# M110C Week 2

**Goals:**

- Recap, warm-up.
- Equations of Lines in Space.
- Cross Product revisited.
  - Matrix Determinants
  - Properties
- Equations of Planes in Space.
- Distance Between Point and Plane.
- Distance Between Point and Line.
- Distance Between Two Skew Lines.
- Practice Problems (OpenStax Sec 2.5; Stewart, Sec 12.5).

# Recap and Warm Up.

What did we see last week?

Vectors: Magnitude, Direction; Arithmetic.

Dot Product.

$$< v_1, v_2, v_3 > \cdot < u_1, u_2, u_3 >$$
$$= v_1u_1 + v_2u_2 + v_3u_3$$

What's this good for?

Angles between vectors!

Projections! $\mathbf{proj}_{\mathbf{u}}(\mathbf{v}) = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}$

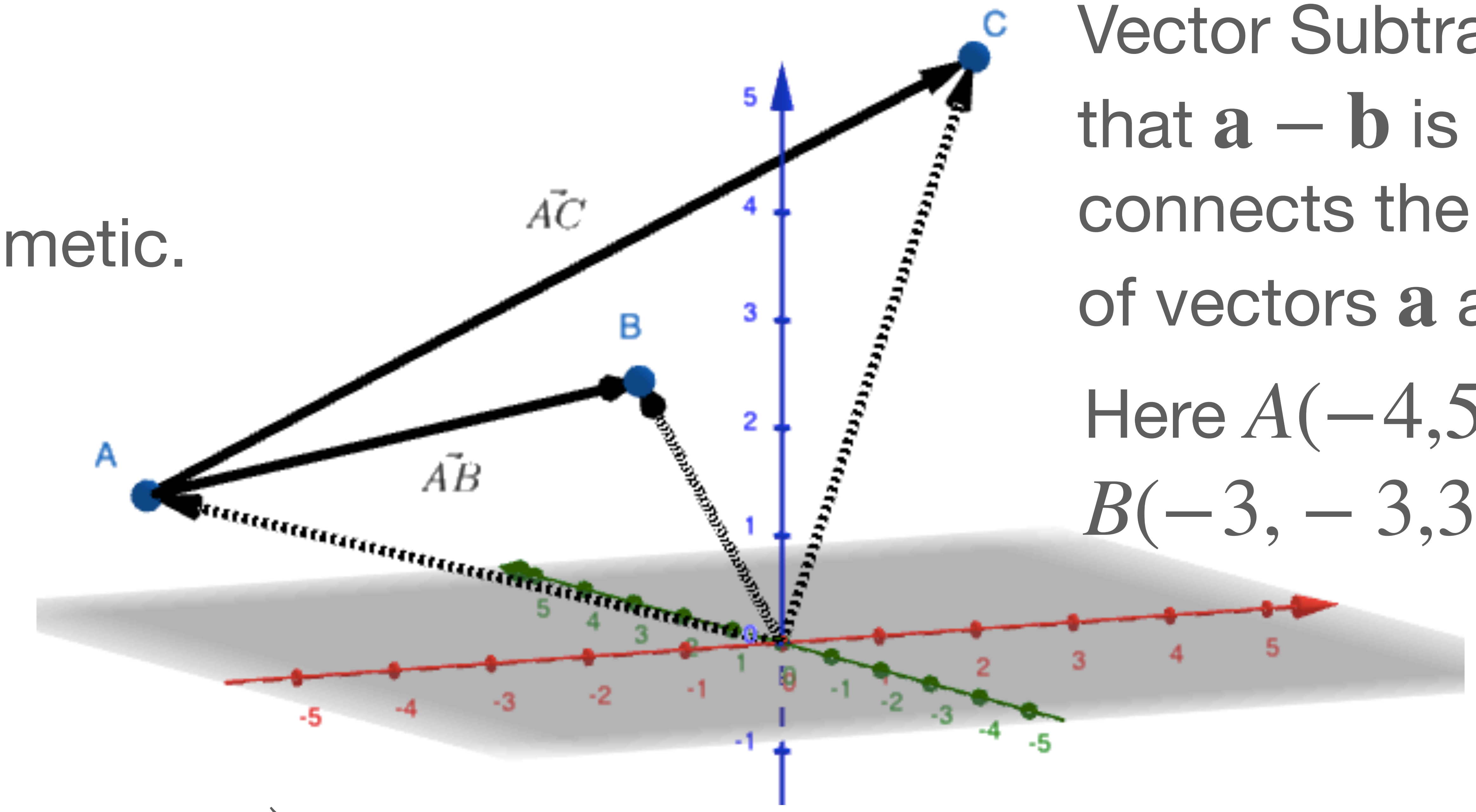
$$\mathbf{proj}_{\mathbf{u}}\mathbf{v} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|^2}\mathbf{u} = \left(\frac{\mathbf{u}}{|\mathbf{u}|}\cdot\mathbf{v}\right)\frac{\mathbf{u}}{|\mathbf{u}|} = (\hat{\mathbf{u}}\cdot\mathbf{v})\hat{\mathbf{u}}$$

Cross Product.

$$\mathbf{u}\times\mathbf{v} = < u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1 >$$

What's this good for?

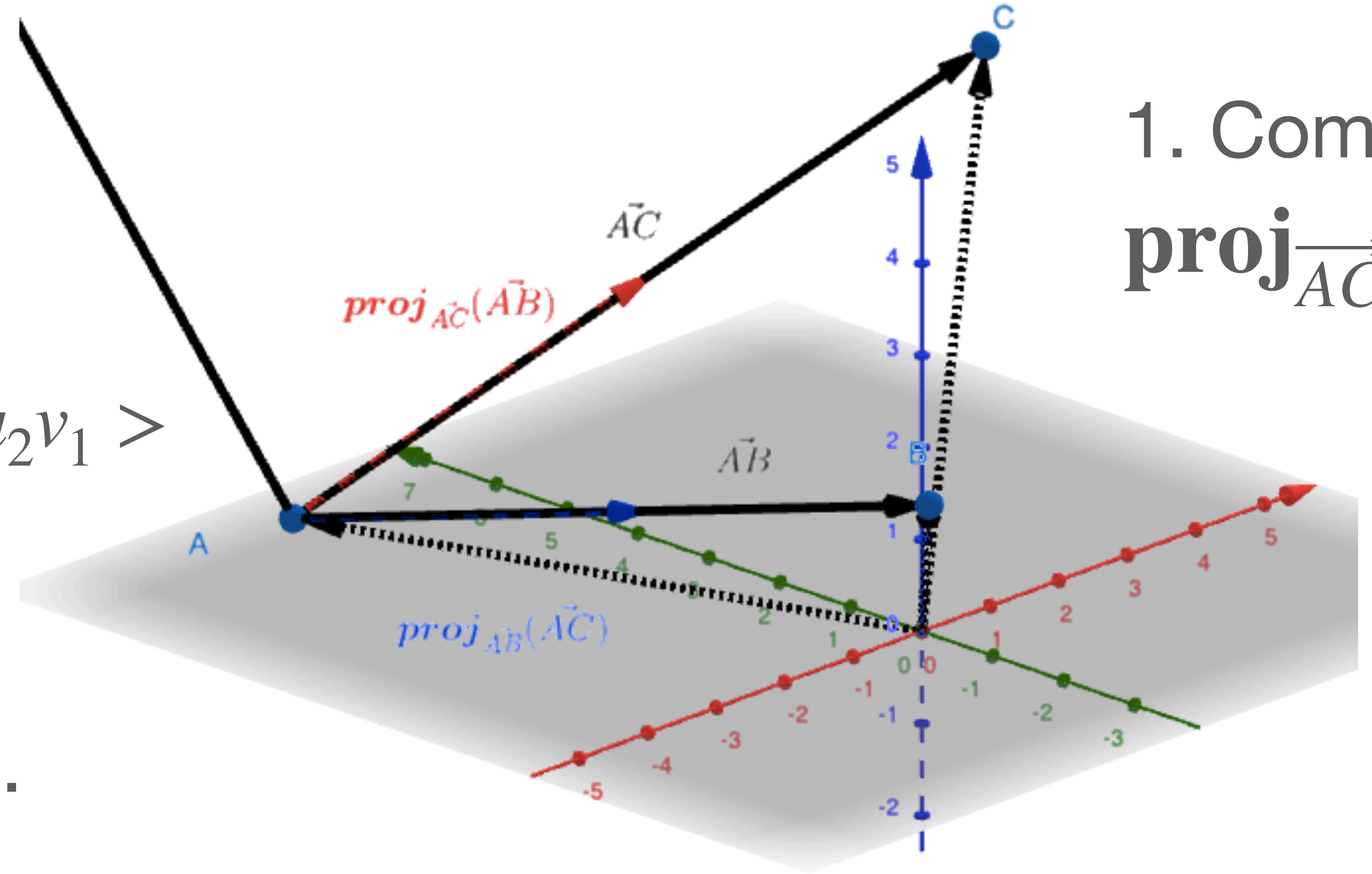
The cross product gives a vector that is perpendicular to both of the factors.




Vector Subtraction: we saw that $\mathbf{a} - \mathbf{b}$ is a vector that connects the terminal points of vectors $\mathbf{a}$ and $\mathbf{b}$.

Here $A(-4,5,1)$
$B(-3,-3,3)$, $C(3,2,5)$

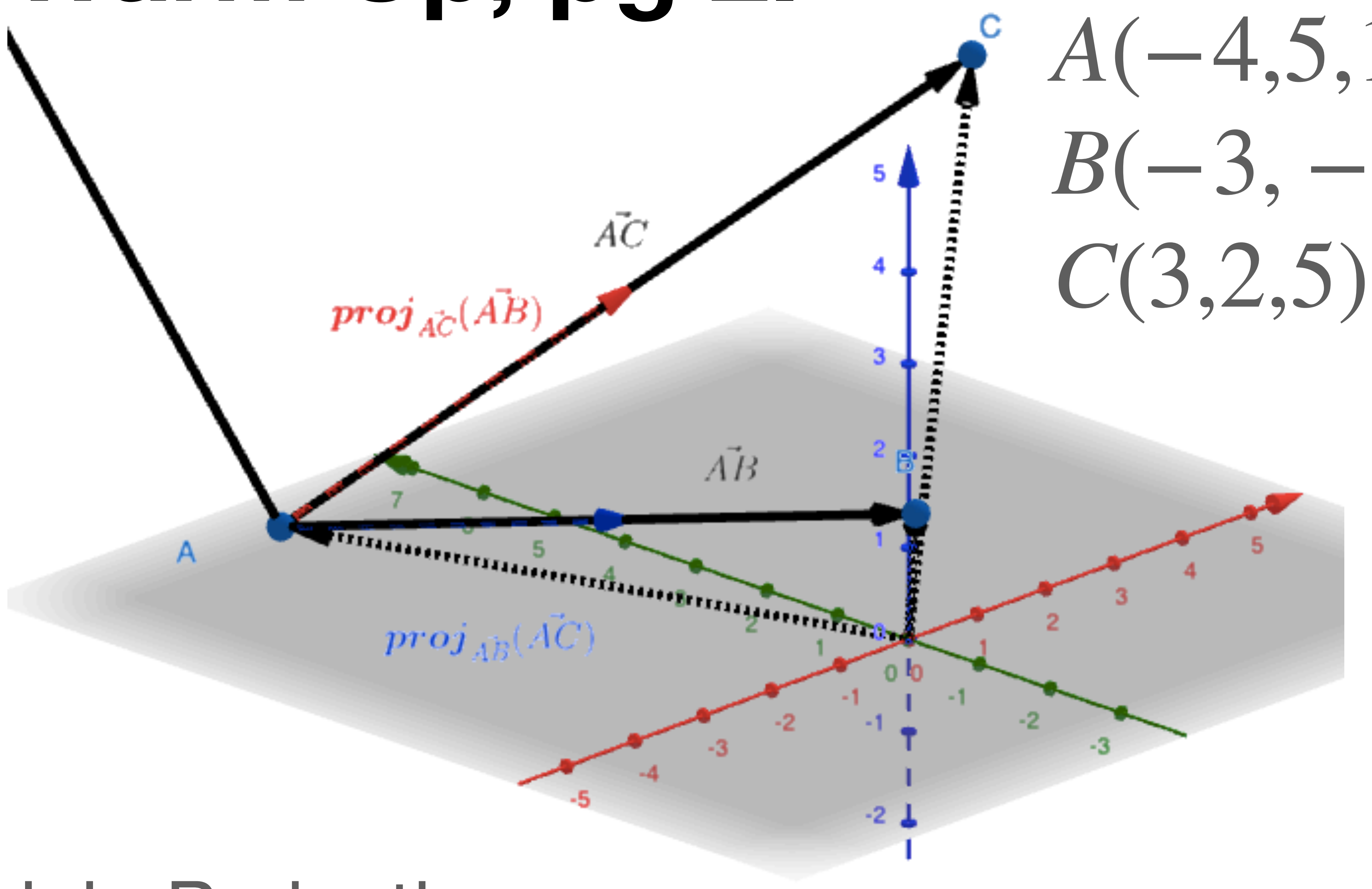
$$\overrightarrow{AC} = \mathbf{c} - \mathbf{a} = < 3,2,5 > - < -4,5,1 > = < 7,-3,4 >$$
$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = < -3,-3,3 > - < -4,5,1 > = < 1,-8,2 >$$




1. Compute projections $\mathbf{proj}_{\overrightarrow{AC}}(\overrightarrow{AB})$, $\mathbf{proj}_{\overrightarrow{AB}}(\overrightarrow{AC})$

2. Find a vector $\mathbf{w}$ perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

# Warm Up, pg 2.




$A(-4,5,1)$,
$B(-3,-3,3)$,
$C(3,2,5)$

1. Compute projections
$\mathbf{proj}_{\overrightarrow{AC}}(\overrightarrow{AB})$, $\mathbf{proj}_{\overrightarrow{AB}}(\overrightarrow{AC})$

2. Find a vector $\mathbf{w}$ perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

Link: Projections

$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = <1,-8,2>$

$\overrightarrow{AC} = \mathbf{c} - \mathbf{a} = <7,-3,4>$

$$\mathbf{proj}_{\overrightarrow{AC}}(\overrightarrow{AB}) = \left(\frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{\overrightarrow{AC}\cdot\overrightarrow{AC}}\right)\overrightarrow{AC}$$

$$= \frac{39}{74} <7,-3,4>$$

$$\approx <3.69,\ -1.58,\ 2.11>$$

$$\mathbf{proj}_{\overrightarrow{AB}}(\overrightarrow{AC}) = \left(\frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{\overrightarrow{AB}\cdot\overrightarrow{AB}}\right)\overrightarrow{AB}$$

$$= \frac{39}{69} <1,-8,2>$$

$$\approx <0.57,\ -4.52,\ 1.13>$$

2. A vector perpendicular to both $\mathbf{u} = \overrightarrow{AB}$ and $\mathbf{v} = \overrightarrow{AC}$ is

$$\mathbf{w} = \mathbf{u}\times\mathbf{v} = <u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1>$$

$$= <-8\cdot 4 - 2\cdot(-3), 2\cdot 7 - 1\cdot 4, 1\cdot(-3) - (-8)\cdot 7>$$

$$= <-26,10,53>$$

check perpendicularity using the dot product.

(the dot products of $\mathbf{w}$ with $\mathbf{u}$ and $\mathbf{v}$ should be 0.)

$$\mathbf{w}\cdot\mathbf{u} = <-26,10,53>\cdot<1,-8,2>$$

$$= -26 - 80 + 106 = 0 \qquad \text{check!}$$

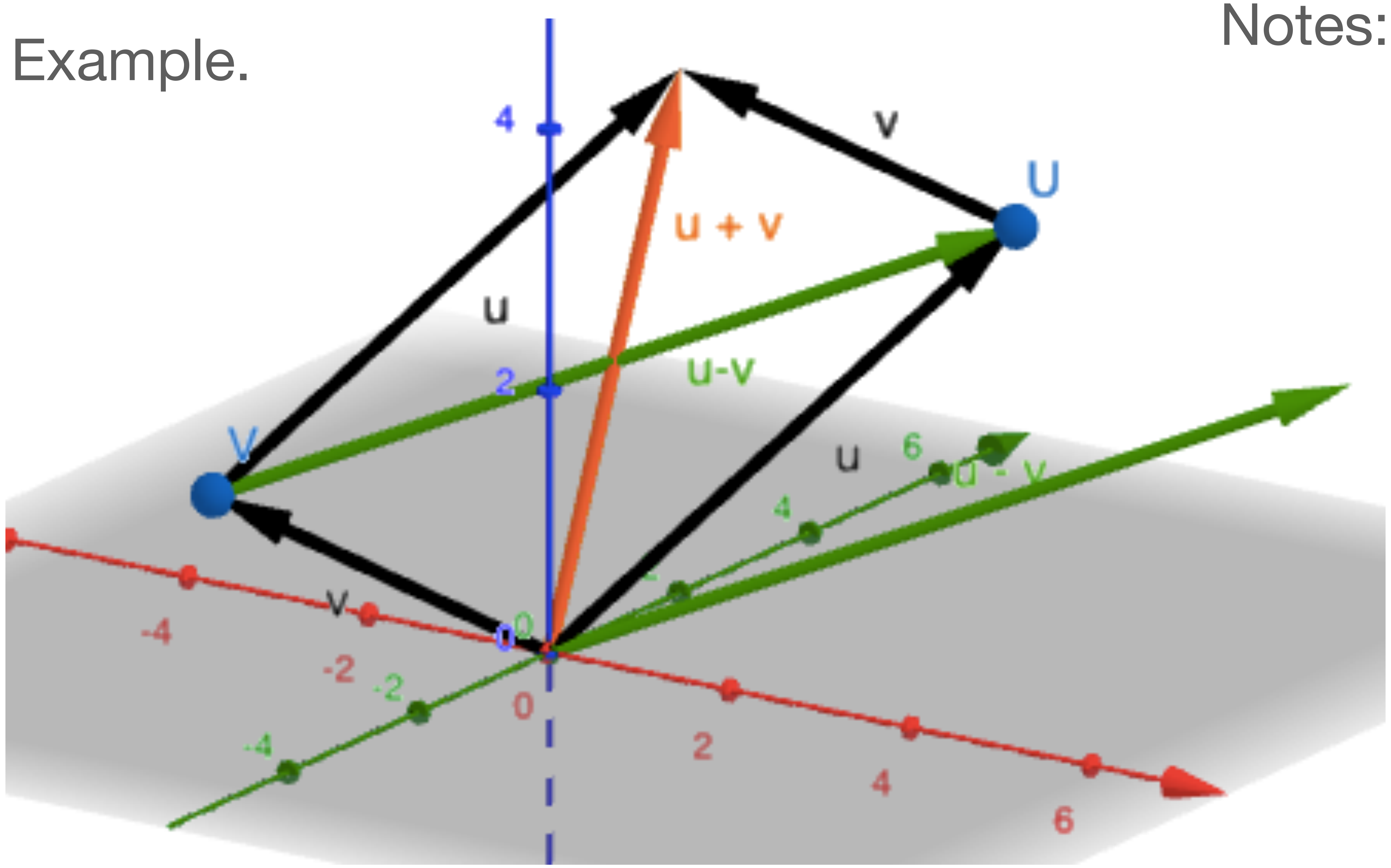
$$\mathbf{w}\cdot\mathbf{v} = <-26,10,53>\cdot<7,-3,4>$$

$$= -182 - 30 + 212 = 0 \qquad \text{check!}$$

# Equations of Lines in Space, pg 1: 3D addition.

The geometry of arithmetic that we saw in 2D also works in 3D.

Example.




Link: 3DVectorArithmetic

$\mathbf{u} = <3,3,3>$

$\mathbf{v} = <-3,-1,1>$

$\mathbf{u}+\mathbf{v} = <0,2,4>$

$\mathbf{u}-\mathbf{v} = <6,4,2>$

Notes: 1. Thinking of vectors as displacement…

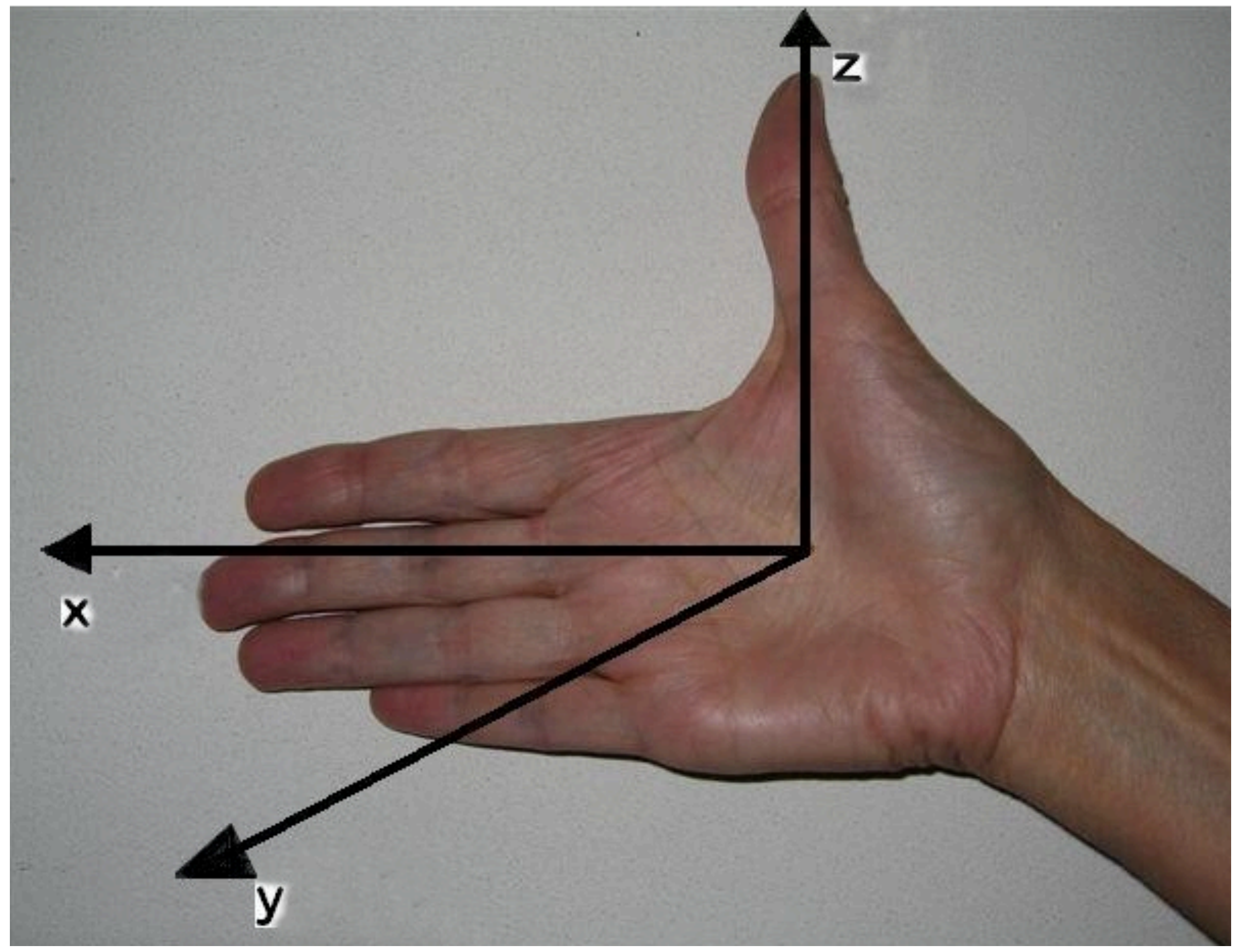
$\mathbf{u}+\mathbf{v}$ is the net displacement of $\mathbf{u}$ and $\mathbf{v}$.

$\mathbf{u}-\mathbf{v}$ is the vector from the terminal point of $\mathbf{v}$ to the terminal point of $\mathbf{u}$;

$\mathbf{u}-\mathbf{v}$ is the displacement needed to get from the terminal point of $\mathbf{v}$ to the terminal point of $\mathbf{u}$

2. Scalar multiplication, $k\mathbf{u}$, stretches $\mathbf{u}$ if $k > 1$, shrinks $\mathbf{u}$ if $0 < k < 1$, and flips $\mathbf{u}$ if $k < 0$.

3. The x,y and z axes are oriented by the right hand rule.

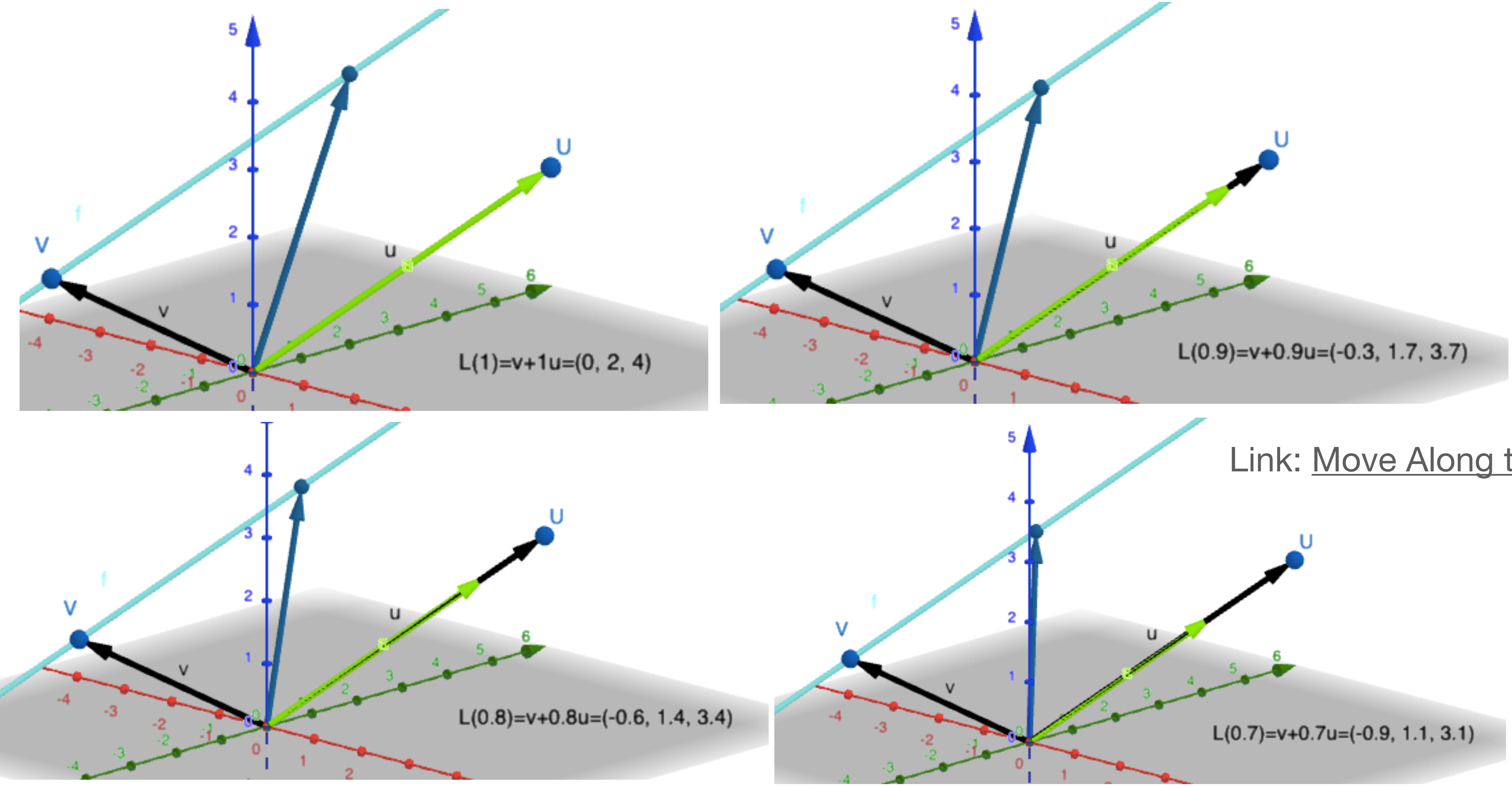



Link: wikimediacommons

# Equations of Lines in Space, pg 2: Making a Line.

We can get a line by adding all multiples of one vector to another.

Example. $\mathbf{u} = < 3,3,3 >$ , $\mathbf{v} = < -3, -1,1 >$




Link: Move Along the Line

# Equations of Lines in Space, pg 3.

We can describe a line in space algebraically like this:

$\mathbf{L}(t) = \mathbf{v} + t\mathbf{u}$

With the initial point of $\mathbf{v}$ at the origin, $\mathbf{L}(t)$ is the line that goes through the terminal point of $\mathbf{v}$, and that goes parallel to $\mathbf{u}$.

In our example, $\mathbf{u} = <3,3,3>$ , $\mathbf{v} = <-3,-1,1>$

$$\mathbf{L}(t) = <-3,-1,1> + t<3,3,3>$$
$$= <-3,-1,1> + <3t,3t,3t>$$
$$= <-3+3t,\ -1+3t,\ 1+3t>$$

You can express this with some *scalar equations*:

$$x(t) = -3+3t$$
$$y(t) = -1+3t$$
$$z(t) = 1+3t$$

You can also express this with *symmetric equations*, by expressing t in terms of x, y and z.

$$t = \frac{x+3}{3} = \frac{y+1}{3} = \frac{z-1}{3}$$

Note that the same line can be described with many different vectors $\mathbf{u}$

For example, the line through the point $(-3,-1,1)$ which is parallel to the vector $\mathbf{w} = <1,1,1>$ has the exact same points on it as our line $\mathbf{L}(t)$

…but its algebraic description is different:

$$\mathbf{K}(t) = <-3,-1,1> + t<1,1,1>$$
$$= <-3+t,-1+t,1+t>$$

$\mathbf{K}(t)$ achieves the exact same points at $\mathbf{L}(t)$ does, but at different times!

e.g. $<0,2,4> = \mathbf{L}(1) = \mathbf{K}(3)$

Also, the same line could be described with a different starting point.

$$\mathbf{Q}(t) = <0,2,4> + t<1,1,1>$$

$\mathbf{Q}$ describes the same line as $\mathbf{L}$ and $\mathbf{K}$.

# Equations of Lines in Space, pg 4.

Try these. Find the equation of the line which …

1. (S12.5 #2) … goes through the point $(6,5,-2)$ and is parallel to the vector $1\mathbf{i}+3\mathbf{j}-2/3\mathbf{k} = <1,3,-2/3>$.
(recall $\mathbf{i} = <1,0,0>$, $\mathbf{j} = <0,1,0>$, $\mathbf{k} = <0,0,1>$)

2. (S12.5 #4) … goes through the point $(0,14,-10)$ and parallel to the line given by
$x = -1+2t, \quad y = 6-3t, \quad z = 3+9t$.

3. (OX2.5 #246)
… goes through the points $(7,-2,6)$ and $(-3,0,6)$.

1. $\mathbf{L}(t) = \mathbf{v} + t\mathbf{u}$

$$\mathbf{L}(t) = <6,5,-2> + t<1,3,-2/3>$$
$$= <6+t, 5+3t, -2-2t/3>$$

$$x = 6+t$$
$$y = 5+3t$$
$$z = -2-2t/3 \qquad t = x-6 = \frac{y-5}{3} = \frac{-3(z+2)}{2}$$

2. $$\mathbf{L}(t) = <0,14,-10> + t<2,-3,9>$$
$$= <2t, 14-3t, -10+9t>$$

$$x = 2t$$
$$y = 14-3t$$
$$z = -10+9t$$

$$t = \frac{x}{2} = \frac{14-y}{3} = \frac{z+10}{9}$$

3. $\mathbf{v} = <7,-2,6>$

$$\mathbf{u} = <-3,0,6> - <7,-2,6>$$
$$= <-10,2,0>$$

$$\mathbf{L}(t) = <7,-2,6> + t<-10,2,0>$$

$$x = 7-10t$$
$$y = -2+2t$$
$$z = 6$$

$$t = \frac{x-7}{-10} = \frac{y+2}{2}; \quad z = 6$$

# Matrices, pg 1.

Here's a 2x2 matrix:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

where $a_{ij} \in \mathbf{R}$

The *determinant* of a 2x2 matrix is

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Example.

$$\det \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} = (1)(1) - (2)(1) = -1$$

Here's a 3x3 matrix:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$ where $a_{ij} \in \mathbf{R}$

The *determinant* of a 3x3 matrix is $\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \ldots$

$$\ldots = a_{11} \det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12} \det \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13} \det \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

Example.

$$\det \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \ldots$$

$$= 1 \det \begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix} - 2 \det \begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix} + 3 \det \begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix}$$

$$= 1(-3) - 2(-6) + 3(-3) = 0$$

Regarding Notation. You may also see this for a determinant:

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

# Matrices, pg 2. Cross Product.

We can use the 3x3 determinant concept to compute cross products of vectors.

$\mathbf{u} = < u_1, u_2, u_3 >$

$\mathbf{v} = < v_1, v_2, v_3 >$

Last week we computed a cross product like this:

$$\mathbf{u} \times \mathbf{v} = < u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1 >$$

$$\begin{aligned} = \quad & (u_2v_3 - u_3v_2) < 1,0,0 > \\ & -(u_1v_3 - u_3v_1) < 0,1,0 > \\ & +(u_1v_2 - u_2v_1) < 0,0,1 > \end{aligned}$$

$$\begin{aligned} = \quad & (u_2v_3 - u_3v_2)\mathbf{i} \\ & -(u_1v_3 - u_3v_1)\mathbf{j} \\ & +(u_1v_2 - u_2v_1)\mathbf{k} \end{aligned}$$

(Each of these components can be expressed as a 2x2 determinant….)

$$\ldots = \mathbf{i} \det \begin{pmatrix} u_2 & u_3 \\ v_2 & v_3 \end{pmatrix} - \mathbf{j} \det \begin{pmatrix} u_1 & u_3 \\ v_1 & v_3 \end{pmatrix} + \mathbf{k} \det \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}$$

$$= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}$$

←Concise formula for $\mathbf{u} \times \mathbf{v}$

Example1. $\mathbf{u} = < 1,1,2 >$

$\mathbf{v} = < -1,1,1 >$

$$\mathbf{u} \times \mathbf{v} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 2 \\ -1 & 1 & 1 \end{pmatrix}$$

$$= \mathbf{i} \det \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} - \mathbf{j} \det \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} + \mathbf{k} \det \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \mathbf{i}(-1) - \mathbf{j}(3) + \mathbf{k}(2)$$

$$= < -1, -3,2 >$$

remember the cross product should be perpendicular to both of the factors $\mathbf{u}$, $\mathbf{v}$.

check: $< -1, -3,2 > \cdot \mathbf{u} = 0$ check!

$< -1, -3,2 > \cdot \mathbf{v} = 0$ check!

# Matrices, pg 3. Cross Product Examples.

Example2.

$$\mathbf{u} = <-1,2,-3>,$$
$$\mathbf{v} = <1,4,1>$$

$$\mathbf{u}\times\mathbf{v} = \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & -3 \\ 1 & 4 & 1 \end{pmatrix}$$

$$= \mathbf{i}(14) - \mathbf{j}(2) + \mathbf{k}(-6)$$

$$= <14,-2,-6>$$

Example 3.

$$\mathbf{u} = <2,1,-3>,$$
$$\mathbf{v} = <-4,-2,6>$$

$$\mathbf{u}\times\mathbf{v} = \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -3 \\ -4 & -2 & 6 \end{pmatrix}$$

$$= \mathbf{i}(0) - \mathbf{j}(0) + \mathbf{k}(0)$$

$$= <0,0,0>$$

Compute some Cross Product.

$$\mathbf{a} = <1,2,3>$$
$$\mathbf{b} = <2,1,0>$$
$$\mathbf{c} = <1,-1,1>$$

1. $\mathbf{a}\times\mathbf{b} = ?$
2. $(2\mathbf{a})\times\mathbf{b} = ?$
3. $\mathbf{a}\times\mathbf{c} = ?$
4. $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = ?$

1. $\mathbf{a}\times\mathbf{b} = \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 2 & 1 & 0 \end{pmatrix}$

$$= <-3,6,-3>$$
$$= -3<1,-2,1>$$

2. $(2\mathbf{a})\times\mathbf{b} = \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 4 & 6 \\ 2 & 1 & 0 \end{pmatrix}$

$$= <-6,12,-6>$$
$$= -6<1,-2,1>$$
$$= 2(\mathbf{a}\times\mathbf{b})$$

3. $\mathbf{a}\times\mathbf{c} = \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 1 & -1 & 1 \end{pmatrix}$

$$= <5,2,-3>$$

4. $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 3 & 0 & 1 \end{pmatrix}$

$$= <2,8,-6>$$
$$= 2<1,4,-3>$$
$$= \mathbf{a}\times\mathbf{b} + \mathbf{a}\times\mathbf{c}$$

# Properties of Cross Product, pg1.

$\mathbf{u} \times \mathbf{v} = < u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1 >$

$$\mathbf{u} \times \mathbf{v} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}$$

1. $\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v}$
2. $k\mathbf{u} \times \mathbf{v} = k(\mathbf{u} \times \mathbf{v}) = \mathbf{u} \times k\mathbf{v}$ for all $k \in \mathbf{R}$
3. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
4. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$
5. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$
6. $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin(\theta)$ where $\theta$ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

6.5 $\mathbf{u} \times \mathbf{v} = 0$ exactly when $\mathbf{v} = k\mathbf{u}$ for some $k \in \mathbf{R}$, i.e. when $\mathbf{u}$ and $\mathbf{v}$ are *parallel.*

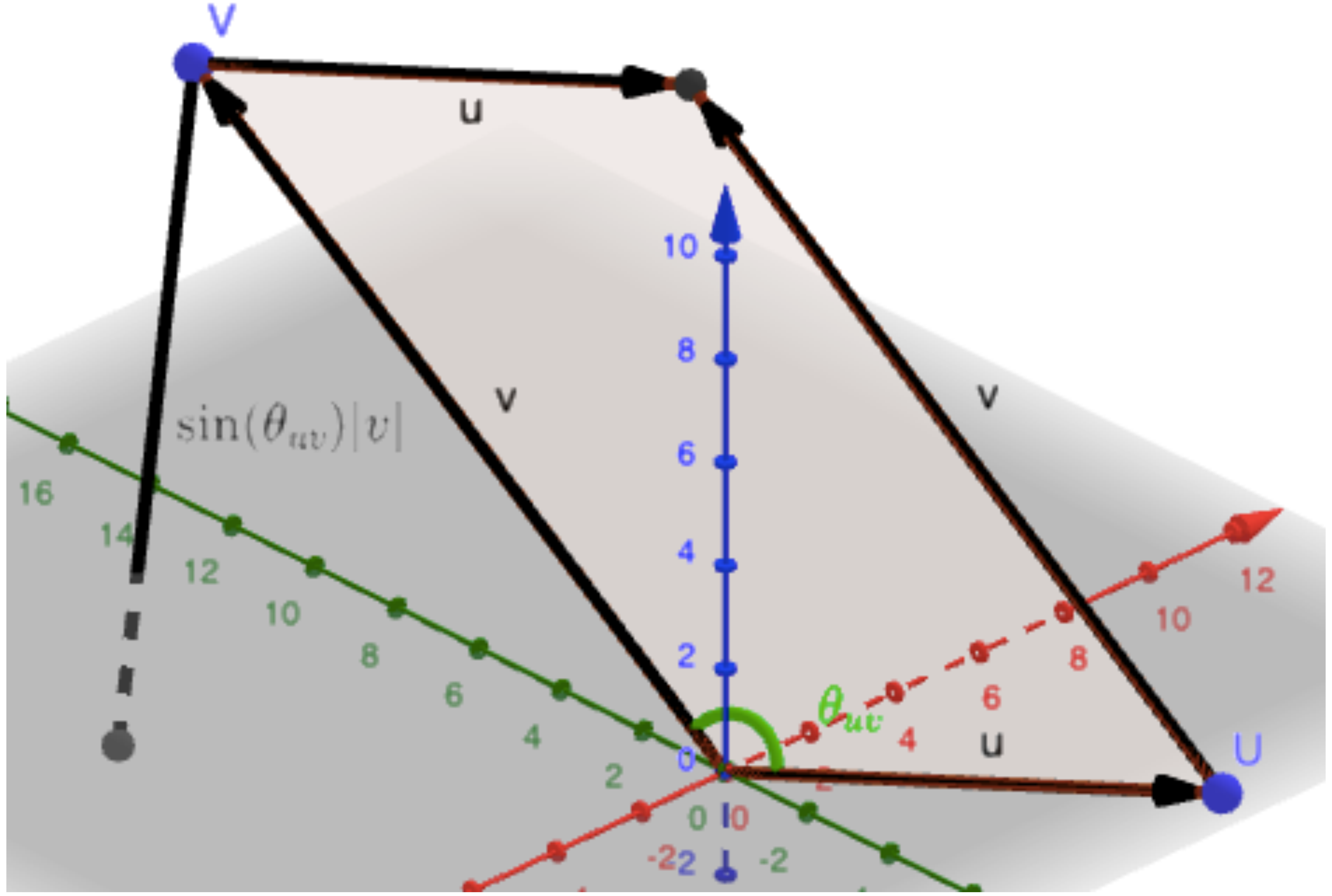
Regarding property 6:

$$\begin{gathered}
|\mathbf{u}|^2|\mathbf{v}|^2\sin^2(\theta) = \\
|\mathbf{u}|^2|\mathbf{v}|^2(1 - \cos^2(\theta)) = \\
|\mathbf{u}|^2|\mathbf{v}|^2 - (\mathbf{u} \cdot \mathbf{v})^2 = \\
(u_1^2 + u_2^2 + u_3^2) \cdot (v_1^2 + v_2^2 + v_3^2) - (u_1v_1 + u_2v_2 + u_3v_3)^2 = \\
u_1^2v_1^2 + u_1^2v_2^2 + u_1^2v_3^2 + \\
u_2^2v_1^2 + u_2^2v_2^2 + u_2^2v_3^2 + \\
u_3^2v_1^2 + u_3^2v_2^2 + u_3^2v_3^2 + \\
-(u_1^2v_1^2 + u_2^2v_2^2 + u_3^2v_3^2 \\
+2u_2u_3v_2v_3 + 2u_1u_3v_1v_3 + 2u_1u_2v_1v_2) = \\
u_2^2v_3^2 - 2u_2u_3v_2v_3 + u_3^2v_2^2 + \\
u_3^2v_1^2 - 2u_1u_3v_1v_3 + u_1^2v_3^2 + \\
u_1^2v_2^2 - 2u_1u_2v_1v_2 + u_2^2v_1^2 \\
= (u_2v_3 - u_3v_2)^2 + (u_3v_1 - u_1v_3)^2 + (u_1v_2 - u_2v_1)^2 \\
= |\mathbf{u} \times \mathbf{v}|^2
\end{gathered}$$

# Properties of Cross Product, pg2.

6. $|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin(\theta)$ where $\theta$ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

There's some geometry here…



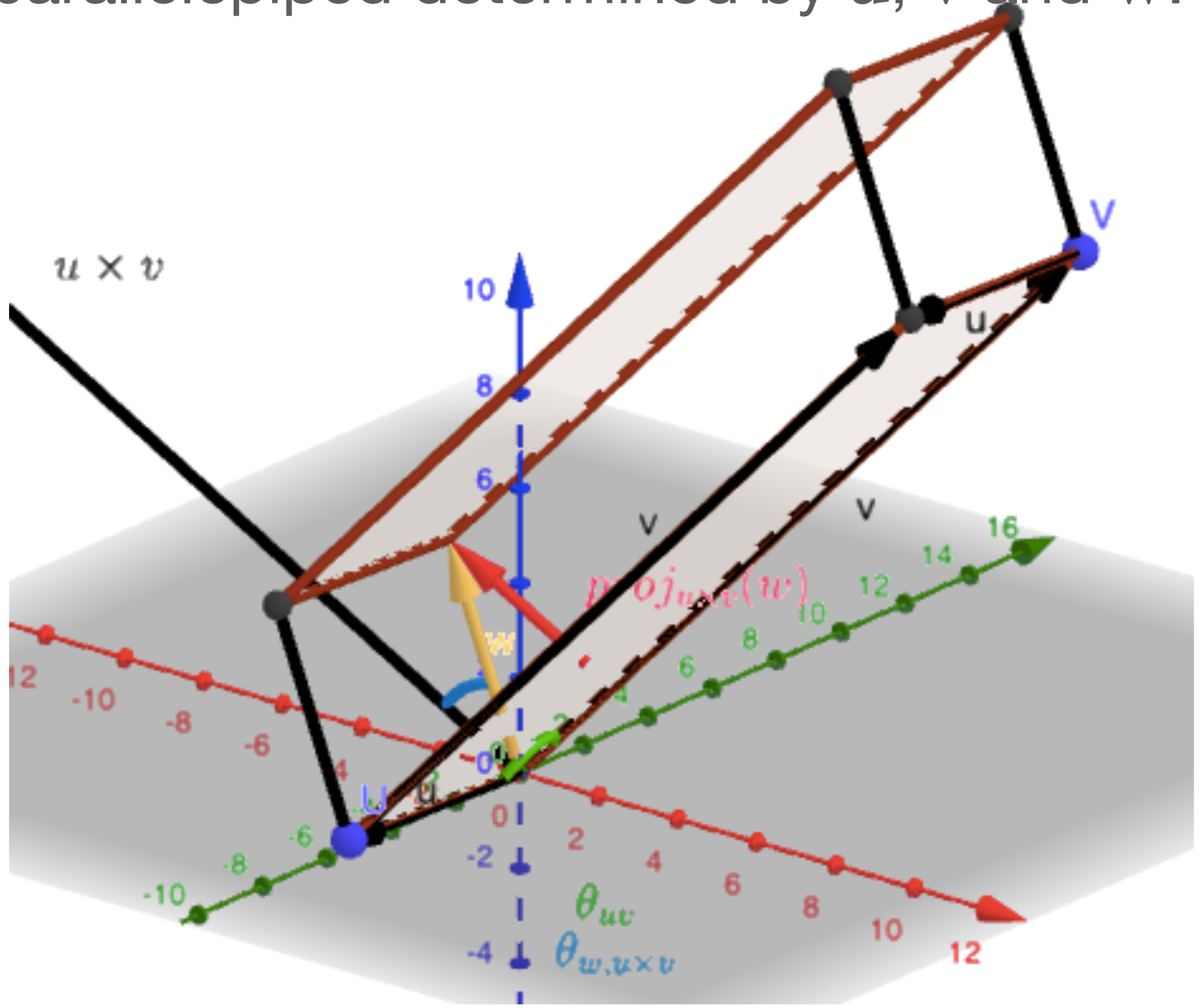

$|\mathbf{u}||\mathbf{v}|\sin(\theta)$ is the area of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$

7. $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}$ is a *scalar triple product.*

Its magnitude is $|(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}| = |\mathbf{u}\times\mathbf{v}||\mathbf{w}||\cos(\theta)|$

$\theta$ is the angle between $\mathbf{u}\times\mathbf{v}$ and $\mathbf{w}$

$|\mathbf{w}||\cos(\theta)| = |\mathbf{proj}_{\mathbf{u}\times\mathbf{v}}(\mathbf{w})|$ is the height of the parallelepiped determined by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.




$|(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}| = |\mathbf{u}\times\mathbf{v}||\mathbf{w}||\cos(\theta)|$
is the volume of this parallelepiped.

# Properties of Cross Product, parallelepiped.

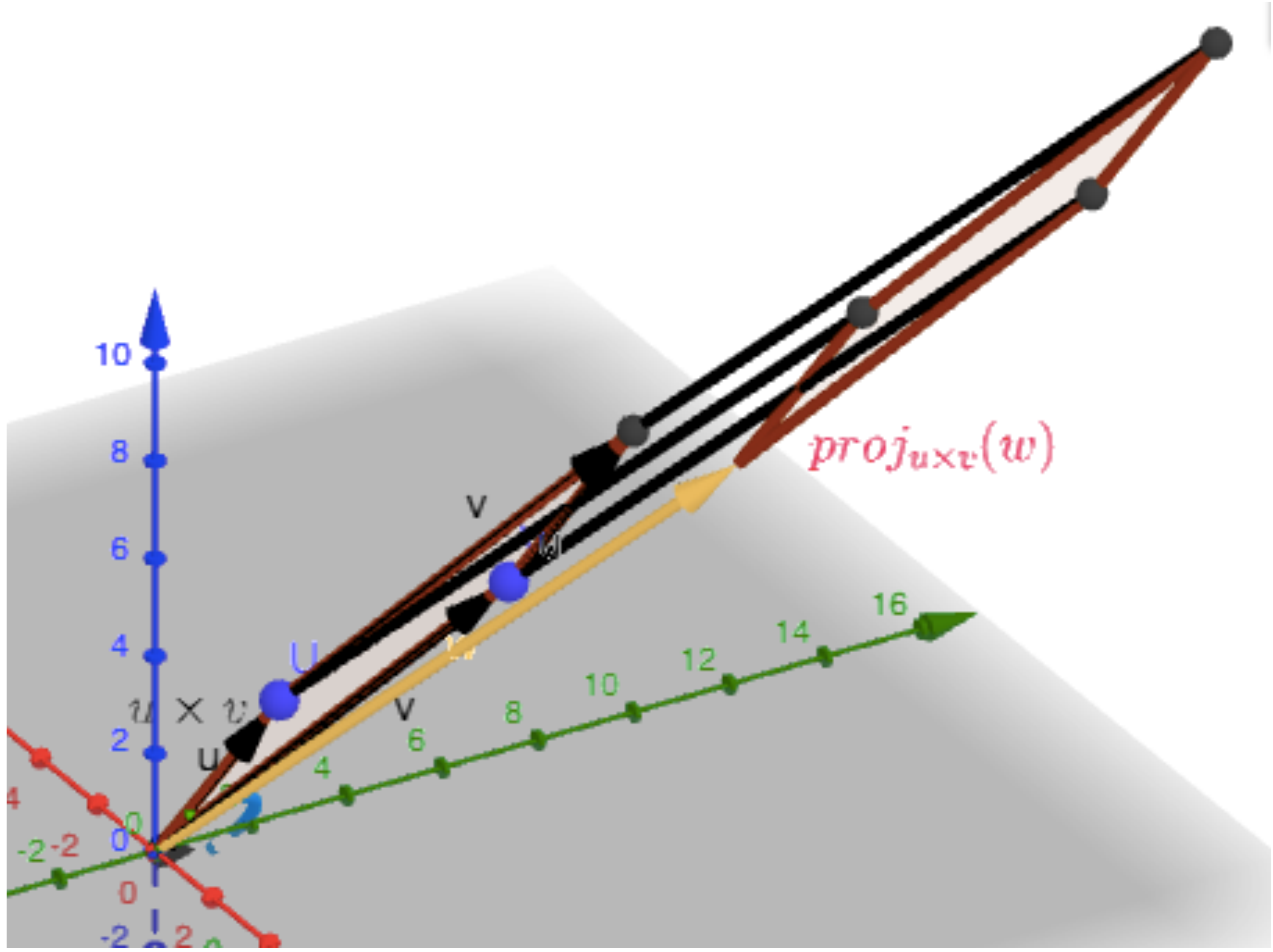
the volume of the parallelepiped determined by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ is

$$|\mathbf{u}\times\mathbf{v}||\mathbf{w}||\cos(\theta)| = |(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}|$$

$$= |w_1(u_2v_3 - u_3v_2) + w_2(u_3v_1 - u_1v_3) + w_3(u_1v_2 - u_2v_1)|$$

$$= |w_1(u_2v_3 - u_3v_2) - w_2(u_1v_3 - u_3v_1) + w_3(u_1v_2 - u_2v_1)|$$

$$= \left|\det\begin{pmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}\right|$$



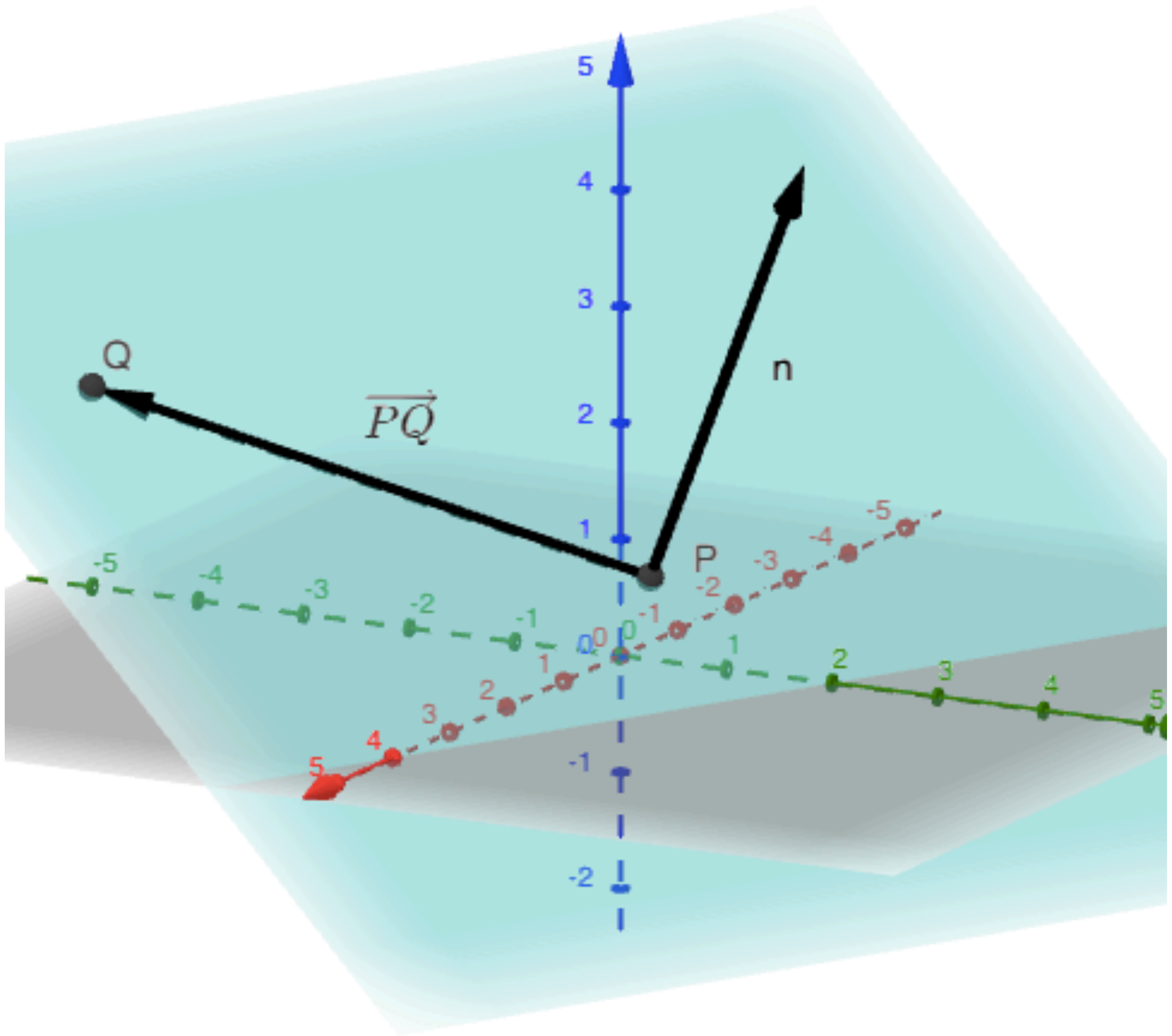

Examples.

$$\det\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \ldots = 0$$

Remark: Of course it's zero, because $\mathbf{w} = 2\mathbf{u} - \mathbf{v}$ lies in the same plane as $\mathbf{u}$ and $\mathbf{v}$.

The volume of the parallelepiped determined by the vector $\mathbf{w} = <1,2,3>$, $\mathbf{u} = <4,5,6>$, $\mathbf{v} = <7,8,9>$ is 0.

# Equations of Planes, pg 1.




Suppose we know a vector $\mathbf{n} = < a, b, c >$ that is perpendicular to a plane at the point $P(x_0, y_0, z_0)$.

Is there a condition for a point $Q(x, y, z)$ to lie on the plane?

Yep!

Since $\mathbf{n}$ is perpendicular to the plane, it will be perpendicular to any vector that lies in the plane.

So $\mathbf{n}$ is perpendicular to the vector $\overrightarrow{PQ}$ with initial point $P$ and terminal point $Q$,

$$0 = \mathbf{n} \cdot \overrightarrow{PQ} = < a, b, c > \cdot < x - x_0, y - y_0, z - z_0 >$$

$$= a(x - x_0) + b(y - y_0) + c(z - z_0)$$

$$= ax + by + cz - d$$

The equation of the plane is $ax + by + cz = d$ where $d = ax_0 + by_0 + cz_0$

Note 1: $\mathbf{n}$ is called the *normal vector* to the plane.

Note 2: Some textbooks write $ax + by + cz + d = 0$ with $d = -ax_0 - by_0 - cz_0$.

# Equations of Planes, pg 2.

Find the equation of the plane which …

1. … is perpendicular to $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and goes through the point $P(4,5,6)$

The perpendicular vector is $< 1,2,3 >$ .

The equation is…

$1(x-4) + 2(y-5) + 3(z-6) = 0$

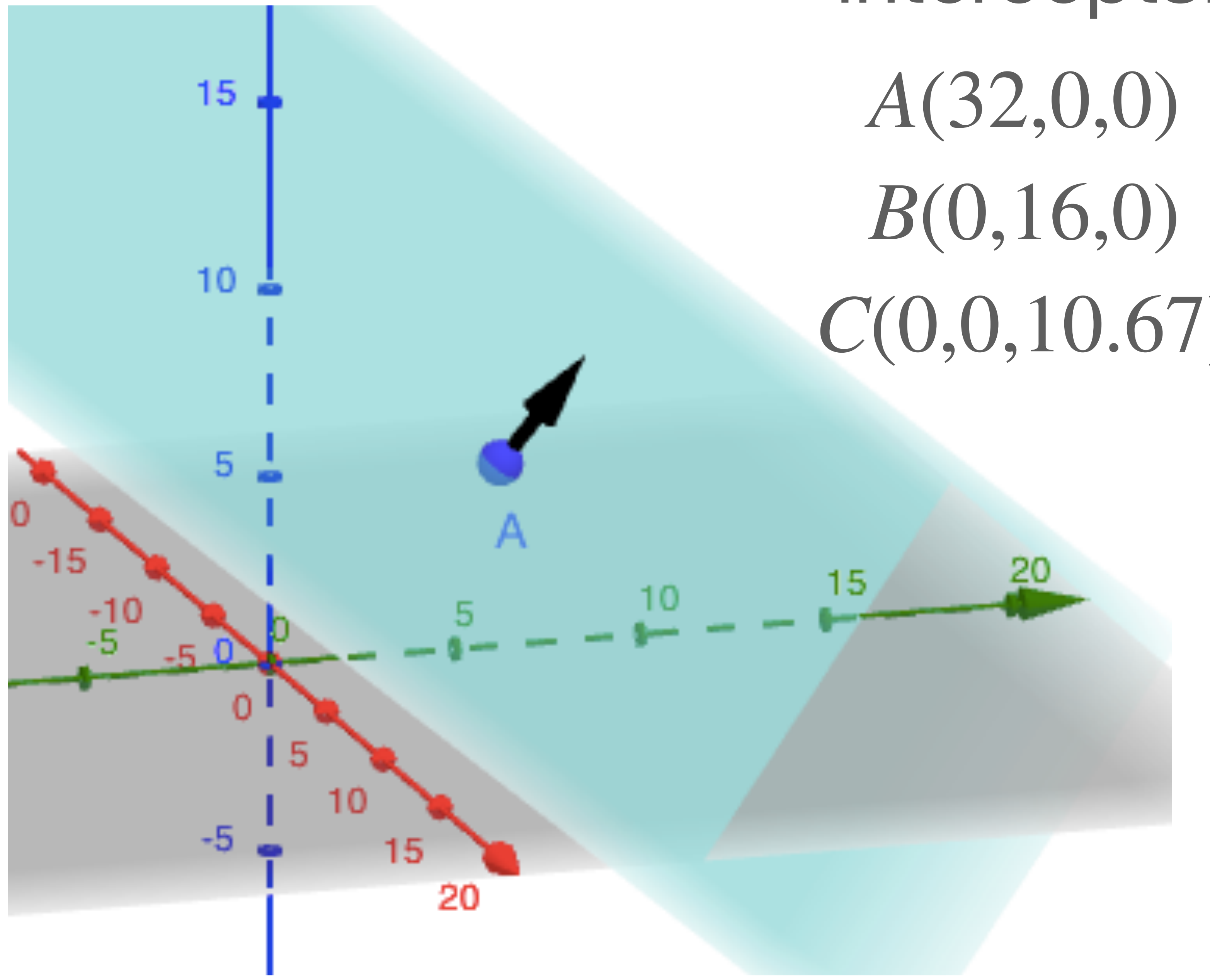
$x + 2y + 3z = 32$

intercepts:

$A(32,0,0)$

$B(0,16,0)$

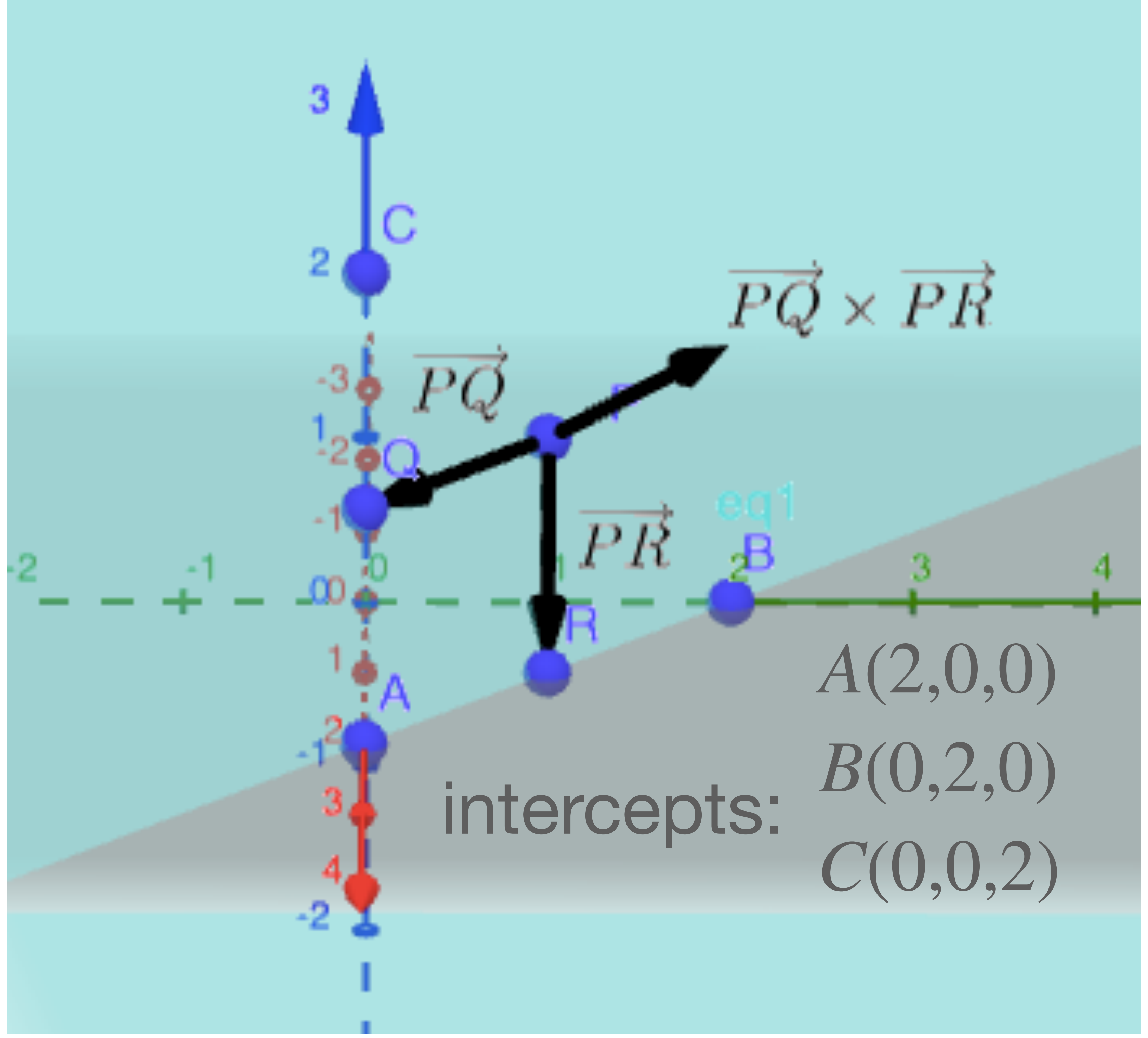
$C(0,0,10.67)$




2. … goes through the points $P(0,1,1), Q(1,0,1), R(1,1,0)$.

we get two vectors that lie on the plane: $\overrightarrow{PR} = < 1,0,-1 >$ $\overrightarrow{PQ} = < 1,-1,0 >$

(This means that if their initial point is placed on the plane, then their terminal points are also on the plane.)




intercepts:

$A(2,0,0)$

$B(0,2,0)$

$C(0,0,2)$

So a vector perpendicular to the plane is

$\overrightarrow{PQ} \times \overrightarrow{PR} = \mathbf{i} + \mathbf{j} + \mathbf{k} = < 1,1,1 >$

The equation is

$1(x-0) + 1(y-1) + 1(z-1) = 0$

$x + y + z = 2$

# Equations of Planes, pg 3. Examples.

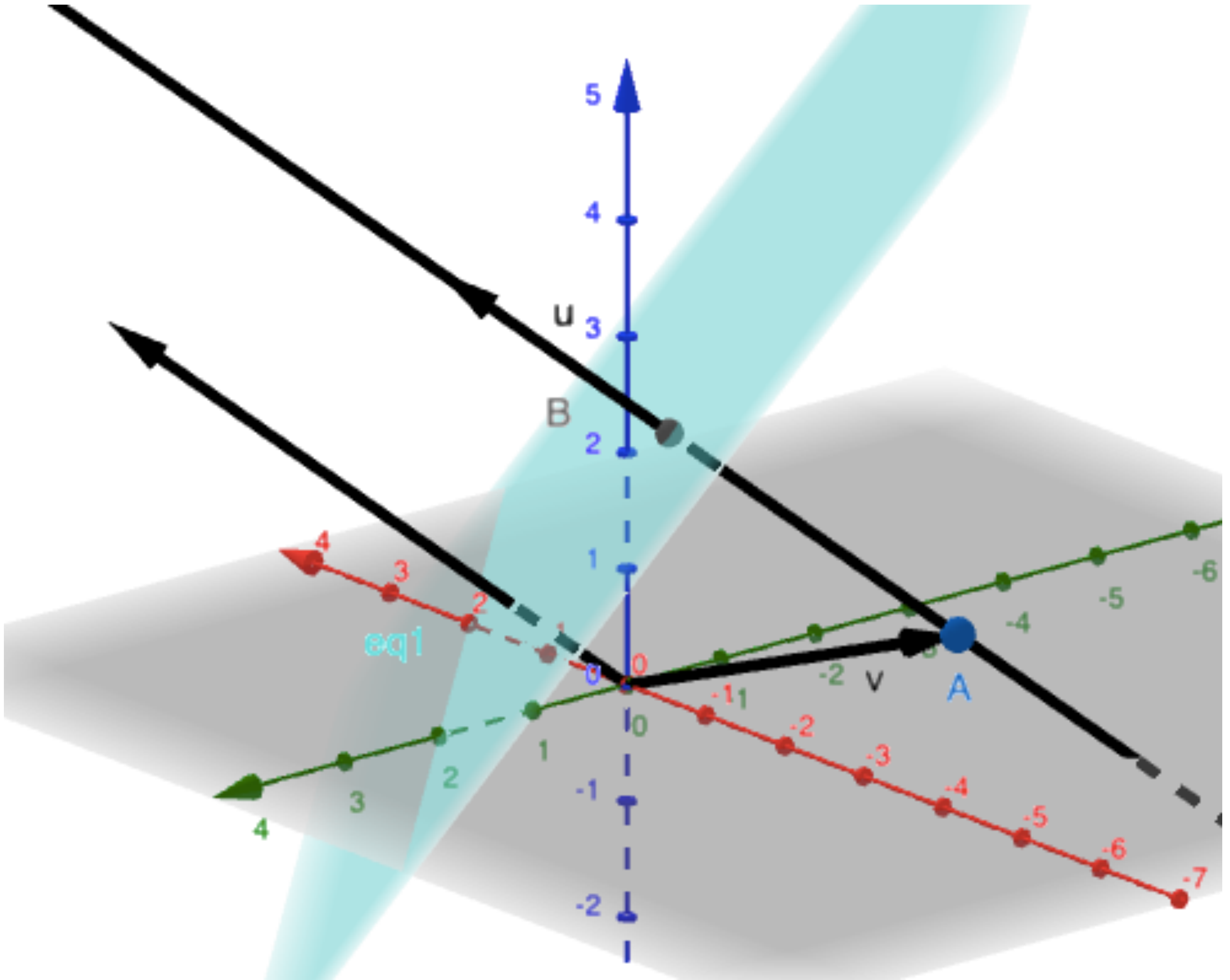
**1.** Find the point where the line $\mathbf{L}(t) = < -3, -1, 1 > + t < 3,3,3 >$ intersects the plane $x + y + z = 2$

If they intersect, then for some value of $t$, the terminal point of $\mathbf{L}(t)$ will lie on the plane.

When (if ever), does the point $(-3 + 3t, -1 + 3t, 1 + 3t)$ lie on the plane?

$$2 = x + y + z = -3 + 3t + -1 + 3t + 1 + 3t = -3 + 9t \;\ldots\; t = \frac{5}{9}$$

The line and plane intersect at the point $\mathbf{L}\left(\frac{5}{9}\right) = \left(-\frac{4}{3}, \frac{2}{3}, \frac{8}{3}\right)$




**2.** Find the equation of the line where the planes $x + y + z = 2$ and $x + 2y + 3z = 32$ intersect.

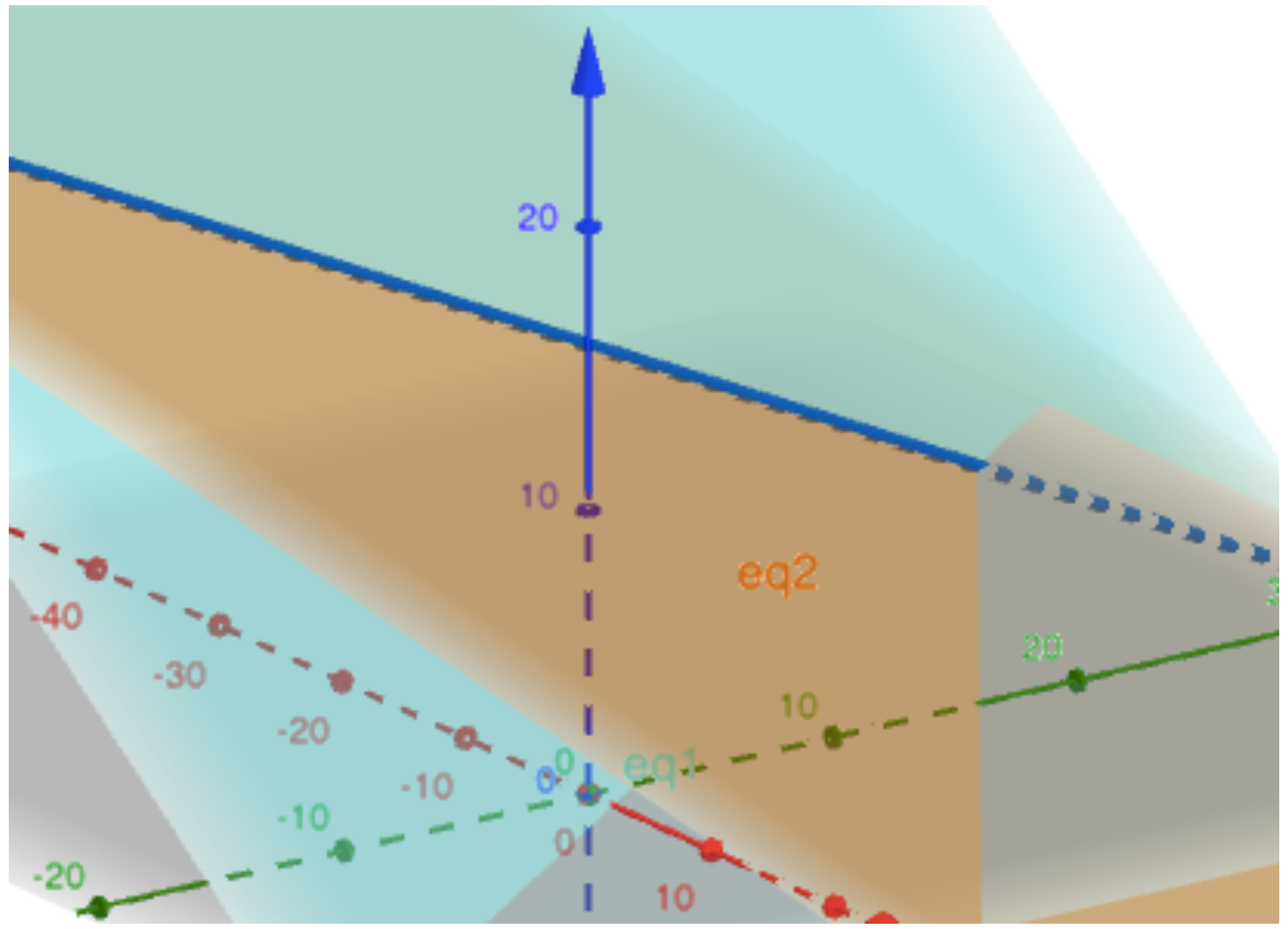
The vector to determine the line's direction lies in both planes.

This vector is perpendicular to both $< 1,1,1 >$ and $< 1,2,3 >$

$$< 1,1,1 > \times < 1,2,3 > = \mathbf{i} - 2\mathbf{j} + \mathbf{k} = < 1, -2, 1 >$$

A point on both planes is $(-28, 30, 0)$ (you can find this by setting z=0.)

$$\mathbf{L}(t) = < -28, 30, 0 > + t < 1, -2, 1 > = < -28 + t, +30 - 2t, t >$$



# Plane Practice.

1. (OX 2.5#281 )Find the equation of the plane through the points $P(1,1,1), Q(2,4,3), R(-1,-2,-1)$.

2. (S12.5 #39) Find the equation of the plane that goes through $(1,5,1)$ and is perpendicular to both $2x + y - 2z = 2$ and $x + 3z = 4$.

3. (S12.5 #65) Find the equation of the line that goes through $(0,1,2)$, is parallel to the plane $x + y + z = 2$, and perpendicular to the line $x = 1 + t, y = 1 - t, z = 2t$.

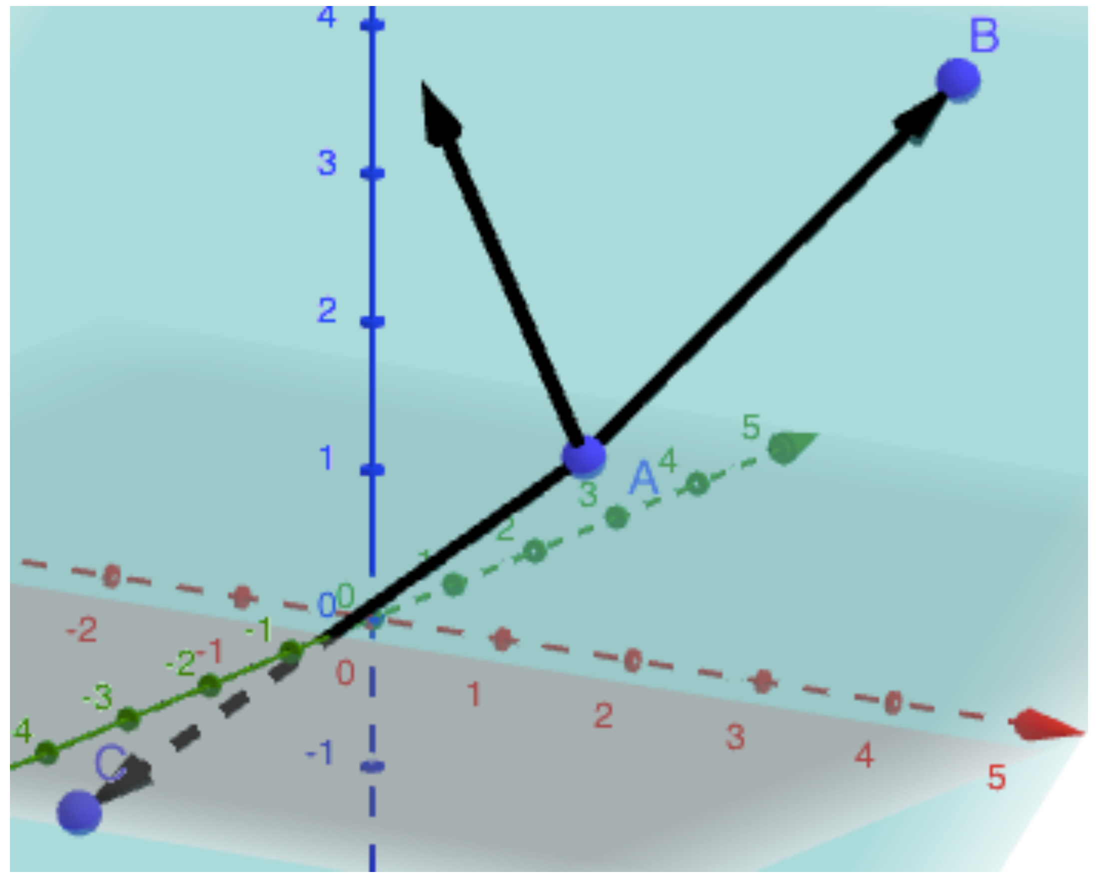
1. $\overrightarrow{PQ} = < 1,3,2 >$

$\overrightarrow{PR} = < -2,-3,-2 >$

$\overrightarrow{PQ} \times \overrightarrow{PR} = < 0,-2,3 >$

$0(x-1) - 2(y-1) + 3(z-1) = 0$

$-2y + 3z = 1$



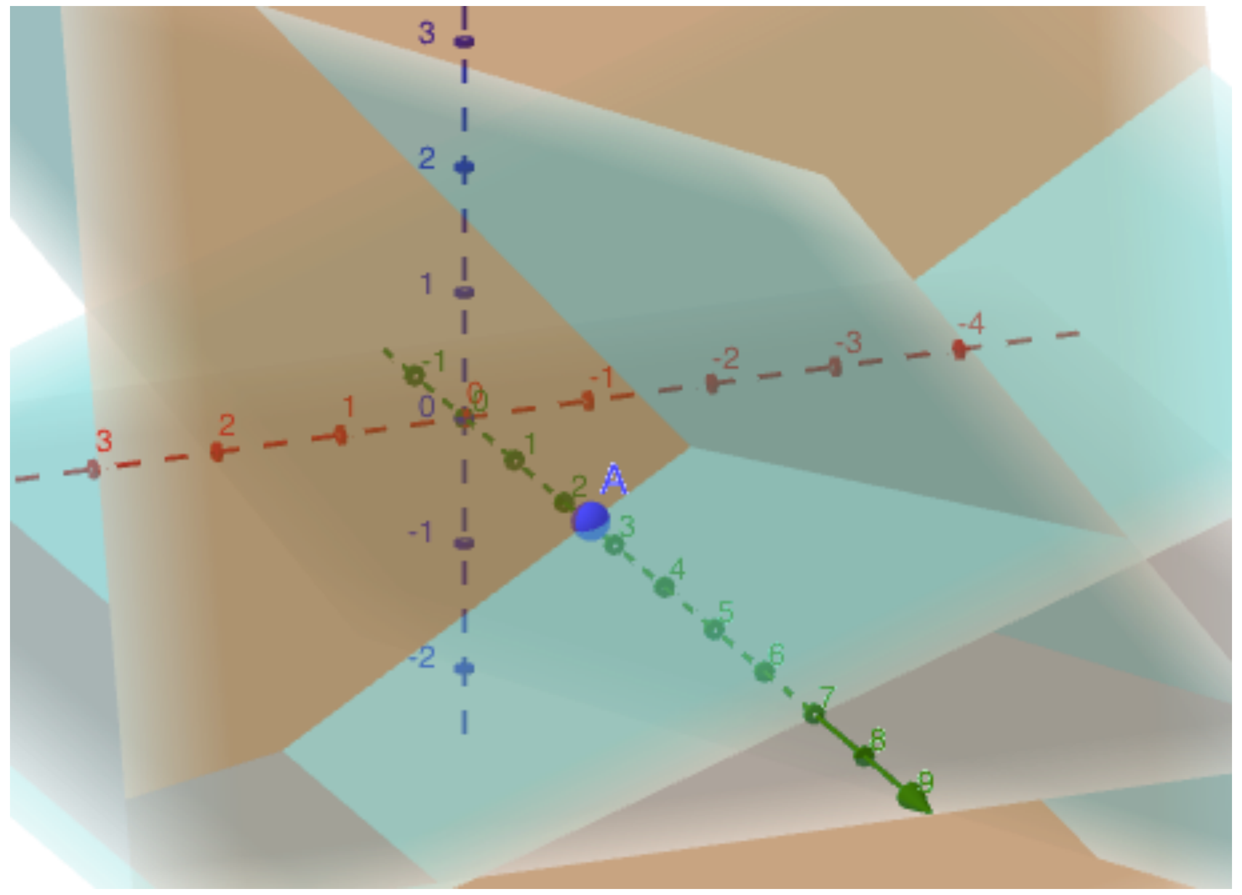

2. Both $< 2,1,-2 >$ and $< 1,0,3 >$ lie in the plane,

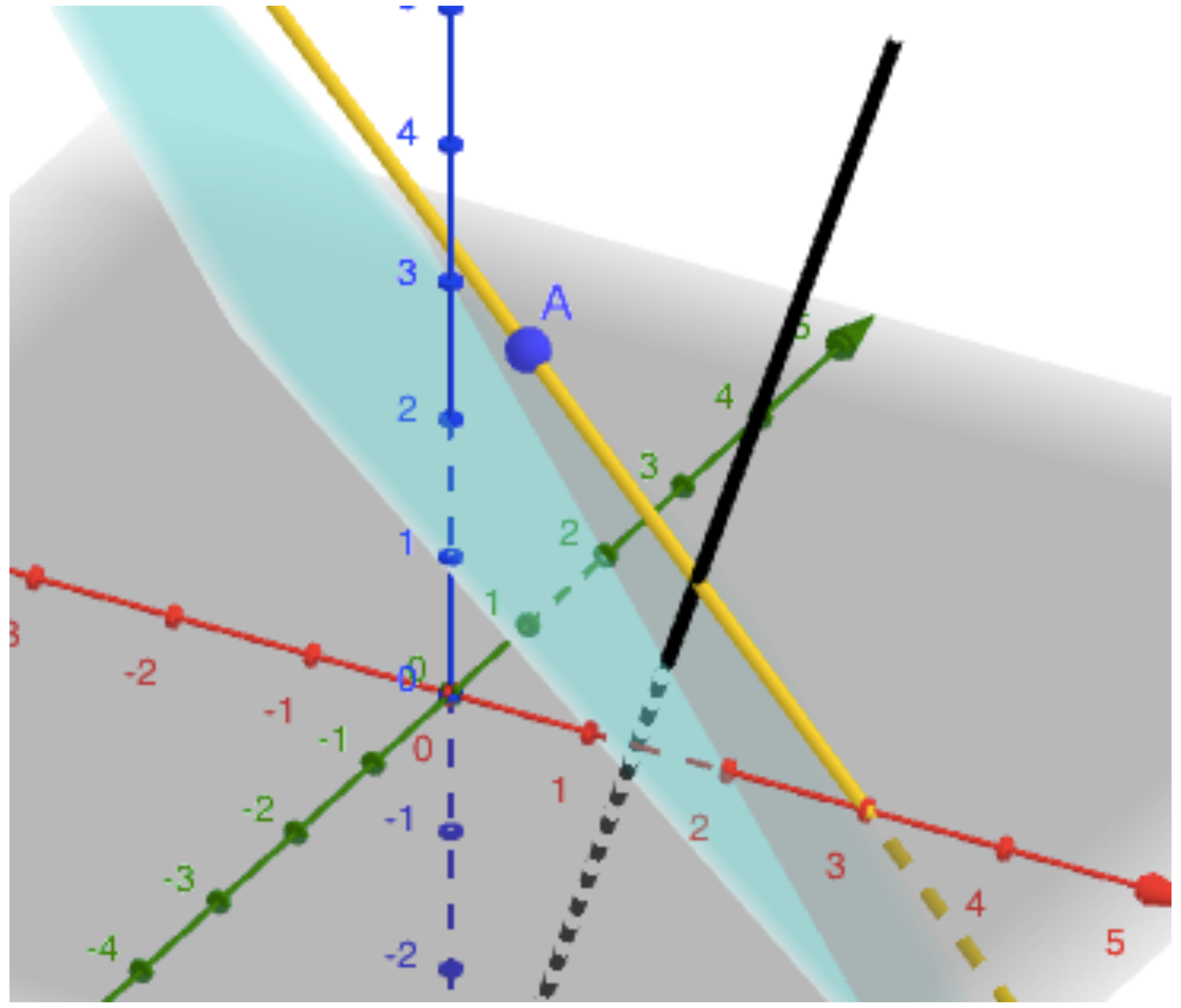
$\mathbf{n} = < 2,1,-2 > \times < 1,0,3 > = < 3,-8,-1 >$

$3(x-1) - 8(y-5) - 1(z-1) = 0$

#2 $3x - 8y - z = -38$

#3 $\mathbf{L}(t) = < 3t, 1-t, 2-2t >$




3. $\mathbf{L}(t) = \mathbf{v} + t\mathbf{u}$

$\mathbf{u}$ is perpendicular to both $< 1,1,1 >$ and $< 1,-1,2 >$

$\mathbf{u} = < 1,1,1 > \times < 1,-1,2 >$

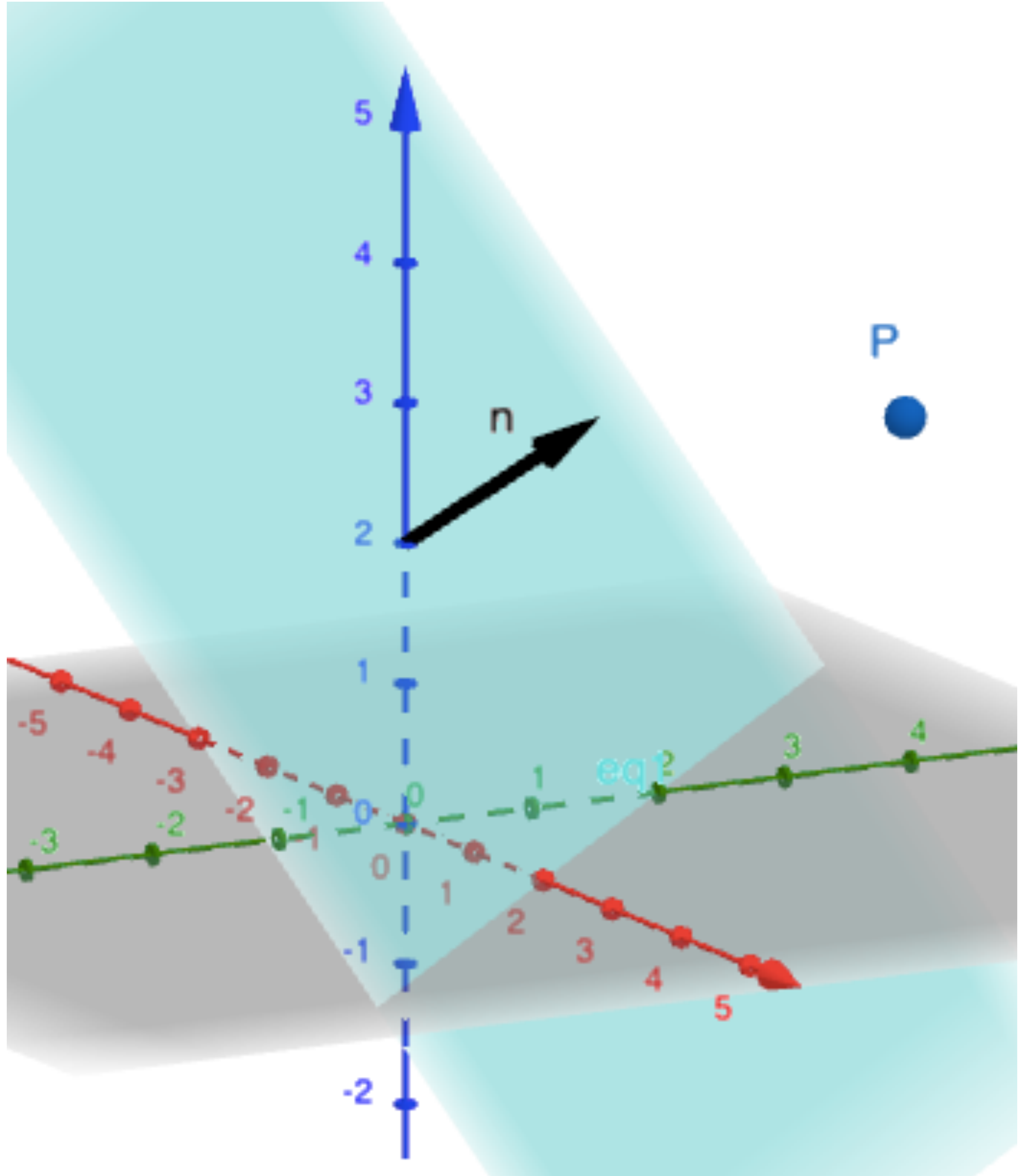
$= < 3,-1,-2 >$

$\mathbf{L}(t) = < 0,1,2 > + t < 3,-1,-2 >$

$= < 3t, 1-t, 2-2t >$

# Point-Plane Distance.

How far is a point from a plane?




Say the plane has equation

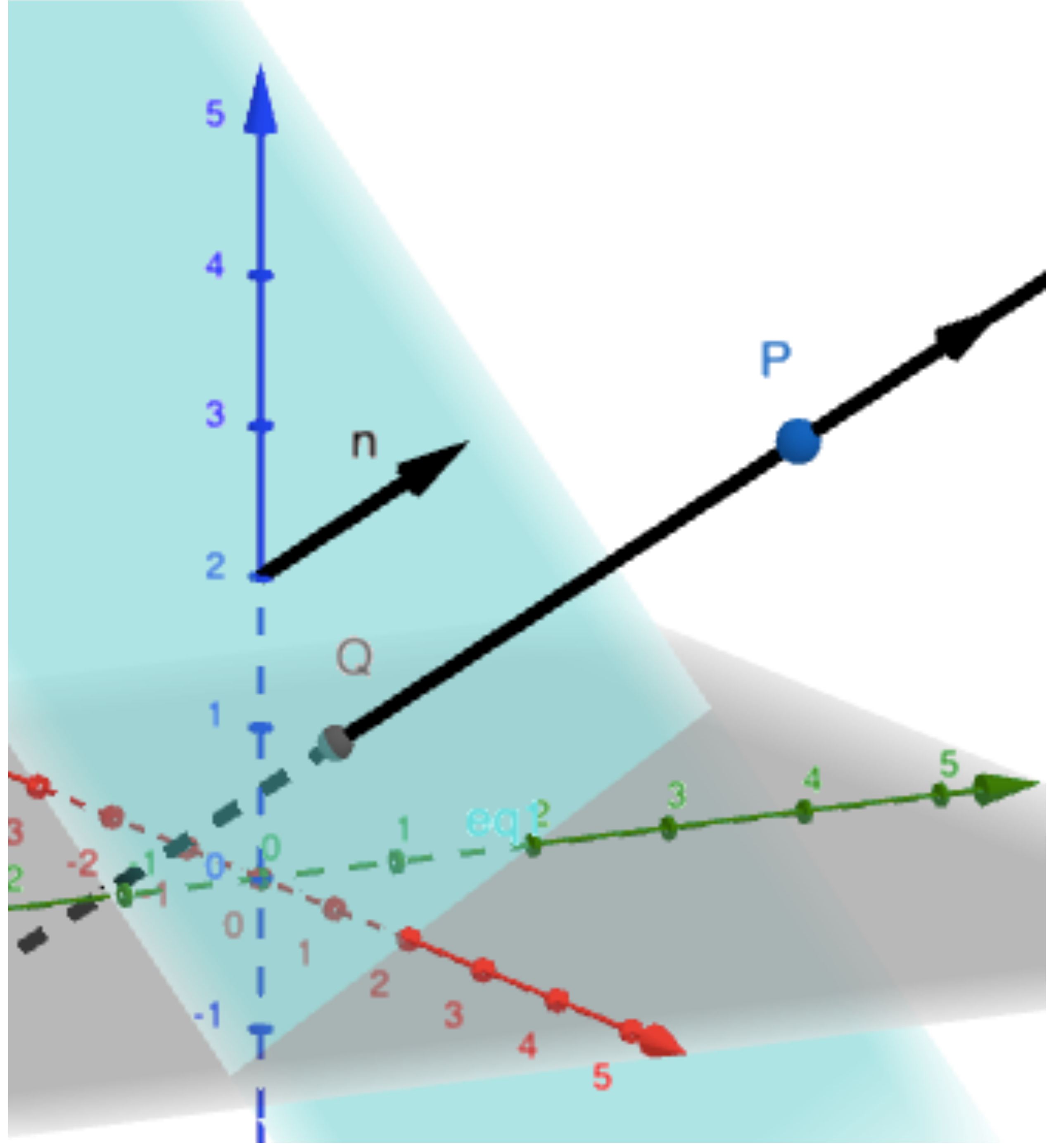
$$ax + by + cz + d = 0.$$

Then the perpendicular vector is $\mathbf{n} = < a, b, c >$

We're given a point $P(x_0, y_0, z_0)$. We can describe the line through this point parallel to $\mathbf{n}$, like this

$$\mathbf{L}(t) = \mathbf{P} + t\mathbf{n}$$
$$= < x_0 + at, y_0 + bt, z_0 + ct >$$

The line intersects the plane at a point $Q = (x_1, y_1, z_1)$




The line, parallel to $\mathbf{n}$, is perpendicular to the plane

The distance we're looking for is $|\overrightarrow{PQ}|$.

$Q$ happens when the terminal point of $\mathbf{L}$, $(x_0 + at, y_0 + bt, z_0 + ct)$, lies on the plane.

$$0 = a(x_0 + at) + b(y_0 + bt) + c(z_0 + ct) + d$$

$$0 = d + ax_0 + by_0 + cz_0 + t(a^2 + b^2 + c^2)$$

$$t = \frac{-d - (ax_0 + by_0 + cz_0)}{a^2 + b^2 + c^2} =: t_Q$$

$$\overrightarrow{PQ} = \mathbf{L}(t_Q) - \mathbf{L}(\mathbf{0})$$

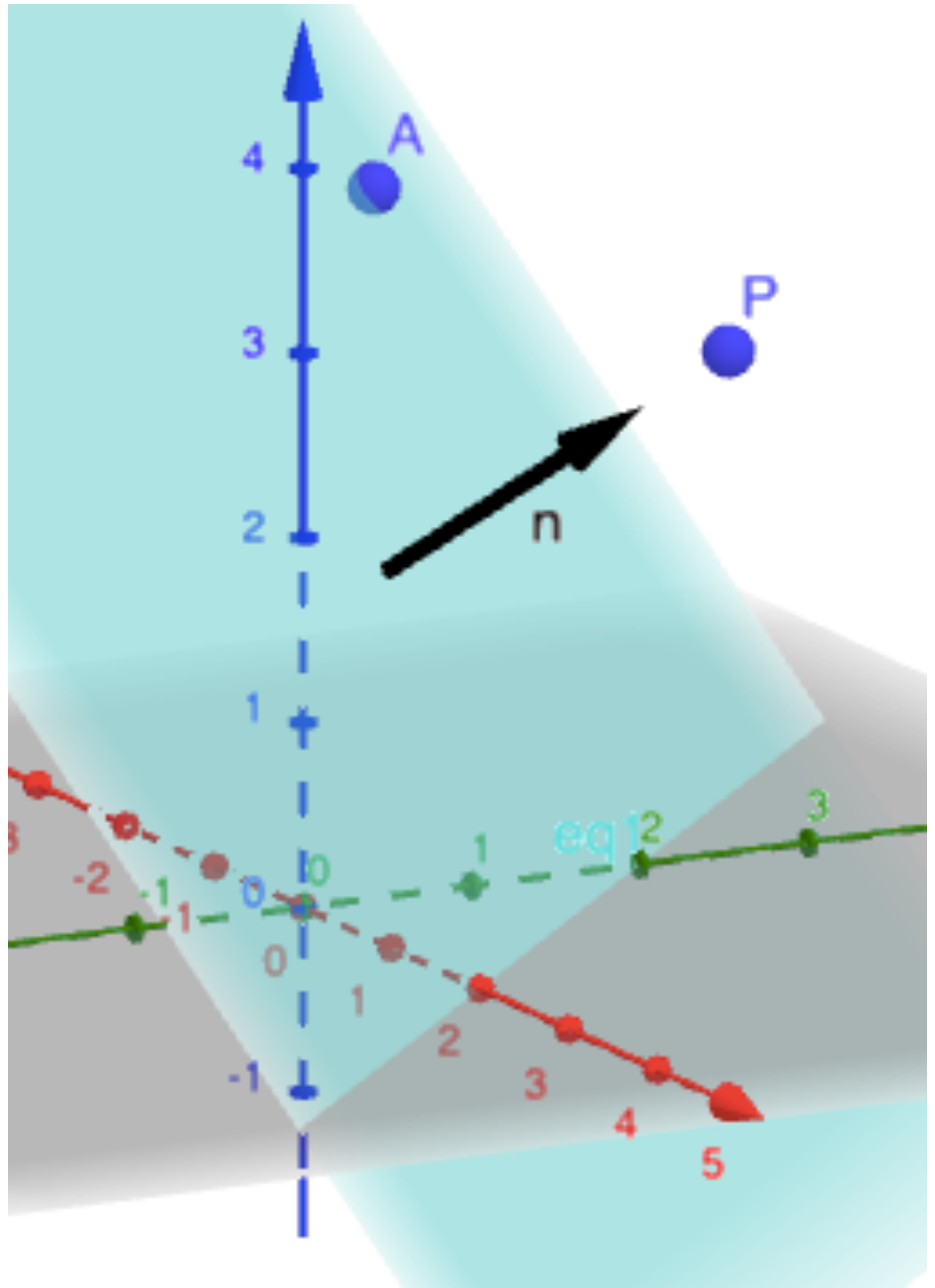
$$= < x_0 + at_Q, y_0 + bt_Q, z_0 + ct_Q > - < x_0, y_0, z_0 >$$

$$= < at_Q, bt_Q, ct_Q > = t_Q < a, b, c >$$

$$|\overrightarrow{PQ}| = |t_Q|\,| < a, b, c > | = |t_Q|\sqrt{a^2 + b^2 + c^2}$$

$$= \frac{|d + (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}}$$

# Point-Plane Distance, pg 2. A second derivation.




We're given $\mathbf{n} = < a, b, c >$ and $P(x_0, y_0, z_0)$ as before.

Say $A(x_1, y_1, z_1)$ is some point on the plane.

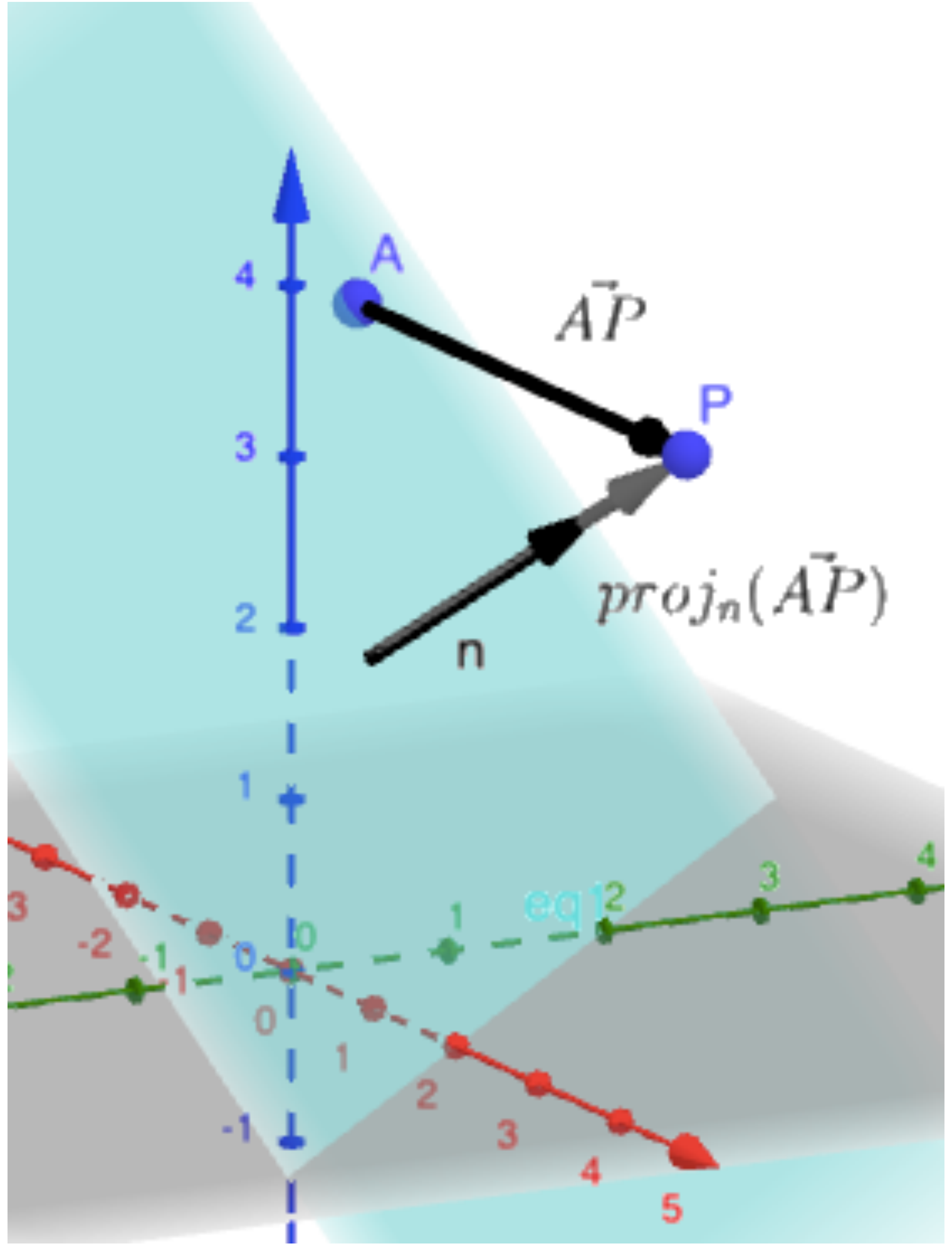
So $ax_1 + by_1 + cz_1 + d = 0$

i.e. $d = -ax_1 - by_1 - cz_1$

Project the vector

$\overrightarrow{AP} = < x_0 - x_1, y_0 - y_1, z_0 - z_1 >$

onto the vector $\mathbf{n}$.




The distance from the plane to $P$ is $|\mathbf{proj_n}(\overrightarrow{AP})|$

$$|\mathbf{proj_n}(\overrightarrow{AP})| = \left| \left( \frac{\mathbf{n} \cdot \overrightarrow{AP}}{\mathbf{n} \cdot \mathbf{n}} \right) \mathbf{n} \right|$$

$$= \frac{|\mathbf{n} \cdot \overrightarrow{AP}|}{|\mathbf{n} \cdot \mathbf{n}|} |\mathbf{n}|$$

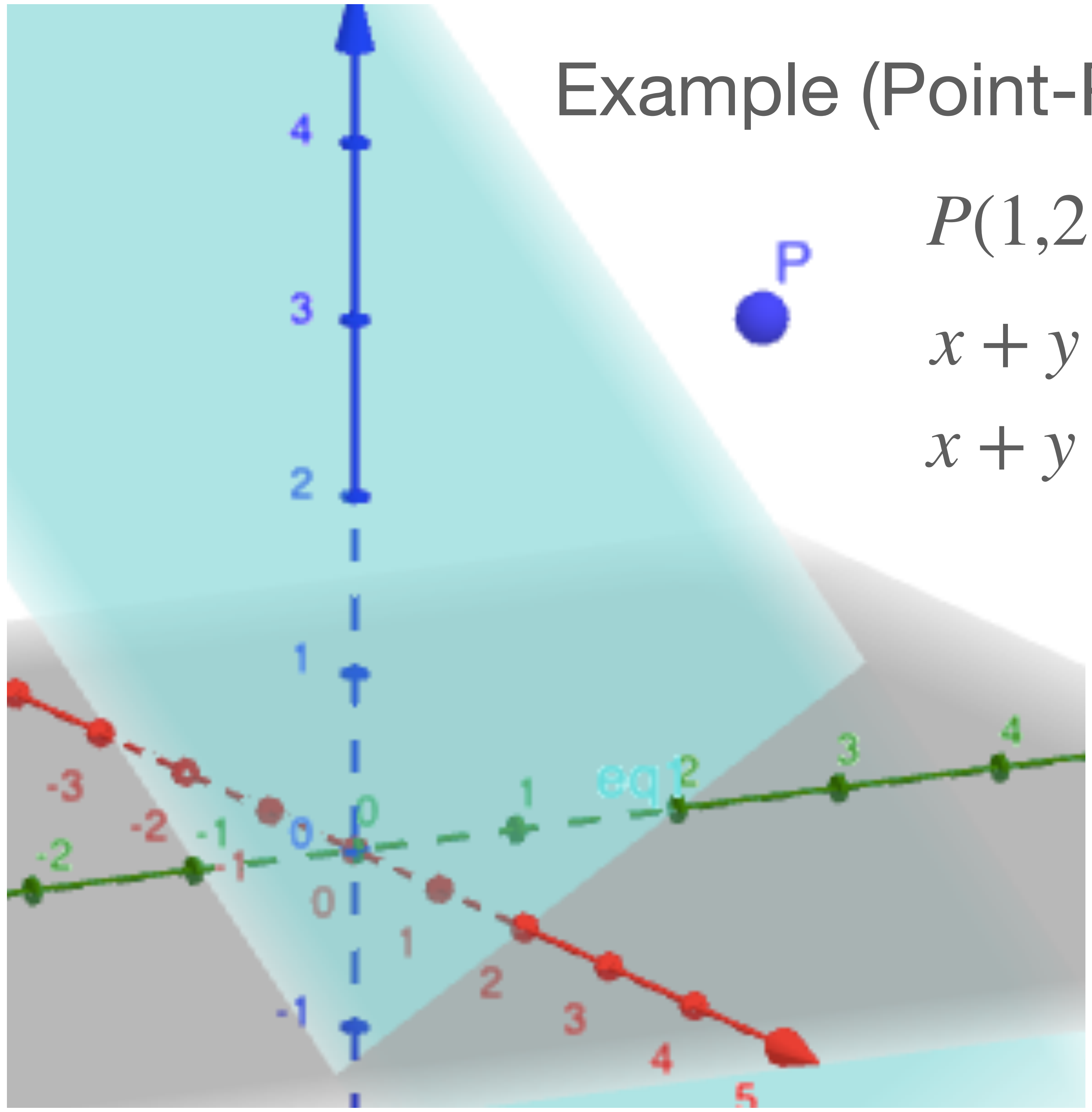
$$= \frac{|\mathbf{n} \cdot \overrightarrow{AP}|}{|\mathbf{n}|}$$

$$= \frac{|a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1)|}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{|ax_0 + by_0 + cz_0 - ax_1 - by_1 - cz_1|}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$ (as before)

# Distances: Plane-Plane; Point-Line.




Example (Point-Plane Distance).

$P(1,2,3)$

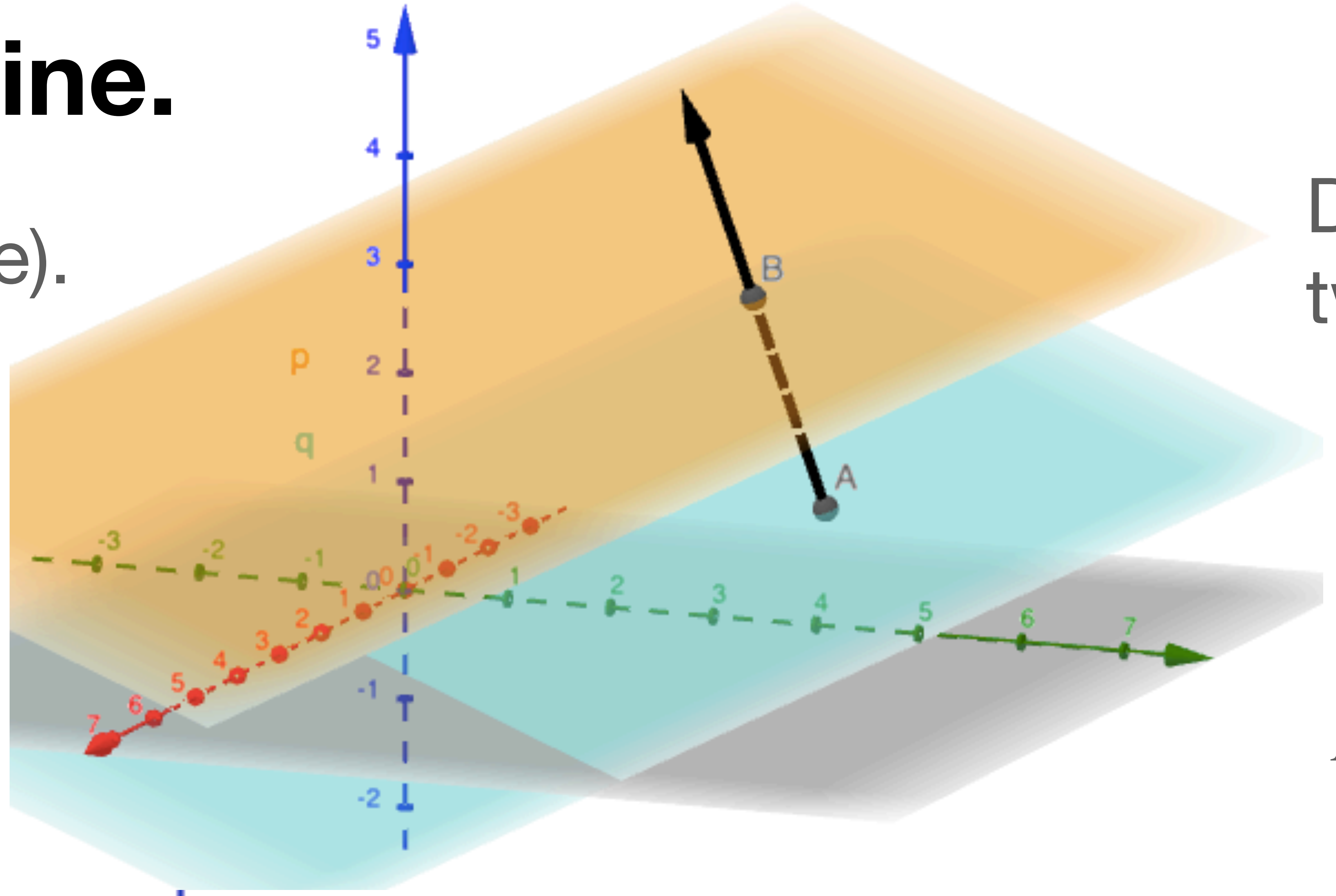
$x+y+z=2$

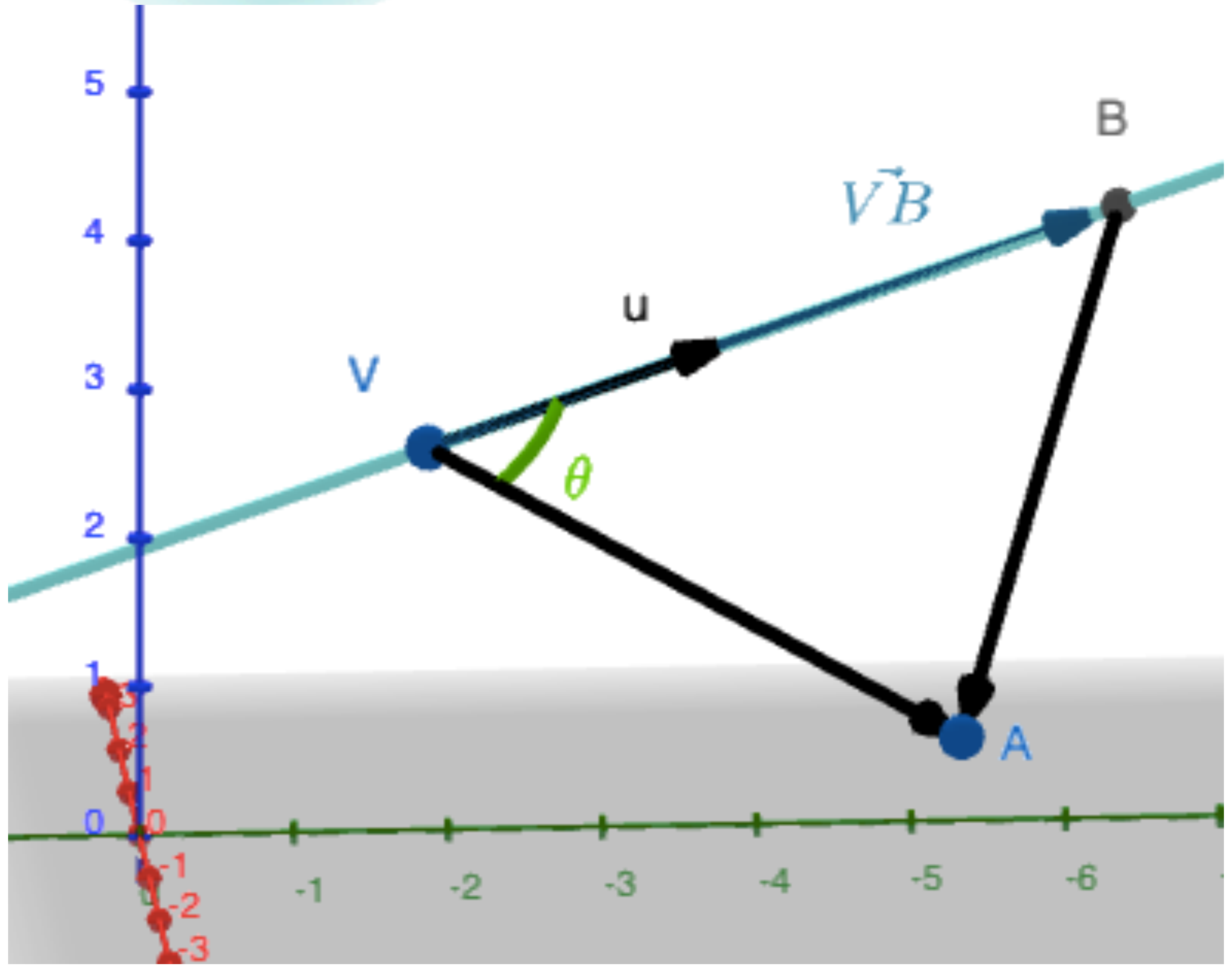
$x+y+z-2=0$

$$D = \frac{|1(1)+1(2)+1(3)-2|}{\sqrt{1^2+1^2+1^2}} = \frac{4}{\sqrt{3}} \approx 2.31 \text{ units}$$

(or, take a point on the plane, like A(2,0,0))

$$D = \frac{|\mathbf{n}\cdot\overrightarrow{AP}|}{|\mathbf{n}|} = \frac{|<1,1,1>\cdot<-1,2,3>|}{\sqrt{3}} = \frac{4}{\sqrt{3}}$$




Link: ParallelPlanes'Distance

Link: Point Distance Line

More Formulae!

Distance between two parallel planes:

$ax+by+cz=d_1$

$ax+by+cz=d_2$

$$D = \frac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}$$

Distance from a point, A, to a line:

$\mathbf{L}(t)=\mathbf{v}+t\mathbf{u}$

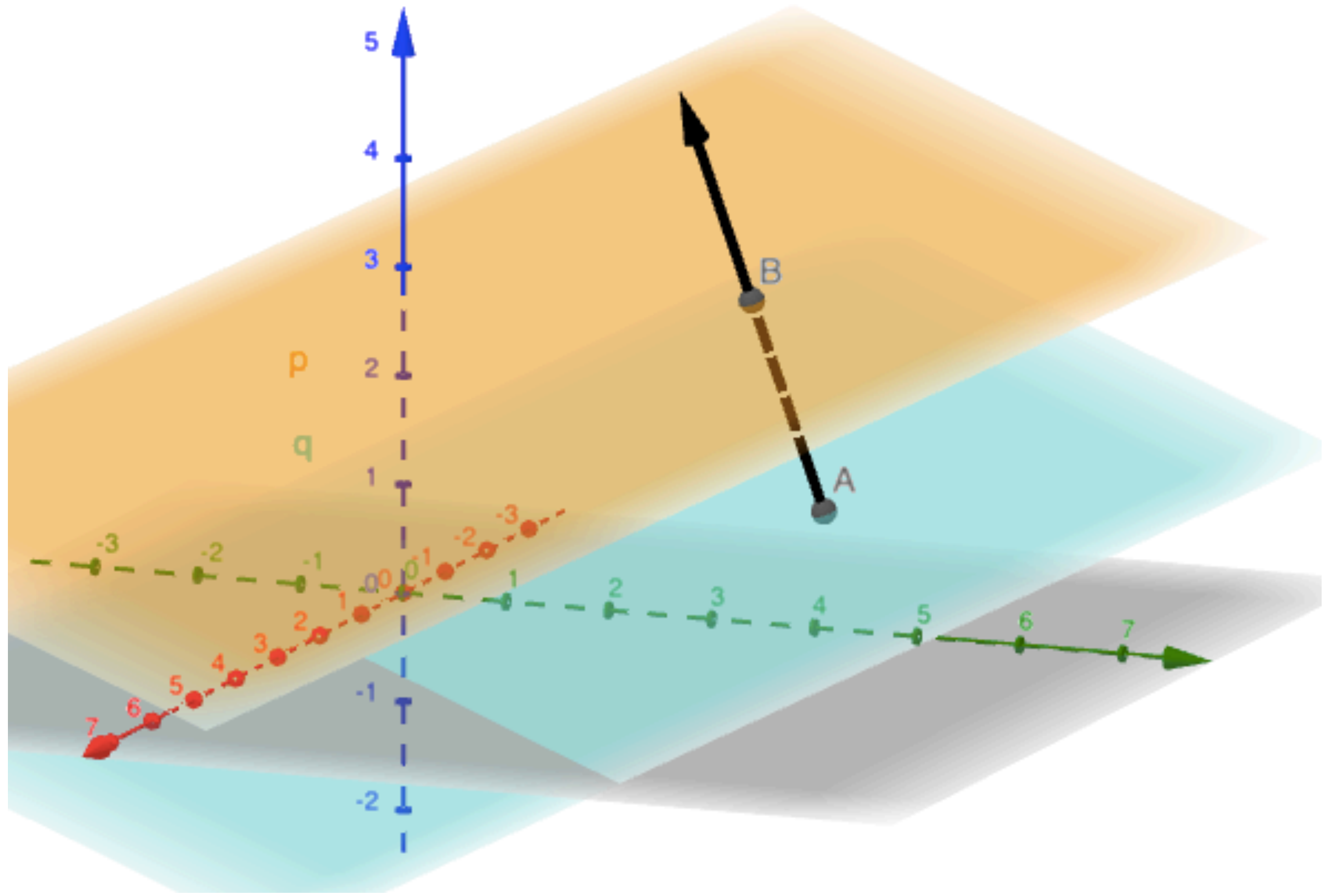
V is any point on the line.

$$D = |\overrightarrow{VA}|\sin(\theta)$$

$$= |\mathbf{u}||\overrightarrow{VA}|\sin(\theta)/|\mathbf{u}|$$

$$= \frac{|\overrightarrow{VA}\times\mathbf{u}|}{|\mathbf{u}|}$$

# Distances: Plane-Plane; Point-Line, pg 2.




$$x - y + 4z = 2$$
$$x - y + 4z = 11$$

Distance:

$$D = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{|2 - 11|}{1^2 + (-1)^2 + 4^2}$$

$$\approx 2.12$$

Note: Here's another way to approach the distance from a point to a line.

Given $\mathbf{L}(t) = \mathbf{v} + t\mathbf{u}$ and $A$

Pick a point V on $\mathbf{L}$.

Compute $\overrightarrow{VA}$

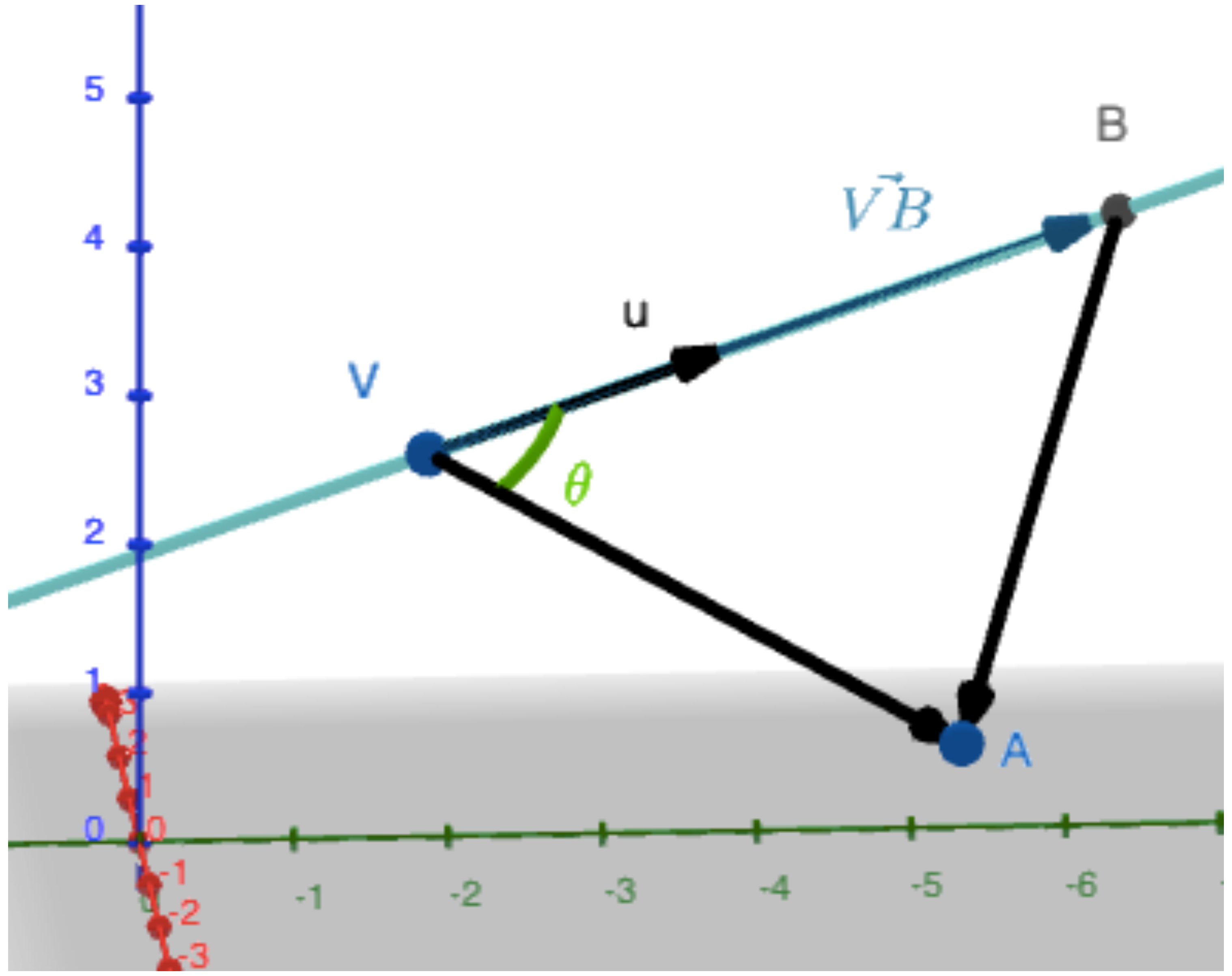
Project $\overrightarrow{VA}$ onto $\mathbf{u}$ to get $\overrightarrow{VB}$

Compute $\overrightarrow{BA} = \overrightarrow{VA} - \overrightarrow{VB}$

It should be that $\overrightarrow{BA} \perp \mathbf{L}$

i.e., B is the point on $\mathbf{L}$ that is closest to the given point A.

The desired distance is $|\overrightarrow{BA}|$




$$\mathbf{L}(t) = < 2, -2,2 > + t < -1, -2,1 >$$

$$V = (2, -2,2), \;\; \mathbf{u} = < -1, -2,1 >$$

$$A = (-5, -5,2) \quad \overrightarrow{VA} = < -7, -3,0 >$$

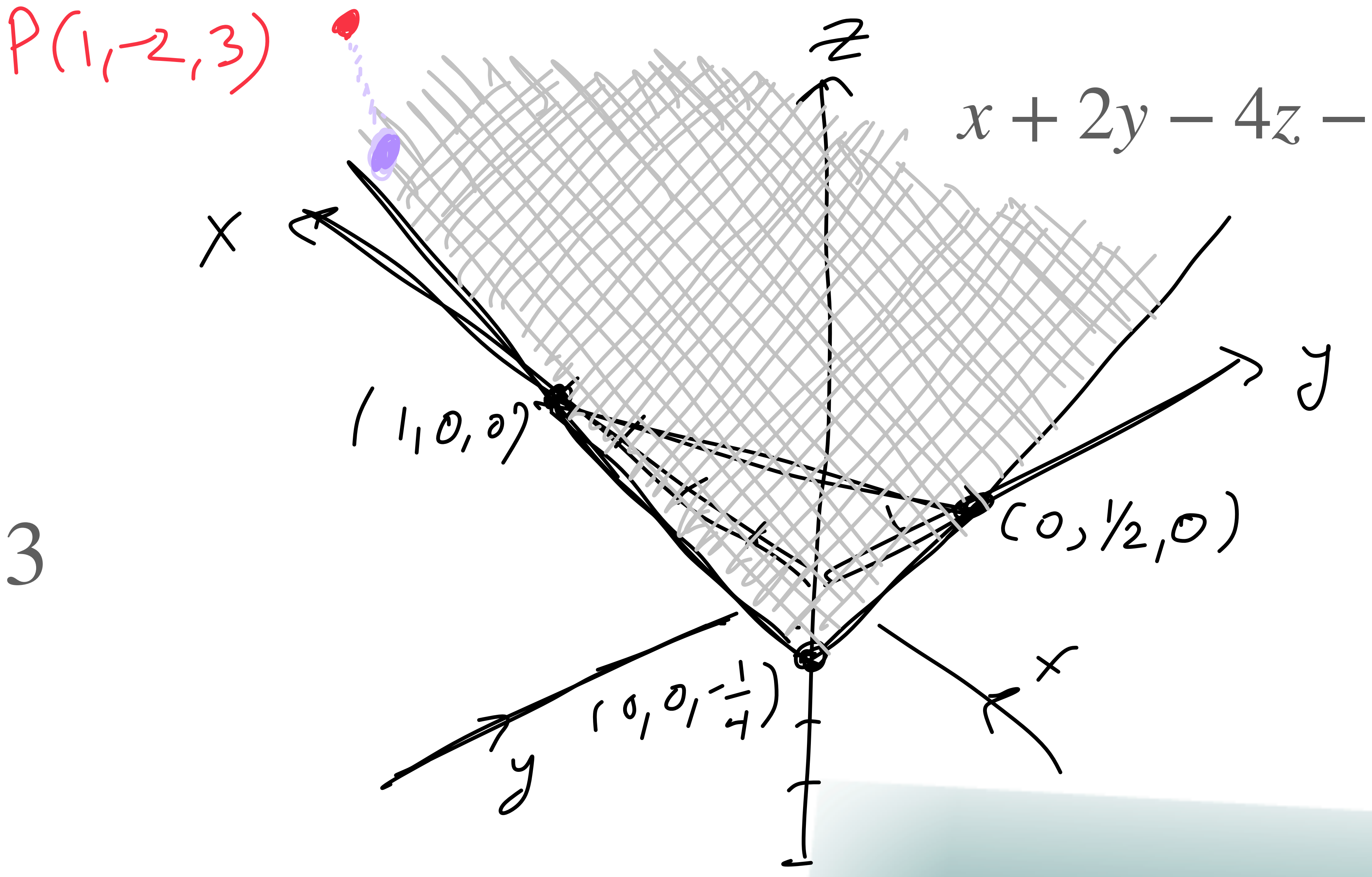
$$D = \frac{|\overrightarrow{VA} \times \mathbf{u}|}{|\mathbf{u}|} = \frac{|< -3,7,11 >|}{|< -1, -2,1 >|} \approx 5.46$$

xc: Use this approach to get a second derivation of the formula we used (on the left).

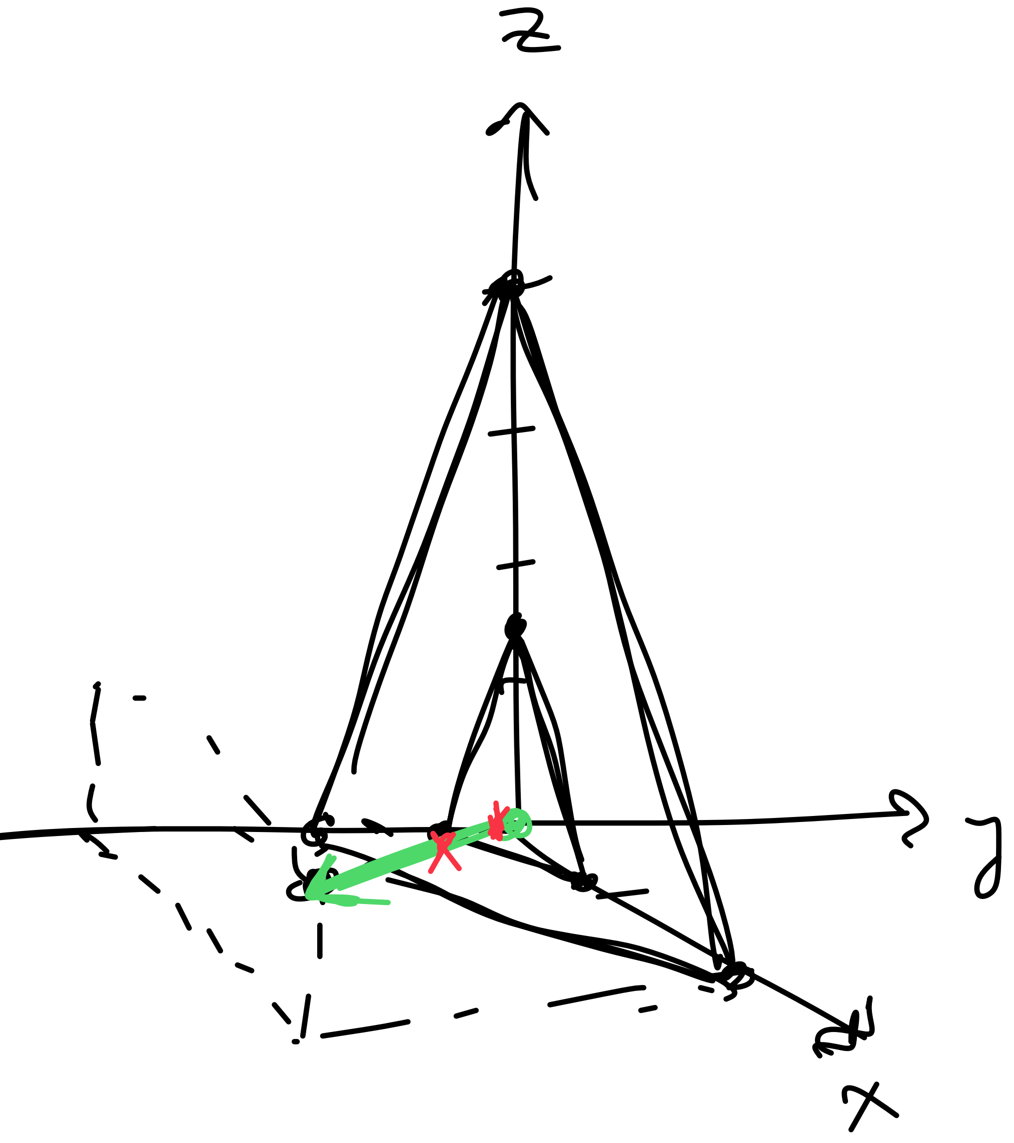
# More Distance Formulae. You try.

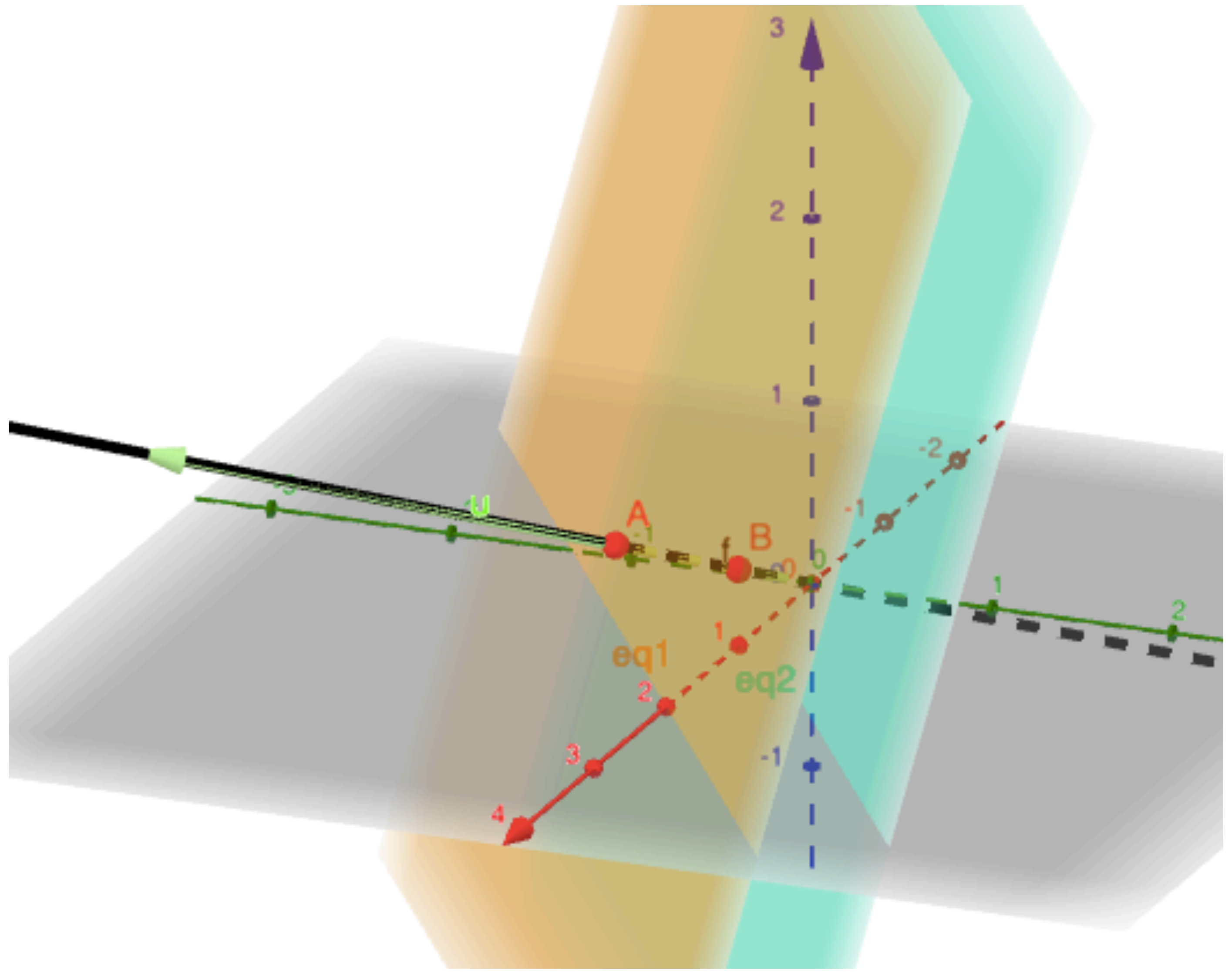
1. (OX2.5 #290) Find the distance from $(1, -2, 3)$ to the plane $(x-3) + 2(y+1) - 4z = 0$.

2. (S12.5 #73) Find the distance between the parallel planes $2x - 3y + z = 4$ and $4x - 6y + 2z = 3$




$x + 2y - 4z - 1 = 0$

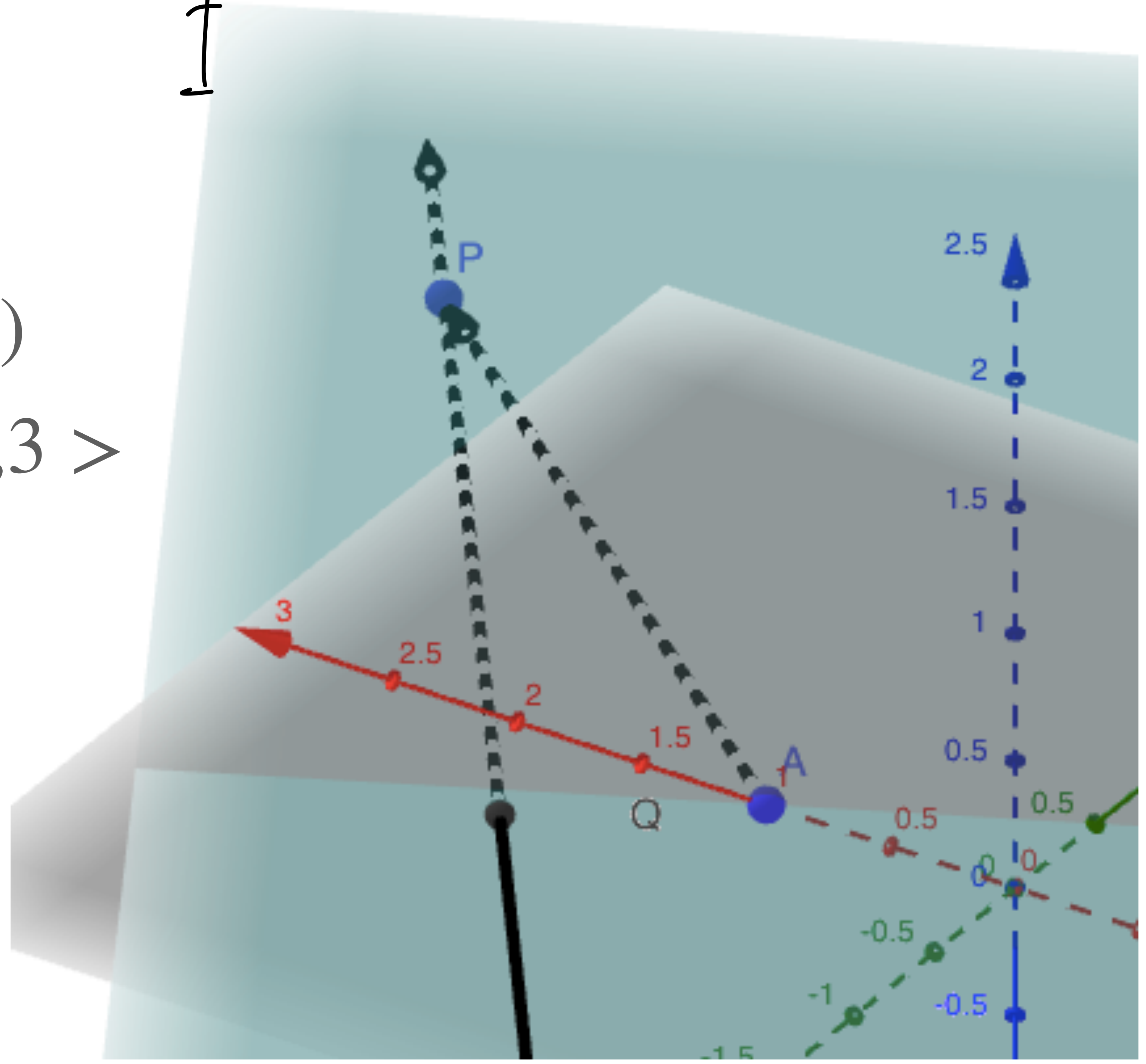



1. $A = (1,0,0)$

$P = (1, -2, 3)$

$\overrightarrow{AP} = < 0, -2, 3 >$

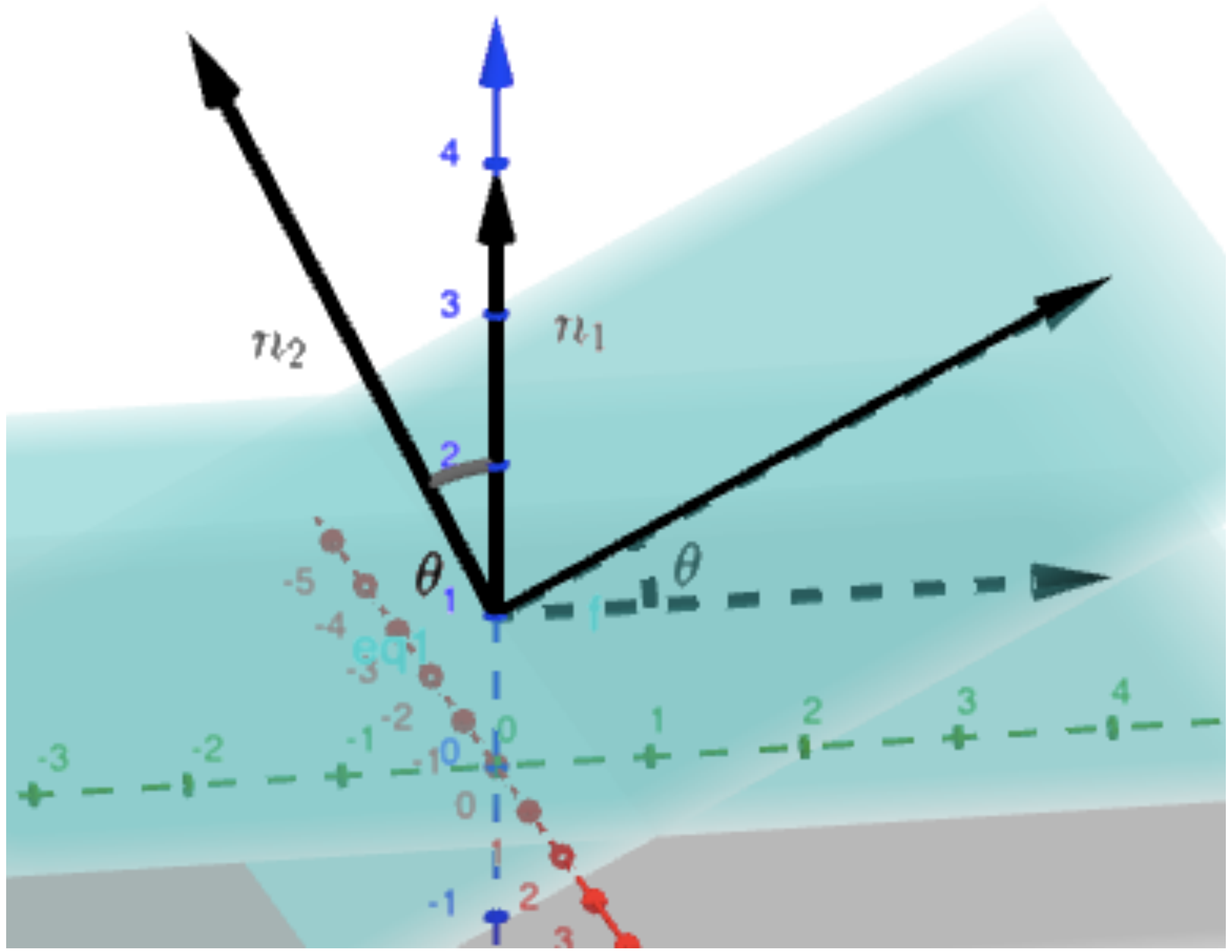



$$D = \frac{|\mathbf{n} \cdot \overrightarrow{AP}|}{|\mathbf{n}|}$$

$$= \frac{|< 1, 2, -4 > \cdot < 0, -2, 3 >|}{\sqrt{1^2 + 2^2 + (-4)^2}} = \frac{16}{\sqrt{21}} \approx 3.49$$

2. $$D = \frac{\left|4 - \frac{3}{2}\right|}{\sqrt{4 + 9 + 1}} = \frac{2.5}{\sqrt{14}} \approx 0.67$$

# Planes' angles; Skew Lines' Distance.




The angle between two planes is the same as the angle between the planes' normal vectors.

$$\cos(\theta) = \frac{\mathbf{n_1} \cdot \mathbf{n_2}}{|\mathbf{n_1}||\mathbf{n_2}|}$$ This angle is acute $\leq 90°$

with appropriate choices of $\mathbf{n_1}$ and $\mathbf{n_2}$.

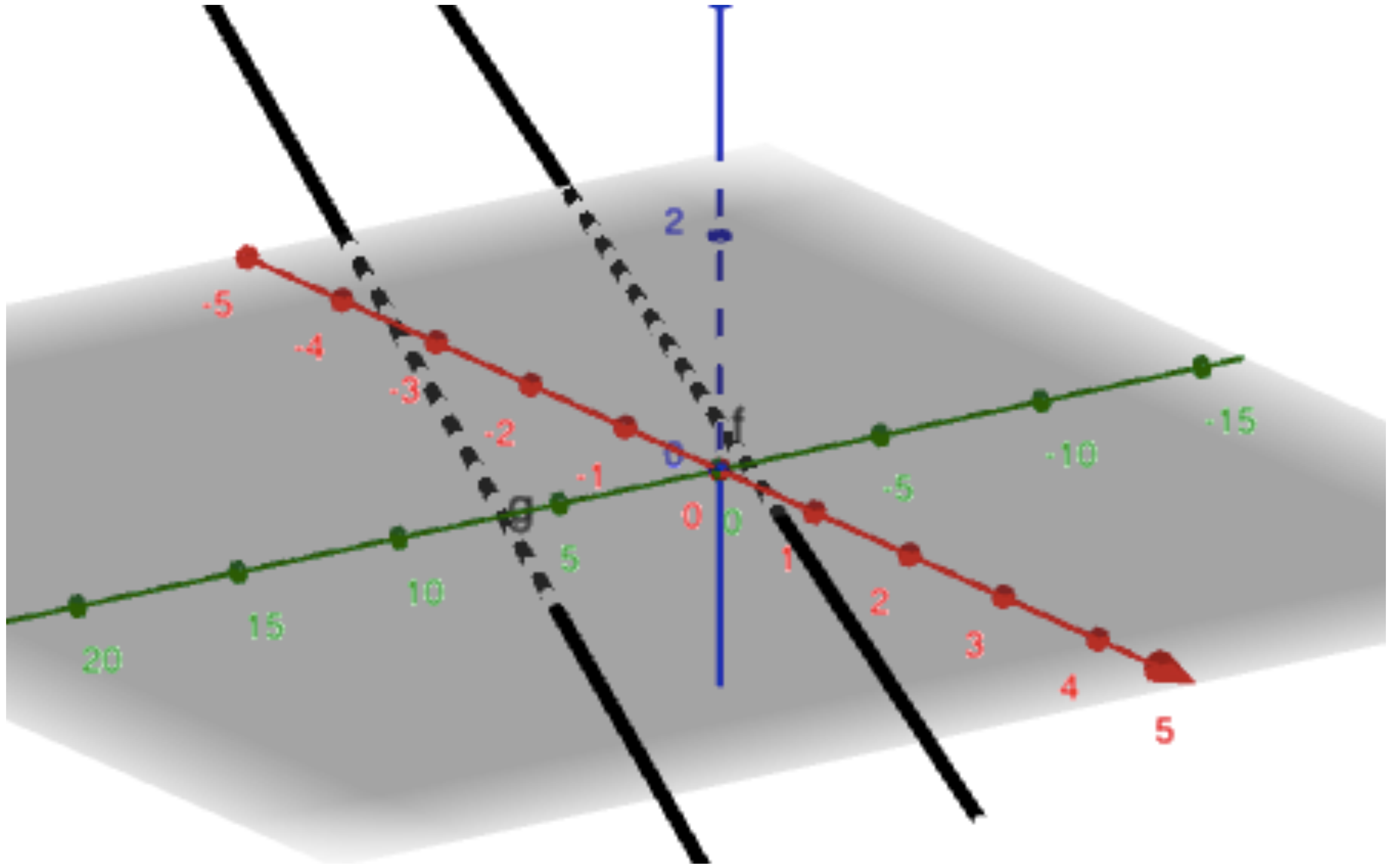
Lines in 3D might not intersect, even if they're not parallel. Such lines are *skew.*

Example. (S12.5 #78)

Find the distance between the lines

$$\mathbf{L}(t) = < 1 + t, 1 + 6t, 2t >$$

$$\mathbf{K}(s) = < 1 + 2s, 5 + 15s, -2 + 6s >$$



# Skew Lines' Distance, pg2.

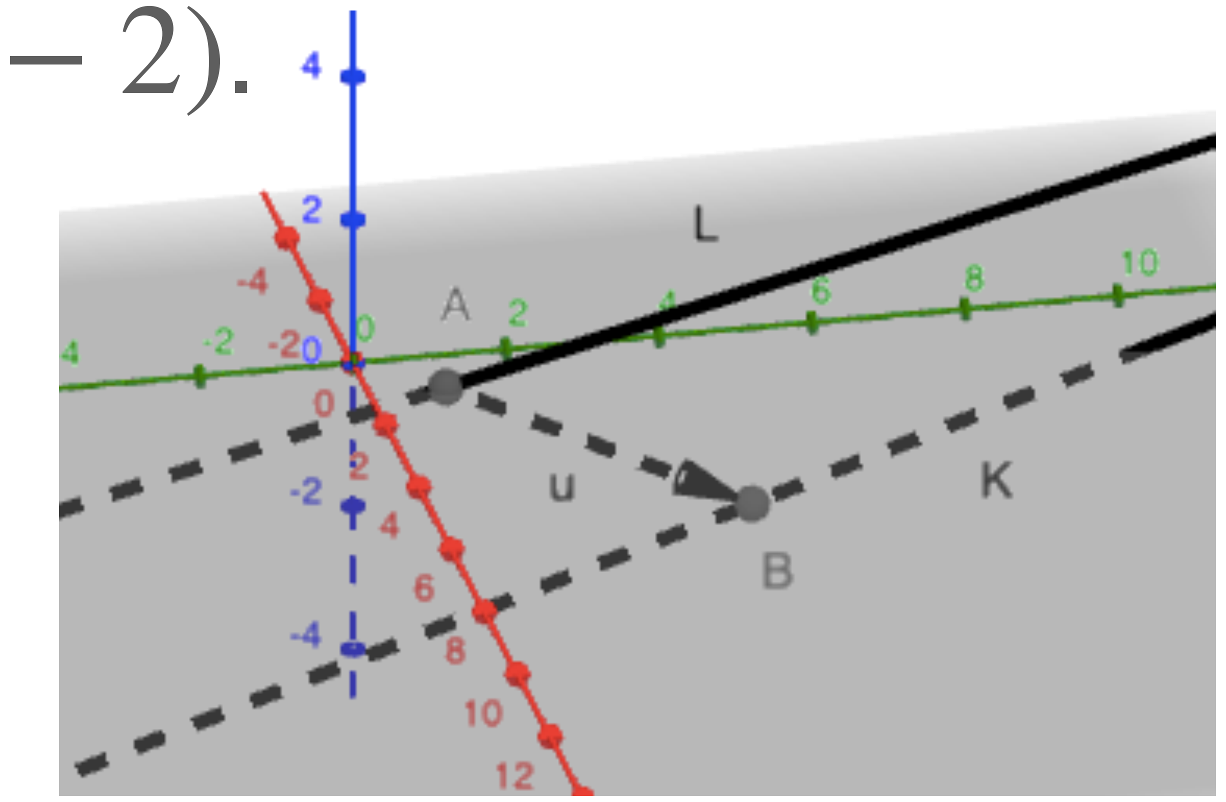
$\mathbf{L}(t) = < 1 + t,\ 1 + 6t,\ 2t >$

$\mathbf{K}(s) = < 1 + 2s,\ 5 + 15s,\ -2 + 6s >$

Warm-up1: Find the distance between the initial points $(1,1,0)$ and $(1,5,-2)$.

$$\sqrt{(1-1)^2 + (1-5)^2 + (0 - -2)^2}$$

$$= \sqrt{20} \approx 4.47 \text{ units}$$




$$\overrightarrow{VA} \times \mathbf{u} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -4 & 2 \\ 2 & 15 & 6 \end{pmatrix}$$

$$= < -54,4,8 >$$

$$\frac{|\overrightarrow{VA} \times \mathbf{u}|}{|\mathbf{u}|} = \frac{\sqrt{2996}}{\sqrt{265}} \approx 3.36 \text{ units}$$

but what about the distance between lines themselves?

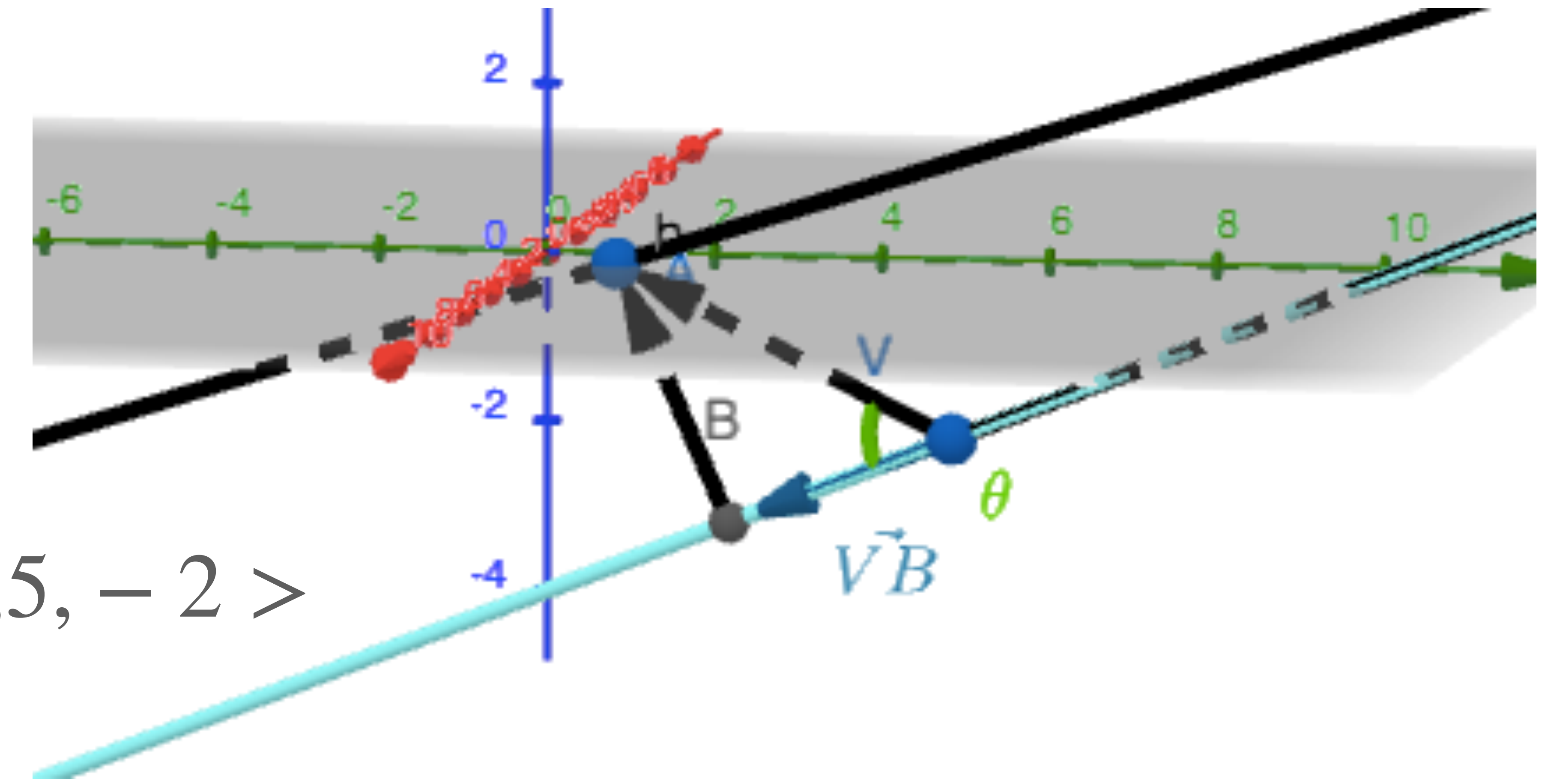
Warm-up2: Find the distance between the point $(1,1,0)$ and the line $\mathbf{K}$.

$$D = \frac{|\overrightarrow{VA} \times \mathbf{u}|}{|\mathbf{u}|}$$

$\mathbf{u} = < 2,15,6 >$

$\overrightarrow{VA} = < 1,1,0 > - < 1,5,-2 >$

$= < 0,-4,2 >$




Method 1.

Find the distance between the point $(1 + t,\ 1 + 6t,\ 2t)$ on $\mathbf{L}$ and the line $\mathbf{K}$, then minimize the resulting function of $t$.

$$\overrightarrow{V\mathbf{L}(t)} = < 1 + t, 1 + 6t, 2t > - < 1,5,-2 >$$

$$= < t, -4 + 6t, 2 + 2t >$$

# Skew Lines' Distance, pg3.

$$\overrightarrow{V\mathbf{L}(t)} = <1+t, 1+6t, 2t> - <1,5,-2>$$
$$= <t, -4+6t, 2+2t>$$

$$\overrightarrow{V\mathbf{L}(t)} \times \mathbf{u} = \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & -4+6t & 2+2t \\ 2 & 15 & 6 \end{pmatrix}$$

$$= <-54+6t,\ 4-2t,\ 8+3t>$$

$$\frac{|\overrightarrow{V\mathbf{L}(t)} \times \mathbf{u}|}{|\mathbf{u}|} = \frac{\sqrt{49t^2 - 616t + 2996}}{\sqrt{265}} = D(t)$$

This function is minimized when…

$$t = \frac{-b}{2a} = \frac{616}{98} = \frac{44}{7}$$

the minimum distance is $D\left(\frac{44}{7}\right) = 2$

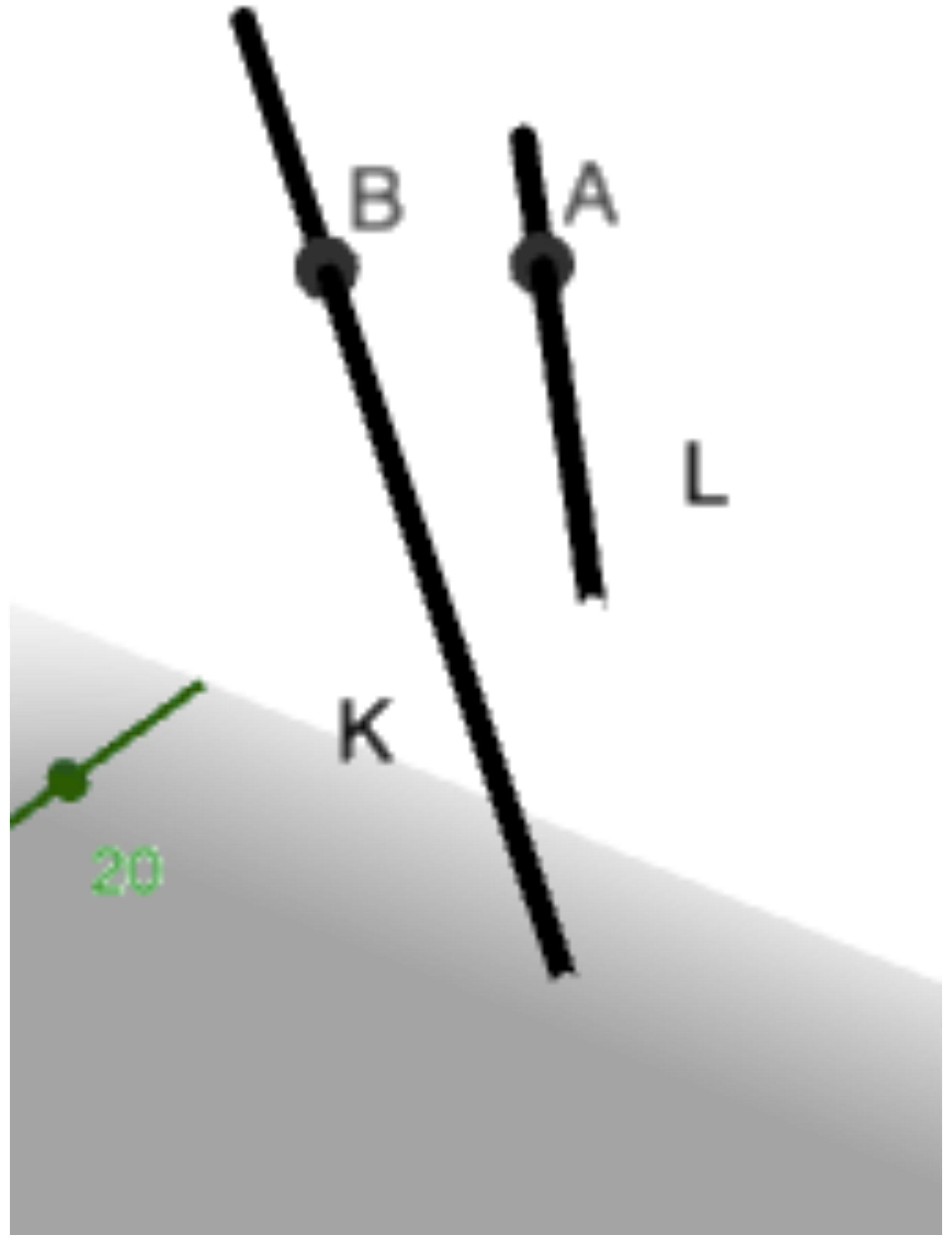
Method 2. Find the closest possible pair of points on each line.

The optimal pair of points happens when

$\overrightarrow{AB} \perp \mathbf{L}, \quad \overrightarrow{AB} \perp \mathbf{K}$

$A(1+t, 1+6t, 2t)$

$B(1+2s, 5+15s, -2+6s)$




$\overrightarrow{AB} = <2s-t, 15s-6t+4, 6s-2t-2>$

$\overrightarrow{AB} \perp \mathbf{L}$ means

$$1(2s-t) + 6(15s-6t+4) + 2(6s-2t-2) = 0$$

$\overrightarrow{AB} \perp \mathbf{K}$ means

$$2(2s-t) + 15(15s-6t+4) + 6(6s-2t-2) = 0$$

$$\begin{cases} 1(2s-t) + 6(15s-6t+4) + 2(6s-2t-2) = 0 \\ 2(2s-t) + 15(15s-6t+4) + 6(6s-2t-2) = 0 \end{cases}$$

…solve this for s and t!

# Skew Lines' Distance, pg 4.

$$\mathbf{L}(t) = <1+t,\ 1+6t,\ 2t>$$

$$\mathbf{K}(s) = <1+2s,\ 5+15s,\ -2+6s>$$

$$\begin{cases} 1(2s-t)+6(15s-6t+4)+2(6s-2t-2)=0 \\ 2(2s-t)+15(15s-6t+4)+6(6s-2t-2)=0 \end{cases}$$

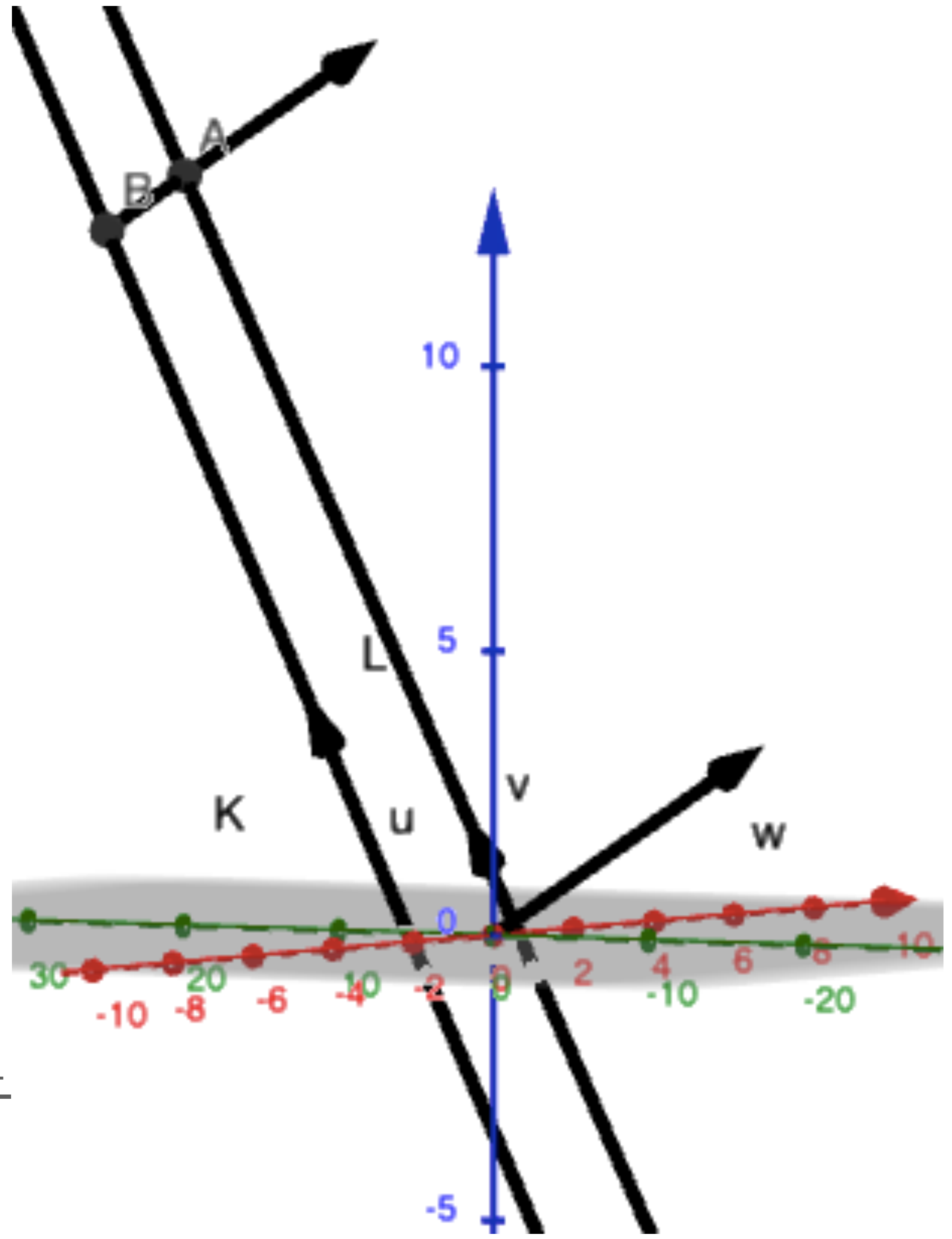
work, work, work….

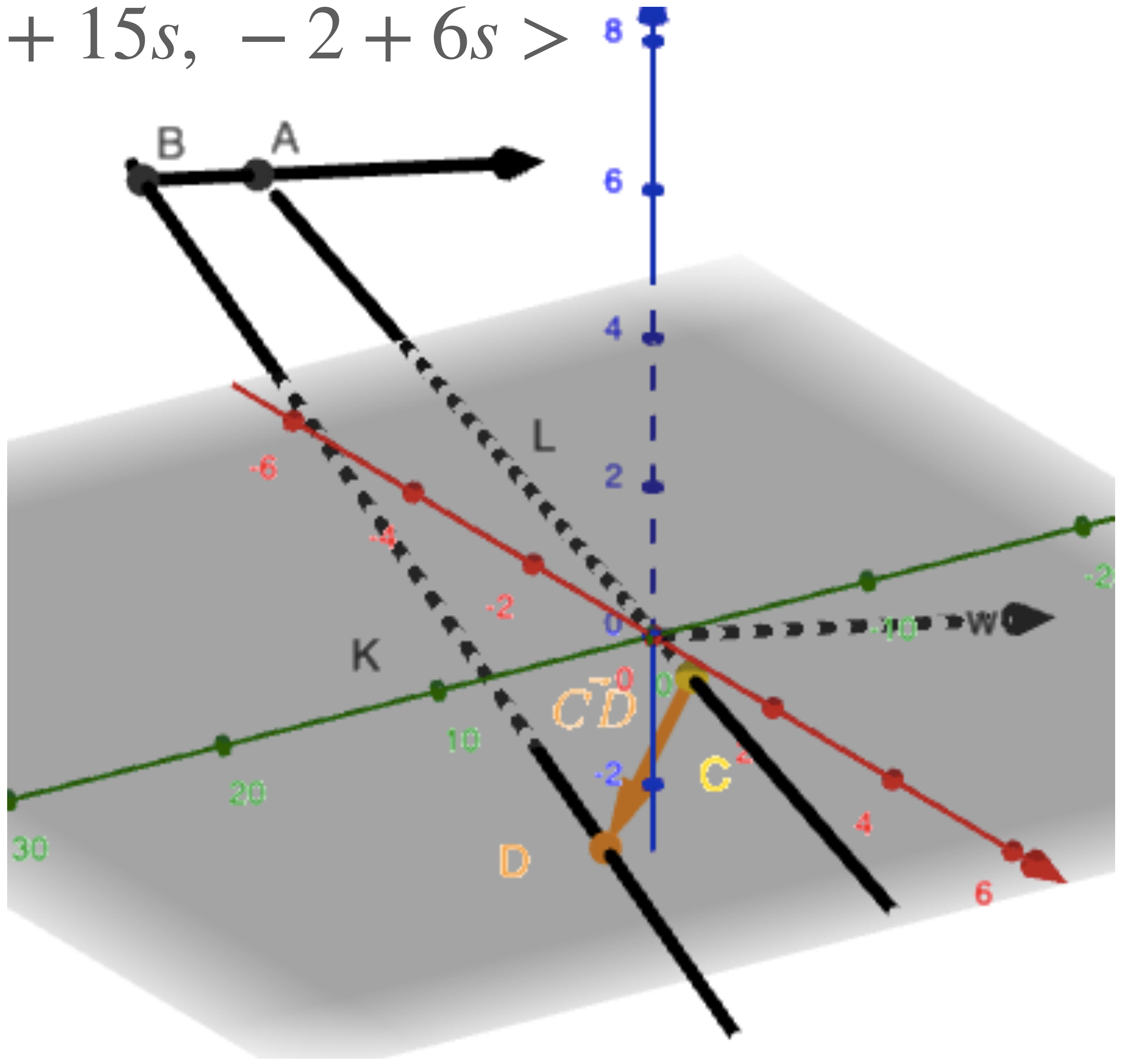
$$s = \frac{16}{7}, \quad t = \frac{44}{7}$$

The distance between the optimal points is:

$$|\overrightarrow{AB}| = |<2s-t, 15s-6t+4, 6s-2t-2>| \Big|_{s=\frac{16}{7}, t=\frac{44}{7}}$$

$$= 2$$




Method 3: Project onto a mutually perpendicular vector, $\mathbf{w}$.

We get such a $\mathbf{w}$ with a cross product of the direction vectors of each line.

Then project a random vector $\overrightarrow{CD}$, between points on each line onto $\mathbf{w}$. The magnitude of this projection is the distance we want.

$$\mathbf{u} = <2,15,6>$$

$$\mathbf{v} = <1,6,2>$$

$$\mathbf{w} = \mathbf{u} \times \mathbf{v} = <-6,2,-3>$$

$$\overrightarrow{CD} = \mathbf{K}(0) - \mathbf{L}(0)$$

$$= <1,5,-2> - <1,1,0>$$

$$= <0,4,-2>$$

$$\mathbf{proj}_{\mathbf{w}}(\overrightarrow{CD}) = \frac{\overrightarrow{CD} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}} \mathbf{w}$$

$$|\mathbf{proj}_{\mathbf{w}}(\overrightarrow{CD})| = \frac{|\overrightarrow{CD} \cdot \mathbf{w}|}{|\mathbf{w}|^2} |\mathbf{w}|$$

$$= \frac{|\overrightarrow{CD} \cdot \mathbf{w}|}{|\mathbf{w}|}$$

$$= 14/\sqrt{49} \quad = 2$$

# Skew Lines' Distance, pg 5. Practice.

1. (OX2.5 #295) Show that the lines with equations $\mathbf{L}(t) = < t, 1+t, 2+t >$ and $\dfrac{x}{2} = \dfrac{y-1}{3} = z - 3$ are skew. Find the distance between these two lines.

1. The second line given has vector equation $K(s) = < 2s, 1+3s, 3+s >$

Is there a point that's on both lines $L$ and $K$ ?

$$\begin{cases} t = 2s \\ 1+t = 1+3s \\ 2+t = 3+s \end{cases}$$

there's no solution here, since the first two equations require $s = t = 0$, but then the third equation says 2=3!

We can find the distance between the lines by projecting a vector $\overrightarrow{L(0)K(0)}$ onto $\mathbf{w} = \mathbf{u} \times \mathbf{v}$.

$$\overrightarrow{L(0)K(0)} = < 0,1,3 > - < 0,1,2 > = < 0,0,1 >$$

$$\mathbf{w} = < 1,1,1 > \times < 2,3,1 > = < -2,1,1 >$$

$$|\,\mathbf{proj_w}(\overrightarrow{L(0)K(0)}\,)\,| = \frac{< 0,0,1 > \cdot < -2,1,1 >}{|< -2,1,1 >|} = \frac{1}{\sqrt{6}} \approx 0.41$$

2. (S12.5 #79) Find the distance between the line that goes through (0,0,0), (2,0,-1) and the line that goes through (1,-1,1) and (4,1,3).

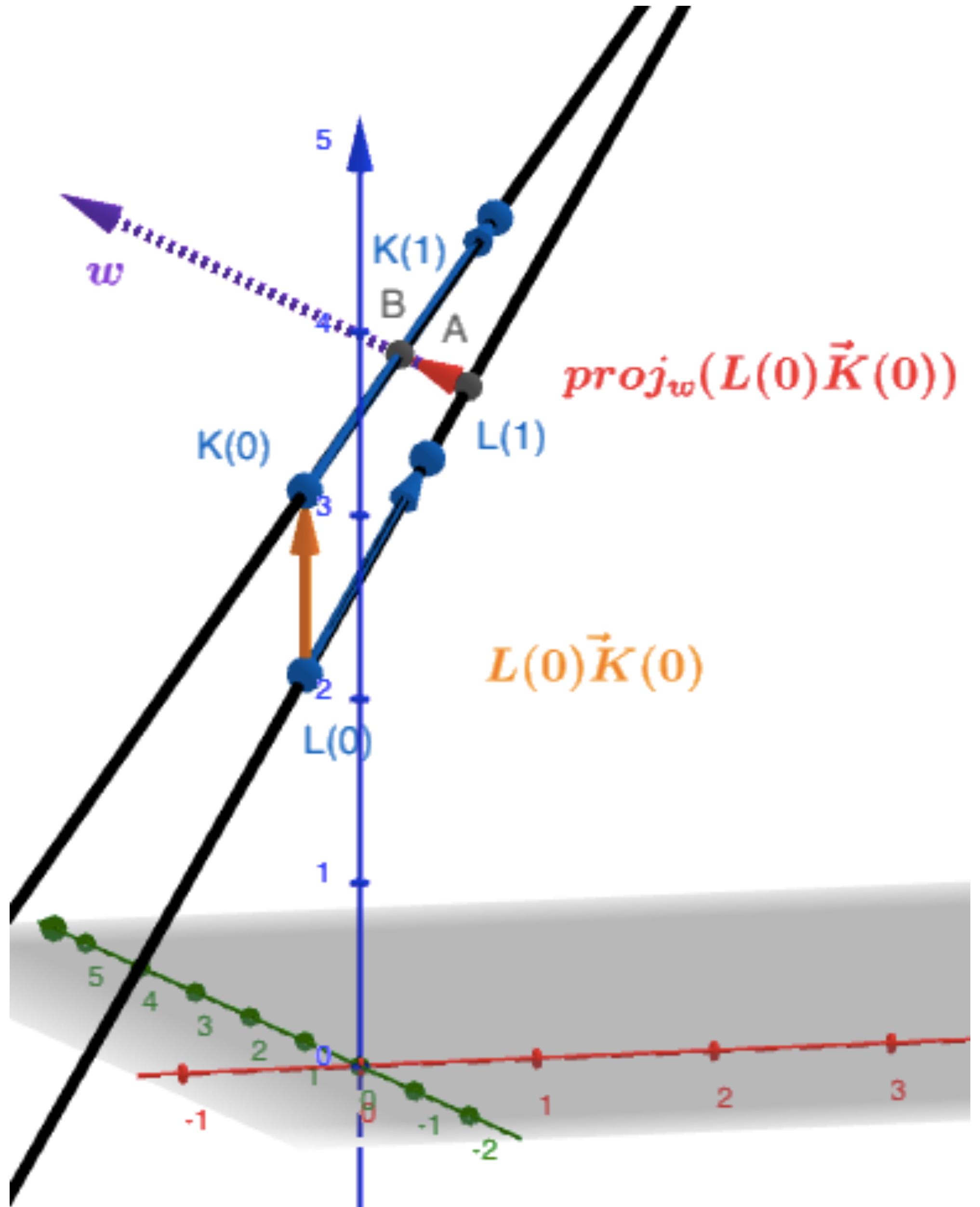
$$\mathbf{L}_1(t) = \overrightarrow{P}_1 + t \cdot \overrightarrow{P_1Q_1} = < 2t, 0, -t >$$

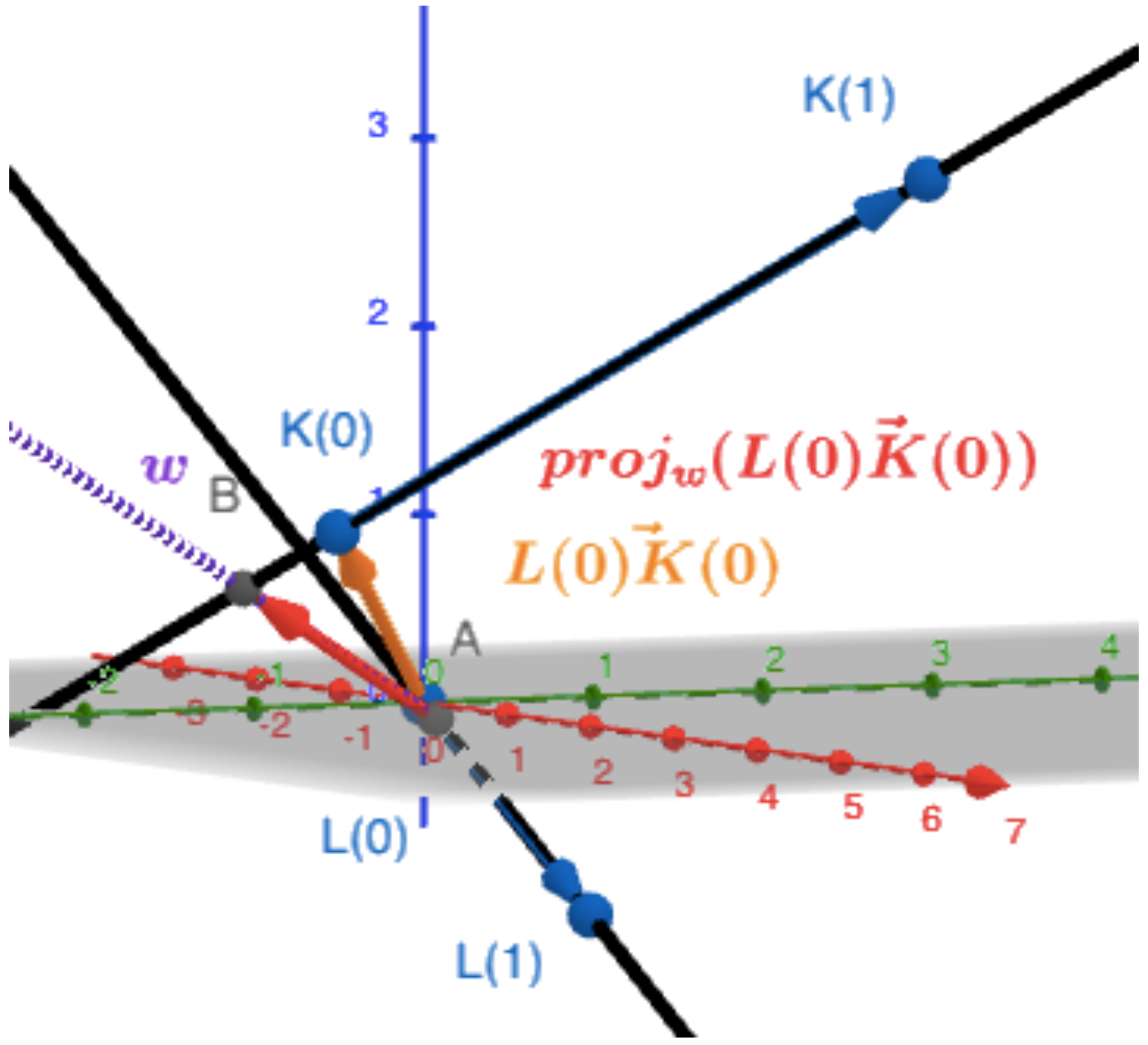
$$\mathbf{L}_2(t) = \overrightarrow{P}_2 + t \cdot \overrightarrow{P_2Q_2} = < 1+3t, -1+2t, 1+2t >$$

$$\mathbf{w} = < 2,0,-1 > \times < 3,2,2 > = < 2,-7,4 >$$

$$\overrightarrow{\mathbf{L}(0)\mathbf{K}(0)} = \overrightarrow{P_1P_2} = < 1,-1,1 >$$

$$D = \frac{|\,\overrightarrow{\mathbf{L}(0)\mathbf{K}(0)} \cdot \mathbf{w}\,|}{|\,\mathbf{w}\,|} = \frac{13}{\sqrt{69}} \approx 1.57$$




Link:SkewLines'Distance

# Practice Problems from OpenStax 2.5 and Stewart Section 12.5

1. (OX2.5 #291). Two planes are given by $x + y + z = 0, \quad 2x - y + z - 7 = 0.$
Are these planes parallel? Perpendicular? What is the angle between them?

2. (OX2.5 #297) You have a point C(-3,2,4) and a plane $2x + 4y - 3z = 8.$
a) What's the radius of the sphere centered at C, tangent to the plane?
b) What is the point of tangency?

3. (OX2.5 #296)
Show that the lines with equations $\mathbf{L}(t) = \langle -1 + t, -2 + t, 3t \rangle$
and $\mathbf{K}(s) = \langle 5 + s, \ -8 + 2s, \ 7s \rangle$ are skew. Find the distance between these lines.

4. (S12.5#80) $P_1$ is the plane $x - y + 2z + 1 = 0$. $P_2$ is the plane through the points $A(3,2,-1), B(0,0,1), C(1,2,1)$. $L_1$ is the line through the points $P(1,2,6)$ and $Q(2,4,8)$. $L_2$ is the line of intersection of $P_1$ and $P_2$. Find the distance between $L_1$ and $L_2$.

5. (S12.5 #82) Give a geometric description of each family of planes for $c \in R$ and $\theta \in [0,2\pi)$
$a)\ x + y + z = c \qquad b)\ x + y + cz = 1 \qquad c)\ \cos(\theta)y + \sin(\theta)z = 1$

# Solutions to Practice Problems, pg 1.

1. (OX2.5 #291). Two planes are given by $x + y + z = 0, \quad 2x - y + z - 7 = 0.$
Are these planes parallel? Perpendicular? What is the angle between them?

The first plane has a normal vector of $\mathbf{n}_1 = < 1,1,1 >$

The second plane's normal vector is $\mathbf{n}_2 = < 2, -1,1 >$

These normal vectors aren't parallel because one is not a multiple of the other.

(For example a vector parallel to $\mathbf{n}_1$ would look like $< j,j,j > = j < 1,1,1 >$)

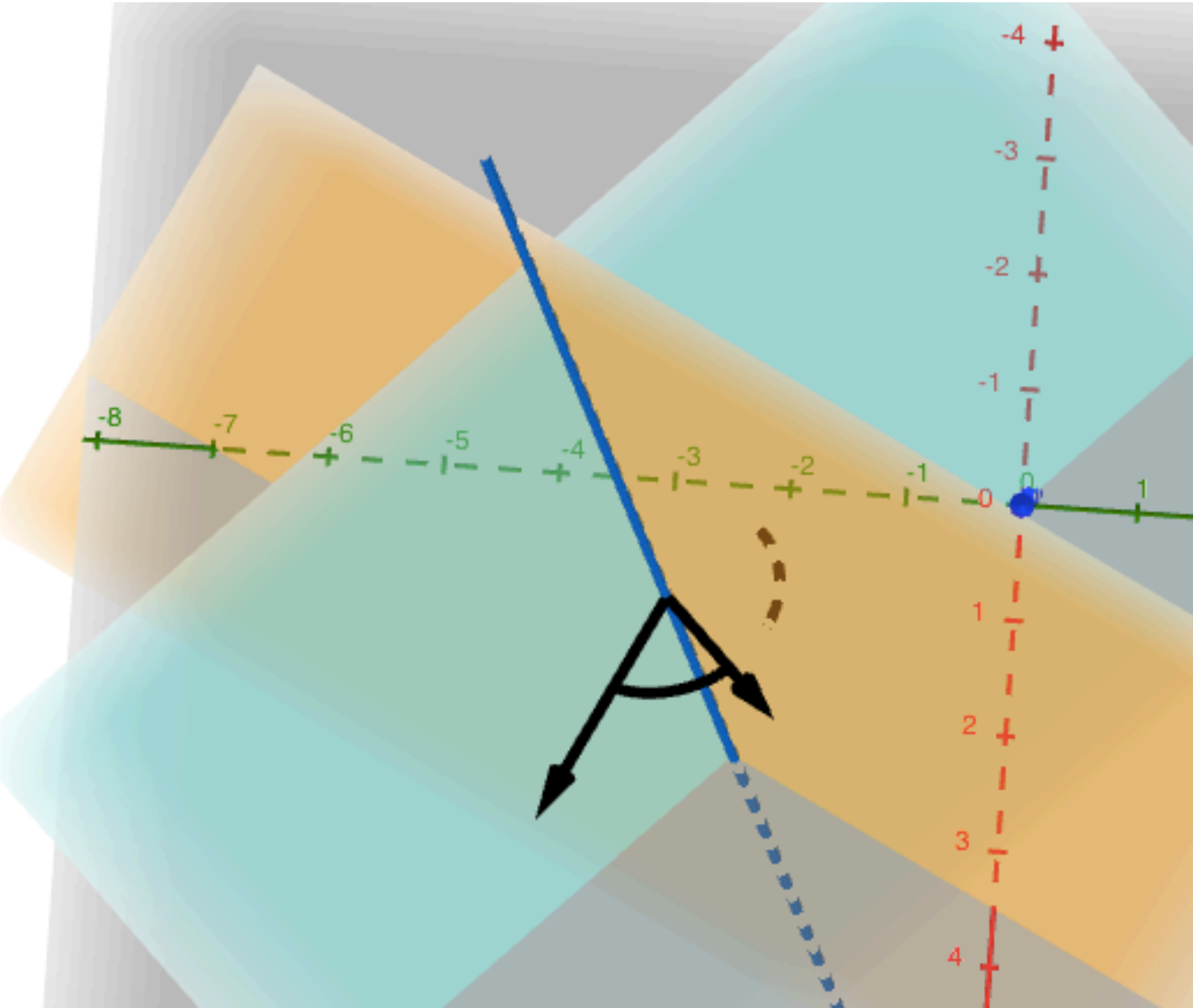
Since the normal vectors aren't parallel, the planes are not parallel

Similarly, the planes aren't perpendicular because their normal vectors aren't, which you can see with a dot product: $\mathbf{n}_1 \cdot \mathbf{n}_2 \neq 0$

The angle between the planes is
the angle between their normal vectors:

$$\theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right)$$

$$= \cos^{-1}\left(\frac{2}{\sqrt{3}\sqrt{6}}\right) \approx 1.08 \text{ rad} \approx 61.87°$$



# Solutions to Practice Problems, pg 2.

2. (OX2.5 #297) You have a point C(-3,2,4) and a plane $2x + 4y - 3z = 8$.

a) What's the radius of the sphere centered at C, tangent to the plane?

b) What is the point of tangency?

a) This is a fancy way of asking how far the point C is from the given plane. We know how to compute this:

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2+b^2+c^2}} = \frac{|2(-3)+4(2)-3(4)-8|}{\sqrt{2^2+4^2+(-3)^2}} = \frac{18}{\sqrt{29}} \approx 3.34$$

Or, you can compute the magnitude of the projection of $\overrightarrow{AC}$ onto $\mathbf{n} = <2,4,-3>$, for any point $A$ on the plane.

e.g. $A = (4,0,0)$ $\quad \overrightarrow{AC} = < -3,2,4 > - < 4,0,0 > = < -7,2,4 >$

$$D = \frac{|\mathbf{n}\cdot\overrightarrow{AC}|}{|\mathbf{n}|} = \frac{|<2,4,-3>\cdot<-7,2,4>|}{\sqrt{29}} = \frac{18}{\sqrt{29}}$$

b) Find a point that is on the plane and on the line, parallel to $\mathbf{n}$, that goes through C….

$(b)\ldots$ $\mathbf{L}(t) = < -3,2,4 > + t < 2,4,-3 >$

$= < -3+2t,\ 2+4t,\ 4-3t >$

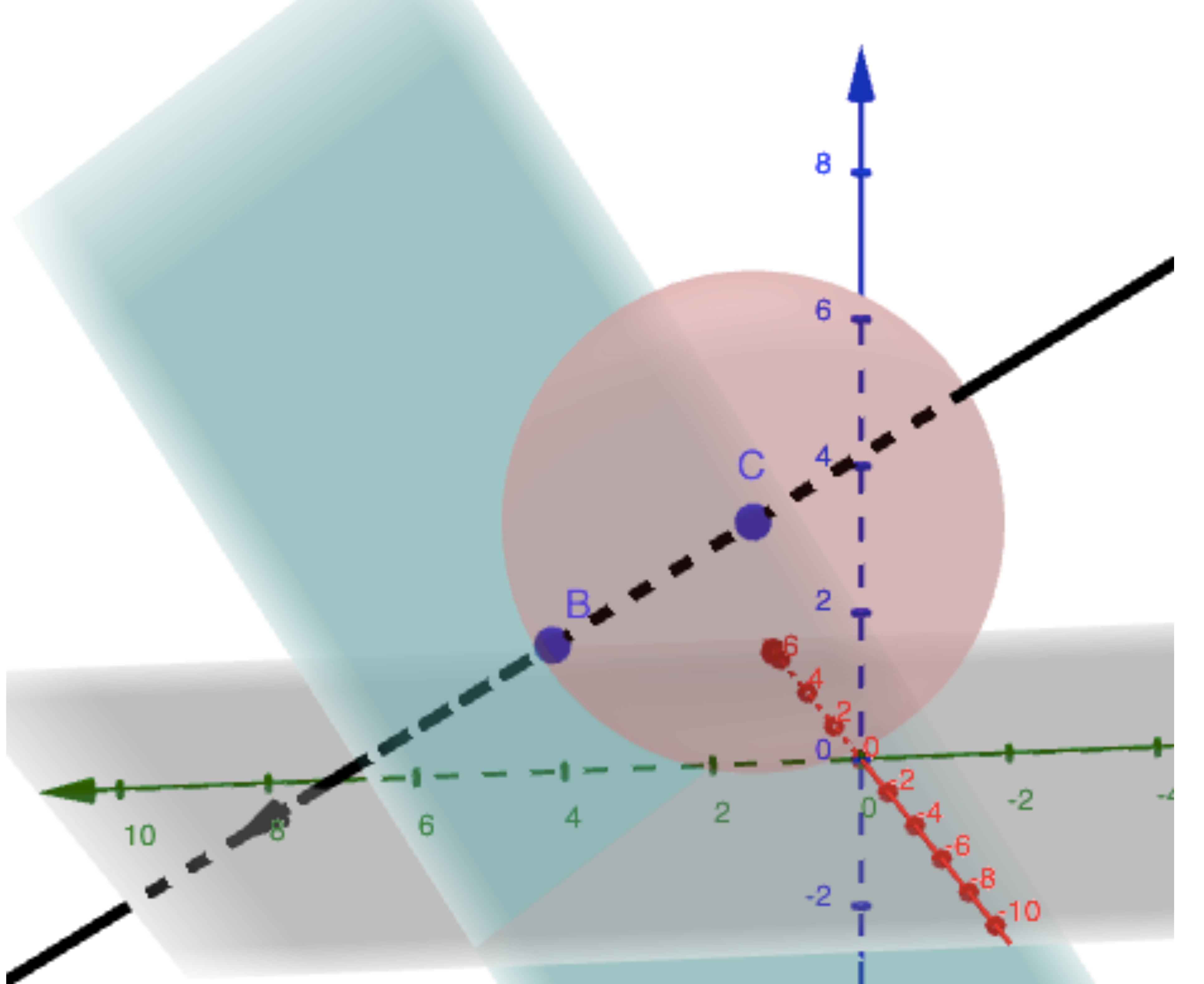
$2(-3+2t)+4(2+4t)-3(4-3t) = 8$

$\ldots t_B = 18/29$

$$\mathbf{L}(t_B) = \frac{1}{29} < -51,130,62 >$$

$\approx < -1.76,\ 4.48,\ 2.14 >$

the point of tangency is ~$B(-1.76, 4.48, 2.14)$



# Solutions to Practice Problems, pg 3.

3. (OX2.5 #296) Show that the lines with equations $\mathbf{L}(t) = < -1+t, -2+t, 3t >$ and $\mathbf{K}(s) = < 5+s, \ -8+2s, 7s >$ are skew. Find the distance between these lines.

$\mathbf{L}(t) = < -1+t, -2+t, 3t >$

$\mathbf{K}(s) = < 5+s, \ -8+2s, 7s >$

The lines would intersect if for some value of $s$ and t all components were equal

$$\begin{cases} -1+t = 5+s \\ -2+t = -8+2s \\ 3t = 7s \end{cases} \rightarrow \begin{cases} t = 6+s \\ t = -6+2s \\ t = 7s/3 \end{cases}$$

No solution!

Hence the lines don't intersect.

Also, the lines aren't parallel, since their direction vectors $< 1,1,3 >$ and $< 1,2,7 >$ aren't.

So the lines are skew.

$\mathbf{L}$ starts at $L(0) = (-1, -2, 0)$ and is parallel to $\mathbf{u} = < 1,1,3 >$.

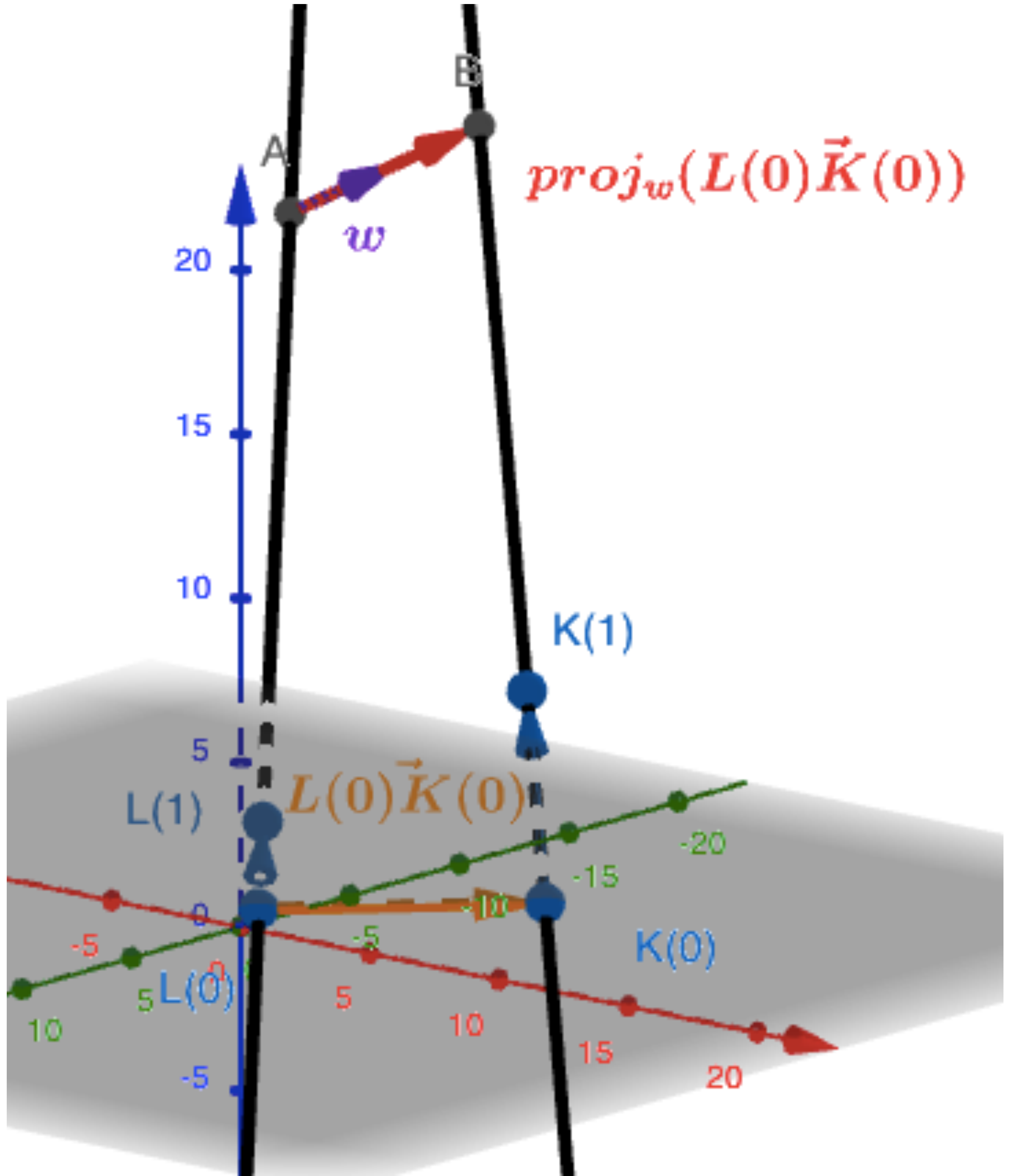
$\mathbf{K}$ starts at $K(0) = (5, -8, 0)$ and is parallel to $\mathbf{v} = < 1,2,7 >$.

A vector perpendicular to both lines is $\mathbf{w} = \mathbf{u} \times \mathbf{v} = < 1, -4, 1 >$

$$|\,\mathbf{proj}_{\mathbf{w}}(\overrightarrow{L(0)K(0)})\,|$$

$$= \frac{|< 6, -6, 0 > \cdot < 1, -4, 1 >|}{|< 1, -4, 1 >|}$$

$$= \frac{30}{\sqrt{18}} \approx 7.07$$



# Solutions to Practice Problems, pg 4.

4. (S12.5 #80) $P_1$ is the plane $x - y + 2z + 1 = 0$. $P_2$ is the plane through the points $A(3,2,-1), B(0,0,1), C(1,2,1)$. $L_1$ is the line through the points $P(1,2,6)$ and $Q(2,4,8)$. $L_2$ is the line of intersection of $P_1$ and $P_2$. Find the distance between $L_1$ and $L_2$.

Finding $P_2$: $\overrightarrow{BA} = < 3,2,-2 >$, $\overrightarrow{BC} = < 1,2,0 >$ $\mathbf{n} = \overrightarrow{BA} \times \overrightarrow{BC} = < 4,-2,4 >$

$4(x-0) - 2(y-0) + 4(z-1) = 0, \qquad 4x - 2y + 4z = 4, \qquad 2x - y + 2z = 2$

Finding $L_1$: $\mathbf{u} = \overrightarrow{PQ} = < 1,2,2 >$ $\mathbf{L_1}(t) = < 1,2,6 > + t < 1,2,2 > = < 1+t, 2+2t, 6+2t >$

Finding $L_2$: $\mathbf{n_1} = < 4,-2,4 >$, $\mathbf{n_2} = < 1,-1,2 >$ $\mathbf{u} = \mathbf{n_1} \times \mathbf{n_2} = < 0,-4,-2 >$

A point on both planes happens when $z = 0$, $\begin{cases} 4x - 2y = 4 \\ x - y = -1 \end{cases}$ $(x, y, z) = (3,4,0)$

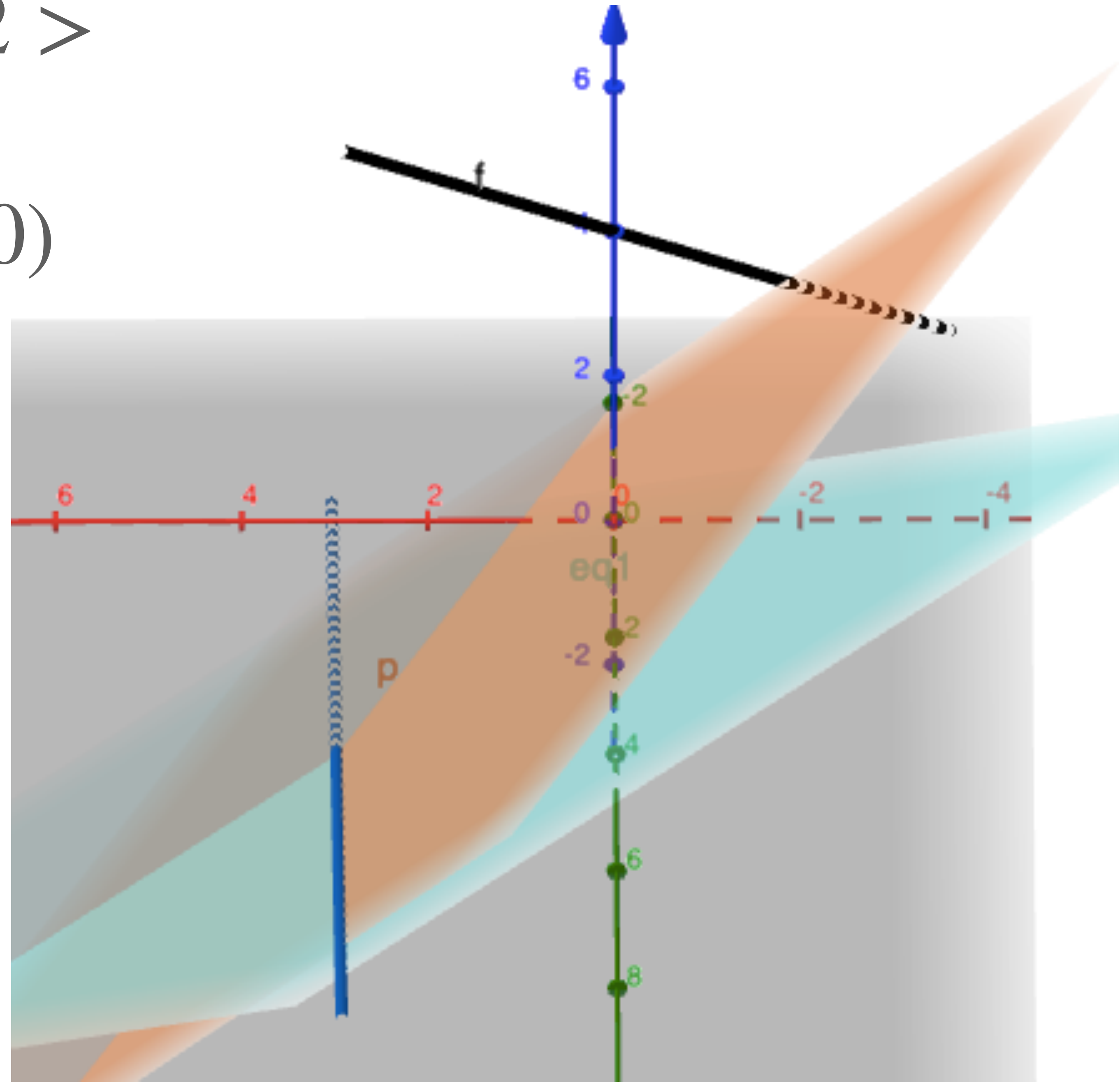
$\mathbf{L_2}(t) = < 3,4,0 > + t < 0,-4,-2 > = < 3, 4-4t, -2t >$

Distance between the lines: $\mathbf{w} = < 0,-4,-2 > \times < 1,2,2 > = < -4,-2,4 >$

$C(1,2,6), \ D(3,4,0)$

$\overrightarrow{CD} = < 2,2,-6 >$

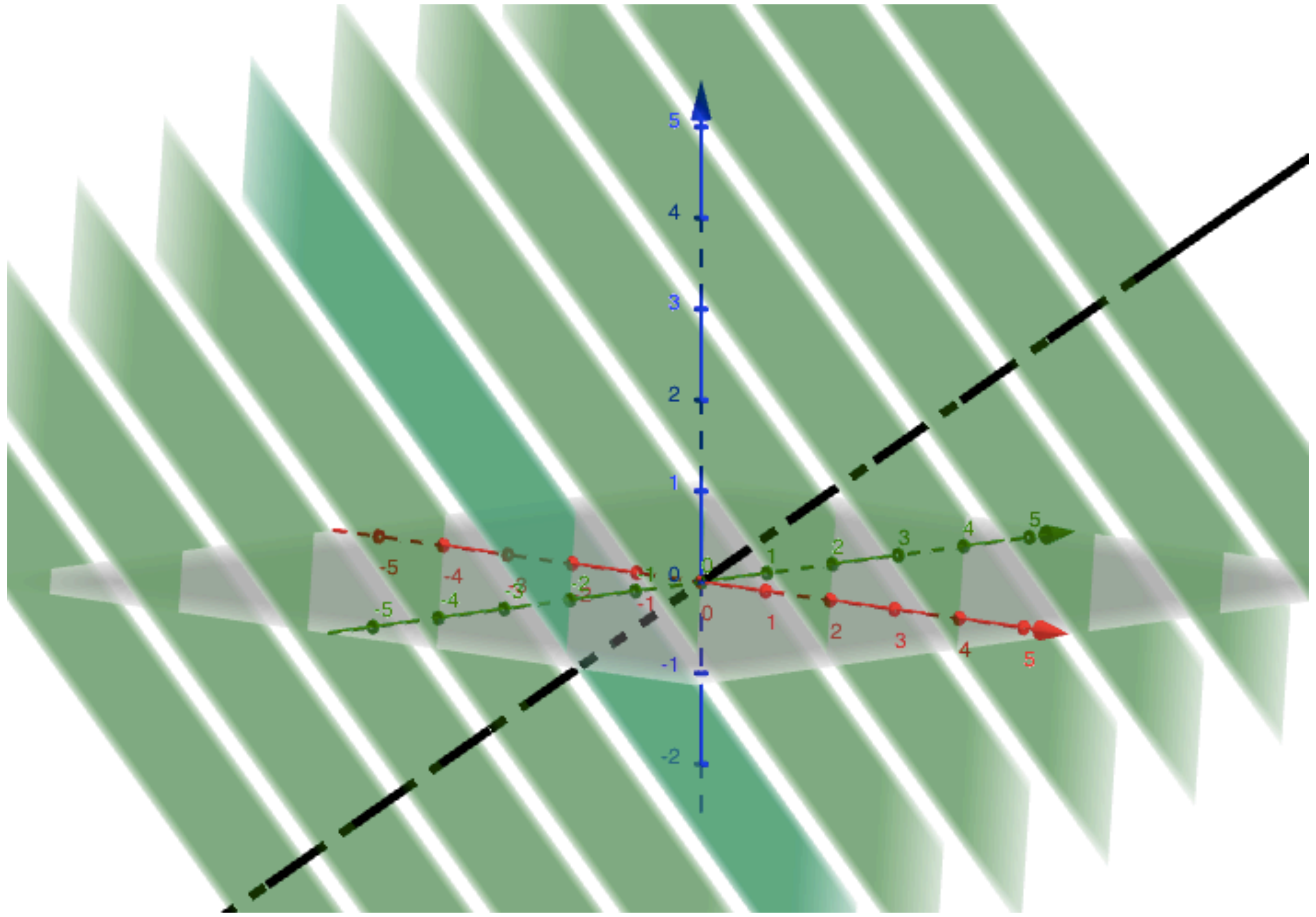
$$|\mathbf{proj_w}(\overrightarrow{CD})| = \frac{|\overrightarrow{CD} \cdot \mathbf{w}|}{|\mathbf{w} \cdot \mathbf{w}|} |\mathbf{w}| = \frac{|\overrightarrow{CD} \cdot \mathbf{w}|}{|\mathbf{w}|} = \frac{36}{\sqrt{36}} = 6$$



# Solutions to Practice Problems, pg 5.

5. (S12.5 #82) Give a geometric description of each family of planes for $c \in R$ and $\theta \in [0,2\pi)$

$a)\ x+y+z=c \qquad b)\ x+y+cz=1 \qquad c)$ (next page)

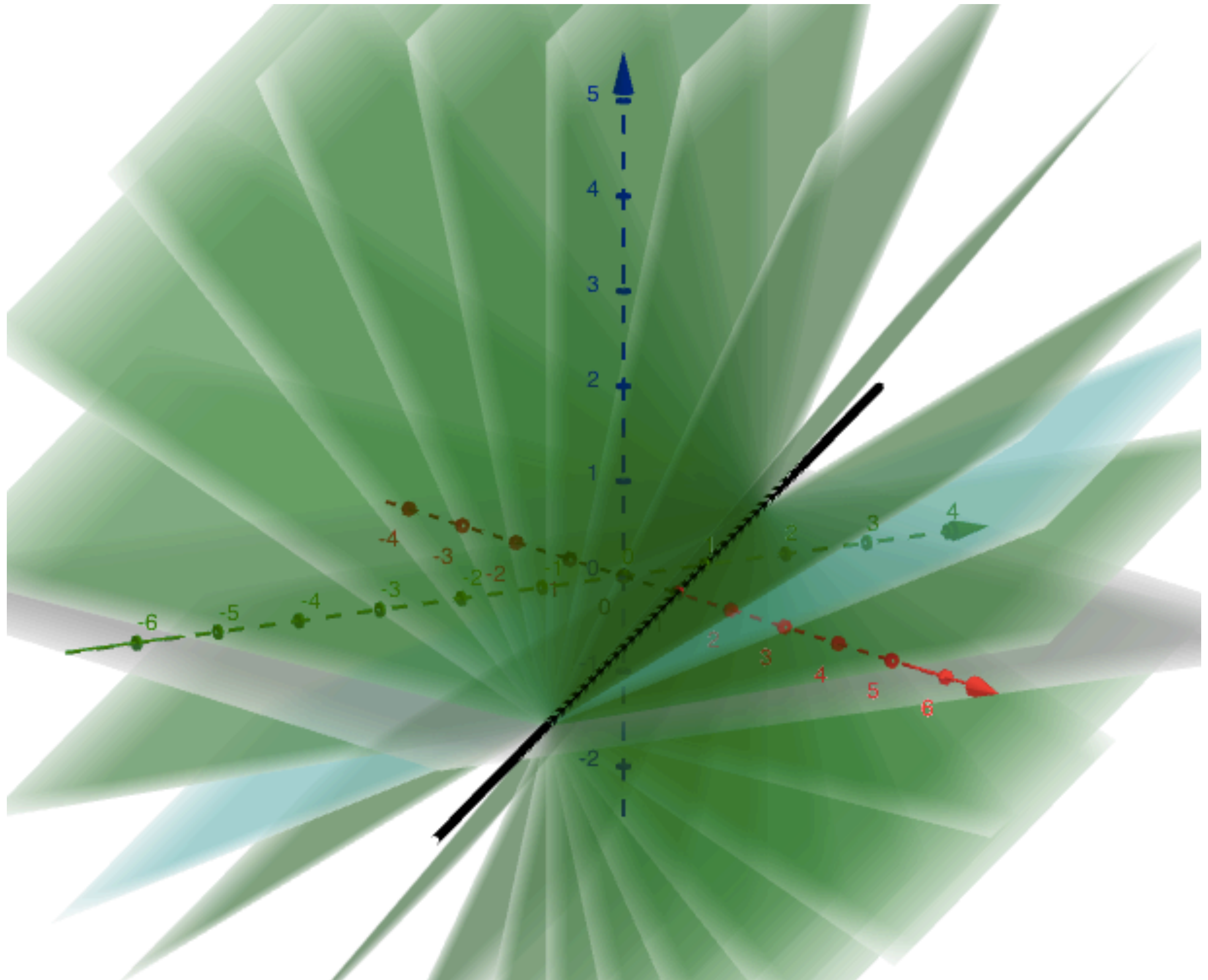


a) Look at the intercepts for different values of c.

| c | x-int | y-int | z-int |
|---|---|---|---|
| 1 | (1,0,0) | (0,1,0) | (0,0,1) |
| 2 | (2,0,0) | (0,2,0) | (0,0,2) |
| 3 | (3,0,0) | (0,3,0) | (0,0,3) |

In general, the plane $x+y+z=c$ goes through the points $(c,0,0), (0,c,0), (0,0,c)$.

a) You could also say $x+y+z=c$ is the set of planes that perpendicularly intersect $L(t) = < t, t, t >$ over all values of $t$.



b) The z intercept varies with $c$: In general it's $(0,0,1/c)$. But The x-intercepts and y-intercepts are always $(1,0,0)$ and $(0,1,0)$. i.e. every one of these planes contains the line $\mathbf{L}(t) = < 1,0,0 > + t < 1, -1,0 >$ . $x+y+cz=1$ is the set of all planes, except the xy plane, which contain the line $\mathbf{L}$.

# Solutions to Practice Problems, pg 6.

5. (S12.5 #82) Give a geometric description of each family of planes for $c \in R$ and $\theta \in [0,2\pi)$

$a)\ x + y + z = c \qquad b)\ x + y + cz = 1 \qquad c)\ \cos(\theta)y + \sin(\theta)z = 1$

c) This set of planes contains $y = 1$ (when $\theta = 0$), $z = 1$ (when $\theta = \dfrac{\pi}{2}$), also the planes $y = -1$ and $z = -1$.

Every one of these planes is parallel to the x axis.

Every of the planes is a distance of $1$ away

from the origin, using $\dfrac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$

Conversely, if you had a plane which was parallel to the x axis, and was one unit from $(0,0,0)$, must it be one of the planes $\cos(\theta)y + \sin(\theta)z = 1$ ?

Yes! *Can you argue why?*