mathbf{b}$?

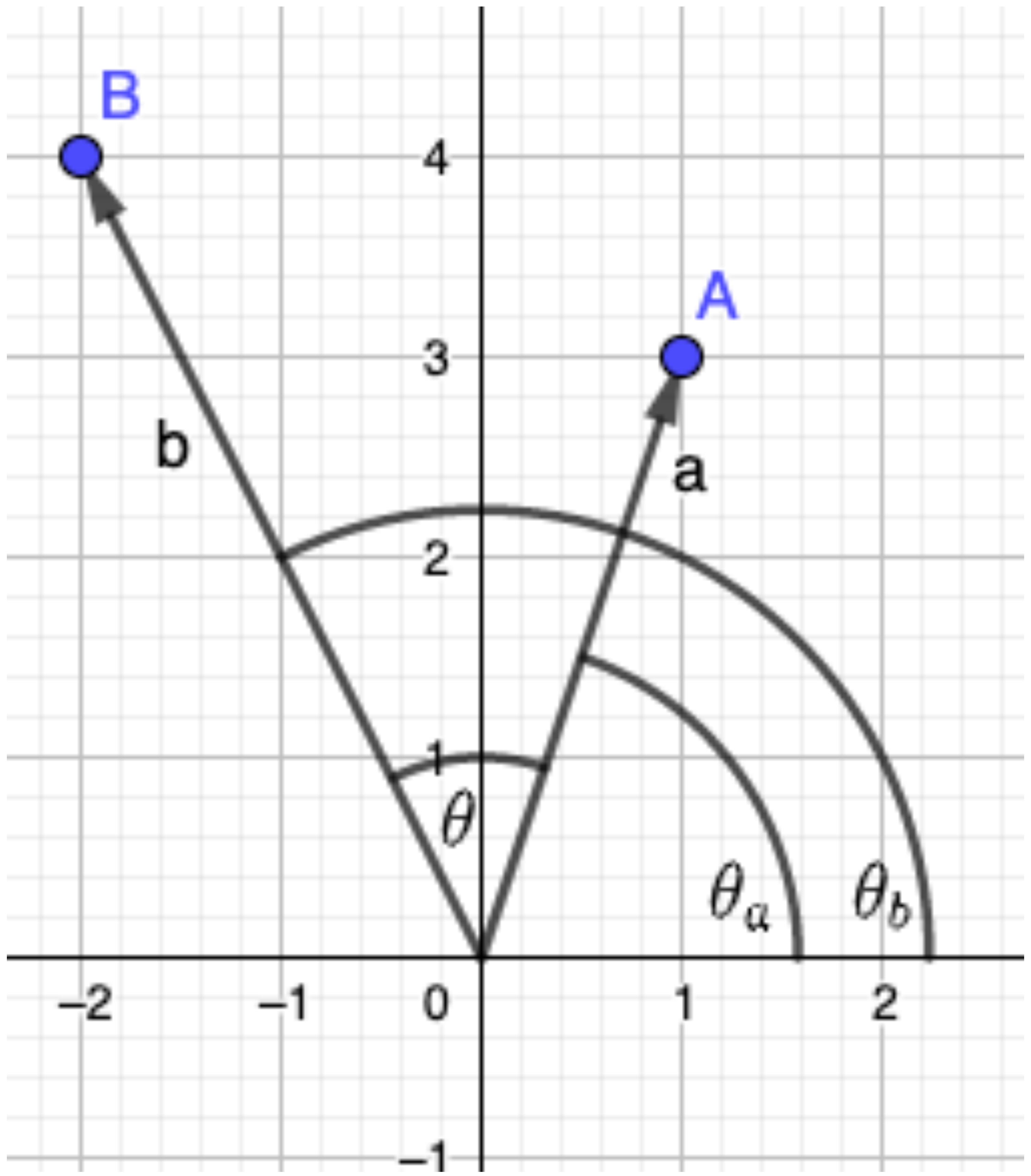
Method 1: $\theta = \theta_b - \theta_a$

$$\theta_a = \tan^{-1}\left(\frac{3}{1}\right) \approx 1.25\text{rad} \approx 71.57^\circ$$

$$\theta_b = \tan^{-1}\left(-\frac{4}{2}\right) + \pi \approx 2.03\text{rad} \approx 116.57^\circ$$

$$\theta = \theta_b - \theta_a = 45^\circ$$

(rounded to the nearest hundredth
at the very end of the calculation.)



Another method:

Suppose $\mathbf{a} = < a_1, a_2 >$
and $\mathbf{b} = < b_1, b_2 >$

$$\theta = \theta_b - \theta_a$$

$$\cos(\theta) = \cos(\theta_b - \theta_a)$$

$$= \cos(\theta_b)\cos(\theta_a) + \sin(\theta_b)\sin(\theta_a)$$

$$= \frac{b_1}{|\mathbf{b}|}\cdot\frac{a_1}{|\mathbf{a}|} + \frac{b_2}{|\mathbf{b}|}\cdot\frac{a_2}{|\mathbf{a}|}$$

$$= \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

In our specific example… $\cos(\theta) = \dfrac{10}{\sqrt{10}\sqrt{20}} = \dfrac{1}{\sqrt{2}}$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}\text{rad} = 45^\circ$$

Dot Product: The Angle between two vectors.

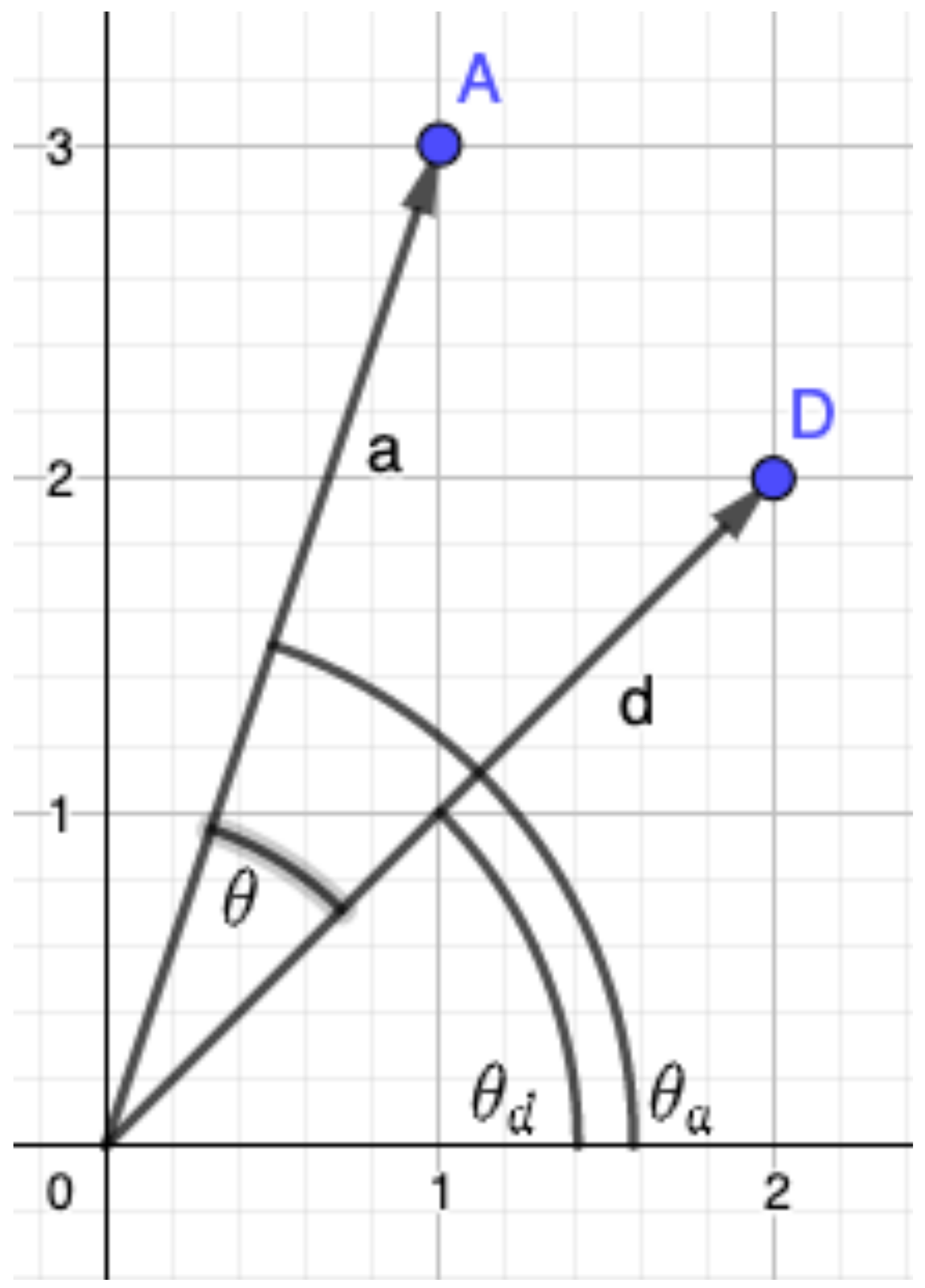
Helpful formula: if $\theta \in [0°, 180°]$ is the angle between vectors $\mathbf{a}$ and $\mathbf{b}$, then $\cos(\theta) = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$

Equivalently, $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos(\theta)$

This explains the perpendicularity rule we saw earlier:

$\mathbf{a}$ and $\mathbf{b}$ are perpendicular when $\theta = 90°$, which happens when $\cos(\theta) = 0$, which happens when $\mathbf{a} \cdot \mathbf{b} = 0$, and conversely.

Wanna try?



$$\mathbf{a} = <1,3>, \quad \mathbf{d} = <2,2>$$

Find the angle between $\mathbf{a}$ and $\mathbf{d}$.

Using our formula:

$$\cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{d}}{|\mathbf{a}||\mathbf{d}|} = \frac{8}{\sqrt{10}\sqrt{8}} = \frac{2\sqrt{5}}{5}$$

$$\theta = \cos^{-1}\left(\frac{2\sqrt{5}}{5}\right) \approx 26.57°$$

Or computing each angle separately:

$$\theta_a = \tan^{-1}\left(\frac{3}{1}\right) \approx 1.25\text{rad} \approx 71.57°$$

$$\theta_d = \tan^{-1}\left(\frac{2}{2}\right) = 45°$$

$$\theta = \theta_a - \theta_d \approx 26.57°$$

Dot Product: Projecting one vector onto another. Derivation.

Say $\mathbf{a} = < a_1, a_2 >$ and $\mathbf{b} = < b_1, b_2 >$

We want to compute the *projection of* $\mathbf{b}$ *onto* $\mathbf{a}$, written $\mathbf{proj_a(b)}$

Imagine the sun shining beams of light in a direction perpendicular to $\mathbf{a}$. Then $\mathbf{proj_a(b)}$ is the shadow of $\mathbf{b}$ on the line along $\mathbf{a}$.

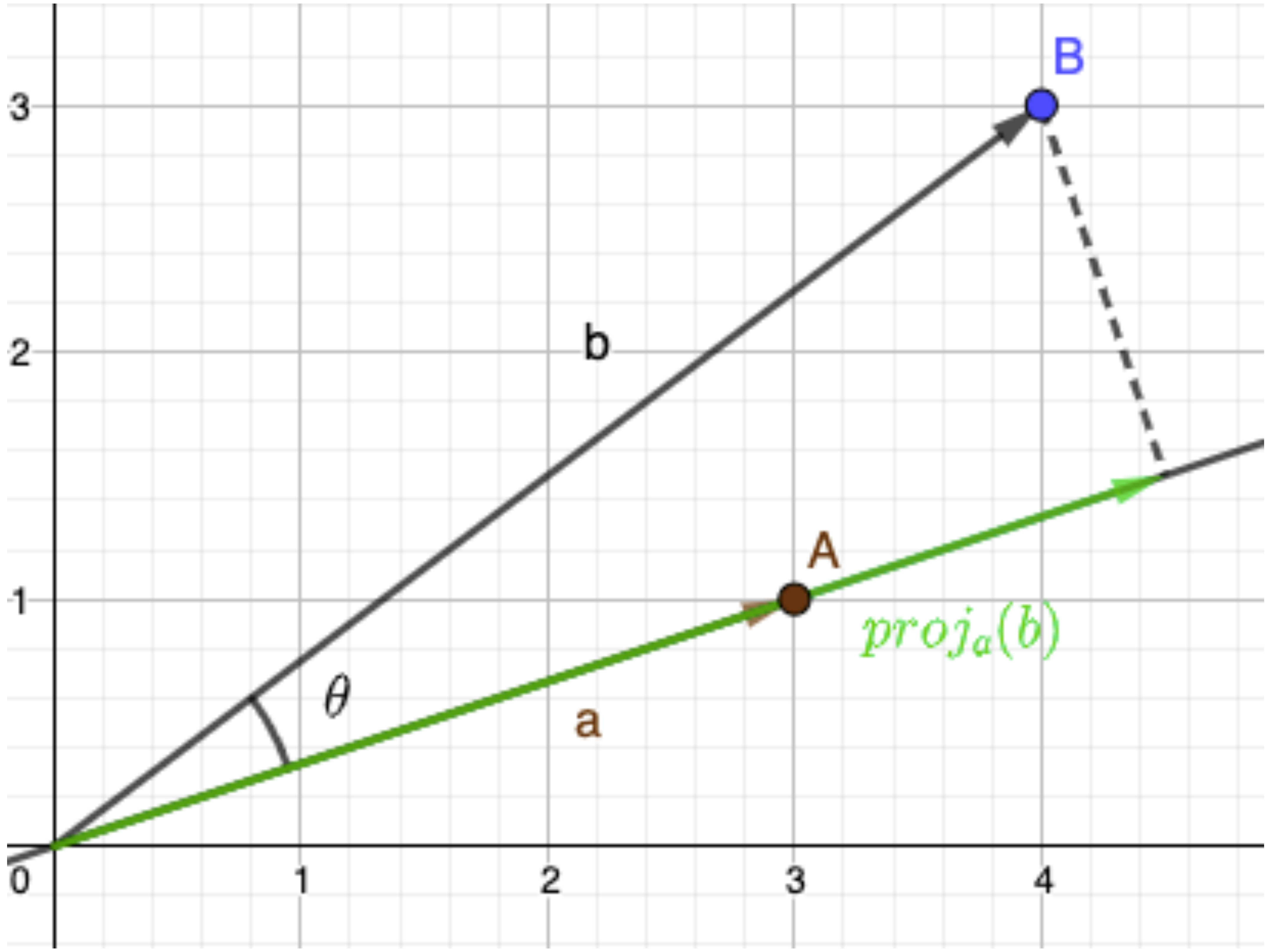
As this vector lies on the same line as $\mathbf{a}$, it must be that $\mathbf{proj_a(b)} = k\mathbf{a}$ for some scalar k.

What's k? How much of $\mathbf{a}$ goes into $\mathbf{proj_a(b)}$?

The requirement that the projection needs to fulfill has to do with perpendicularity:

The vector which completes the triangle from $\mathbf{proj_a(b)}$ to $\mathbf{b}$ should be perpendicular to $\mathbf{a}$.

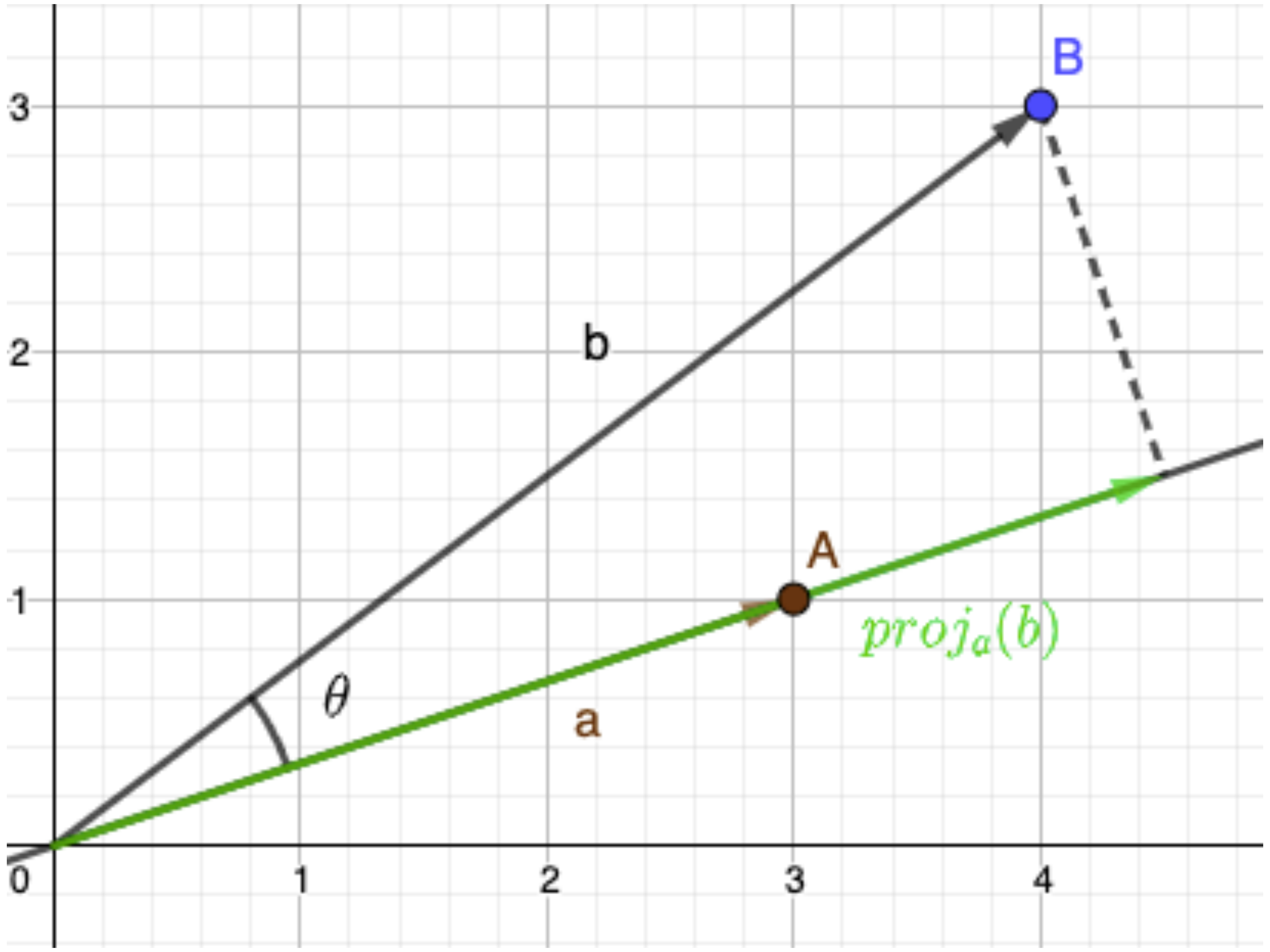
$$0 = (\mathbf{b} - \mathbf{proj_a(b)}) \cdot \mathbf{a} = (\mathbf{b} - k\mathbf{a}) \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{a} - k\mathbf{a} \cdot \mathbf{a}$$



This tells us what k needs to be. $k = \dfrac{\mathbf{b} \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{a}}$

$$\mathbf{proj_a(b)} = k\mathbf{a} = \left(\frac{\mathbf{b} \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{a}}\right)\mathbf{a}$$

Dot Product: Projecting one vector onto another, Example.



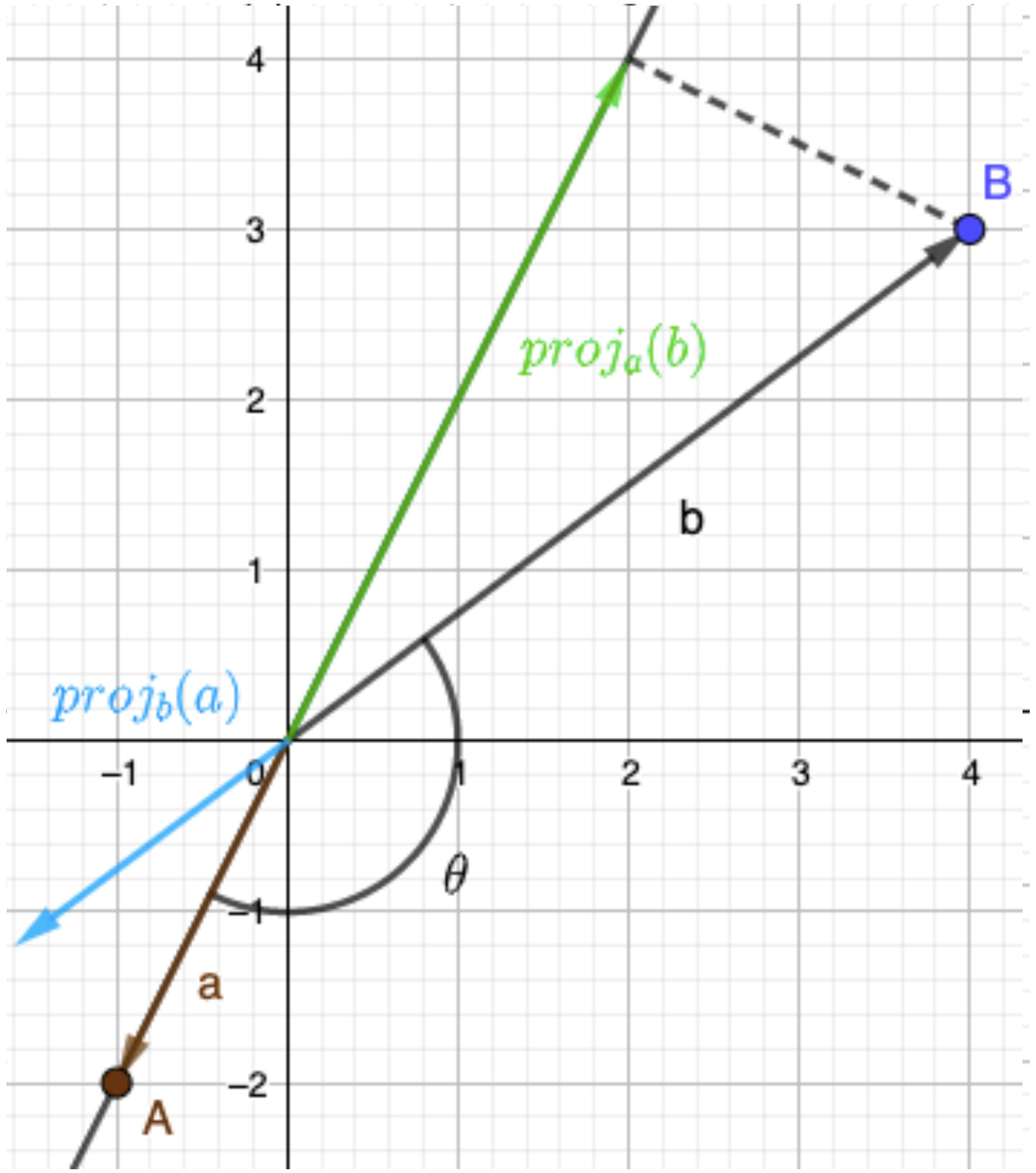
$\mathbf{a} = <3,1>, \quad \mathbf{b} = <4,3>$

$$\left(\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\right) = \frac{15}{10} = 1.5$$

$$\mathbf{proj_a(b)} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a} = 1.5\mathbf{a} = <4.5,1.5>$$

You try.

$\mathbf{a} = <-1,-2>, \quad \mathbf{b} = <4,3>$



Find $\mathbf{proj_a(b)}$

$$\left(\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\right) = \frac{-10}{5} = -2$$

$$\mathbf{proj_a(b)} = -2\mathbf{a} = <2,4>$$

What about $\mathbf{proj_b(a)}$??

(Try computing it, and draw a picture!

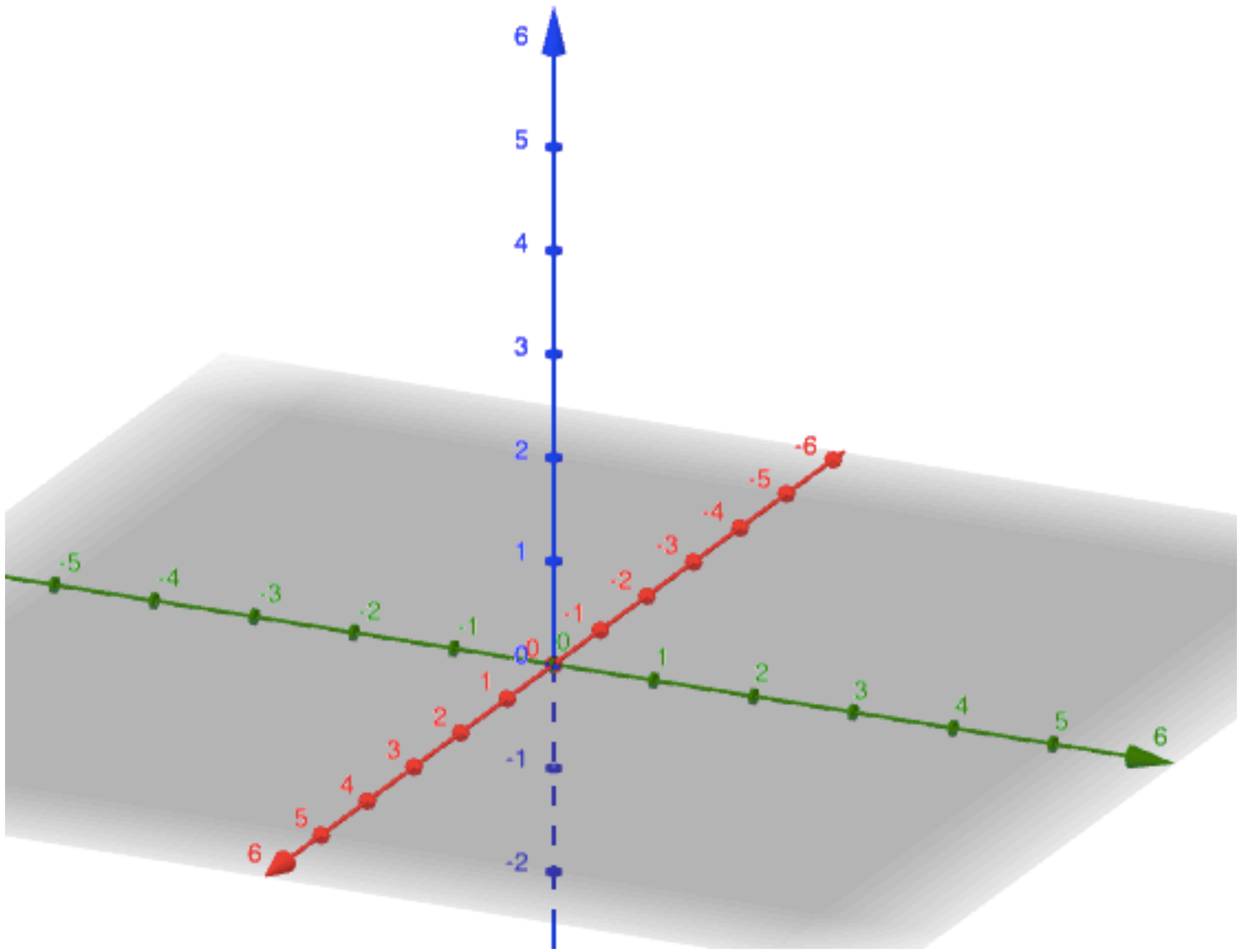
$$\mathbf{proj_b(a)} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\right)\mathbf{b}$$

$$= \frac{-10}{25}\mathbf{b}$$

$$= <-1.6,-1.2>$$

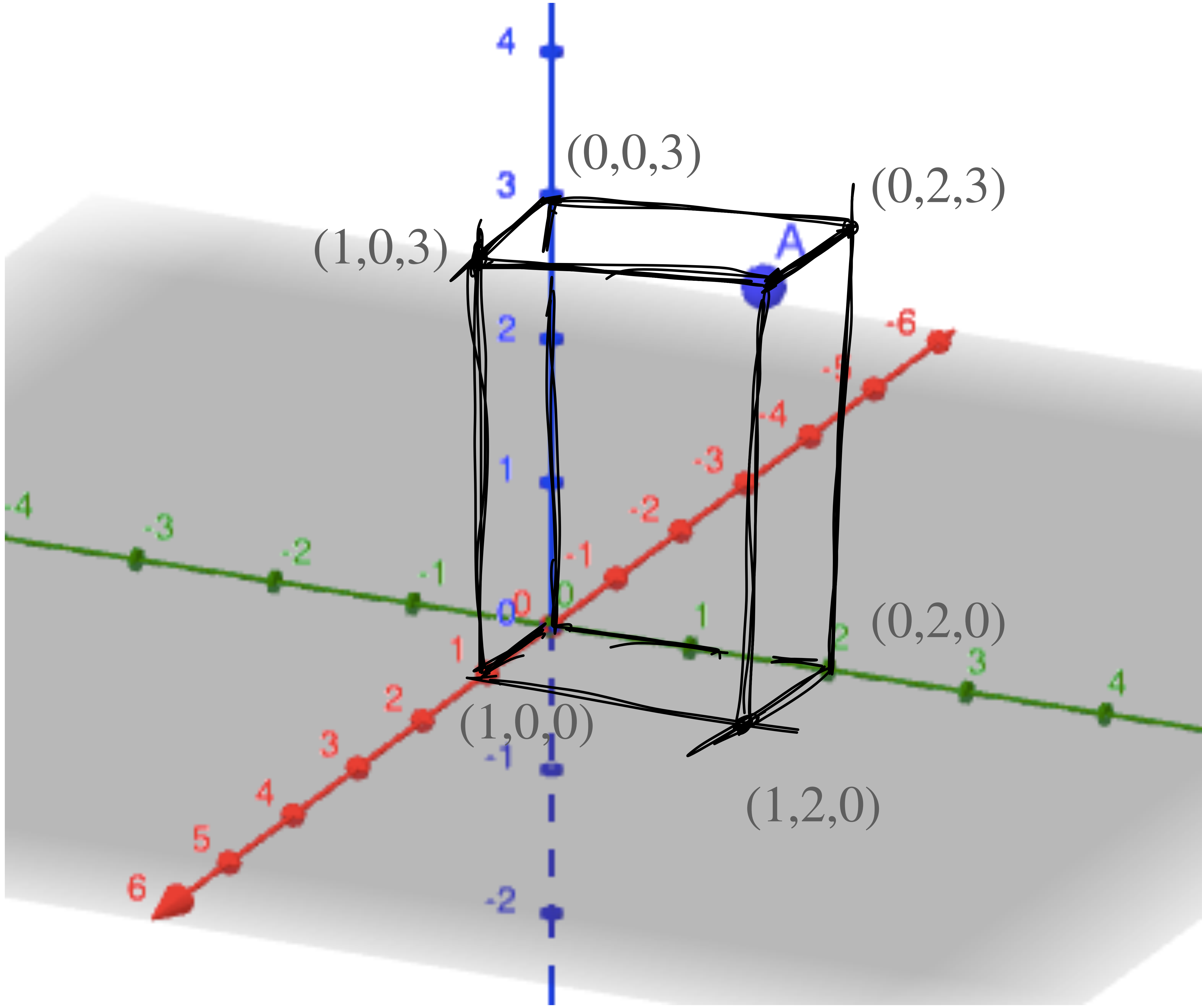
Three Dimensional Space (3D)

Now there are three axes,
an x-axis, y-axis and z-axis.



How do we plot points?

$A(1,2,3)$ is the point where
$x = 1$, $y = 2$ and $z = 3$

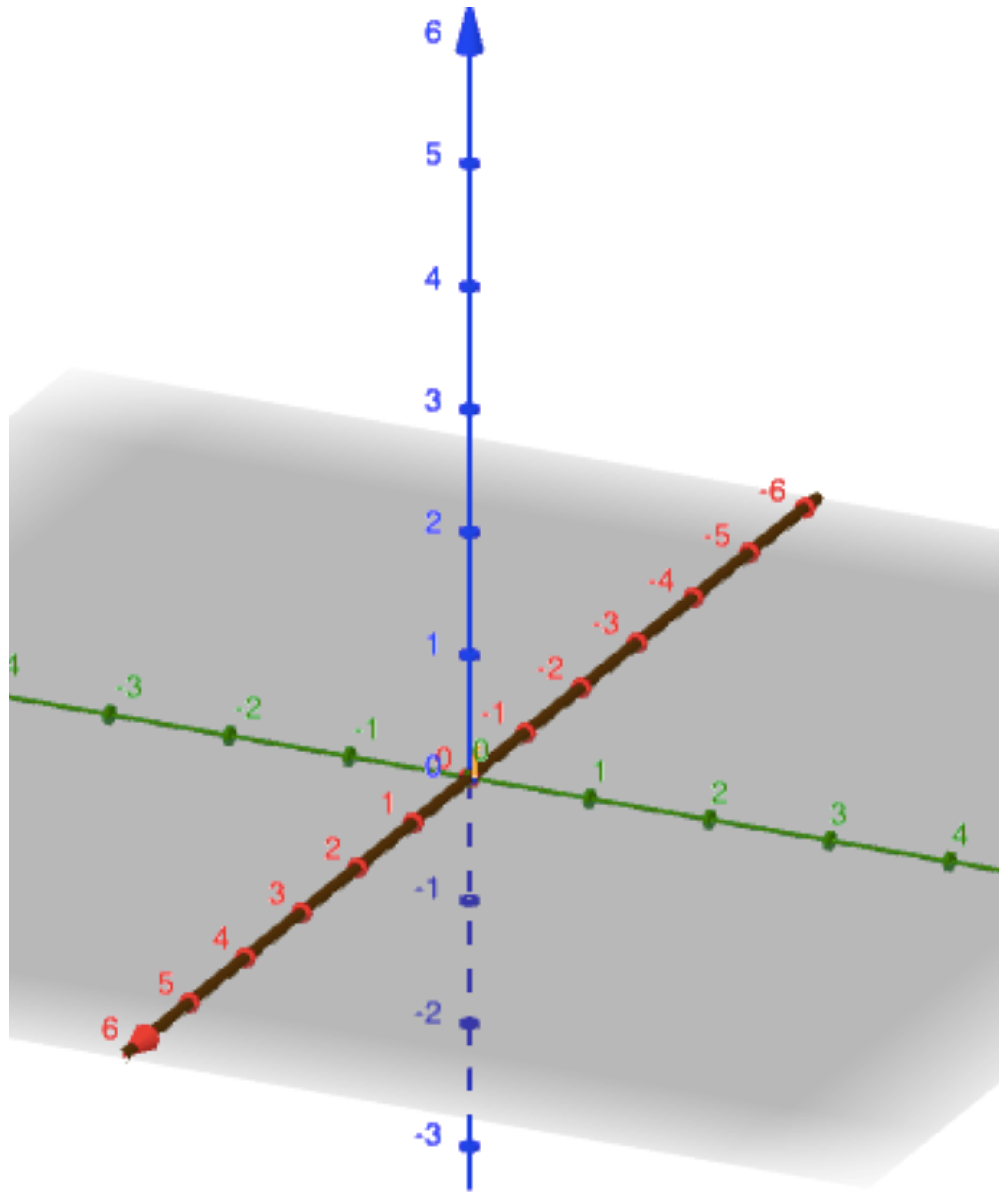


Some (3D) Equations, and their graphs.

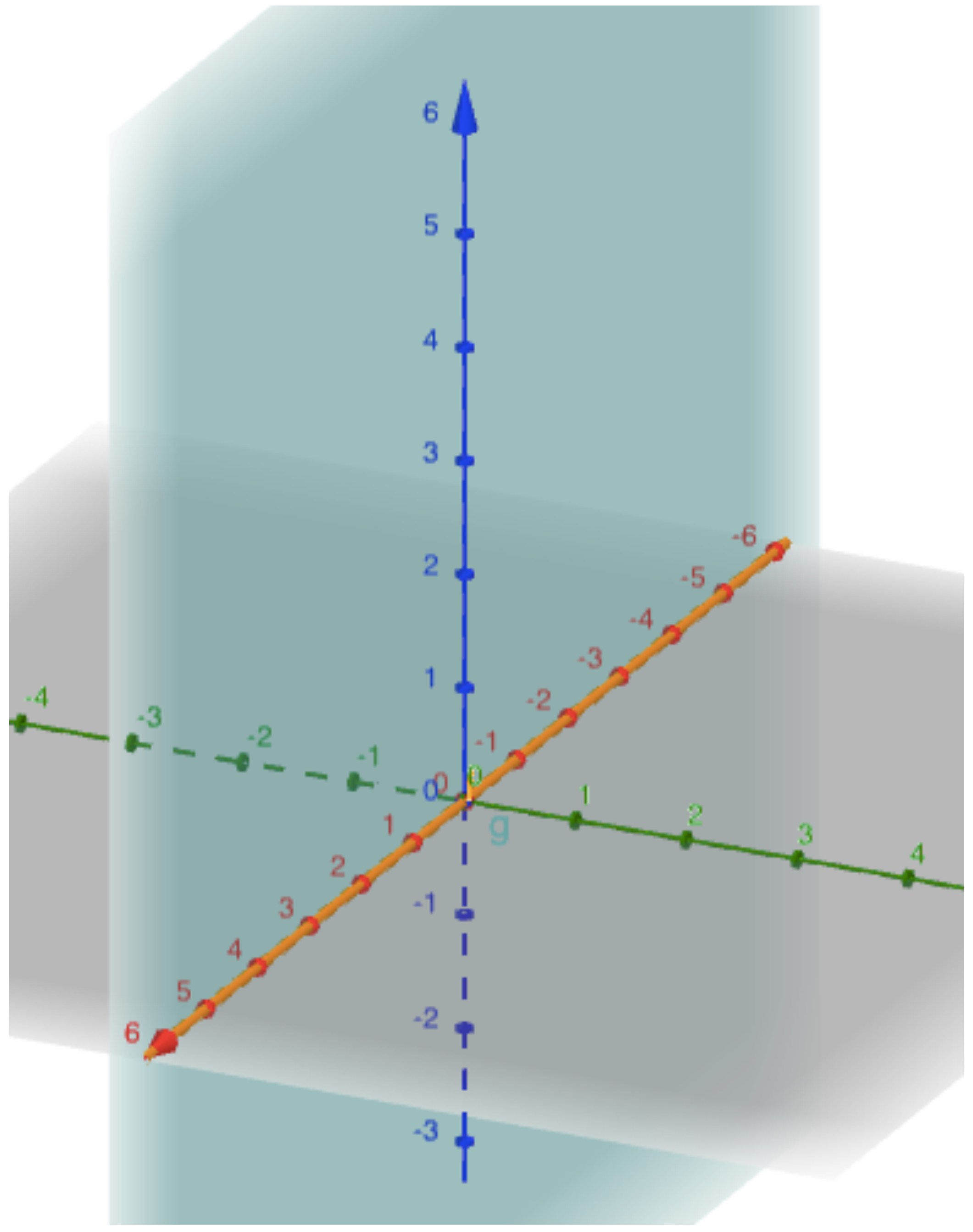
Plot the points which satisfy…

1. $y = 0$

(2D version)

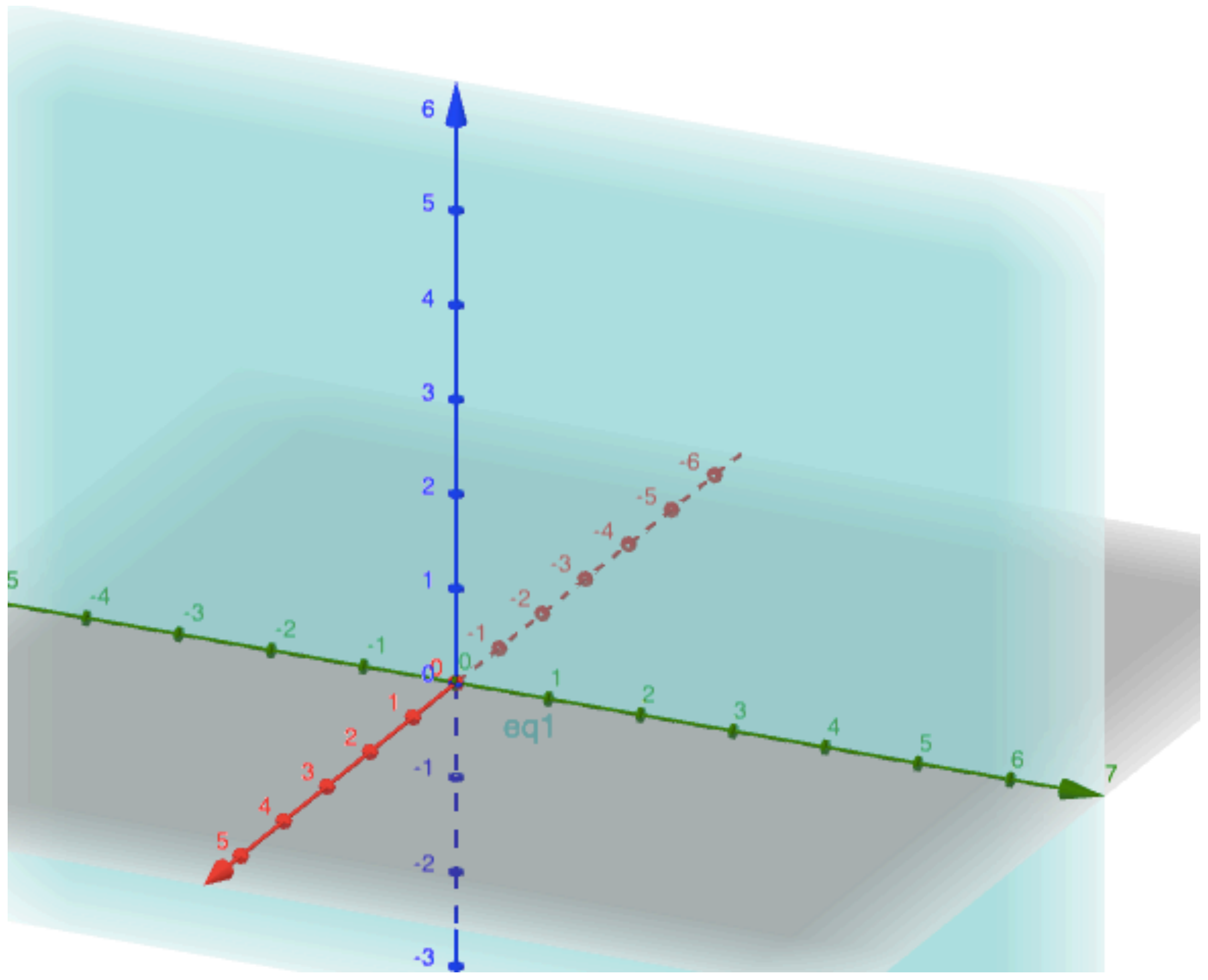


In 3D, the graph looks like this:

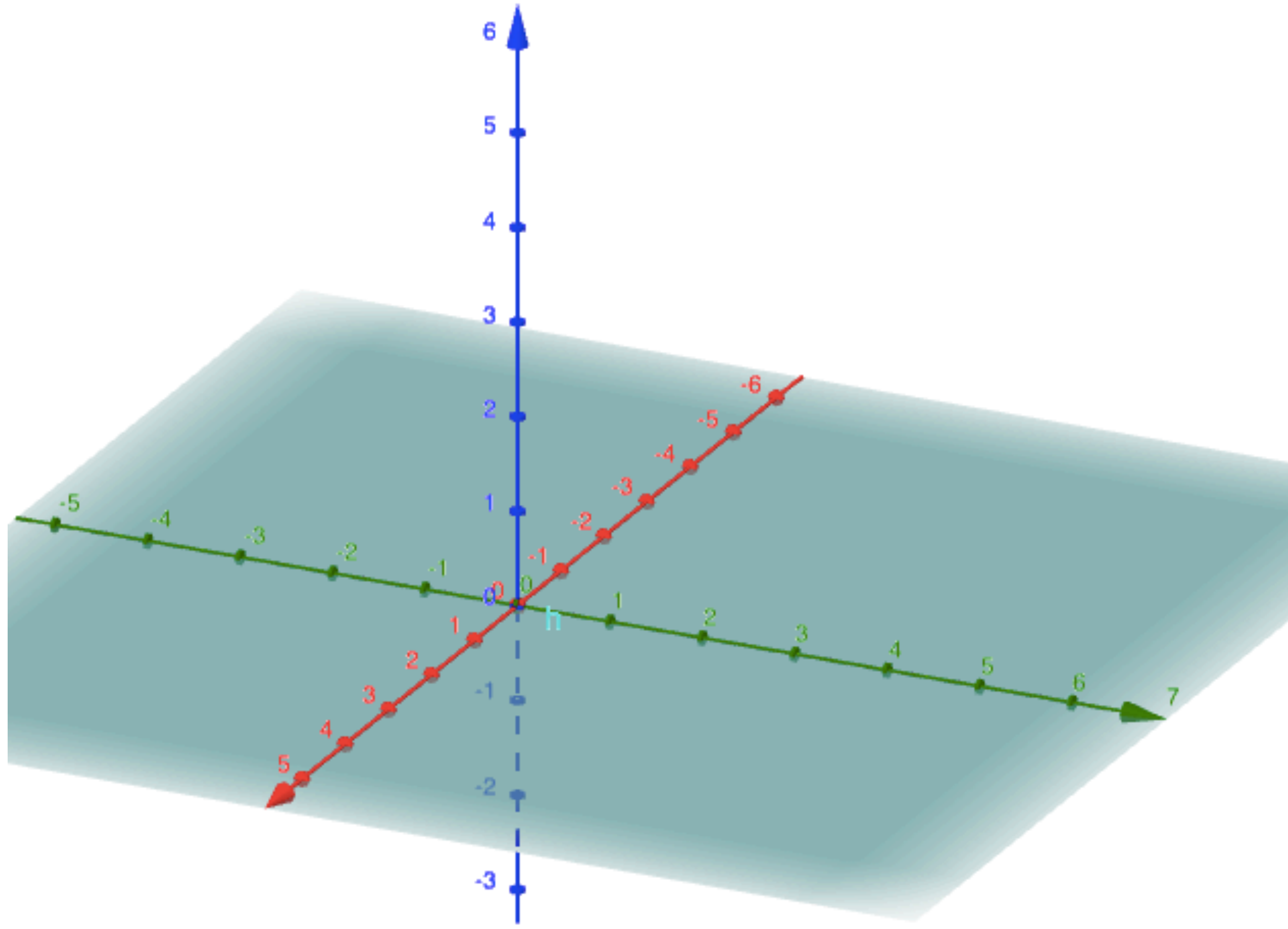


In the equation $y = 0$, the variables x and z are *free.* They can be anything.

2. $x = 0$



3. $z = 0$



Some (3D) Equations, and their graphs, pg 2.

Plot the points which satisfy…

What about these?

1. $y = z$

It's not just this:

where we only have points like

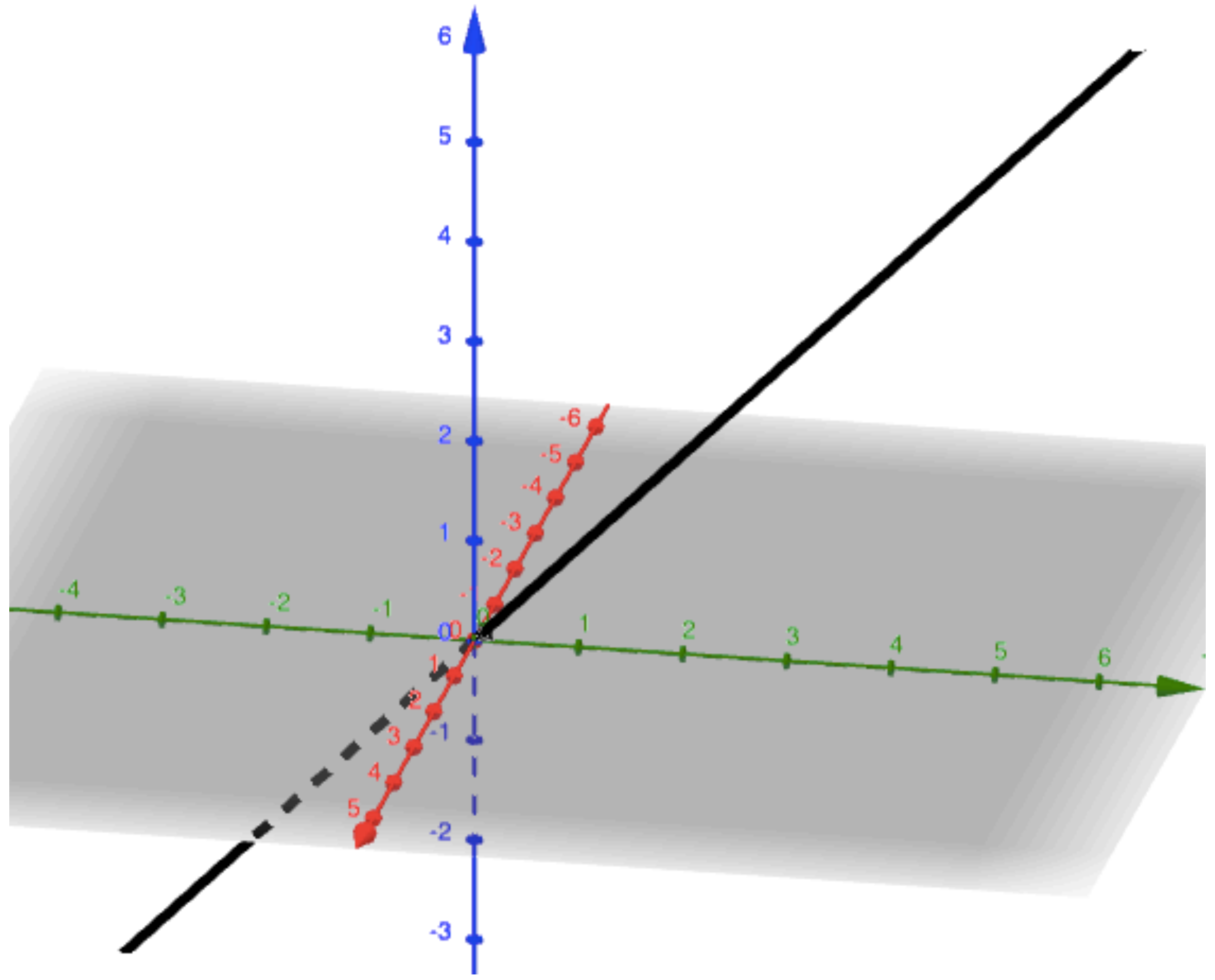
$\dots$

$(0,1,1)$

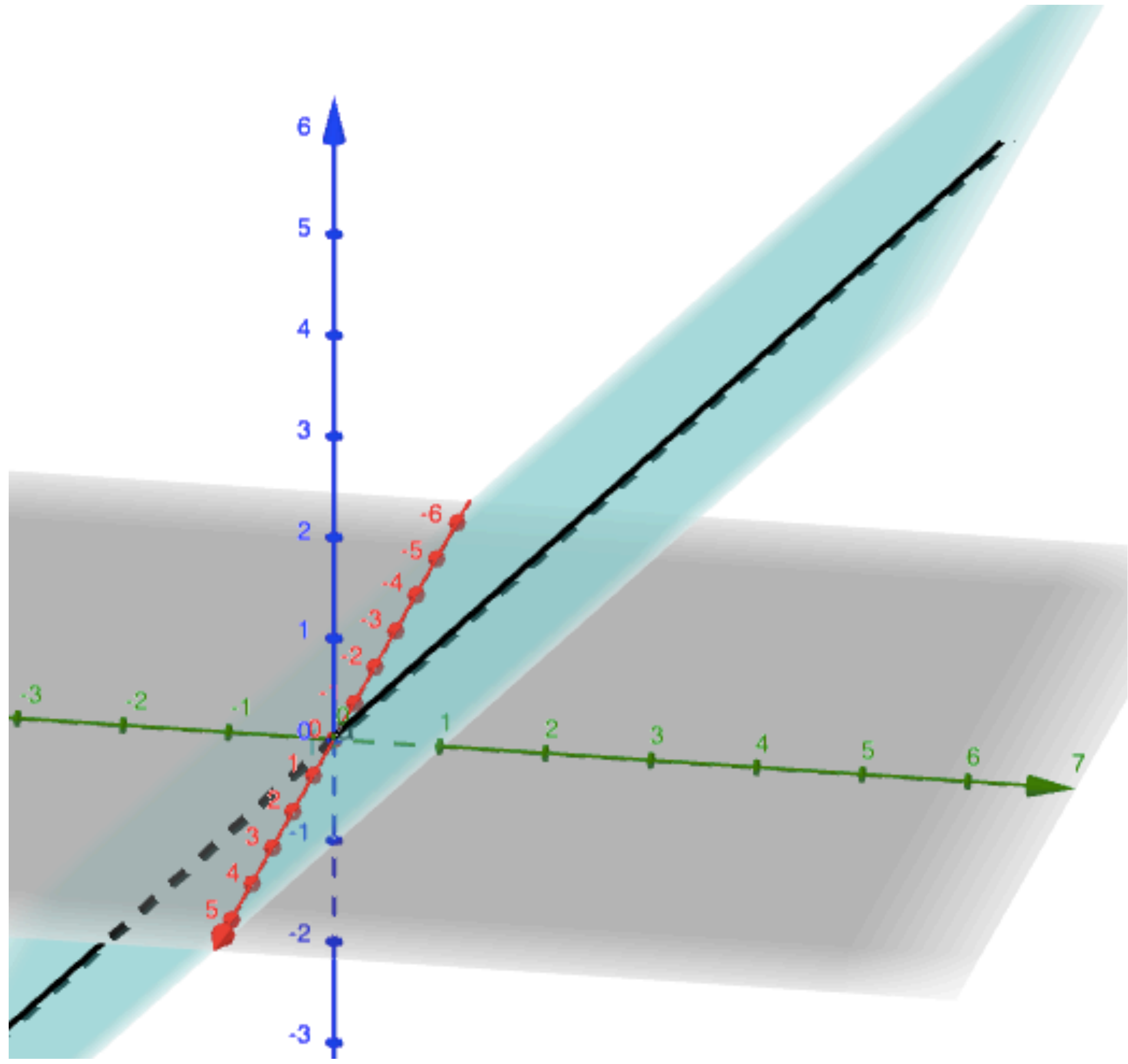
$(0,2,2)$

$(0,3,3)$

$\dots$



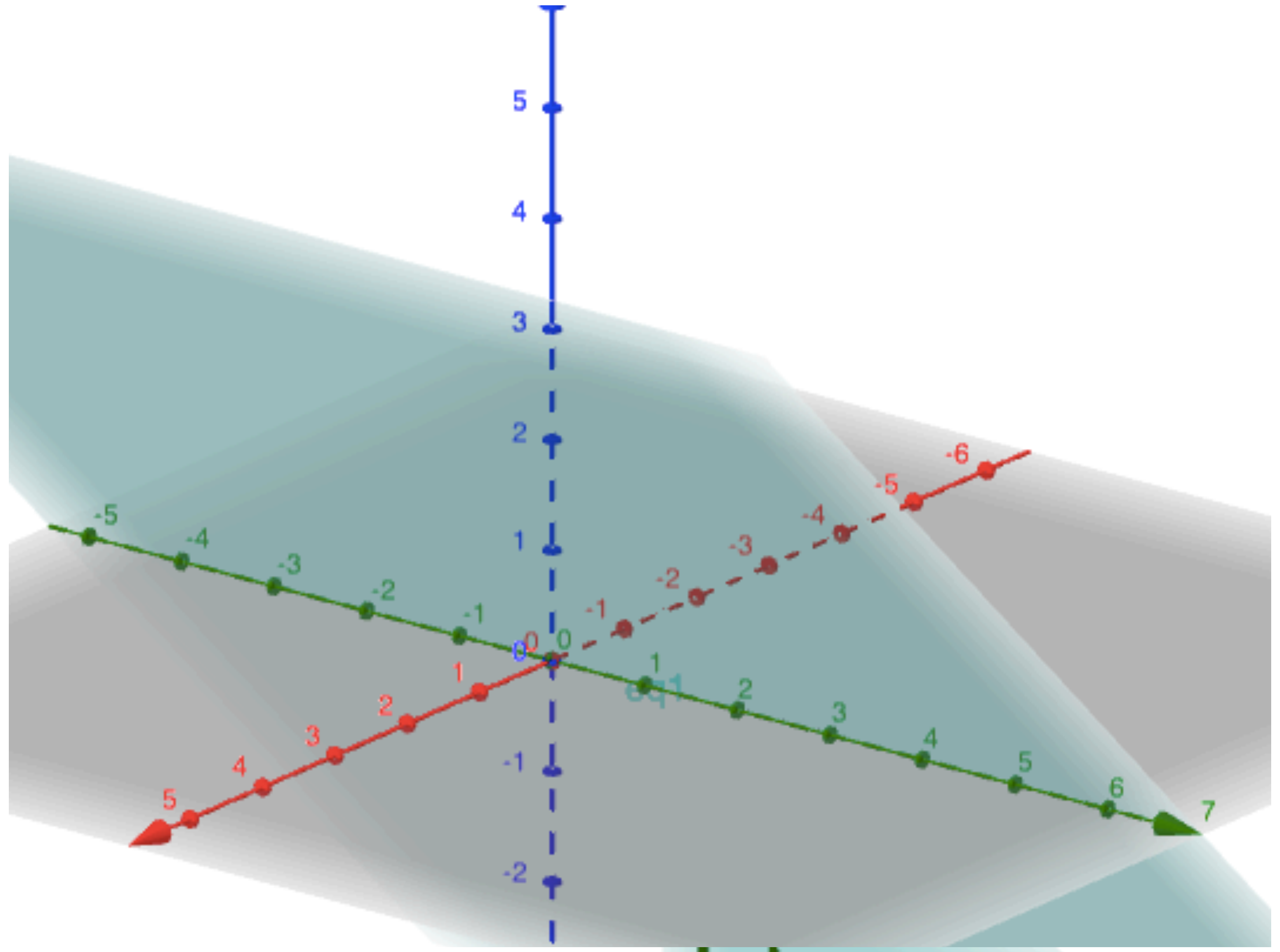
The graph is this:



In the equation $y = z$, the variable x is free. x can be anything.

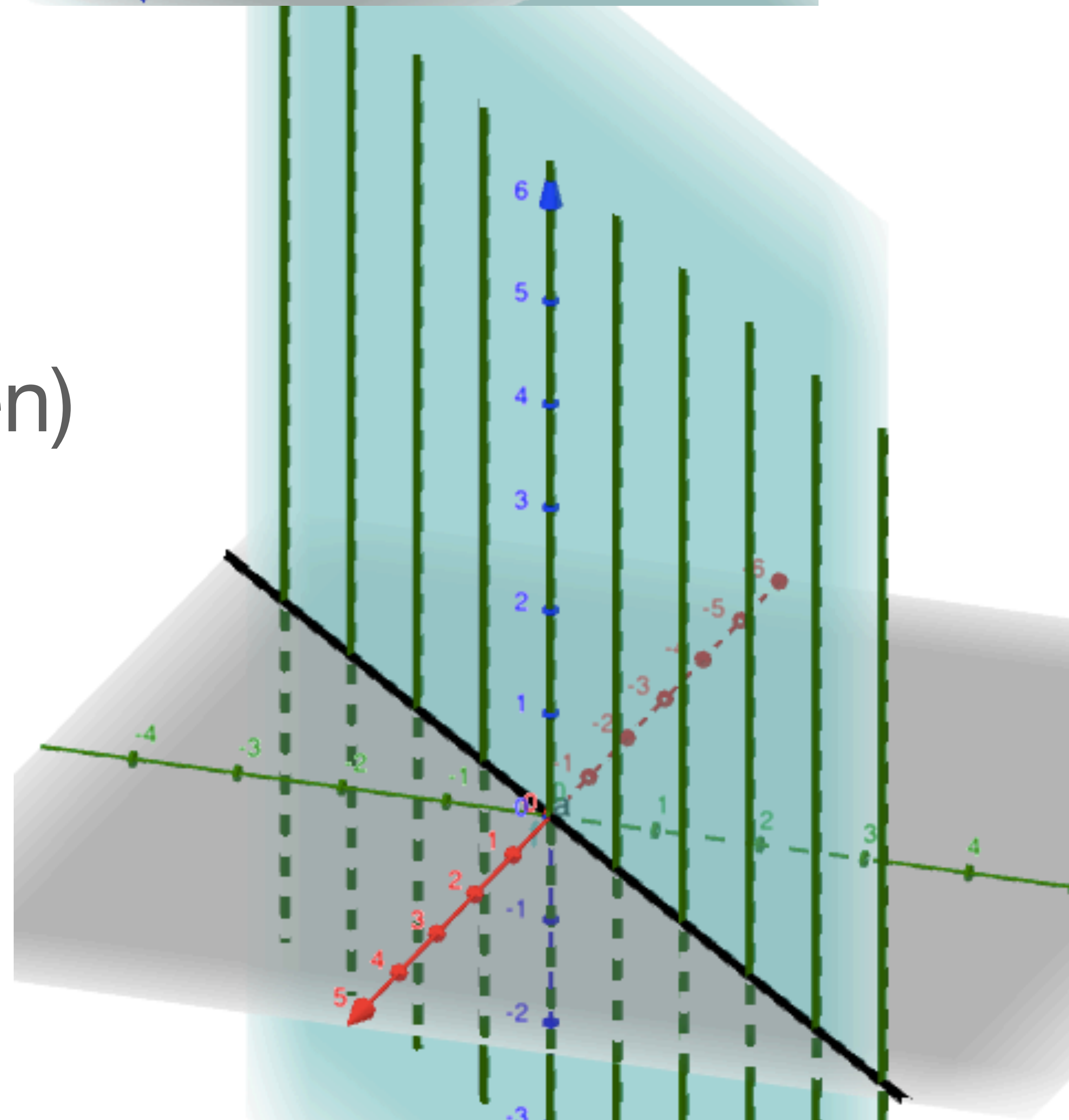
e.g. the graph includes points $(x,1,1)$ for all $x \in \mathbf{R}$

2. $x = z$



3. $y = x$

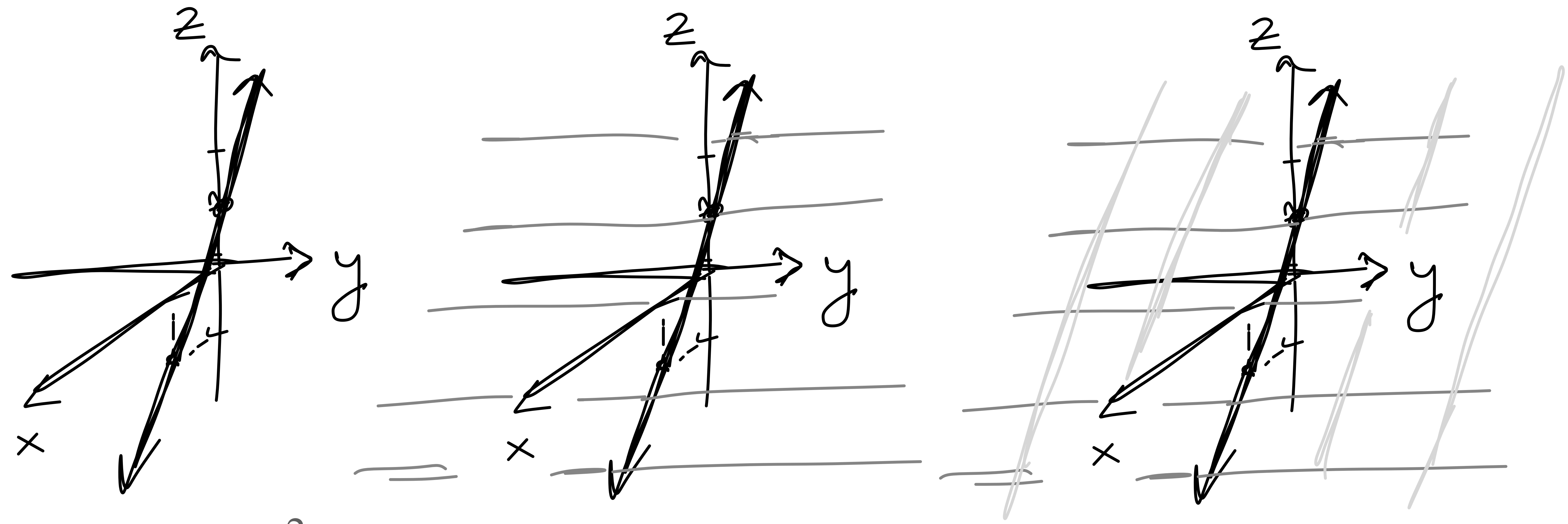
Additional illustration to 3: all of the lines (in green) through a point on the 2D line $y = x$ (in black) parallel to the z-axis lie on the graph of $y = x$.



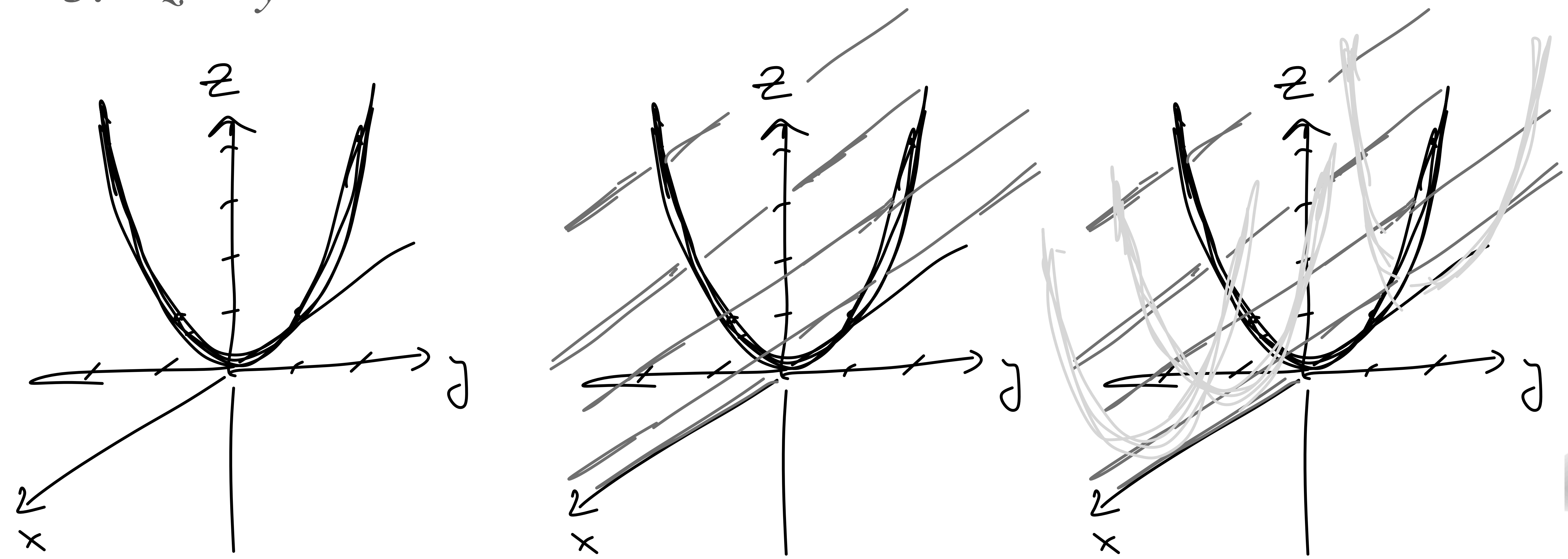
Some (3D) Equations, and their graphs, pg 3.

More examples.

4. $z = -2x + 1$



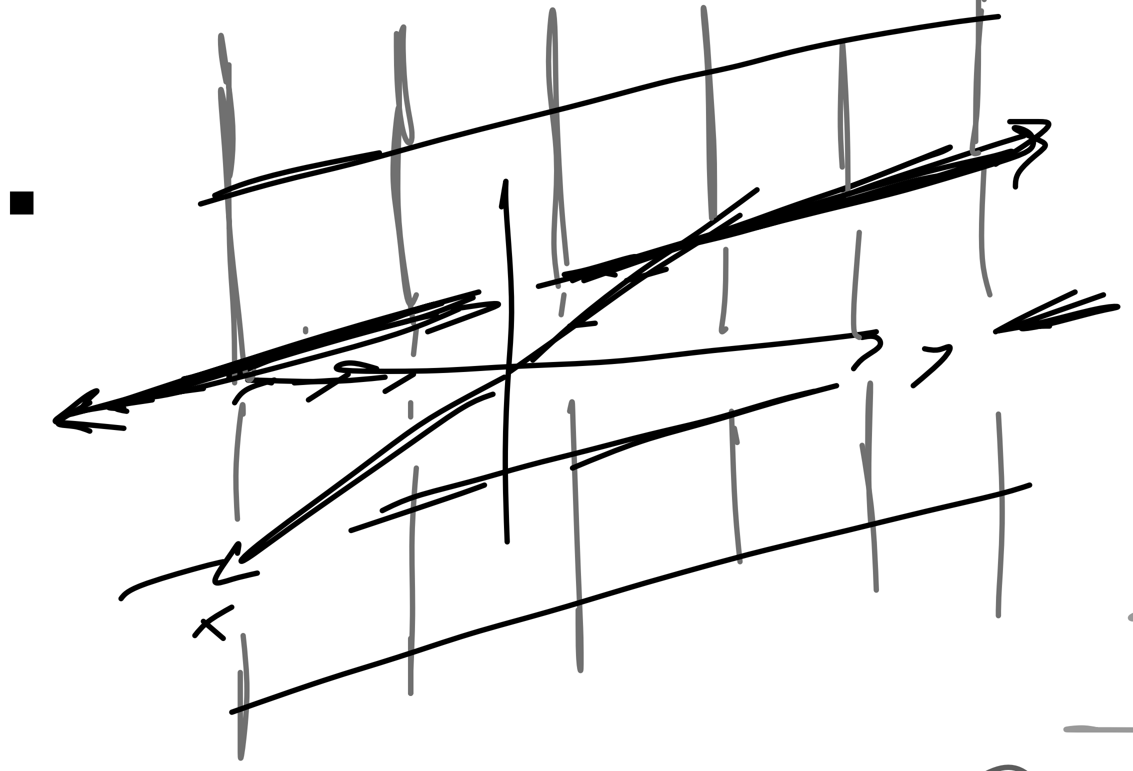
5. $z = y^2$



You try:

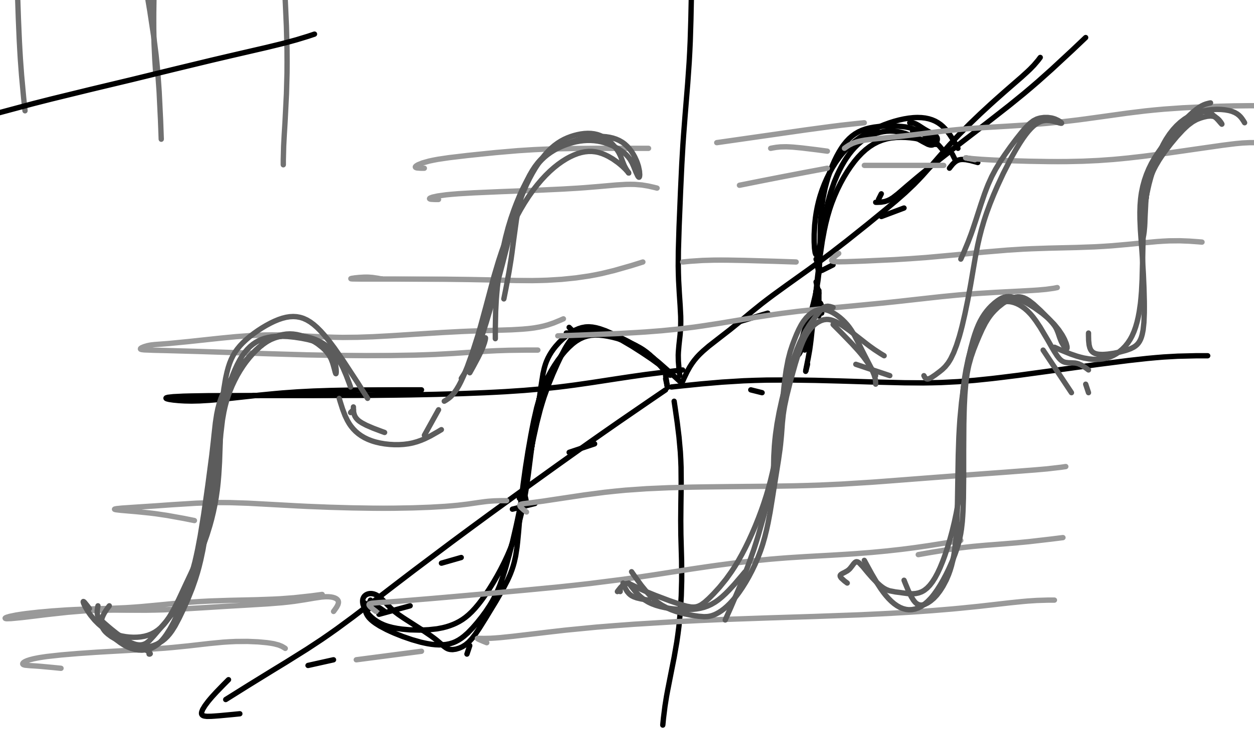
6. $y = -x - 3$

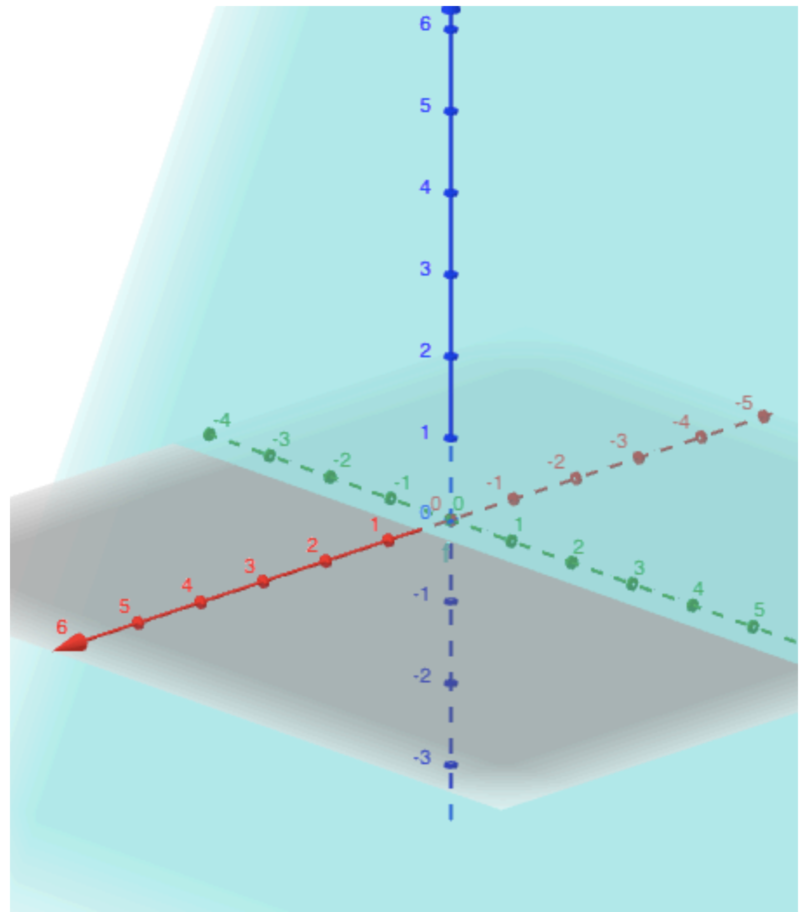
7. $z = \sin(x)$

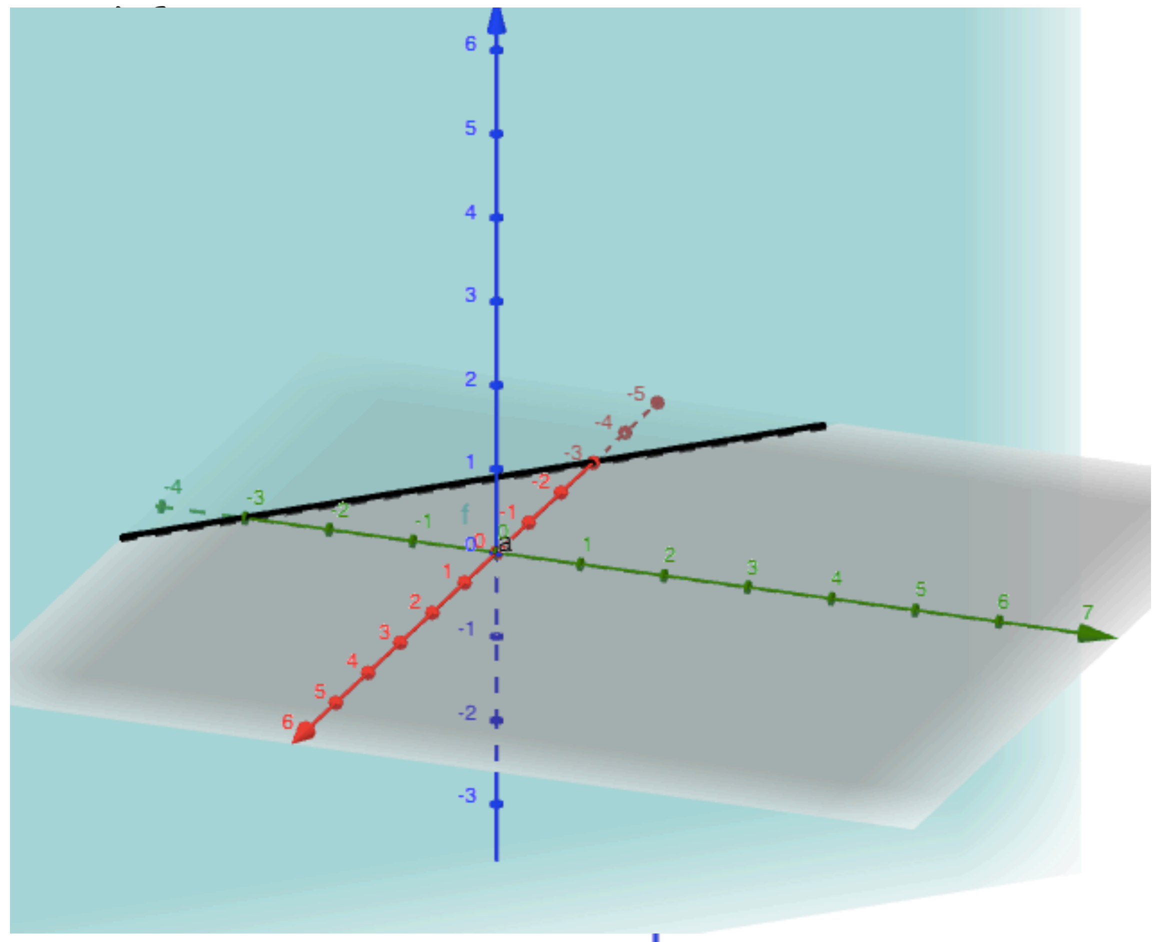


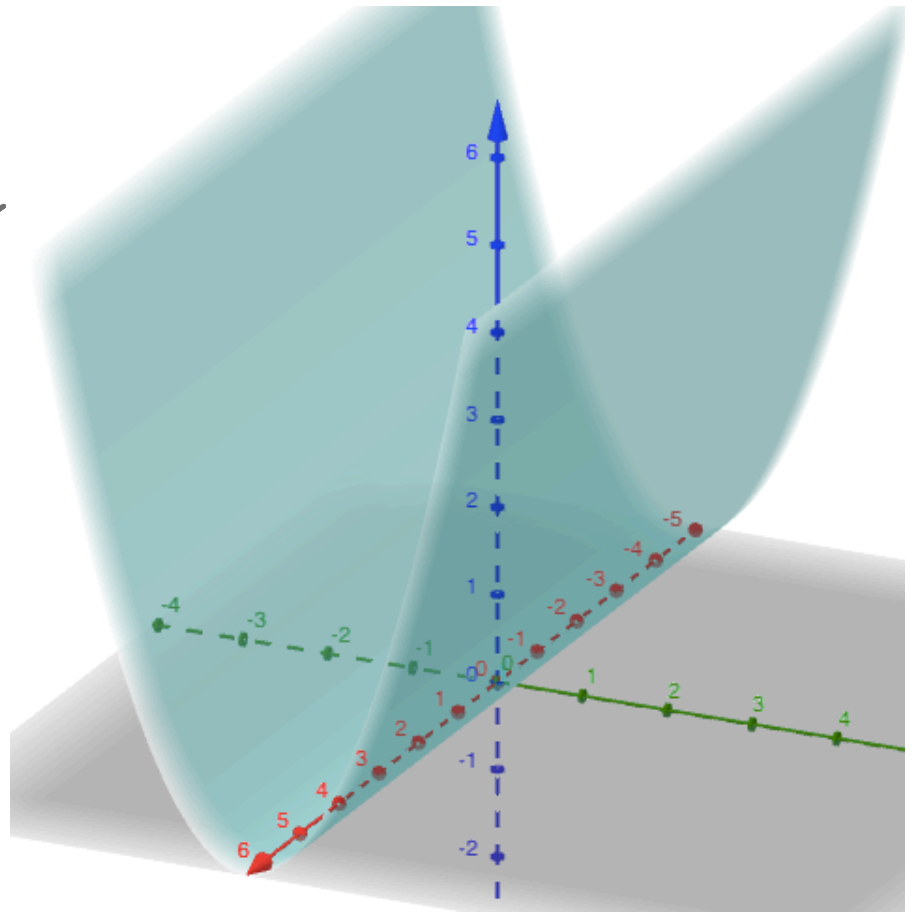
$\leftarrow y = -x - 3$

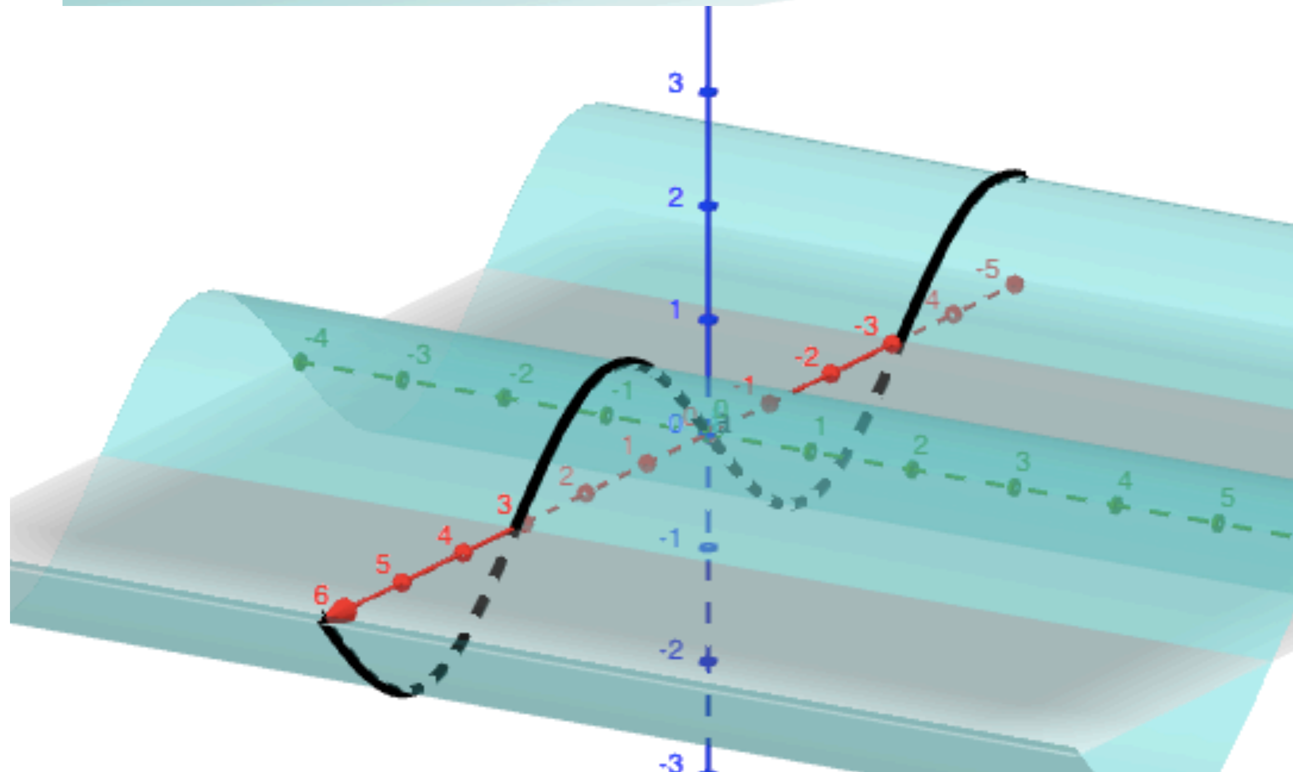
$\downarrow z = \sin(x)$





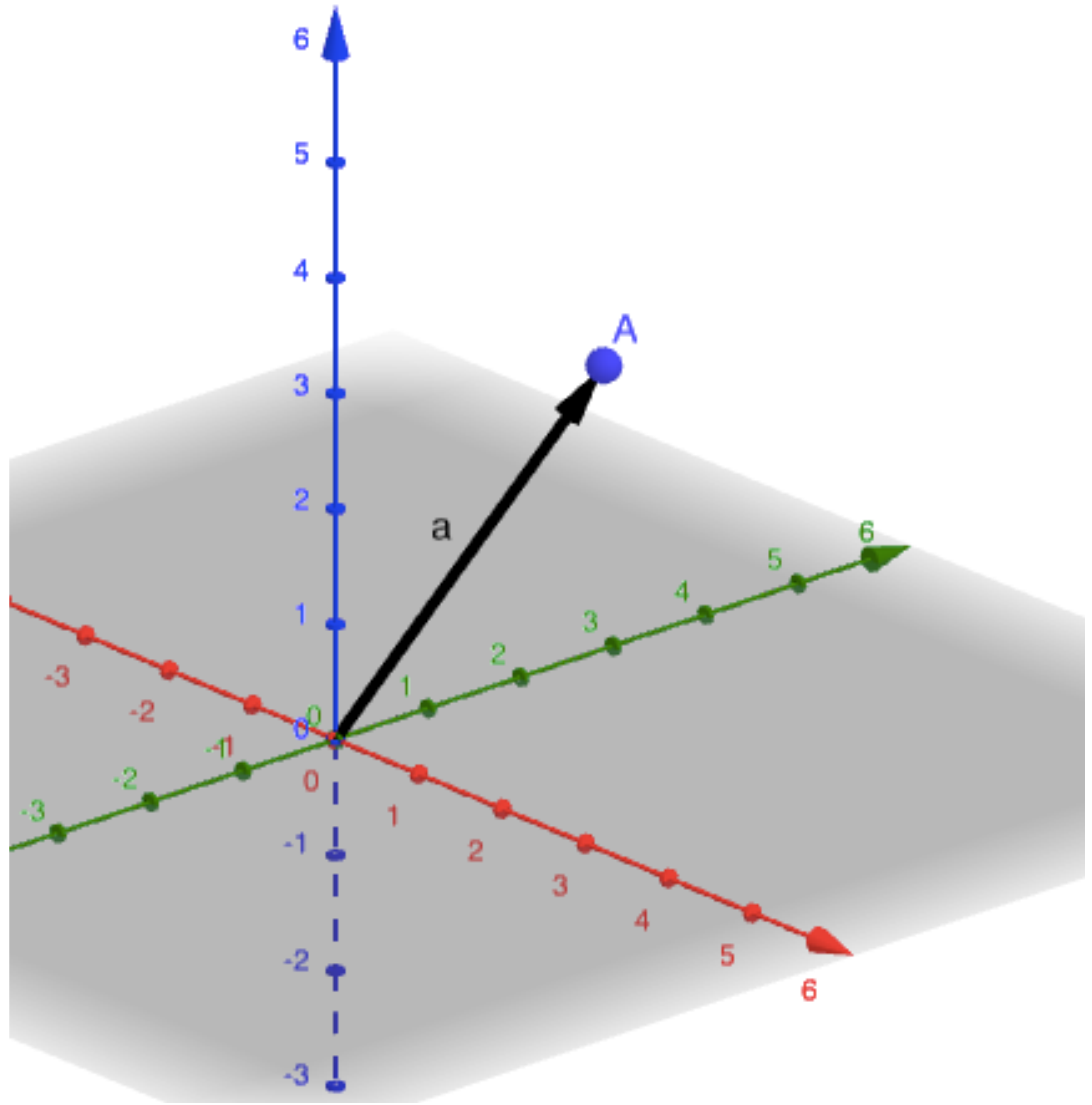


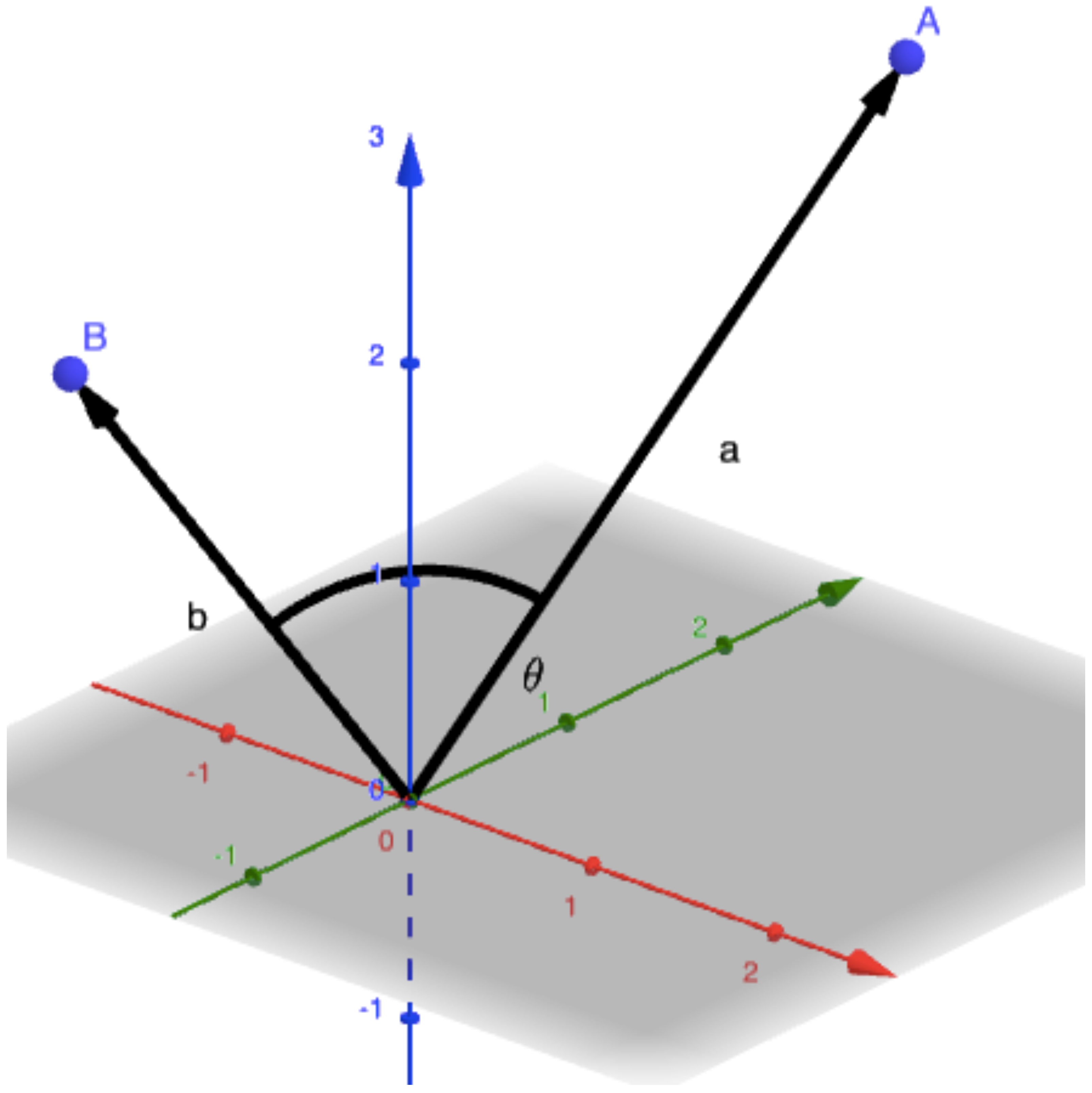




3D vectors and their angles.

Vectors in 3D work the same way as in 2D.





$$\mathbf{a} = < a_1, a_2, a_3 >$$

magnitude: $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

direction: $\hat{\mathbf{a}} = \dfrac{\mathbf{a}}{|\mathbf{a}|}$

Note: the formula

$$\cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

still works in 3D.

Example.

$$\mathbf{a} = < 1,2,3 >$$

$$\mathbf{b} = < -1, -1,2 >$$

$$\mathbf{a} \cdot \mathbf{b} = 1(-1) + 2(-1) + 3(2)$$
$$= 3$$

$$|\mathbf{a}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$

$$|\mathbf{b}| = \sqrt{(-1)^2 + (-1)^2 + 2^2} = \sqrt{6}$$

$$\theta = \cos^{-1}\left(\frac{3}{\sqrt{14}\sqrt{6}}\right) \approx 70.89^\circ$$

Angles of 3D vectors, pg2.

Another example of angle.

$\mathbf{a} = < -1,2,2 >$

$\mathbf{b} = < 2, -1, -1 >$

$\theta = \ ???$

$$\cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

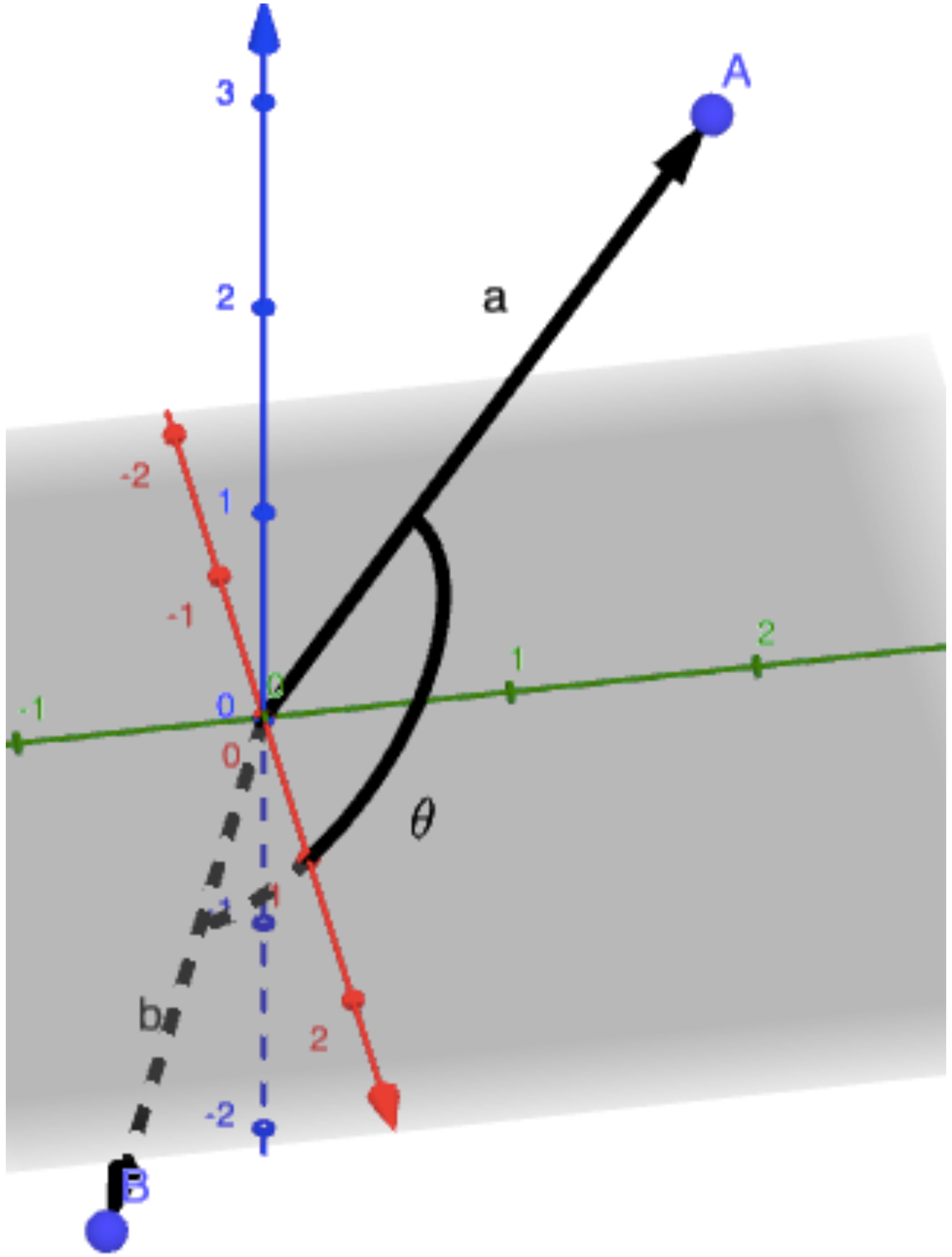
$\mathbf{a} \cdot \mathbf{b} = \ -6$

$|\mathbf{a}| = \sqrt{9} = 3$

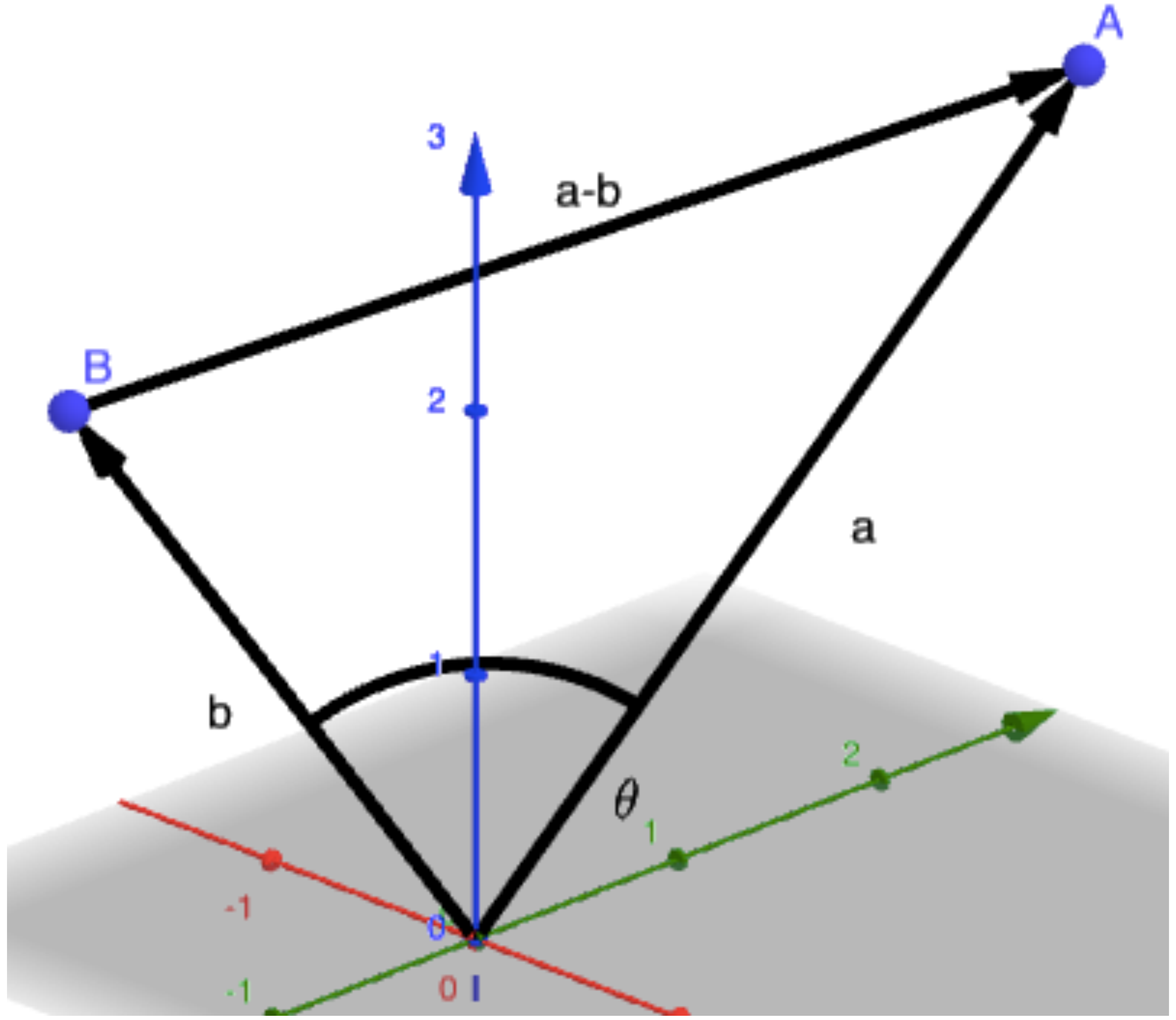
$|\mathbf{b}| = \sqrt{6}$

$$\theta = \cos^{-1}\left(\frac{-6}{3\sqrt{6}}\right)$$

$\approx 144.74°$



Why does the formula $\cos(\theta) = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$ still work in 3D?



Hint: $|\mathbf{a} - \mathbf{b}|^2 = \ldots$

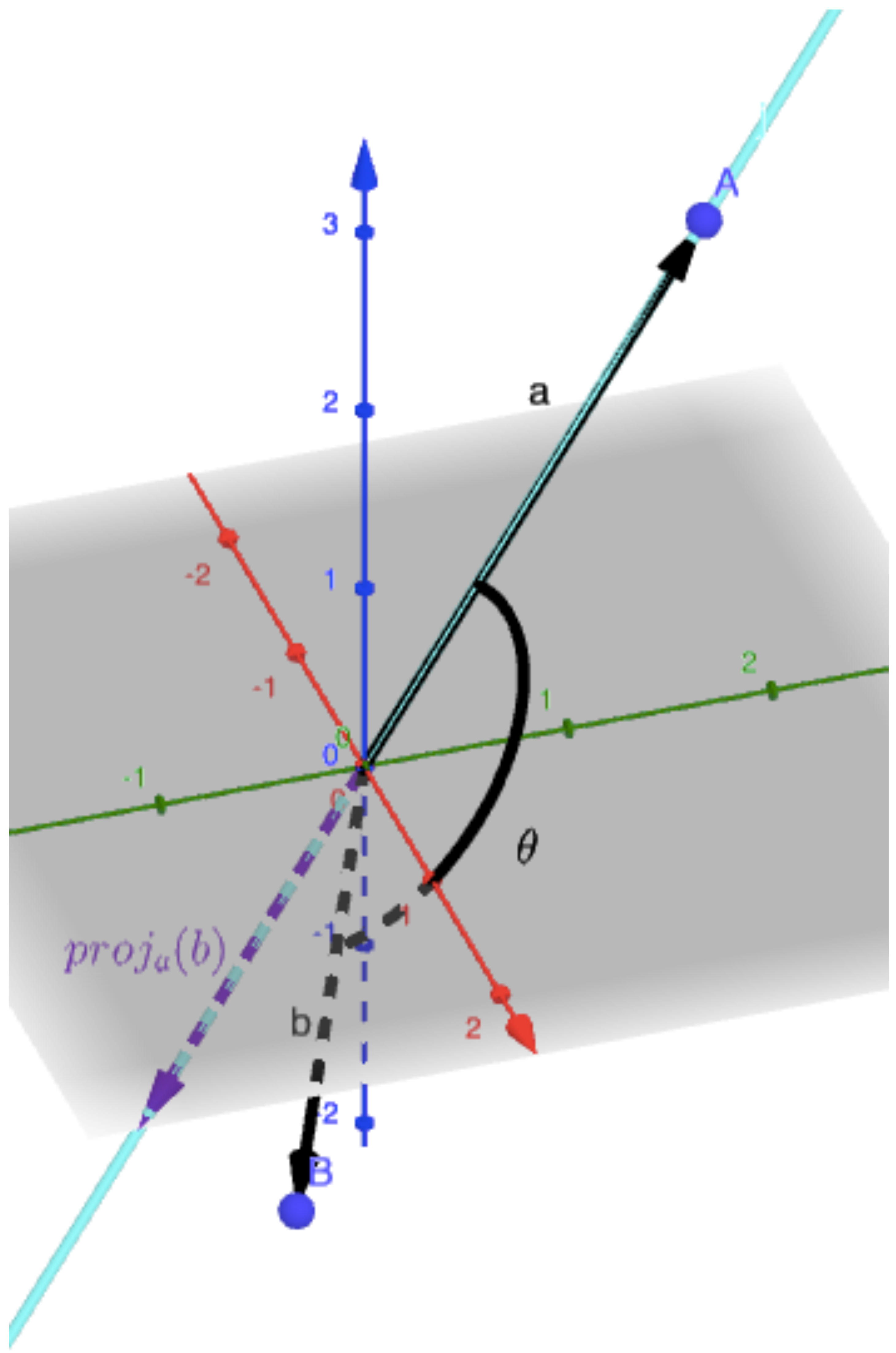
1. $(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b})$

2. (Law of Cosines): $|\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos(\theta)$.

3D Projection of Vectors

Example.

$\mathbf{a} = < -1,2,2 > \quad \mathbf{b} = < 2, -1, -1 >$



The formula $\mathbf{proj_a(b)} = \left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a}$ still applies.

$$\mathbf{proj_a(b)} = \frac{-6}{9}\mathbf{a} = -\frac{2}{3} < -1,2,2 > = \left\langle \frac{2}{3}, -\frac{4}{3}, -\frac{4}{3} \right\rangle$$

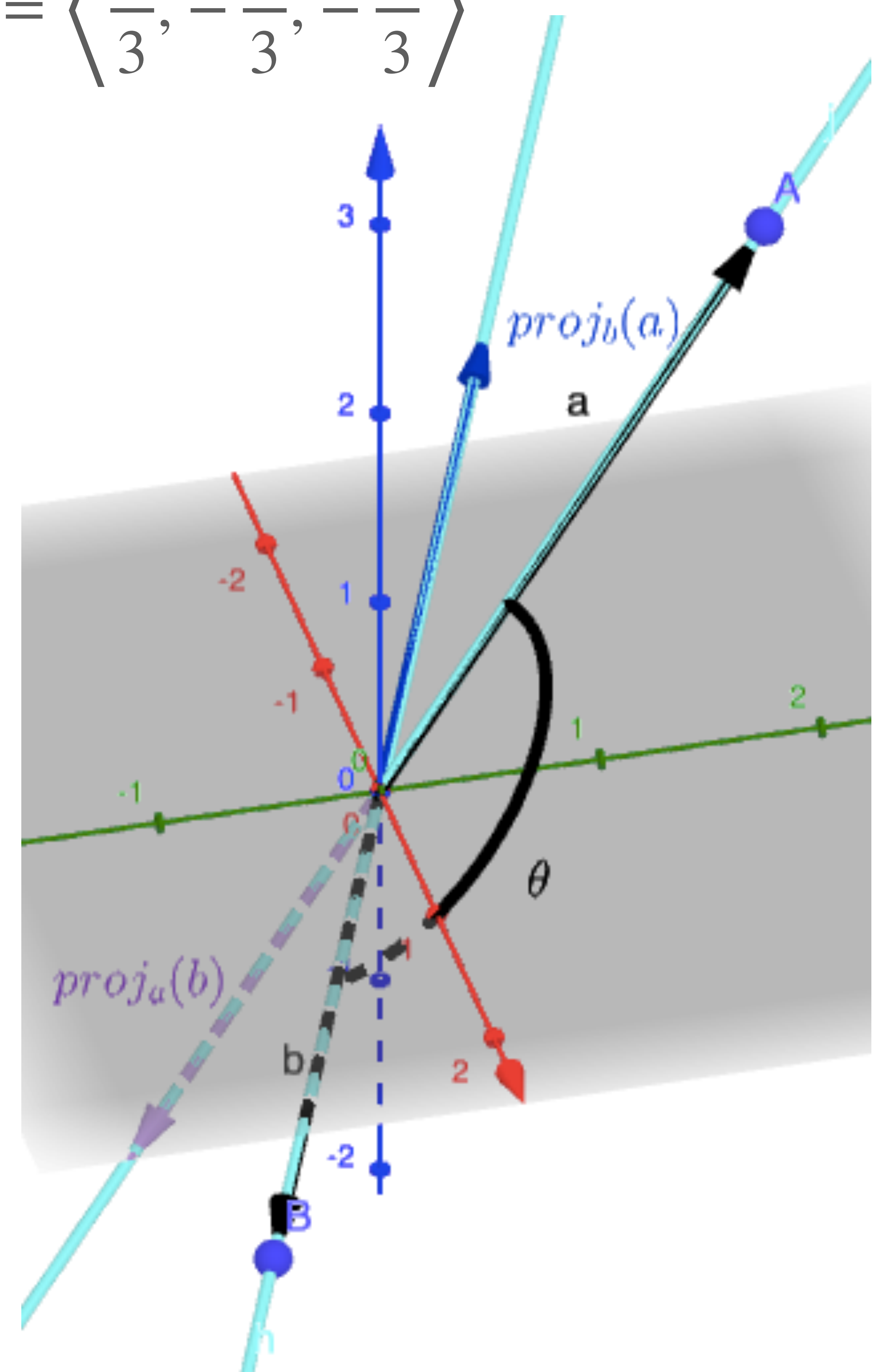
Try computing $\mathbf{proj_b(a)}$.

$$\mathbf{proj_b(a)} = \left(\frac{\mathbf{b}\cdot\mathbf{a}}{\mathbf{b}\cdot\mathbf{b}}\right)\mathbf{b}$$

$$= \frac{-6}{6}\mathbf{b}$$

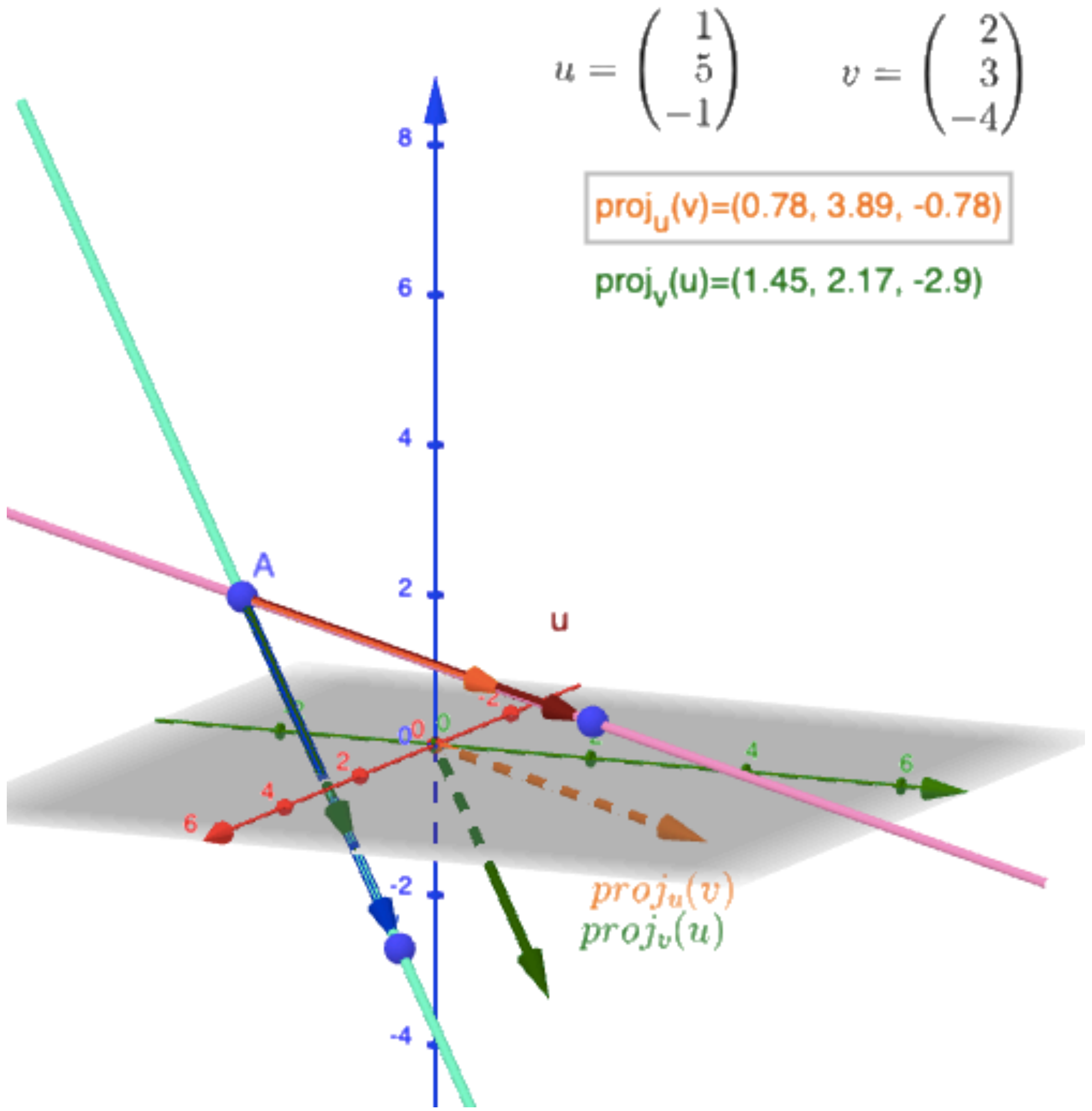
$$= - < 2, -1, -1 >$$

$$= < -2,1,1 >$$



3D Projection of Vectors, pg 2.

Projections don't need to be based at the origin.



$A(1,-2,2),\ \ U(2,3,1),\ \ V(3,1,-2)$

Note: you can get the vector with initial point A and terminal point U just by measuring x,y and z displacement….

… which you can get by subtracting coordinates.

$$\mathbf{u} = \overrightarrow{AU} = <2,3,1> - <1,-2,2> = <1,5,-1>$$

$$\mathbf{v} = \overrightarrow{AV} = <3,1,-2> - <1,-2,2> = <2,3,-4>$$

$$\mathbf{proj_u(v)} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\cdot\mathbf{u}$$

$$= \frac{21}{27}\mathbf{u} = \frac{7}{9}\mathbf{u} = \left\langle \frac{7}{9}, \frac{35}{9}, -\frac{7}{9} \right\rangle$$

$$\mathbf{proj_v(u)} = ???$$

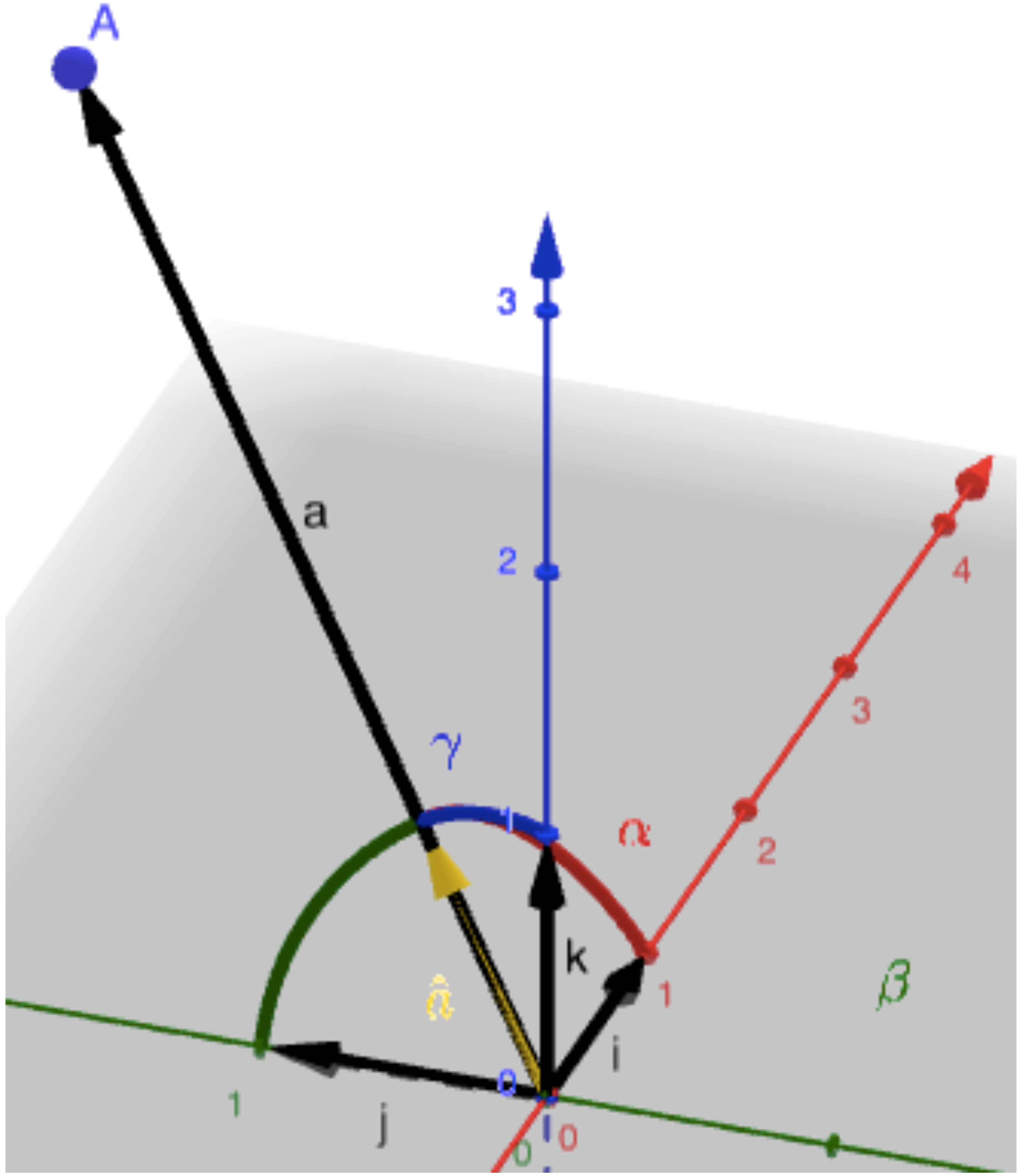
$$= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\cdot\mathbf{v}$$

$$= \frac{21}{29}\cdot\mathbf{v} = \left\langle \frac{42}{29}, \frac{63}{29}, -\frac{84}{29} \right\rangle = \frac{1}{29} <42,63,-84>$$

Link: [3DVectorProjections]

Direction Angles of a Vector.

The 3D direction of a vector can be measured using the angles off of each axis.



Say $\mathbf{a} = < a_1, a_2, a_3 >$. Then $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

The *unit vector* in the direction of $\mathbf{a}$ is …

$$\hat{\mathbf{a}} = \frac{1}{|\mathbf{a}|}\mathbf{a} = \frac{1}{|\mathbf{a}|} < a_1, a_2, a_3 > = \left\langle \frac{a_1}{|\mathbf{a}|}, \frac{a_2}{|\mathbf{a}|}, \frac{a_3}{|\mathbf{a}|} \right\rangle$$

$$= \left\langle \frac{\mathbf{a} \cdot < 1,0,0 >}{|\mathbf{a}||< 1,0,0 >|}, \frac{\mathbf{a} \cdot < 0,1,0 >}{|\mathbf{a}||< 0,1,0 >|}, \frac{\mathbf{a} \cdot < 0,0,1 >}{|\mathbf{a}||< 0,0,1 >|} \right\rangle$$

$$= \left\langle \frac{\mathbf{a} \cdot \mathbf{i}}{|\mathbf{a}||\mathbf{i}|}, \frac{\mathbf{a} \cdot \mathbf{j}}{|\mathbf{a}||\mathbf{j}|}, \frac{\mathbf{a} \cdot \mathbf{k}}{|\mathbf{a}||\mathbf{k}|} \right\rangle = < \cos(\alpha), \cos(\beta), \cos(\gamma) >$$

where α, β, γ are the angles between $\mathbf{a}$ and the positive x-axis,y-axis, and z-axis respectively.

α, β, γ are the *direction angles* of the vector $\mathbf{a}$.

Two vectors, $\mathbf{a}$ and $\mathbf{b}$, have the same direction if their direction angles are equal.

said differently, $\mathbf{a}$ and $\mathbf{b}$ have the same direction if their unit vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ are equal.

Direction Angles of a Vector, pg 2.

Example1.

$\mathbf{a} = < 1,4,2 >$

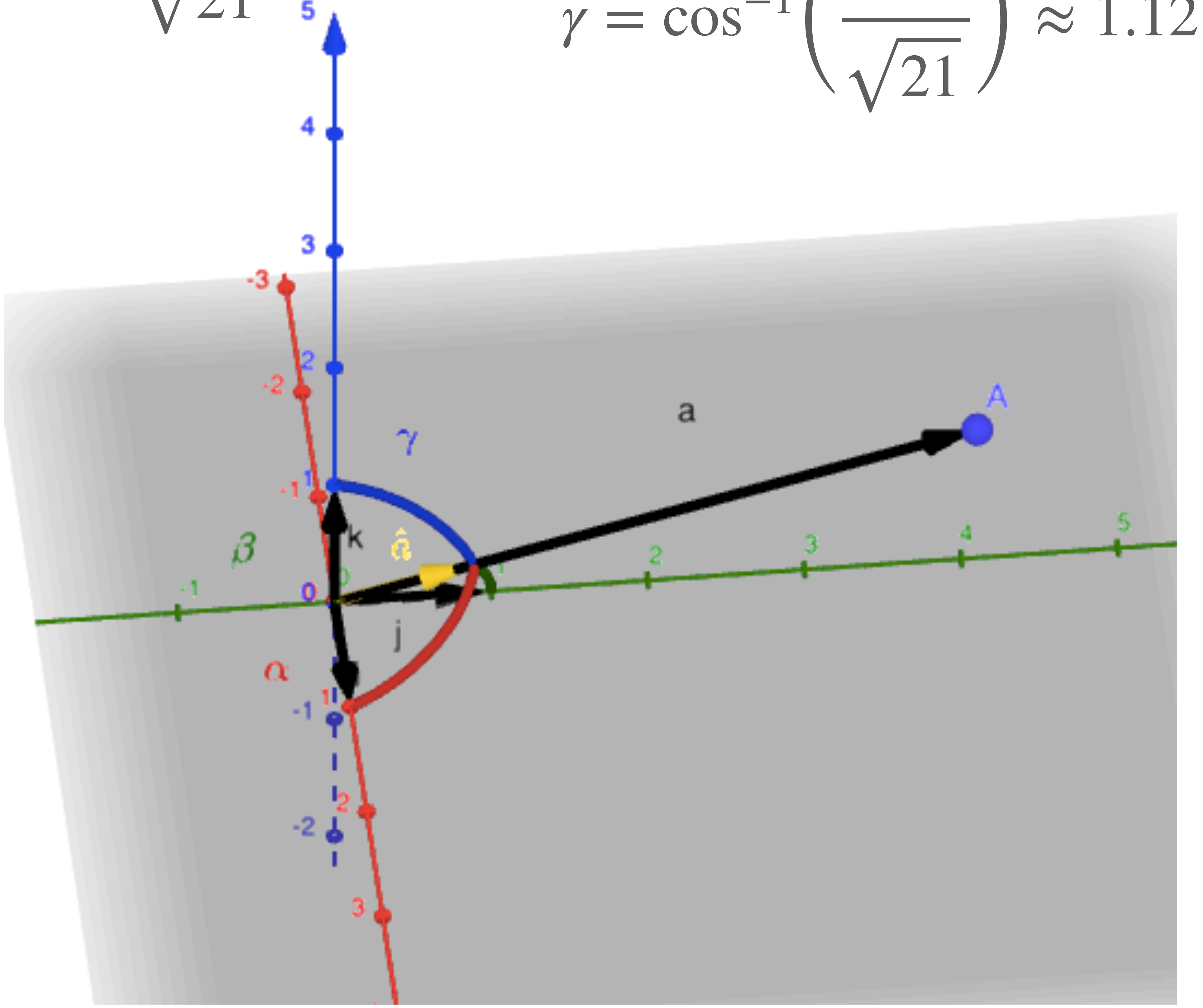
$|\mathbf{a}| = \sqrt{21}$

$\hat{\mathbf{a}} = \dfrac{1}{\sqrt{21}} < 1,4,2 >$

$\alpha = \cos^{-1}\left(\dfrac{1}{\sqrt{21}}\right) \approx 1.35 \text{ rad} \approx 77.40°$

$\beta = \cos^{-1}\left(\dfrac{4}{\sqrt{21}}\right) \approx 0.51 \text{ rad} \approx 29.21°$

$\gamma = \cos^{-1}\left(\dfrac{2}{\sqrt{21}}\right) \approx 1.12 \text{ rad} \approx 64.12°$



Example2. $\mathbf{a} = < 1, -2, -3 >$

$|\mathbf{a}| = \sqrt{14}$

$\hat{\mathbf{a}} = \dfrac{1}{\sqrt{14}} < 1, -2, -3 >$

$= \left\langle \dfrac{1}{\sqrt{14}}, -\dfrac{2}{\sqrt{14}}, -\dfrac{3}{\sqrt{14}} \right\rangle$

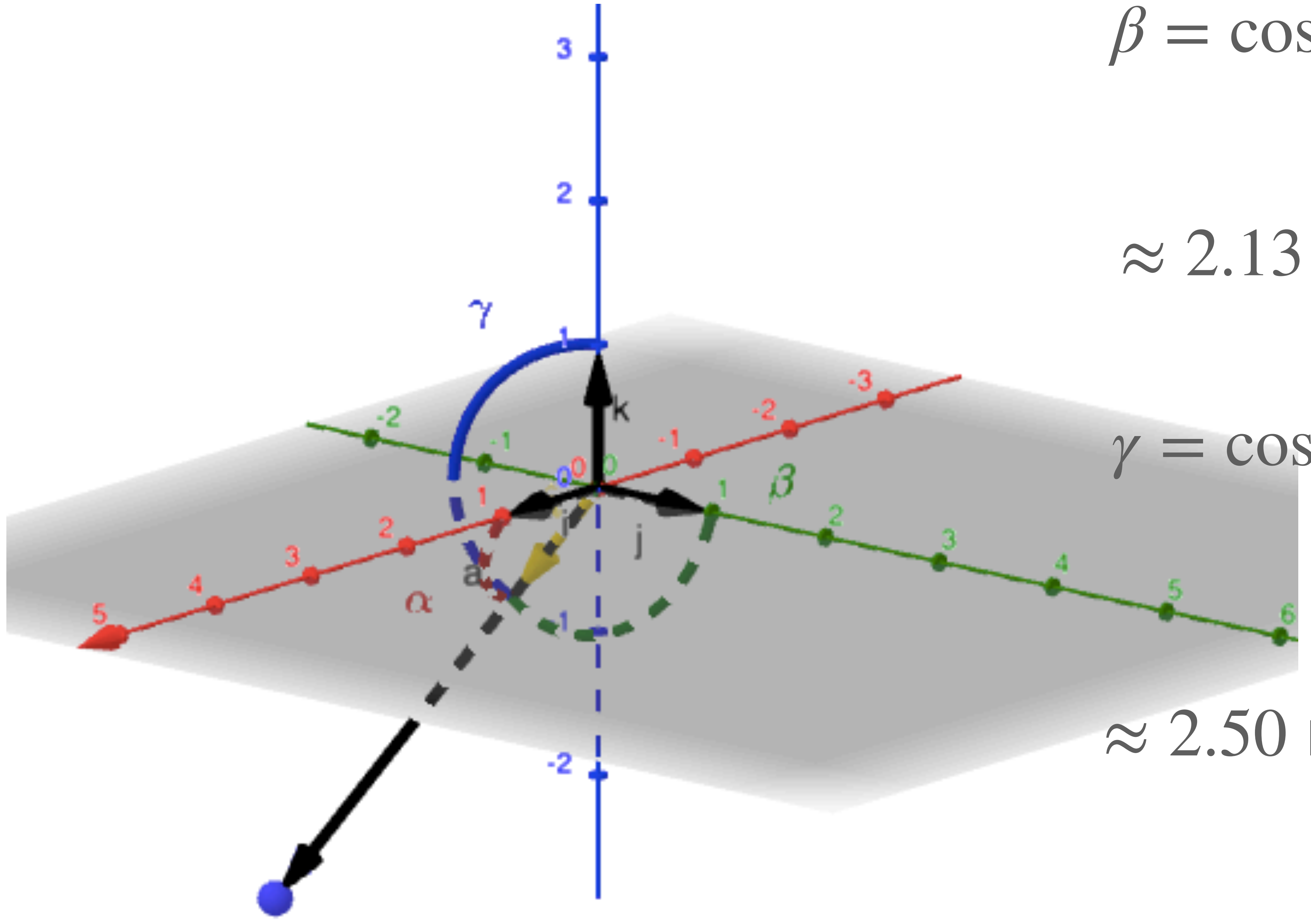
$\alpha = \cos^{-1}\left(\dfrac{1}{\sqrt{14}}\right) \approx 1.3 \text{ rad} \approx 74.50°$

$\beta = \cos^{-1}\left(-\dfrac{2}{\sqrt{14}}\right)$

$\approx 2.13 \text{ rad} \approx 122.31°$

$\gamma = \cos^{-1}\left(-\dfrac{3}{\sqrt{14}}\right)$

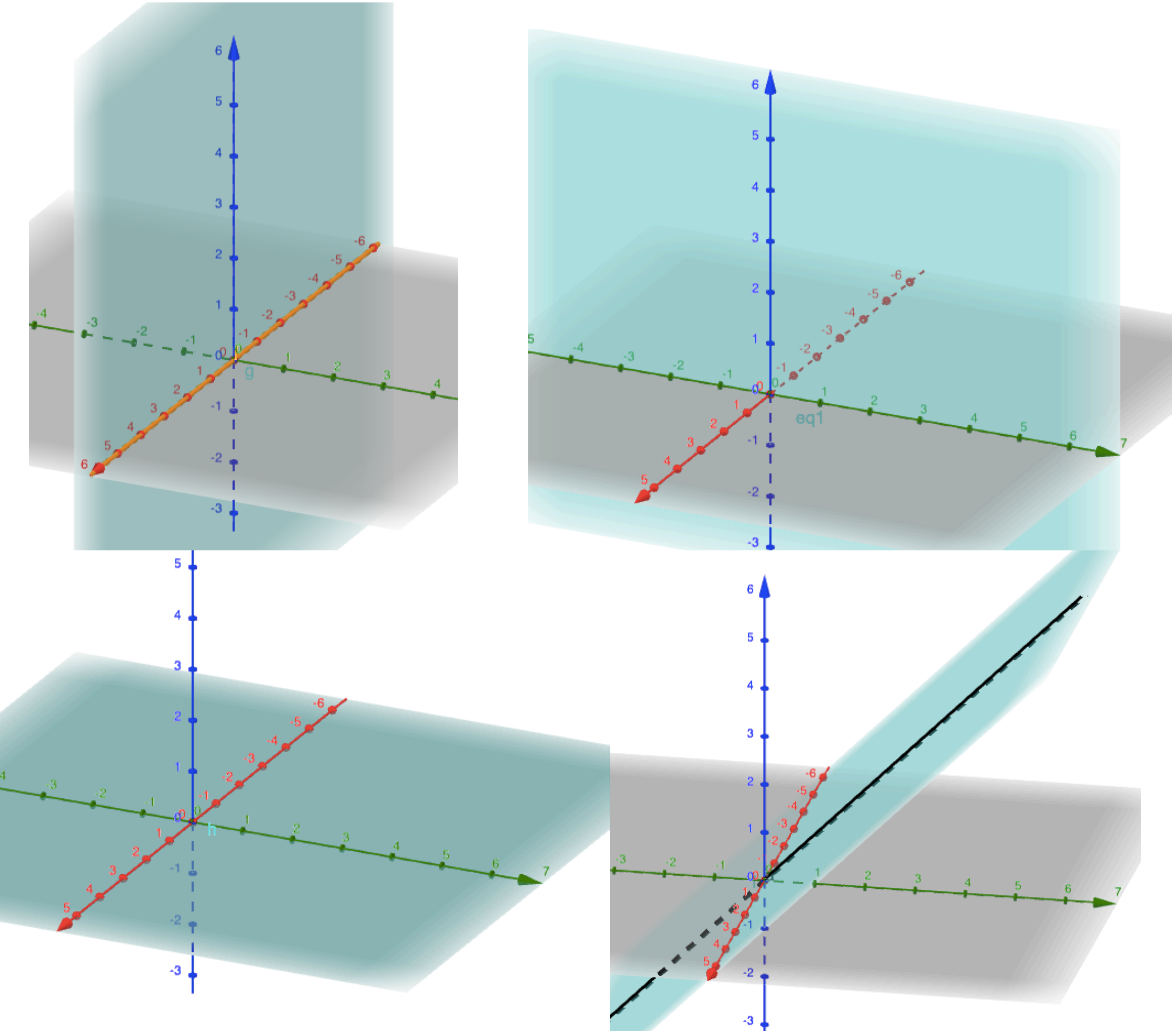
$\approx 2.50 \text{ rad} \approx 143.30°$



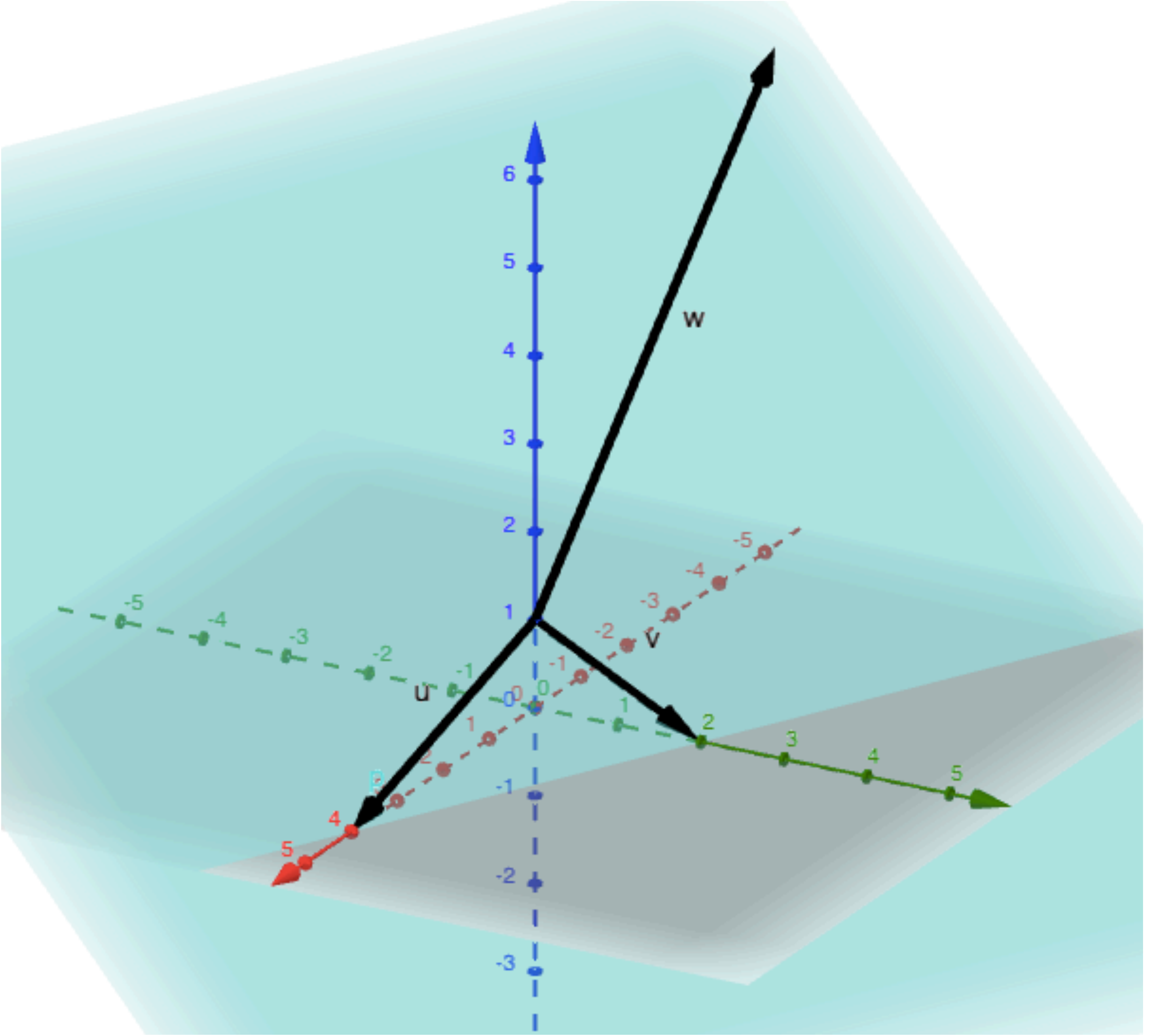
Link: DirectionAngles

The cross product, pg 1.

We've learned a little about planes like these:



What about this plane?



This is the plane determined by the three points $(4,0,0)$, $(0,2,0)$, and $(0,0,1)$.

To find the equation of such a plane we need…

the Cross Product of two 3D vectors!

The cross product, pg 2.

Given two vectors $\mathbf{u}$ and $\mathbf{v}$, can you find a non-zero vector $\mathbf{w}$ that is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$?

(Our answer to this question will be the cross product $\mathbf{u} \times \mathbf{v}$.)

Specific example.

$\mathbf{u} = < 1,1,2 >$

$\mathbf{v} = < -1,1,1 >$

$\mathbf{w} = < a, b, c > \neq < 0,0,0 >$

For $\mathbf{w}$ and $\mathbf{u}$ to be perpendicular, we need $0 = \mathbf{w} \cdot \mathbf{u} = a + b + 2c$

For $\mathbf{w}$ and $\mathbf{v}$ to be perpendicular, we need $0 = \mathbf{w} \cdot \mathbf{v} = -a + b + c$

Solve the system $\begin{cases} 1. \quad 0 = a + b + 2c \\ 2. \quad 0 = -a + b + c \end{cases}$

Usually, with three unknowns, but only two equations, there are infinitely many non-zero solutions.

Equation 2 says $a = b + c$

Plug into equation 1 to get
$0 = (b + c) + b + 2c = 2b + 3c$

So $b = -\frac{3}{2}c$ and $a = -\frac{1}{2}c$

$$\mathbf{w} = < a, b, c > = \left\langle -\frac{1}{2}c, -\frac{3}{2}c, c \right\rangle = c\left\langle -\frac{1}{2}, -\frac{3}{2}, 1 \right\rangle$$

The vector $\left\langle -\frac{1}{2}, -\frac{3}{2}, 1 \right\rangle$ is a solution to our problem.

so is any multiple of this vector, such as $< -1, -3, 2 >$.

The cross product, pg 3.

More generally, given nonzero $\mathbf{u}$ and $\mathbf{v}$

$$\mathbf{u} = < u_1, u_2, u_3 >$$

$$\mathbf{v} = < v_1, v_2, v_3 >$$

We want $\mathbf{w} = < a, b, c >$ so that both …

$$0 = \mathbf{w} \cdot \mathbf{u} = au_1 + bu_2 + cu_3$$

$$0 = \mathbf{w} \cdot \mathbf{v} = av_1 + bv_2 + cv_3$$

We solve the following system

$$\begin{cases} 1. \quad 0 = au_1 + bu_2 + cu_3 \\ 2. \quad 0 = av_1 + bv_2 + cv_3 \end{cases}$$

Using 2. we get $a = \dfrac{-bv_2 - cv_3}{v_1}$

Then in 1. we get…

$$0 = \frac{-bu_1v_2 - cu_1v_3}{v_1} + \frac{bu_2v_1 + cu_3v_1}{v_1}$$

$$= \frac{b(u_2v_1 - u_1v_2) + c(u_3v_1 - u_1v_3)}{v_1}$$

We get $b(u_1v_2 - u_2v_1) = c(u_3v_1 - u_1v_3)$

There are many solutions…

Take $b = u_3v_1 - u_1v_3$ and $c = u_1v_2 - u_2v_1$

Then $a = \ldots$

$$a = \frac{-(u_3v_1 - u_1v_3)v_2 - (u_1v_2 - u_2v_1)v_3}{v_1}$$

$$= \frac{-u_3v_1v_2 + u_2v_1v_3}{v_1} = u_2v_3 - u_3v_2$$

$$\mathbf{w} = < u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1 >$$

here we have got the *cross product*, $\mathbf{w} = \mathbf{u} \times \mathbf{v}$

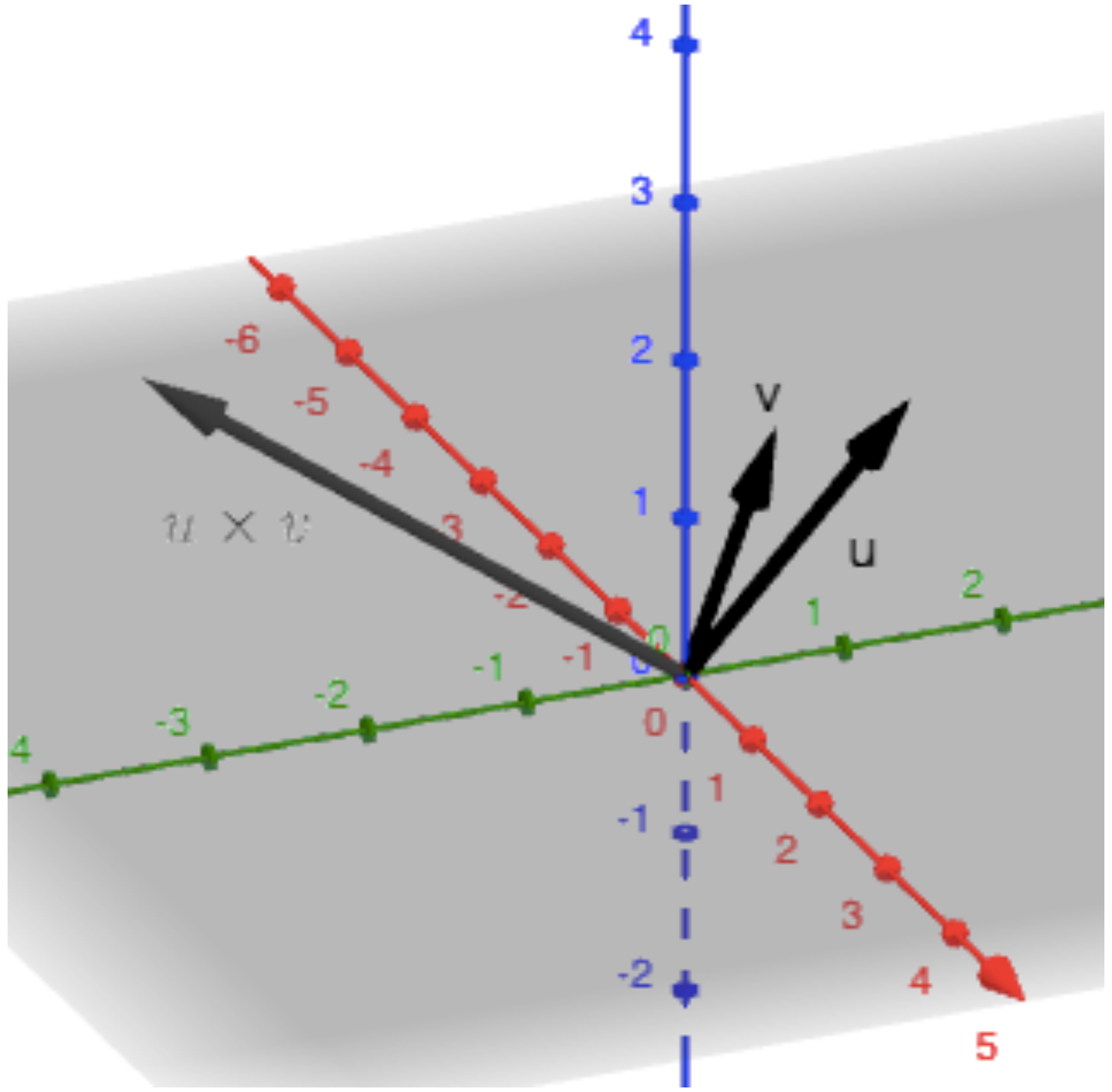
Cross Product Examples. $\mathbf{u} \times \mathbf{v} = < u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1 >$

Example 1.

$\mathbf{u} = < 1,1,2 >$

$\mathbf{v} = < -1,1,1 >$

$\mathbf{u} \times \mathbf{v} = < -1, -3,2 >$



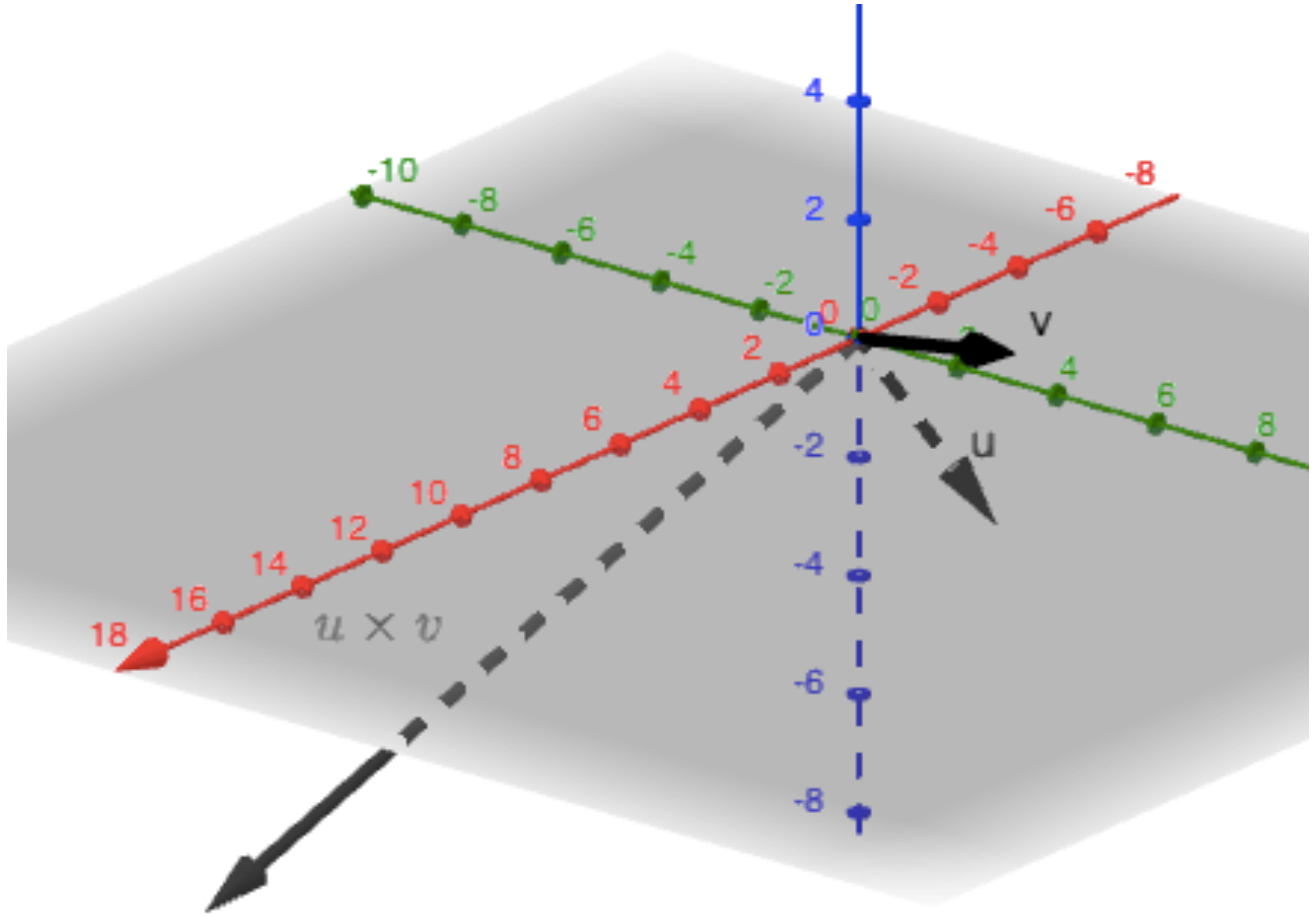
Example 2.

$\mathbf{u} = < -1,2, -3 >$

$\mathbf{v} = < 1,4,1 >$

$\mathbf{u} \times \mathbf{v} = < 14, -2, -6 >$

$\mathbf{v} \times \mathbf{u} = < -14,2,6 >$



Remember the cross product should be perpendicular to each of the vector factors.

Check perpendicularity using the dot product:

$< -1, -3,2 > \cdot < 1,1,2 > = ?$

$= 0$ check!

$< -1, -3,2 > \cdot < -1,1,1 > = ?$

$= 0$ check!

$< 14, -2, -6 > \cdot < -1,2, -3 > = ?$

$= 0$ check!

$< 14, -2, -6 > \cdot < 1,4,1 > = ?$

$= 0$ check!