M110C Week5

Goals:

-Recap.

-Multivariable Calculus.

  Functions of Two Variables.

  Contour Maps.

  Limits.

  Continuity.

  Partial Derivatives.

A Broad Recap.

What wonderful things have we seen in our course so far?

– Vectors.

- Vector Operations.

  Arithmetic Operations.

  Addition, Subtraction, Scalar Multiplication.

  Dot Product.

  Cross Product.

- Applications.

  Lines.

  Planes.

  Distances.

– Vector-Valued Functions.

- Planar Curves.
- Curves in Space.

  Parameterization Dependent:

  Velocity, Acceleration.

  Parameterization Independent:

  Arc-Length.

  $\mathbf{T}, \mathbf{N}, \mathbf{B}, \kappa, \tau$.

- A famous differential equation:

$$\begin{pmatrix} \mathbf{T}'(s) \\ \mathbf{N}'(s) \\ \mathbf{B}'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}(s) \end{pmatrix}$$

- Kepler's Laws (we didn't see these.)

How can we explore these paths further?

Here are some references:

For more on curves, try "Elements of Differential Geometry" Millman, Parker

For more on differential Equations, try "Elementary Differential Equations" Boyce, DiPrima or "Elementary Differential Equations" Trench

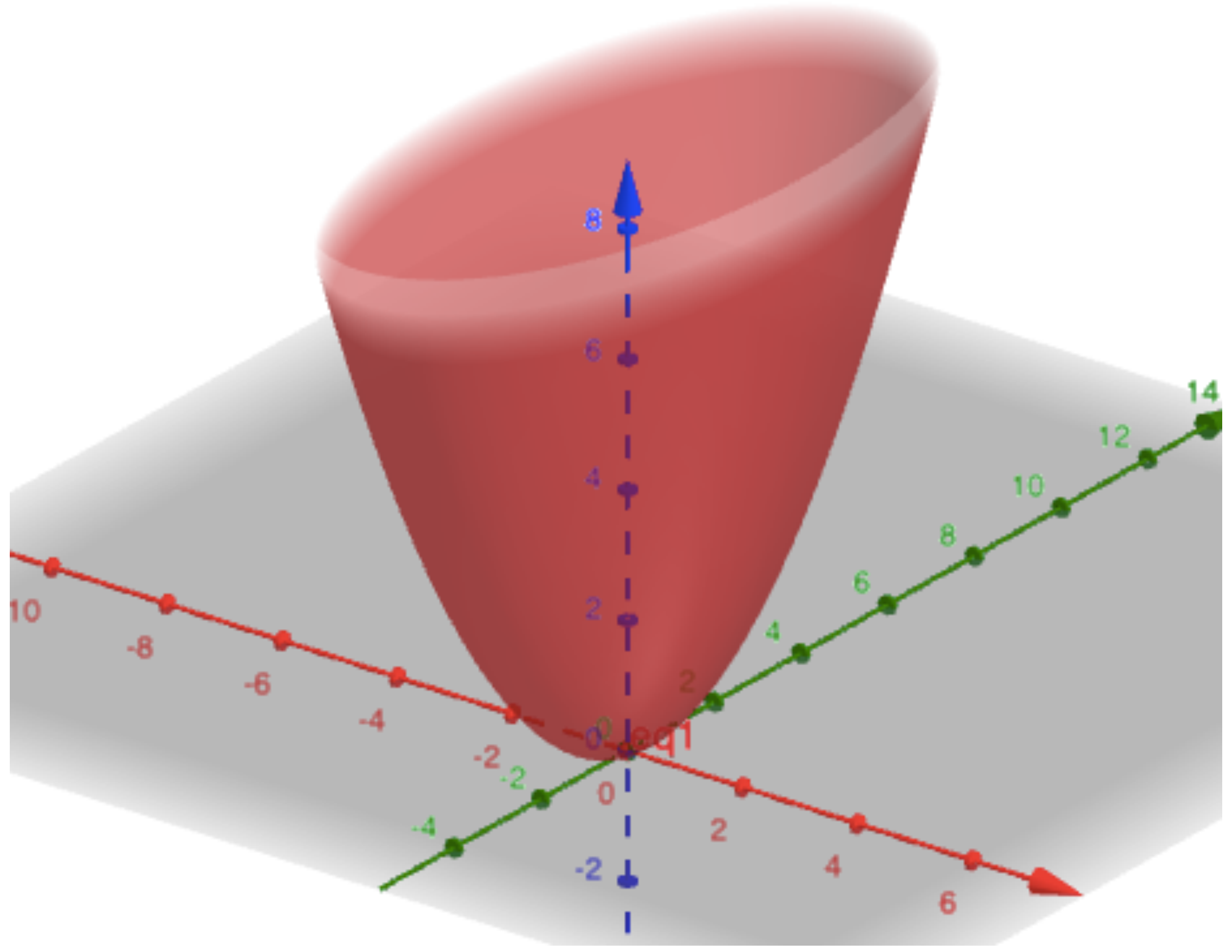
Where are we going now?

Multivariable Functions!

Multivariable Derivatives!

Multivariable Integrals!

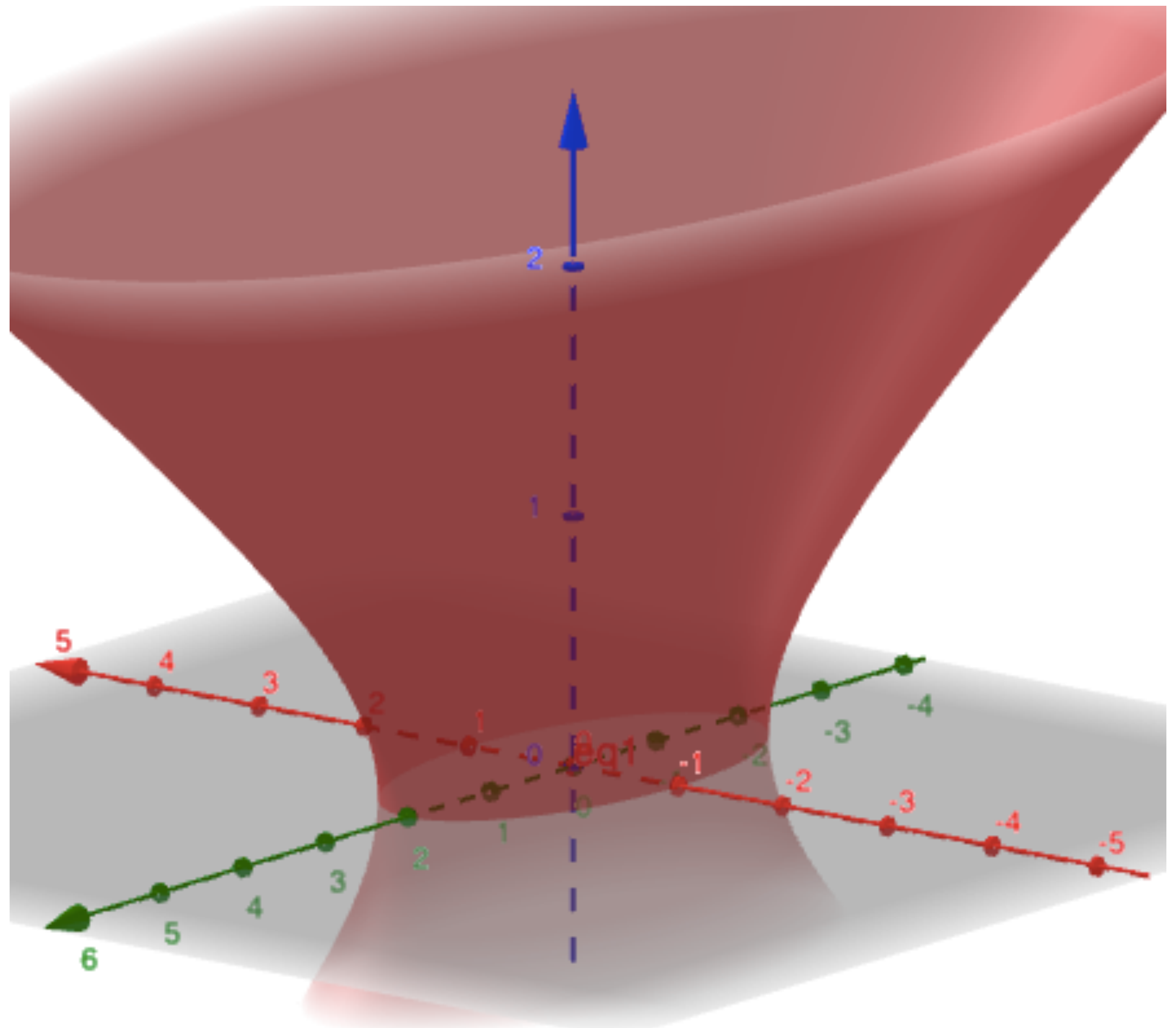
Quadric Surfaces, reminder.



Many of these can be described as a function of two variables, with $f(x,y) = z$

1. $\dfrac{z}{c} = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$

1. $f(x,y) = \dfrac{cx^2}{a^2} + \dfrac{cy^2}{b^2}$

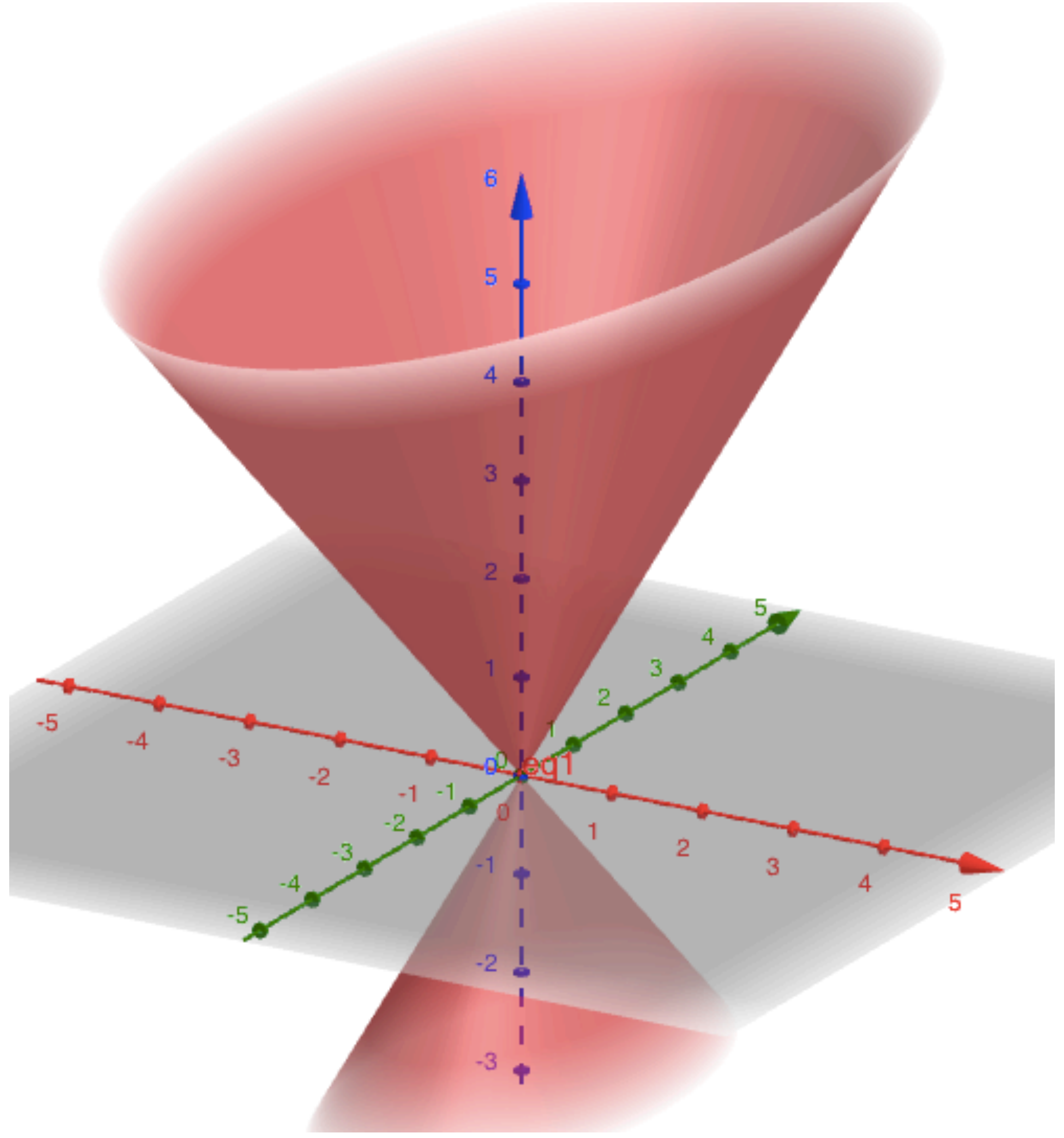


2. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$

2. $z^2 = c^2\left(\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1\right)$

$$f_+(x,y) = c\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1}$$

$$f_-(x,y) = -c\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1}$$

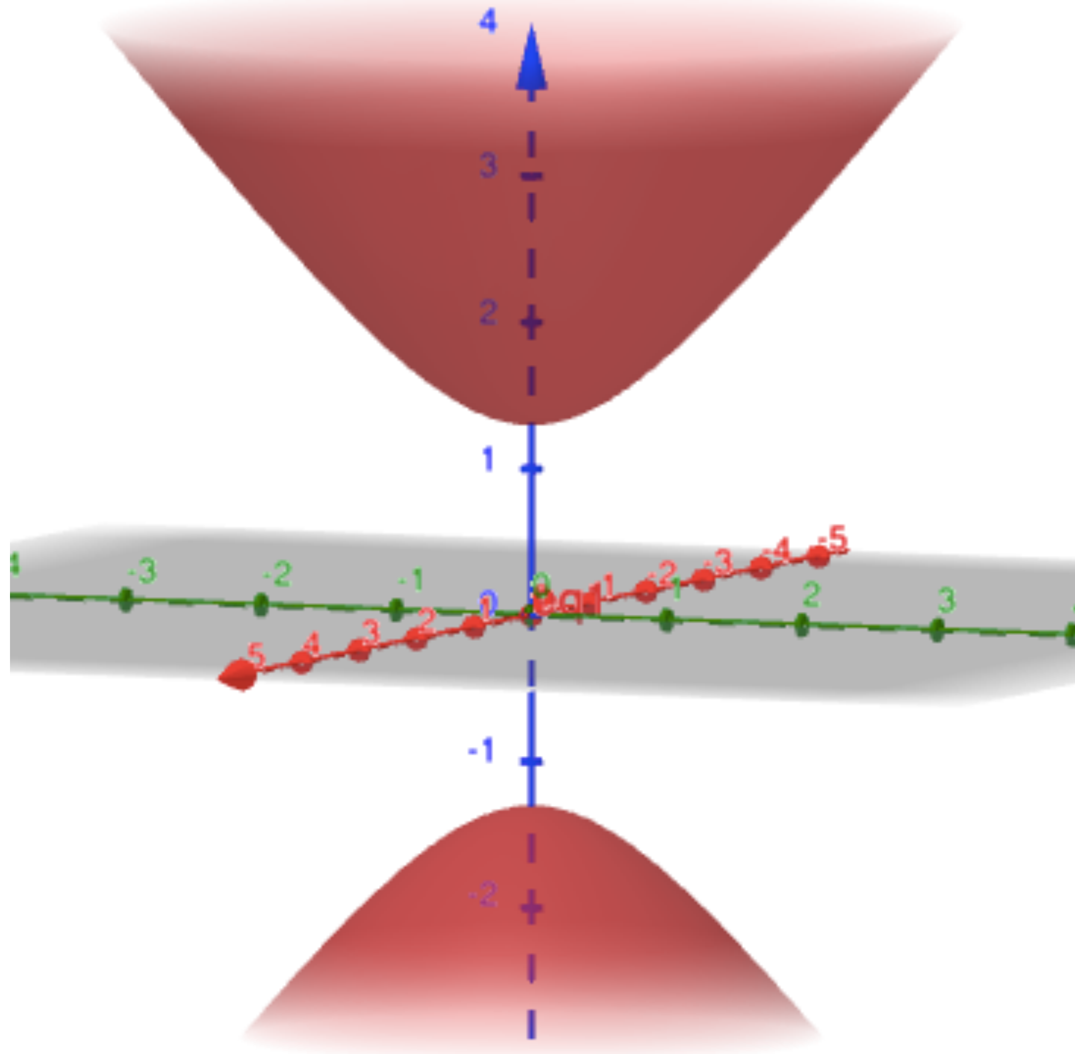


3. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 0$

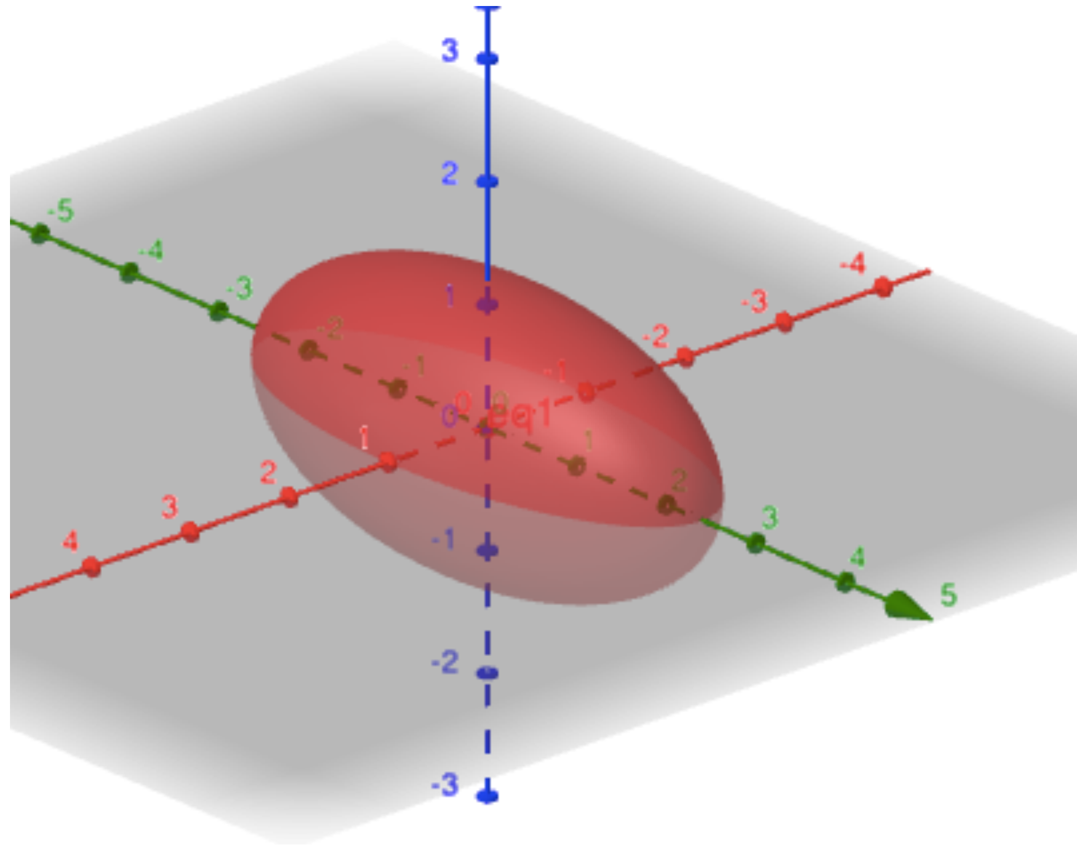
3. $z^2 = c^2\left(\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}\right)$

$$f_+(x,y) = c\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2}}$$

$$f_-(x,y) = -c\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2}}$$



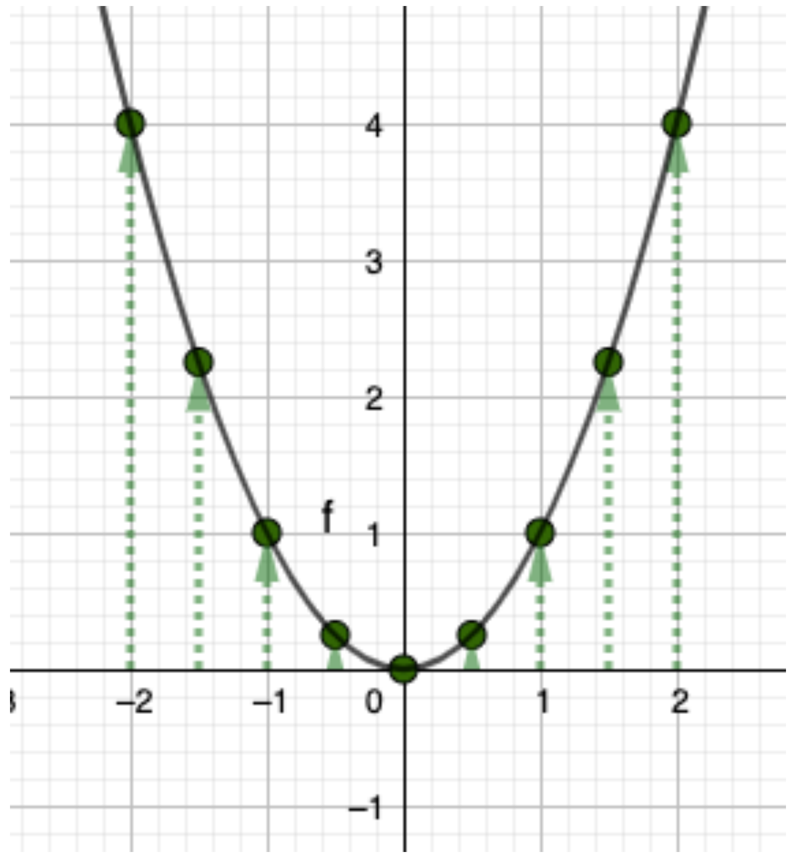
4. $-\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$



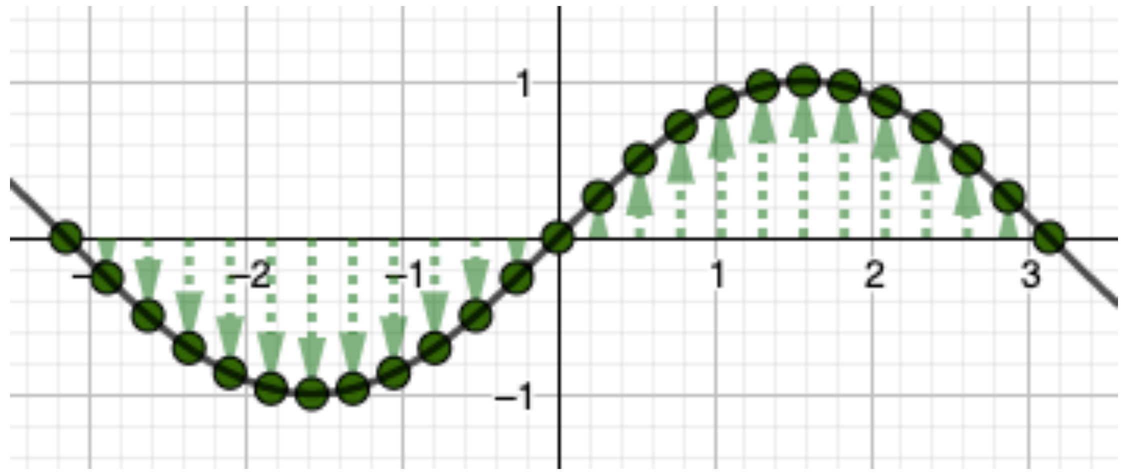
5. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

Multivariable Calculus. Functions of Two Variables.

We know functions of a single variable.



$$f(x) = x^2$$
$$f(0) = 0$$
$$f(1) = 1$$
$$f(2) = 4$$



$$g(x) = \sin(x)$$
$$g(0) = 0$$
$$g(\pi/6) = 0.5$$
$$g(\pi/2) = 1$$

Now we will study real-valued functions whose domain is in $\mathbf{R}^2$.

$$\mathbf{R}^2 = \{(x, y) : x \in \mathbf{R}, y \in \mathbf{R}\}$$

The domain will lie on the xy plane.

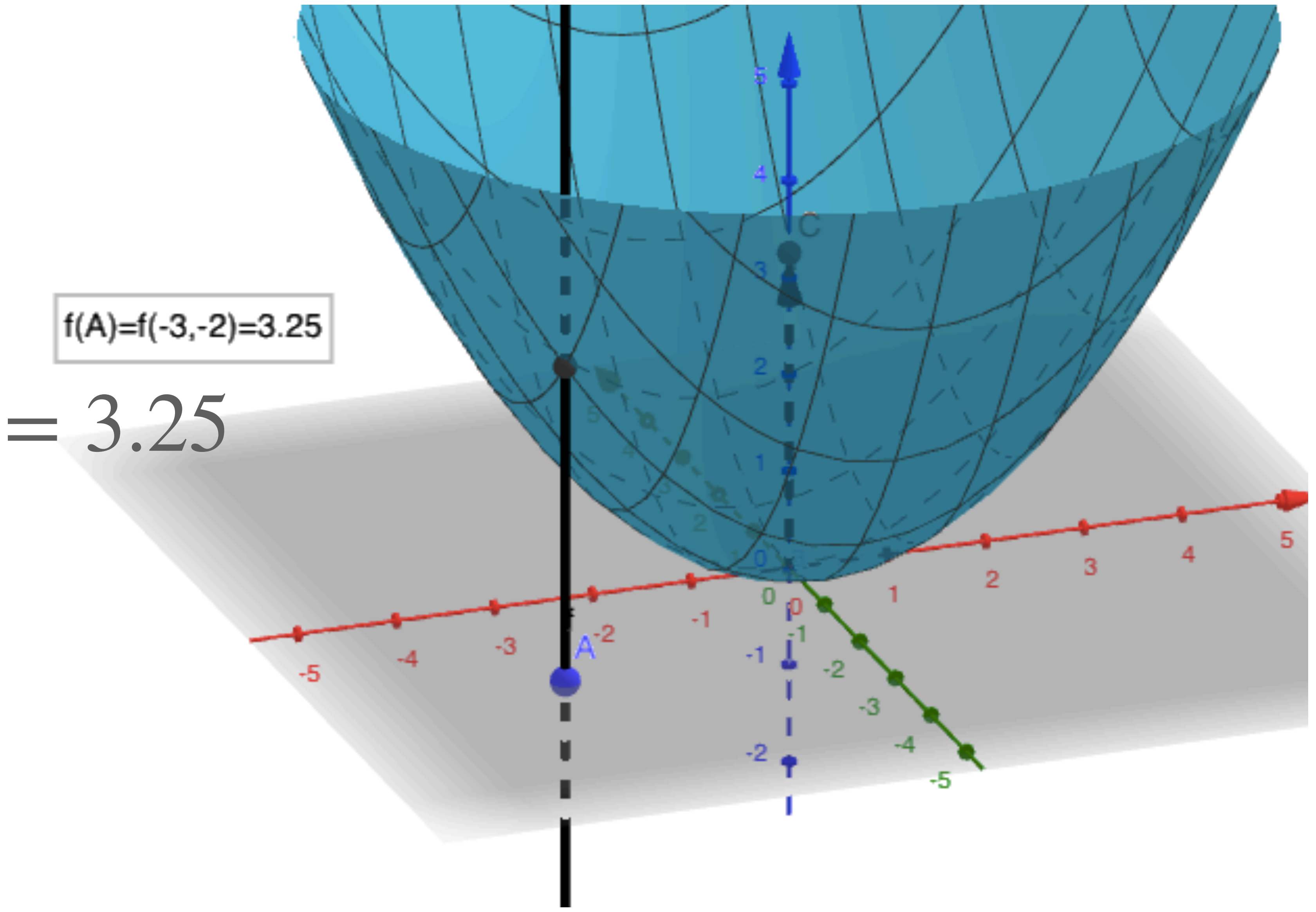
The values of the function, elements of the range, will be measured by height on the z axis.

Example. $f(x, y) = \dfrac{x^2 + y^2}{4}$

$$f(-3, -2) = \frac{(-3)^2 + (-2)^2}{4} = 3.25$$

$$f(-1,3) = \frac{(-1)^2 + 3^2}{4} = 2.5$$

$$f(2,2) = \frac{2^2 + 2^2}{4} = 2$$



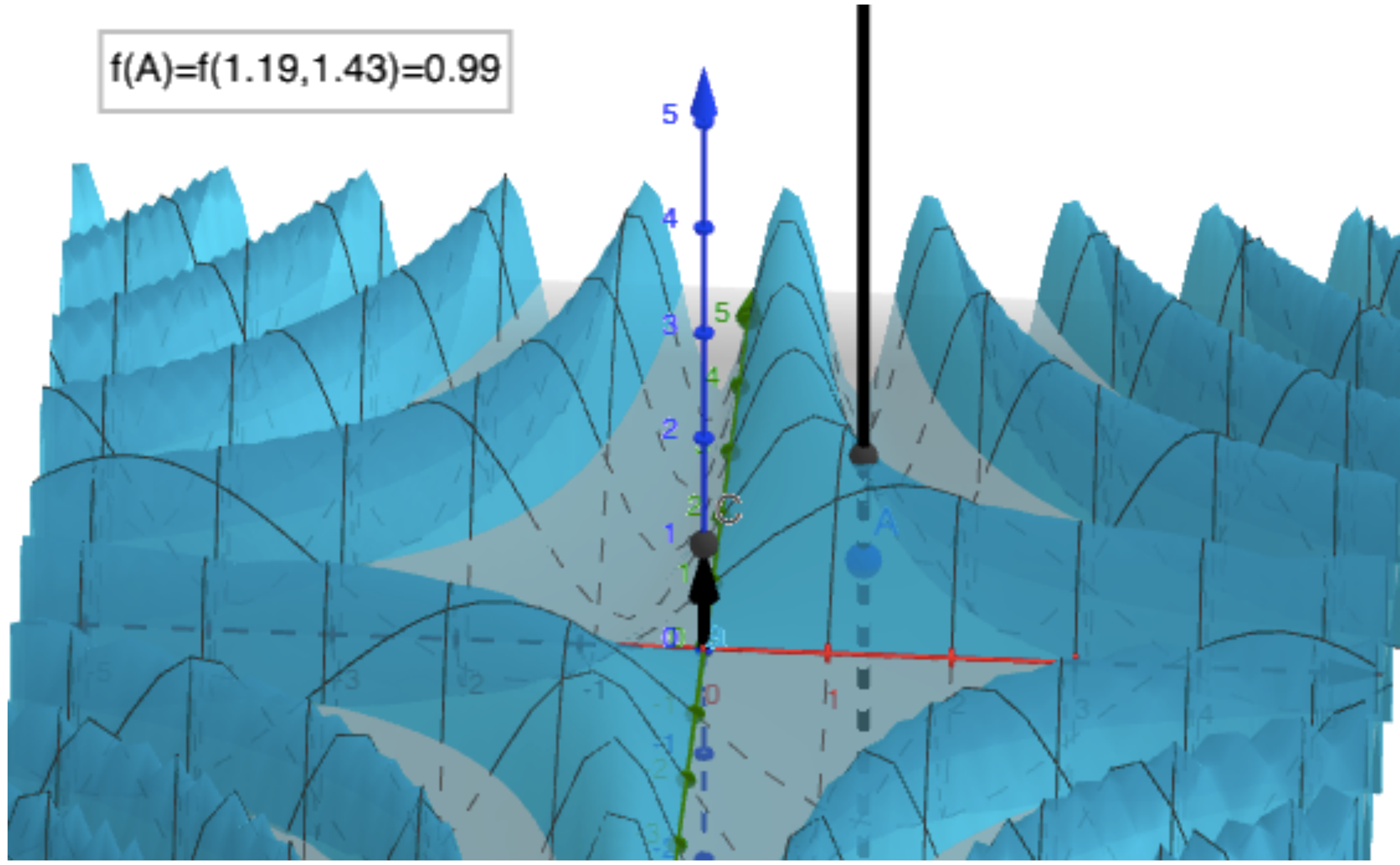
Example.

$$g(x, y) = \sin(xy)$$

$$g(\pi/2,1) = \sin(\pi/2) = 1$$

$$g(\pi/2,2) = \sin(\pi) = 0$$

$$g(\pi/2,3) = \sin(3\pi/2) = -1$$

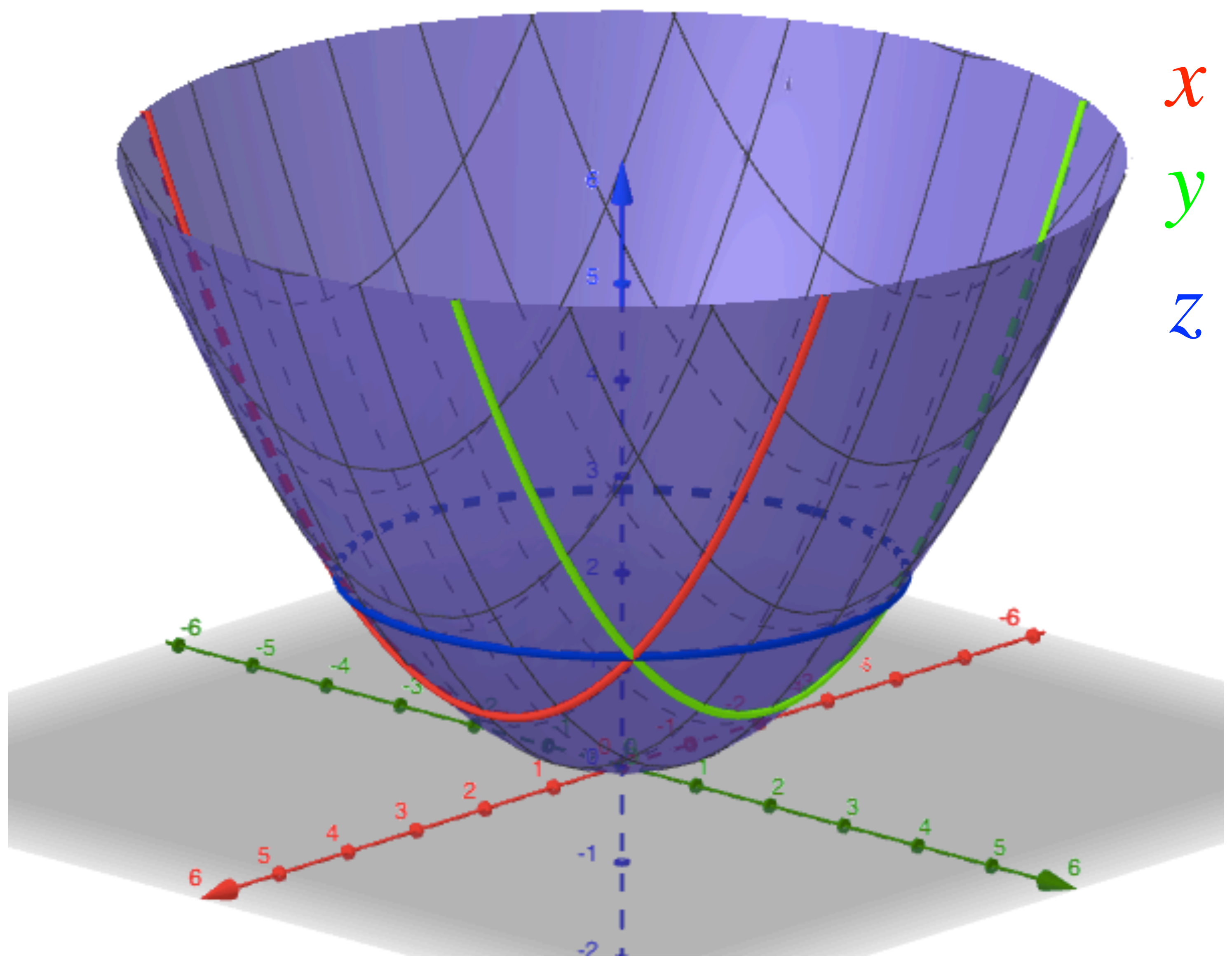


Link: Multivariable Input

Functions of Two Variables, pg 2. Traces.

As we did with quadric surfaces, you can graph *traces* of a multivariable function along any axis.

$$f(x,y) = \frac{x^2+y^2}{4}$$



Here are three traces

$x = 2$

$y = 2$

$z = 2$

$$x = 2: \qquad z = f(2,y) = \frac{4+y^2}{4} = 1 + \frac{y^2}{4}$$

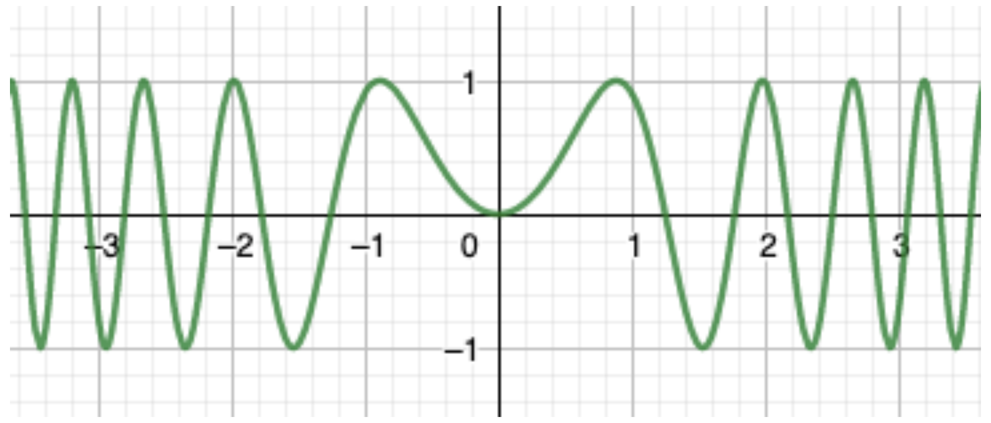
$$y = 2: \qquad z = f(x,2) = \frac{x^2+4}{4} = \frac{x^2}{4} + 1$$

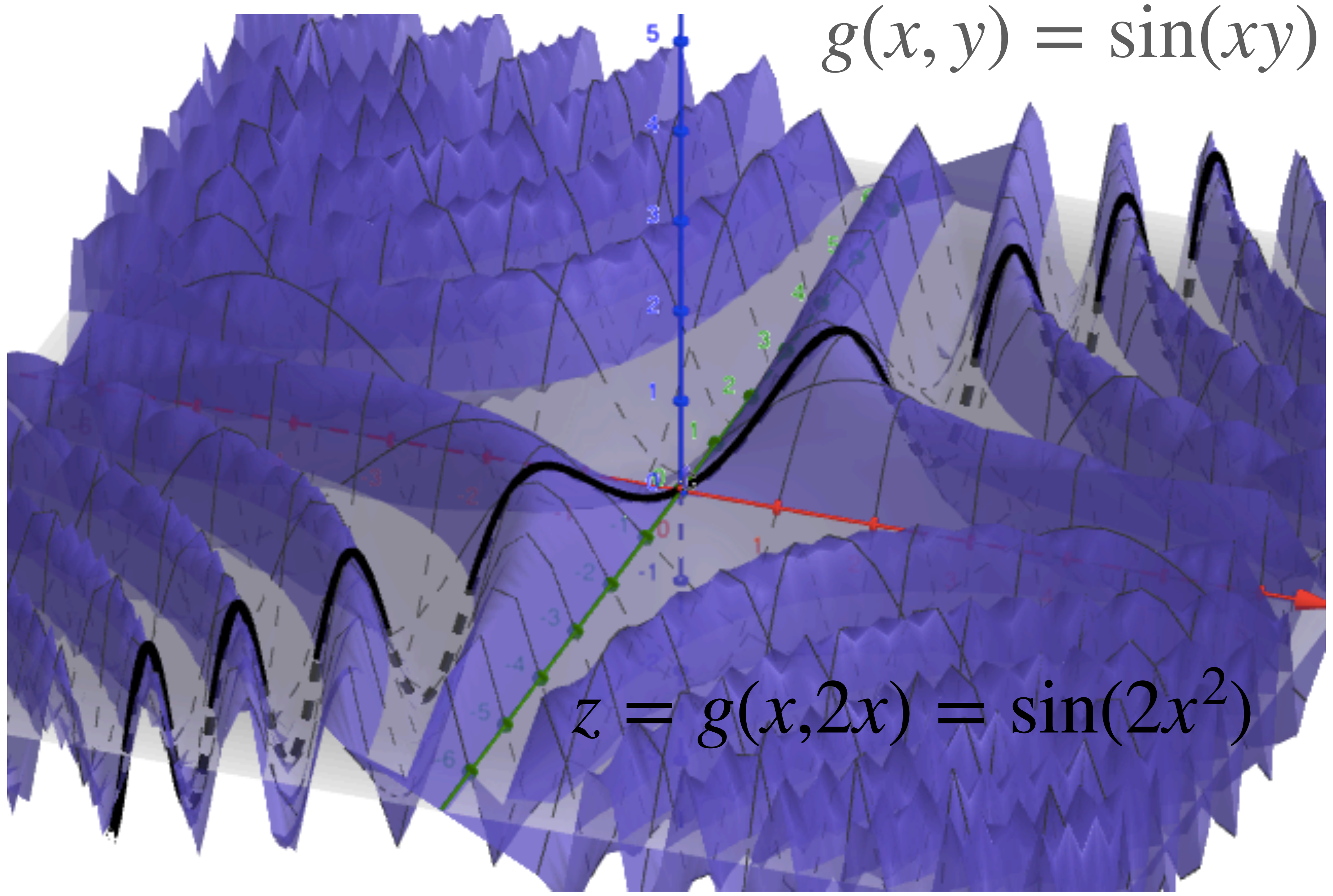
$$z = 2: \qquad 2 = \frac{x^2+y^2}{4}, \quad x^2+y^2 = 8$$

You can get very creative with tracing.

$g(x,y) = \sin(xy)$

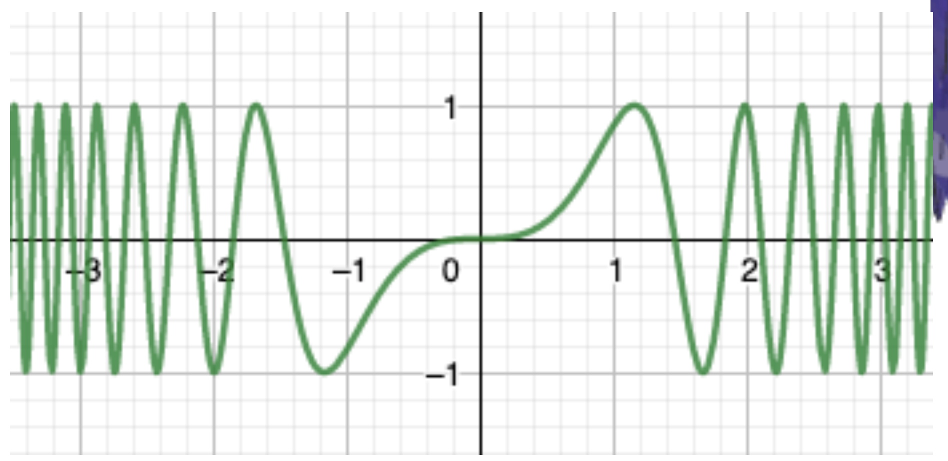
A trace along the line $y = 2x$

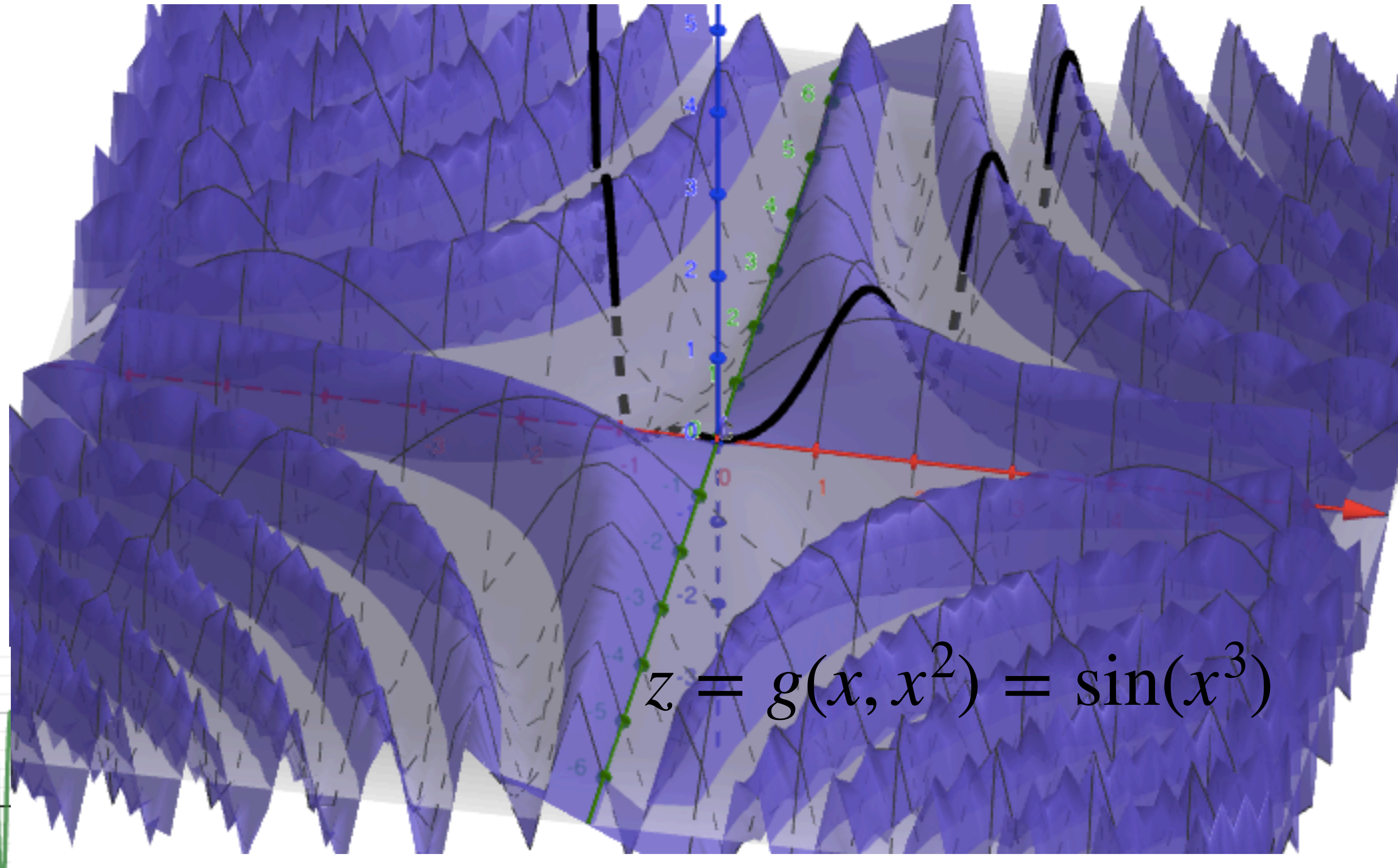




$z = g(x,2x) = \sin(2x^2)$

A trace along the parabola $y = x^2$





$z = g(x,x^2) = \sin(x^3)$

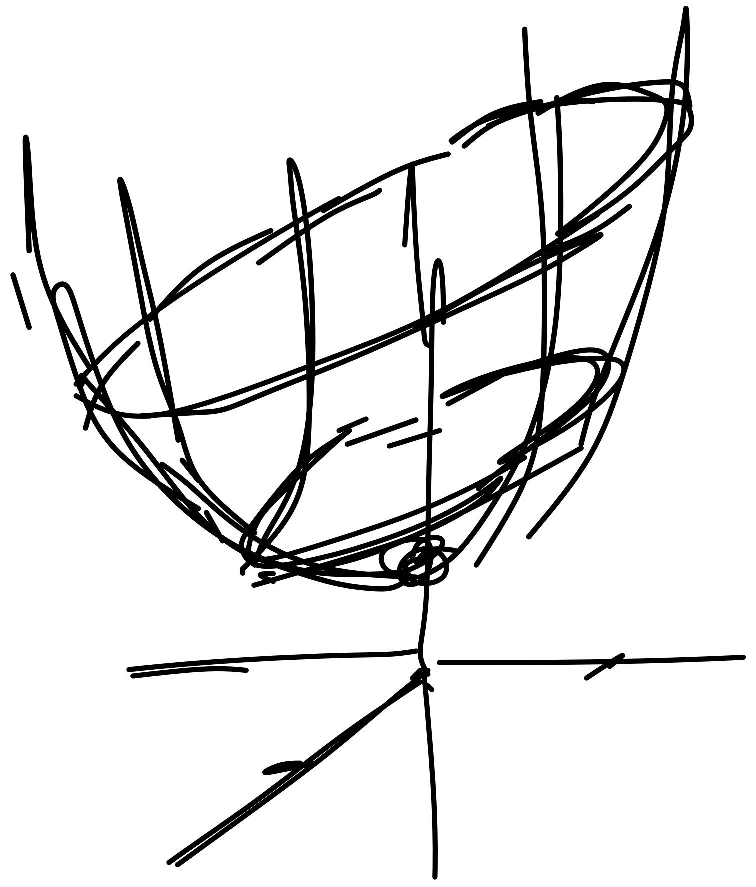
Link: MultiVariableTraces

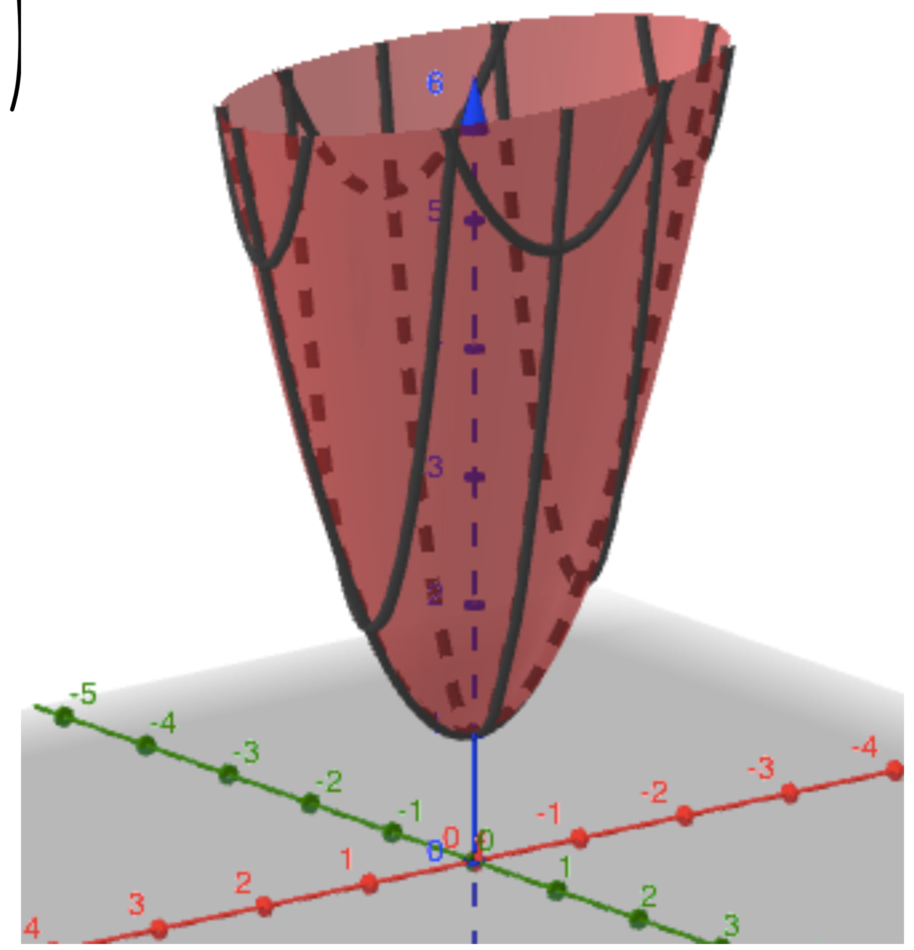
Functions of Two Variables, pg 3. Practice.

Sketch some traces e.g. $x = \pm 2, \pm 1, 0$; $y = \pm 2, \pm 1, 0$; $z = \pm 2, \pm 1, 0$; $y = kx$

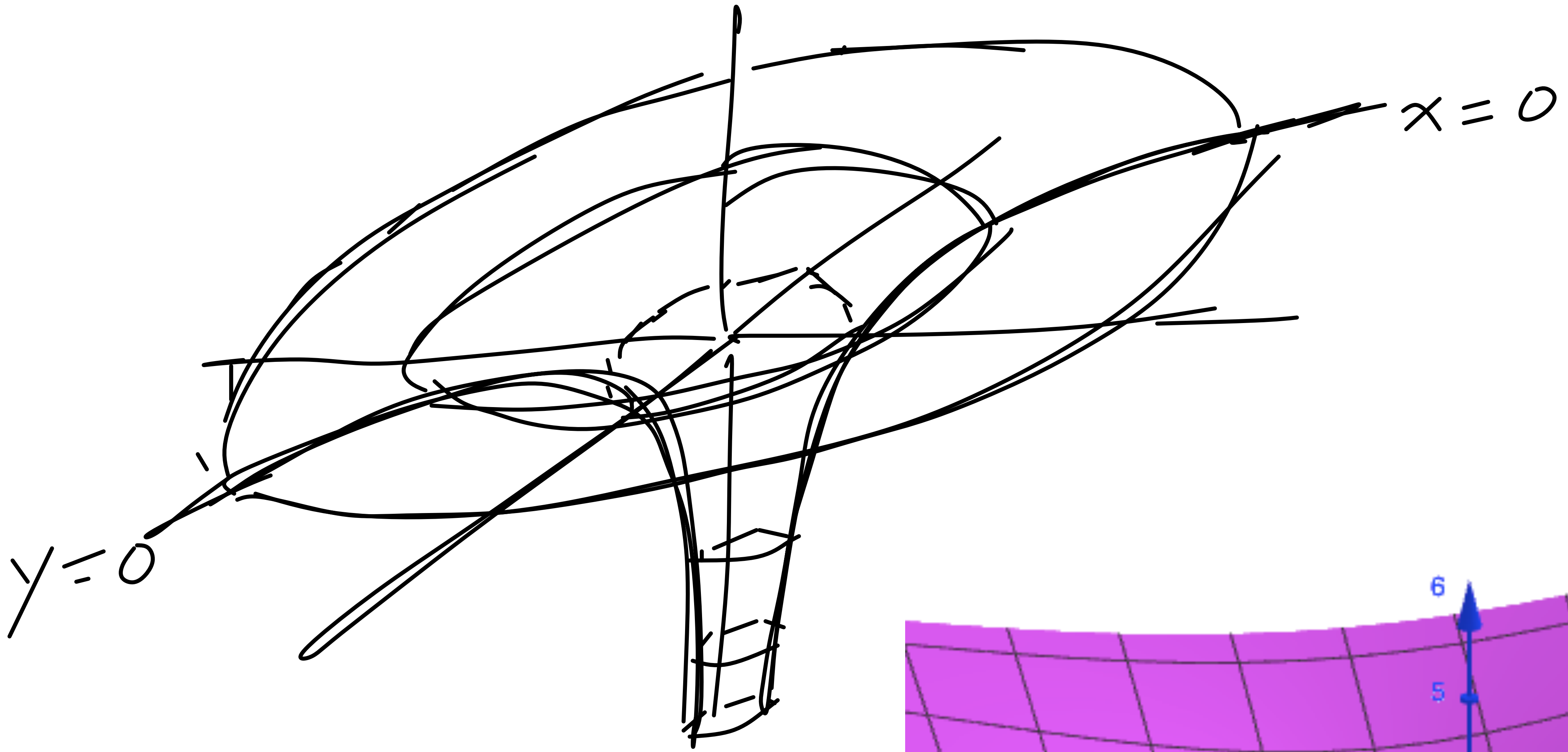
Try to sketch a graph of $f(x, y)$.

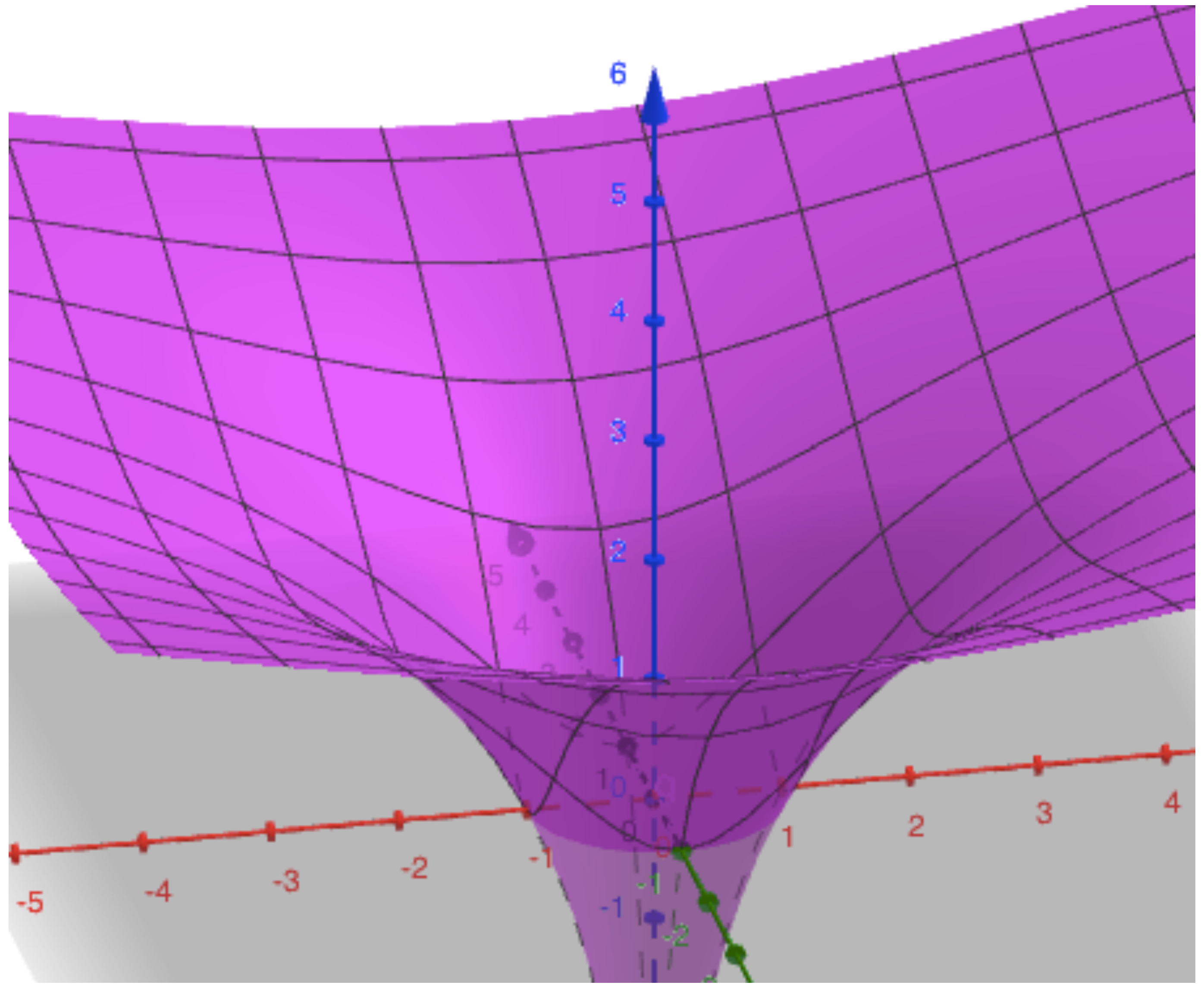
1.Sec 14.1 #29 $f(x, y) = x^2 + 4y^2 + 1$





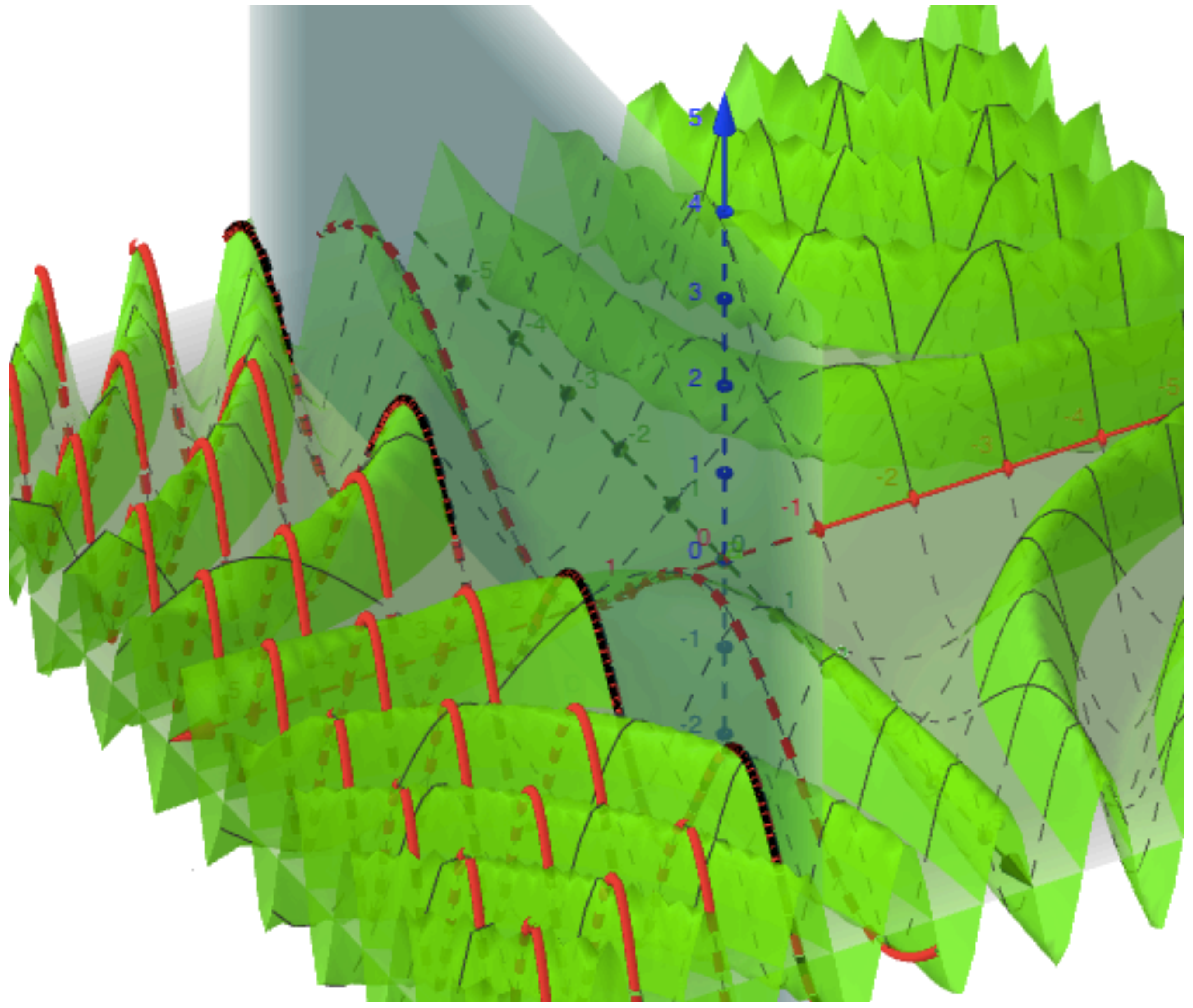
2.Sec14.1 #32c $f(x, y) = \ln(x^2 + y^2)$





Traces, Level Curves.

We have seen traces of a function z=f(x,y) by plugging in specific values of x,y, or z. The resulting curve is a trace of the function.

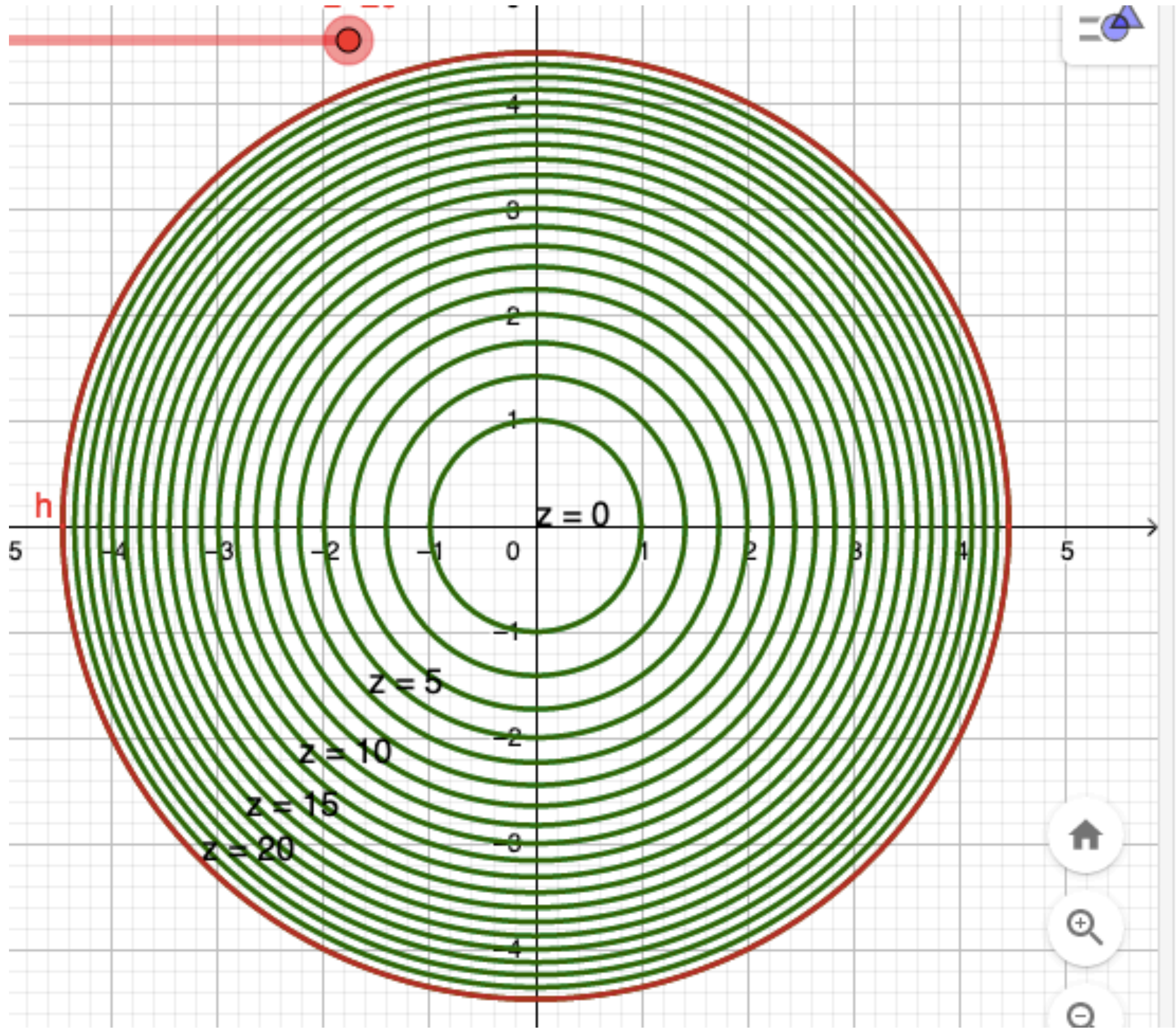


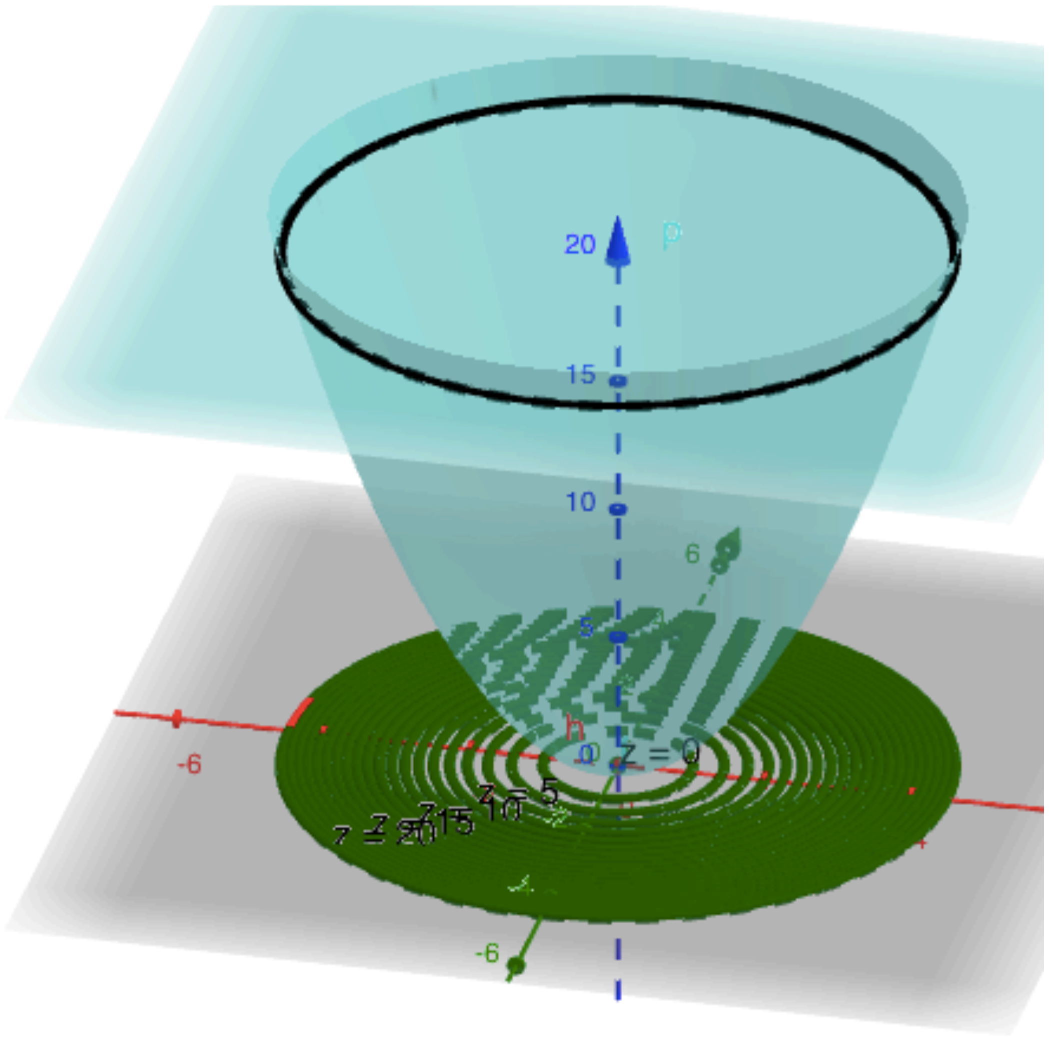
Example. $f(x, y) = \sin(xy)$

A trace along the x axis at $x = k$ results in a sine curve $f(k, y) = \sin(ky)$, or $z = \sin(ky)$.

Level Curves of a function $f(x, y)$ are traces along the z axis.

These are curves where the output value of the function, z, is fixed .





When placed on a single plane (the xy plane), we get an idea of how the function looks. The result is sometimes called a *contour map* of f.

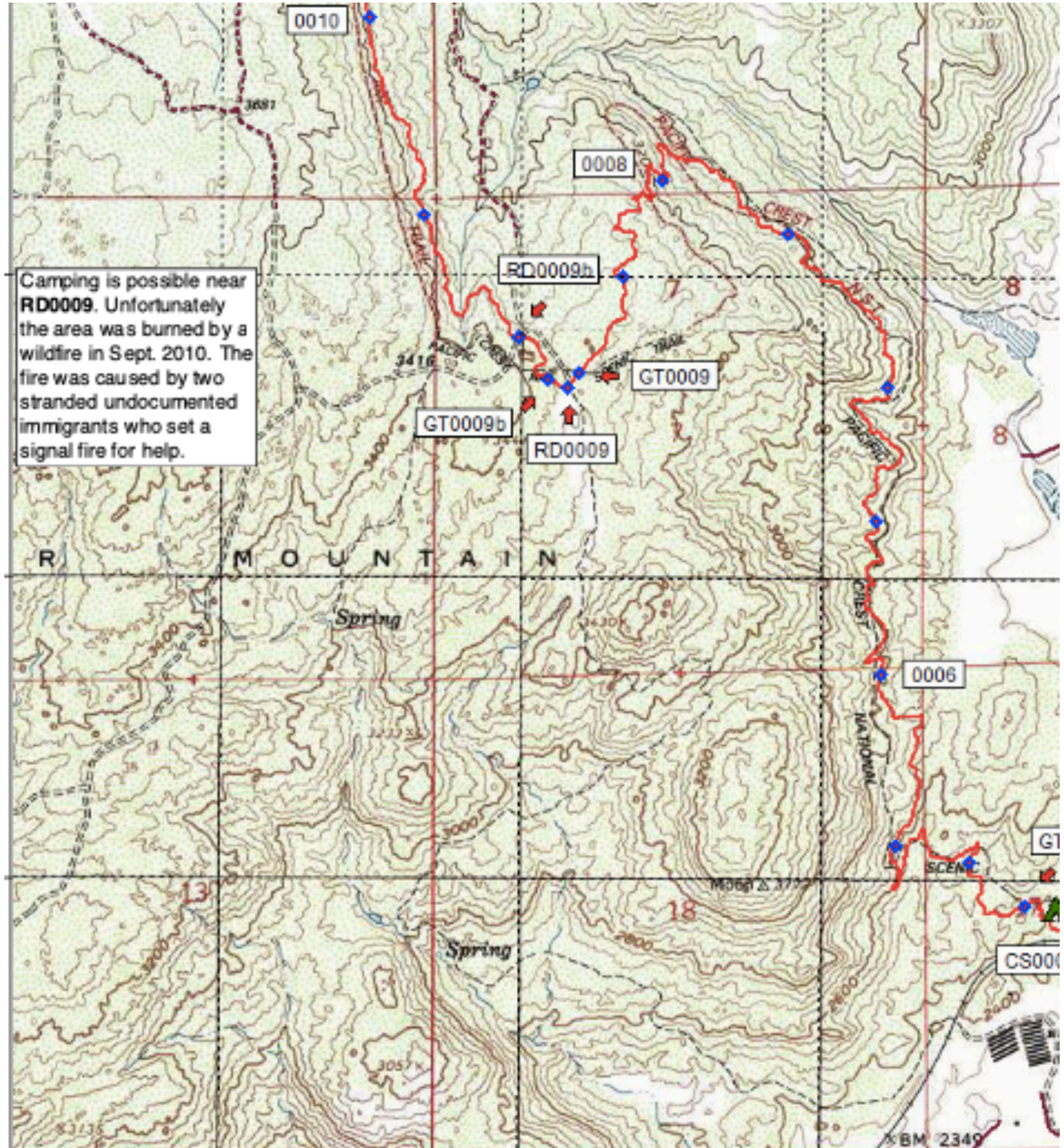
Example. $z = f(x, y) = x^2 + y^2$

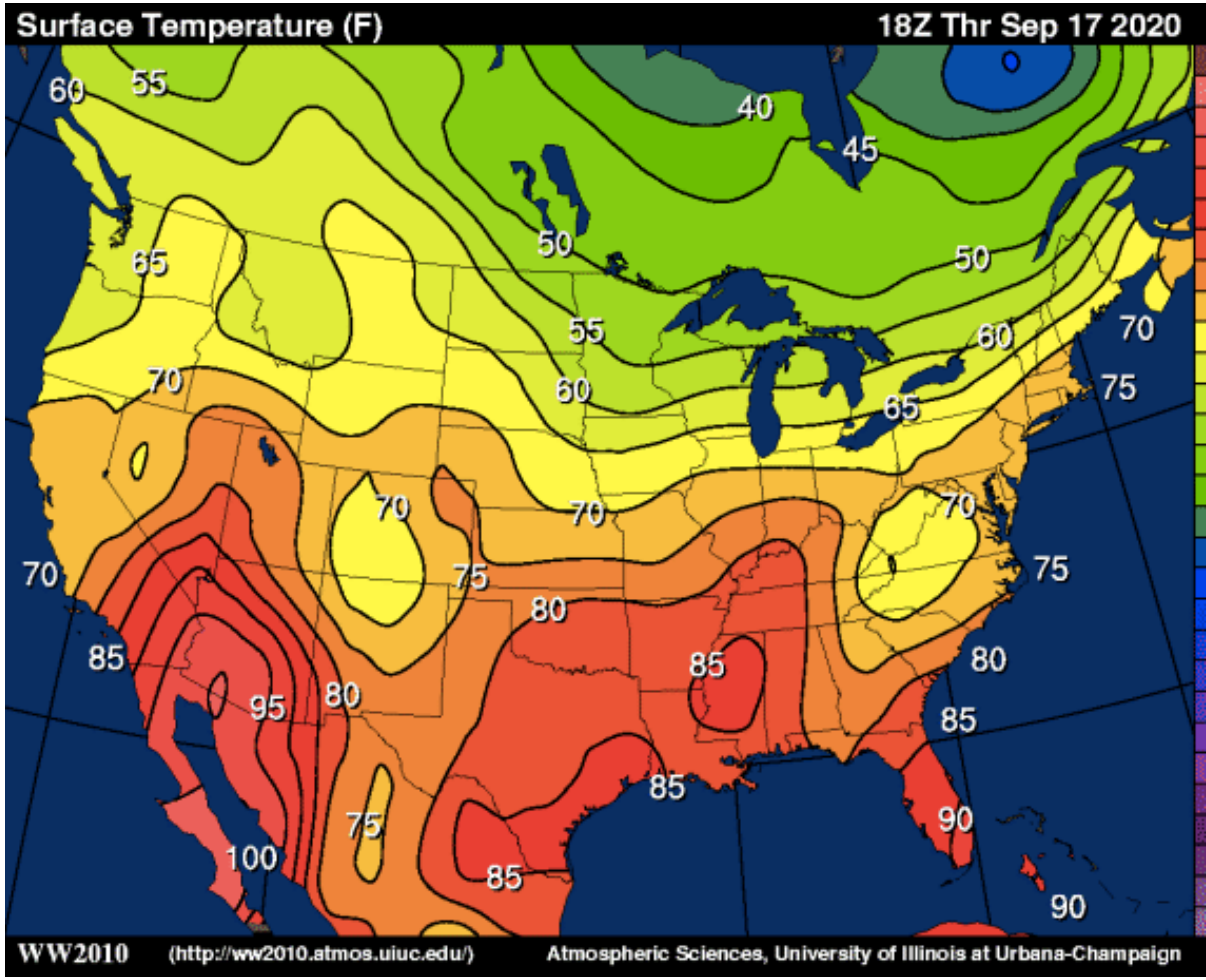
Level curves are circles, $k = x^2 + y^2$

Link: ContourMap Paraboloid

Contour Maps for real!

Contour Maps are not uncommon.

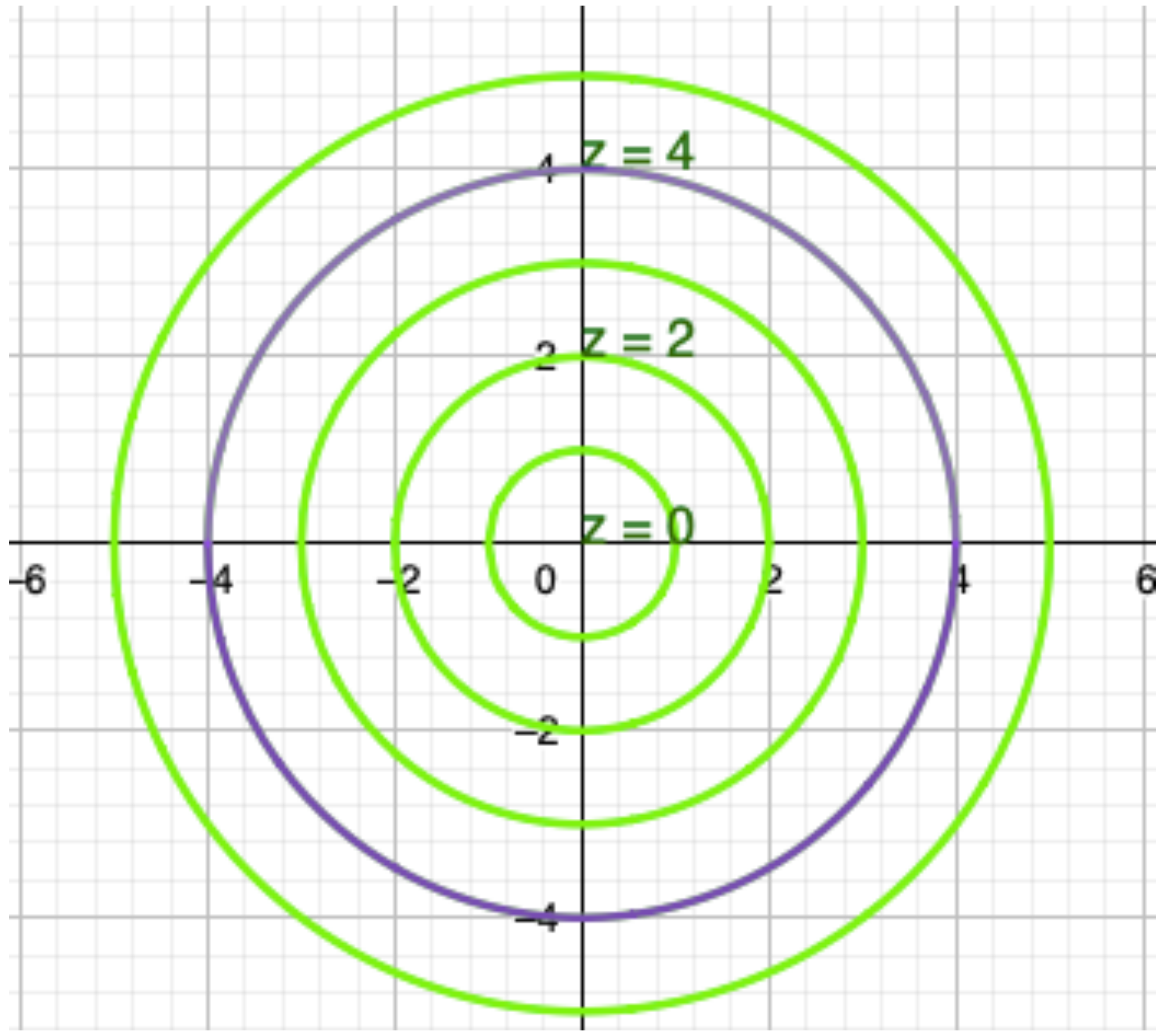


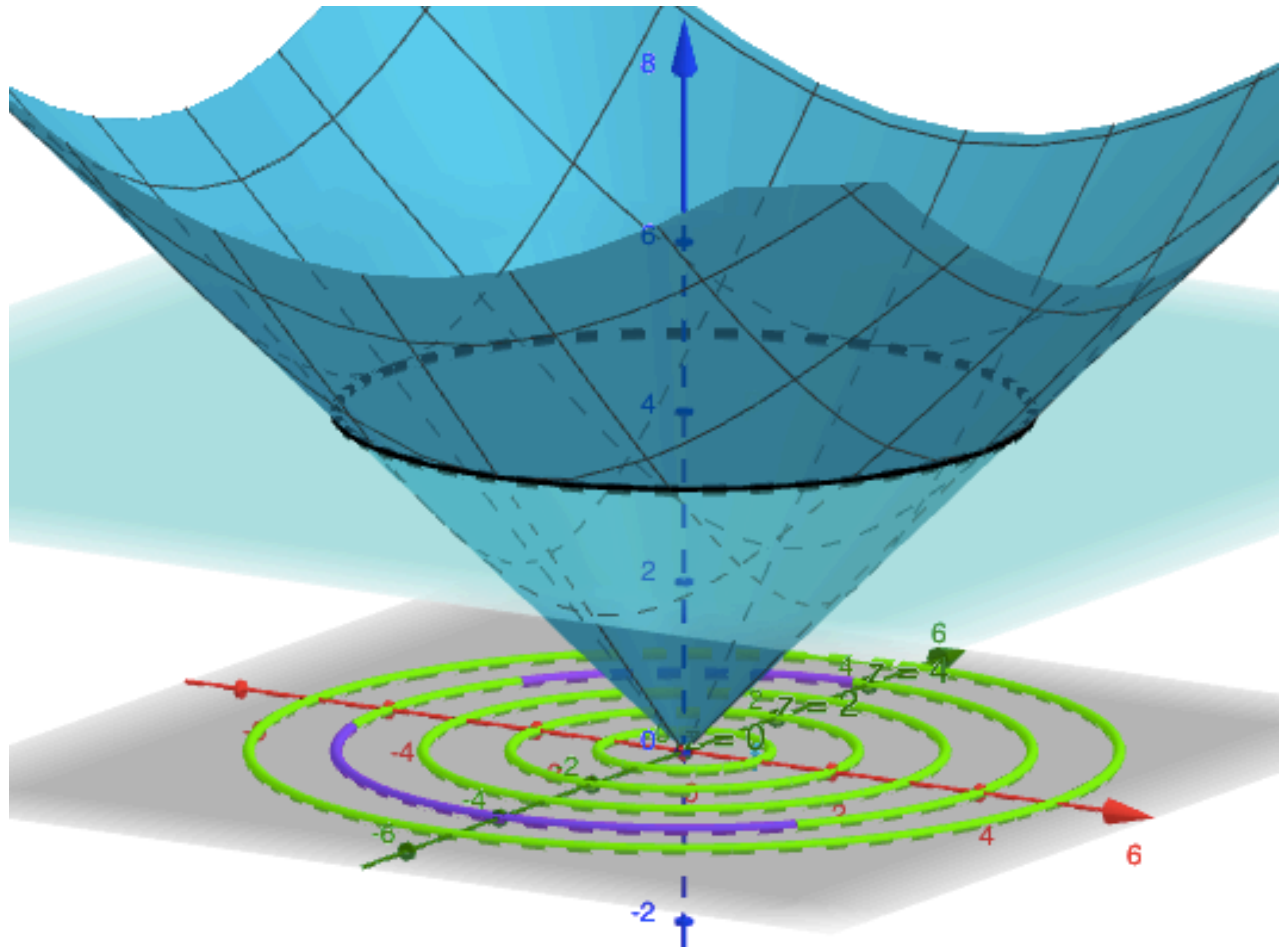


Source: http://ww2010.atmos.uiuc.edu/(Gh)/guides/maps/sfc/temp/sfctmp.rxml

Traces, Level Curves, pg 2.

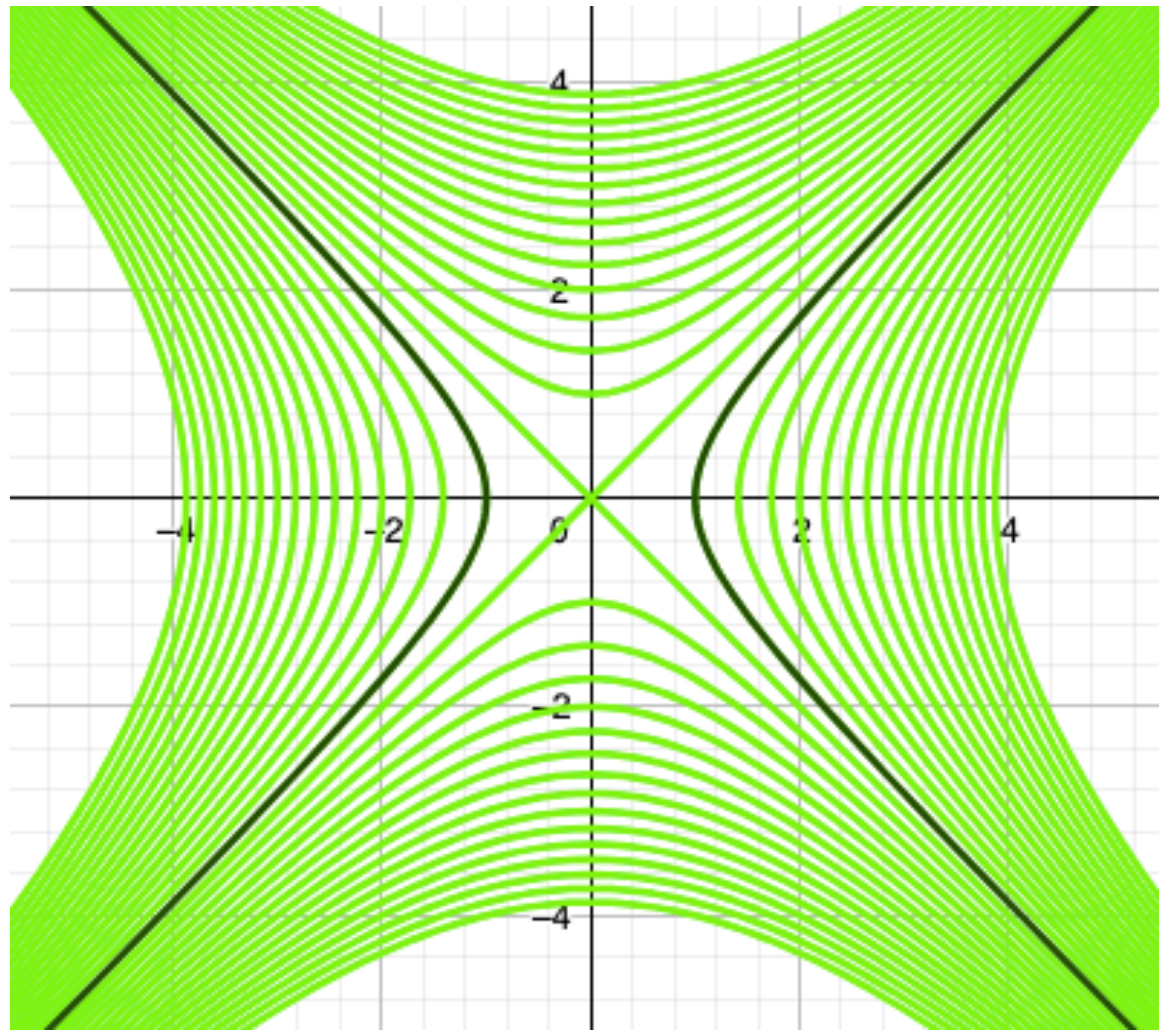
What function has this kind of contour map?

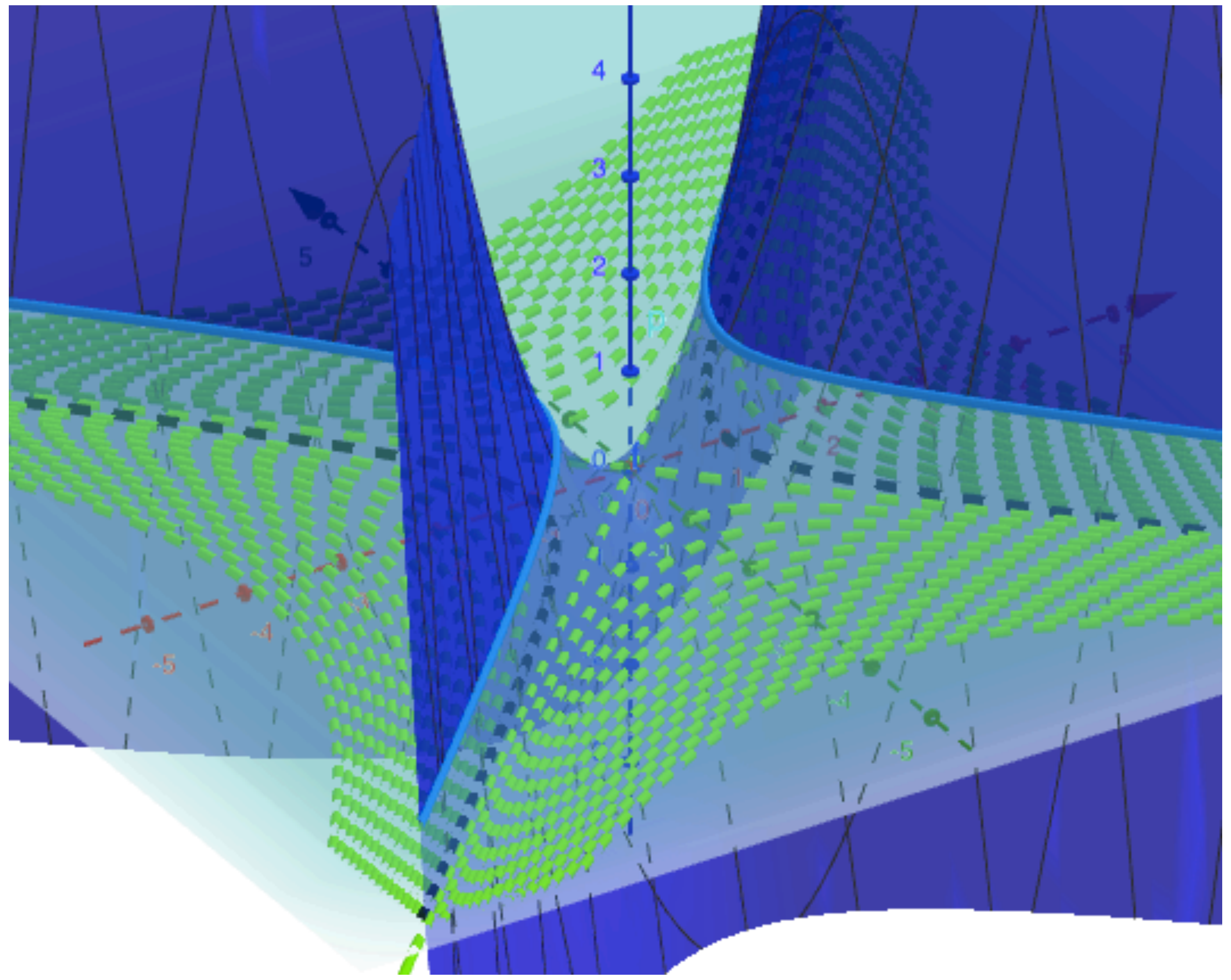




$$z = f(x, y) = \sqrt{x^2 + y^2}$$

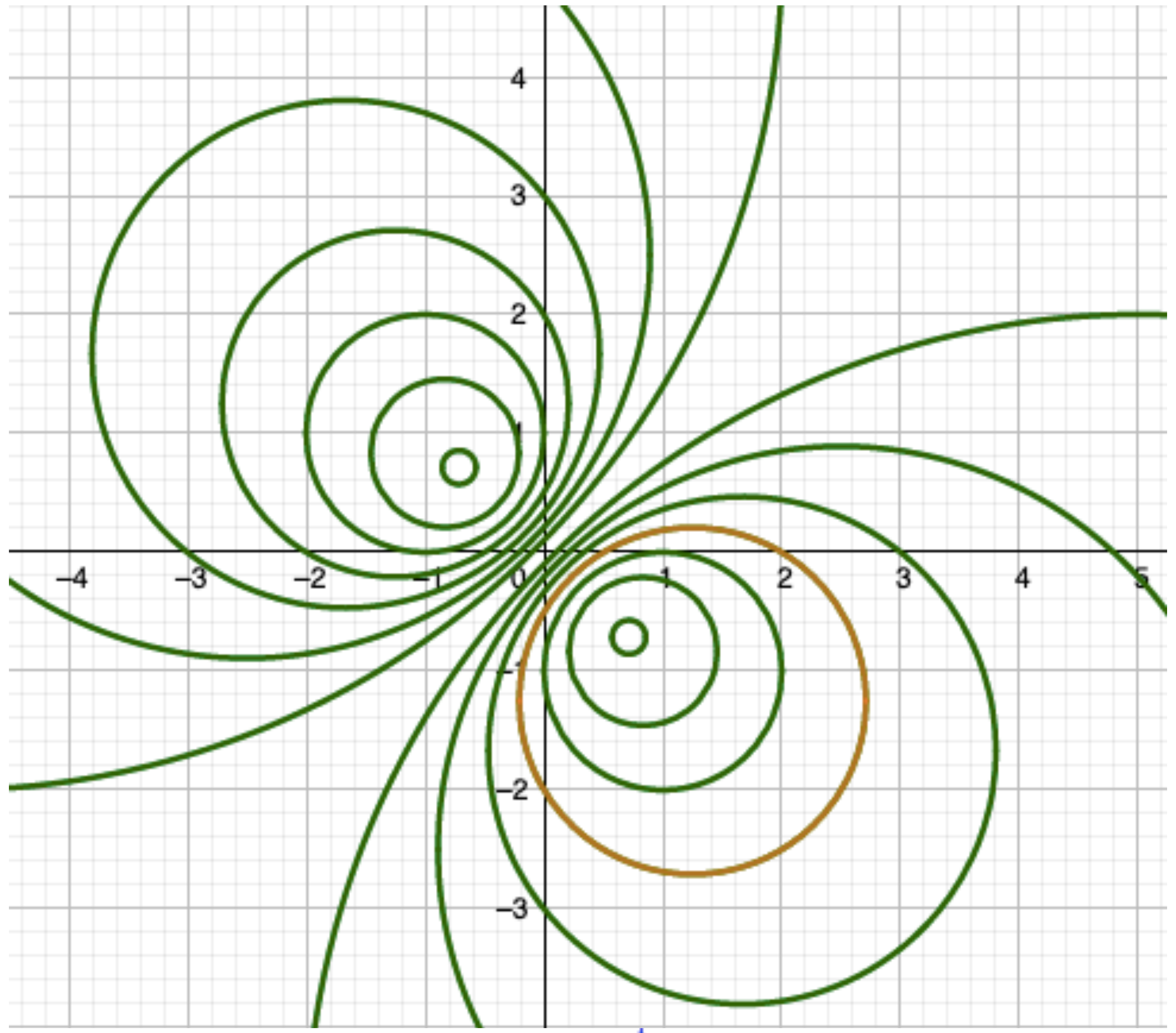
Link: ContourMap of a Cone

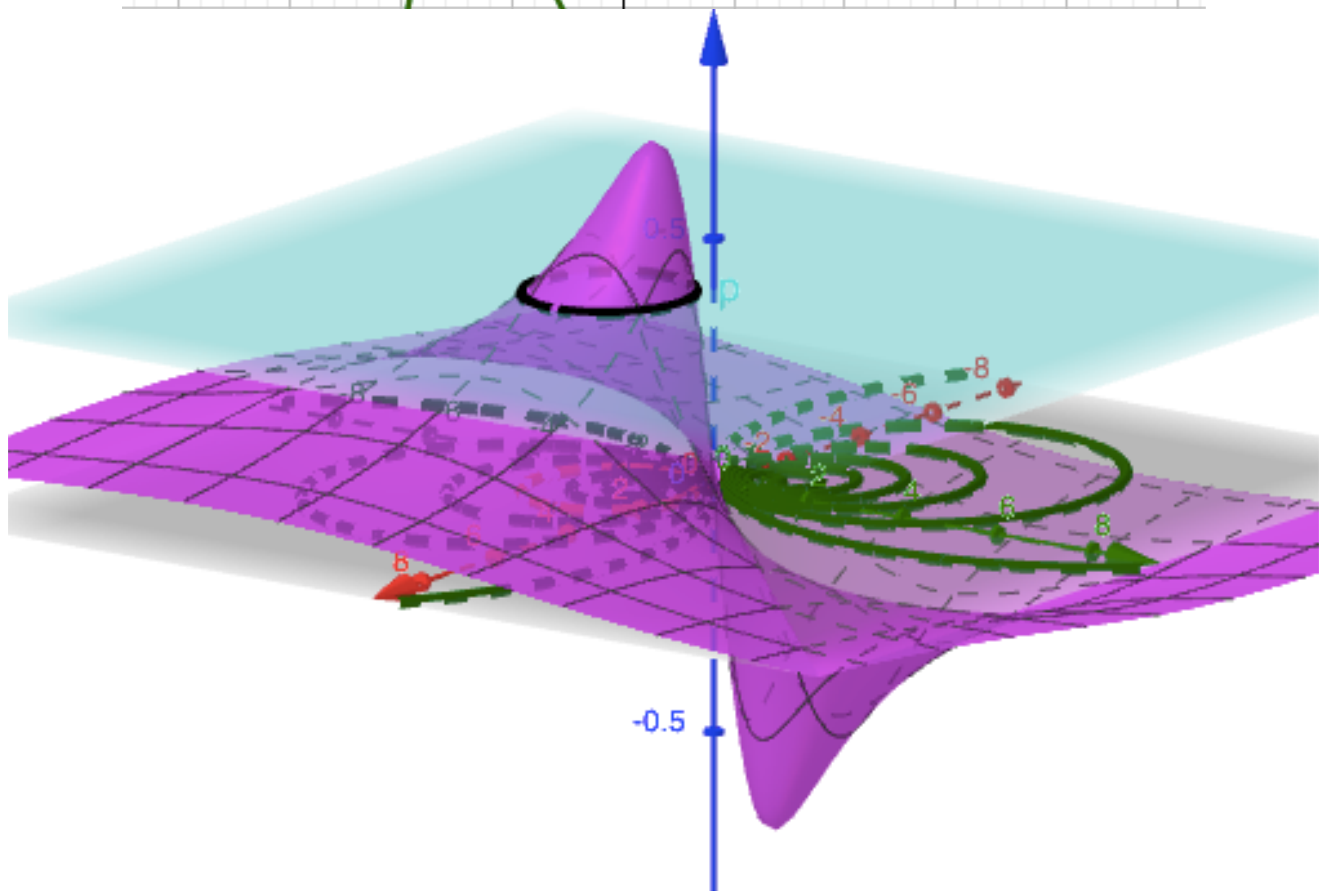




$$z = f(x, y) = x^2 - y^2$$

Link: Saddle Contour

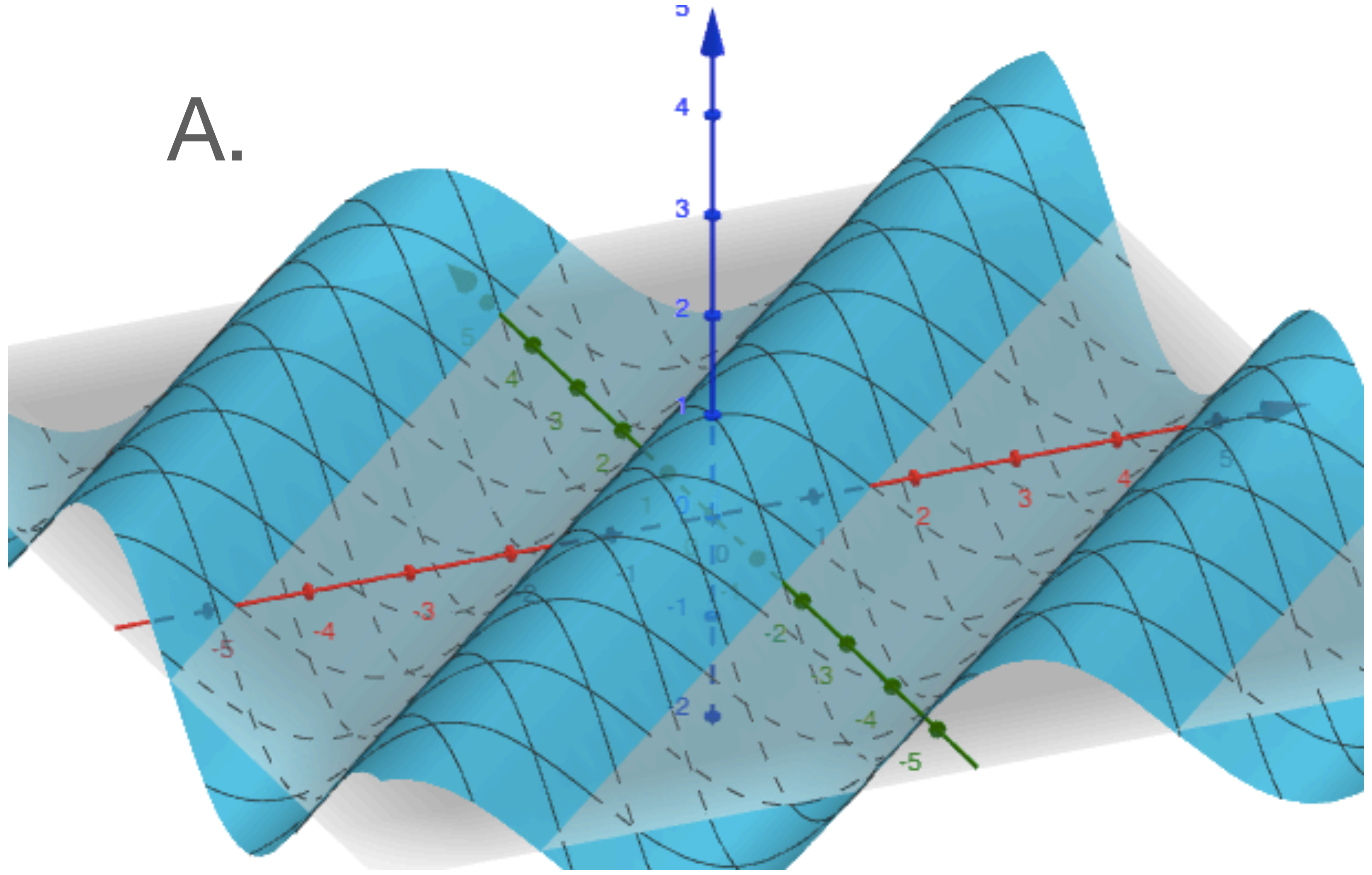


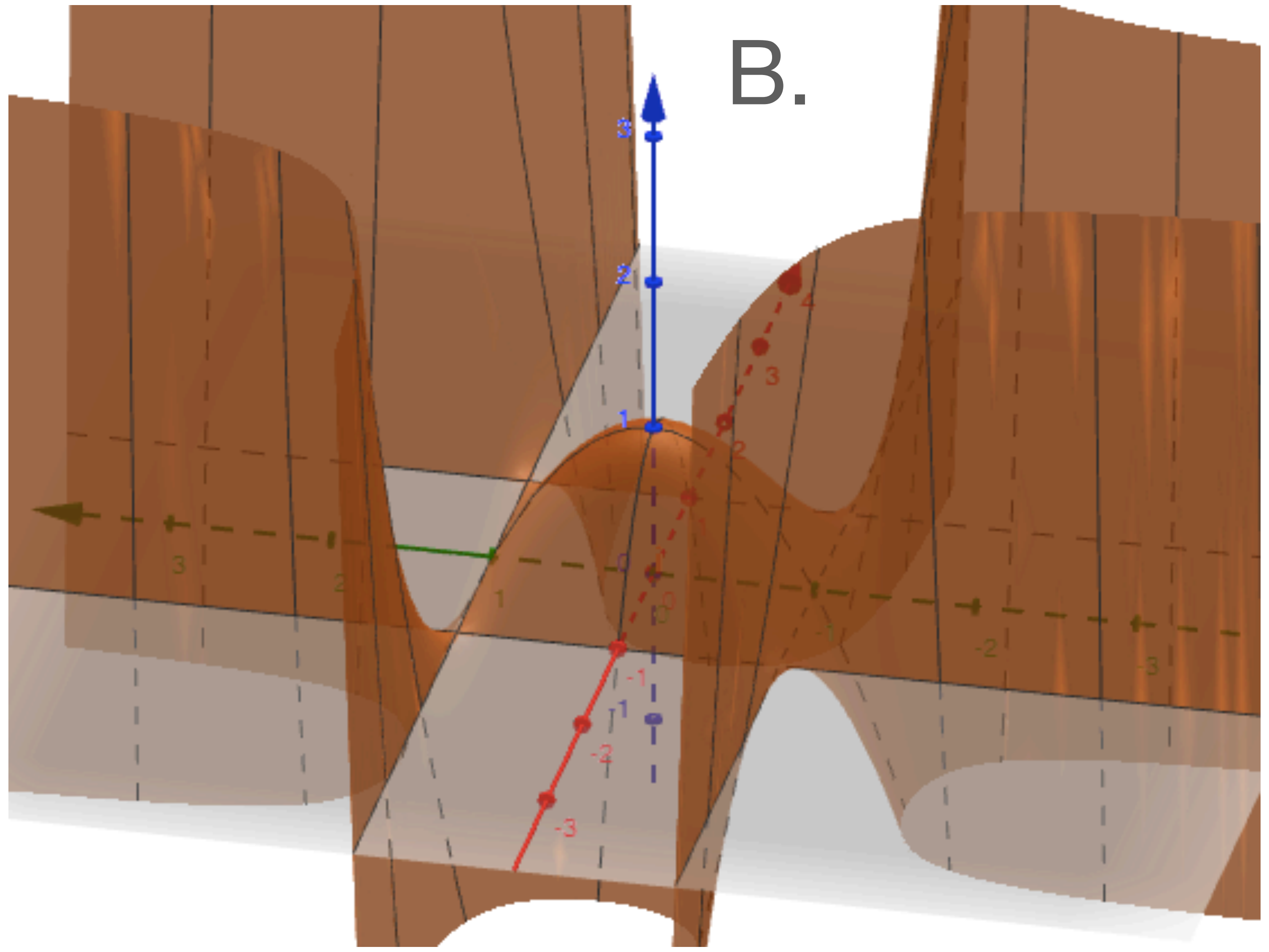


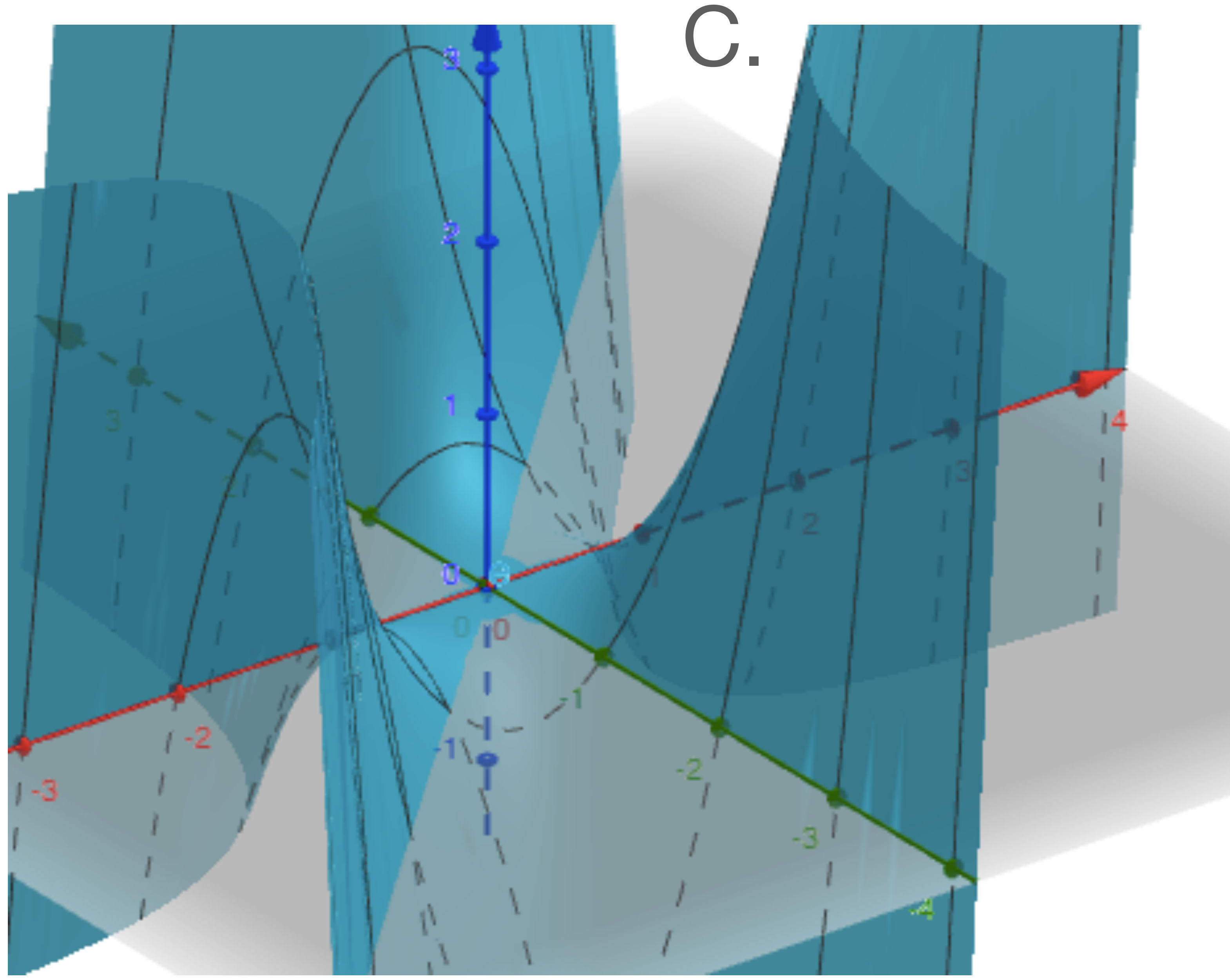
$$z = f(x, y) = \frac{x - y}{1 + x^2 + y^2}$$

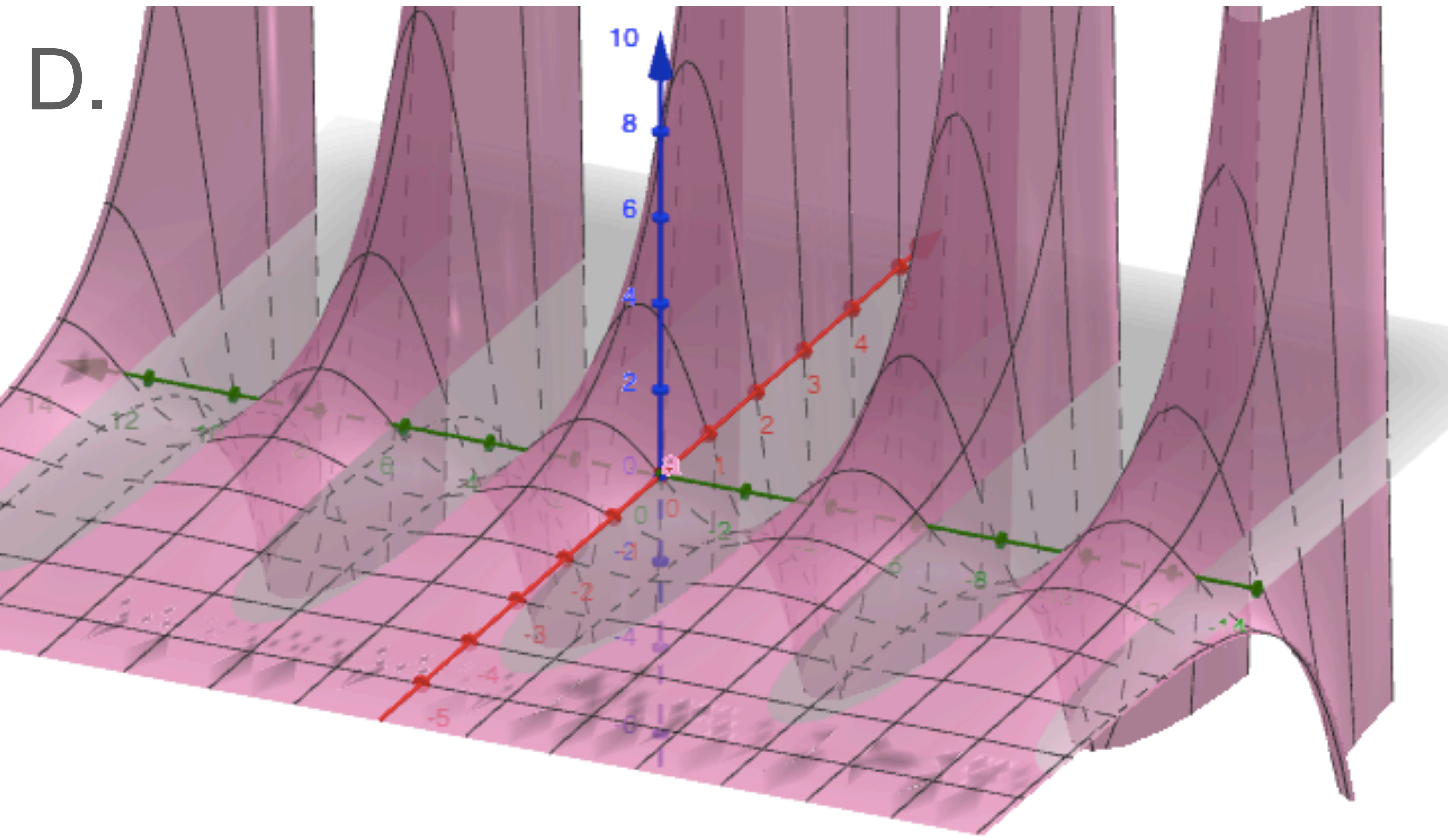
Link: WeirdContourMap

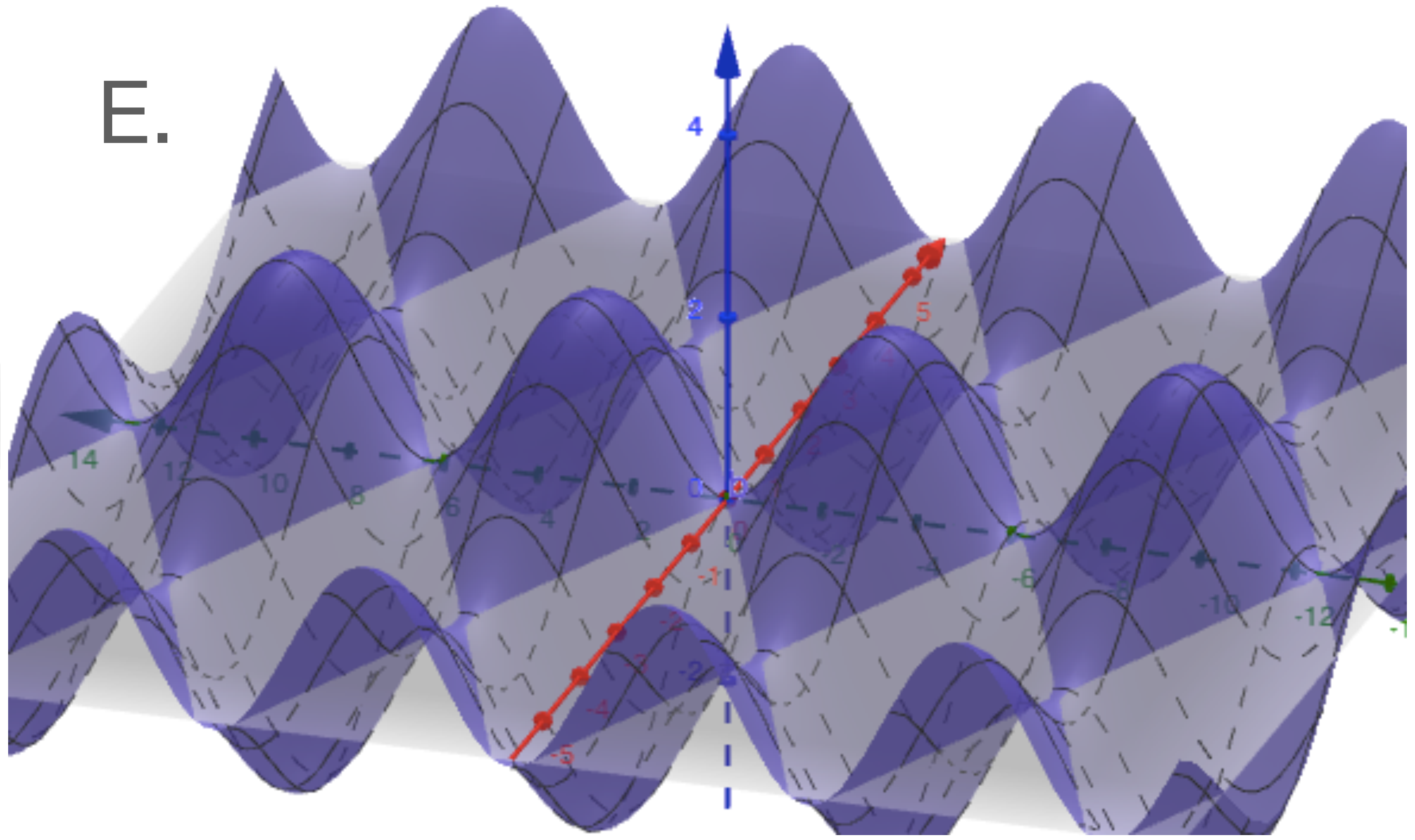
Graphs and Level Curves, pg 3. Practice.











B. $z = (1 - x^2)(1 - y^2)$

A. $z = \cos(x - y)$

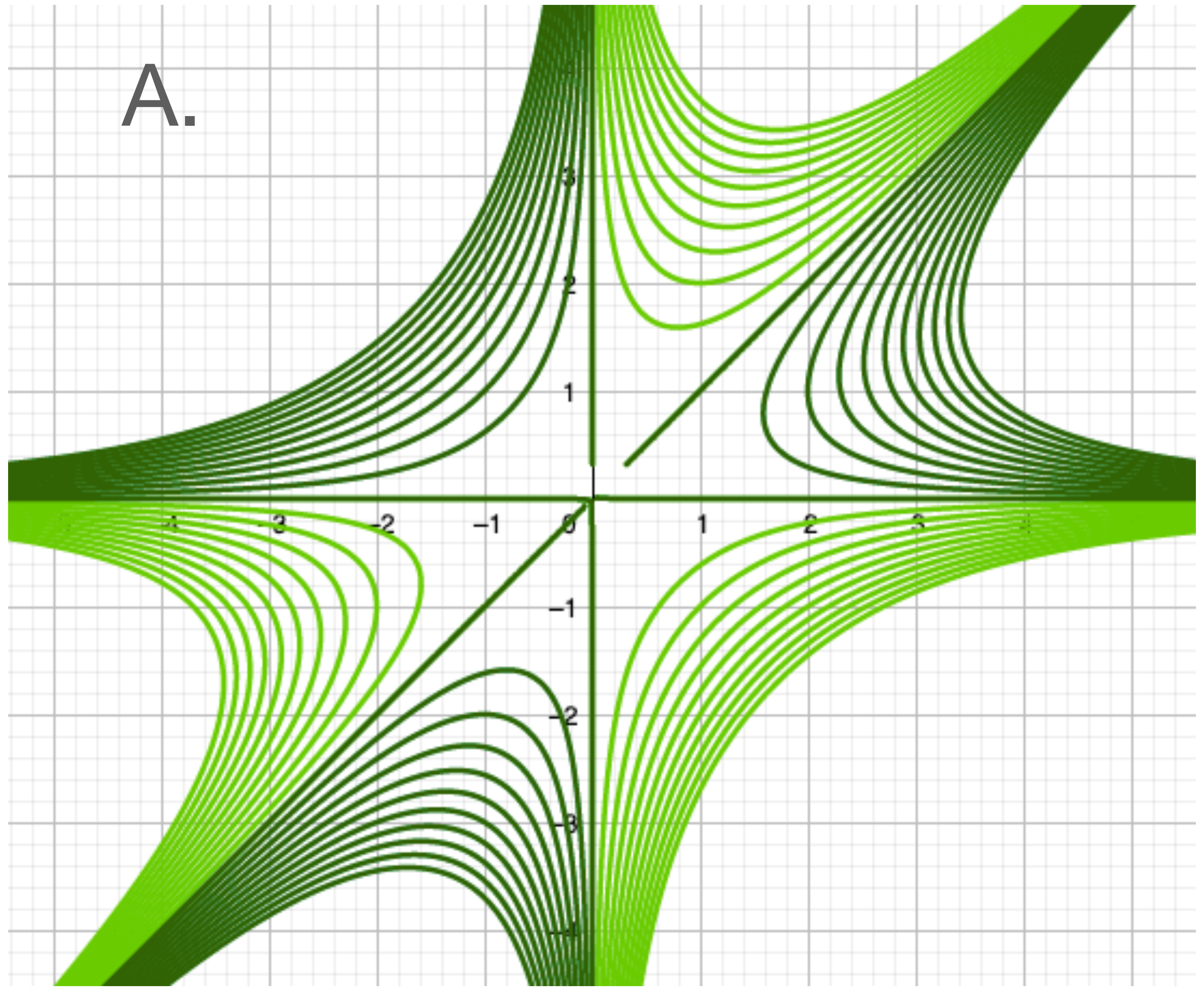
D. $z = e^x \sin(y)$

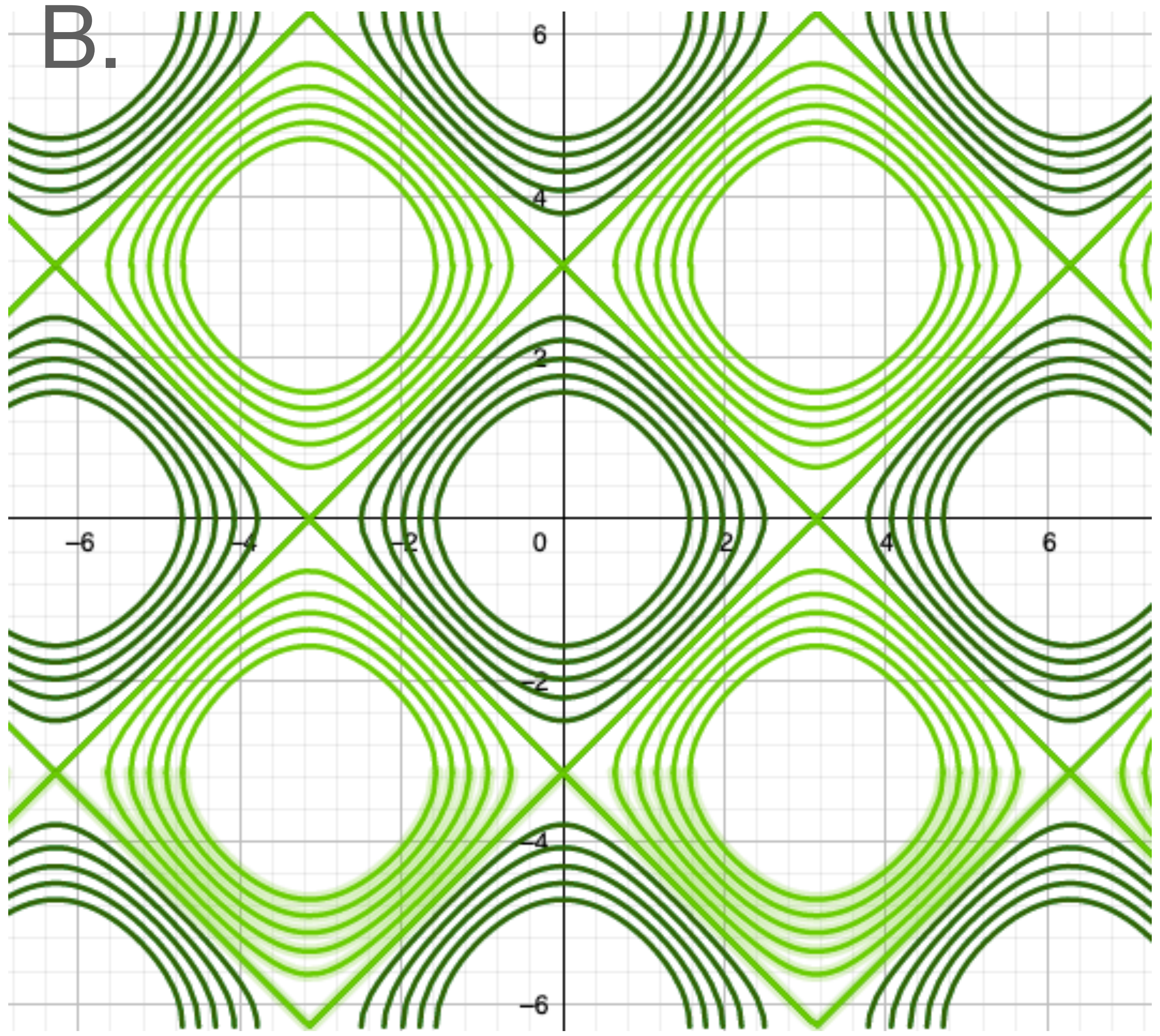
E. $z = \cos(x) - \cos(y)$

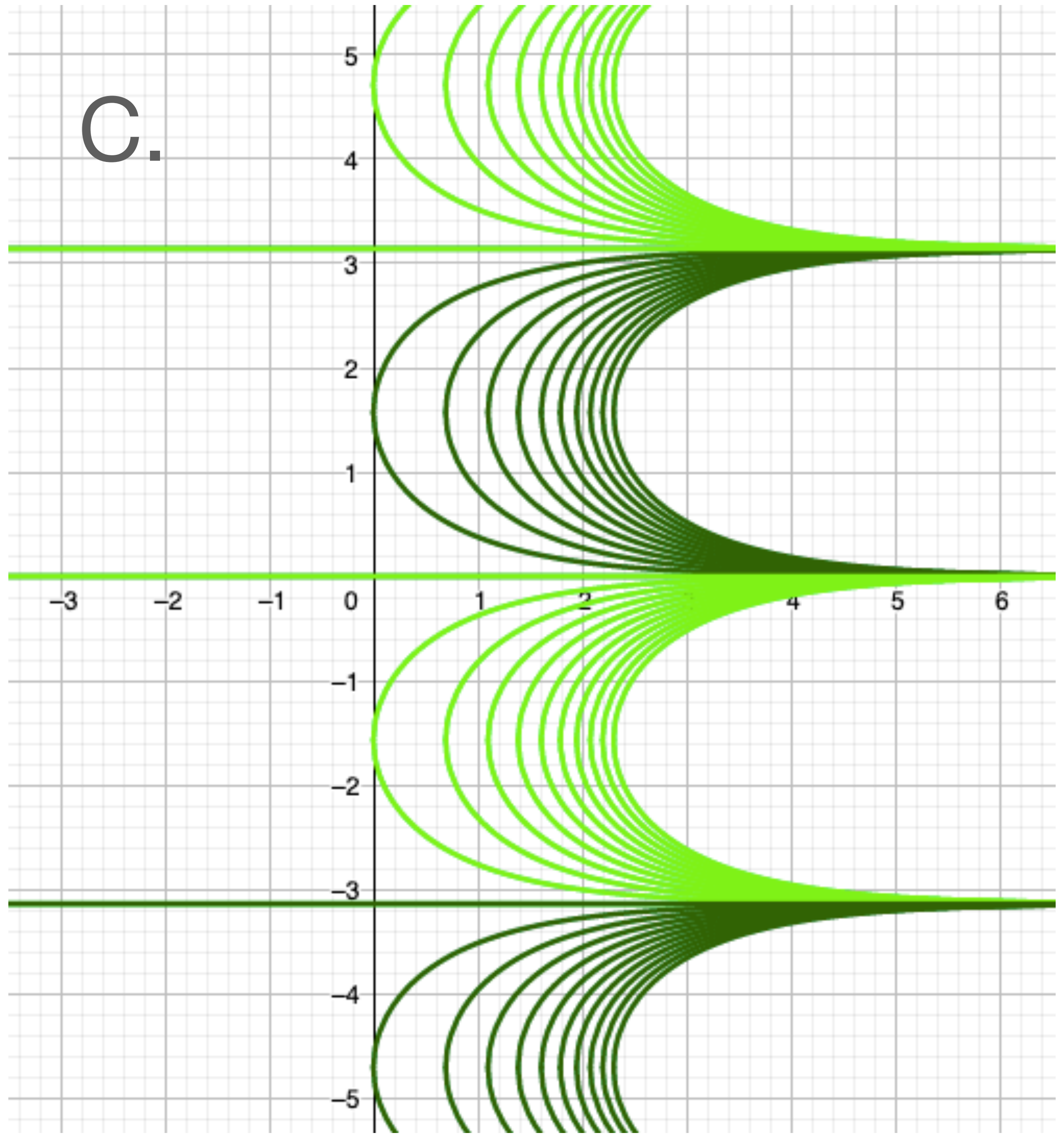
C. $z = xy^2 - yx^2$

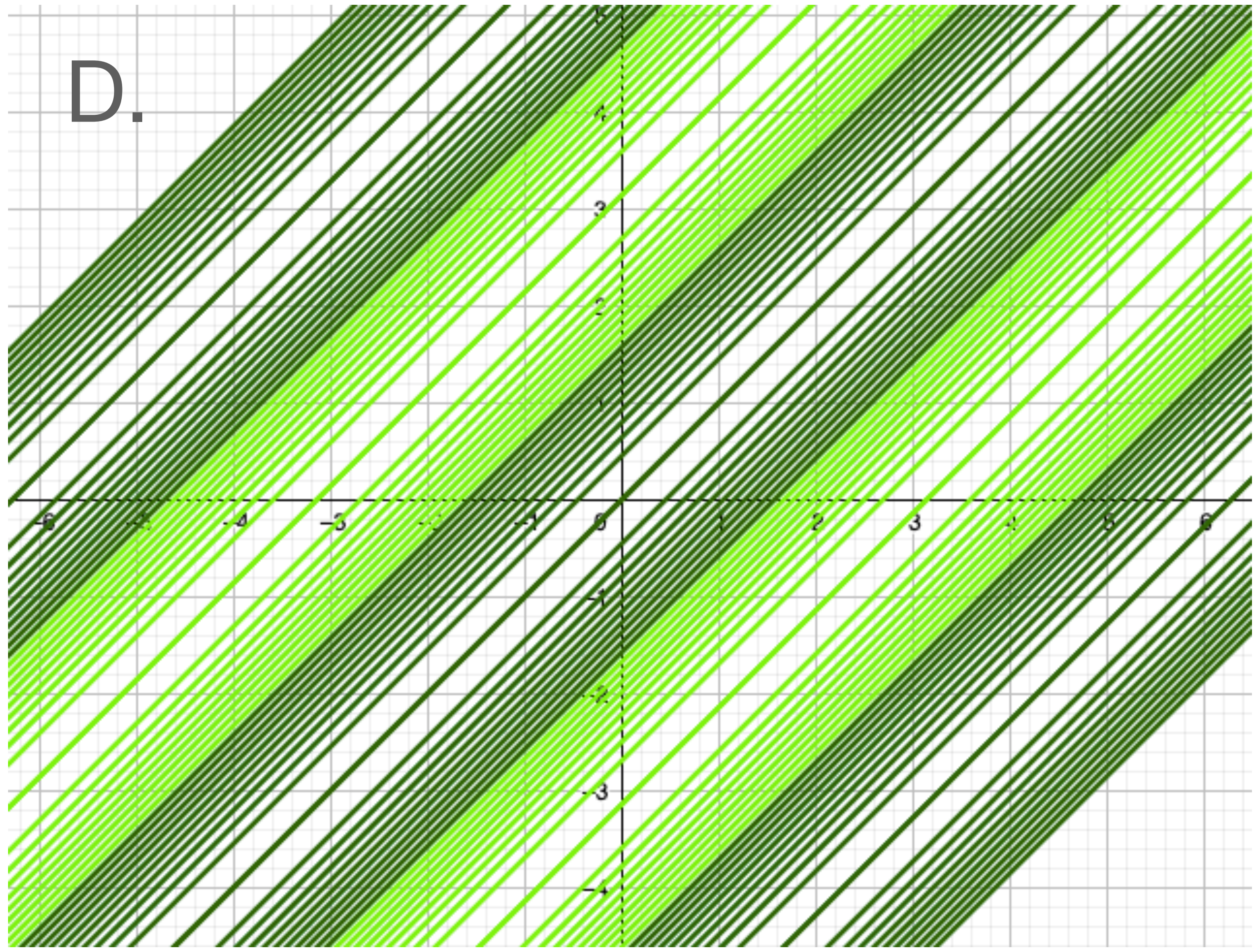
Graphs and Level Curves, pg 4. Practice.

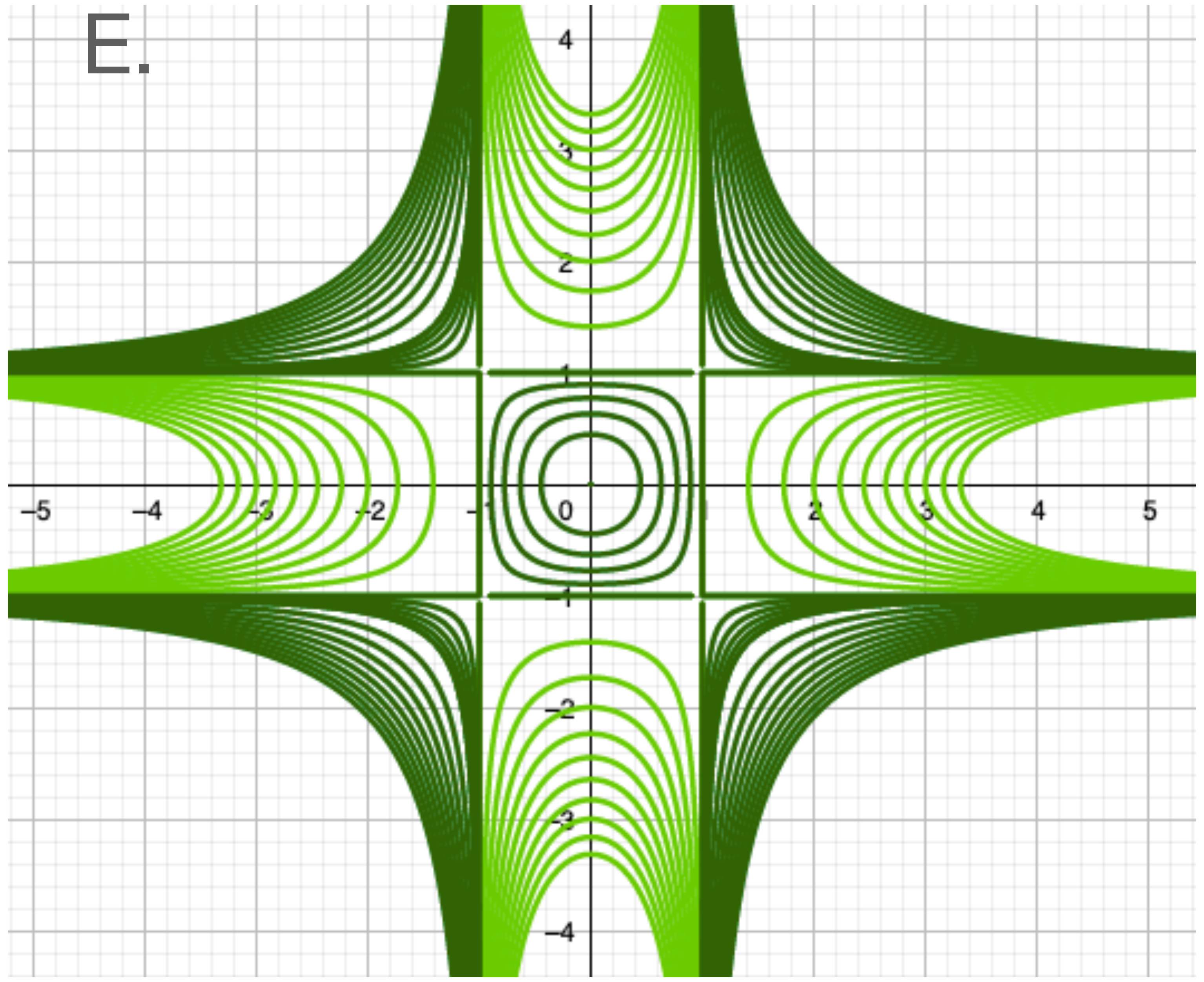
Match the Contours to the functions.











E. $z = (1 - x^2)(1 - y^2)$

D. $z = \cos(x - y)$

C. $z = e^x \sin(y)$

B. $z = \cos(x) - \cos(y)$

A. $z = xy^2 - yx^2$

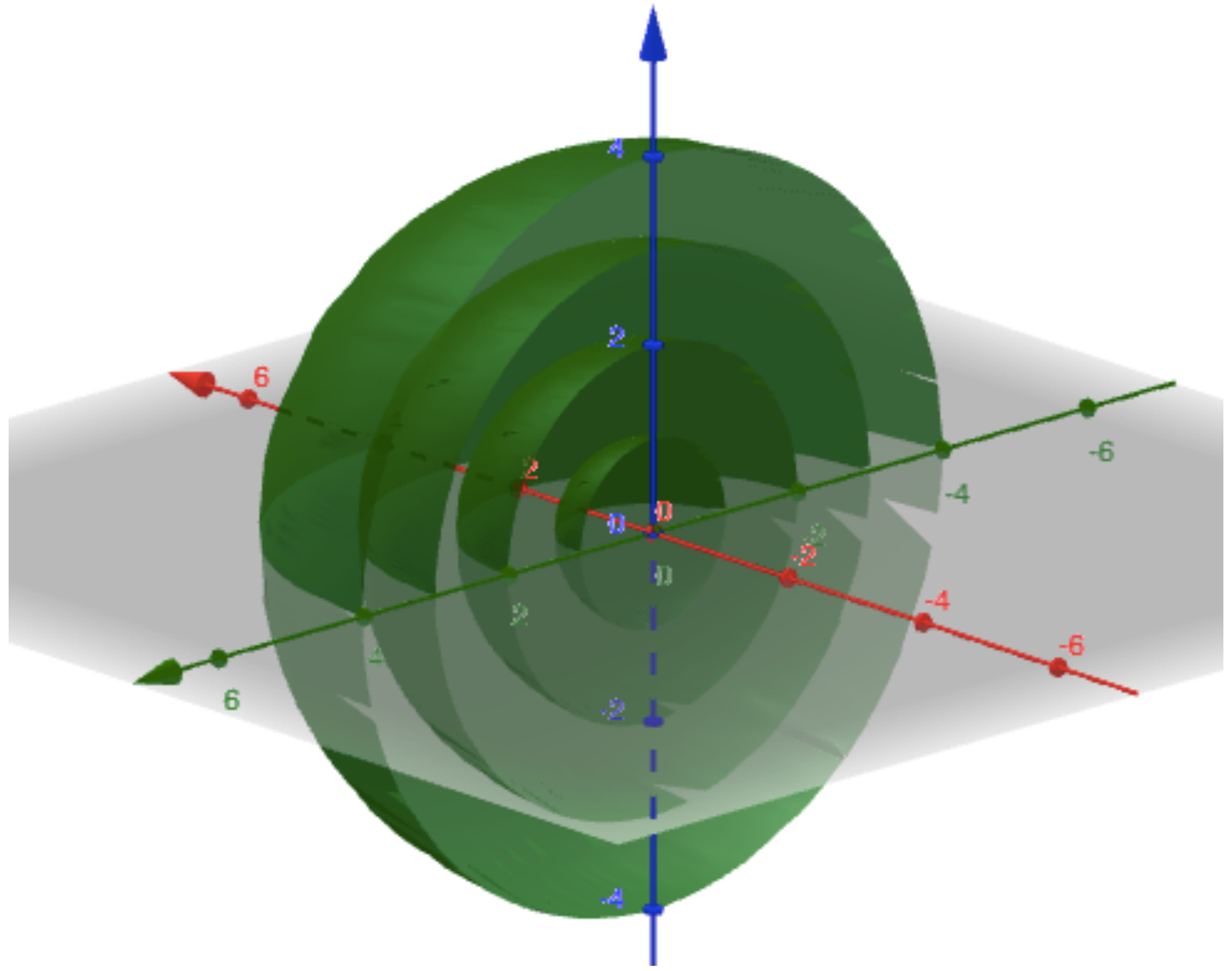
Higher Dimensional "Level Curves"

Level Curves on the xy plane help us visualize the different z-traces of a function $z = f(x, y)$.

Level Surfaces in xyz space help us visualize the different w-traces of a function $w = f(x, y, z)$.

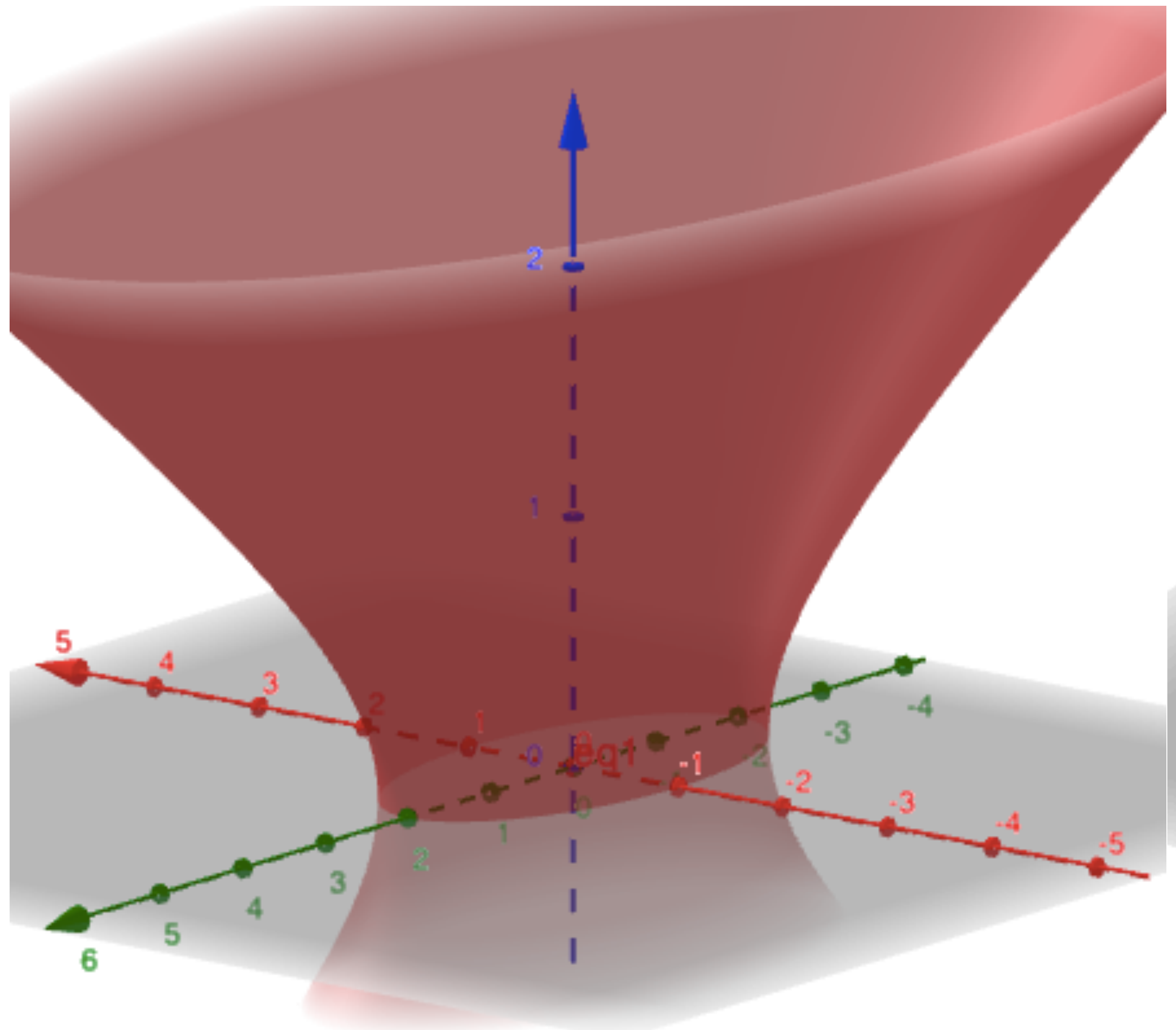
Examples.

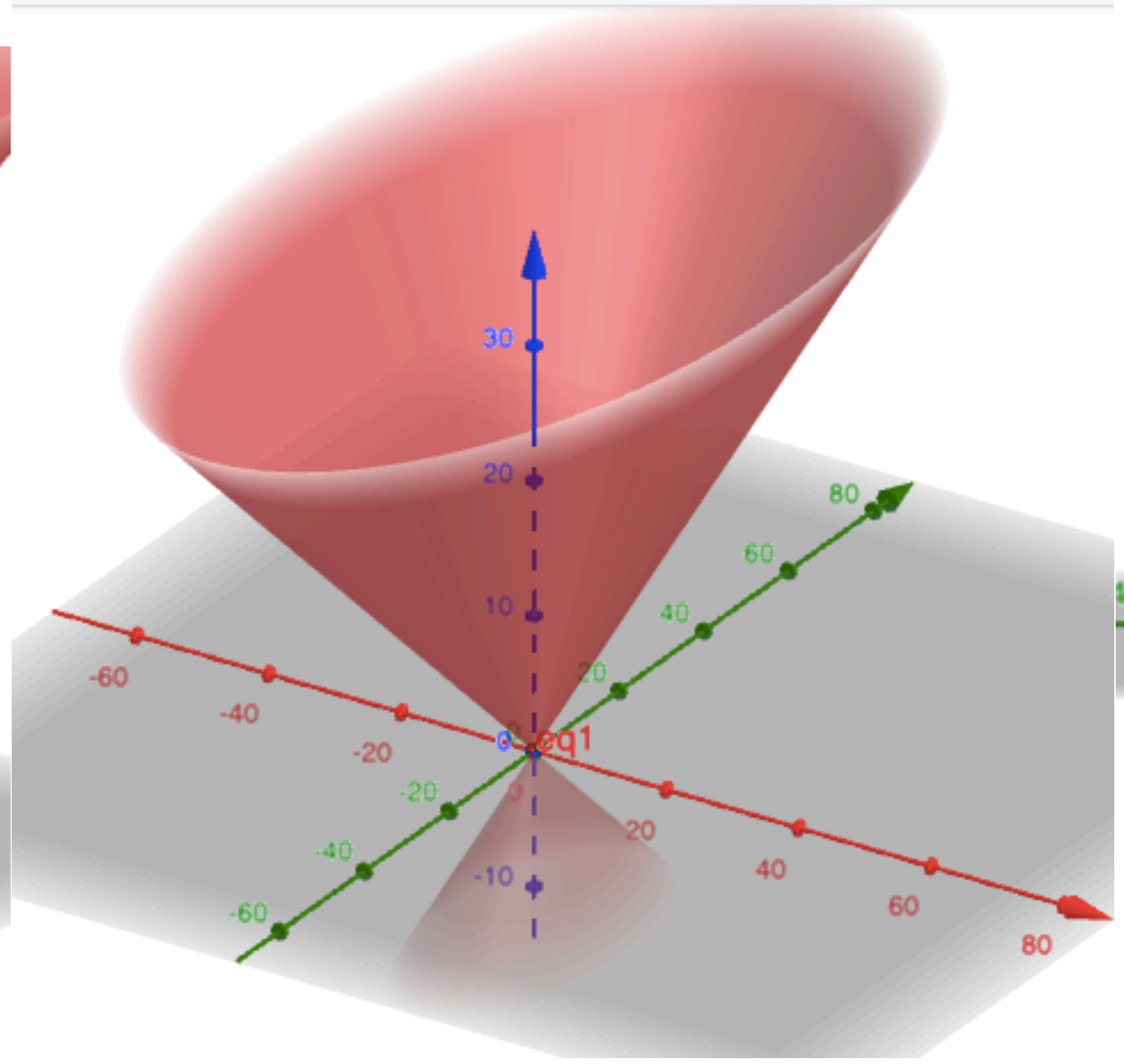
$$w = f(x, y, z) = x^2 + y^2 + z^2$$

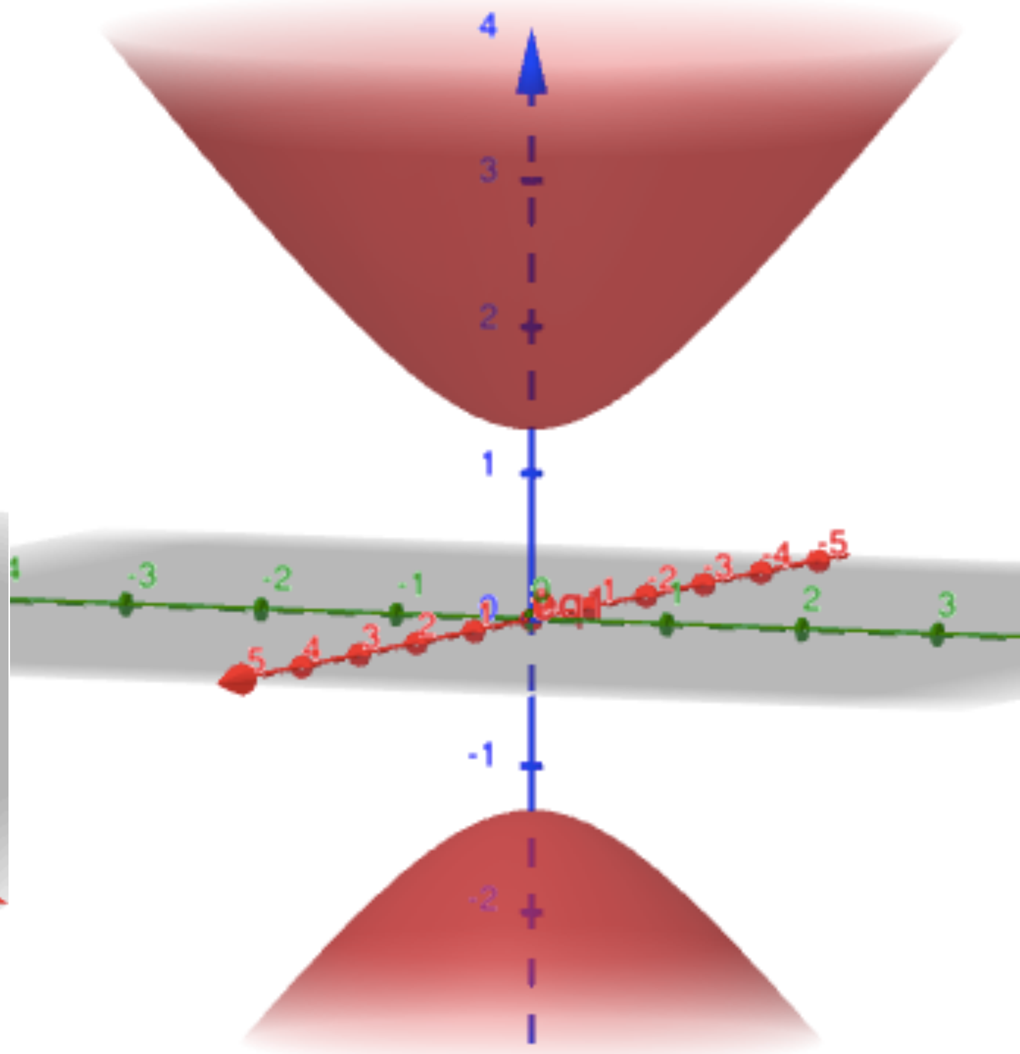


Each sphere is a level surface of the function $w = f(x, y, z)$.

Shown $w = 1{,}2{,}3{,}4$







The cone and hyperboloids are all levels surfaces of a single function, $w = x^2 + y^2 - z^2$

One-sheeted hyperboloid (on the left): w=1

Cone (middle): w=0

Two-sheeted hyperboloid (on the right): w=-1

You can make a movie of this "4D" function with geogebra.

Limits, pg 1.

Limits are part of theoretical foundation of calculus.

They are important (in theory)!

Examples.

1. $\lim\limits_{x\to 5} x^2 - x + 1 = 25 - 5 + 1 = 21$

2. $\lim\limits_{x\to 5} \dfrac{x^2 - 25}{x - 5} = \dfrac{0}{0} = ????$

$= \lim\limits_{x\to 5} \dfrac{(x+5)(x-5)}{x-5} = 10$

3. $\lim\limits_{(x,y)\to(2,-1)} \dfrac{x^2y + xy^2}{x^2 - y^2}$

$= \dfrac{2^2(-1) + 2(-1)^2}{2^2 - (-1)^2} = \dfrac{-2}{3}$

4. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - 3y^2}{x^2 + y^2}$

Ex 4: In order for a limit to exist, $f(x, y)$ must approach the same number L no matter how (x, y) approaches (a, b).

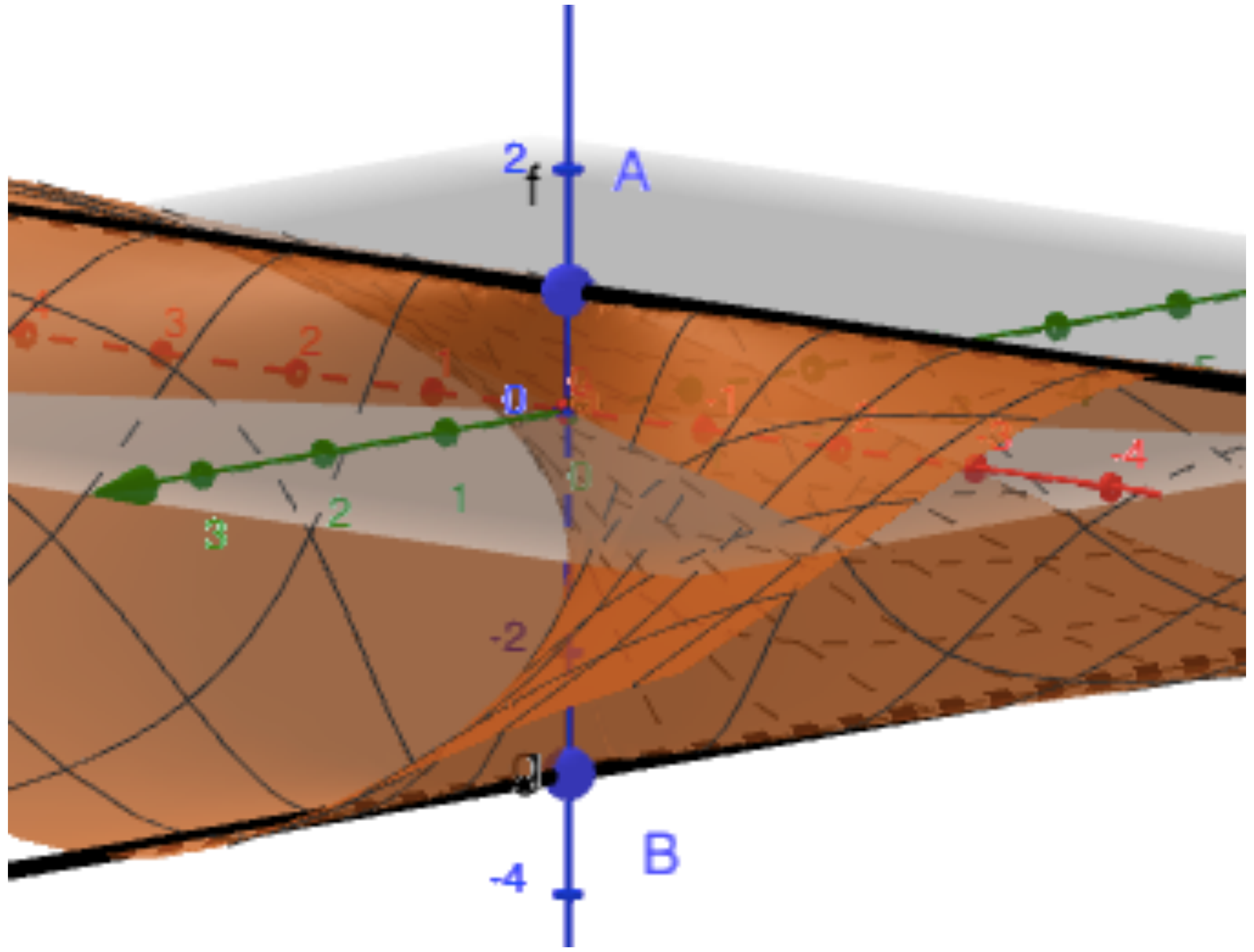
Suppose (x, y) approaches $(0,0)$ along the line $x = 0$...

4. $\lim\limits_{(0,y)\to(0,0)} \dfrac{0^2 - 3y^2}{0^2 + y^2} = \lim\limits_{y\to 0} \dfrac{-3y^2}{y^2} = -3$

But if (x, y) approaches $(0,0)$ along the line $y = 0$...

4. $\lim\limits_{(x,0)\to(0,0)} \dfrac{x^2 - 3(0)^2}{x^2 + 0^2} = \lim\limits_{x\to 0} \dfrac{x^2}{x^2} = 1$

Because $-3 \neq 1$,
the limit doesn't exist.



Limits, pg 2.

There are many ways to seriously define the limit of a function.

Formal Definition#1 (sequence definition):

$\lim_{(x,y)\to(a,b)} f(x, y) = L$ means that there exists a number L so that $\lim_{n\to\infty} f(x_n, y_n) = L$ where (x_n, y_n) is *any* sequence of points that converges to the point (a, b).

Formal Definition#2 (ϵ and δ):

$\lim_{(x,y)\to(a,b)} f(x, y) = L$ means that for every $\epsilon > 0$, there exists a $\delta > 0$ so that $|f(x, y) - L| < \epsilon$ whenever $|< x, y > - < a, b >| < \delta$.

Formal Definition#3 (ϵ and δ, multidimensional!):

$\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x}) = L$ means that for every $\epsilon > 0$, there exists a $\delta > 0$ so that $|f(\mathbf{x}) - L| < \epsilon$ whenever $|\mathbf{x} - \mathbf{c}| < \delta$.

Showing that a limit *does not* exist is sometimes easy:

We just show that there are two different directions toward the point (a, b) by which $f(x, y)$ yields different values of L.

Formally, this entails finding two different sequences, both of which converge to (a, b), but which yield different values of $\lim_{n\to\infty} f(x_n, y_n)$, thereby contradicting definition#1.

But if a limit actually exists, how do we prove it?

A formal proof goes according to one of the definitions. This inevitably involves some ϵ and δ, and often is not easy!

Limits, pg 3. More examples.

1. $$\lim_{(x,y)\to(0,0)} \frac{xy^2\cos(y)}{x^2+y^4}$$

If we approach $(0,0)$ along the line $x=0$, we get…

$$\lim_{(0,y)\to(0,0)} \frac{0y^2\cos(y)}{0^2+y^4} = \lim_{y\to 0}\frac{0}{y^4} = 0$$

If we approach $(0,0)$ along the line $y=0$, we get…

$$\lim_{(x,0)\to(0,0)} \frac{x0^2\cos(0)}{x^2+0^4} = \lim_{x\to 0}\frac{0}{x^2} = 0$$

If we approach $(0,0)$ along the parabola $x=y^2$, we get…

$$\lim_{(y^2,y)\to(0,0)} \frac{y^4\cos(y)}{y^4+y^4} = \lim_{y\to 0}\frac{\cos(y)}{2} = 0.5$$

Hence the limit in example 1 does not exist.

2. $$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2} = \frac{x}{(x/y)^2+1}, \quad \left|\frac{x}{(x/y)^2+1}\right| \le |x| \to 0$$

A different argument

If we approach $(0,0)$ from many different directions, we always get 0.

This doesn't prove, but it suggests, that the limit exists, and that the limit is 0.

Proof! Let $\epsilon > 0$. We need to find $\delta > 0$ so that

If $|<x,y> - <0,0>| = \sqrt{x^2+y^2} < \delta$ then $\left|\frac{xy^2}{x^2+y^2} - 0\right| < \epsilon$

We will take $\delta = \epsilon/2$. Then…

$$|x| = \sqrt{x^2} \le \sqrt{x^2+y^2} < \delta = \epsilon/2 \le \epsilon/2\left(1+\frac{y^2}{x^2}\right)$$

$$|x^3| < \epsilon/2(x^2+y^2), \quad \frac{|x^3|}{x^2+y^2} < \epsilon/2$$ Finally,

$$\left|\frac{xy^2}{x^2+y^2}\right| = \left|x - \frac{x^3}{x^2+y^2}\right| \le |x| + \left|\frac{x^3}{x^2+y^2}\right| \le 2(\epsilon/2) = \epsilon$$

QED.

Limits, pg 3. Practice.

1. $$\lim_{(x,y)\to(0,0)} \frac{y^2\sin^2(x)}{x^4+y^4}$$

2. $$\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz}{x^2+y^2+z^2}$$

3. $$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$$

1. Along the line $x = 0$, we get

$$\lim_{(x,y)\to(0,0)} \frac{y^2\sin^2(x)}{x^4+y^4} = \lim_{(0,y)\to(0,0)} \frac{y^2\sin^2(0)}{0^4+y^4} = \lim_{y\to 0}\frac{0}{y^4} = 0$$

1. Along the line $y = x$, we get

$$\lim_{(x,y)\to(0,0)} \frac{y^2\sin^2(x)}{x^4+y^4} = \lim_{(x,x)\to(0,0)} \frac{x^2\sin^2(x)}{x^4+x^4} = \lim_{x\to 0}\frac{\sin^2(x)}{2x^2} = 0.5$$

2. $$\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz}{x^2+y^2+z^2}$$

Approaching $(0,0,0)$ along the line $(0,0,z)$

$$\frac{xy+yz}{x^2+y^2+z^2} = \lim_{z\to 0}\frac{0}{z^2} = 0$$

Approaching $(0,0,0)$ along the line (z,z,z)

$$\frac{xy+yz}{x^2+y^2+z^2} = \lim_{z\to 0}\frac{2z^2}{3z^2} = \frac{2}{3}$$

3. $$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} = 2$$

$$\frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} = \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}\cdot\frac{\sqrt{x^2+y^2+1}+1}{\sqrt{x^2+y^2+1}+1}$$

$$= \frac{(x^2+y^2)(\sqrt{x^2+y^2+1}+1)}{x^2+y^2}$$

$= \sqrt{x^2+y^2+1}+1 \to 2$ as $(x,y)\to(0,0)$

Continuity.

It is a good day when you can compute $\lim_{(x,y)\to(a,b)} f(x,y)$ just by plugging in $x = a$ and $y = b$ into f!

This is what is means for a function to be continuous.

Definition. A function $f : \mathbf{R}^2 \to \mathbf{R}$ is *continuous* at a point (a, b) if …

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

This definition says three things.

1. (a, b) is in the domain of f

2. $\lim_{(x,y)\to(a,b)} f(x,y)$ exists.

3. the limit equals the function's value at (a,b).

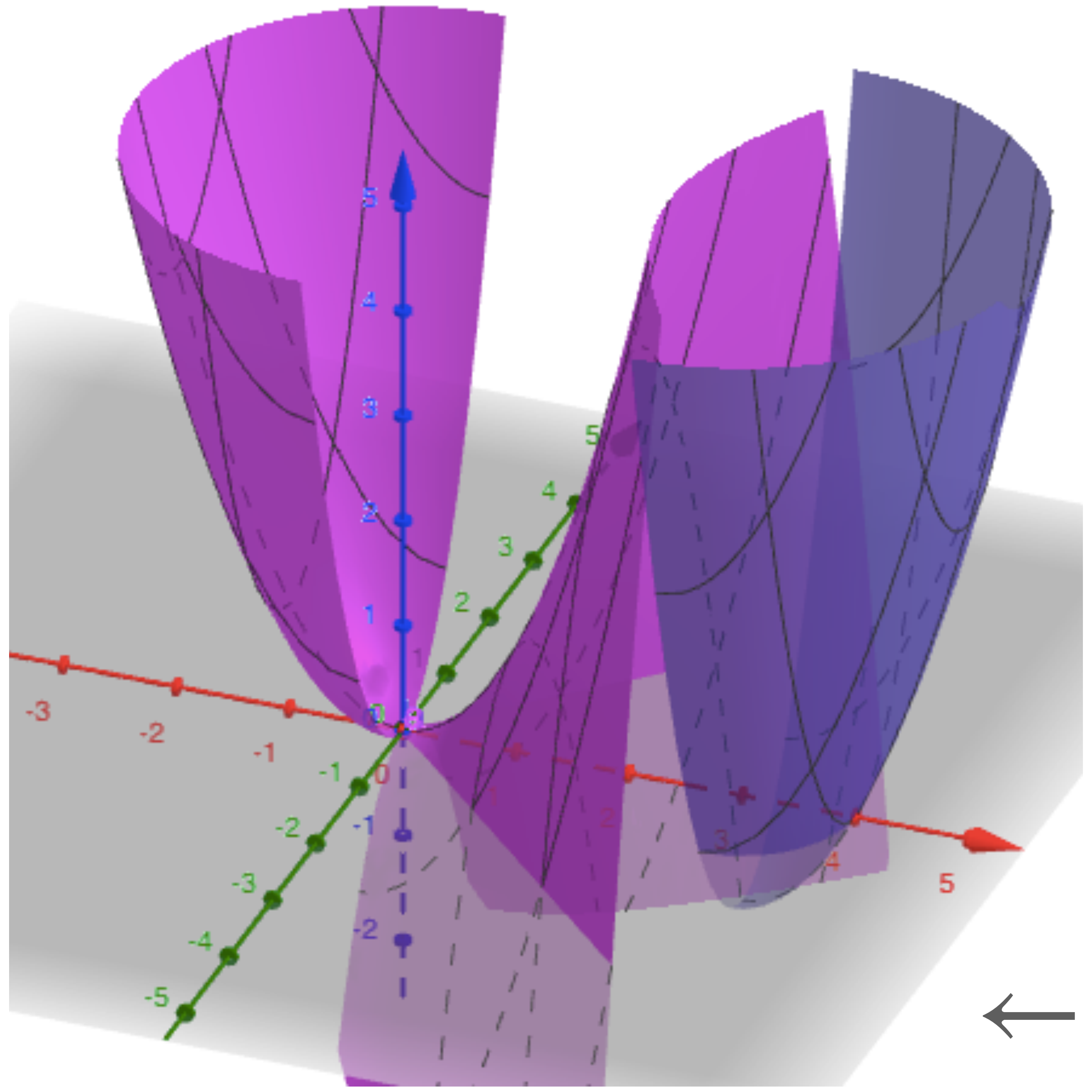
You may remember an informal idea for continuity of a function defined on $\mathbf{R}$:

It's continuous if you can draw its graph without picking up your pencil.

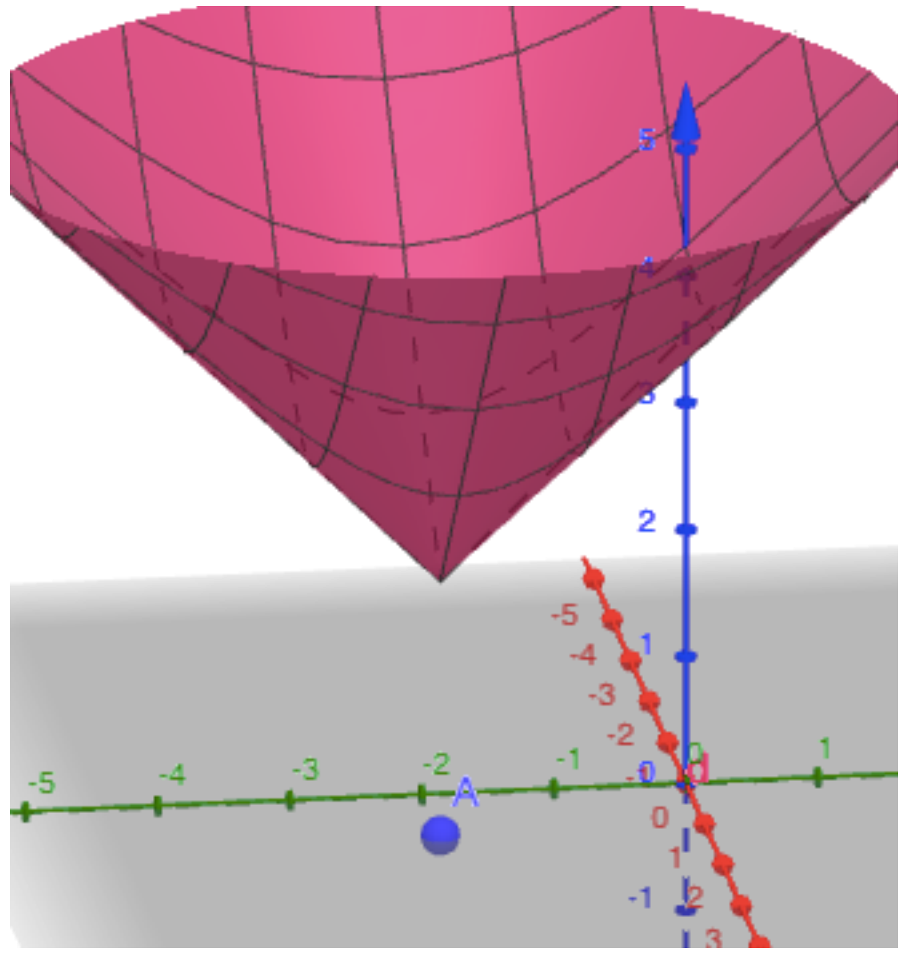
For functions defined on $\mathbf{R}^2$, you might say

It's continuous if you can shade the surface of its graph without picking up your pencil.

That is to say, the graph has no breaks, tears or jumps.



← not continuous when x=0 and x=3



←continuous everywhere, even at A.

Approximating derivatives in many directions.

You may remember a problem like:

x rad	sin(x)
-1.0	-0.84
-0.9	-0.78
-0.8	-0.72
-0.7	-0.64
-0.6	-0.56
-0.5	-0.48
-0.4	-0.39
-0.3	-0.30
-0.2	-0.20
-0.1	-0.10

Estimate the rate of change of $\sin(x)$ when x = -0.5 rad

The slope of the segment just before x=-0.5 is

$$\frac{-0.48 - -0.56}{-0.5 - -0.6} = 0.8$$

The slope of the segment just after x=-0.5 is

$$\frac{-0.39 - -0.48}{-0.4 - -0.5} = 0.9$$

The approximate rate of change is 0.85

(The exact rate of change is $\cos(-0.5) = 0.878$.)

sin(xy)	x=0.6	0.7	0.8	0.9
y=-0.7	-0.41	-0.47	-0.53	-0.59
-0.6	-0.35	-0.41	-0.46	-0.51
-0.5	-0.30	-0.34	-0.39	-0.43
-0.4	-0.24	-0.28	-0.31	-0.35
-0.3	-0.18	-0.21	-0.24	-0.27
-0.2	-0.12	-0.14	-0.16	-0.18
-0.1	-0.06	-0.07	-0.08	-0.09
0.0	0.00	0.00	0.00	0.00
0.1	0.06	0.07	0.08	0.09
0.2	0.12	0.14	0.16	0.18

How quickly does $\sin(xy)$ change at the point (0.8,-0.6)?

a) *in the x-direction*?

From x=0.7 to 0.8:

$$\frac{-0.46 - -0.41}{0.8 - 0.7} = -0.5$$

From x=0.8 to 0.9:

$$\frac{-0.51 - -0.46}{0.9 - 0.8} = -0.5$$

In the x-direction the rate of change is about -0.5

b) What about *in the y-direction*?

$$\frac{-0.46 - -0.53}{-0.6 - -0.7} = 0.7$$

$$\frac{-0.39 - -0.46}{-0.5 - -0.6} = 0.7$$

(The exact rates of change are -0.53 and 0.71. We will learn how to compute these!)

Approximating derivatives in many directions, practice.

Link: Estimating a Directional Derivative

f(x,y)=x^2+y^2	y									
x		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
	0.1	0.02	0.05	0.1	0.17	0.26	0.37	0.5	0.65	0.82
	0.2	0.05	0.08	0.13	0.2	0.29	0.4	0.53	0.68	0.85
	0.3	0.1	0.13	0.18	0.25	0.34	0.45	0.58	0.73	0.9
	0.4	0.17	0.2	0.25	0.32	0.41	0.52	0.65	0.8	0.97
	0.5	0.26	0.29	0.34	0.41	0.5	0.61	0.74	0.89	1.06
	0.6	0.37	0.4	0.45	0.52	0.61	0.72	0.85	1	1.17
	0.7	0.5	0.53	0.58	0.65	0.74	0.85	0.98	1.13	1.3
	0.8	0.65	0.68	0.73	0.8	0.89	1	1.13	1.28	1.45
	0.9	0.82	0.85	0.9	0.97	1.06	1.17	1.3	1.45	1.62

Estimate the derivatives of $f(x,y) = x^2 + y^2$ at the point $(0.4, 0.7)$

a) in the x direction.

b) in the y direction.

a) From $x = 0.3$ to $x = 0.4$, the function changes at a rate of

$$\frac{0.65 - 0.58}{0.4 - 0.3} = 0.7$$

From $x = 0.4$ to $x = 0.5$, the function changes at a rate of

$$\frac{0.74 - 0.65}{0.5 - 0.4} = 0.9$$

The rate of change in the x direction is about

$(0.9 + 0.7)/2 = 0.8$

b) From $y = 0.6$ to $y = 0.7$ the rate of change is about

$$\frac{0.65 - 0.52}{0.7 - 0.6} = 1.3$$

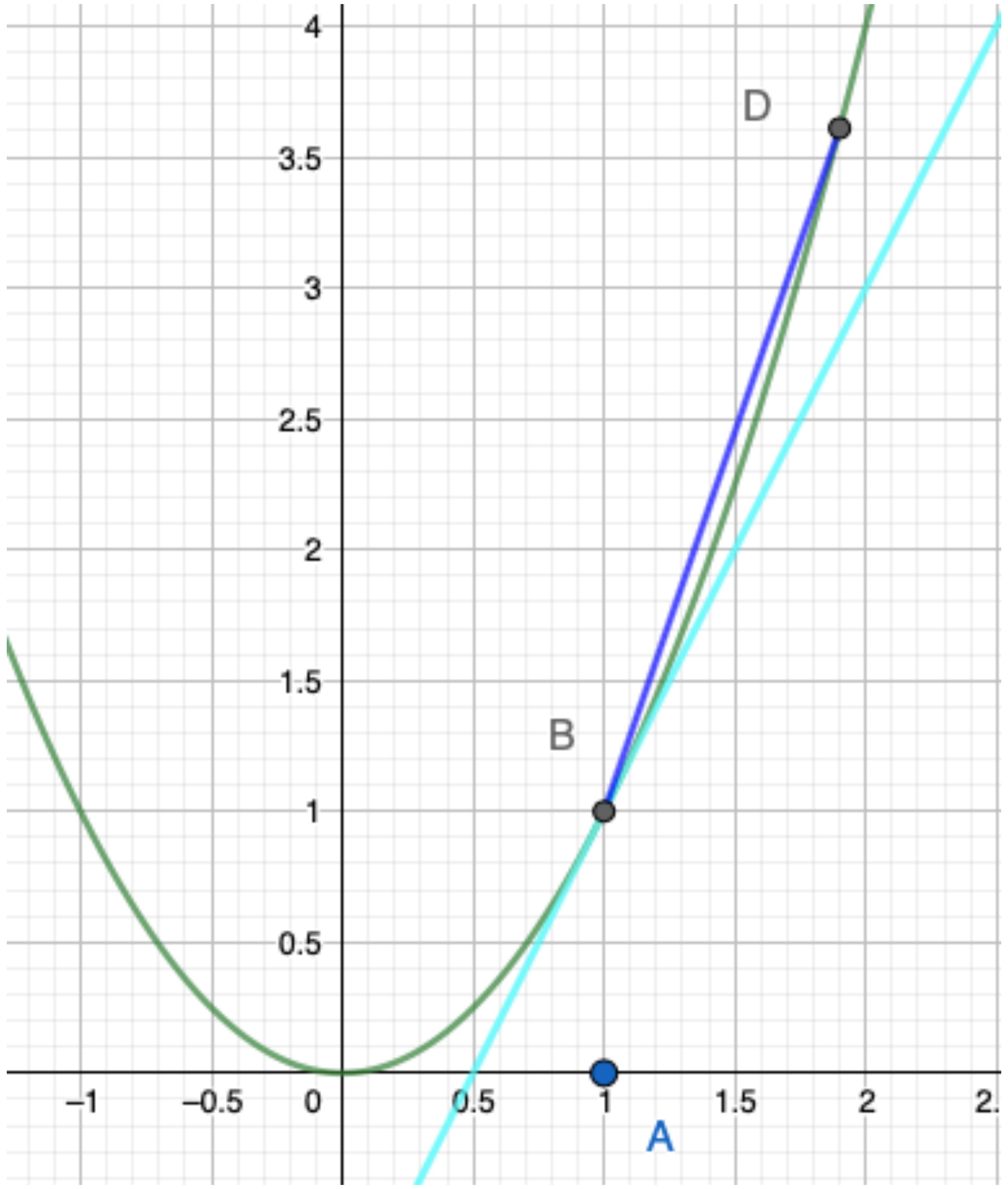
From $y = 0.7$ to $y = 0.8$ the rate of change is about

$$\frac{0.8 - 0.65}{0.8 - 0.7} = 1.5$$

The average of these two is an approximation of rate of change in the y direction: $(1.5 + 1.3)/2 = 1.4$

Partial Derivatives, pg 1.

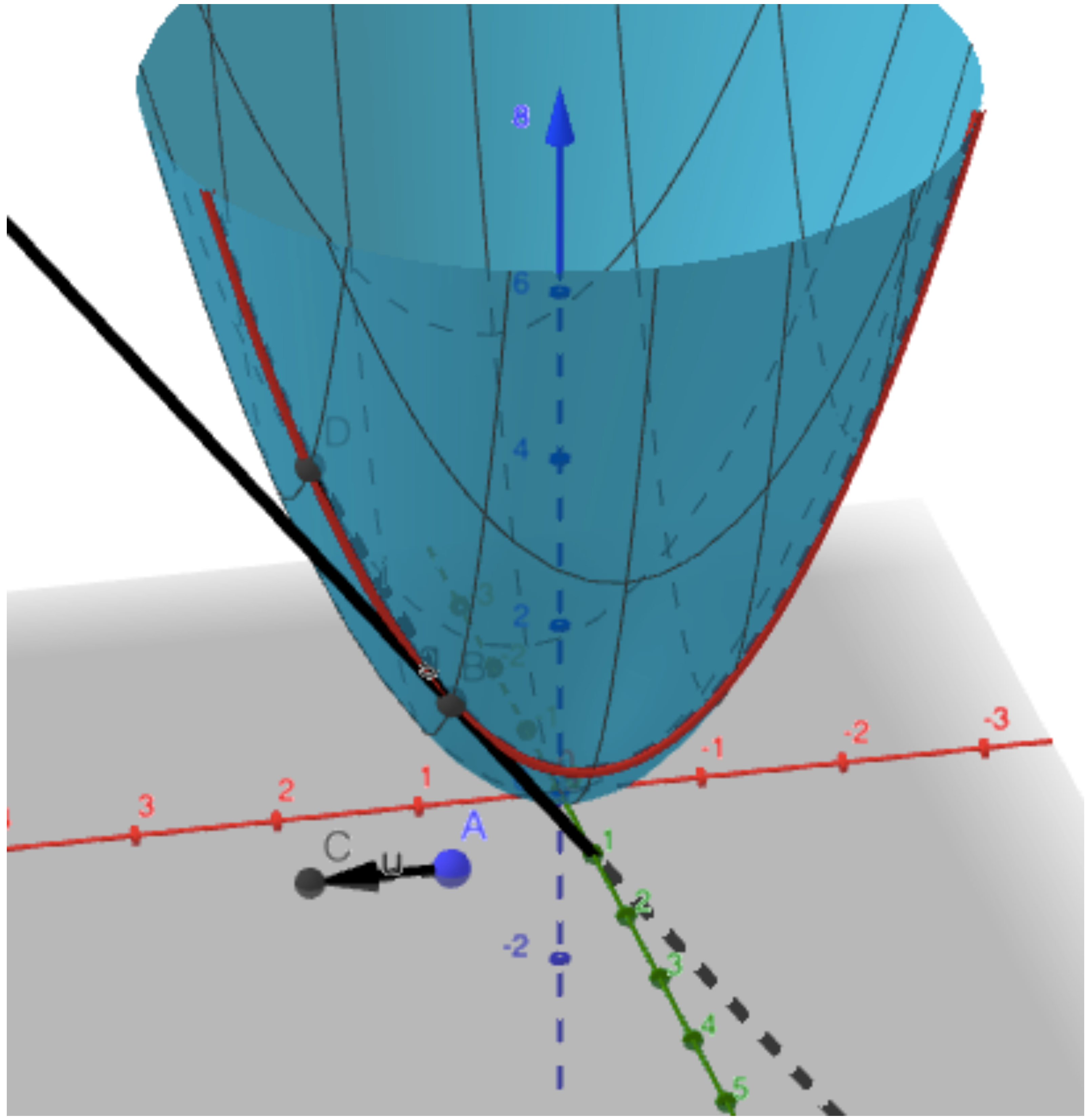
Remember how the derivative $f'(x)$ was defined.



Link: Derivative f'(x)

$$f'(x) := \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

You can compute a multidimensional derivative similarly. But now there are many directions in which you can approach a point (a, b) in the domain of $f(x, y)$. There is a derivative in each direction!



Link: DirectionalDerivatives

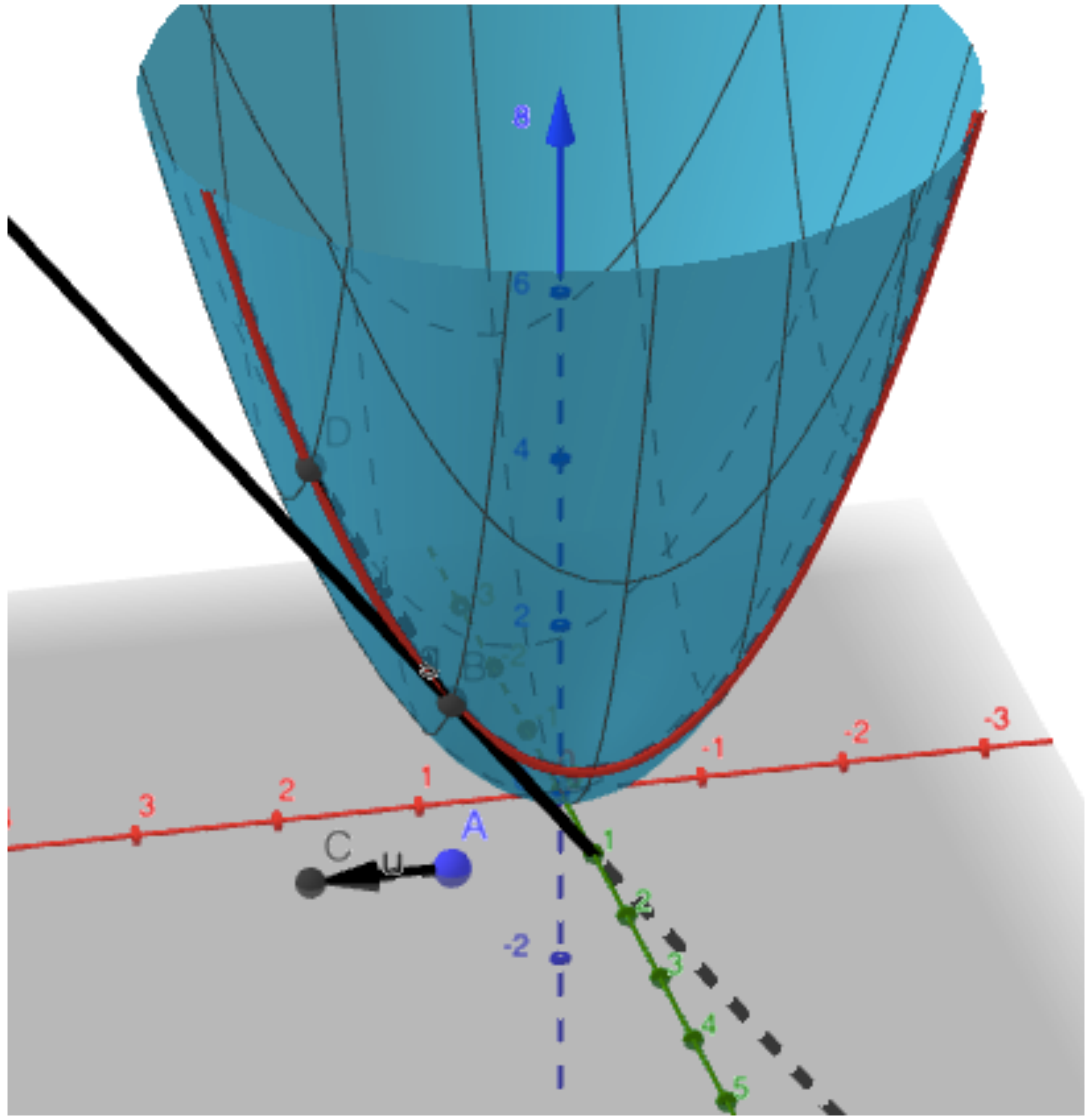
How shall we measure the instantaneous rate of change of $f(x, y)$ at a point $A(a, b)$?

One possible approach:

Find the curve $g(t)$ on the graph's surface that goes through $g(0) = (a, b, f(a, b))$ in the given direction .

The rate of change of the curve's z coordinate when the curve goes through the point $A(a, b)$ is the derivative that we want. This is $z'(0)$.

Partial Derivatives, pg 2.



$f(x, y) = x^2 + y^2$

We'll compute the derivative at the point (1,1), in the x direction.

The curve (in red) is

$g(t) = < x(t), y(t), z(t) >$

$g(t) = < 1 + t,\ 1,\ (1 + t)^2 + 1^2 >$

The slope of the tangent line is $z'(0)$.

$z'(t) = 2(1 + t)$

$z'(0) = 2$

Terminology and Notation:

This is called the *partial derivative* of f with respect to x.

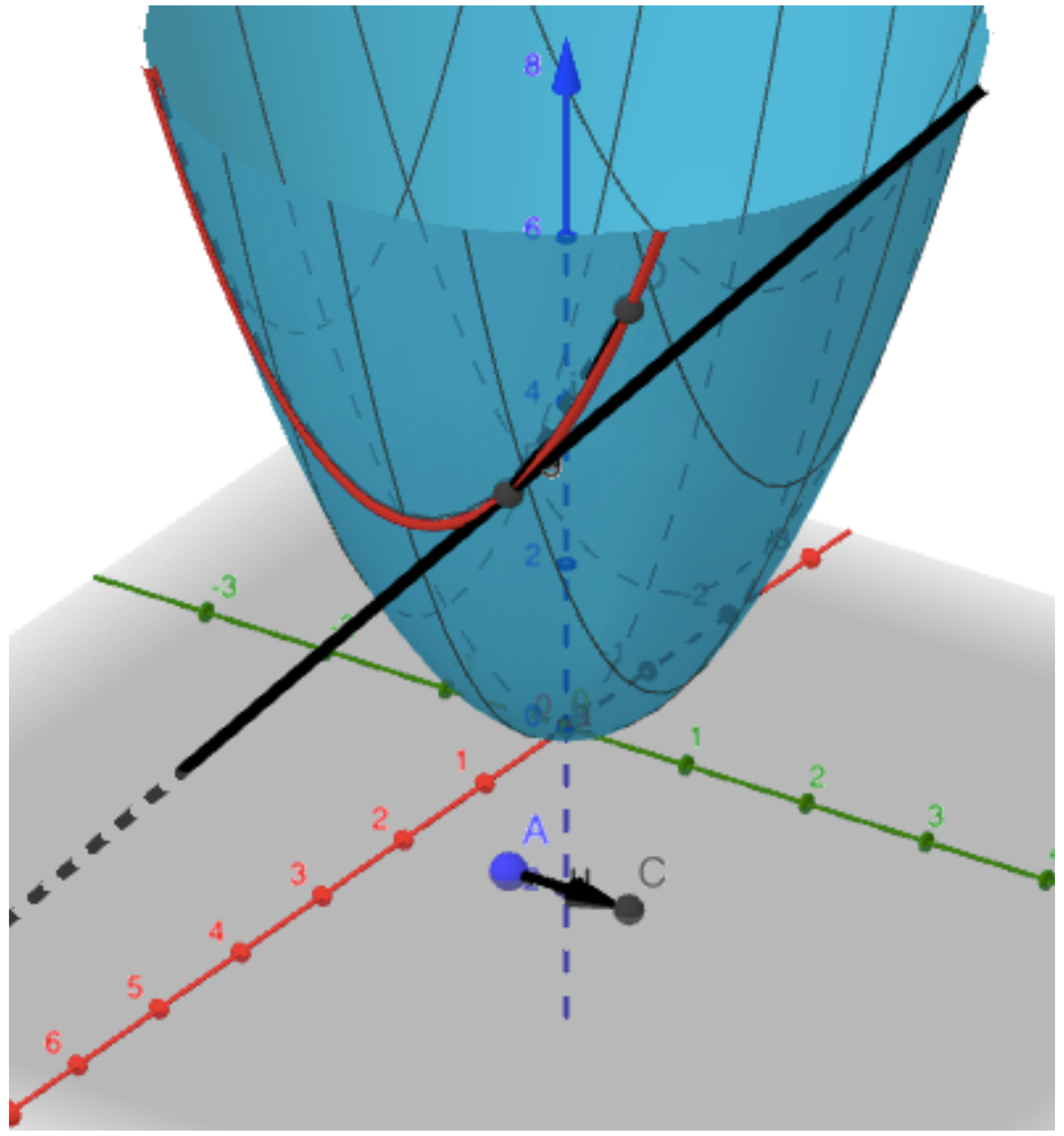
It is denoted many ways …

$\dfrac{\partial f}{\partial x}$ or $\dfrac{\partial}{\partial x}(f)$ or f_x or $D_x(f)$

A partial derivative is a function of x and y (even if x and y often aren't written there explicitly.)

We just computed $f_x(1,1) = 2$.

Let's do another!!!



Compute $f_y(2,1)$.

$g(t) = < 2, 1 + t, (2)^2 + (1 + t)^2 >$

$z'(t) = 2(1 + t)$

$z'(0) = 2$ $\qquad f_y(2,1) = 2$

Compute $f_x(2,1)$.

$g(t) = < 2 + t,\ 1,\ (2 + t)^2 + (1)^2 >$

$z'(t) = 2(2 + t)$

$z'(0) = 4$ $\qquad f_x(2,1) = 4$

Partial Derivatives, pg 3. Examples 2,3.

$f(x, y) = x^2 + y^2$

We can compute some
general partial derivatives at (a,b):

$g_x(t) = < a + t,\ b,\ (a+t)^2 + b^2 >$

$z'(t) = 2(a + t)$

$z'(0) = 2a$

$f_x(a, b) = 2a$

we often write

$f_x(x, y) = 2x$

Similarly,

$g_y(t) = < a,\ b + t,\ a^2 + (b+t)^2 >$

$z'(t) = 2(b + t)$

$z'(0) = 2b$

$f_y(a, b) = 2b$

or

$f_y(x, y) = 2y$

Another example. $f(x, y) = x^2y^2$

Compute $f_x(a, b)$ and $f_y(a, b)$.

$g_x(t) = < a + t,\ b,\ (a+t)^2b^2 >$

$z'(t) = 2(a + t)b^2$

$z'(0) = 2ab^2 = f_x(a, b)$

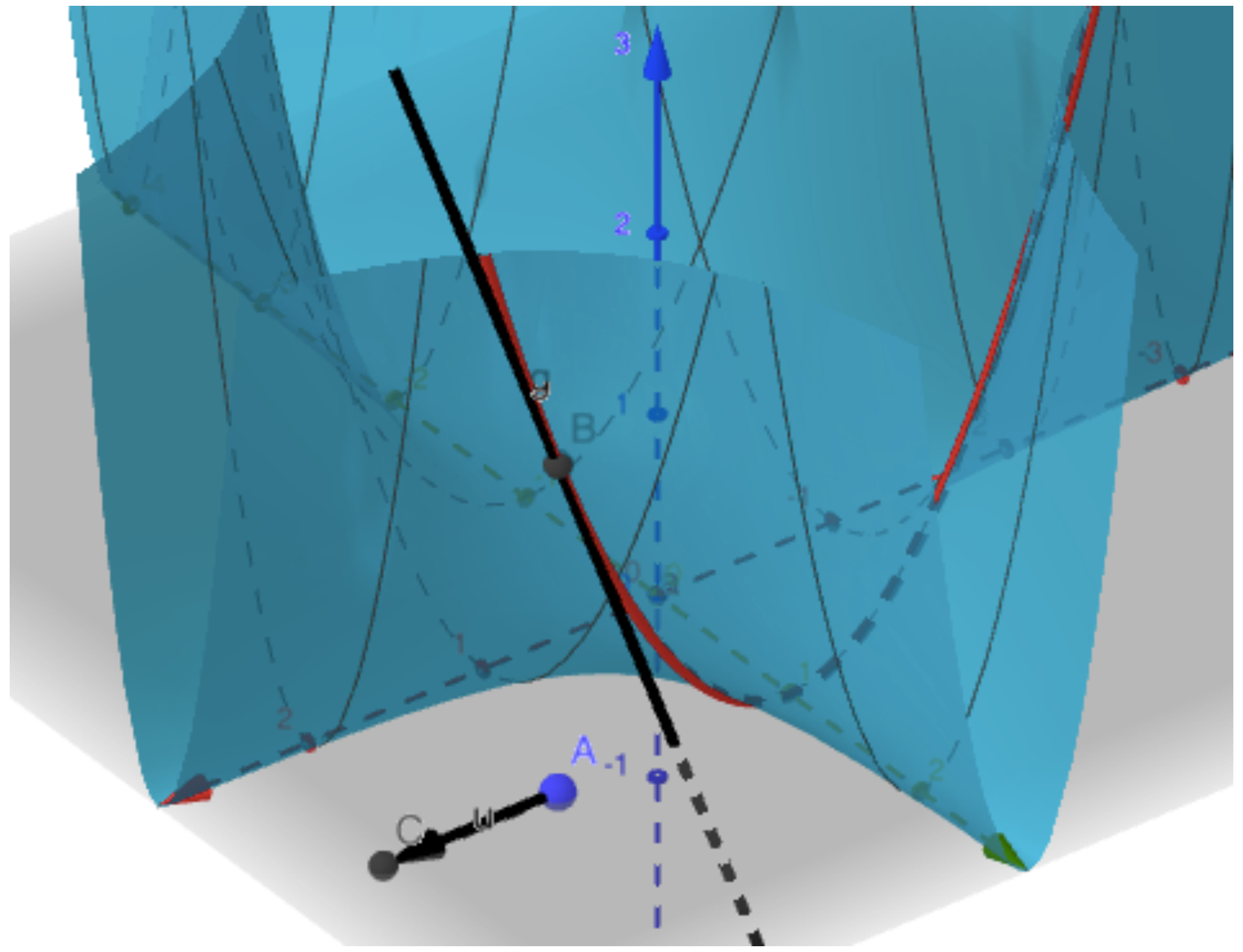
$g_y(t) = < a, b + t, a^2(b+t)^2 >$

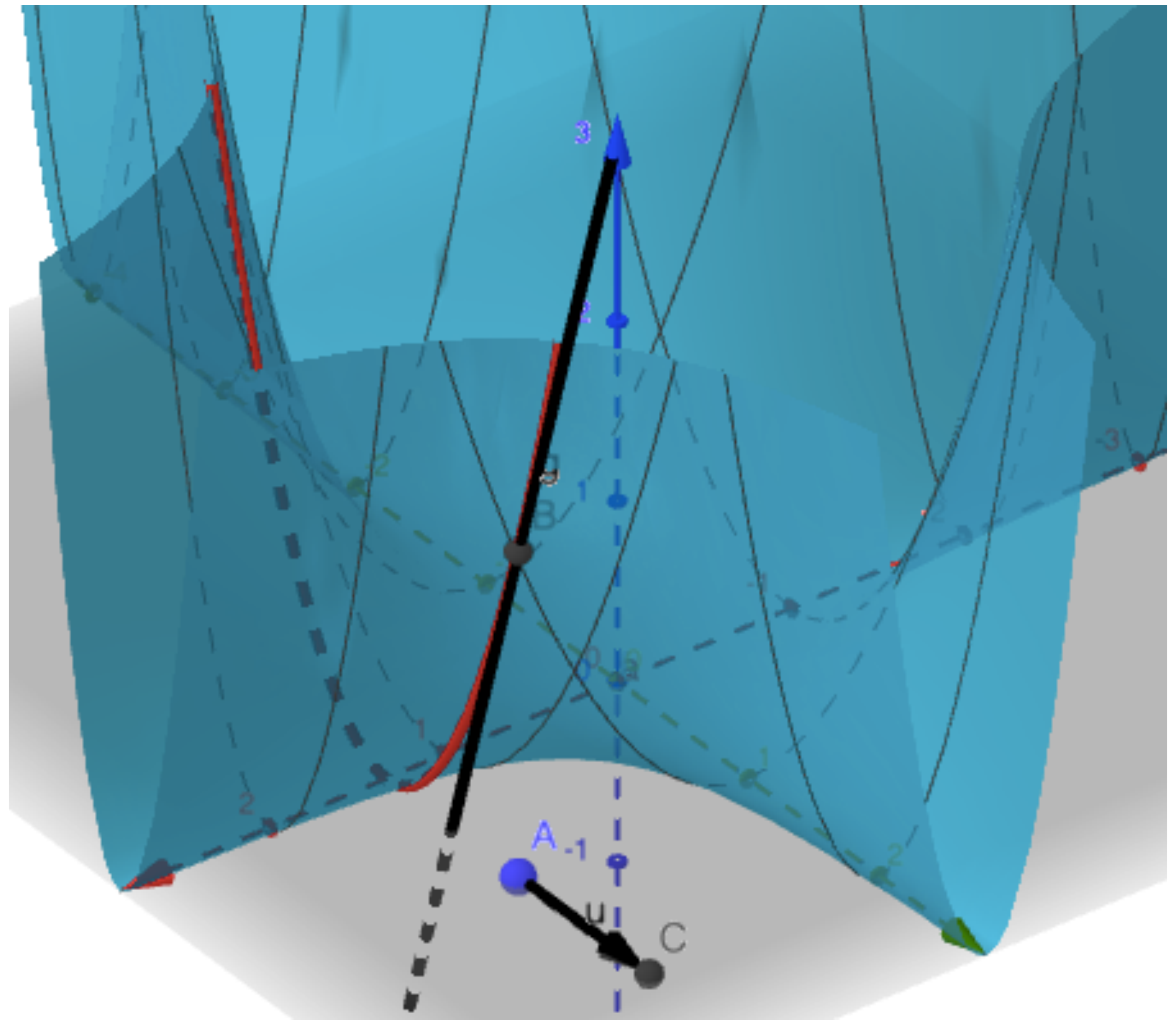
$z'(t) = 2a^2(b + t)$

$z'(0) = 2a^2b = f_y(a, b)$

$f_x(x, y) = 2xy^2$

$f_y(x, y) = 2x^2y$





Even more generally, given any function $f(x, y)$ (whose partial derivatives exist), you can find f_x by treating y like a constant, and differentiating the result with respect to x. Similarly f_y can be computed by treating x like a constant, and differentiating with respect to y. Example4 →

Partial Derivatives, pg 4. Example 4.

$f(x, y) = \sin(xy)$

Compute $f_x(a, b)$ and $f_y(a, b)$.

$g_x(t) = < a + t,\ b,\ \sin((a + t)b) >$

$z'(t) = b\cos((a + t)b)$

$z'(0) = b\cos(ab)$

$f_x(x, y) = y\cos(xy)$

$g_y(t) = < a,\ b + t,\ \sin(a(b + t)) >$

$z'(t) = a\cos(a(b + t))$

$z'(0) = a\cos(ab)$

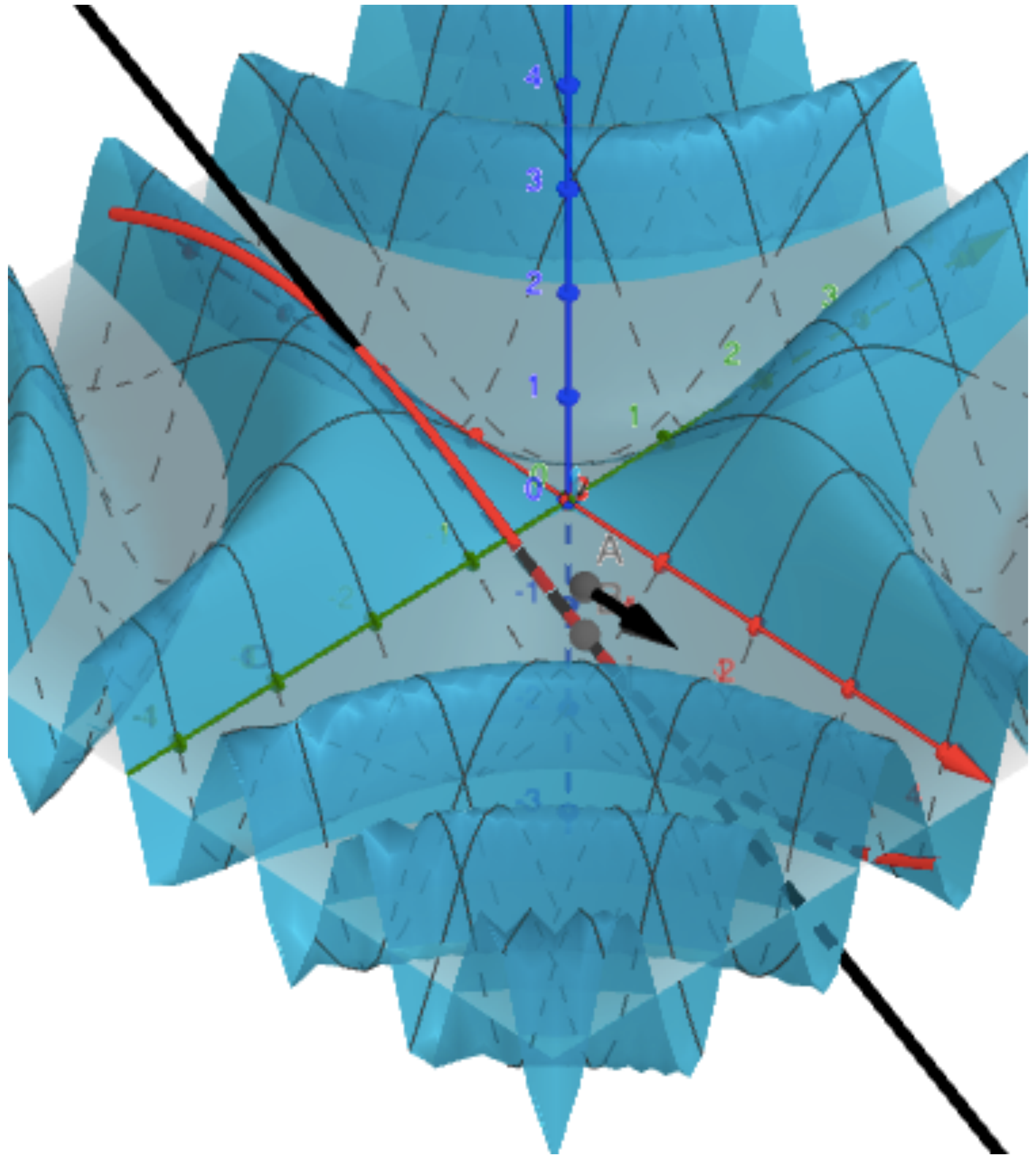
$f_y(x, y) = x\cos(xy)$

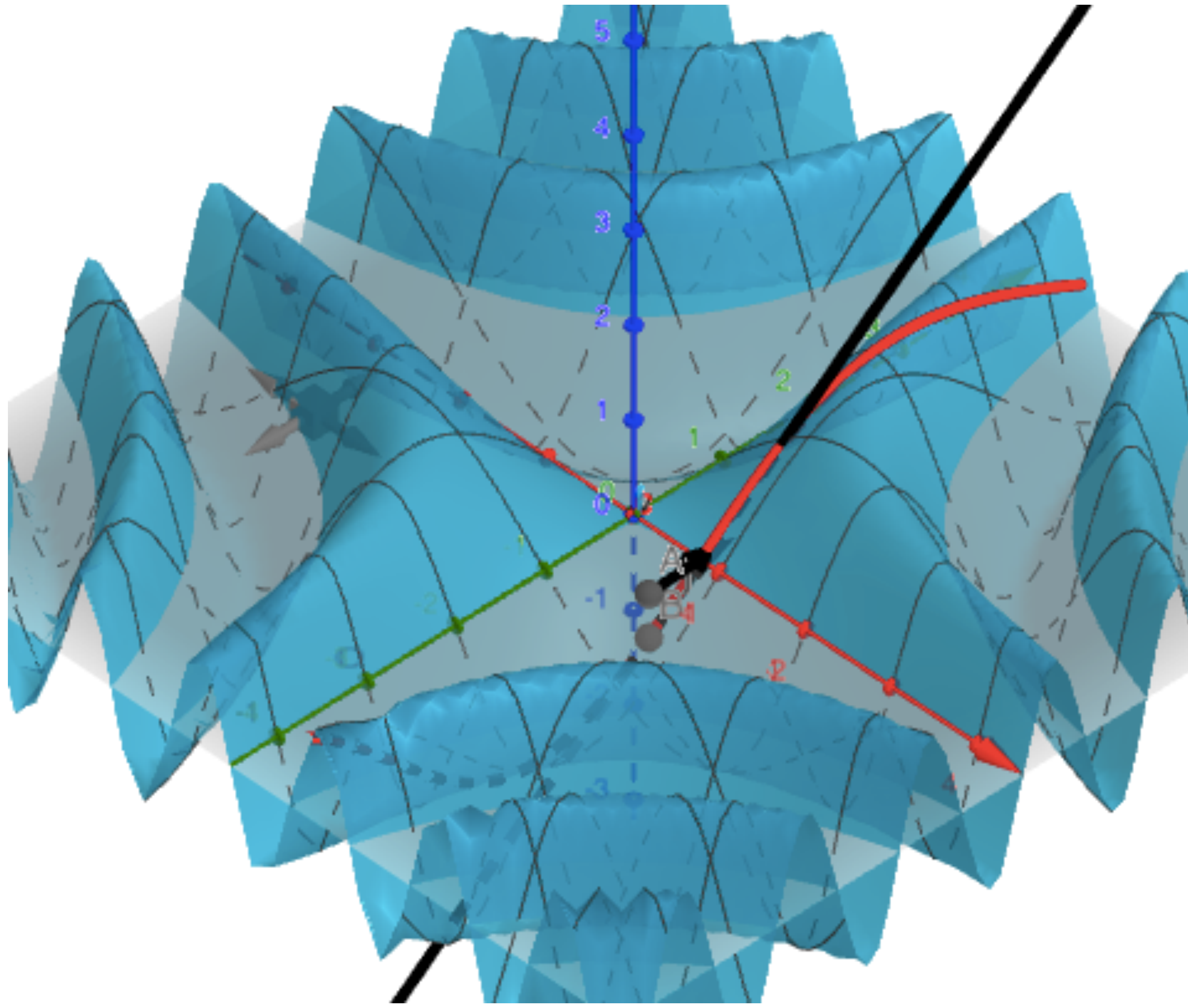
Note1: when $f(x, y) = \sin(xy)\ldots$

$f_x(0.8, -0.6) = -0.6\cos(-0.48) \approx -0.53$

$f_y(0.8, -0.6) = 0.8\cos(-0.48) \approx 0.71$

We estimated these values earlier.
(Illustrations on the right.)





Note2: Again we see that these partial derivatives can be computed by treating one of the variables as though it were a constant.

$$\frac{\partial}{\partial x}(\sin(xy)) = y\cos(xy)$$

$$\frac{\partial}{\partial y}(\sin(xy)) = x\cos(xy)$$

Partial Derivatives, pg 5. Practice.

Compute $f_x(x,y)$ and $f_y(x,y)$.

1. $f(x,y) = xy^2 - yx^2$
2. $f(x,y) = x^4 + 5xy^3$
3. $f(x,y) = \cos(xy)$
4. $f(x,y) = \dfrac{x-y}{1+x^2+y^2}$
5. $f(x,y) = \sin(x+y)\cdot\cos(y)$

	$f_x(x,y)$	$f_y(x,y)$
#1	$y^2 - 2yx$	$2xy - x^2$
2	$4x^3 + 5y^3$	$15xy^2$
3	$-y\sin(xy)$	$-x\sin(xy)$
4	$\dfrac{1-x^2+y^2+2xy}{(1+x^2+y^2)^2}$	$\dfrac{-1-x^2+y^2-2xy}{(1+x^2+y^2)^2}$
5	$\cos(x+y)\cos(y)$	$\cos(x+y)\cos(y)$ $-\sin(x+y)\sin(y)$

Some details to number 4:

$$f_x(x,y) = \frac{\frac{\partial}{\partial x}(x-y)\cdot(1+x^2+y^2) - (x-y)\cdot\frac{\partial}{\partial x}(1+x^2+y^2)}{(1+x^2+y^2)^2} = \frac{1(1+x^2+y^2)-(x-y)(2x)}{(1+x^2+y^2)^2}$$

$$f_y(x,y) = \frac{\frac{\partial}{\partial y}(x-y)\cdot(1+x^2+y^2) - (x-y)\cdot\frac{\partial}{\partial y}(1+x^2+y^2)}{(1+x^2+y^2)^2} = \frac{-1(1+x^2+y^2)-(x-y)(2y)}{(1+x^2+y^2)^2}$$