M110C Week7

Goals:
Recap, Warm Up.
Tangent Planes to Level Surfaces.
2nd derivatives in a given direction.
Differentiability.
Chain Rule.
Applied derivatives:
  depth.
  temperature.
  temporal wave.

Recap, Warm-Up, pg 1.

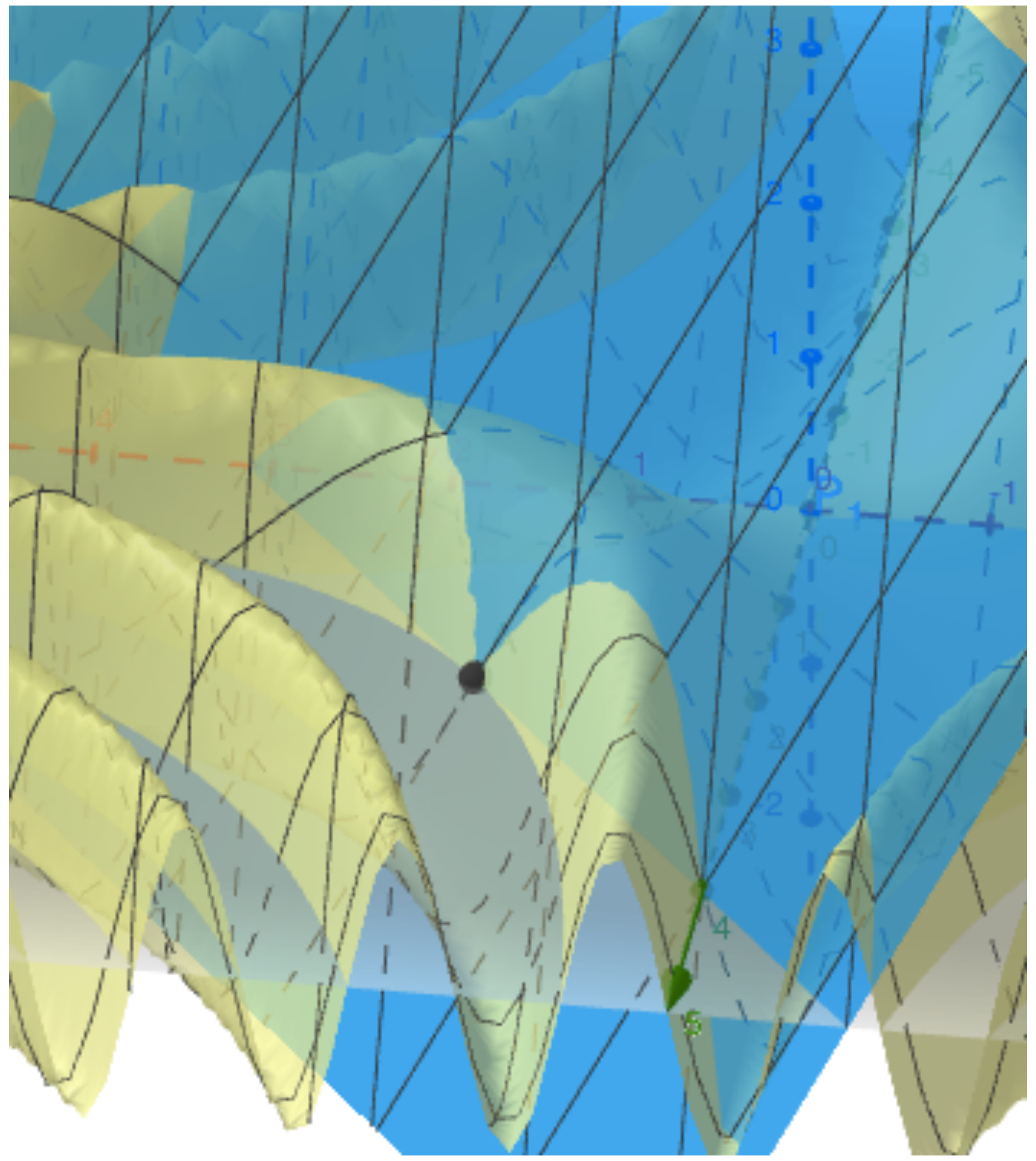
What did we see last time?

Higher order partial derivatives.

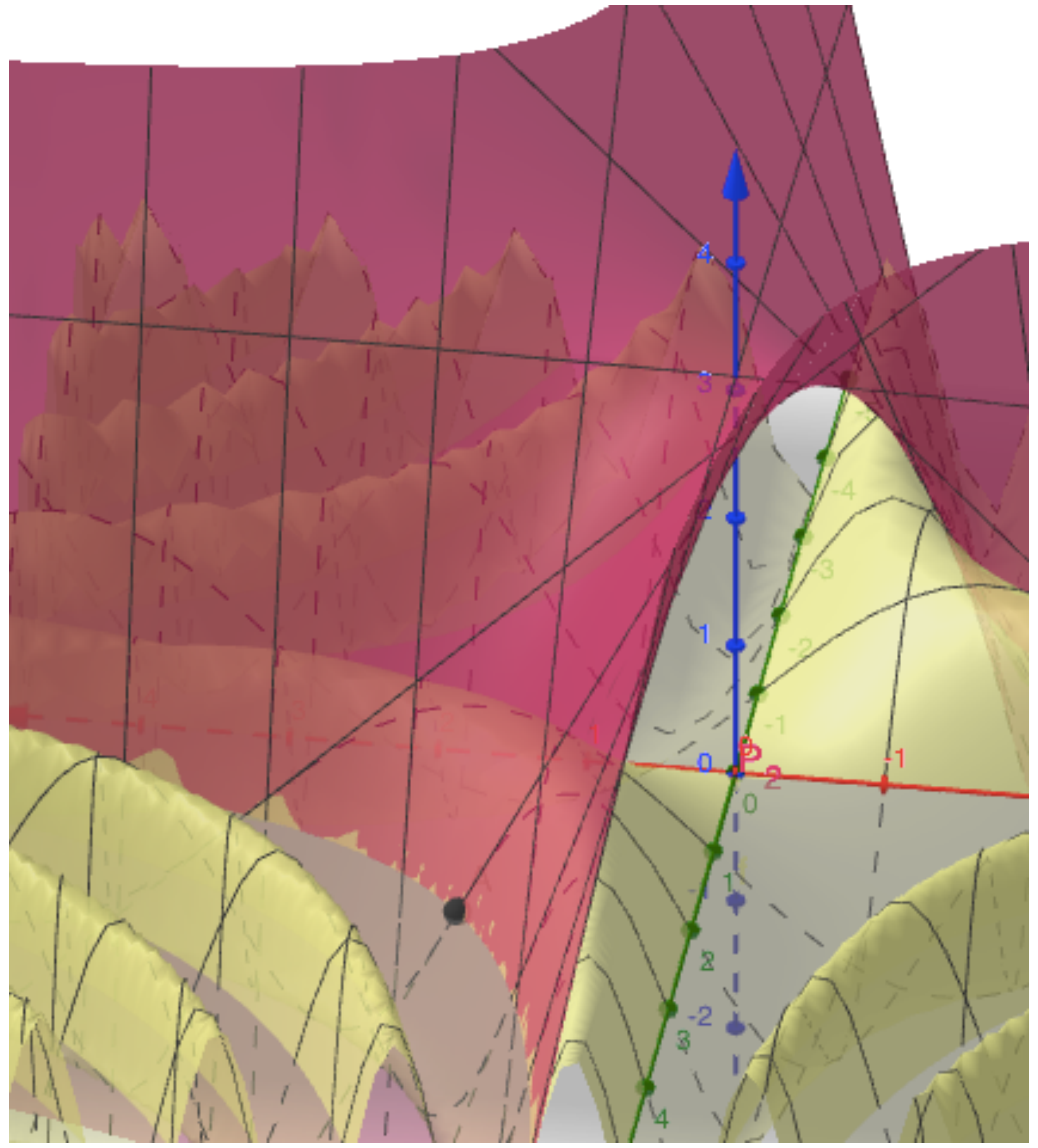
Taylor Series for multivariable functions!!!

We have Taylor Polynomial approximations of degree n to any function whose (partial) derivatives exist up to degree n.

$f(x, y) = \sin(xy)$



1st degree approximation



2nd degree approximation.

$$T_1(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

$$T_2(x, y) = T_1(a, b) + \frac{1}{2!}\big(f_{xx}(a, b)(x - a)^2 + 2f_{xy}(a, b)(x - a)(y - b) + f_{yy}(a, b)(y - b)^2\big)$$

Find the first and second degree polynomial approximations to

$f(x, y) = \ln(xy + 1)$ at $P(1,2)$

$$f_x(x, y) = \frac{1}{xy + 1} \cdot y = \frac{y}{xy + 1} = y(xy + 1)^{-1} \qquad f_x(1,2) = 2/3$$

$$f_y(x, y) = \frac{1}{xy + 1} \cdot x = \frac{x}{xy + 1} = x(xy + 1)^{-1} \qquad f_y(1,2) = 1/3$$

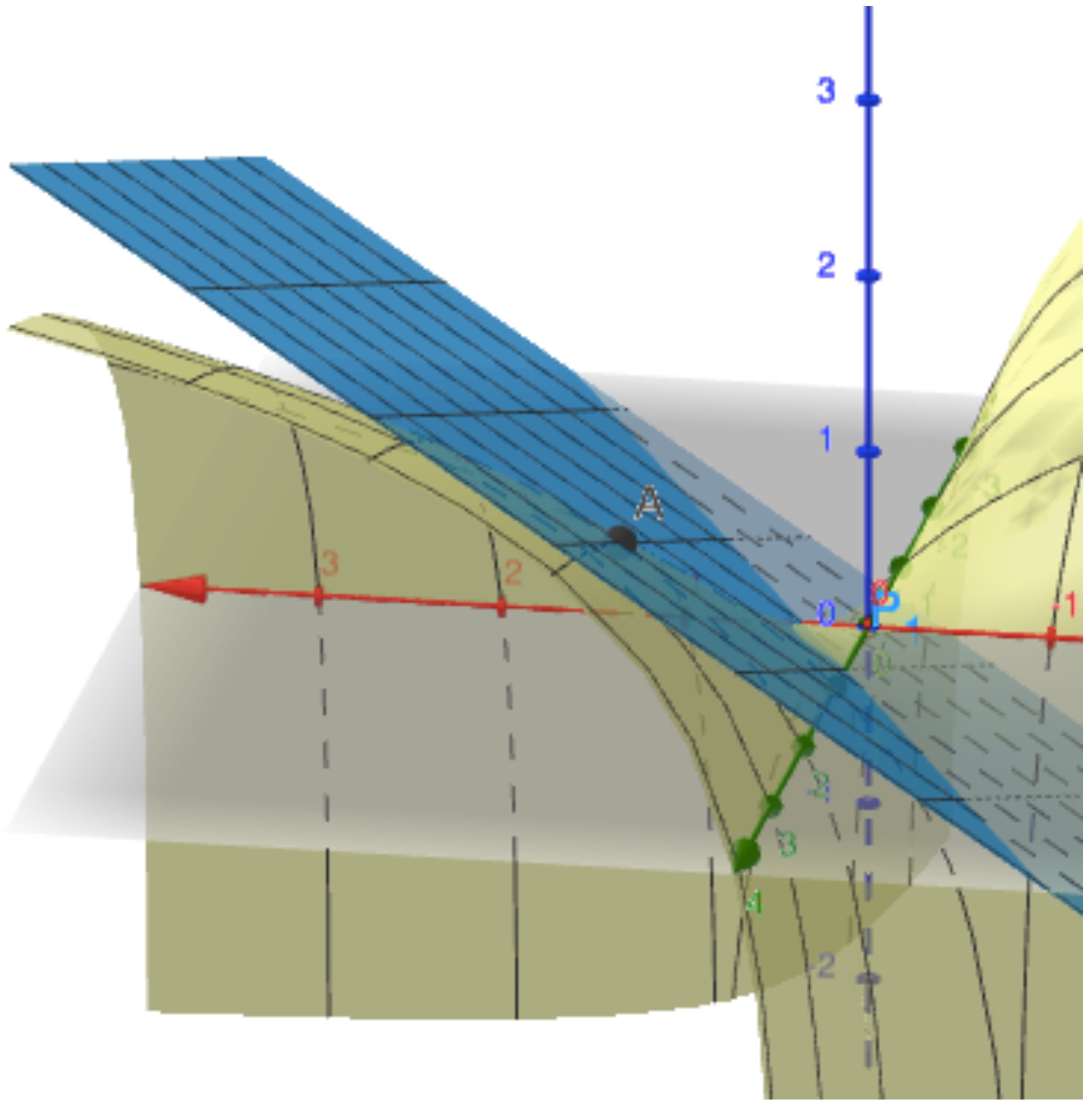
$$f_{xx}(x, y) = -y(xy + 1)^{-2} \cdot y = -\frac{y^2}{(xy + 1)^2} \qquad f_{xx}(1,2) = -4/9$$

$$f_{yy}(x, y) = -x(xy + 1)^{-2} \cdot x = -\frac{x^2}{(xy + 1)^2} \qquad f_{xy}(1,2) = 1/9$$

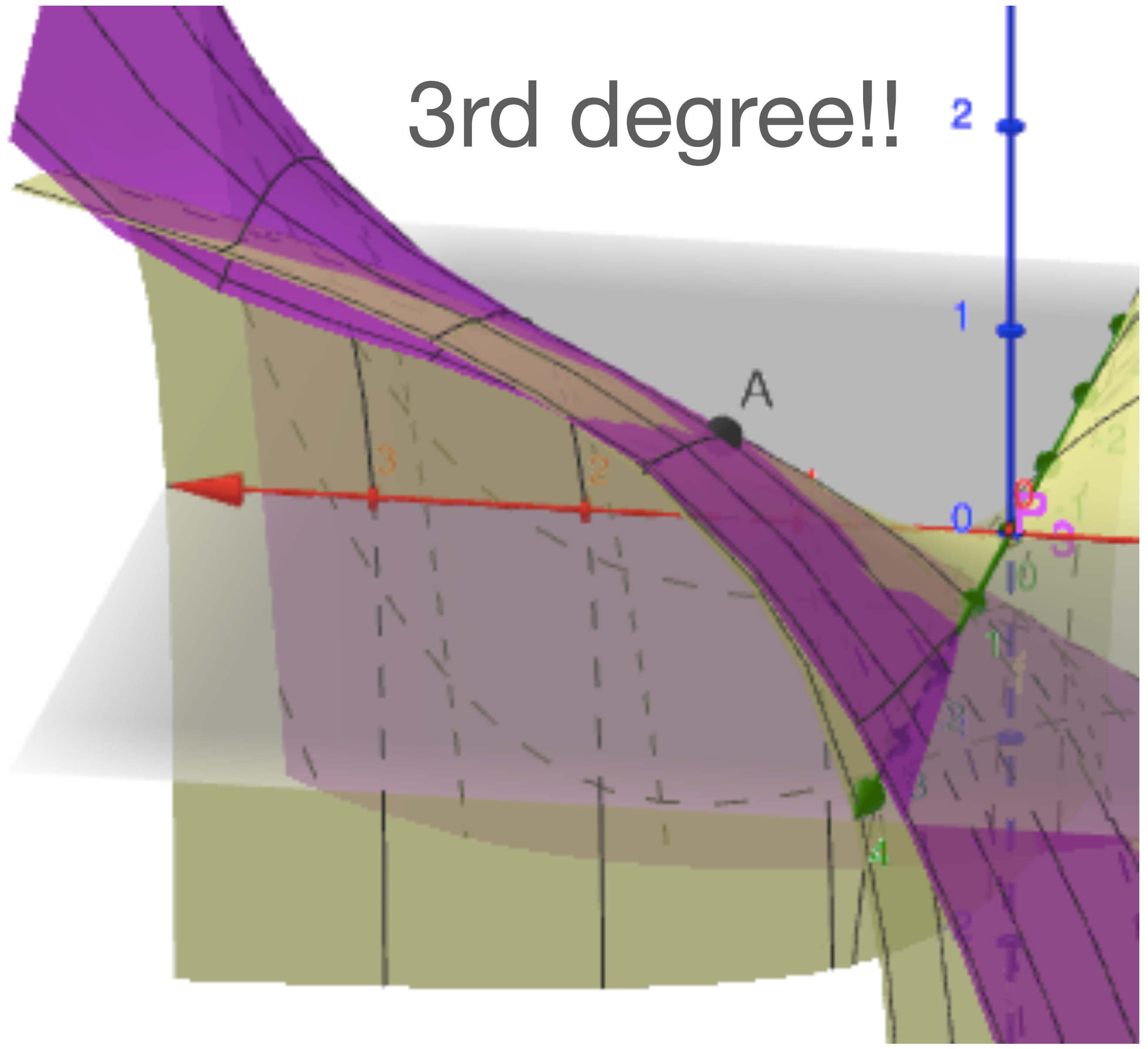
$$f_{xy}(x, y) = \frac{1(xy + 1) - y \cdot x}{(xy + 1)^2} = \frac{1}{(xy + 1)^2} \qquad f_{yy}(1,2) = -1/9$$

Warm-Up, pg 2. $f(x, y) = \ln(xy + 1)$ at $P(1,2)$

1st degree approximation.



3rd degree!!



$$f_{xxx}(x, y) = 2y^3(xy + 1)^{-3} \qquad f_{xxx}(1,2) = 16/27$$

$$f_{xxy}(x, y) = -2y(xy + 1)^{-3} \qquad f_{xxy}(1,2) = -4/27$$

$$f_{yyx}(x, y) = -2x(xy + 1)^{-3} \qquad f_{xyy}(1,2) = -2/27$$

$$f_{yyy}(x, y) = 2x^3(xy + 1)^{-3} \qquad f_{yyy}(1,2) = 2/27$$

$$T_3(x, y) = T_2(x, y)$$

$$+\frac{1}{3!}\Big(f_{xxx}(a, b)(x - a)^3 + 3f_{xxy}(a, b)(x - a)^2(y - b)$$

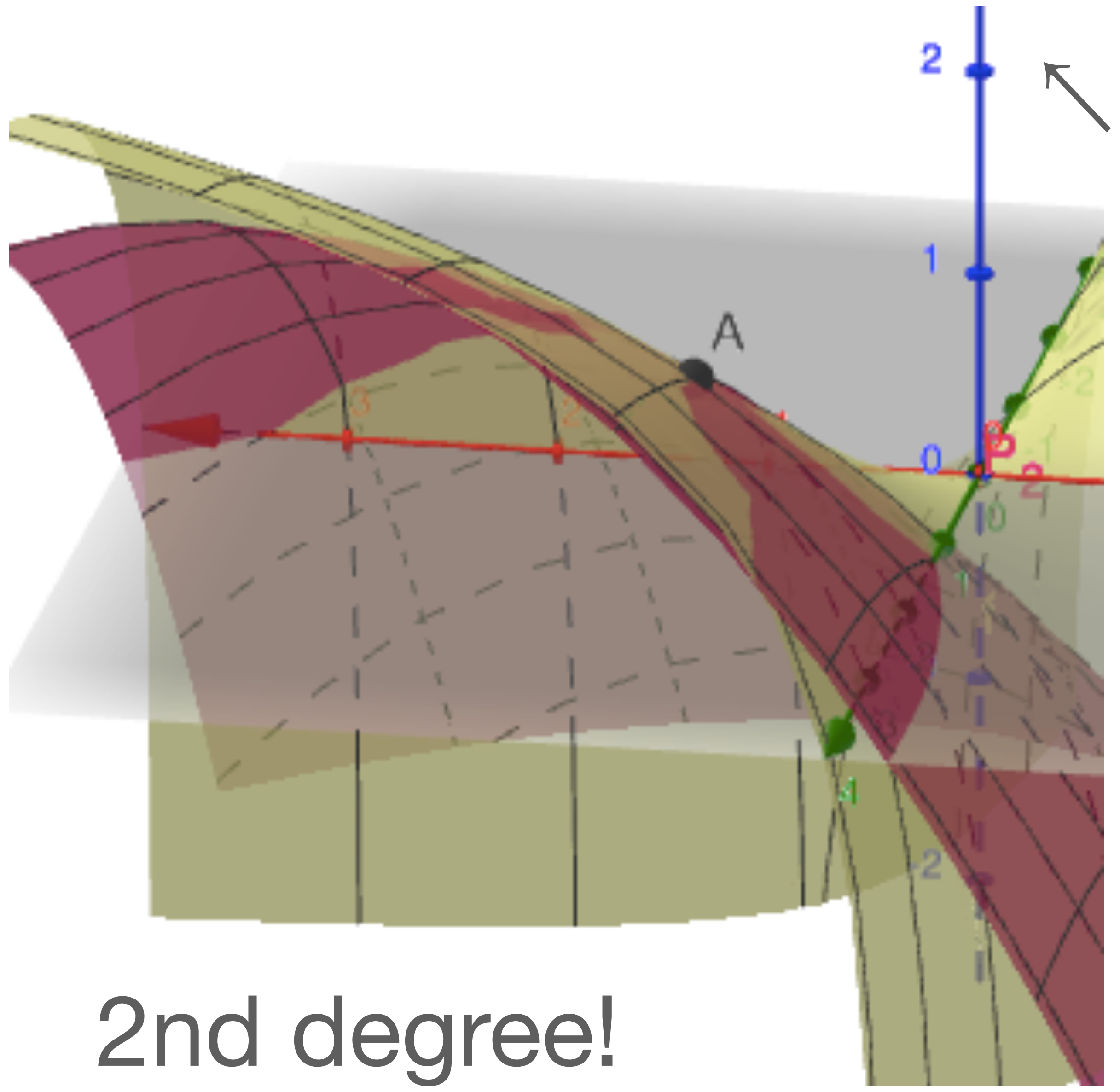
$$+3f_{xyy}(a, b)(x - a)(y - b)^2 + f_{yyy}(a, b)(y - b)^3\Big)$$

$$= \ln(3) + \frac{2}{3}(x - 1) + \frac{1}{3}(y - 2)$$

$$+\frac{1}{2!}\left(-\frac{4}{9}(x - 1)^2 + \frac{2}{9}(x - 1)(y - 2) - \frac{1}{9}(y - 2)^2\right)$$

$$+\frac{1}{3!}\Big(\frac{16}{27}(x - 1)^3 - \frac{12}{27}(x - 1)^2(y - 2)$$

$$-\frac{6}{27}(x - 1)(y - 2)^2 + \frac{2}{27}(y - 2)^3\Big)$$



$\nwarrow$ $T_1(x, y) = \ln(3) + \frac{2}{3}(x - 1) + \frac{1}{3}(y - 2)$

$\leftarrow$ $T_2(x, y) = \ln(3) + \frac{2}{3}(x - 1) + \frac{1}{3}(y - 2)$

$$+\frac{1}{2!}\left(-\frac{4}{9}(x - 1)^2 + \frac{2}{9}(x - 1)(y - 2) - \frac{1}{9}(y - 2)^2\right)$$

2nd degree!

Recap, Warm-Up, pg 3.

What else did we see last time?

The gradient of a function $f(x,y)$: $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$

Directional derivatives can be expressed with a dot product.

$$f_u(a,b) = f_x(a,b)\hat{u}_x + f_y(a,b)\hat{u}_y$$
$$= \nabla f(a,b) \cdot \hat{u}$$

Extreme derivatives happen when $\hat{u}$ is parallel to ∇f

Maximum: $\hat{u} = \dfrac{\nabla f(a,b)}{|\nabla f(a,b)|}$

Minimum: $\hat{u} = -\dfrac{\nabla f(a,b)}{|\nabla f(a,b)|}$

Example.

$f(x,y) = (x-1)^2 + (y-2)^2$

a) Compute $f_u(5,5)$ when $\mathbf{u} = < 2,7 >$.

b) Find the extreme values of $f_u(5,5)$ over all $\mathbf{u}$

$a)\ \nabla f(x,y) = < 2x-2, 2y-4 > \quad \nabla f(5,5) = < 8, 6 >$

$$\hat{\mathbf{u}} = \frac{1}{\sqrt{53}} < 2,7 > = < 2/\sqrt{53}, 7/\sqrt{53} >$$

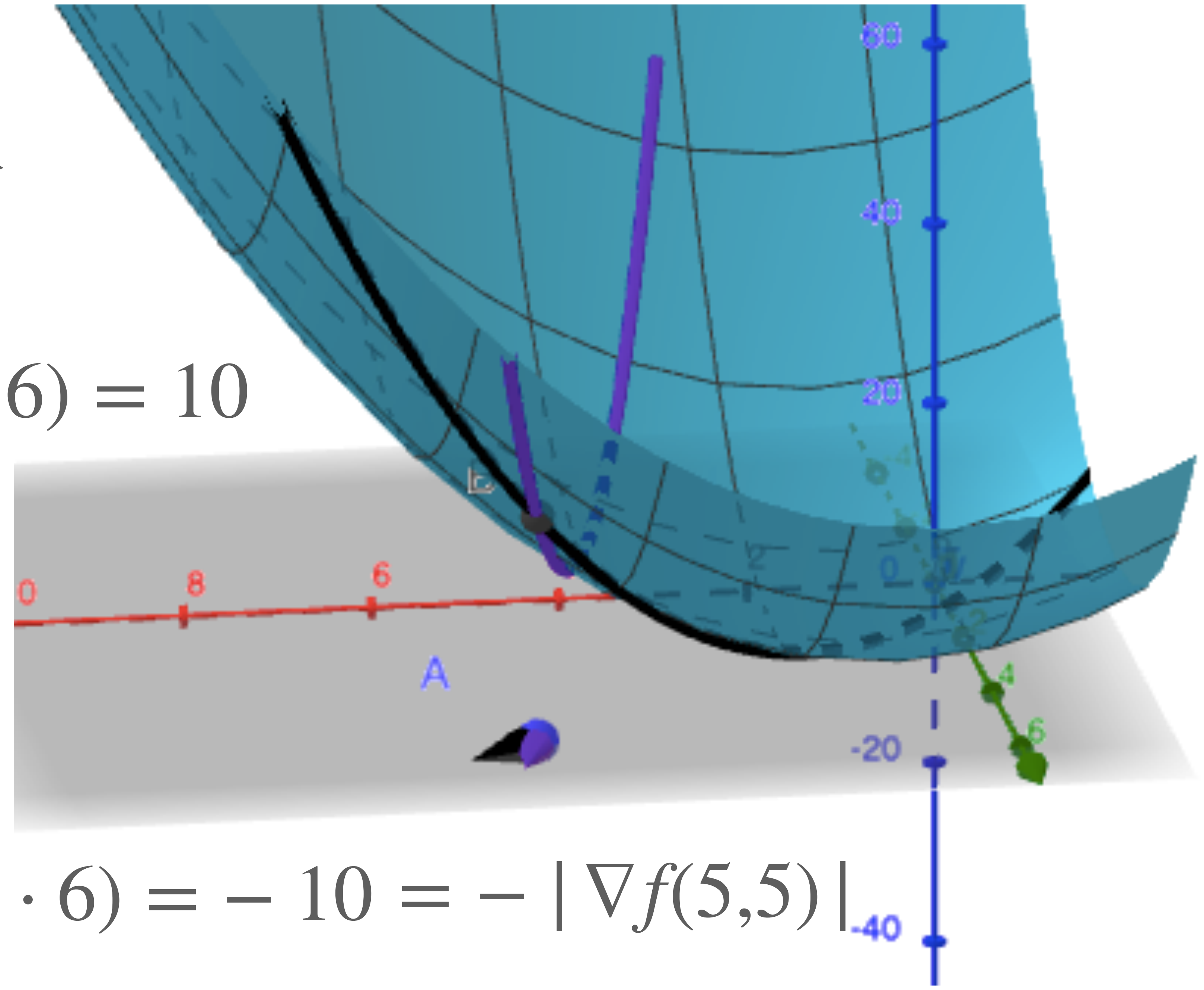
$$f_u(5,5) = 8 \cdot \frac{2}{\sqrt{53}} + 6 \cdot \frac{7}{\sqrt{53}} = \frac{56}{\sqrt{53}} \approx 7.69$$

b) The max derivative happens when $\hat{u}$ points in the same direction as the gradient.

$$\hat{\mathbf{u}}_{max} = \frac{1}{10} < 8,6 >$$

$$\hat{\mathbf{u}}_{min} = -\frac{1}{10} < 8,6 >$$

$$f_{u_{max}} = \frac{1}{10}(8 \cdot 8 + 6 \cdot 6) = 10$$
$$= |\nabla f(5,5)|$$



$$f_{u_{min}} = -\frac{1}{10}(8 \cdot 8 + 6 \cdot 6) = -10 = -|\nabla f(5,5)|$$

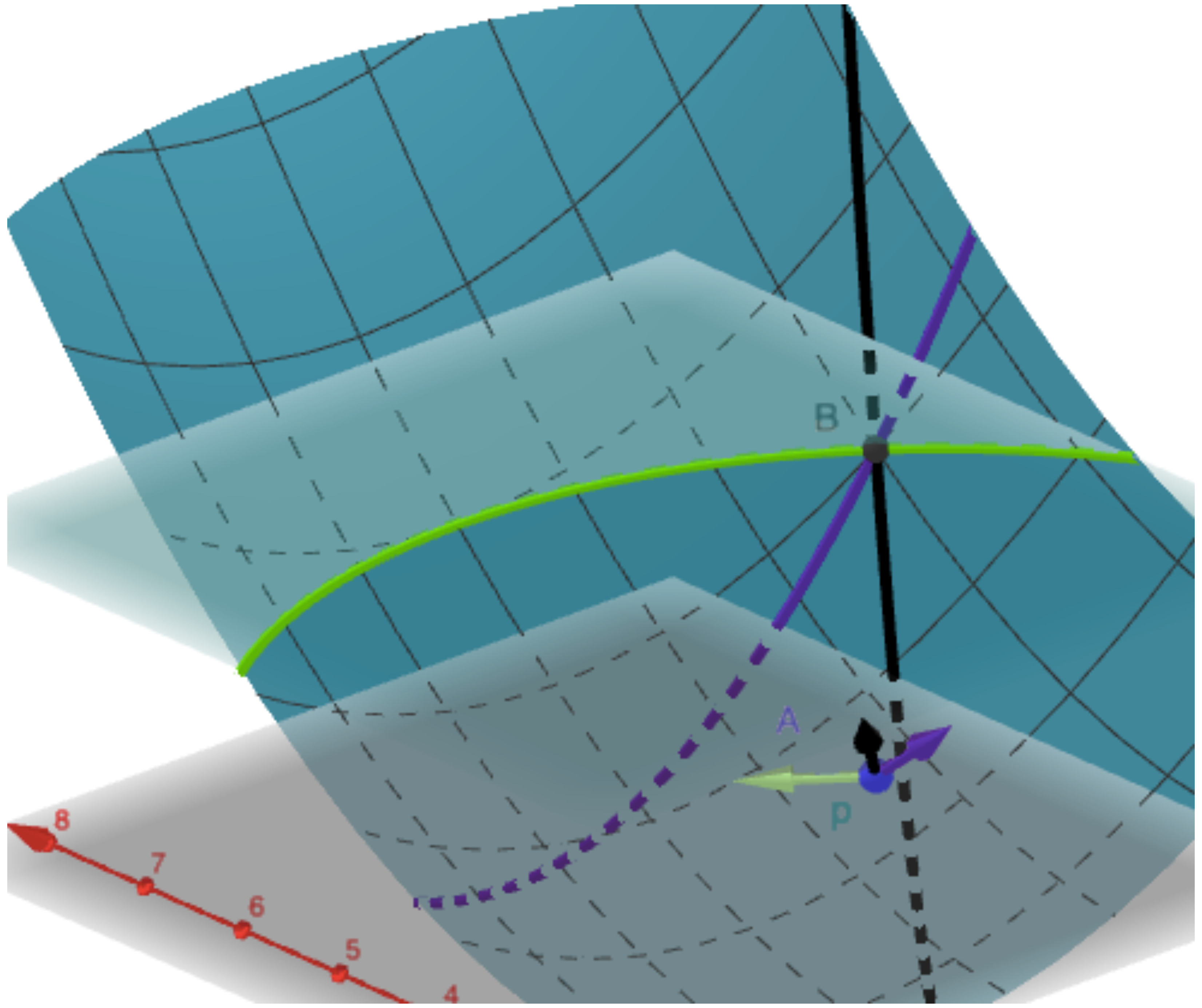
Tangent Planes to Level Surfaces, pg 1

Which $\hat{\mathbf{u}}$ is the directional derivative equal to 0?!

1. When the direction $\hat{\mathbf{u}}$ is perpendicular to $\nabla f(a,b)$.

$$0 = f_u(a,b) = \nabla f(a,b) \cdot \hat{\mathbf{u}}$$

2. When the direction $\hat{\mathbf{u}}$ is tangent to a level curve.



Putting 1 and 2 together, we get an important idea:

$\nabla f(a,b)$ is perpendicular to the level curve given by $z = f(a,b)$.

In the example we had previously $f(x,y) = (x-1)^2 + (y-2)^2$ at $(a,b) = (5,5)$

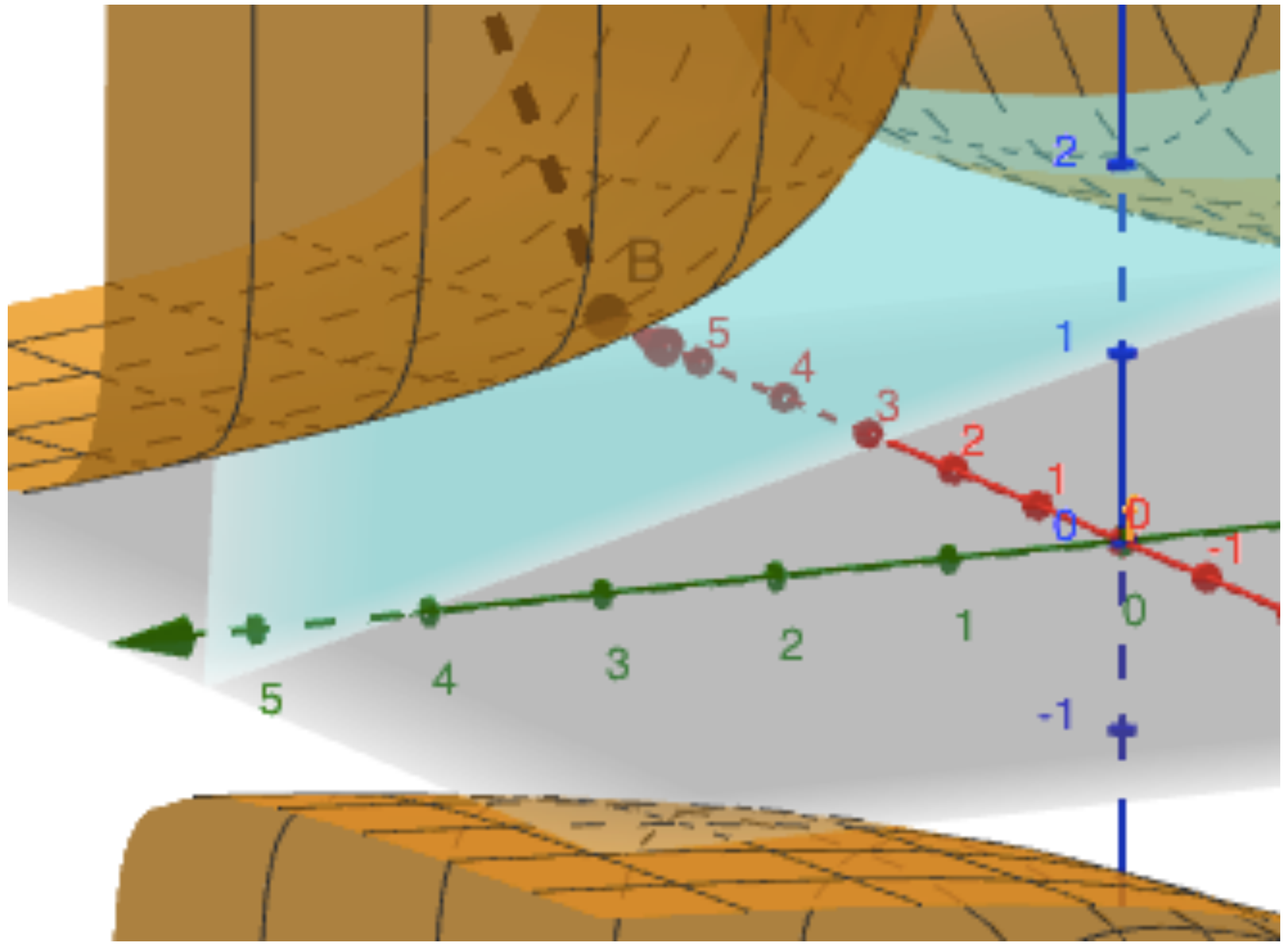
$$\hat{\mathbf{u}}_{\nabla f} = \frac{1}{10} < 8{,}6 > \qquad \hat{u}_{level} = \frac{1}{10} < 6, -8 >$$

The perpendicularity of the gradient to a level curve applies to functions with any number of input variables, i.e. functions whose "level curves" are surfaces of 2 or more dimensions.

Example (S14.6 #43). Find the the tangent plane to the surface $xy^2z^3 = 8$ at the point $(2{,}2{,}1)$.

$xy^2z^3 = 8$ is a level curve of the function $w(x,y,z) = xy^2z^3$.

This function has gradient $\nabla w = < y^2z^3, 2xyz^3, 3xy^2z^2 >$



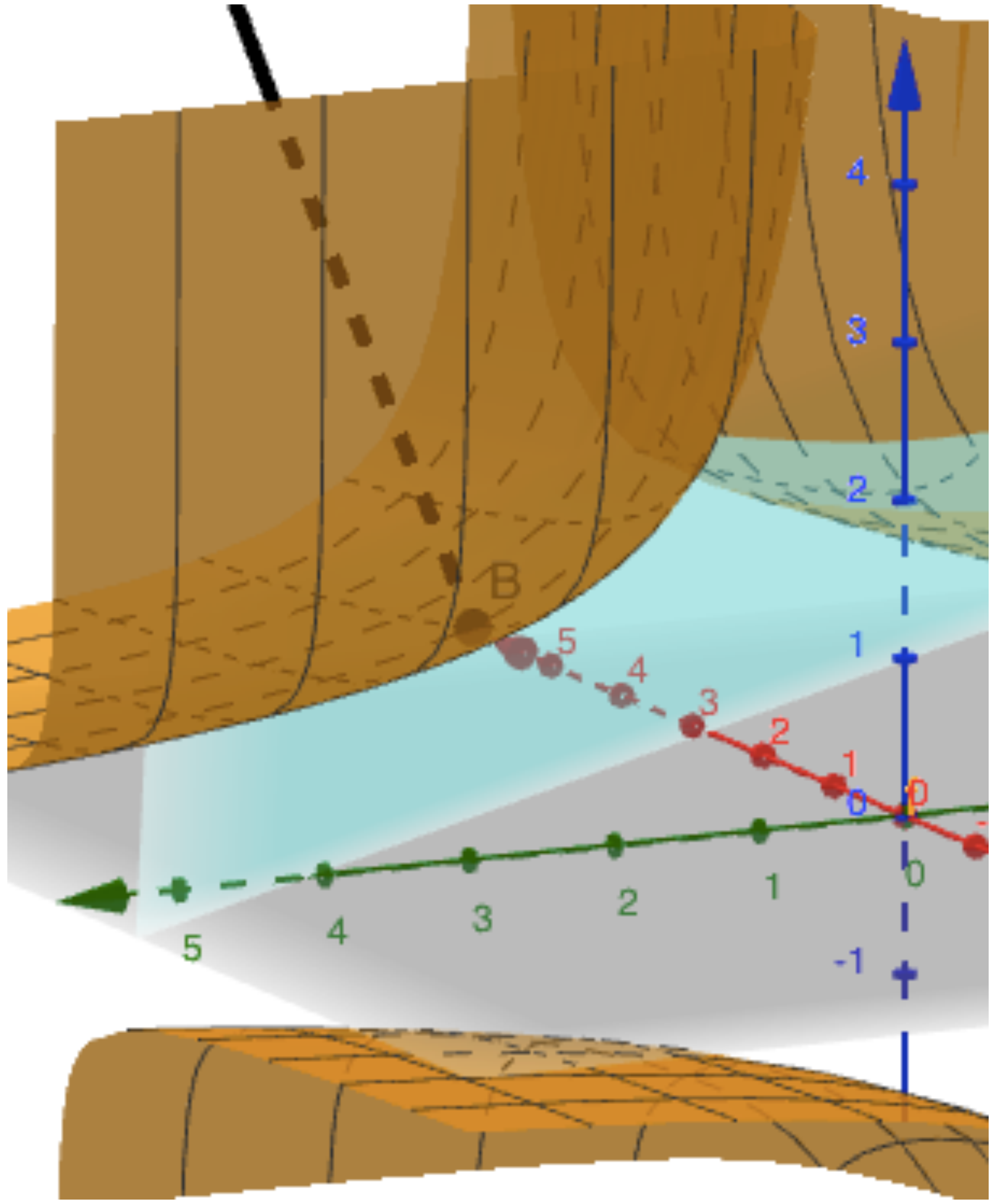
Tangent Planes to Level Surfaces, pg 2.

$w(x, y, z) = xy^2z^3 \quad \nabla w = < y^2z^3, 2xyz^3, 3xy^2z^2 >$

At every point (x, y, z), this gradient is perpendicular to corresponding level surface.

In particular, $\nabla w(2,2,1)$ is perpendicular to the level curve when $w = w(2,2,1) = 8$

$\nabla w(2,2,1) = < 4,8,24 > = 4 < 1,2,6 >$



The tangent plane is:

$1(x-2)+$
$2(y-2)+$
$6(z-1) = 0$

$x + 2y + 6z = 12$

Link: SurfaceTangentPlane

Try this. (OX sec 4.6# 303). Find the tangent plane to $xy + xz + yz = 5$ at the point $(2,1,1)$.

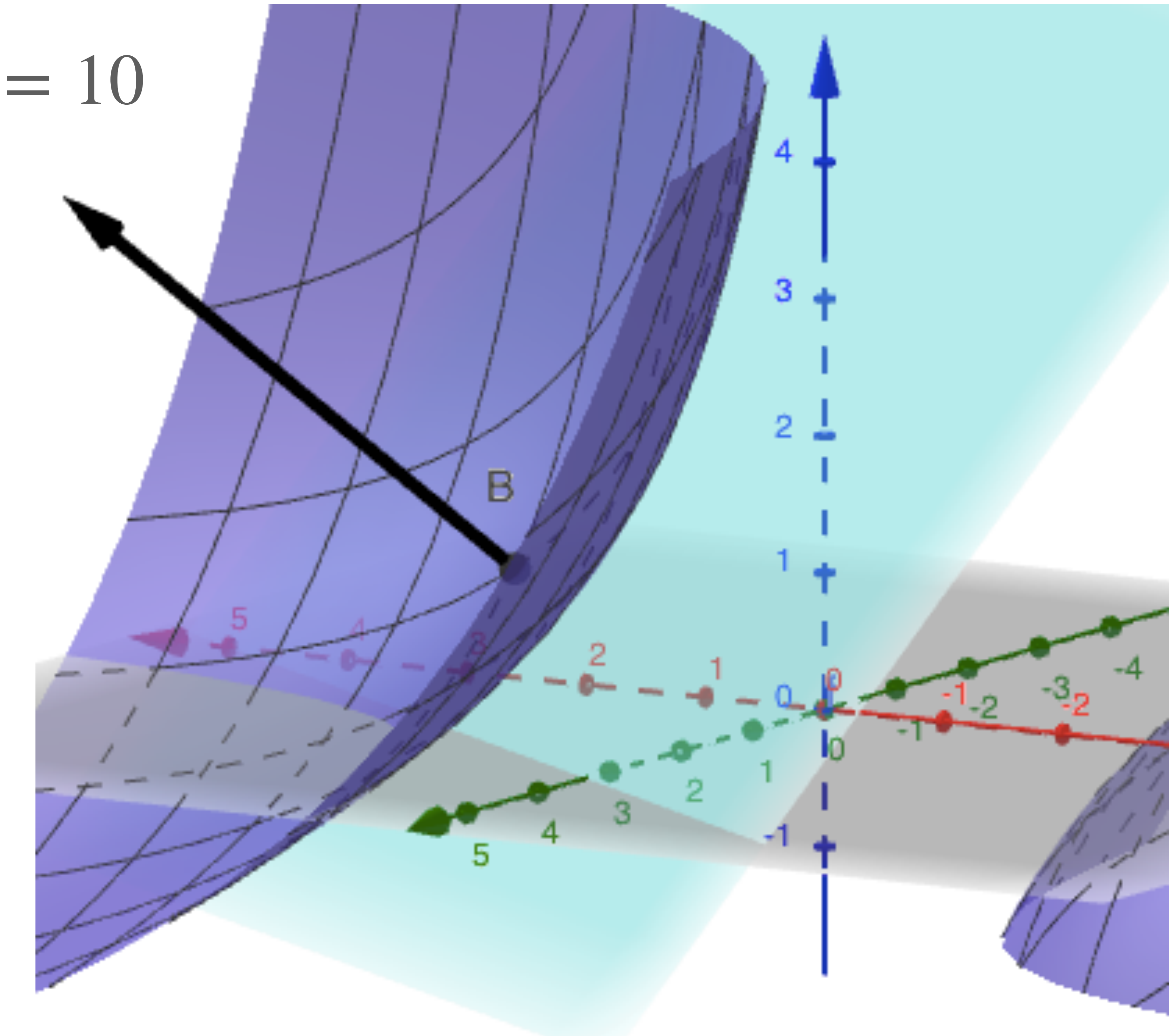
$w(x, y, z) = xy + xz + yz$

$\nabla w(x, y, z) = < y + z, x + z, x + y >$

$\nabla w(2,1,1) = < 2, 3, 3, >$

$2(x-2) + 3(y-1) + 3(z-1) = 0$

$2x + 3y + 3z = 10$



2nd degree directional derivatives.

$f(x, y) = \ln(xy + 1)$ at $P(1,2)$

Find the second-degree directional derivative of f at P in the direction of $\mathbf{u} = < 3,4 >$

Method1: Use the curve on the surface of the graph in the direction of $\mathbf{u}$.

Make $\mathbf{u}$ a unit vector.

$$\hat{\mathbf{u}} = \frac{1}{|\mathbf{u}|}\mathbf{u} = \frac{1}{5} < 3,4 >$$

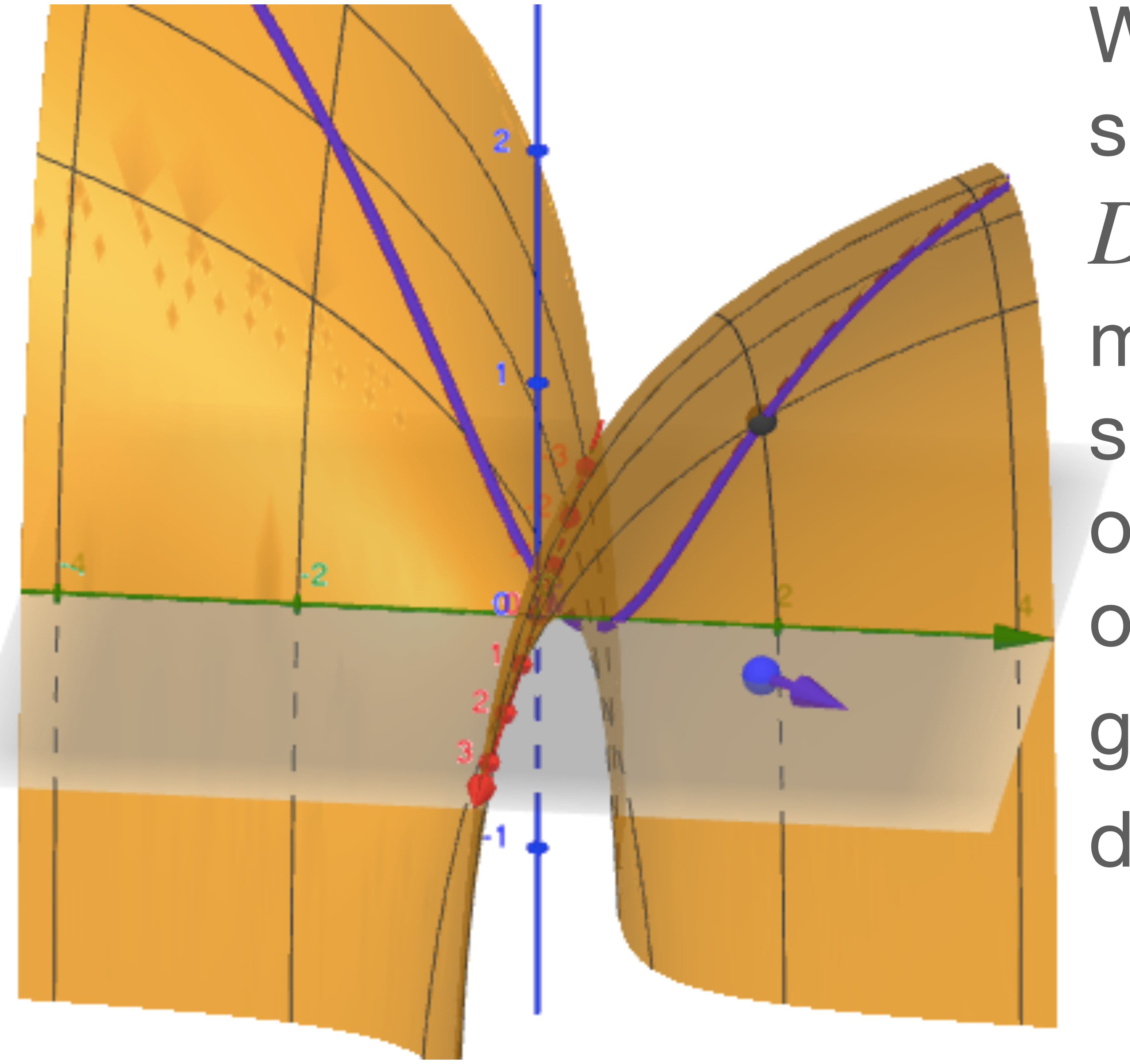
$$g(t) = < x(t), y(t), z(t) >$$
$$= < a + \hat{u}_x t,\ \ b + \hat{u}_y t,\ \ f(a + \hat{u}_x t, b + \hat{u}_y t) >$$
$$= < 1 + 3t/5, 2 + 4t/5,\ \ln((1 + 3t/5)(2 + 4t/5) + 1) >$$

$$z(t) = \ln((1 + 3t/5)(2 + 4t/5) + 1)$$
$$= \ln\left(\frac{12t^2 + 50t + 75}{25}\right) = \ln\left(12t^2 + 50t + 75\right) - \ln(25)$$

$$z'(t) = \frac{24t + 50}{12t^2 + 50t + 75}$$

$$z''(t) = \frac{24(12t^2 + 50t + 75) - (24t + 50)(24t + 50)}{(12t^2 + 50t + 75)^2}$$

$$z''(0) = \frac{24 \cdot 75 - 50 \cdot 50}{75^2} = -\frac{28}{225} \approx -0.124$$

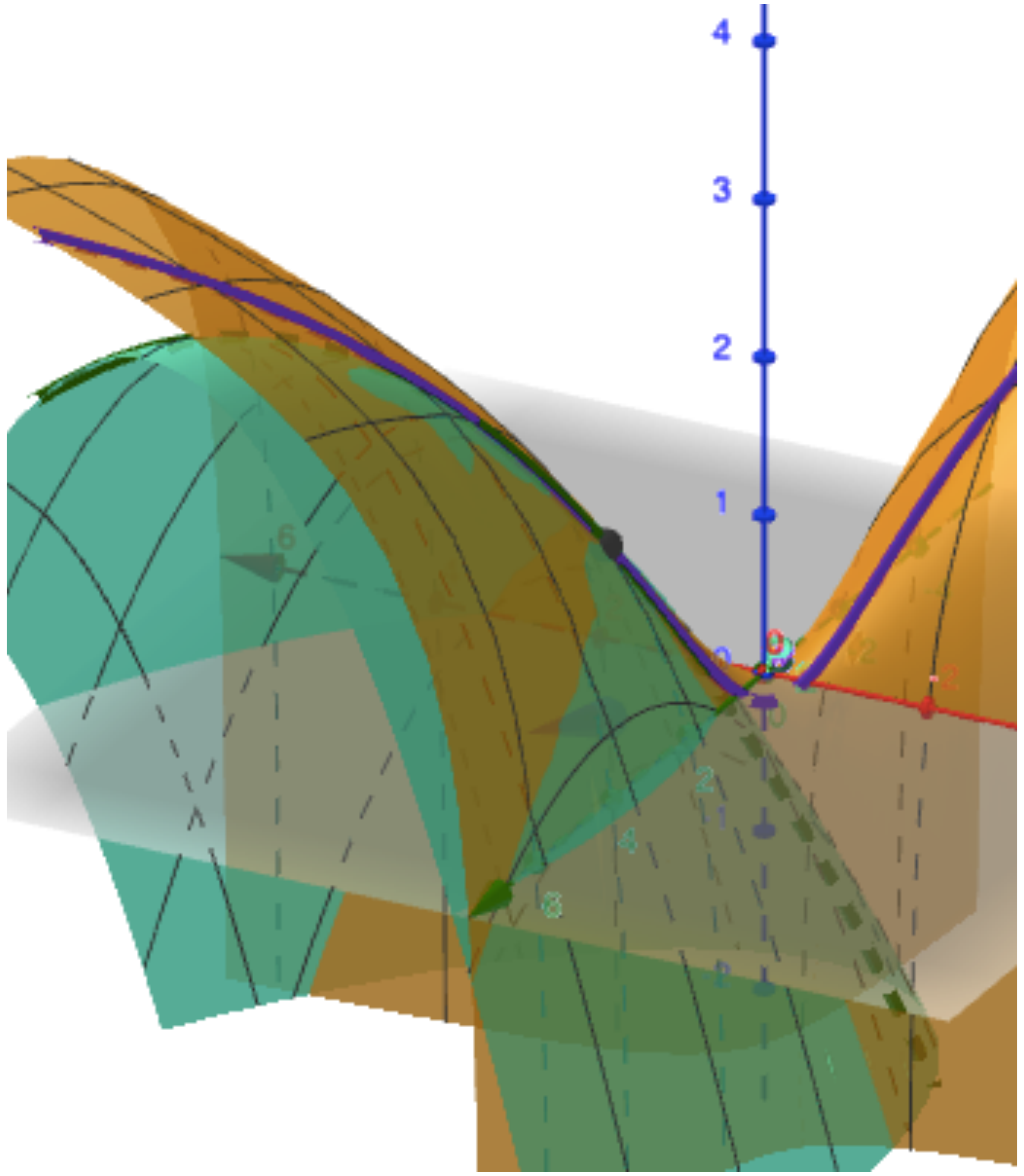


We've computed a second derivative, $D_u^2(f)(a, b)$, by means of the second derivative of the z component of the curve on the graph of f, in the direction of $\mathbf{u}$.

2nd degree directional derivatives, pg 2.

Method 2: use the curve on the surface of the tangent quadric of f at P.

This curve is not the same as the curve on the surface of f's graph, but it will have the same second derivative at P!



$$g(t) = < x(t), y(t), z(t) >$$
$$= < a + \hat{u}_x t,\ \ b + \hat{u}_y t,\ \ T_2(a + \hat{u}_x t, b + \hat{u}_y t) >$$

$$z(t) = T_2(a + \hat{u}_x t, b + \hat{u}_y t)$$
$$= f(a,b) + f_x(a,b)\hat{u}_x t + f_y(a,b)\hat{u}_y t$$
$$+\frac{1}{2!}\left(f_{xx}(a,b)(\hat{u}_x t)^2 + 2f_{xy}(a,b)(\hat{u}_x t)(\hat{u}_y t) + f_{yy}(a,b)(\hat{u}_y t)^2\right)$$

$$z''(t) = \ldots = f_{xx}(a,b)\hat{u}_x^2 + 2f_{xy}(a,b)\hat{u}_x\hat{u}_y + f_{yy}(a,b)\hat{u}_y^2 = z''(0)$$

In our example:

$$f(x,y) = \ln(xy+1)$$
$$(a,b) = (1,2)$$
$$\mathbf{u} = < 3,4 >$$
$$\hat{\mathbf{u}} = \frac{1}{5} < 3,4 >$$

$$f_{xx}(1,2) = -4/9$$
$$f_{xy}(1,2) = 1/9$$
$$f_{yy}(1,2) = -1/9$$

$$z''(0) = -\frac{4}{9}\cdot\frac{9}{25} + 2\cdot\frac{1}{9}\cdot\frac{12}{25} - \frac{1}{9}\cdot\frac{16}{25}$$
$$= -\frac{28}{225} \approx -0.124$$

Formula! $D_u^2(f)(a,b) = f_{xx}(a,b)\hat{u}_x^2 + 2f_{xy}(a,b)\hat{u}_x\hat{u}_y + f_{yy}(a,b)\hat{u}_y^2$

2nd degree directional derivatives. Practice.

Find the second derivative of f at P, in the direction of $\mathbf{u}$.

1. $f(x,y) = e^{-x^2-y^2}$ $\quad P(0,0) \quad \hat{\mathbf{u}} = < 1/\sqrt{10}, 3/\sqrt{10} >$

$f_x = -2xe^{-x^2-y^2}$ $\quad f_x(0,0) = 0$

$f_y = -2ye^{-x^2-y^2}$ $\quad f_y(0,0) = 0$

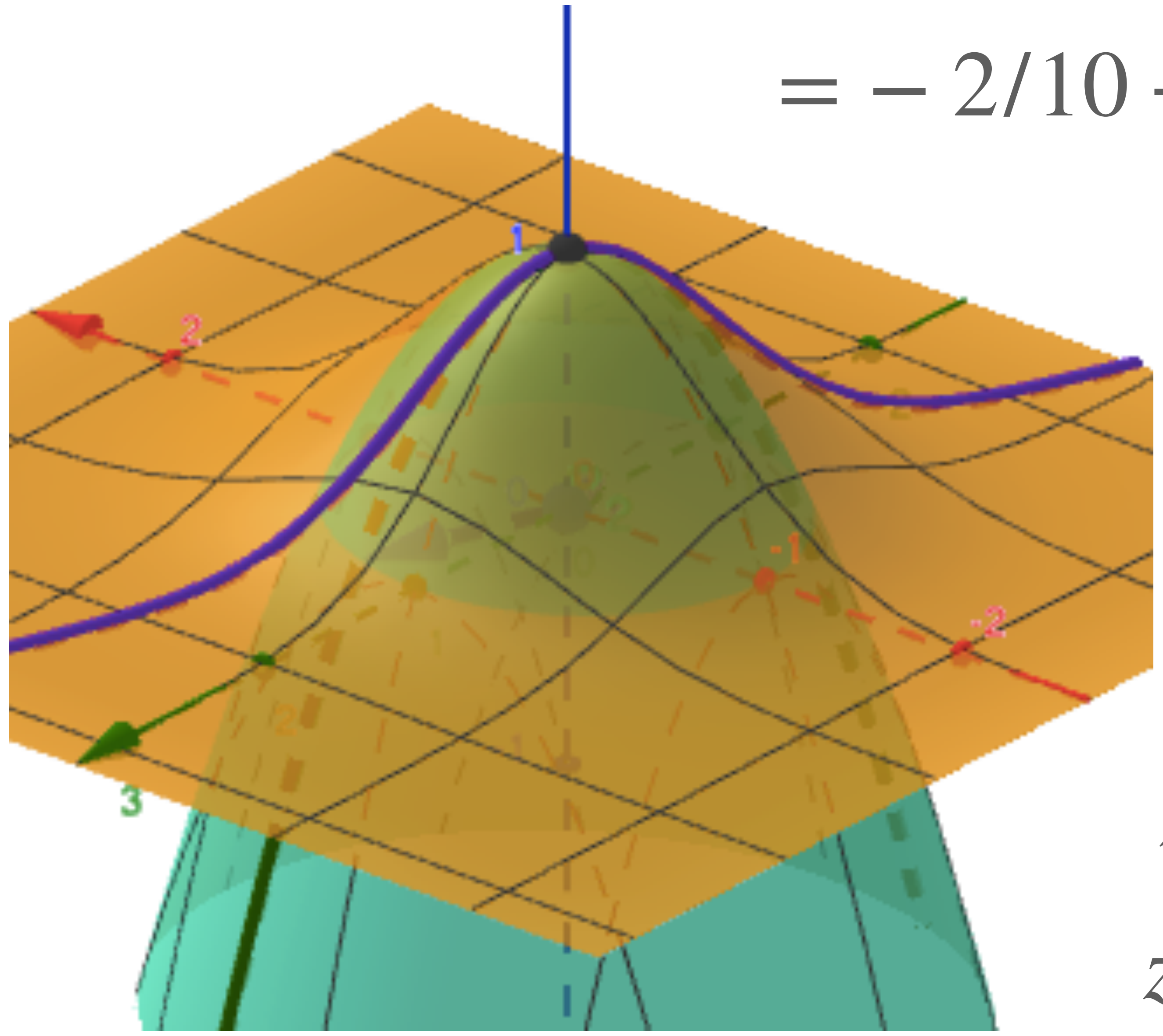
$f_{xx} = (4x^2-2)e^{-x^2-y^2}$ $\quad f_{xx}(0,0) = -2$

$f_{xy} = f_{yx} = 4xye^{-x^2-y^2}$ $\quad f_{xy}(0,0) = 0$

$f_{yy} = (4y^2-2)e^{-x^2-y^2}$ $\quad f_{yy}(0,0) = -2$

$$D_u^2(f)(a,b) = f_{xx}(a,b)\hat{u}_x^2 + 2f_{xy}(a,b)\hat{u}_x\hat{u}_y + f_{yy}(a,b)\hat{u}_y^2$$

$$= -2/10 + 0 - 2 \cdot 9/10 = -2$$



You should get the same answers using the curve method, e.g.

$$z(t) = e^{-(0+t/\sqrt{10})^2+(0+3t/\sqrt{10})^2} = e^{-t^2}$$

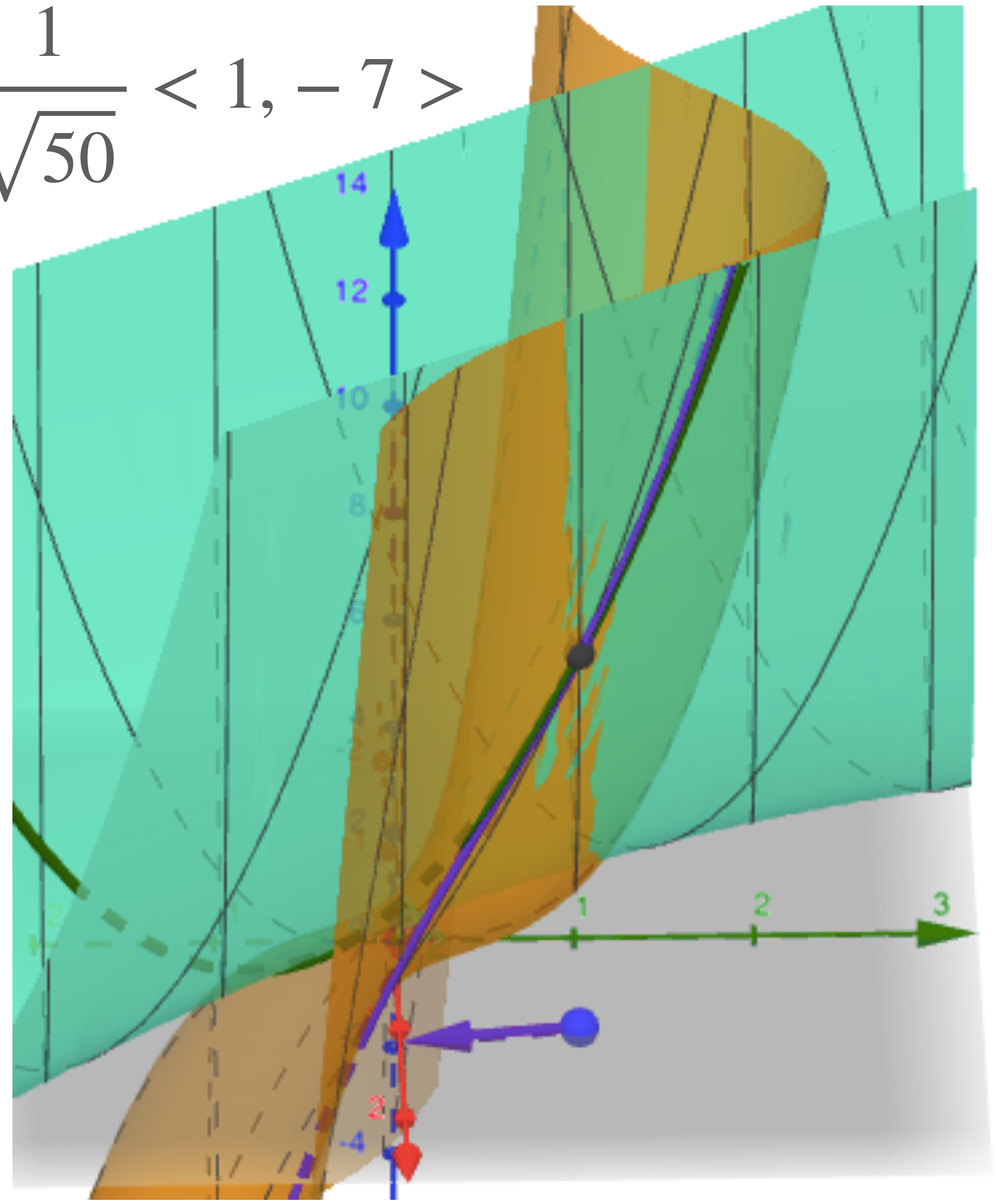
$$z''(t) = -2e^{-t^2} - 2t(-2t)e^{-t^2}$$

$$z''(0) = -2$$

2. $f(x,y) = x^3 + 5x^2y + y^3$

$P(1,1) \quad \mathbf{u} = < 1, -7 >$

$$\hat{\mathbf{u}} = \frac{1}{\sqrt{50}} < 1, -7 >$$



$f_x = 3x^2 + 10xy$

$f_y = 5x^2 + 3y^2$

$f_{xx} = 6x + 10y$

$f_{xy} = f_{yx} = 10x$

$f_{yy} = 6y$

$f_x(1,1) = 13$

$f_y(1,1) = 8$

$f_{xx}(1,1) = 16$

$f_{xy}(1,1) = 10$

$f_{yy}(1,1) = 6$

$$f_{xx}(a,b)\hat{u}_x^2 + 2f_{xy}(a,b)\hat{u}_x\hat{u}_y + f_{yy}(a,b)\hat{u}_y^2$$

$$= 16 \cdot 1/50 + 20 \cdot -7/50 + 6 \cdot 49/50$$

$$= 170/50 = 3.4$$

Differentiability, pg1. Remember the definition(s) of differentiability with single-variable functions.

Def1: A function $f : \mathbf{R} \to \mathbf{R}$ is differentiable at $x = a$ when this limit exists: $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

Def2: A function $f : \mathbf{R} \to \mathbf{R}$ is differentiable at $x = a$
when you can approximate f with a linear function near $x = a$: $f(x) = f(a) + c(x - a) + \epsilon(x)(x - a)$
for some constant c, and some function $\epsilon(x)$ for which $\epsilon(x) \to 0$ as $x \to a$

Note:

These definitions are equivalent.

Suppose definition 1 holds.

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = L$$

Can you show that definition 2 also holds? Yep!

Let $c = L$

Let $\epsilon(x) = \frac{f(x) - f(a)}{x - a} - L$

Then $\lim_{x \to a} \epsilon(x) = 0$

Also $(x - a) \cdot \epsilon(x) + L \cdot (x - a) + f(a) = f(x)$

Conversely, suppose definition 2 holds.
Can you show that definition1 holds?

We have some constant c and some function $\epsilon(x)$ so that

$$f(x) = f(a) + c(x - a) + \epsilon(x) \cdot (x - a)$$

Does $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exist? Yep! Subtract $f(a)$, and divide by $(x - a)$.

$$\frac{f(x) - f(a)}{x - a} = c + \epsilon(x)$$

Take the limit as $x \to a$ of both sides. This limit exists on the right side, hence also on the left side.

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} (c + \epsilon(x)) = c + 0 = c \checkmark$$

Differentiability, pg2. Multivariable differentiability is defined in the flavor of definition 2.

A function $f : \mathbf{R}^2 \to \mathbf{R}$ is *differentiable* at the point $(a, b) \in \mathbf{R}^2$ if there are numbers c and d and functions $\epsilon_1(x, y)$ and $\epsilon_2(x, y)$ such that $\lim_{(x,y)\to(a,b)} \epsilon_1(x, y) = \lim_{(x,y)\to(a,b)} \epsilon_2(x, y) = 0$ and

$$f(x, y) = f(a, b) + c \cdot (x - a) + d \cdot (y - b) + \epsilon_1(x, y) \cdot (x - a) + \epsilon_2(x, y) \cdot (y - b)$$

Notes:

1. The numbers c and d must be the partial derivatives of f: $c = f_x(a, b)$, $d = f_y(a, b)$

e.g. if (x, y) approaches the point (a, b) along the line $y = b$, then the definition says that

$$f(x, b) = f(a, b) + c \cdot (x - a) + \epsilon_1(x, b) \cdot (x - a)$$

$$c = \frac{f(x, b) - f(a, b)}{x - a} - \epsilon_1(x, b) \to f_x(a, b) \text{ as } x \to a$$

2. You might think of this definition as saying that the tangent plane is a good approximation of f near (a, b). The error is measured by

$$\epsilon_1(x, y) \cdot (x - a) + \epsilon_2(x, y) \cdot (y - b)$$

which approaches 0 as $(x, y) \to (a, b)$.

In particular, if f has higher-order partial derivatives, then we can express the error as part of a Taylor polynomial approximation.

e.g. $f(x, y) - T_1(a, b) =$

$$\frac{1}{2!}\left(f_{xx}(a, b)(x - a)^2 + 2f_{xy}(a, b)(x - a)(y - b) + f_{yy}(a, b)(y - b)^2\right)$$

$$+\frac{1}{3!}\left(f_{xxx}(a, b)(x - a)^3 + \ldots\right) + \ldots$$

$$= (x - a)\left(\frac{1}{2!}f_{xx}(a, b)(x - a) + f_{xy}(a, b)(y - b) + \frac{1}{3!}f_{xxx}(a, b)(x - a)^2 + \ldots\right.$$

$$= (y - b)\left(\frac{1}{2!}f_{yy}(a, b)(y - b) + f_{yx}(a, b)(x - a) + \frac{1}{3!}f_{yyy}(a, b)(y - b)^2 + \ldots\right.$$

$$= (x - a) \cdot \epsilon_1(x, y) + (y - b) \cdot \epsilon_2(x, y)$$

The Chain Rule, pg 1.

The definitions with ϵ are useful if you want to prove the chain rule.

Single Variable.

Suppose $f(x)$ is differentiable at $x = a$, and $x(t)$ is differentiable at $t = b$, when $x(b) = a$

Then $g(t) = (f \circ x)(t) = f(x(t))$ is differentiable at $t = b$ with derivative $f'(a) \cdot x'(b)$. but why?!

Assuming f differentiable at $x = a$ gives us

$f(x) = f(a) + f'(a)(x - a) + \epsilon(x) \cdot (x - a)$ Then…

$$\frac{g(t) - g(b)}{t - b} = \frac{f(x) - f(a)}{t - b} = f'(a)\frac{x - a}{t - b} + \epsilon(x) \cdot \frac{x - a}{t - b}$$

$\longrightarrow f'(a)x'(b) + 0 \cdot x'(b) = f'(a) \cdot x'(b)$ as $t \to b$

$\epsilon(x) \to 0$ as $t \to b$ because $x \to a$ as $t \to b$

Multivariable Variable.

Suppose $f(x, y)$ is differentiable at $(x, y) = (a, b)$, and $x(t), y(t)$ are differentiable at $t = c$, with $(x(c), y(c)) = (a, b)$

Then $g(t) = f(x(t), y(t))$ is differentiable at $t = c$ with derivative $f_x(a, b)x'(c) + f_y(a, b)y'(c)$. Why?

Assuming differentiability at $(x, y) = (a, b)$ gives us

$$f(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$
$$+\epsilon_1(x, y) \cdot (x - a) + \epsilon_2(x, y) \cdot (y - b)$$ Then…

$$\frac{g(t) - g(c)}{t - c} = \frac{f(x, y) - f(a, b)}{t - c} = f_x(a, b)\frac{x - a}{t - c} + f_y(a, b)\frac{y - b}{t - c}$$
$$+\epsilon_1(x, y) \cdot \frac{x - a}{t - c} + \epsilon_2(x, y) \cdot \frac{y - b}{t - c}$$

As $t \to c$, $$\frac{g(t) - g(c)}{t - c} \longrightarrow f_x(a, b)x'(c) + f_y(a, b)y'(c) + 0 \cdot x'(c) + 0 \cdot y'(c)$$
$$= f_x(a, b)x'(c) + f_y(a, b)y'(c)$$

The Chain Rule, pg 2. Examples.

Express $\dfrac{dz}{dt}$ or $\dfrac{df}{dt}$ as a function of t.

Example1. (S14.5 #3)

$z = \sin(x)\cos(y) \quad x = \sqrt{t}, y = t^{-1}$

$$\frac{dz}{dt} = z_x(x,y) \cdot x'(t) + z_y(x,y) \cdot y'(t)$$

$$= \cos(x)\cos(y)\frac{1}{2}t^{-1/2} - \sin(x)\sin(y)(-t^{-2})$$

$$= \frac{1}{2\sqrt{t}}\cos\left(\sqrt{t}\right)\cos\left(t^{-1}\right) + \frac{1}{t^2}\sin\left(\sqrt{t}\right)\sin\left(t^{-1}\right)$$

You could get this result directly:

$z(t) = \sin(\sqrt{t})\cos(t^{-1})$

$$z'(t) = \cos\left(\sqrt{t}\right)\frac{1}{2}t^{-1/2}\cos\left(t^{-1}\right)$$

$$+\sin\left(\sqrt{t}\right) \cdot -\sin\left(t^{-1}\right) \cdot -t^{-2}$$

Example2. (S14.5 #5)

$f(x,y,z) = xe^{y/z}$

$x = t^2, \quad y = 1 - t, \quad z = 1 + 2t$

$$\frac{df}{dt} = f_x(x,y) \cdot x'(t) + f_y(x,y) \cdot y'(t) + f_z(x,y) \cdot z'(t)$$

$$= e^{y/z} \cdot 2t + xe^{y/z} \cdot \frac{1}{z} \cdot (-1) + xe^{y/z} \cdot \frac{-y}{z^2} \cdot (2)$$

$$= 2te^{\frac{1-t}{1+2t}} - \frac{t^2}{1+2t}e^{\frac{1-t}{1+2t}} - \frac{2t^2 - 2t^3}{(1+2t)^2}e^{\frac{1-t}{1+2t}}$$

$$= e^{\frac{1-t}{1+2t}}\left(2t - \frac{3t^2}{(1+2t)^2}\right)$$ or: start with $f(t) = t^2e^{\frac{1-t}{1+2t}}$

$$f'(t) = 2te^{\frac{1-t}{1+2t}} + t^2e^{\frac{1-t}{1+2t}} \cdot \left(\frac{-(1+2t) - (1-t) \cdot 2}{(1+2t)^2}\right)$$

$$= e^{\frac{1-t}{1+2t}}\left(2t - \frac{t^2}{1+2t} - \frac{2t^2(1-t)}{(1+2t)^2}\right) = e^{\frac{1-t}{1+2t}}\left(2t - \frac{3t^2}{(1+2t)^2}\right)$$

Chain Rule pg 3. Practice (round1).

Compute $\dfrac{dz}{dt}$ or $\dfrac{df}{dt}$

(S14.5 #1)

$$z = xy^3 - x^2y \qquad x = t^2+1, \quad y = t^2-1$$

$$\frac{dz}{dt} = z_x(x,y)x'(t) + z_y(x,y)y'(t)$$

$$= (y^3 - 2xy)\cdot 2t + (3xy^2 - x^2)\cdot 2t$$

$$= 2t\big[(t^2-1)^3 - 2(t^2+1)(t^2-1) + 3(t^2+1)(t^2-1)^2 - (t^2+1)^2\big]$$

(S14.5 #2)

$$z = \frac{x-y}{x+2y} \quad x = e^{\pi t}, \quad y = e^{-\pi t}$$

$$\frac{dz}{dt} = z_x(x,y)x'(t) + z_y(x,y)y'(t) = \dots$$

$$\dots = \frac{1\cdot(x+2y) - (x-y)\cdot 1}{(x+2y)^2}\cdot \pi e^{\pi t} + \frac{-1\cdot(x+2y) - (x-y)\cdot 2}{(x+2y)^2}\cdot -\pi e^{-\pi t}$$

$$= \frac{\pi}{(x+2y)^2}\big[x(x+2y) - x(x-y) + y(x+2y) + 2y(x-y)\big]$$

$$= \frac{6xy\pi}{(x+2y)^2} = \frac{6\pi}{(e^{\pi t} + 2e^{-\pi t})^2}$$

#3. $\displaystyle f(w,x,y,z) = \frac{w^2x^3}{yz} \qquad w = t^{1/2} \quad x = 2t \quad y = t^2 \quad z = t^3$

$$f'(t) = f_w\cdot w_t + f_x\cdot x_t + f_y\cdot y_t + f_z\cdot z_t$$

$$= \frac{2wx^3}{yz}\cdot\frac{1}{2}t^{-1/2} + \frac{3w^2x^2}{yz}\cdot 2 - \frac{w^2x^3}{y^2z}\cdot 2t - \frac{w^2x^3}{yz^2}\cdot 3t^2$$

$$= \frac{8t^3}{t^2t^3} + \frac{6t\cdot 4t^2}{t^2t^3} - \frac{2t\cdot t\cdot 8t^3}{t^4t^3} - \frac{3t^2\cdot t\cdot 8t^3}{t^2t^6} = -\frac{8}{t^2}$$

The Chain Rule, pg 4. Even More variables!

$f(x,y)$ is a function of x and y, each of which are functions of two or more variables! $x = x(s,t), \quad y = y(s,t)$

Then through composition, f is a function of all s and t.

Then what are $f_t = \dfrac{\partial f}{\partial t}(s,t)$ and $f_s = \dfrac{\partial f}{\partial s}(s,t)$?

$$f_t(s,t) = \frac{\partial f}{\partial t}(s,t) = f_x(x,y)\cdot x_t(s,t) + f_y(x,y)\cdot y_t(s,t)$$

$$f_s(s,t) = \frac{\partial f}{\partial s}(s,t) = f_x(x,y)\cdot x_s(s,t) + f_y(x,y)\cdot y_s(s,t)$$

Even more generally, suppose we had $f : \mathbf{R}^n \to \mathbf{R}$.

$f(x_1, x_2, x_3, \ldots x_n)$

and each of the variables x_i themselves are functions of many variables. Say $x_i = x_i(t_1, t_2, t_3, \ldots t_m)$

Then through composition, f is a function of all those t's.

So what are the $f_{t_i}(t_1, t_2, \ldots t_m)$?

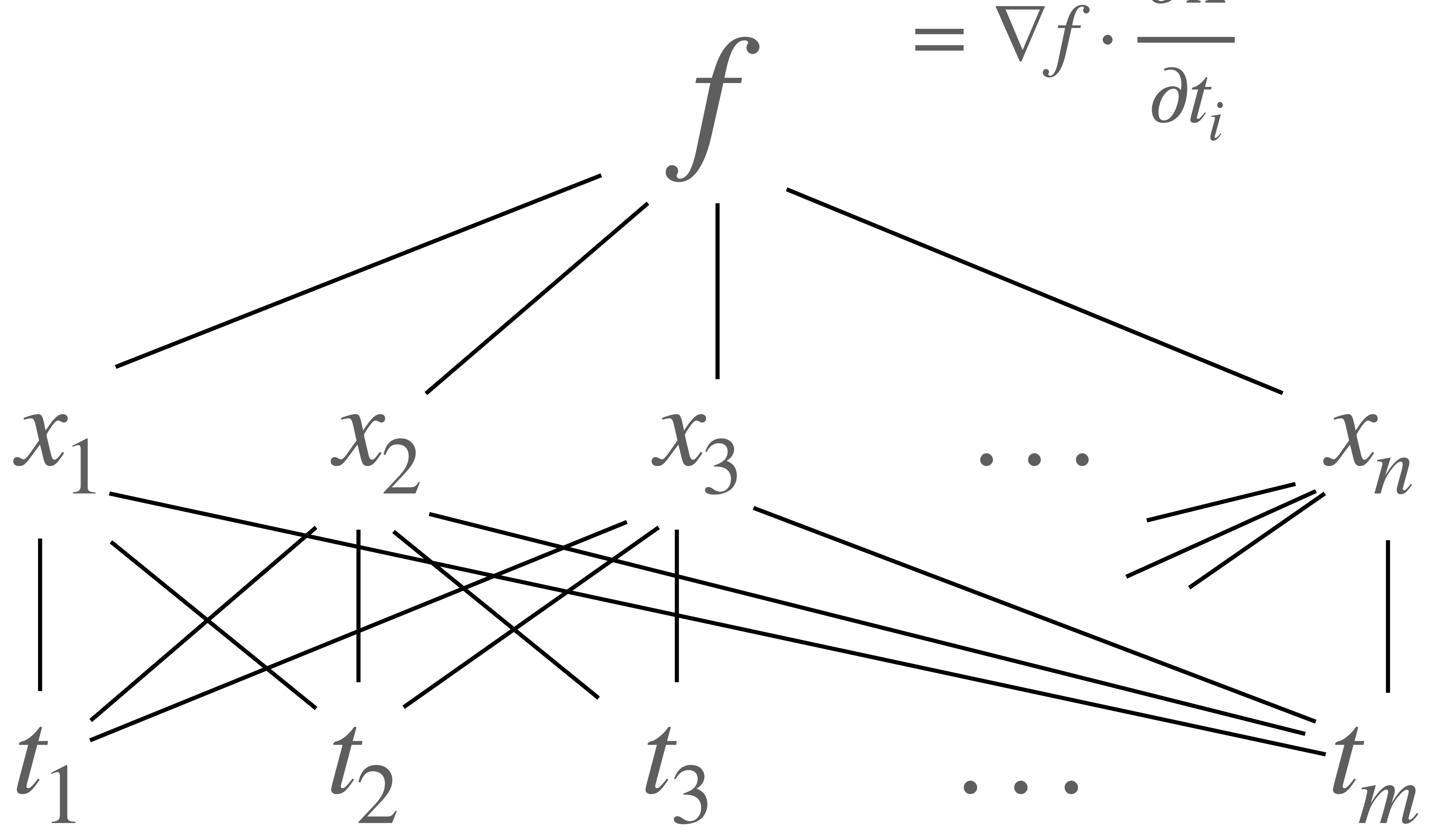
$$f_{t_i}(t_1, t_2, \ldots t_m) = f_{x_1}(x_1, x_2, \ldots x_n)\cdot\frac{\partial x_1}{\partial t_i}$$

$$+ f_{x_2}(x_1, x_2, \ldots x_n)\cdot\frac{\partial x_2}{\partial t_i} + f_{x_3}(x_1, x_2, \ldots x_n)\cdot\frac{\partial x_3}{\partial t_i}$$

$$+ f_{x_4}(x_1, x_2, \ldots x_n)\cdot\frac{\partial x_4}{\partial t_i} + \ldots + f_{x_n}(x_1, x_2, \ldots x_n)\cdot\frac{\partial x_n}{\partial t_i}$$

$$= f_{x_1}\cdot x_{t_i} + f_{x_2}\cdot x_{t_i} + \ldots + f_{x_n}\cdot x_{t_i} = \sum_{j=1}^{j=n} f_{x_j}\cdot x_{t_i}$$

$$= \nabla f \cdot \frac{\partial \mathbf{x}}{\partial t_i}$$



Each path from f through the x's to one of the t_i yields a product that is part of the derivative according to the chain rule.

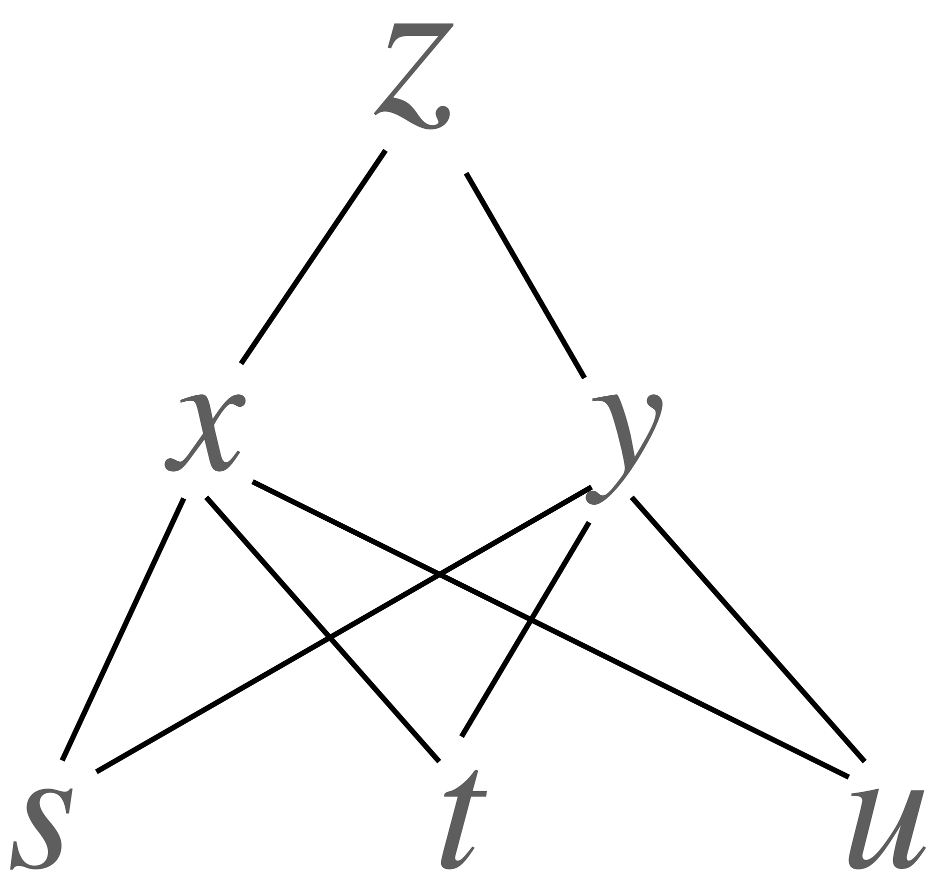
The Chain Rule, pg 5. More Examples.

Example 1. (S14.5 #21)

$z = x^4 + x^2 y \qquad x = s + 2t - u$

$\qquad\qquad\qquad y = s \cdot t \cdot u^2$

a) Compute $\dfrac{\partial z}{\partial s}, \dfrac{\partial z}{\partial t}, \dfrac{\partial z}{\partial u}$



$$\frac{\partial z}{\partial s} = z_x(x,y)x_s(s,t,u) + z_y(x,y)y_s(s,t,u)$$
$$= (4x^3 + 2xy) \cdot 1 + x^2 \cdot t \cdot u^2$$
$$= 4(s+2t-u)^3 + 2(s+2t-u)(s \cdot t \cdot u^2) + (s+2t-u)^2 \cdot t \cdot u^2$$

$$\frac{\partial z}{\partial t} = z_x(x,y)x_t(s,t,u) + z_y(x,y)y_t(s,t,u)$$
$$= (4x^3 + 2xy) \cdot 2 + x^2 \cdot s \cdot u^2$$
$$= 2 \cdot (4(s+2t-u)^3 + 2(s+2t-u)(s \cdot t \cdot u^2)) + (s+2t-u)^2 \cdot s \cdot u^2$$

$$\frac{\partial z}{\partial u} = z_x(x,y)x_u(s,t,u) + z_y(x,y)y_u(s,t,u)$$
$$= (4x^3 + 2xy) \cdot (-1) + x^2 \cdot 2s \cdot t \cdot u$$
$$= -1 \cdot (4(s+2t-u)^3 + 2(s+2t-u)(s \cdot t \cdot u^2)) + (s+2t-u)^2 \cdot 2s \cdot t \cdot u$$

b) Compute $\dfrac{\partial z}{\partial s}, \dfrac{\partial z}{\partial t}, \dfrac{\partial z}{\partial u}$ at the point $(s,t,u) = (4,2,1)$

$x(4,2,1) = 7, \quad y(4,2,1) = 8$

$\frac{\partial z}{\partial s}(4,2,1) = 1582$

$\frac{\partial z}{\partial t}(4,2,1) = 3164$

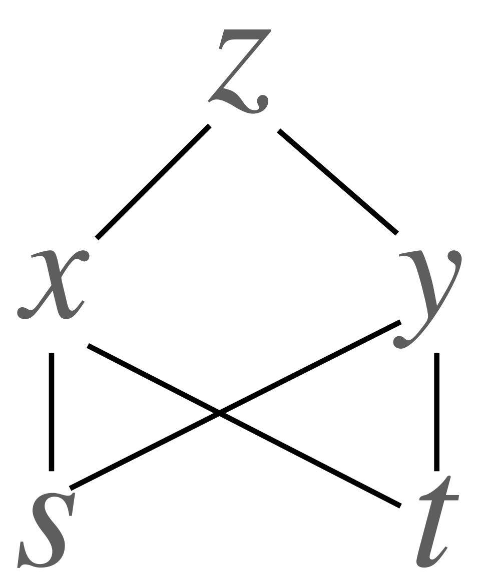
$\frac{\partial z}{\partial u}(4,2,1) = -700$

The Chain Rule, pg 6. Practice (round2).

Compute the partial derivatives at the given points.

$\approx$ Sec 15.4 #10

$$z = \sqrt{x}e^{xy}, \;\; x = 1 + st, \;\; y = s^2 - t^2$$



Find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$ when $(s,t) = (1,0)$

$$\frac{\partial z}{\partial s} = z_x(x,y)\cdot x_s(s,t) + z_y(x,y)\cdot y_s(s,t)$$

$$= z_x \cdot x_s + z_y \cdot y_s$$

$$= \left(\frac{1}{2}x^{-1/2}e^{xy} + x^{1/2}ye^{xy}\right)\cdot t + x^{3/2}e^{xy}\cdot 2s$$

$$\frac{\partial z}{\partial t} = z_x(x,y)\cdot x_t(s,t) + z_y(x,y)\cdot y_t(s,t)$$

$$= z_x \cdot x_t + z_y \cdot y_t$$

$$= \left(\frac{1}{2}x^{-1/2}e^{xy} + x^{1/2}ye^{xy}\right)\cdot s + x^{3/2}e^{xy}\cdot(-2t)$$

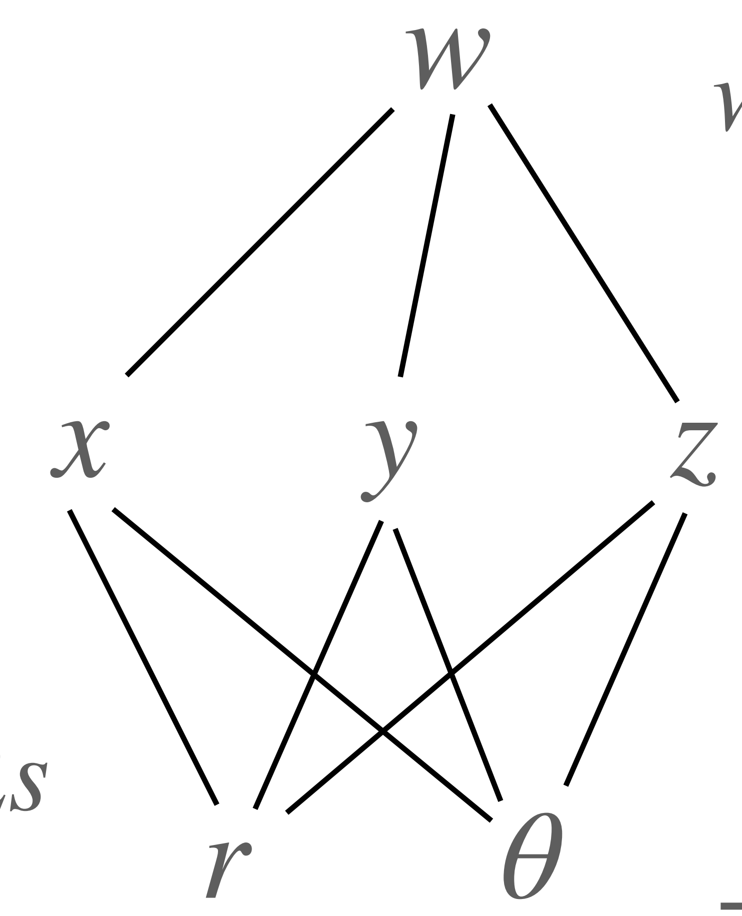
when $(s,t) = (1,0)$, $x(1,0) = 1$, $y(1,0) = 1$

$$\frac{\partial z}{\partial s}(1,0) = \left(\frac{1}{2}\cdot 1\cdot e + 1\cdot e\right)\cdot 0 + 1\cdot e\cdot 2 = 2e$$

$$\frac{\partial z}{\partial t}(1,0) = \left(\frac{1}{2}\cdot 1\cdot e + 1\cdot e\right)\cdot 1 + 1\cdot e\cdot 0 = 1.5e$$

Sec 14.5 #23

$$w = xy + xz + yz, \quad x = r\cos(\theta) \quad y = r\sin(\theta) \quad z = r\theta$$



Compute $\dfrac{\partial w}{\partial \theta}$ and $\dfrac{\partial w}{\partial r}$ at $(r,\theta) = (2,\ \pi/2)$

$$\frac{\partial w}{\partial \theta} = w_x\cdot x_\theta + w_y\cdot y_\theta + w_z\cdot z_\theta$$

$$= (y+z)\cdot(-r\sin(\theta)) + (x+z)\cdot(r\cos(\theta)) + (x+y)\cdot r$$

$$= (2+\pi)\cdot(-2) + (0+\pi)\cdot 0 + (0+2)\cdot 2 = -2\pi$$

$$\frac{\partial w}{\partial r} = w_x\cdot x_r + w_y\cdot y_r + w_z\cdot z_r$$

$$= (y+z)\cdot\cos(\theta) + (x+z)\cdot\sin(\theta) + (x+y)\cdot\theta$$

$$= (2+\pi)\cdot 0 + (0+\pi)\cdot 1 + (0+2)\cdot\pi/2 = 2\pi$$

Connecting Methods 1 and 2 using the chain rule.

The chain rule gives us a simple theoretical foundation for directional derivatives.

$$g(t) = < a + \hat{u}_x t, b + \hat{u}_y t, f(a + \hat{u}_x t,\ b + \hat{u}_y t) >$$

We have measured 1st and 2nd derivatives of f in the direction of $\hat{\mathbf{u}} = < \hat{u}_x, \hat{u}_y >$ by computing the derivatives of the z coordinate of the curve on the graph of f in the direction of $\mathbf{u}$ at time 0.

$$z = f(x, y) = f(a + \hat{u}_x t, b + \hat{u}_y t)$$

$$\begin{tikzcd} & z \arrow[dl, dash] \arrow[dr, dash] & \\ x \arrow[dr, dash] & & y \arrow[dl, dash] \\ & t & \end{tikzcd}$$

$$x(t) = a + \hat{u}_x t$$

$$y(t) = b + \hat{u}_y t$$

$$z'(t) = z_x \cdot x_t + z_y \cdot y_t$$

$$= f_x(a + \hat{u}_x t, b + \hat{u}_y t)\hat{u}_x + f_y(a + \hat{u}_x t, b + \hat{u}_y t)\hat{u}_y$$

$$z'(0) = f_x(a, b) \cdot \hat{u}_x + f_y(a, b) \cdot \hat{u}_y$$

And the second derivative?

$$z''(t) = \frac{d}{dt} z'(t) = \frac{\partial z'}{\partial x} \cdot x_t + \frac{\partial z'}{\partial y} \cdot y_t$$

$$= f_{xx}(a + \hat{u}_x t, b + \hat{u}_y t)\hat{u}_x\hat{u}_x + f_{yx}(a + \hat{u}_x t, b + \hat{u}_y t)\hat{u}_y\hat{u}_x$$

$$+ f_{xy}(a + \hat{u}_x t, b + \hat{u}_y t)\hat{u}_x\hat{u}_y + f_{yy}(a + \hat{u}_x t, b + \hat{u}_y t)\hat{u}_y\hat{u}_y$$

$$z''(0) = f_{xx}(a, b)\hat{u}_x^2 + f_{yx}(a, b)\hat{u}_y\hat{u}_x + f_{xy}(a, b)\hat{u}_x\hat{u}_y + f_{yy}(a, b)\hat{u}_y^2$$

$$= f_{xx}(a, b)\hat{u}_x^2 + 2f_{xy}(a, b)\hat{u}_x\hat{u}_y + f_{yy}(a, b)\hat{u}_y^2$$

Similarly…

$$z'''(0) = f_{xxx}(a, b)\hat{u}_x^3 + 3f_{xxy}(a, b)\hat{u}_x^2 u_y + 3f_{xyy}(a, b)\hat{u}_x\hat{u}_y^2 + f_{yyy}(a, b)\hat{u}_y^3$$

Also note: some derivatives can be described with matrices!

$$z'(0) = < f_x(a, b), f_y(a, b) > \cdot < \hat{u}_x, \hat{u}_y >$$

$$z''(0) = (u_x \quad u_y) \cdot \begin{pmatrix} f_{xx}(a, b) & f_{xy}(a, b) \\ f_{yx}(a, b) & f_{yy}(a, b) \end{pmatrix} \cdot \begin{pmatrix} u_x \\ u_y \end{pmatrix}$$

$$z'''(0) = ?!?!$$

⟵You can learn much more about matrices in *Linear Algebra*!!!

Applications. Depth, pg 1.

Example 1. The depth of a lake is

$d(x, y) = 200 + 0.02x^2 - 0.001y^3$

The x axis points north, the y axis points west. Units of x,y and $d(x, y)$ are meters.

Find and interpret the first and second derivatives of d at (50,50) when directed

a) north. b) east. c) $S30°W$

$d_x(x, y) = 0.04x$ $\quad d_x(50,50) = 0.04(50) = 2$

$d_y(x, y) = -0.003y^2$ $\quad d_y(50,50) = -0.003(50^2) = -7.5$

$d_{xx}(x, y) = 0.04$ $\quad d_{xx}(50,50) = 0.04$

$d_{xy}(x, y) = 0$ $\quad d_{xy}(50,50) = 0$

$d_{yy}(x, y) = -0.006y$ $\quad d_{yy}(50,50) = -0.3$

$a)\ \ \mathbf{u} = <1,0> = \hat{\mathbf{u}}$

$f_{north}(50,50) = d_x(50,50) \cdot 1 + d_y(50,50) \cdot 0 = d_x(50,50) = 2$

As we move north from (50,50), the lake gets deeper at a rate of 2 meters (vertically) per meter north.

$b)\ \ \hat{\mathbf{u}} = <0, -1>$

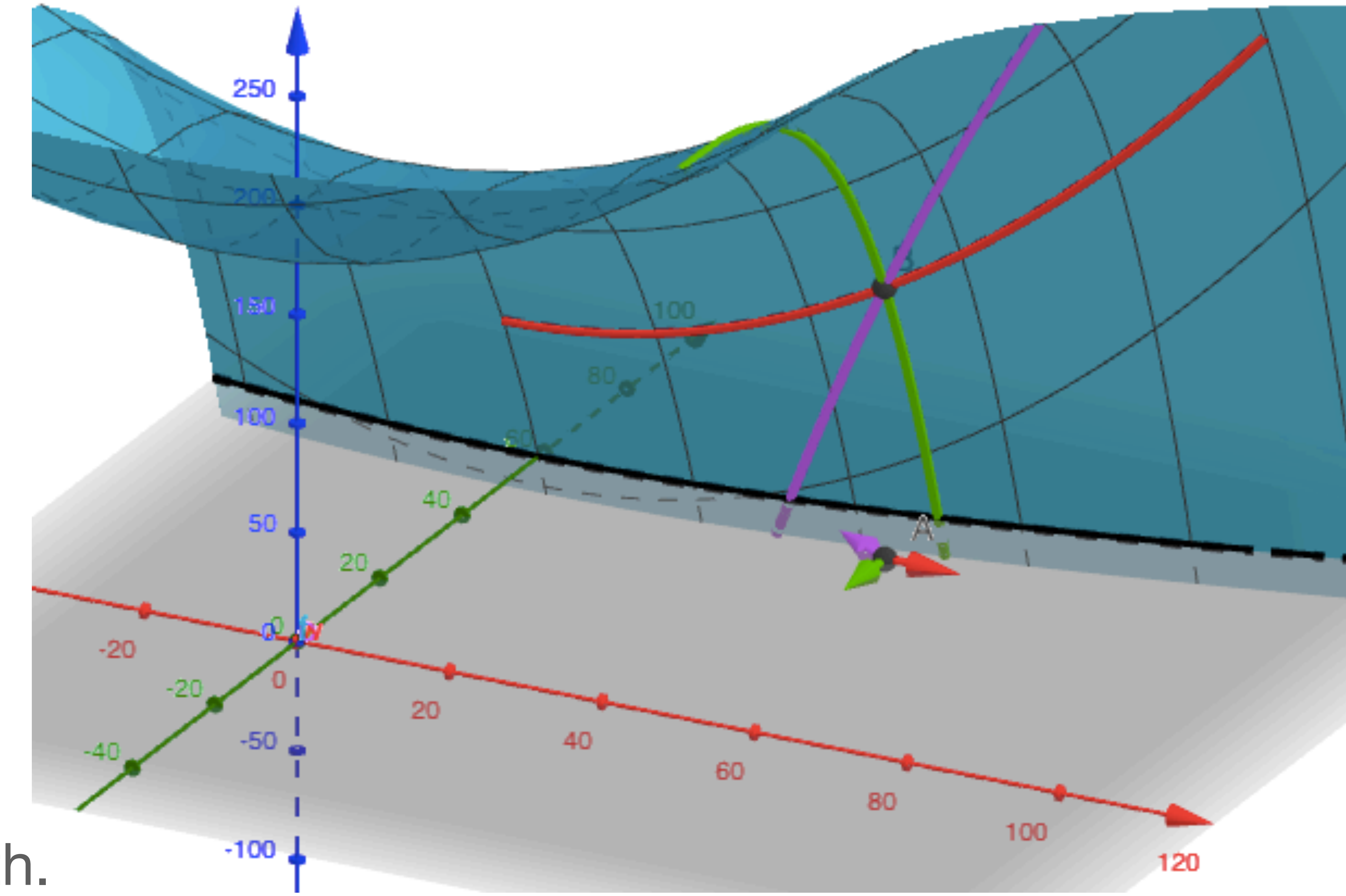
$f_{east}(50,50) = d_x(50,50) \cdot 0 + d_y(50,50) \cdot (-1) = -d_y(50,50) = 7.5$

As we move east from (50,50), the lake gets deeper at a rate of 7.5 meters (vertically) per meter east.

$c)\ \ \hat{\mathbf{u}} = <\cos(150°), \sin(150°)> = <-\sqrt{3}/2, 1/2>$

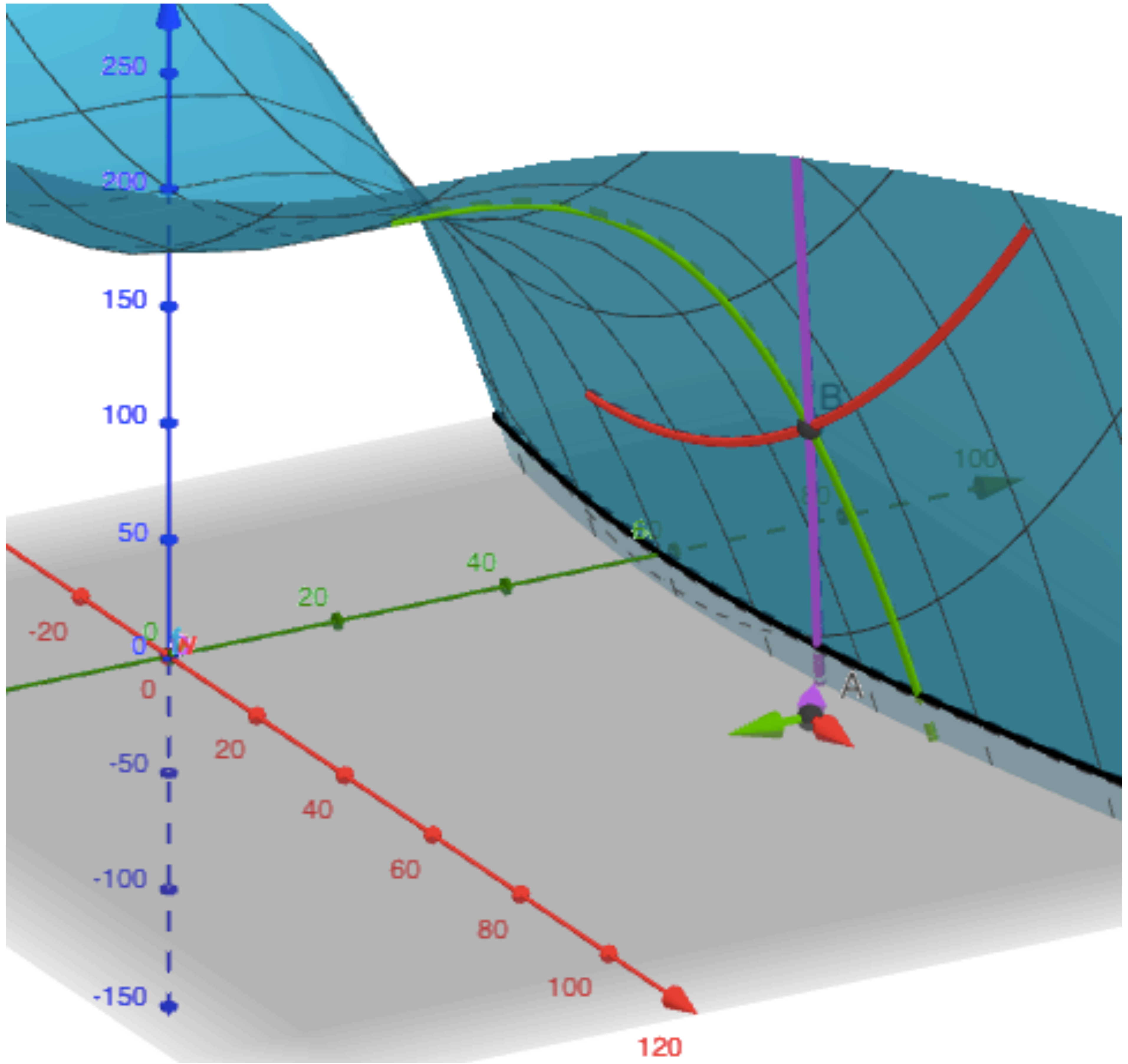
$d_{S30°W}(50,50) = 2 \cdot (-\sqrt{3}/2) + -7.5 \cdot (1/2) \approx -5.48$

As we move $S30°W$ from (50,50), the lake gets shallower at a rate of 5.48 meters (vertically) per meter $S30°W$.



Depth, pg 2.

2nd derivatives?



a) north $\hat{\mathbf{u}} = <1{,}0>$ b) east $\hat{\mathbf{u}} = <0,-1>$

c) $S30^\circ W$ $\quad \hat{\mathbf{u}} = <-\sqrt{3}/2, 1/2>$

$d_{xx}(x,y) = 0.04$ $\qquad d_{xx}(50{,}50) = 0.04$

$d_{xy}(x,y) = 0$ $\qquad d_{xy}(50{,}50) = 0$

$d_{yy}(x,y) = -0.006y$ $\qquad d_{yy}(50{,}50) = -0.3$

a) $$D^2_{north}(d)(50{,}50) = d_{xx}(50{,}50)\cdot 1^2 + 2d_{xy}(50{,}50)\cdot 1\cdot 0 + d_{yy}(50{,}50)\cdot 0^2$$
$$= d_{xx}(50{,}50) = 0.04$$

At the point (50,50), in the direction of <1,0> (i.e. north), the rate of change of depth is increasing by 0.04 meters deep per meter north, per meter north, or by 0.04 meters deep per meter north squared.

b) $$D^2_{east}(d)(50{,}50)$$
$$= d_{xx}(50{,}50)\cdot 0^2 + 2d_{xy}(50{,}50)\cdot 0\cdot(-1) + d_{yy}(50{,}50)\cdot(-1)^2$$
$$= d_{yy}(50{,}50) = -0.3$$

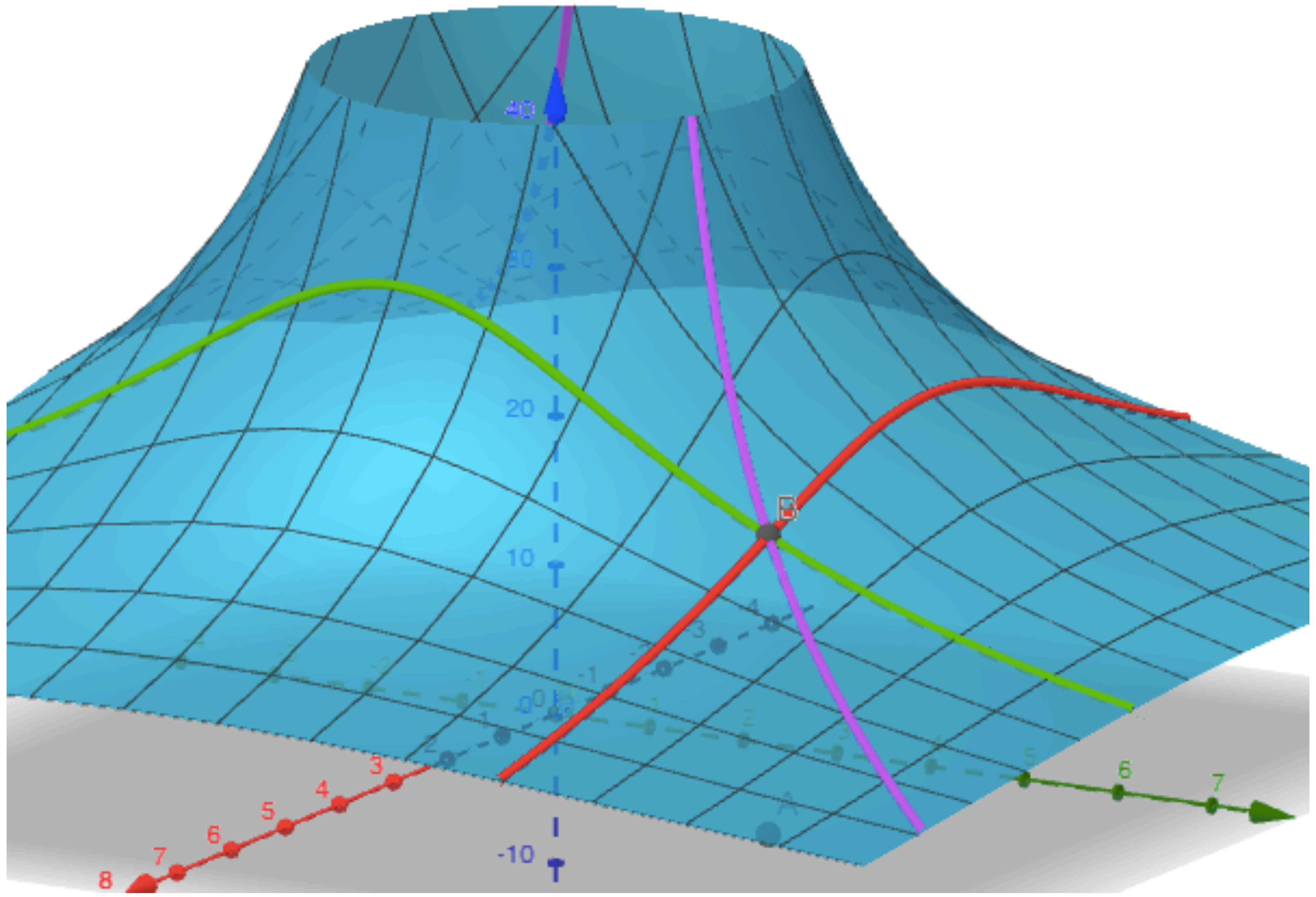
At the point (50,50), in the direction of <0,-1> (i.e. east), the rate of change of depth is decreasing by 0.3 meters deep per meter east squared.

c) $$D^2_{S30^\circ W}(d)(50{,}50)$$
$$= d_{xx}(50{,}50)\cdot(-\sqrt{3}/2)^2 + 2d_{xy}(50{,}50)\cdot(-\sqrt{3}/2)\cdot(1/2)$$
$$+ d_{yy}(50{,}50)\cdot(1/2)^2 \approx -0.045$$

At the point (50,50), in the direction of $\hat{\mathbf{u}} = <-\sqrt{3}/2,\ 1/2>$, the rate of change of depth is decreasing by 0.045 meters deep per meter squared in the direction $S30^\circ W$.

Applications. Temperature.

Say temperature in a plane is inversely proportional to the distance from the origin. We measure the temperature a specific point, $T(3,4) = 20^\circ$ Centigrade.



$$T(x,y) = \frac{k}{\sqrt{x^2+y^2}} \quad 20 = T(3,4) = \frac{k}{5},\ k = 100$$

$$T(x,y) = \frac{100}{\sqrt{x^2+y^2}} = 100(x^2+y^2)^{-1/2}$$

Compute and interpret the first and second derivatives of T at the point $(3,4)$ in the directions…

a) the x-axis. b) the y-axis.

c) the vector $\mathbf{u} = < -2, -2 >$.

$$T_x(x,y) = \frac{-100x}{(x^2+y^2)^{3/2}}$$

$$T_y(x,y) = \frac{-100y}{(x^2+y^2)^{3/2}}$$

$$T_{xx}(x,y) = \frac{100(2x^2-y^2)}{(x^2+y^2)^{5/2}}$$

$$T_{xy}(x,y) = \frac{300xy}{(x^2+y^2)^{5/2}}$$

$$T_{yy}(x,y) = \frac{100(2y^2-x^2)}{(x^2+y^2)^{5/2}}$$

$$T_x(3,4) = -2.4$$
$$T_y(3,4) = -3.2$$
$$T_{xx}(3,4) = 0.064$$
$$T_{xy}(3,4) = 1.152$$
$$T_{yy}(3,4) = 0.736$$

a)

$$T_{<1,0>}(3,4) = -2.4 \cdot 1 + -3.2 \cdot 0$$
$$= -2.4 \text{ deg / meter}$$

$$D^2_{<1,0>}(T)(3,4) = 0.064 \cdot 1^2 + 1.152 \cdot 1 \cdot 0 + 0.736 \cdot 0^2$$
$$= 0.064 \text{ deg/meter/meter}$$

b)

$$T_{<0,1>}(3,4) = -2.4 \cdot 0 + -3.2 \cdot 1$$
$$= -3.2 \text{ deg / meter}$$

$$D^2_{<0,1>}(T)(3,4) = 0.064 \cdot 0^2 + 1.152 \cdot 0 \cdot 1 + 0.736 \cdot 1^2$$
$$= 0.736 \text{ deg/meter/meter}$$

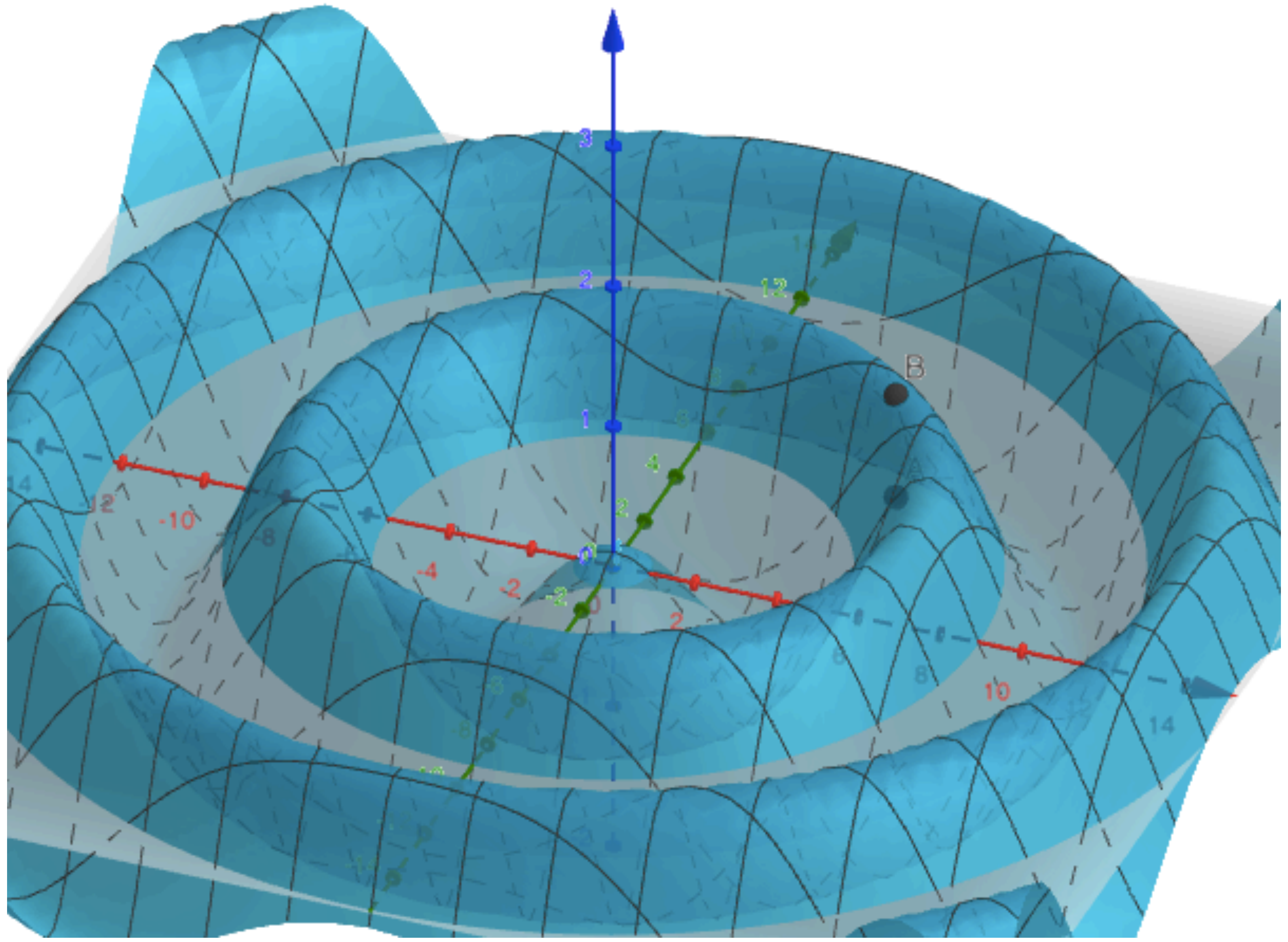
c) $\hat{\mathbf{u}} = -\frac{1}{\sqrt{2}} < 1,1 >$

$$T_{\mathbf{u}}(3,4) = -2.4 \cdot -1/\sqrt{2} + -3.2 \cdot -1/\sqrt{2}$$
$$\approx 3.96 \text{ deg/meter}$$

$$D^2_{\mathbf{u}}(T)(3,4)$$
$$= 0.064 \cdot (-1/\sqrt{2})^2 + 2 \cdot 1.152 \cdot -1/\sqrt{2} \cdot -1/\sqrt{2} + 0.736 \cdot (-1/\sqrt{2})^2$$
$$= 1.552 \text{ deg/m}^2$$

Applications. A wave in time.

Link: [WaveInSpaceAndTime]



$$h(x,y,t) = e^{-0.1t}\sin\left(\sqrt{x^2+y^2+t^2}\right)$$

units of x,y and h are cm.
units of t are sec.

Find the first and second derivatives at the point (5,5,3) in the directions

$a)\ \ \hat{\mathbf{u}} = <1{,}0{,}0> \quad b)\ \ \hat{\mathbf{u}} = <0{,}1{,}0>$

$c)\ \ \hat{\mathbf{u}} = <0{,}0{,}1> \quad d)\ \ \hat{\mathbf{u}} = <2/3,\ 2/3,\ 1/3>$

$$h_x(x,y,t) = \frac{xe^{-0.1t}\cos\left((x^2+y^2+t^2)^{1/2}\right)}{(x^2+y^2+t^2)^{1/2}}$$

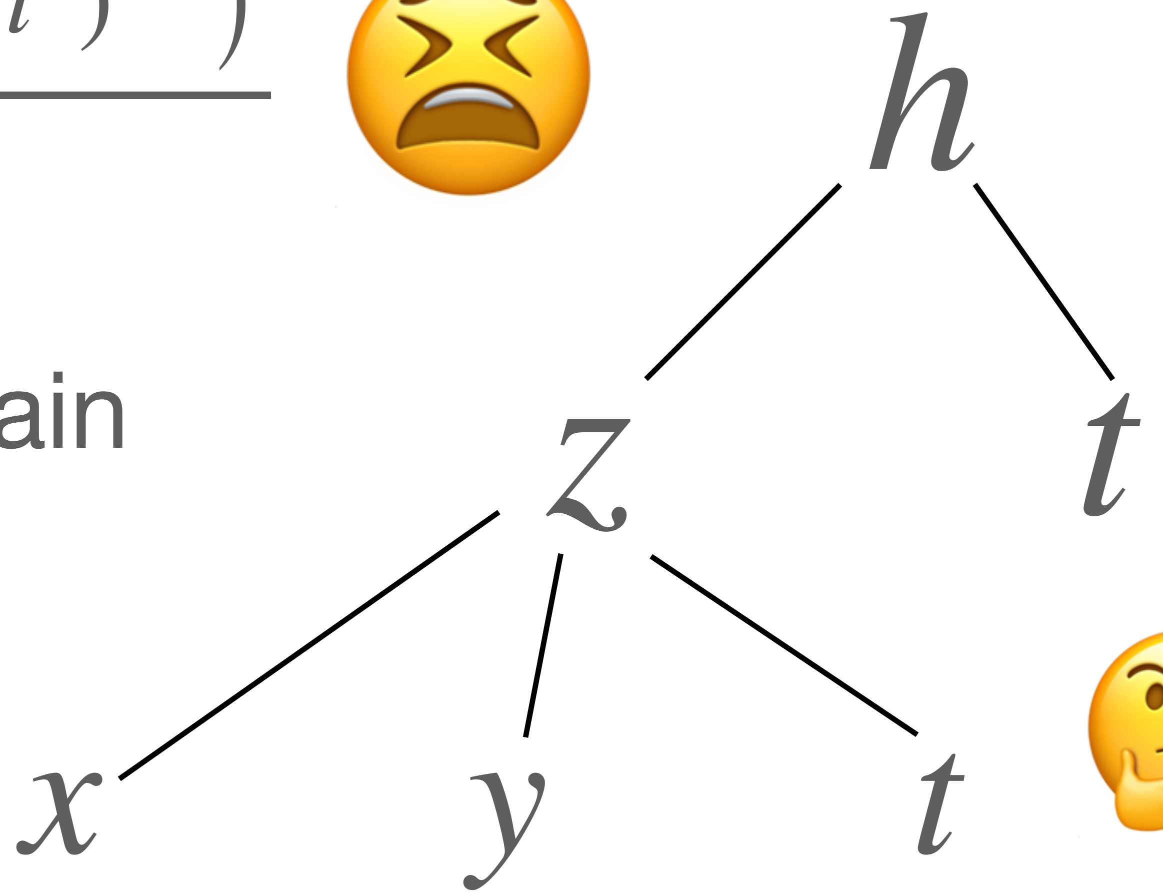
$$h_y(x,y,t) = \frac{ye^{-0.1t}\cos\left((x^2+y^2+t^2)^{1/2}\right)}{(x^2+y^2+t^2)^{1/2}}$$

$$h_t(x,y,t) = \frac{te^{-0.1t}\cos\left((x^2+y^2+t^2)^{1/2}\right)}{(x^2+y^2+t^2)^{1/2}} - 0.1e^{-0.1t}\sin\left((x^2+y^2+t^2)^{1/2}\right)$$

$$h_{xx}(x,y) = \frac{e^{-0.1t}\cos\left((x^2+y^2+t^2)^{1/2}\right)}{(x^2+y^2+t^2)^{1/2}} - \frac{x^2e^{-0.1t}\sin\left((x^2+y^2+t^2)^{1/2}\right)}{x^2+y^2+t^2} - \frac{x^2e^{-0.1t}\cos\left((x^2+y^2+t^2)^{1/2}\right)}{(x^2+y^2+t^2)^{3/2}}$$

Regarding derivatives. The chain rule might make life easier.

$$h(z,t) = e^{-0.1t}\sin(z)$$
$$z = (x^2+y^2+t^2)^{1/2}$$

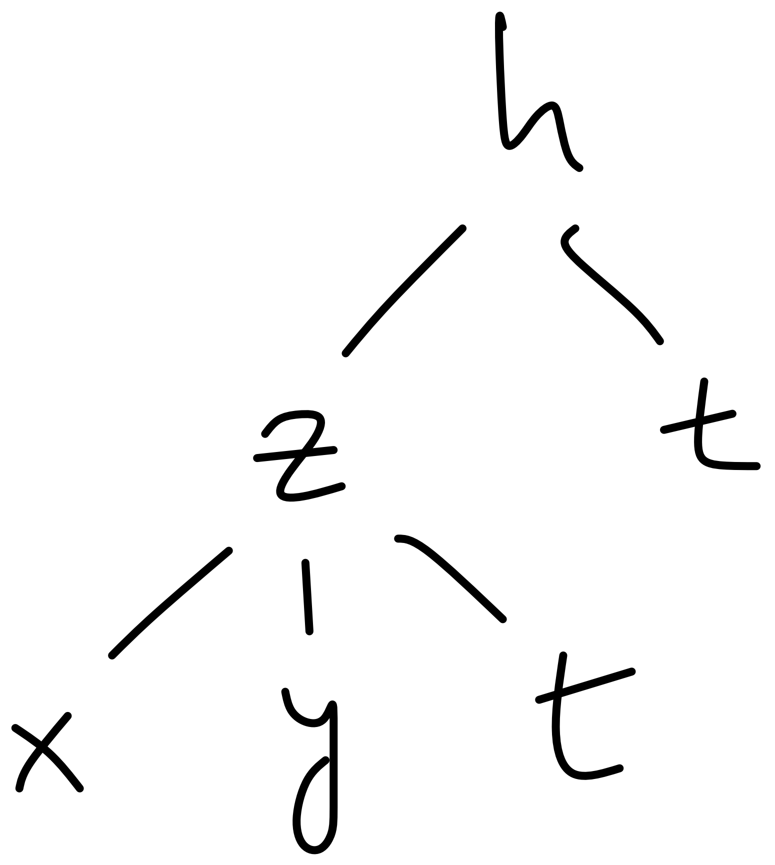


A wave in time, pg 2.

$$h(x,y,t) = e^{-0.1t}\sin\left(\sqrt{x^2+y^2+t^2}\right)$$

$$h(z,t) = e^{-0.1t}\sin(z)$$

where $z = (x^2+y^2+t^2)^{1/2}$



$$z_x = \frac{x}{(x^2+y^2+t^2)^{1/2}} = \frac{x}{z}$$

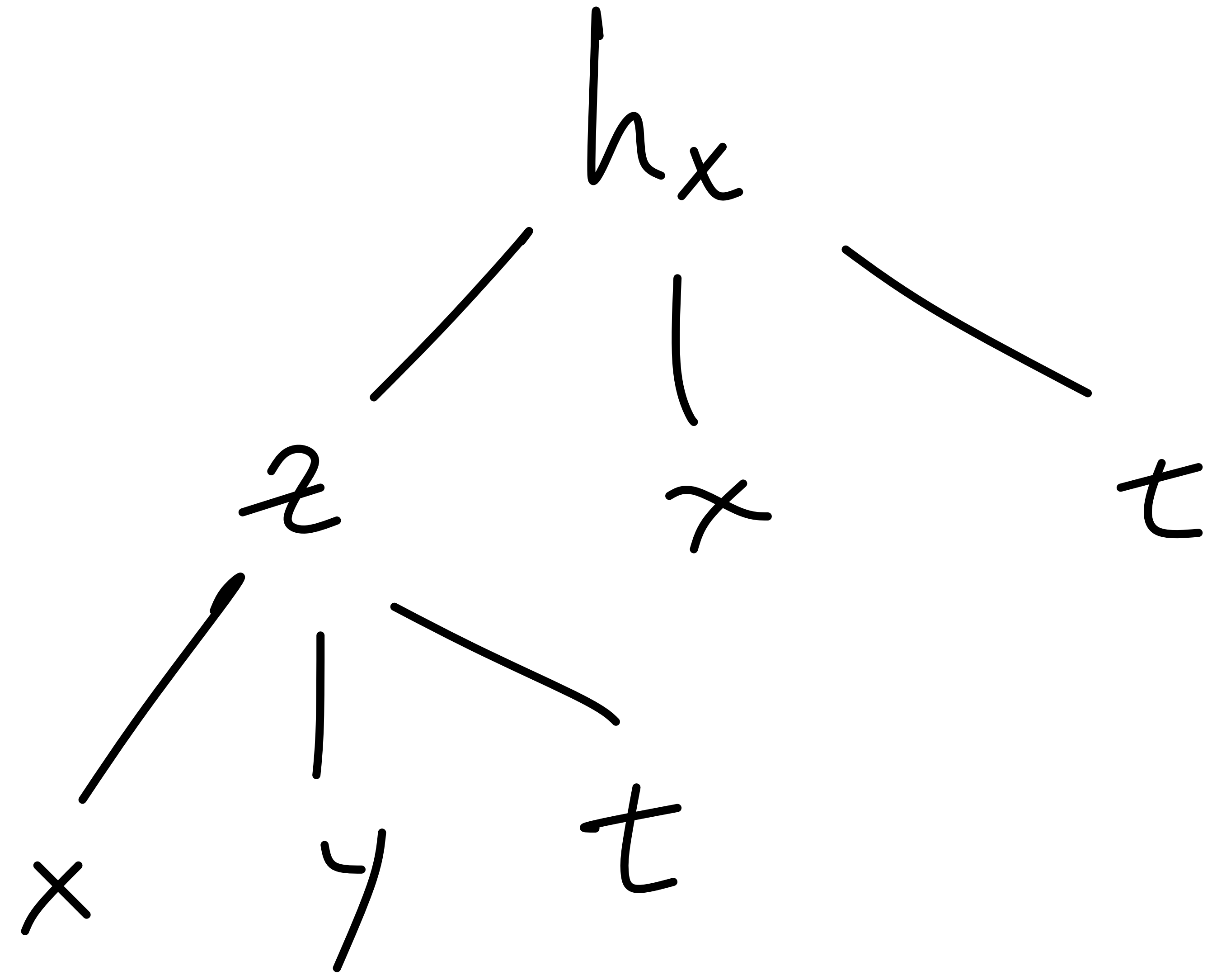
$$z_y = \frac{y}{(x^2+y^2+t^2)^{1/2}} = \frac{y}{z}$$

$$z_t = \frac{t}{(x^2+y^2+t^2)^{1/2}} = \frac{t}{z}$$

$$\begin{aligned} h_x &= h_z \cdot z_x + h_t \cdot t_x \\ &= e^{-0.1t}\cos(z)\cdot\frac{x}{z} + h_t \cdot 0 \\ &= \frac{xe^{-0.1t}\cos(z)}{z} \\ &= \frac{xe^{-0.1t}\cos\left((x^2+y^2+t^2)^{1/2}\right)}{(x^2+y^2+t^2)^{1/2}} \end{aligned}$$

$$\begin{aligned} h_y &= h_z \cdot z_y + h_t \cdot t_y \\ &= e^{-0.1t}\cos(z)\cdot\frac{y}{z} \\ &= \frac{ye^{-0.1t}\cos(z)}{z} \end{aligned}$$

$$\begin{aligned} h_t &= h_z \cdot z_t + h_t \cdot 1 \\ &= e^{-0.1t}\cos(z)\cdot\frac{t}{z} + -0.1e^{-0.1t}\sin(z) \\ &= \frac{te^{-0.1t}\cos(z)}{z} - 0.1e^{-0.1t}\sin(z) \end{aligned}$$



$$\begin{aligned} h_{xx} &= (h_x)_z \cdot z_x + (h_x)_x \cdot x_x + (h_x)_t \cdot t_x \\ &= (h_x)_z \cdot z_x + (h_x)_x \cdot 1 + (h_x)_t \cdot 0 \\ &= \frac{e^{-0.1t}\cos(z)}{z} \\ &\quad + \left(\frac{-xe^{-0.1t}\sin(z)}{z} - \frac{xe^{-0.1t}\cos(z)}{z^2}\right)\cdot\frac{x}{z} \\ &= \frac{e^{-0.1t}\cos(z)}{z} - \frac{x^2e^{-0.1t}\sin(z)}{z^2} - \frac{x^2e^{-0.1t}\cos(z)}{z^3} \end{aligned}$$

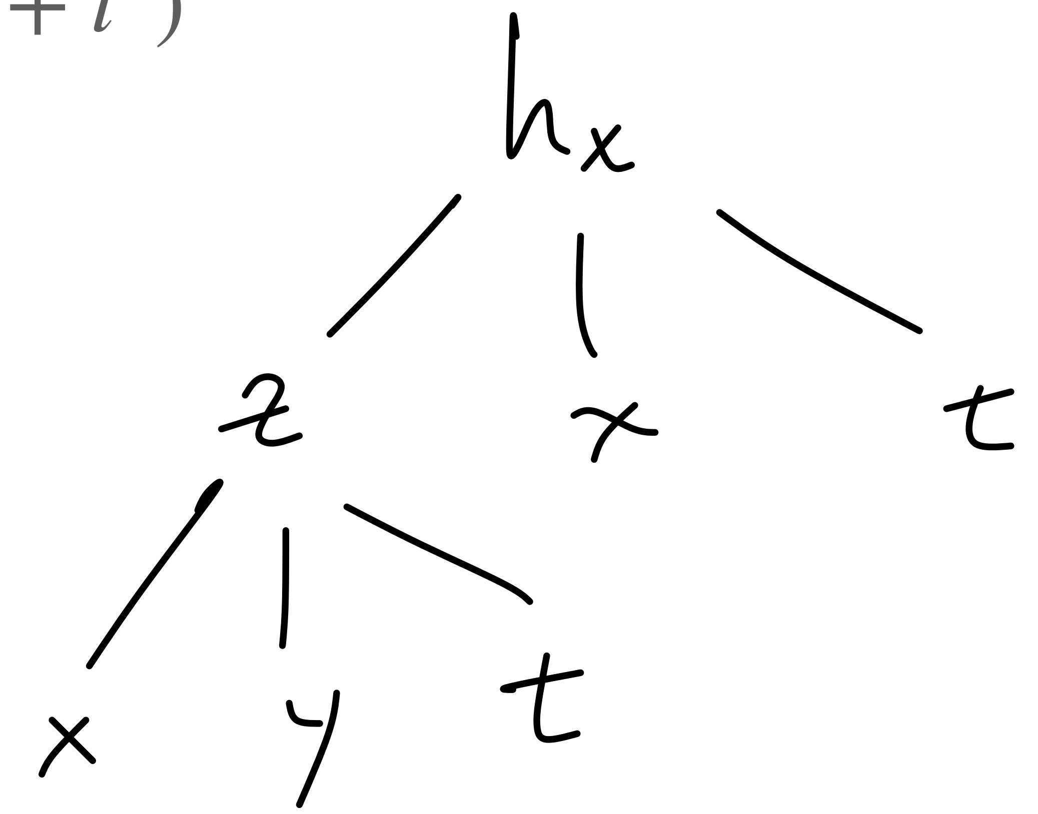
(Next slide: h_{yy} and h_{tt} . . .

A wave in time, pg 3.

$$h(x,y,t) = e^{-0.1t}\sin\left(\sqrt{x^2+y^2+t^2}\right)$$

$$h(z,t) = e^{-0.1t}\sin(z)$$

$$z = (x^2+y^2+t^2)^{1/2}$$



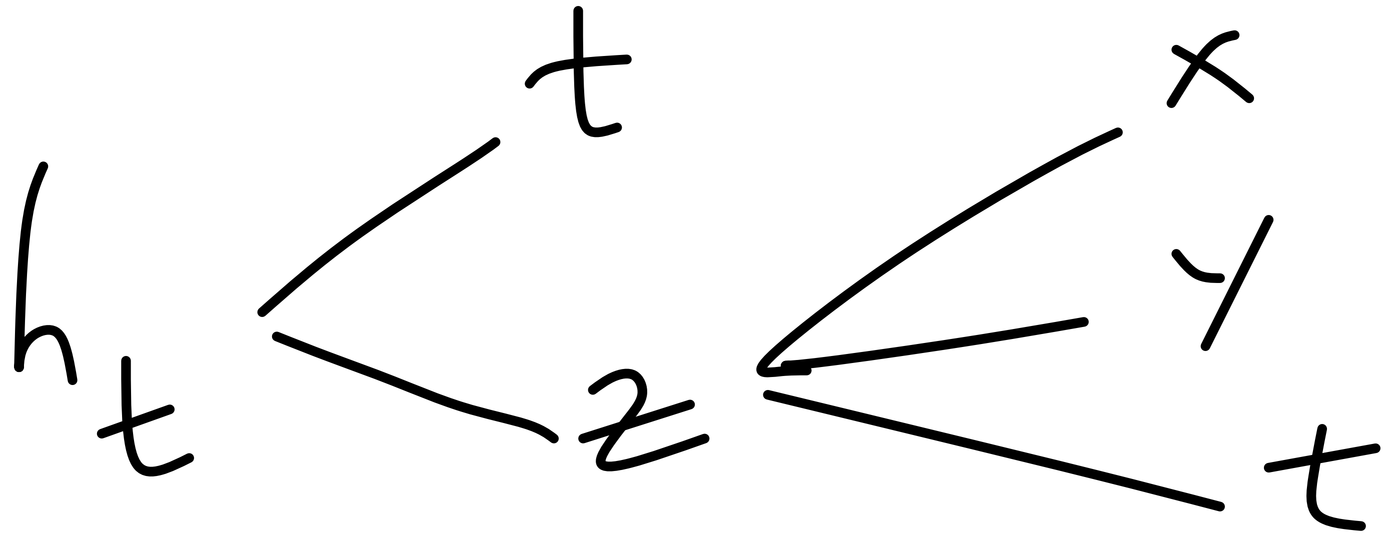
$$h_x = \frac{xe^{-0.1t}\cos(z)}{z}$$

$$h_y = \frac{ye^{-0.1t}\cos(z)}{z}$$

$$h_t = \frac{te^{-0.1t}\cos(z)}{z} - 0.1e^{-0.1t}\sin(z)$$

$$h_{xx} = \frac{e^{-0.1t}\cos(z)}{z} - \frac{x^2e^{-0.1t}\sin(z)}{z^2} - \frac{x^2e^{-0.1t}\cos(z)}{z^3}$$

$$h_{yy} = \frac{e^{-0.1t}\cos(z)}{z} - \frac{y^2e^{-0.1t}\sin(z)}{z^2} - \frac{y^2e^{-0.1t}\cos(z)}{z^3}$$



$$h_{tt} = (h_t)_z \cdot z_t + (h_t)_t \cdot 1 = \left(\frac{-te^{-0.1t}\sin(z)}{z} + \frac{-te^{-0.1t}\cos(z)}{z^2} - 0.1e^{-0.1t}\cos(z)\right)\cdot\frac{t}{z}$$

$$+\frac{e^{-0.1t}\cos(z) - 0.1te^{-0.1t}\cos(z)}{z} + 0.01e^{-0.1t}\sin(z)$$

$$h_{tt} = e^{-0.1t}\left[\cos(z)\left(\frac{-t^2}{z^3} + \frac{-0.2t+1}{z}\right) + \sin(z)\left(\frac{-t^2}{z^2} + \frac{1}{100}\right)\right]$$

We also need the mixed partials h_{xy} , h_{xt} , h_{yt}

$$h_{xy} = (h_x)_z \cdot z_y + (h_x)_x \cdot x_y + (h_x)_t \cdot t_y = (h_x)_z \cdot z_y + (h_x)_x \cdot 0 + (h_x)_t \cdot 0$$

$$= \left(\frac{-xe^{-0.1t}\sin(z)}{z} - \frac{xe^{-0.1t}\cos(z)}{z^2}\right)\cdot\frac{y}{z} = -xye^{-0.1t}\left(\frac{\sin(z)}{z^2} + \frac{\cos(z)}{z^3}\right)$$

$$h_{xt} = -xe^{-0.1t}\left(\frac{0.1\cos(z)}{z} + \frac{t\sin(z)}{z^2} + \frac{t\cos(z)}{z^3}\right)$$

$$h_{yt} = -ye^{-0.1t}\left(\frac{0.1\cos(z)}{z} + \frac{t\sin(z)}{z^2} + \frac{t\cos(z)}{z^3}\right)$$

A wave in time, pg 4.

$$h(x,y,t) = e^{-0.1t}\sin\left(\sqrt{x^2+y^2+t^2}\right)$$

Find the first and second derivatives at the point (5,5,3) in the directions:

$a)\ \hat{\mathbf{u}} = <1,0,0>$

$b)\ \hat{\mathbf{u}} = <0,1,0>$

$c)\ \hat{\mathbf{u}} = <0,0,1>$

$d)\ \hat{\mathbf{u}} = <2/3,2/3,1/3>$

	a) **u** = <1,0,0>	b) **u** = <0,1,0>	c) **u** = <0,0,1>	d) **u** = 1/3<2,2,1>
$h_u(5,5,3)$	0.0829	0.0829	-0.0232	0.1028
$D_u^2(h)(5,5,3)$	-0.2997	-0.2997	-0.0999	-0.7347

At the point (5,5,3), $z = \sqrt{59}$

$h_x(5,5,3) = 0.0829$

$h_y(5,5,3) = 0.0829$

$h_t(5,5,3) = -0.0232$

$h_{xx}(5,5,3) = -0.2997$

$h_{xy}(5,5,3) = -0.3163$

$h_{xt}(5,5,3) = -0.1980$

$h_{yy}(5,5,3) = -0.2997$

$h_{yt}(5,5,3) = -0.1980$

$h_{tt}(5,5,3) = -0.0999$

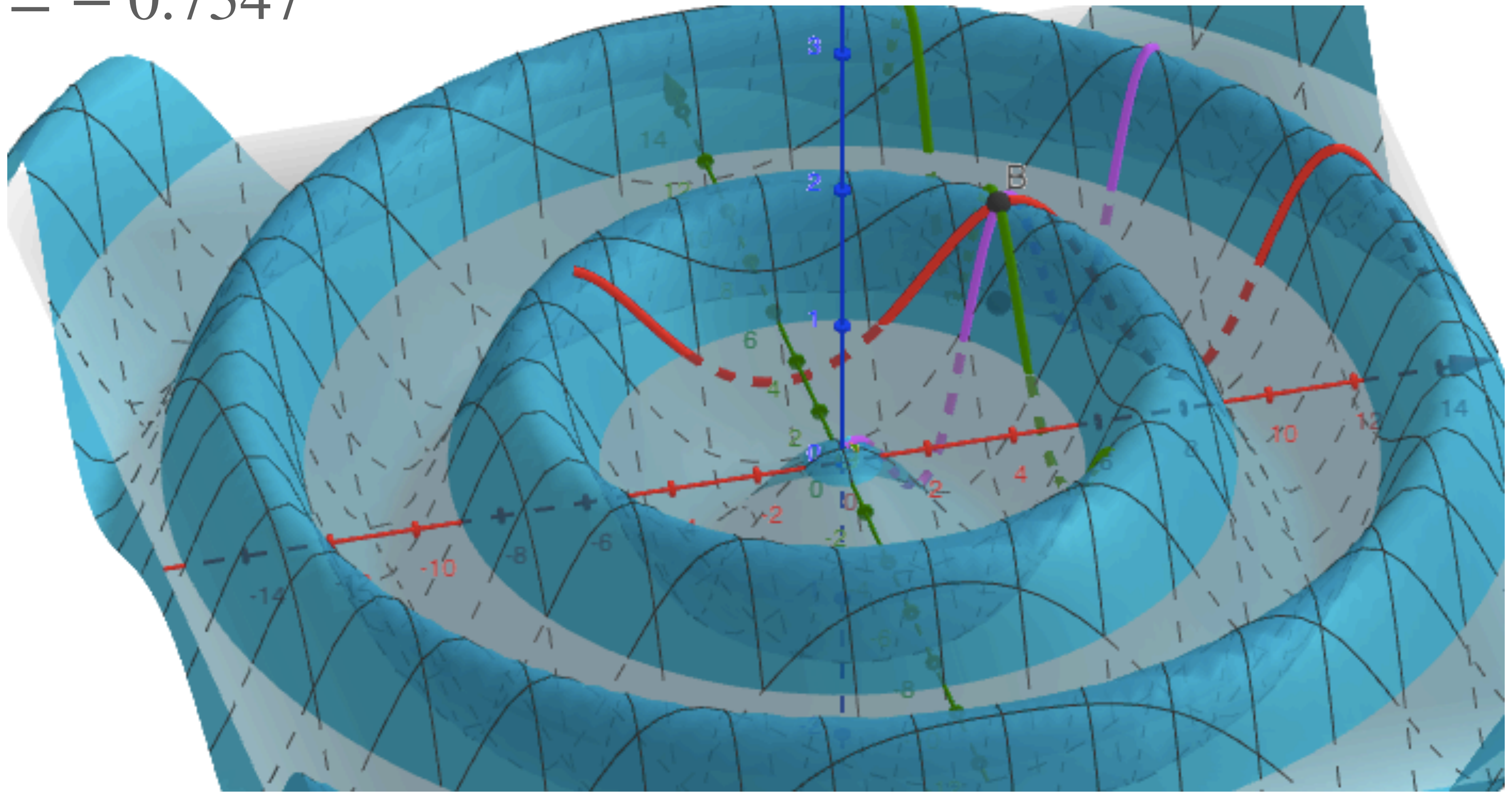
$d)\ \hat{\mathbf{u}} = <2/3,2/3,1/3>\ \ h_u = <h_x(5,5,3), h_y(5,5,3), h_t(5,5,3)> \cdot <2/3,2/3,1/3>$

$= 0.0829 \cdot 2/3 + 0.0829 \cdot 2/3 - 0.0232 \cdot 1/3 = 0.1028$

$$D_u^2(h)(5,5,3) = h_{xx}\cdot\hat{u}_x^2 + h_{yy}\cdot\hat{u}_y^2 + h_{tt}\cdot\hat{u}_t^2 + 2h_{xy}\cdot\hat{u}_x\hat{u}_y + 2h_{xt}\cdot\hat{u}_x\hat{u}_t + 2h_{yt}\cdot\hat{u}_y\hat{u}_t$$

$= -0.2997 \cdot 4/9 + -0.2997 \cdot 4/9 + -0.0999 \cdot 1/9 + 2(-0.3163 \cdot 4/9 + -0.1980 \cdot 2/9 + -0.1980 \cdot 2/9)$

$= -0.7347$



You can see the curves in space, their rates of change and their concavity at $(x,y) = (5,5)$

But can you see those derivatives in space and time?!

Link: WaveInSpaceAndTime