09/09/2020

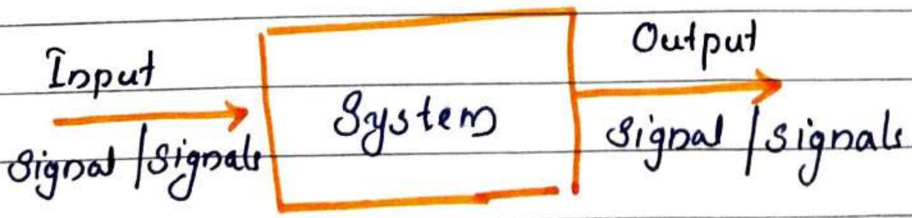
EC 2014D SIGNALS & SYSTEMS

Signals : A signal is formally defined as a function of one or more variables that conveys information on the nature of a physical phenomenon.

* 1-dimensional signals
  Eg: speech signal

* Multi-dimensional signals
  Eg: Images, video signals

System : A system is formally defined as an entity that manipulates one or more signals to accomplish a function thereby yielding new signals



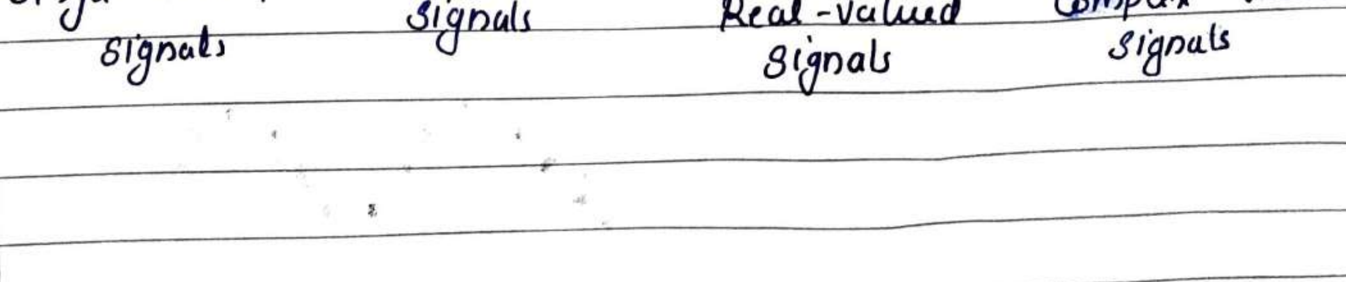
A System is characterized by its inputs, its outputs and rules of operation which describes the nature of the system.

Using these rules, we derive mathematical equations relating the inputs and outputs. These equations give mathematical model of the system.

Applications

* Machine Learning
* Communication Systems
* Control Systems
* Remote Sensing
* Image / Video Processing.
* Computer Vision
* Biomedical Signal Processing
* Artificial Intelligence
* Many more

CLASSIFICATION OF SIGNALS

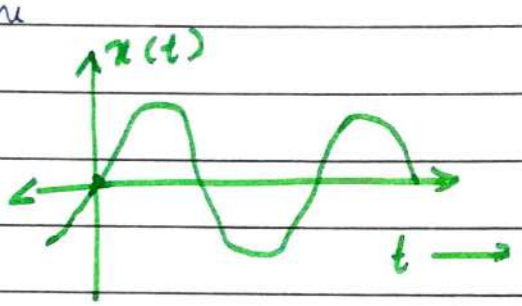


Based on different features, signals can be classified into five Categories.

1. Continous-time and Discrete-time Signals

A signal, $x(t)$ is said to be continous if it is defined for all time

Eg: $A \sin \omega t$



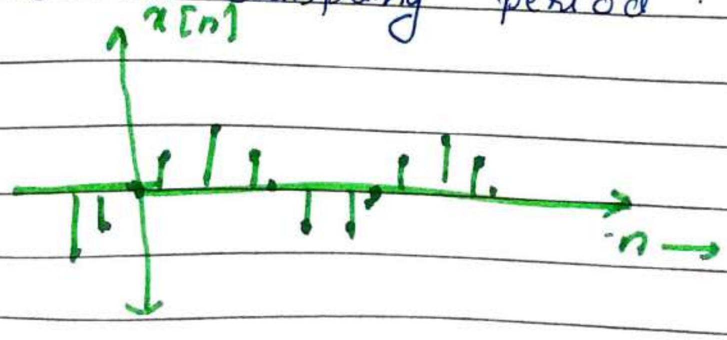
Discrete-time signals are defined only at discrete instances of time; thus the variables have discrete values, which are uniformly spaced.

* Discrete-time signal is often derived from a continous-time signal by sampling it at uniform rate.

Eg: $x[n] = A \sin(n \omega T_s)$

where $n = 0, \pm 1, \pm 2, \ldots$

and T_s is the sampling period.



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2. ## Even and Odd Signals

A continous - time signal $x(t)$ is said to be an even signal if

$$\boxed{x(-t) = x(t), \quad \forall t}$$

The signal $x(t)$ is said to be odd signal if,

$$\boxed{x(-t) = -x(t), \quad \forall t}$$

* Even signal $\longrightarrow$ symmetrical about vertical axis

* Odd signal $\longrightarrow$ asymmetrical about vertical axis

Suppose we have an arbitary signal $x(t)$.

It is possible to develop an even - odd decomposition of $x(t)$

Let us assume that,

$$x(t) = x_e(t) + x_o(t)$$

Define $x_e(t)$ to be even and $x_o(t)$ to be odd

$\therefore \quad x_e(-t) = x_e(t) \quad$ and $\cancel{x_o(t) =}$ $x_o(-t) = -x_o(t)$

Then,

$$x(-t) = x_e(-t) + x_o(-t)$$

$$= \cancel{x_e(t) + ?} = x_e(t) - x_o(t)$$

Then,

$$x_e(t) = \frac{1}{2}\left[x(t) + x(-t)\right]$$

and $$x_o(t) = \frac{1}{2}\left[x(t) - x(-t)\right]$$

Q1.

Let $$x(t) = \begin{cases} \sin\left(+\frac{\pi t}{T}\right), & -T \le t \le T \\ 0, & \text{otherwise} \end{cases}$$

Is the signal $x(t)$ an even or odd function of t.

Ans:

$$x(-t) = \begin{cases} \sin\left(\frac{-\pi t}{T}\right), & -T \le t \le T \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} -\sin\left(\frac{\pi t}{T}\right), & -T \le t \le T \\ 0, & \text{otherwise} \end{cases}$$

$$= -x(t)$$

$\therefore$ $x(t)$ is an odd signal

Q2 Find the even and odd components of the signal:

$$x(t) = e^{-2t}\cos t$$

Ans:

$$x(-t) = e^{2t}\cos(-t) = e^{2t}\cos t$$

Then, $x_e(t) = \frac{1}{2}\left[x(t) + x(-t)\right]$

$$= \frac{1}{2}\left[e^{-2t}\cos t + e^{2t}\cos t\right]$$

$$= \cosh(2t)\cos t$$

Odd signal, $x_o(t) = \frac{1}{2}\left[e^{-2t}\cos t - e^{2t}\cos t\right]$

$$= -\sinh(2t)\cos t.$$

Note: In the case of a complex-valued signal, we may speak of conjugate symmetry. A complex-valued signal $x(t)$ is said to be conjugate symmetric if

$$\boxed{x(-t) = x^*(t).}$$

Let $x(t) = a(t) + j\,b(t)$, then

$$x^*(t) = a(t) - j\,b(t) \quad \text{——— (1)}$$

$$x(-t) = a(-t) + j\,b(-t) \quad \text{——— (2)}$$

To satisfy the condition, from (1) and (2)

$$x(-t) = x^*(t)$$

$$\Rightarrow a(-t) + j\,b(-t) = a(t) - j\,b(t)$$

$$\Rightarrow a(-t) = a(t) \text{ and } b(-t) = -b(t)$$

Thus, a complex valued signal $x(t)$ is conjugate symmetric if its real part is even and its imaginary part is odd.

3. Periodic and non-periodic Signals

A continuous-time periodic signal $x(t)$ is a function of time that satisfies the condition

$$\boxed{x(t) = x(t+T), \forall t} \rightarrow (A)$$

where T is a +ve constant

The smallest value of T which satisfies the above condition is known as fundamental period, T_0

$$\therefore \quad T = T_0, 2T_0, 3T_0, \ldots$$

* T_0 denotes the duration of one complete cycle of $x(t)$.
* The reciprocal of T_0 is called the fundamental frequency of periodic signal

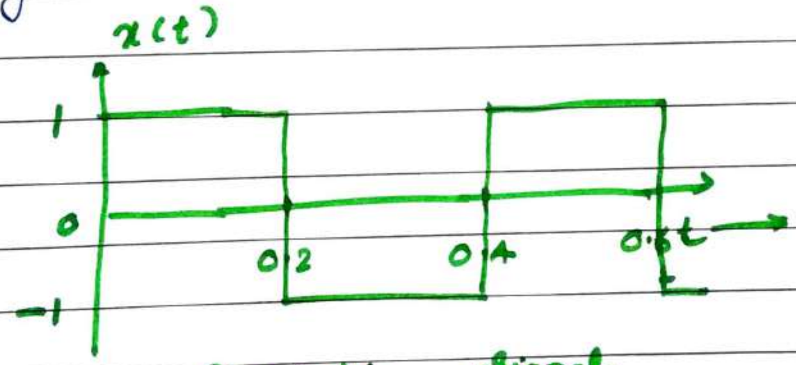
$$\boxed{f = 1/T_0 \; ; \; Hz.}$$

The angular frequency, measured in radians per second

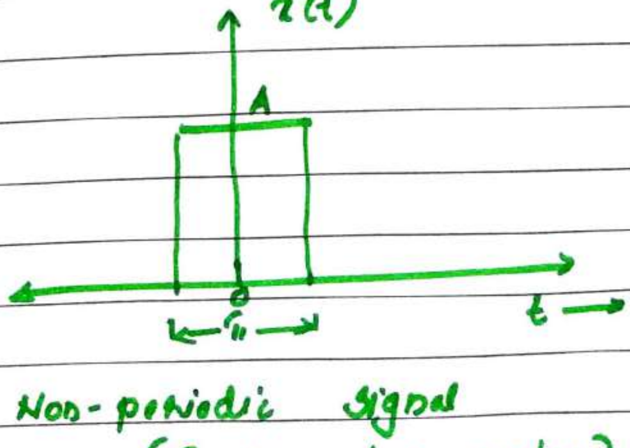
ie $\boxed{\omega = 2\pi f = \frac{2\pi}{T_0}}$, where 2π radians

represents one complete cycle.

Any signal $x(t)$ for which no value of T satisfies Eq. (A) is called aperiodic or non-periodic signal.



Periodic Signal



Non-periodic signal
(Rectangular pulse)

A discrete signal $x[n]$ is said to be periodic if $\boxed{x[n] = x[n+N]}$, where N is a +ve integer.

* The smallest N that satisfies this condition is called fundamental period of $x[n]$.

* The fundamental angular frequency is defined as,

$$\Omega = \frac{2\pi}{N} \text{ radians.}$$

4 Deterministic and Random Signals.

A deterministic signal is a signal about which there is no uncertainity with respect to its value at any instant of time.

Eg: Square wave.

A random signal is a signal about which there is an uncertainity before it occurs.

Eg: Noise signal, EEG

5. Energy Signals and Power Signals

How can a signal that exists over a certain time interval with varying amplitude be measured by one number that will indicate the signal size and strength ?.

Such a measure must consider not only the signal amplitude but also its duration.

a. Signal Energy

* Consider the area under a signal $x(t)$ as a possible measure of its size. Is it a good measure ?.

No, Even for large signal $x(t)$, its positive and negative areas cancel each other indicating a signal of small size.

* How can we solve this issue ?.

By defining the signal size as the area under $x^2(t)$, which is always positive. This is called signal energy.

Signal Energy, $\boxed{E_x = \int_{-\infty}^{\infty} x^2(t)\, dt}$

For complex valued signal, $x(t)$,

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt.$$

b, Signal Power

* The signal energy must be finite for it to be a meaningful measure of signal size.

* The sufficient condition:
$x(t) \longrightarrow 0$ as $t \longrightarrow \infty$.

* But if $x(t) \neq 0$ when $t \longrightarrow \infty$, the signal energy is infinite.

* In such cases, the time average of energy is more meaningful and is called power.

* For a signal $x(t)$, we define the power

$$P_x = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t) \, dt$$

For complex-valued signals.

$$P_x = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt$$

Mean Squared value of $x(t)$.

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* This is satisfied when the signal is having a periodical or structural similarity.

* Ramp signal $\longrightarrow$ no energy, no power
* Unit step signal $\longrightarrow$ no energy, defined power

* A signal with finite energy is called energy signal and a signal with finite and non-zero power is known as power signal.

* A signal cannot both be an energy signal or power signal.

* Periodic and random signals are viewed as power signals, whereas signals that are both deterministic and non-periodic are usually viewed as energy signals.

For discrete case,

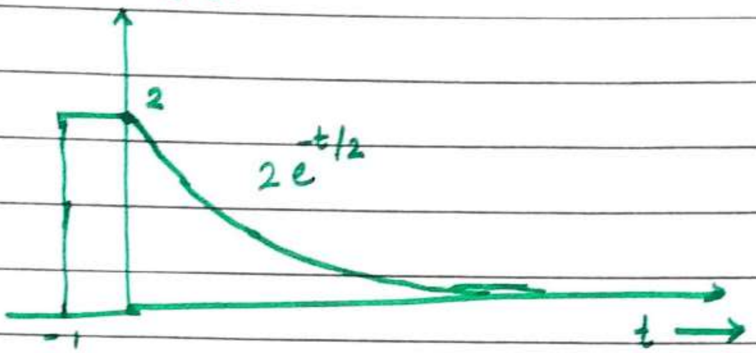
$$E_x = \sum_{n=-\infty}^{\infty} x^2[n] \quad \text{and}$$

$$P_x = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} x^2[n]$$

* ———— * ———— * ————

Q1 Determine the suitable measures of the following signals.

a)



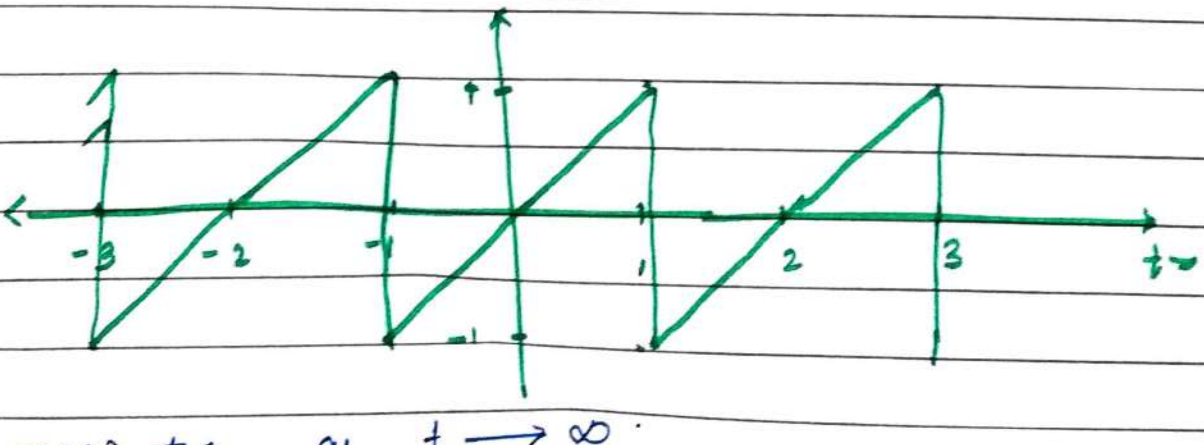
As $t \longrightarrow \infty$, $x(t) \longrightarrow 0$

Hence Energy is defined.

$$E_x = \int_{-\infty}^{\infty} (x(t))^2 dt = \int_{-1}^{0} (2)^2 dt + \int_{0}^{\infty} 4e^{-t} dt$$

$$= 4 + 4 = \underline{\underline{8}}$$

b)



$x(t) \neq 0$, as $t \longrightarrow \infty$.

However $x(t)$ is a periodic signal ; power exists

$$P_x = \frac{1}{2}\int_{-1}^{1} x^2(t) dt = \frac{1}{2}\int_{-1}^{1} t^2 dt = \underline{\underline{1/3}}$$

The RMS value of the signal is $\underline{\underline{1/\sqrt{3}}}$

Class - 2

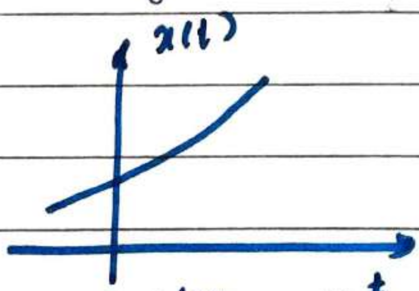
SIGNAL CLASSIFICATIONS: FEW MORE

a. **Analog and Digital Signals**

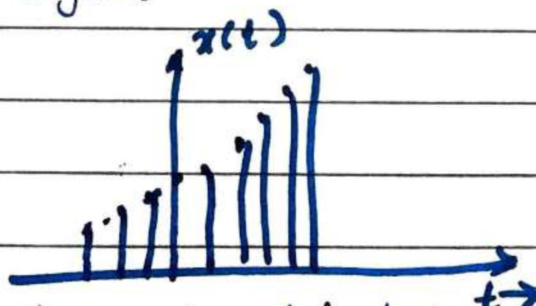
* The terms analog and digital are used to represent the nature of the amplitude (signal value).
* If the amplitude of a signal takes all possible real values in the continous range, the signal is said to be analog signal.
* On the other hand, a digital signal is one whose amplitude take some specific values.

Based upon this discussion, 4 combinations are possible.

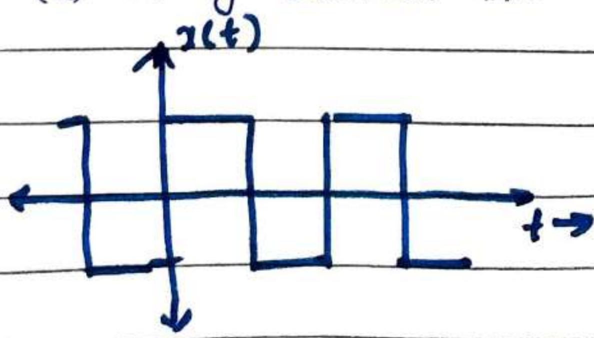
* analog, continous-time signal
* analog, discrete-time signal
* digital, continous-time signal
* digital, discrete-time signal



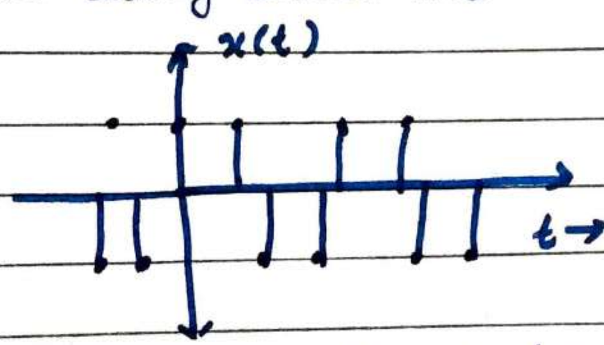
(a) analog - continous - time



(b) analog discrete-time



(c) Digital, continous-time



(d) Digital, discrete-time

b. Multichannel and Multidimensional signals.

Multichannel signals

* Signals generated by multiple sources or multiple channel are represented in vector form. The vector form of is termed as multichannel signal.

Let $x_i(t)$, $i = 1, 2, 3, \cdots$ represents signals from ith source as a function of 't'.

Vector $$X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_i(t) \end{bmatrix}$$

Eg: The output of 3-lead ECG is 3-channel signal.

Multidimensional signals.

* If a signal is function of more than one independent variable, it is called multidimensional signal.

Eg: An image is a 2D signal

Black & White TV picture is a 3D signal.

A color TV picture is 3-channel (RGB) 3D signal

$$I(x, y, t) = \begin{bmatrix} I_{red}(x, y, t) \\ I_{green}(x, y, t) \\ I_{blue}(x, y, t) \end{bmatrix}$$

C. Causal, Anticausal and Non-causal Signals

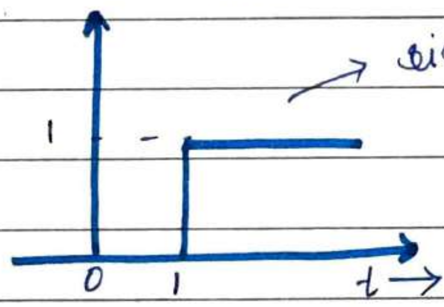
Continous - time case

* A signal $x(t)$ is said to be "causal" if
  $$x(t) = 0 \text{ for } t < 0$$
* A signal $x(t)$ is said to be "anticausal" if $x(t) = 0$ for $t > 0$
* A signal $x(t)$ is said to be "non - causal" if the signal exists for $t > 0$ as well as $t < 0$.

Discrete - time case

* $x[n]$ to be causal if $x[n] = 0$ for $n < 0$.
* $x[n]$ to be anticausal if $x[n] = 0$ for $n > 0$
* $x[n]$ to be non-causal if $x[n]$ exists for $n > 0$ as well as $n < 0$.

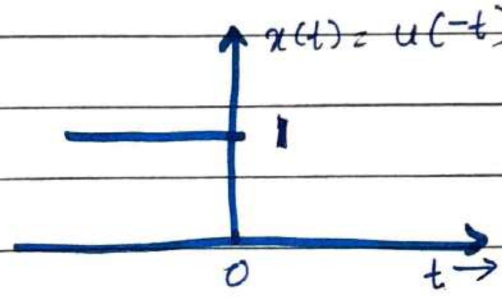
$x(t) = u(t-1)$



→ unit step signal

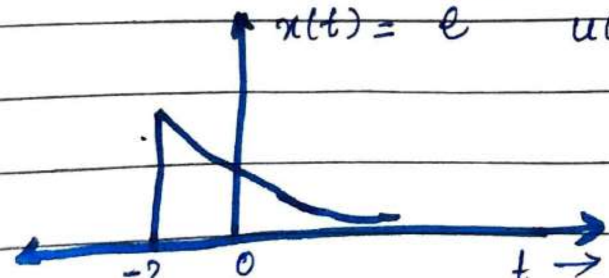
$\Longrightarrow$ causal signal

$x(t) = u(-t)$



$\Longrightarrow$ anticausal signal

$x(t) = e^{-a(t+2)} u(t+2)$



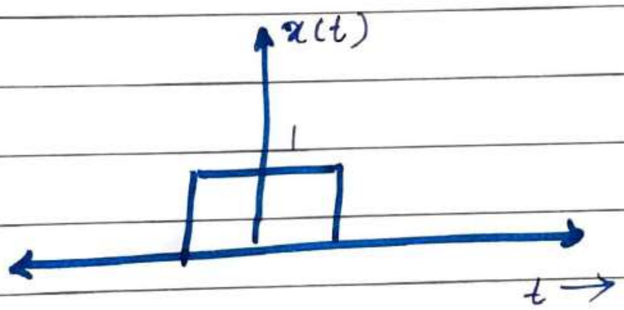
$\Longrightarrow$ Non- causal signal.

d. Bounded and Unbounded Signals

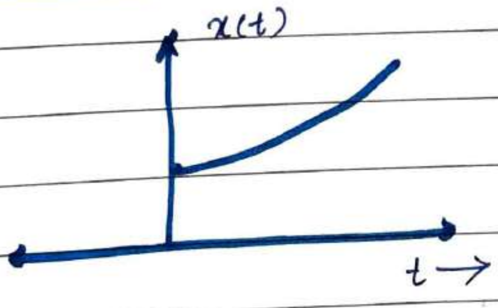
Bounded Signal

* A signal having finite value at any instant of time is called a bounded signal.
* Mathematically,

$|x(t)| \leq M$, where M is a +ve finite value, $\forall t$.



Unbounded Signal



As $t \to \infty$, $x(t) \to \infty$

CLASS - 2

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SIGNAL OPERATIONS

a. **Amplitude Scaling**

Let $x(t)$ be a continuous-time signal, then the amplitude scaled version of $x(t)$ is given as,

$\boxed{y(t) = c\,x(t)}$, where 'c' is the scaling factor

Eg: Amplifier

In discrete case, $y[n] = c\,x[n]$

b. **Addition**

Let $x_1(t)$ and $x_2(t)$ be a pair of continous time signals.

Then, $\boxed{y(t) = x_1(t) + x_2(t)}$

Eg: Audio mixer (voice + music)

Discrete case, $y[n] = x_1[n] + x_2[n]$

c. **Multiplication**

$\boxed{y(t) = x_1(t)\,x_2(t)}$ ie for each prescribed time 't', the value of $y(t)$ is given by the period of corresponding values of $x_1(t)$ and $x_2(t)$.

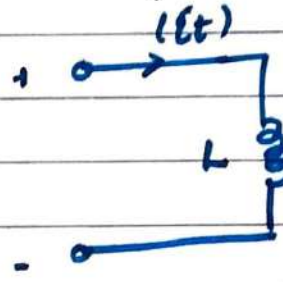
Eg: AM radio signal (Amplitude Modulation)

Discrete case, $y[n] = x_1[n]\,x_2[n]$

d. Differentiation

$$y(t) = \frac{d}{dt} x(t).$$

Eg: Inductor



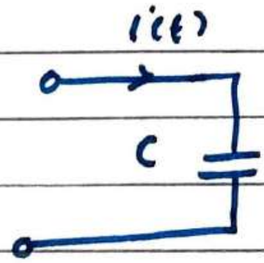
$$v_o(t) = L \frac{d\, i(t)}{dt}$$

Voltage developed across the inductor

e. Integration

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau \quad ; \quad \tau - \text{integration variable}$$

Eg: Capacitor



$$v_c(t) = \frac{1}{C} \int_{-\infty}^{t} i(\tau)\, d\tau$$

Voltage developed across the capacitor.

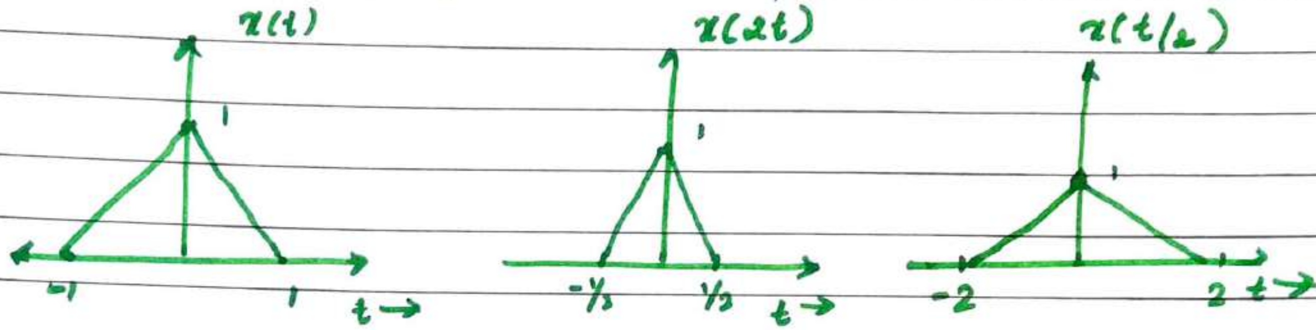
f. Time Scaling

$a > 0$

$$\boxed{y(t) = x(at)}$$

If $a > 1$, $y(t)$ is the compressed version of $x(t)$

If $0 < a < 1$, $y(t)$ is an expanded version of $x(t)$



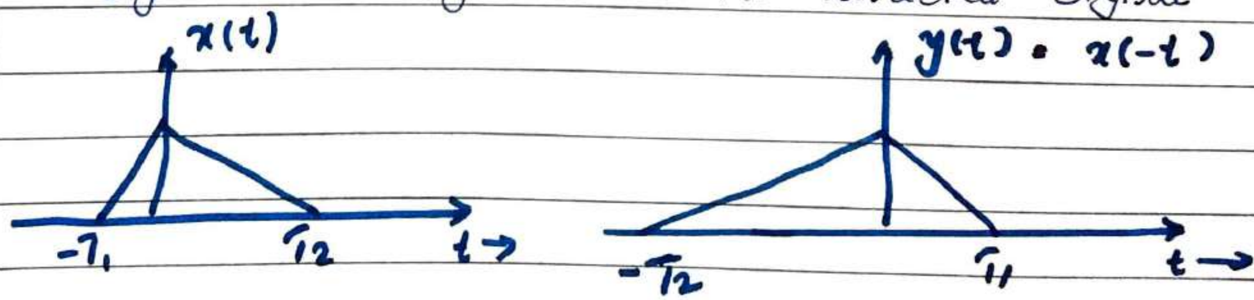
g. <u>Reflection</u>

The signal obtained by replacing 't' with '-t' in $x(t)$ is known as the reflected version of the signal.

$\boxed{y(t) = x(-t)} \Longrightarrow$ $y(t)$ represents the reflected version of $x(t)$ about $t = 0$.

<u>Note</u> * For even signals $x(t) = x(-t)$, hence even signal is same as the reflected signal

* For odd signals, $x(t) = -x(t)$, $\forall t$ ie odd signals are negative of its reflected signal



h. <u>Time Shifting</u>

The time shifted version of $x(t)$ is given as,

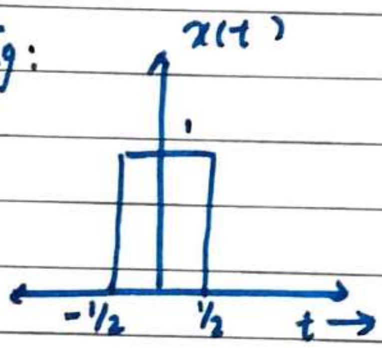
$$\boxed{y(t) = x(t - t_0)}$$

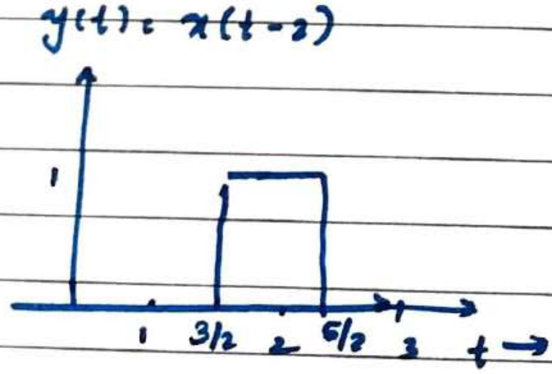
where 't_0' is the time shift.

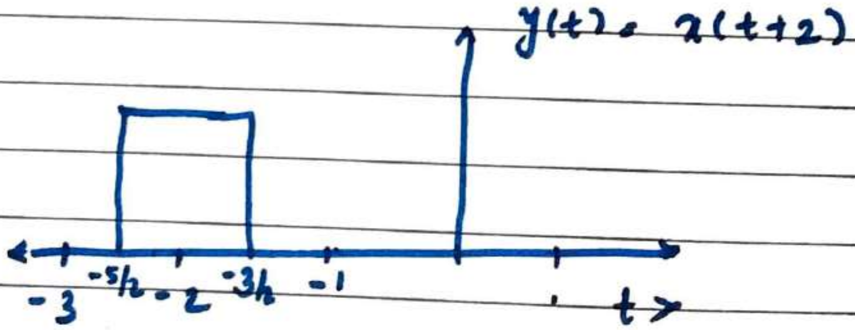
* If $t_0 > 0$, the waveform of $y(t)$ is obtained by shifting $x(t)$ towards right relative to the time axis.

* If $t_0 < 0$, $x(t)$ is shifted to the left.

Eg:







For discrete case, $y[n] = x[n-m]$

Q1.

$$x[n] = \begin{cases} 1, & n = 1, 2 \\ -1, & n = -1, -2 \\ 0, & n = 0 \text{ and } |n| > 2 \end{cases}$$

Find the time-shifted signal $y[n] = x[n+3]$.

Ans:

$$y[n] = \begin{cases} 1 ; & n = -1, -2 \\ -1 ; & n = -4, -5 \\ 0 ; & n = -3, n < -5, n > -1 \end{cases}$$

Combined Operations

The most general form of signal involving the three important operations is $\boxed{x(-at+b)}$.

(i) Method I : shift - scale - reverse
- Shifting by 'b' to obtain $x(t+b)$
- Scaling of shifted version by a to obtain $x(at+b)$.
- Reversal of 't' to obtain $x(-at+b)$

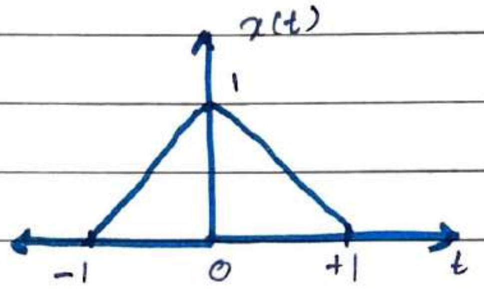
(ii) Method II : scale - shift - reverse

- Scaling by 'a' to obtain $x(at)$
- Shifting of scaled version by b/a to obtain $x\left[a(t+b/a)\right]$
- Reversal of 't' to obtain $x(-at+b)$

These two methods are valid for discrete case also.

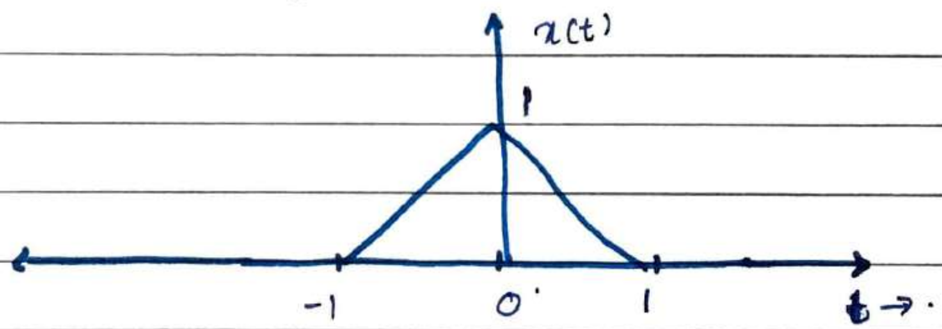
Note: Time reversal and time shifting operations are not commutative.

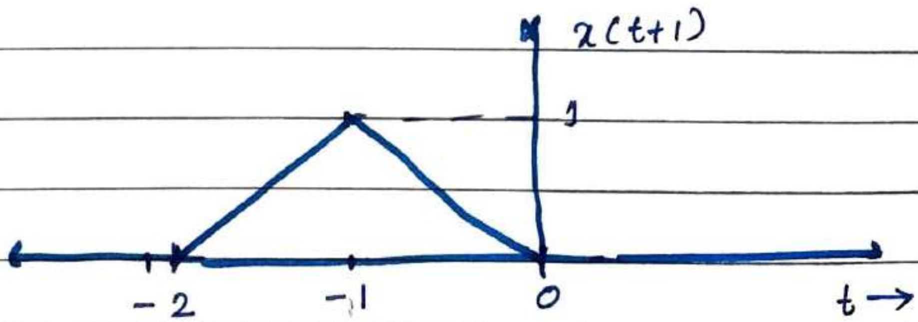
Q1. A triangular pulse is given.

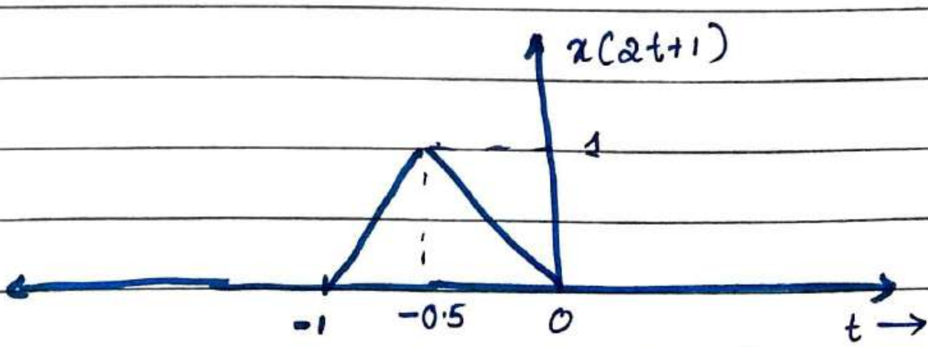


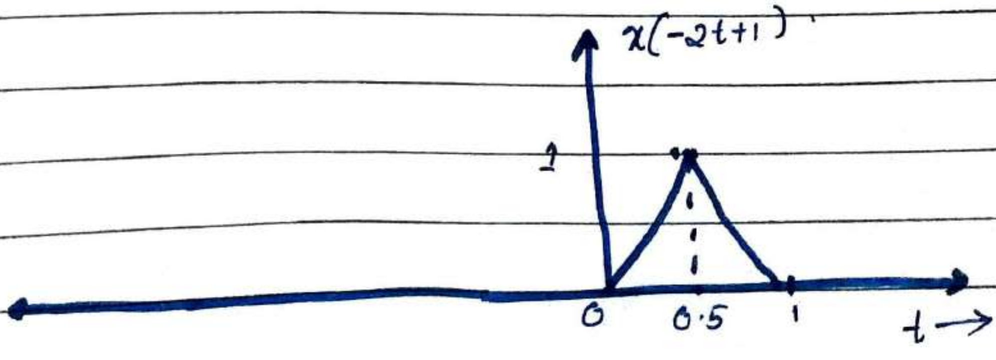
Sketch the signal $x(-2t+1)$.

Ans:

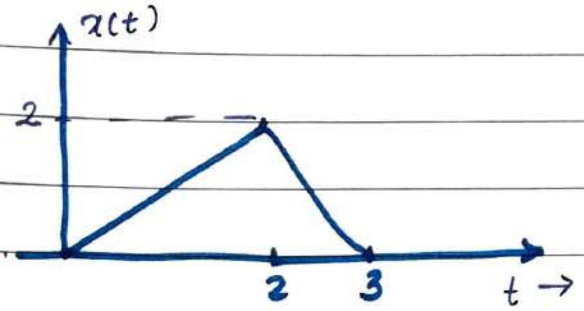






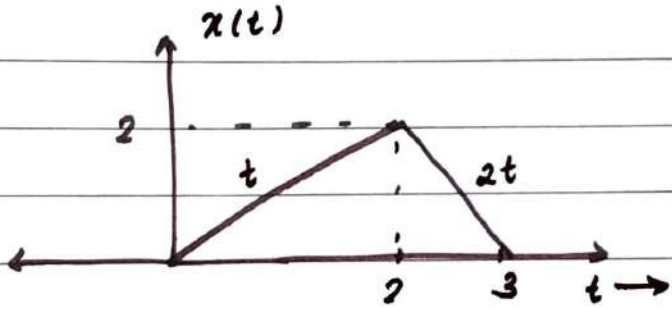


Q2. Sketch the derivative of the given signal.



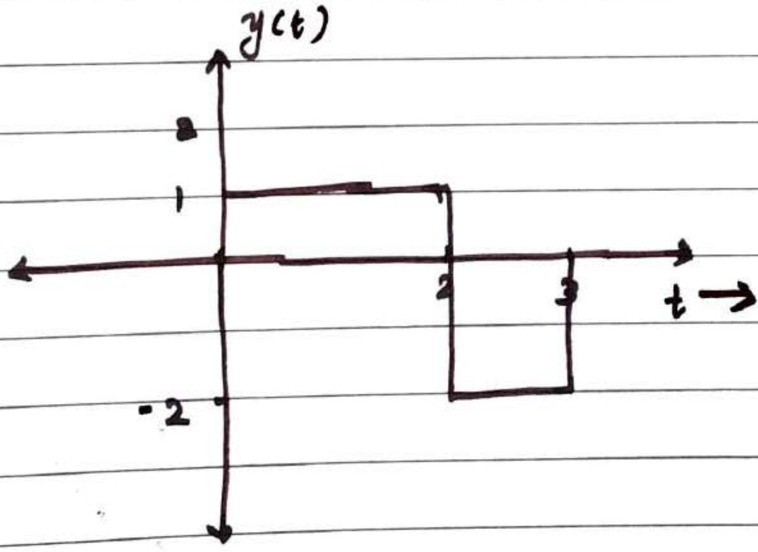
Ans:

$y(t) = \frac{d}{dt} x(t)$.



$$x(t) = \begin{cases} t \; ; & 0 \le t \le 2 \\ 2 - 2(t-2) \; ; & 2 < t \le 3 \end{cases}$$

$$= \begin{cases} t \; ; & 0 \le t \le 2 \\ 2[1-(t-2)] \; ; & 2 < t \le 3 \end{cases}$$



ELEMENTARY SIGNALS

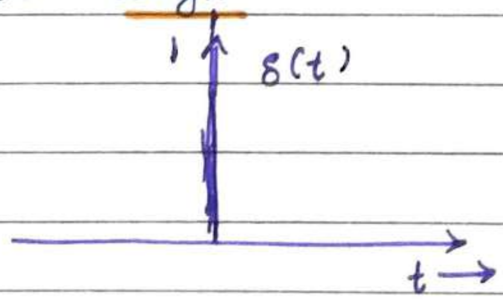
* These signals serve as basic building block for the construction of somewhat more complex signals.
* These elementary signals are also known as basic signals / standard signals.

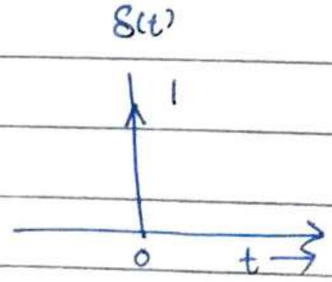
1. Unit Impulse Function

A continuous-time unit impulse function $\delta(t)$, also called dirac-delta function is defined as,

$$\boxed{\delta(t) = \begin{cases} \infty, & t=0 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t)\,dt = 1}$$

* The unit-impulse function is represented by an arrow with strength of '1' which represents its 'area' or 'weight'.

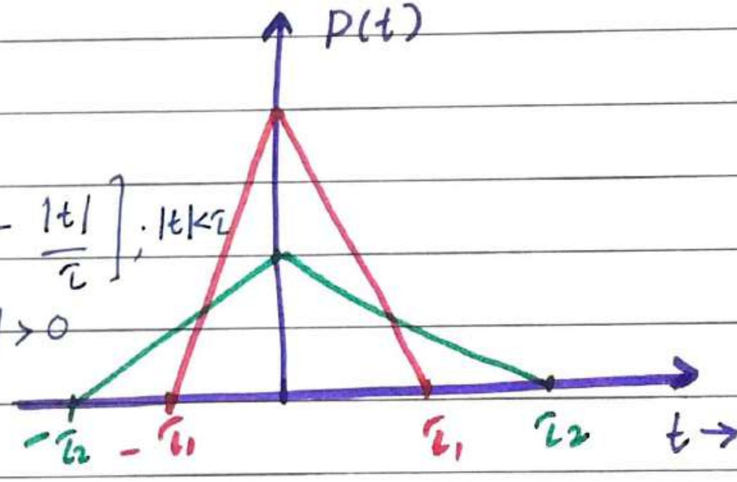




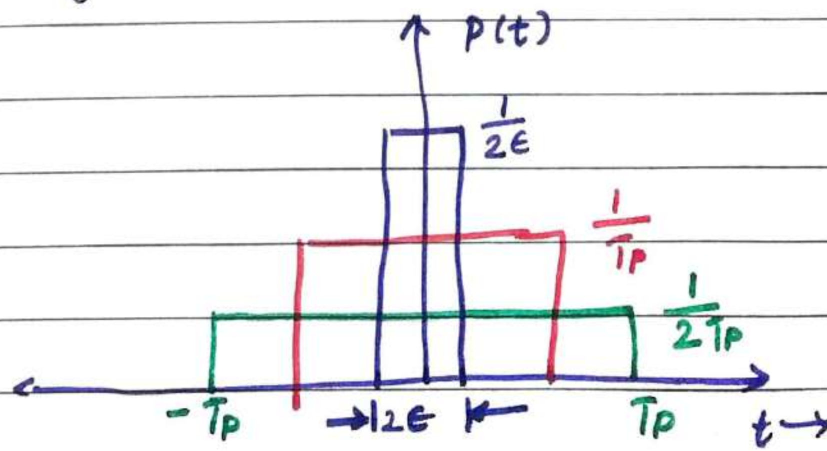
* The definition of impulse function can be generalized as the limiting case of any pulse.
* For example, impulse function can be treated as the limiting case of rectangular pulse, Gaussian pulse, etc. triangular pulse, Exponential pulse, etc.

b)

$$\delta(t) = \begin{cases} \lim\limits_{\tau \to 0} \frac{1}{\tau}\left[1 - \frac{|t|}{\tau}\right] ; & |t| < \tau \\ 0 ; & |t| > 0 \end{cases}$$



a) $\delta(t) = \lim\limits_{\epsilon \to 0} p(t)$



Properties of Continous Time Unit Impulse Function

a) Scaling Property

$$\delta(at) = \frac{1}{|a|}\,\delta(t); \quad a \text{ is a constant.}$$

Proof :

$$\delta(at) = \frac{1}{|a|}\delta(t) \quad \text{——} (A)$$

Integrating Eq (A) on both sides w.r.t 't'.

$$\int_{-\infty}^{\infty} \delta(at)dt = \int_{-\infty}^{\infty} \frac{1}{|a|}\delta(t)\,dt$$

Let $at = \tau$.

$a \cdot dt = d\tau$, 'a' is a constant, +ve or -ve.

$\therefore \ |a| \cdot dt = d\tau$.

Now, $$\int_{-\infty}^{\infty} \delta(at)dt = \int_{-\infty}^{\infty} \delta(\tau) \cdot \frac{d\tau}{|a|} = \int_{-\infty}^{\infty} \frac{1}{|a|} \delta(t) \cdot dt$$

By definition $$\int_{-\infty}^{\infty} \delta(t)dt = \int_{-\infty}^{\infty} \delta(\tau)d\tau = 1$$

Notes:

* $\delta(at \pm b) = \frac{1}{|a|}\delta\left[t \pm \frac{b}{a}\right]$
* $\delta(-t) = \delta(t)$; $\delta(t)$ is an even fn. of time

b) Product property / multiplication property

$$x(t)\,\delta(t - t_0) = x(t_0)\,\delta(t - t_0)$$

Proof :

The function $\delta(t - t_0)$ exists only at $t = t_0$.

Let the signal $x(t)$ be continous at $t = t_0$.

$$\therefore\ x(t)\delta(t-t_0) = x(t)\big|_{t=t_0} \cdot \delta(t-t_0)$$
$$= x(t_0)\,\delta(t-t_0)$$

Note: $x(t)\delta(t) = x(0)\delta(t)$

c) Sampling property

$$\int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)\,dt = x(t_0).$$

Proof:

Using the product property of impulse function
$$x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0)$$
Integrating on both sides w.r.t. 't'
$$\int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)\,dt = \int_{-\infty}^{\infty} x(t_0)\,\delta(t-t_0)\,dt$$
$$= x(t_0)\int_{-\infty}^{\infty} \delta(t-t_0)\,dt = x(t_0)$$

Note $\int_{-\infty}^{\infty} x(t)\,\delta(t)\,dt = x(0)$

d) Shifting property

According to shifting property, any signal can be produced as combination of weighted and

Shifted impulses.

$$\boxed{x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau.}$$

Proof:

Using product property.

$$x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0)$$

Replacing t_0 by τ

$$x(t)\,\delta(t-\tau) = x(\tau)\,\delta(t-\tau)$$

Integrating on both sides w.r.t. 'τ'.

$$\int_{-\infty}^{\infty} x(t)\,\delta(t-\tau)\,d\tau = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

$$\Longrightarrow \quad x(t)\int_{-\infty}^{\infty} \delta(t-\tau)\,d\tau = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

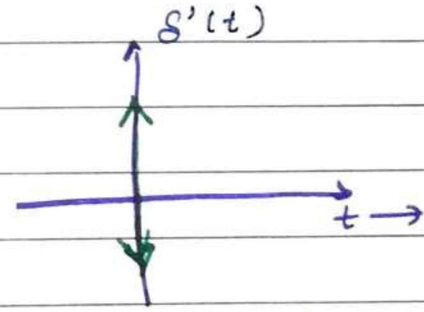
$$\Longrightarrow \quad x(t)\cdot 1 = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

$$\Longrightarrow \quad x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

e) The derivative of impulse function is known as doublet function

$$\delta'(t) = \frac{d}{dt}\delta(t).$$

Graphically,

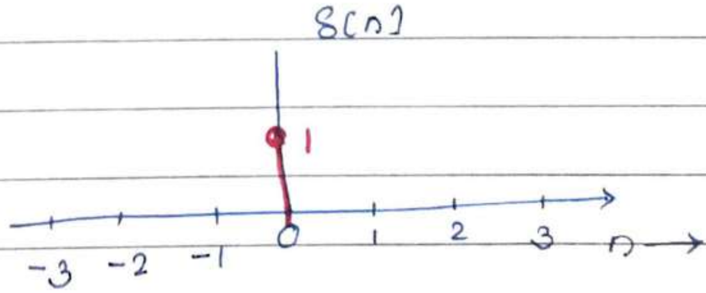


Area under the doublet function is always zero.

UNIT IMPULSE FUNCTION - DISCRETE CASE

The discrete-time unit impulse function $\delta[n]$, also called unit sample sequence or delta sequence is defined as,

$$\delta[n] = \begin{cases} 1 ; & n = 0 \\ 0 ; & \text{otherwise} \end{cases}$$



* This is also known as Kronecker Delta.

Properties

(i) Scaling property

$\delta[kn] = \delta[n]$, k is an integer

(ii) Product Property

$$x[n]\,\delta[n-n_0] = x[n_0]\,\delta[n-n_0]$$

* The impulse has a non-zero value only at $n = n_0$.

(iii) Shifting Property

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

* Any discrete-time signal can be represented as combination of weighted and shifted impulse trains.

Q1. The value of the integral $\int_{-\infty}^{\infty} \delta(t-\frac{\pi}{6})\, 6\sin t \, dt$ is ?.

Ans:

$$x(t) = \int_{-\infty}^{\infty} \delta(t-\pi/6)\, 6\sin t \; dt$$

$$= 1 \cdot 6 \cdot \sin \pi/6 \qquad \because \text{shifting property}$$

$$= 6 \times \frac{1}{2} = \underline{\underline{3}}$$

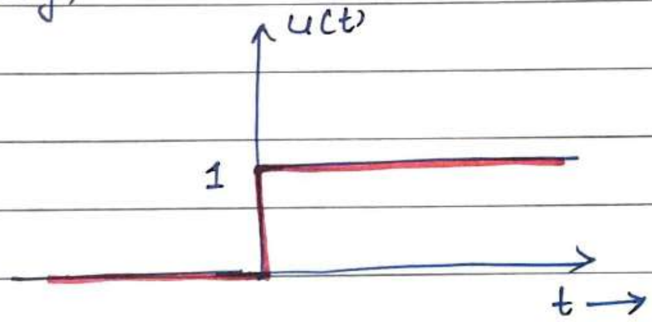
__/__/__

2. UNIT STEP FUNCTION

The continous-time unit step function, also called "Heavyside" unit function is defined as

$$\boxed{u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}}$$

Graphically,



* The function value at $t=0$ is indeterminate (discontinous) → only at $t=0$
$u(0) = 1/2$ (average value)

Properties of unit step function.

(i) The unit step function can be represented as integral of weighted, shifted impulses.

$$\boxed{u(t) = \int_0^{\infty} \delta(t-\tau)\, d\tau}$$

Proof:

According to the shifting property of impulse fns.

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\, \delta(t-\tau)\, d\tau$$

Let $x(t) = u(t)$

$$\therefore \quad u(t) = \int_{-\infty}^{\infty} u(\tau)\,\delta(t-\tau)\,d\tau$$

$$= \int_{0}^{\infty} \delta(t-\tau)\,d\tau \qquad \because \; u(\tau)=0 \; ; \; \tau<0 \quad u(\tau)=1 \; ; \; \tau \geq 0$$

(ii) Scaling Property.

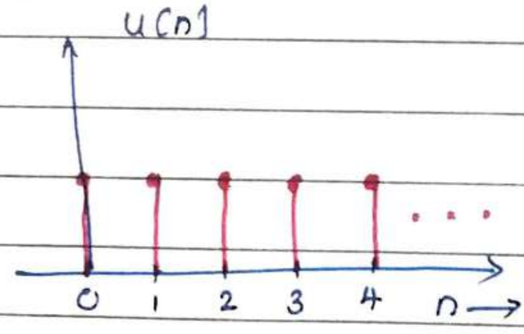
$$\boxed{u(at) = u(t),}$$

Discrete-time case

The discrete-time unit-step sequence $u[n]$ is defined as,

$$\boxed{u[n] = \begin{cases} 1 & ; \; n \geq 0 \\ 0 & ; \; n < 0 \end{cases}}$$

Graphically,



Q1. Show that the first derivative of unit step function results in unit impulse function.

$$\boxed{\delta(t) = \frac{d}{dt} u(t)} \implies \text{self study.}$$

Q2 The first difference of unit step sequence results in unit impulse sequence.

$$\boxed{\delta[n] = u[n] - u[n-1]}$$

Proof:

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k]$$

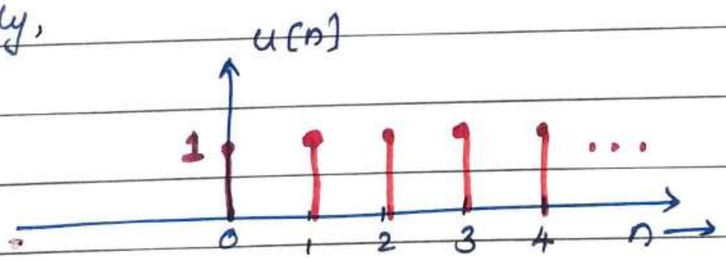
$$= \delta[n] + \sum_{k\geq 1}^{\infty} \delta[n-k]$$

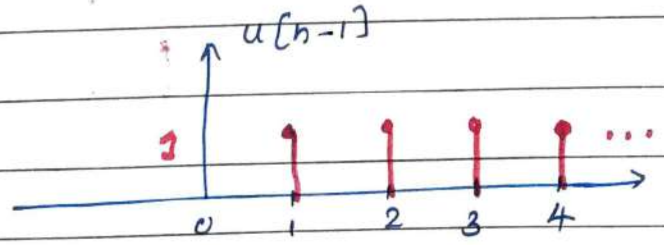
But $$u[n-1] = \sum_{k\geq 1}^{\infty} \delta[n-k]$$

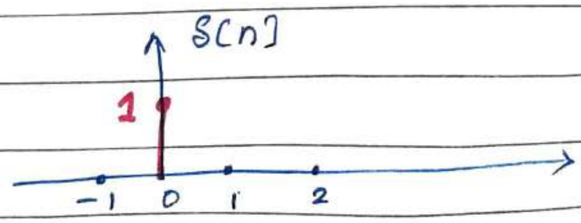
$$\therefore \quad u[n] = \delta[n] + u[n-1]$$

$$\Longrightarrow \quad \delta[n] = u[n] - u[n-1]$$

Graphically,





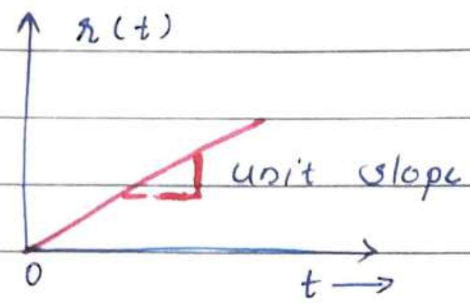


3. UNIT RAMP FUNCTION

A continuous-time unit ramp function is defined as,

$$\boxed{r(t) = \begin{cases} t \; ; & t \geq 0 \\ 0 \; ; & t < 0 \end{cases} \qquad r(t) = t\,u(t).}$$

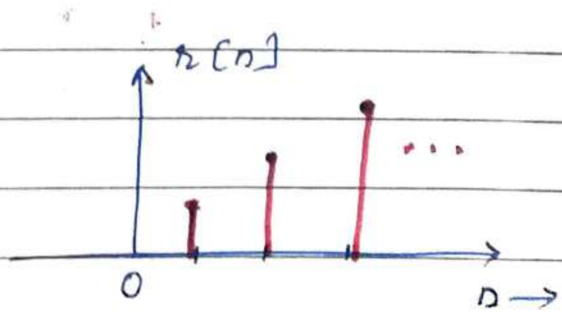
Graphically,



Discrete case

$$r[n] = \begin{cases} n \; ; & n \geq 0 \\ 0 \; ; & n < 0 \end{cases} \implies r[n] = n\,u[n]$$

Graphically,



Note:

$$r(t) = \int_{-\infty}^{\infty} u(t)\,dt = \int_{-\infty}^{t} \int_{-\infty}^{\alpha} \delta(\tau)\,d\tau\,d\alpha \implies r[n] = n\,u[n]$$

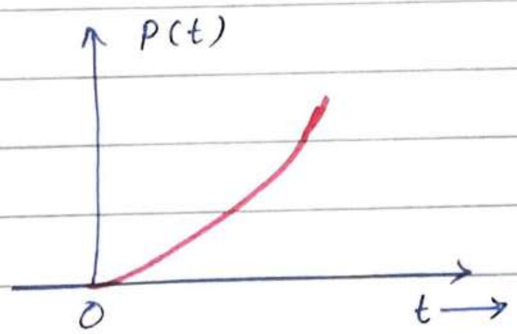
4. Unit Parabolic Function

A continous-time unit parabolic function $p(t)$ (unit acceleration function) is defined as,

$$\boxed{p(t) = \begin{cases} \frac{t^2}{2} & ; \ t \geq 0 \\ 0 & ; \ t < 0 \end{cases}}$$

Also $$p(t) = \frac{t \cdot r(t)}{2} = \frac{t^2}{2} u(t)$$

Graphically,



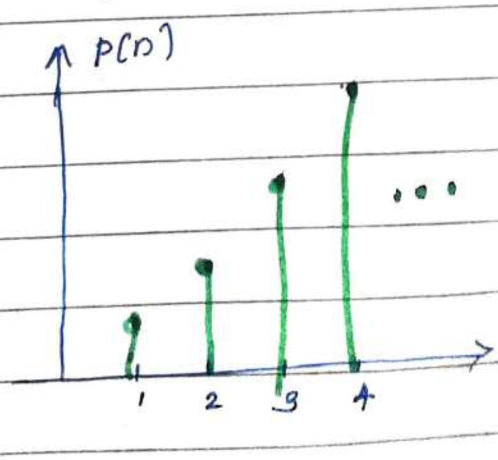
Discrete-time case

$$p[n] = \begin{cases} \frac{n^2}{2} & , \ n \geq 0 \\ 0 & ; \ n < 0 \end{cases}$$

Also

$$p[n] = \frac{n \cdot r[n]}{2} = \frac{n^2 \cdot u[n]}{2}$$

Graphically,



5. SIGNUM FUNCTION

The continuous-time signum function, $sgn(t)$ is defined as,

$$\boxed{sgn(t) = \begin{cases} 1 & ; \ t > 0 \\ -1 & ; \ t < 0 \end{cases}}$$

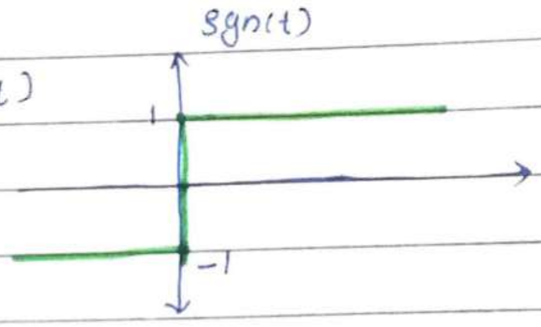
Also

$$u(t) - u(-t) = sgn(t)$$

and

$$u(t) + u(-t) = 1$$

$$\therefore \quad sgn(t) = 2u(t) - 1$$



Discrete-time case

$$sgn[n] = \begin{cases} -1 & ; \ n < 0 \\ 0 & ; \ n = 0 \\ 1 & ; \ n > 0 \end{cases}$$

Also,

$$sgn[n] = u[n-1] - u[-n-1]$$

06. EXPONENTIALS & SINUSOIDAL SIGNALS

A general form of complex exponential signal is

$$\boxed{x(t) = Ce^{\alpha t}}$$

Depending on the values of C and α, we further classify complex exponentials as,

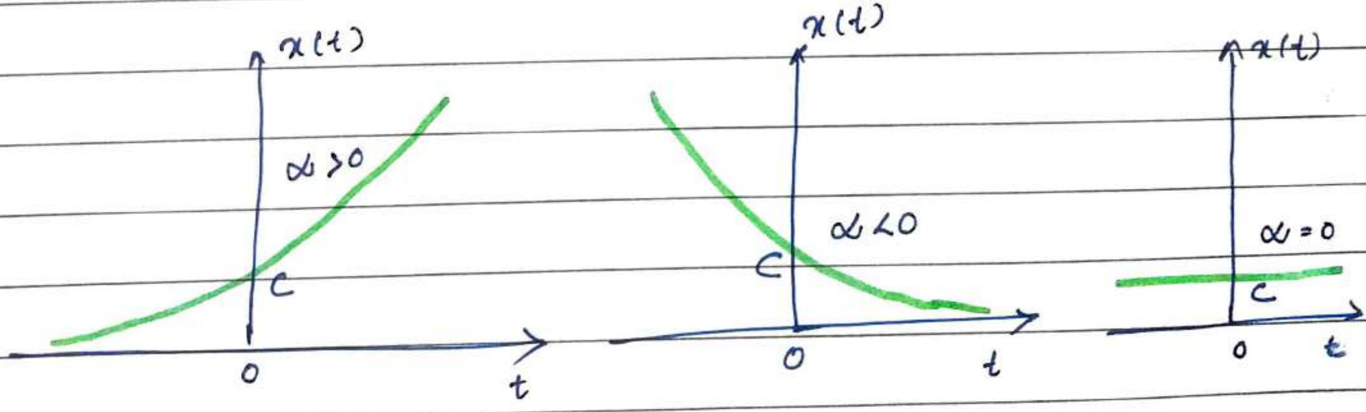
(i) Real exponential : Both C and α are real.
(ii) Periodic complex exponential : C is real and α is purely imaginary.
(iii) Sinusoidal :
(iv) For $\alpha = 0$, $x(t)$ is a constant signal.

(a) Real Exponential Signal

A continuous-time real exponential signal can be defined as,

$$\boxed{x(t) = C e^{\alpha t}} \; ; \text{ both } C \text{ and } \alpha \text{ are real}$$

* If α is +ve, $x(t)$ is growing exponential
* If α is −ve, $x(t)$ is decaying exponential
* If $\alpha = 0$, $x(t)$ is constant signal.



* The discrete-time real exponential sequence is defined as,

$$x[n] = C a^n \; , \; \forall n$$

where $a = e^{\beta}$ and β is real.

HW

Draw the signal waveforms for $x[n] = a^n$ when $a = e^{\beta}$ and β is real.

i) $0 < a < 1$

ii) $a > 1$

iii) $-1 < a < 0$

iv) $a < -1$

(b) Periodic Complex Exponential

The signal $x(t) = Ce^{\alpha t}$ with α be purely imaginary results is periodic complex exponential

ie $\boxed{x(t) = e^{j\omega_0 t}}$ with period T

where the fundamental period of $x(t)$ is T_0 and is given as,

$$T_0 = \frac{2\pi}{\omega_0}$$

Properties:

* The signal $e^{j\omega_0 t}$ is always periodic with period $T = 2\pi/\omega_0$ for any values of ω_0.
* The signal $e^{j\omega_0 t}$ and $e^{-j\omega_0 t}$ have same fundamental period.
* For $\omega_0 = 0$, $x(t)$ is constant.
* Fundamental frequency of a constant signal is zero i.e. constant signal has a zero rate of oscillation.

By using Euler's relation.

$$\boxed{e^{j\omega_0 t} = \cos\omega_0 t + j\sin\omega_0 t}$$

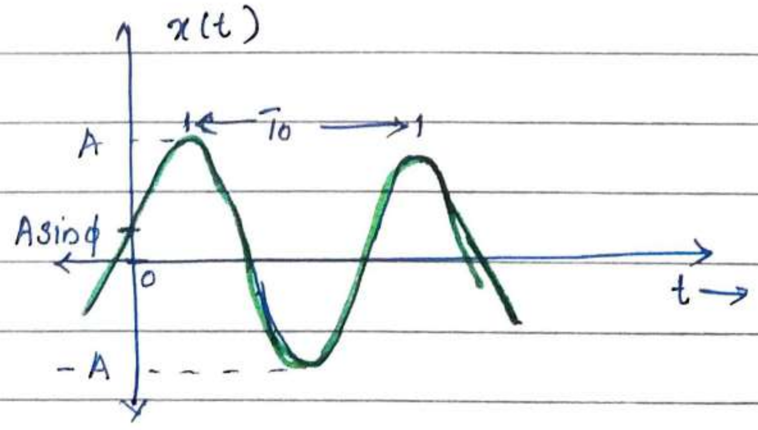
For discrete-time case,

$x[n] = e^{j\Omega_0 n}$; which periodic with period $N = 2\pi/\Omega_0$.

(c) Sinusoidal Signal

* Closely related to continuous-time periodic complex exponential signal.

$$\boxed{x(t) = A\sin(\omega_0 t + \phi)}$$; which is also periodic with fundamental period $T_0 = 2\pi/\omega_0$



For discrete case,

$x[n] = A\sin(\Omega_0 n + \phi)$

with period $N = \frac{2\pi}{\Omega} m$, where Ω is rad/sec and m is an integer.

(d) Complex exponential

The signal $x(t) = C e^{\alpha t}$ with both C and α are complex results is complex exponential.

Let $\quad C = |C| e^{j\phi} \longrightarrow$ polar form

$\qquad \alpha = \sigma + j\omega_0 \longrightarrow$ rectangular form.

Then,

$$x(t) = |C| e^{j\phi} e^{j(\sigma + j\omega_0 t)}$$

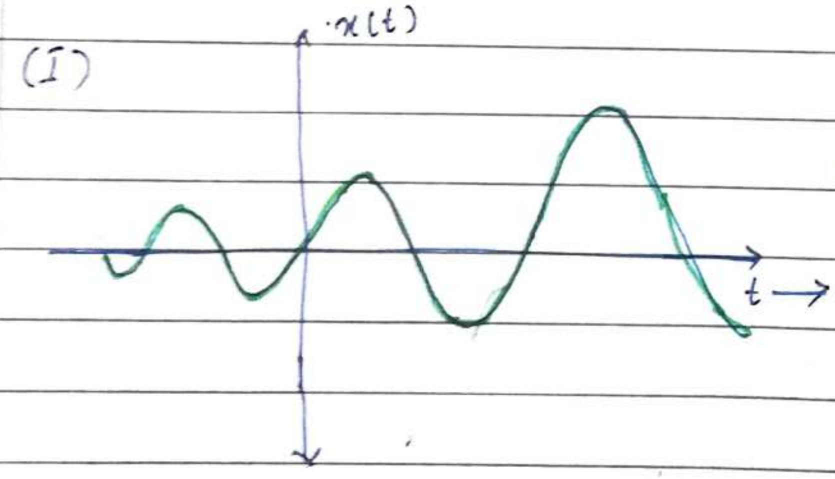
$$= |C| e^{\sigma t} \cdot e^{j(\omega_0 t + \phi)}$$

$$= |C| e^{\sigma t} \left[\cos(\omega_0 t + \phi) + j \sin(\omega_0 t + \phi) \right]$$

(I) * If $\sigma > 0$; sinusoidal signal multiplied by growing exponential.

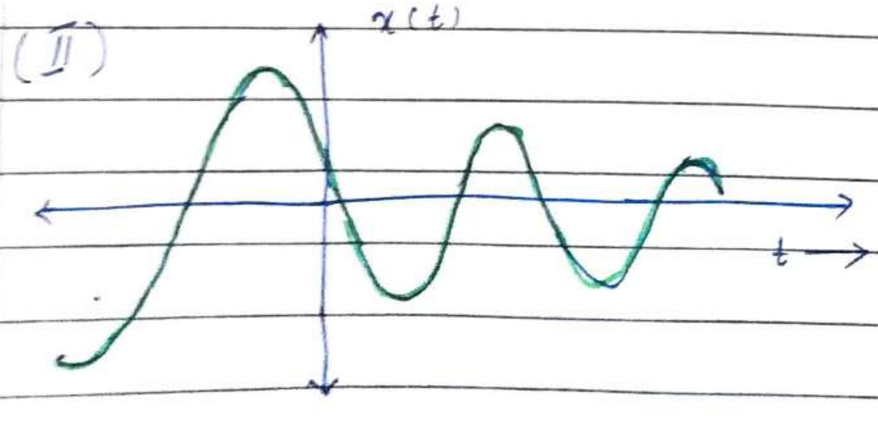
(II) * If $\sigma < 0$; sinusoidal signal multiplied by decaying exponential.

(I)



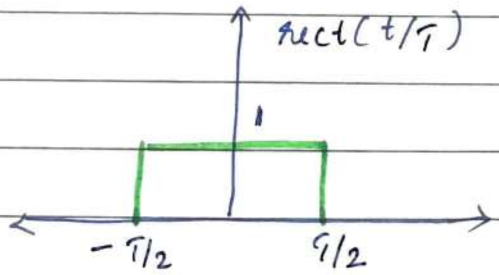
→ unbounded signal

(II)



7. Rectangular Pulse Function

$rect(t/\tau)$ is defined as



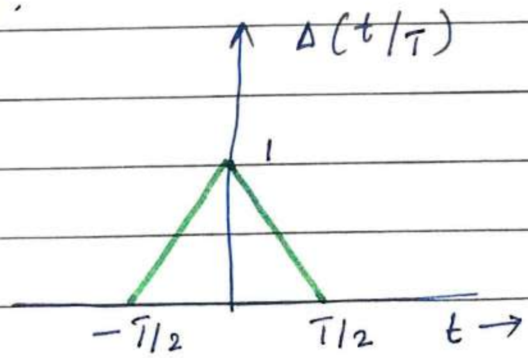
* It is an even function of time and symmetrical about amplitude axis.

8. Triangular Pulse Function

The unit triangular pulse function $\Delta(t/\tau)$ or $tri(t/\tau)$ is defined as,

$$\Delta(t/\tau) = \begin{cases} 1 - \dfrac{2|t|}{\tau} & ; \quad |t| < \tau/2 \\ 0 & ; \quad |t| > \tau/2 \end{cases}$$

Graphically.

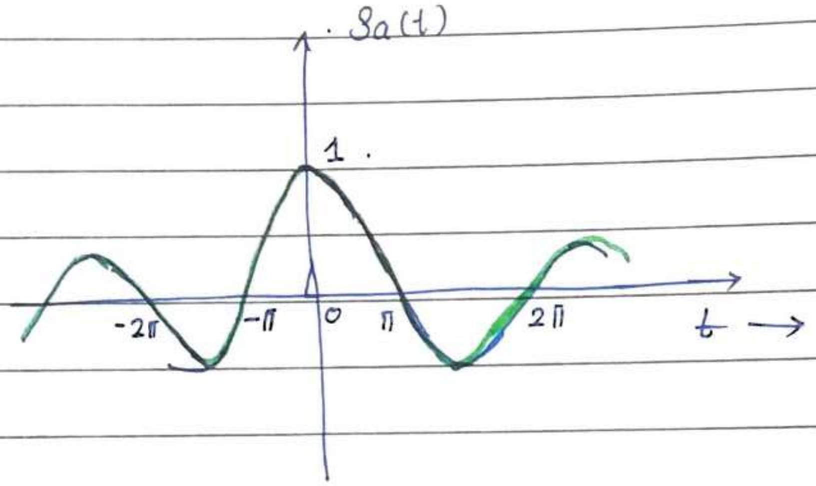


9. Sampling Function

A sampling function $Sa(t)$ is described as,

$$\boxed{Sa(t) = \frac{\sin t}{t}}, \quad \text{for } -\infty < t < \infty$$

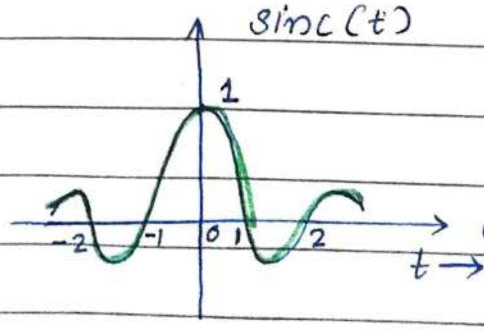
* It is a damped sine wave.



* It is an even function of time and having peak value of '1' at $t = 0$ and zero crossings at $t = \pm n\pi$

* The normalized sampling function is known as sinc function. It is the compressed version of $Sa(t)$.

$$\boxed{Sa(\pi t) = \frac{\sin \pi t}{\pi t} = \text{sinc}(t)}$$

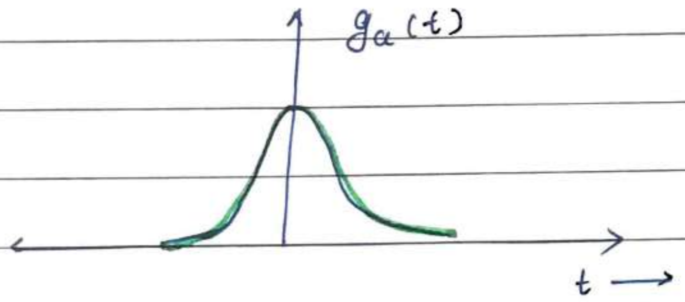


It is also known cardinal sine function.

10. Gaussian Function

The Gaussian function $g_a(t)$ is defined as,

$$g_a(t) = e^{-at^2} \; ; \; \text{for } -\infty < t < \infty$$



* This function has important role in probability and communication theory.

Self Study

Prove that $sgn(t)$ can be represented as limiting case of exponential as,

$$sgn(t) = \lim_{\alpha \to 0} \left[e^{-\alpha t} u(t) - e^{\alpha t} u(-t) \right]$$

CLASS-04

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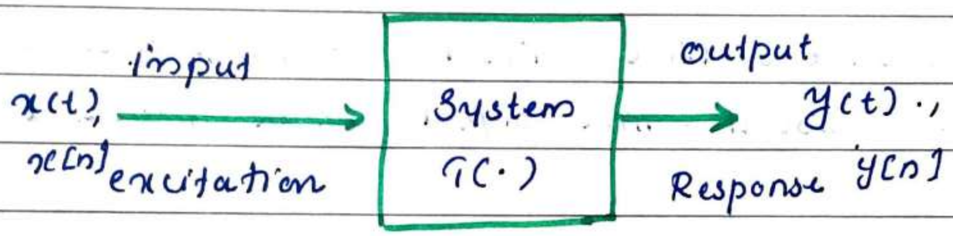
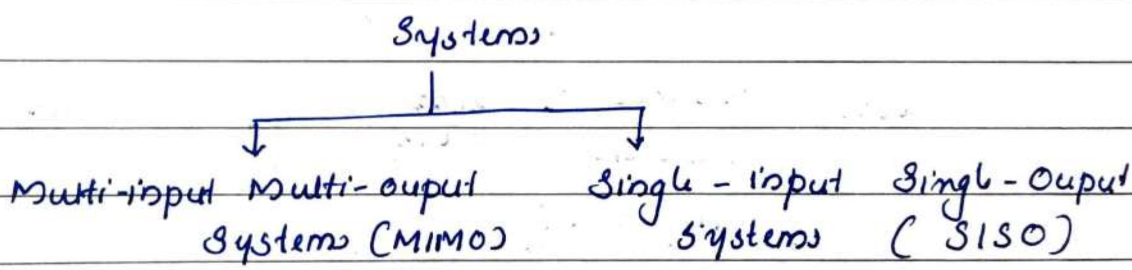
Sudhish N George, NIT Calicut

INTRODUCTION TO SYSTEMS

A system is an operator which maps the relation between input signal and output signal by the process of transformation.

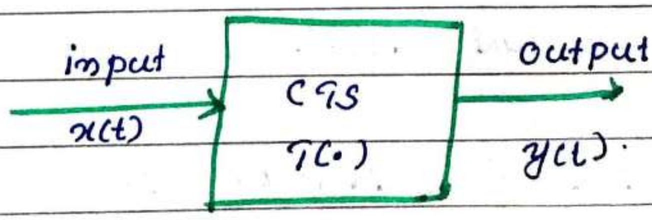
* A system is also defined as set of elements which produces expected output with available input.





Continous-time and Discrete-time Systems

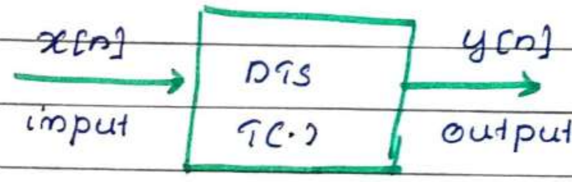
A continous-time system (CTS) is one which continous time input signals are transformed into continous time output signals



$\boxed{y(t) = T(x(t))}$

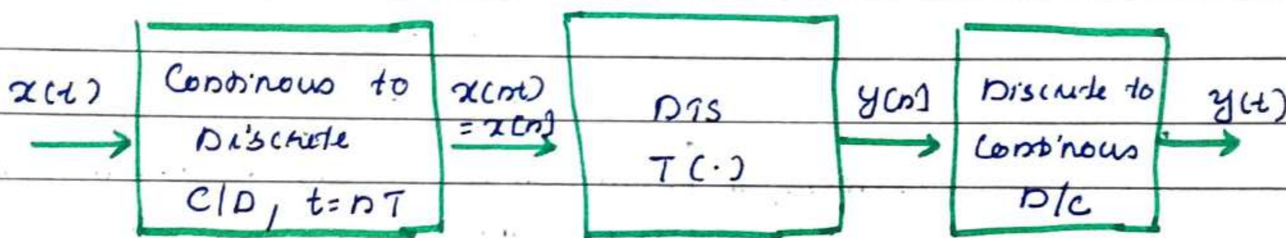
Eg: integrator, differentiator, filters, etc.

A discrete-time system (DTS) ~~it~~ is one which transforms discrete-time input signal into discrete time output signal.



$$\boxed{y[n] = T[x(n)]}$$

* A continous-time signal can be processed by DTS.
* Also discrete-time signal can be processed by CTS



CLASSIFICATION OF SYSTEMS

* Both CTS and DTS are further classified according to the way they interact with input signals.

(a) Linear and Non-Linear Systems

A system is said to be linear if it satisfies the properties

(i) additivity / superposition
(ii) homogeneity (scaling)

Additivity

It states that if an input $x_1(t)$ produces an output $y_1(t)$ and another input $x_2(t)$ also acting along produces output $y_2(t)$; then, when both the signals acting on the system together, produces the output $y_1(t) + y_2(t)$.

If $x_1(t) \xrightarrow{CTST(\cdot)} y_1(t)$ and $x_2(t) \xrightarrow{T(\cdot)} y_2(t)$
then $x_1(t) + x_2(t) \xrightarrow{S} y_1(t) + y_2(t)$

$\|^{ly}$ If $x_1[n] \xrightarrow{DTS,S} y_1[n]$ and $x_2[n] \xrightarrow{S} y_2[n]$.
then $x_1[n] + x_2[n] \xrightarrow{S} y_1[n] + y_2[n]$.

Homogeneity

It states that if the input is scaled by 'c', then output also scaled by some amount.

If $x_1(t) \xrightarrow{S} y_1(t)$
then $c\,x_1(t) \xrightarrow{S} c\,y_1(t)$
where 'c' is real / imaginary number.

$\|^{ly}$, If $x_1[n] \xrightarrow{S} y_1[n]$.
then $c\,x_1[n] \xrightarrow{S} c\,y_1[n]$

These two properties defining a linear system are combined are :

$$\boxed{\begin{aligned} a x_1(t) + b\, x_2(t) &\xrightarrow{S} a y_1(t) + b y_2(t) \\ a x_1[n] + b\, x_2[n] &\xrightarrow{S} a\, y_1[n] + b\, y_2[n] \end{aligned}}$$

Note:

Non-linear operators like sin, cos, tan, $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$, exp, log, modulus, median, square, cube, square root, $u(\cdot)$, $\mathrm{sgn}(\cdot)$, $Sa(\cdot)$, $\mathrm{sinc}(\cdot)$, etc. either on 'x' or 'y' make systems non-linear.

Q1 Check whether the system characterized by $y(t) = a x(t) + b$ is linear or not

Ans:

$$y_1(t) = a x_1(t) + b$$
$$y_2(t) = a x_2(t) + b.$$

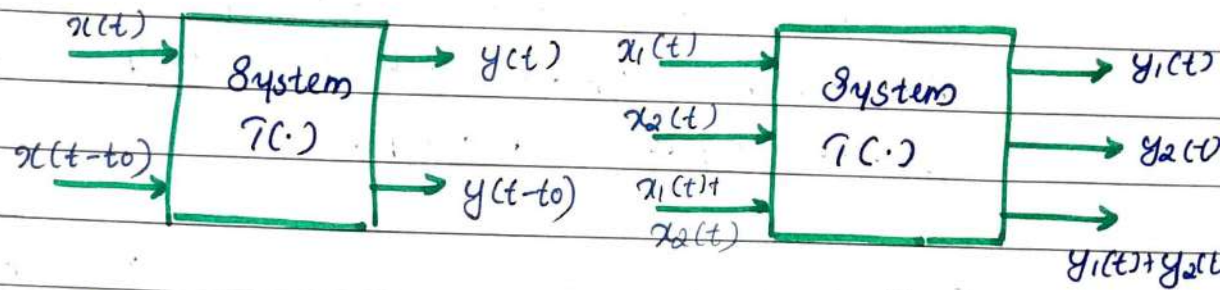
$$x_1(t) + x_2(t) \xrightarrow{S} a\big(x_1(t) + x_2(t)\big) + b$$
$$\neq y_1(t) + y_2(t)$$
$$= a x_1(t) + a x_2(t) + 2b$$

Hence the given s/m is non-linear..

(b) Time Variant and Time Invariant Systems

A s/m is time invariant if its input-output characteristics does not change with time.

In general, if the response of a system remains same irrespective of time.



(a) Time Invariant System.   (b) Linear System.

Note: If delayed input $x(t-t_0)$ produces delayed output $y(t-t_0)$, then the system is said to be time invariant.

Q1. Check whether $y(x) = x(-t)$ is time invariant or not.

Ans:

$$y(t) = x(-t).$$

Then $\quad y_1(t) = x(-t-t_0)$

Then delay O/p by t_0,

let $\quad y_2(t) = y(t-t_0) = x(-(t-t_0))$

$$= x(-t+t_0)$$

$$\therefore \quad y_1(t) \neq y_2(t)$$

Hence the system is time variant.

(C) Causal and Noncausal Systems

A system is causal if the response depends on present inputs, past inputs and past outputs but not on future inputs and outputs.

* Response of non-causal system also depends on future input along present and past input.

Q1 Eg: $y_1[n] = \frac{1}{3}\left[x[n] + x[n-1] + x[n-2]\right]$ → causal system.
(already stored i/ps.

$y_2[n] = \frac{1}{3}\left[\underbrace{x[n+1]}_{\text{future input}} + x[n] + x[n-1]\right]$ → non-causal s/m.

Note: Causality is a mandatory requirement for operating a system in real time.

Q1. Check whether the following systems are causal.
(a) $y(t) = x(\sin t)$ (b) $y(t) = \sin(x(t))$

Ans: (a) $y(t) = x(\sin t)$

Non-causal → Reason → self study.

(Hint : Put $t = -\pi$ and verify the result)

(b) $y(t) = \sin[x(t)]$

Causal as output depends on present input only.

(d) Static and Dynamic System.

A system is static if the present output depends on present input only but not on past or future inputs

* In general, system without memory are called as static or memoryless system.

* Systems having memory are called as dynamic system.

Note: All static systems are causal, but not vice-versa.
All differential terms are corresponding to energy storing elements so they are dynamic.

(e) Stable and unstable System

BIBO stability: If bounded input results in bounded output the system is said to be stable.

Mathematically,

If $|x(t)| \leq M_x < \infty$

then $|y(t)| \leq M_y < \infty$

Then, the system is said to be stable.

Note:

* $u(t)$, is the bounded input in time domain.
* Time operations like scaling, reversal, and shifting do not affect the stability of the system.

Q1 Check whether the system is stable or not.

$y(t) = t\,x(t)$

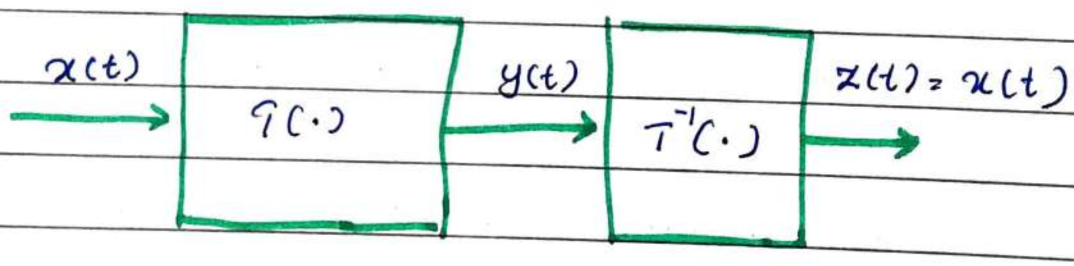
Ans: Let $x(t) = u(t)$ (bounded input)

Then $y(t) = t\,u(t) = r(t)$ (unbounded output)

$\therefore$ It is unstable.

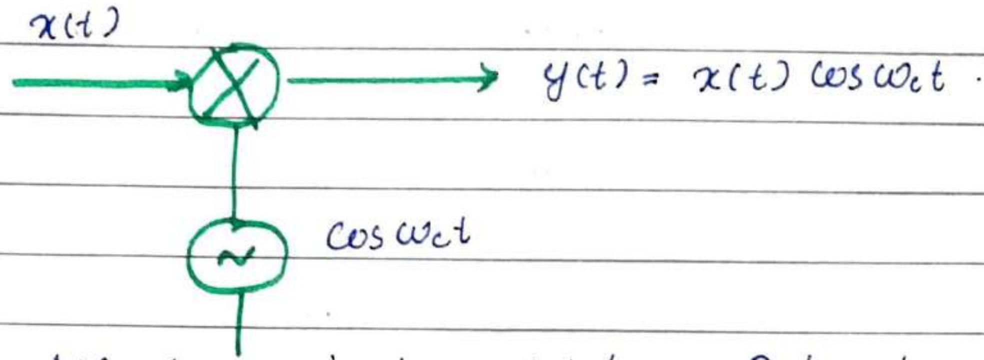
(f) Invertible and Non-invertible Systems

If input can be recovered from the system output, the system is said to be ~~stable~~ invertible.



i.e.
$$y(t) = T(x(t))$$
$$z(t) = T^{-1}(y(t))$$
$$= T^{-1}[T(x(t))]$$
$$= x(t)$$

HW



See the AM communication system. Determine whether the given system is (a) memoryless (b) causal (c) linear (d) time-invariant and (e) stable.

LINEAR TIME-INVARIANT (LTI) SYSTEMS

Systems with two basic properties such as linearity and time invariance are known as LTI systems.

These systems are preferred because of two major reasons.

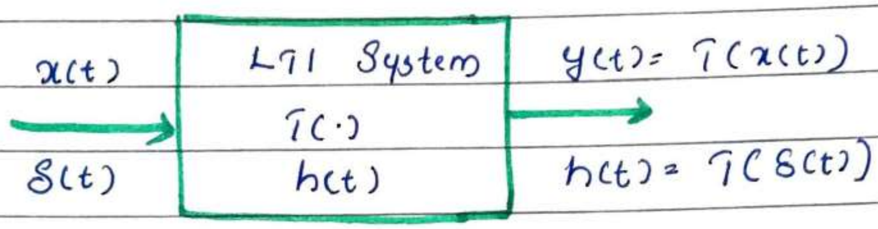
(i) many physical systems possess these properties.
(ii) any LTI system can be represented in terms of impulses.

* LTI systems are defined in terms of impulse response.

Impulse response $\longrightarrow$ Response given by the system when excited with impulse function.

CONTINOUS - TIME LTI SYSTEMS

The Convolution Integral



Mathematically,

Let $x(t) = \delta(t)$. $\rightarrow$ impulse input

Then, $y(t) = h(t) = T(\delta(t))$ $\rightarrow$ impulse response

* The complete characterization of continous-time LTI system is done by unit impulse response.

* According to shifting property

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\, \delta(t-\tau)\, d\tau$$

$$y(t) = T(x(t)) = \int_{-\infty}^{\infty} x(\tau)\, T[\delta(t-\tau)]\, d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$\boxed{\therefore \quad y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau}$$

is known as convolution integral.

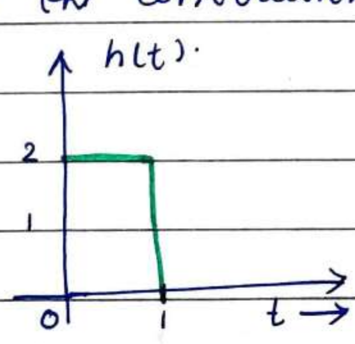
It is symbolically represented as,

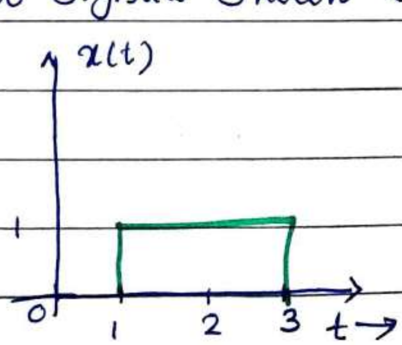
$$\boxed{y(t) = x(t) * h(t)}$$

Methods to perform convolution
(i) Graphical method
(ii) Analytical method.

(a) Steps to perform graphical convolution.
(1) Obtain the limits of $y(t)$:
Sum of lower limits $\leq t \leq$ Sum of upper limits.
(2) Change the axis from 't' to 'τ':
(3) Folding or flipping (either $x(-\tau)$ or $h(-\tau)$)
(4) Shifting (either $x(t-\tau)$ or $h(t-\tau)$)
(5) Multiplication of given signals
($x(t)h(t-\tau)$ or $h(\tau)x(t-\tau)$)
(6) Integrate with limits of $y(t)$.

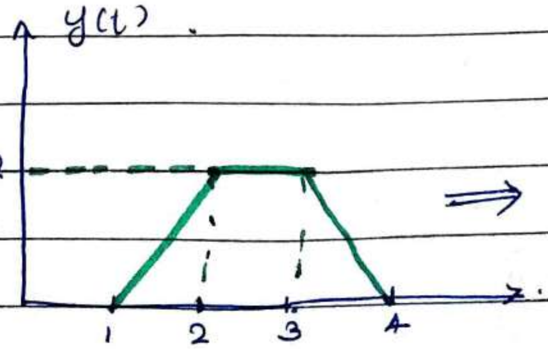
Q1. Find the convolution of two signals shown below





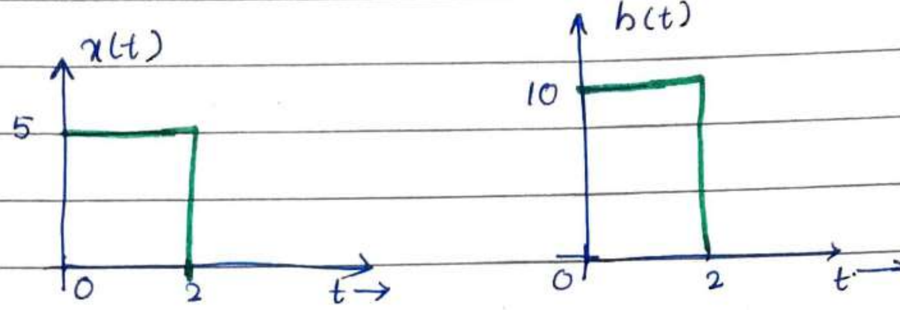
Ans: Self study.

$$y(t) = \begin{cases} 0 & 0 \leq t \leq 1 \\ 2t-2 & 1 < t < 2 \\ 2 & 2 < t < 3 \\ 8-2t & 3 < t < 4 \\ 0 & t > 4 \end{cases}$$



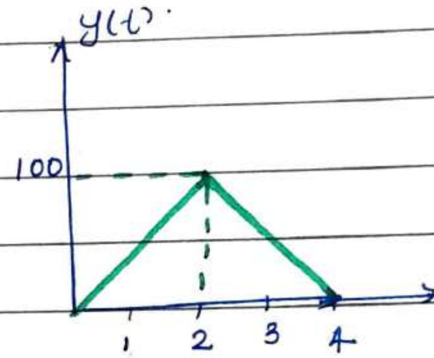
$\Longrightarrow$ trapezoidal wave.

Q2 Find the convolution between the given signals



Ans: Self study.

$$y(t) = \begin{cases} 50t & 0 \le t \le 2 \\ -50t + 200 & 2 \le t \le 3 \\ 0 & t > 4 \end{cases}$$



$\Longrightarrow$ Triangular wave

Properties of Continuous-time Convolution

(i) Commutative Property

It states that response of an LTI system is not affected even if the position of the input and the impulse response of an LTI system are interchanged.

$$\boxed{x(t) * h(t) = h(t) * x(t)}$$

Proof: $x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$

let $t - \tau = z$; thus $\tau = t - z$; $d\tau = -dz$

Then $x(t) * h(t) = \int_{+\infty}^{-\infty} x(t-\tau)\, h(\tau)\, (-d\tau)$

$$= \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau = h(t) * x(t).$$

(ii) Distributive Property

It states that any number of LTI s/ms are connected in parallel can all be combined into a single s/m with an impulse response which is equal to sum of all the individual responses.

$$\boxed{x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t)}$$

Proof: $x(t) * (h_1(t) + h_2(t)) = \int_{-\infty}^{\infty} x(\tau)\, [h_1(t-\tau) + h_2(t-\tau)]\, d\tau$

$$= \int_{-\infty}^{\infty} x(\tau)\, h_1(t-\tau)\, d\tau + \int_{-\infty}^{\infty} x(\tau)\, h_2(t-\tau)\, d\tau$$

$$= x(t) * h_1(t) + x_2(t) * h_1(t)$$

(iii) Associate Property

It states that any number of LTI s/ms connected in cascade can all be combined in single s/m whose impulse response is equal to convolution of individual responses.

$$\boxed{x(t) * [h_1(t) * h_2(t)] = [x(t) * h_1(t)] * h_2(t)}$$

Proof : $x(t) * [h_1(t) * h_2(t)]$

$$= x(t) * \left[\int_{-\infty}^{\infty} h_1(\tau_1) h_2(t-\tau_1) d\tau_1 \right]$$

$$= \int_{-\infty}^{\infty} x(\tau_2) \left[\int_{-\infty}^{\infty} h_1(\tau_1) h_2(t-\tau_1-\tau_2) d\tau_1 \right] d\tau_2$$

Let $\tau_1 + \tau_2 = \alpha$, $\Rightarrow \tau_1 = \alpha - \tau_2$

$\Rightarrow d\tau_1 = d\alpha$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[x(\tau_2) h_1(\alpha - \tau_2) d\tau_2 \right] h_2(t-\alpha) d\alpha$$

$$= \int_{-\infty}^{\infty} \left[x(\alpha) * h_1(\alpha) \right] h_2(t-\alpha) d\alpha$$

$$= \left[x(t) * h_1(t) \right] * h_2(t).$$

(iv) Convolution with an impulse

* The convolution of a function $x(t)$ with unit impulse results in function $x(t)$ itself.

$$\boxed{x(t) * \delta(t) = x(t)}$$

Proof :

$$x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau) \delta(t-\tau) d\tau$$

from sampling property of unit impulse

$$x(t)\,\delta(t-\tau) = x(\tau)\,\delta(t-\tau)$$

$$\therefore \quad x(t) * \delta(t) = \int_{-\infty}^{\infty} x(t)\cdot\delta(t-\tau)\,d\tau$$

$$= x(t)\int_{-\infty}^{\infty} \cancel{\delta(t-\tau)\,d\tau}$$

$$= x(t)\cdot 1 = \underline{\underline{x(t)}}$$

(v) Time shifting property

If $x(t) * h(t) = y(t)$

then
$$x(t) * h(t-t_0) = y(t-t_0)$$
$$x(t-t_0) * h(t) = y(t-t_0)$$
$$x(t-t_1) * h(t-t_2) = y(t-t_1-t_2)$$

Note:
$$x(t) * \delta(t-t_0) = x(t-t_0)$$
$$x(t-t_0) * \delta(t-t_1) = x(t-t_0-t_1)$$

(vi) Time Scaling Property

$$x(at) * h(at) = \frac{1}{|a|}\,y(at)$$

Proof: $x(at) * h(at) = \int_{-\infty}^{\infty} x(a\tau)\,h(a(t-\tau))\,d\tau$

Put $a\tau = z$ ~~ad~~ and $a\,d\tau = dz$

$$\therefore \quad x(at) * h(at) = \int_{-\infty}^{\infty} x(z)\,h(at-z)\,dz/a$$

$$= \frac{1}{a}\,y(at)$$

Since 'a' may be +ve or negative, so general expression given as,

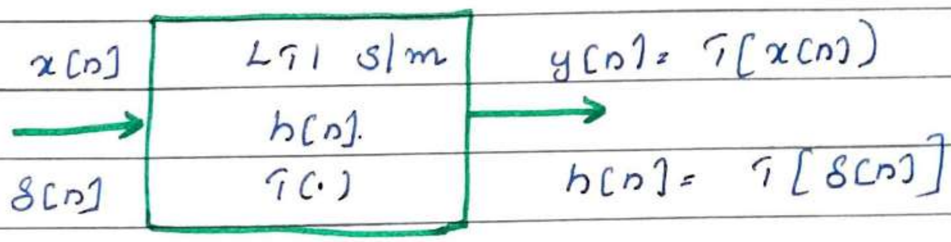
$$x(at) * h(at) = \frac{1}{|a|} y(at)$$

Note: $x(-t) * h(-t) = y(-t)$

Important Points

1. Convolution of two even signal or convolution of two odd signals always results in even signal.
2. Convolution of odd signal and even signal always results in odd signals.
3. The ~~to~~ width of resultant of convolution is sum of width of individual signals to be convolved.
4. The area of resultant of convolution is equal to the product of areas of individual signals being convolved.
5. When two signals are convolved, the lower limit for resultant of convolution is equal to sum of lower limit of individual signals and same is the case of upper limits.
6. Convolution of two causal signals always results in another causal signal.

Discrete-Time LTI Systems



Mathematically,

$$x[n] = \delta[n] \longrightarrow \text{impulse function.}$$

$$y[n] = h[n] = T[\delta[n]]$$

According to shifting property; any signal can be produced as combination of weighted and shifted impulses, that is

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

$$y[n] = T\left[x[n]\right] = \sum_{k=-\infty}^{\infty} x[k]\,T\left[\delta[n-k]\right]$$

$$\boxed{y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]}$$ is known as convolution sum

$$y[n] = x[n] * h[n] = h[n] * x[n]$$

$$= \sum_{k=-\infty}^{\infty} x[k]\,h[n-k] = \sum_{k=-\infty}^{\infty} x[n-k]\,h[k]$$

Methods to perform discrete-time convolution

Graphical method | Analytical method

Graphical Method

1. Represent graphically if signals are given in the form of equation.
2. Obtain the limits of $y[n]$.

   sum of lower limit $\le n \le$ sum of upper limit
3. Change the axis from 't' to 'k'.
4. Folding ($x[-k]$ or $h[-k]$)
5. Shifting (either $x[n-k]$ or $h[n-k]$)
6. Evaluation: Get $x[k]\,h[n-k]$ by varying 'k' and summaries all the results and sketch the results.

HW

Given $x[n] = a^n u[n]$; $0 < a < 1$ and $h[n] = u[n]$, sketch the output.

Ans.

$$y[n] = \begin{cases} 0 & ; \; n < 0 \\ \dfrac{1-a^{n+1}}{1-a} & ; \; n \ge 0 \end{cases}$$

Properties:

(i) $x[n] * h[n] = h[n] * x[n]$

(ii) $x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n]$

(iii) $x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$

(iv) $x[n] * \delta[n] = x[n]$ (v) $x[n-n_1] * h[n-n_2] = y[n-n_1-n_2]$

__/__/____

LTI System Properties

(a) LTI systems with and without memory

A system is said to be memoryless if its output at any time depends only on the values of the input at the same time.

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

$$= \cdots + h[-2]\, x[n+2] + h[-1]\, x[n+1] + h[0]\, x[n]$$

$$+ h[1]\, x[n-1] + h[2]\, x[n+2] + \cdots$$

For a system to be memoryless, $y[n]$ must depend only on $x[n]$ and the only way this can be true if $h[k] = 0$ for $k \neq 0$; thus discrete-time LTI system is memoryless if and only if

$$h[n] = k\,\delta[n] \quad \text{with } k = h[0]$$

Then, the convolution sum

$$y[n] = x[n] * k\,\delta[n]$$

$$\boxed{y[n] = k\,x[n]}$$

Similarly a continuous-time LTI system is memoryless if

$$h(t) = k\,\delta(t)$$

$$h(t) = 0 \text{ for } t \neq 0$$

$\therefore$ The o/p is $\boxed{y(t) = k\,x(t)}$

(b) Causality of LTI System

The output of a causal system depends only on the present and past values of the input.

The convolution sum

$$y(n) = \sum_{k=-\infty}^{-1} h[k]\,x[n-k] + h[0]\,x[n] + \sum_{k=1}^{\infty} h[k]\,x[n-k]$$

future input | present input | past input

We see that past and present values of input are associated with $k \geq 0$ in $h[k]$ and the future values with $k < 0$.

For a causal system, we require $h[k] = 0,\ k < 0$.

∴ For a discrete-time causal system,

$$\boxed{y[n] = \sum_{k=-\infty}^{n} x[k]\,h[n-k] = \sum_{k=0}^{\infty} h[k]\,x[n-k]}$$

For continuous-time causal system,

$$h(t) = 0 \quad ; \text{ for } t < 0$$

$$\therefore\ y(t) = \int_{0}^{\infty} h(\tau)\,x(t-\tau)\,d\tau = \int_{-\infty}^{t} x(\tau)\,h(t-\tau)\,d\tau$$

(C) Stability of LTI Systems

A systems is stable if every bounded input results in bounded output.

Let $x[n]$ that is bounded in magnitude

$$|x[n]| < M < \infty \text{ for all } n.$$

The magnitude of output is given as,

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k]\, x[n-k] \right|$$

* Since the magnitude of sum of numbers is not larger than the sum of the magnitudes of the number, we get,

$$|y[n]| \le \sum_{k=-\infty}^{\infty} |h[k]|\,|x[n-k]|$$

$$\le M \cdot \sum_{k=-\infty}^{\infty} |h[k]|$$

i.e. The impulse response is absolutely summable.

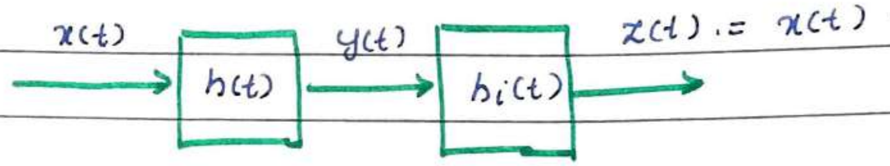
$$\boxed{\sum_{k=-\infty}^{\infty} |h[k]| < \infty}$$

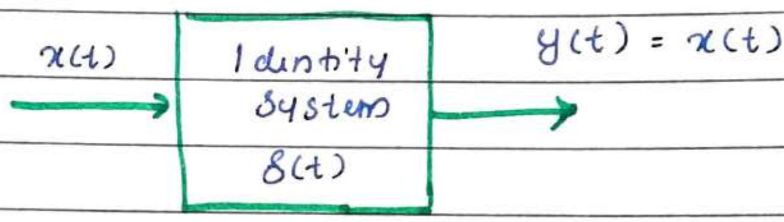
Similarly, a continuous-time LTI system is stable if $|x(t)| < M$ for all t.

then $$\boxed{\int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty}$$

(d) Invertibility of LTI Systems

A system is invertible only if an inverse system exists that, when connected in series with the original system produces an output equal to the input to the first system.





The output of cascaded system is,

$$x(t) * [h(t) * h_i(t)] = x(t)$$

$$\implies x(t) * \delta(t) = x(t).$$

$$\implies \boxed{h(t) * h_i(t) = \delta(t)} \rightarrow \text{Condition for invertibility}$$

Similarly, for discrete-time LTI systems, the condition for invertibility is,

$$\boxed{h[n] * h_i[n] = \delta[n]}$$

Step Response of an LTI System

The unit step response $s(t)$ or $s[n]$ is the response corresponding to unit step that is $x(t) = u(t)$ or $x[n] = u[n]$

$$s(t) = u(t) * h(t)$$

Then, $\frac{ds(t)}{dt} = \frac{du(t)}{dt} * h(t)$

$$= \delta(t) * h(t) = h(t)$$

$\therefore \quad h(t) = \frac{ds(t)}{dt}$ and $\boxed{s(t) = \int_{-\infty}^{t} h(\tau)\,d\tau}$

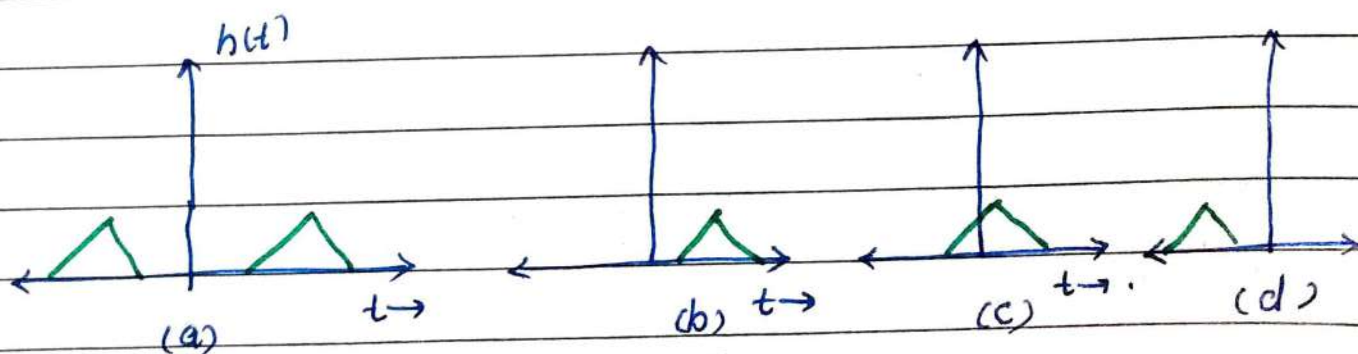
Similarly, for a discrete-time LTI system,

$$s[n] = u[n] * h[n]$$

$$h[n] = s[n] - s[n-1]$$

$$\boxed{s[n] = \sum_{k=-\infty}^{n} h[k] \text{ or } s[n] = \sum_{k=0}^{\infty} h[n-k]}$$

Q1 Which of the following can be impulse response of the causal system ?



Ans: (b) ; $h(t) = 0$ for $t < 0$.

Q2. The impulse response $h[n]$ of an LTI system is $h[n] = u[n+3] + u(n-2) - 2u(n-7)$. Check whether the system is stable and causal.

Ans. ~~For causal system, $h[n] \neq 0$ for $n<0$.~~

Response depends on the future value of input signal u[n+3]

So the system is not causal.

For bounded input, output is bounded.

So the system is stable.

Note:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(t)h(t-\tau)\,d\tau.$$

$$y(t) = x'(t) * s(t) = \int_{-\infty}^{\infty} x'(\tau)s(t-\tau)\,d\tau$$

where $x'(t) = \dfrac{dx(t)}{dt}$.

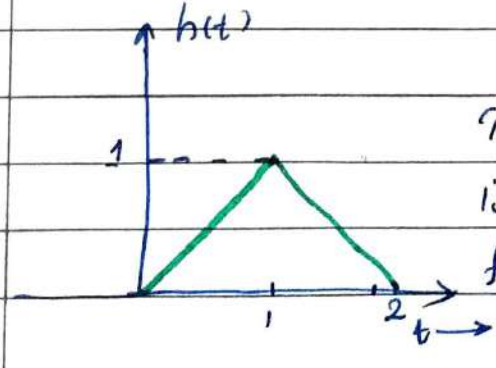
Q3. The response of an LTI system to unit step input is $(\frac{1}{2} - \frac{1}{2}e^{-2t})$. Find out the impulse response of the system.

Ans:

$$h(t) = \frac{d\,s(t)}{dt} = \frac{d}{dt}\left[\frac{1}{2} - \frac{1}{2}e^{-2t}\right]$$

$$= e^{-2t}$$

HW.



The impulse response of an LTI system is shown figure. If the input $x(t) = u(t)$, find out the O/P $y(t)$ at $t = 1.5$ sec.

Continous - Time Fourier Series (CTFS)

Analysis of Signals & Systems

- Time - domain approach (already discussed)
- Frequency domain approach (to be discussed)

Note: * Every signal / system can be represented as a function of time or frequency.

* For ~~thebe~~ the better understanding / analysis, both time domain approach and frequency domain approach are interchangly used.

I Continous - time Fourier Series (CTFS)

Fourier Series is an approximation process where a non-sinusoidal periodic signal is converted into harmonically related sinusoids. It gives us frequency domain representation.

* Thus, a non-sinusoidal periodic signal is represented as the weighted super-position of complex sinusoids.
* Concept of signal space.
* Provides an insightful characterization of signals and systems.

* The study of signals and systems using sinusoidal representation is termed as Fourier analysis, after Joseph Fourier (1768 - 1830) for his development of this theory.

Note: For a signal to have Fourier Series, it is necessary to follow "Principle of orthogonality". It states that the signals $x(t)$ and $g(t)$ are orthogonal if

$$\boxed{\int_{-\infty}^{\infty} x(t)\, g(t)\, dt = 0}$$

Different Forms of Fourier Series

CTFS

- Trignometric Fourier Series
- Polar form
- Exponential Fourier Series.

(a) Trignometric Fourier Series

A periodic signal $x(t)$ can be expressed as infinite sum of sine or cosine functions that are integral multiples of ω_0.

$$\therefore\ x(t) = a_0 + a_1 \cos\omega_0 t + b_1 \sin\omega_0 t + a_2 \cos 2\omega_0 t + b_2 \sin 2\omega_0 t + \cdots$$

$$\therefore\ \boxed{x(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \right]} \longrightarrow (1)$$

where $\omega_0 = \frac{2\pi}{T_0}$ is known as the fundamental frequency (rad/sec) and constant a_0, a_n and b_n are the Fourier coefficients.

Note: The process of determining these coefficients is called Fourier analysis.

Important Trigonometric Integrals.

(i) $\int_0^T \sin n\omega_0 t \, dt = 0$ (ii) $\int_0^T \cos n\omega_0 t \cdot dt = 0$

(iii) $\int_0^T \sin^2 n\omega_0 t \, dt = T/2$ (iv) $\int_0^T \cos^2 n\omega_0 t \cdot dt = T/2$

(v) $\int_0^T \sin n\omega_0 t \cdot \cos n\omega_0 t \, dt = 0$

(vi) $\int_0^T \sin n\omega_0 t \, \cancel{dt} \, \sin m\omega_0 t \, dt = 0 \quad (m \neq n)$

(vii) $\int_0^T \cos n\omega_0 t \cdot \cos m\omega_0 t \, dt = 0 \quad (m \neq n)$

Finding Fourier Coefficients

To find the value of a_0

Integrate Eq. (1) on both sides over a period T.

$$\therefore \int_0^T x(t)\,dt = \int_0^T \left[a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \right] dt$$

$$= \int_0^T a_0 \, dt + \sum_{n=1}^{\infty} \underbrace{\int_0^T a_n \cos n\omega_0 t \, dt}_{0} + \sum_{n=1}^{\infty} \underbrace{\int_0^T b_n \sin n\omega_0 t \, dt}_{0}$$

$$\therefore \int_0^T x(t)\,dt = \int_0^T a_0 \cdot dt.$$

$$\therefore \quad \boxed{a_0 = \frac{1}{T}\int_0^T x(t)\,dt}$$

To find the value of a_n.

Multiply Eq.(1) with $\cos m\omega_0 t$ and integrate it over a period T.

$$\therefore \int_0^T x(t)\cos m\omega_0 t\,dt = \int_0^T \left[a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\right] \times \cos m\omega_0 t\,dt$$

$$= \int_0^T a_0 \cos m\omega_0 t\,dt + \sum_{n=1}^{\infty}\int_0^T a_n \cos n\omega_0 t \cdot \cos m\omega_0 t\,dt$$

$$+ \sum_{n=1}^{\infty}\int_0^T b_n \sin n\omega_0 t \cdot \cos m\omega_0 t\,dt$$

For $m = n$,

$$\int_0^T \cos n\omega_0 t \cdot \cos m\omega_0 t\,dt = T/2$$

$$\therefore \int_0^T x(t)\cos m\omega_0 t\,dt = a_n \cdot T/2$$

$$\therefore \quad \boxed{a_n = \frac{2}{T}\int_0^T x(t)\cos n\omega_0 t \, dt}$$

Similarly, we obtain b_n by multipling both sides of Eq. (1) with $\sin n\omega_0 t$ and integrating over a period T, we get,

$$\boxed{b_n = \frac{2}{T}\int_0^T x(t)\sin n\omega_0 t \, dt}$$

(b) Polar Form of Trigonometric Fourier Series

Fourier series can be expressed in polar form or compact form as,

$$\boxed{x(t) = d_0 + \sum_{n=1}^{\infty} d_n \cos(n\omega_0 t + \theta_n)}$$

where d_n and θ_n are related to a_n and b_n as,

$$d_0 = a_0$$

$$d_n = \sqrt{a_n^2 + b_n^2} \quad \text{and} \quad \theta_n = \tan^{-1}\left[-\frac{b_n}{a_n}\right]$$

or $\quad d_n \angle \theta_n = a_n - jb_n$.

Since the coefficients a_n and b_n are real, the coefficients d_n and θ_n are also real.

(c) Exponential Fourier Series

By using Euler's equality, each of sine and cosine terms in the trigonometric series can be expressed in terms of exponentials $e^{-jn\omega_0 t}$ and $e^{jn\omega_0 t}$ as,

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \left(\frac{e^{-jn\omega_0 t} + e^{jn\omega_0 t}}{2} \right] + b_n \left[\frac{e^{-jn\omega_0 t} - e^{jn\omega_0 t}}{2j} \right) \right]$$

$$= a_0 + \sum_{n=1}^{\infty} \left[e^{jn\omega_0 t} \left[\frac{a_n}{2} + \frac{b_n}{2j} \right] + e^{-jn\omega_0 t} \left[\frac{a_n}{2} - \frac{b_n}{2j} \right] \right]$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left[e^{jn\omega_0 t} \left[\frac{a_n - jb_n}{2} \right] + e^{-jn\omega_0 t} \left[\frac{a_n - jb_n}{2} \right] \right]$$

Let $C_n = \frac{a_n - jb_n}{2}$ and $C_{-n} = \frac{a_n + jb_n}{2}$

$$\therefore x(t) = a_0 + \sum_{n=1}^{\infty} C_n e^{jn\omega_0 t} + \sum_{n=1}^{\infty} C_{-n} e^{-jn\omega_0 t}$$

Put $n = -n$

$$x(t) = a_0 + \sum_{n=1}^{\infty} C_n e^{jn\omega_0 t} + \sum_{n=-\infty}^{-1} C_n e^{jn\omega_0 t}$$

$$\boxed{x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}}$$

when $C_n = \frac{a_n - jb_n}{2} = \frac{1}{T}\int_0^T x(t)\, e^{-jn\omega_0 t} dt$

Fourier Spectrum

$$C_n = \frac{a_n - jb_n}{2}$$

$$|C_n| = \frac{\sqrt{a_n^2 + b_n^2}}{2} = \frac{d_n}{2} = |C_{-n}|$$

$\Longrightarrow$ amplitude spectrum.

$$\angle C_n = \tan^{-1}\left[\frac{-b_n/2}{a_n/2}\right] = \tan^{-1}(-b_n/a_n)$$

~~$\angle C_{-n}$~~

$\Longrightarrow$ phase spectrum

Note: The magnitude spectrum is even symmetric
The phase spectrum is odd symmetric
$\therefore$ C_n is conjugate symmetric.

$$C_{-n} = C_n^*$$

Advantages of exponential Fourier series over Trigonometric Fourier Series

(i) Exponential Fourier series is more compact

(ii) LTI system response to exponential signals is simpler than system response to sinusoids.

(iii) Mathematical manipulation is much easier.

Symmetric Conditions in Fourier Series

* To reduce the calculations.

(i) Even Symmetry

The condition for even symmetric is

$$x(t) = x(-t)$$

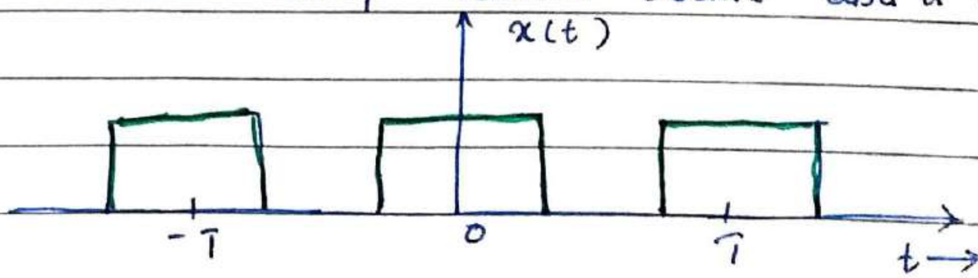
If $x(t)$ is even symmetric, then the trigonometric Fourier series coefficients a_0, a_n and b_n are defined as, (choosing interval $-T/2$ to $T/2$)

$$a_0 = \frac{2}{T}\int_0^{T/2} x(t)\,dt$$

$$a_n = \frac{4}{T}\int_0^{T/2} x(t)\cos n\omega_0 t\,dt \quad \text{and} \quad b_n = 0$$

Note: The Fourier series expansion of an even periodic signal contains only cosine terms and a constant

Eg:



(ii) Odd Symmetry

The condition for odd symmetry is,

$$x(t) = -x(-t)$$

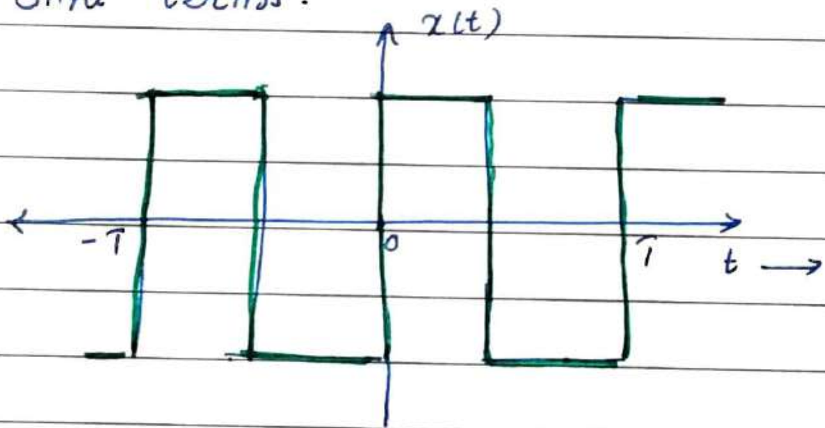
Then, the trigonometric Fourier series coefficients a_0, a_n and b_n are defined as,

$$a_0 = 0$$

$$a_n = 0 \quad \text{and} \quad b_n = \frac{4}{T}\int_0^{T/2} x(t)\cdot \sin(n\omega_0 t)\,dt$$

Note: The Fourier expansion of odd periodic signal contains only sine terms.

Eg:



(iii) Half-wave symmetry

The condition for half-wave symmetry is,

$$x(t) = -x\left[t \pm T/2\right]$$

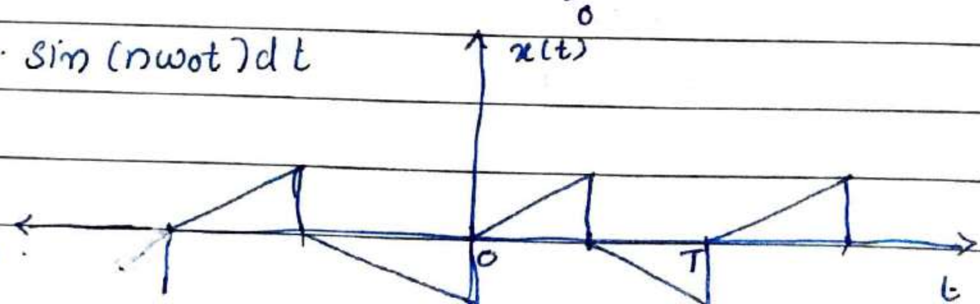
Then, the trigonometric Fourier Series coefficients are

If 'n' is even $\quad a_0 = a_n = b_n = 0$

If 'n' is odd $\quad a_0 = 0 \quad a_n = \frac{4}{T}\int_0^{T} x(t) \cos n\omega_0 t\, dt$

$$b_n = \frac{4}{T}\int_0^{T} x(t)\cdot \sin(n\omega_0 t)\,dt$$

* contains only odd harmonics.



Q1. A waveform is given by $v(t) = 10 \sin(2\pi 100t)$. What will be the magnitude of second harmonic in its Fourier series representation?

Ans: $v(t) = 10 \sin(2\pi \times 100 t)$

Given signal holds in half-wave odd symmetry

$$x(t) = -x\left(t \pm T/2\right)$$

For half-wave symmetry, for $k = 2$ (even)

$$\underline{\underline{a_2 = 0}} \qquad \underline{\underline{b_2 = 0}}$$

DIRICHLET CONDITIONS

If a signal $x(t)$ satisfies certain conditions, its Fourier series is guaranteed to converge point-wise at all points where $x(t)$ is continuous. These conditions are known as Dirichlet conditions.

(i) The signal $x(t)$ must be absolutely integrable

$$\int_0^T |x(t)|\, dt < \infty$$

(ii) It should have finite number of finite discontinuities in one period

(iii) It should have finite number of maxima and minima in one period.

Note: Condition (i) is necessary whereas (ii) and (iii) are sufficient conditions.

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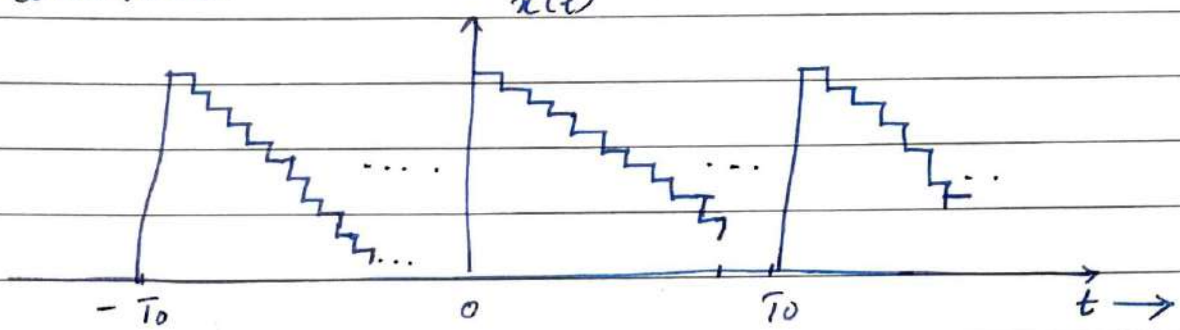
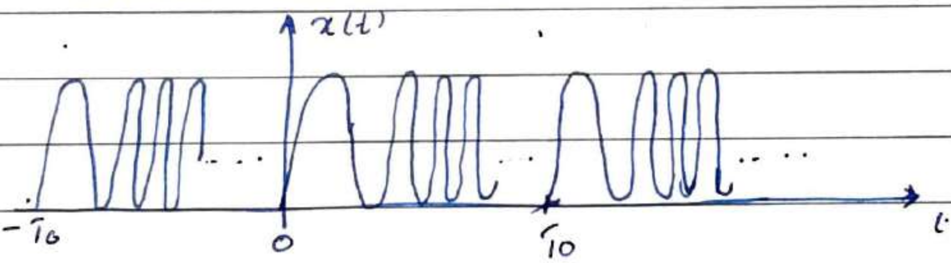
Q1. Comment on existence of Fourier series for the following waveform.

(i)



It has infinite number of discontinuities over one period hence Fourier-series expansion is not possible.

(ii)



As it doesn't have finite number of maxima and minima hence Fourier series expansion is not possible.

Properties of Fourier Series

If Fourier series coefficient of a periodic signal $x(t)$ is C_n, we use notation

$$x(t) \xleftrightarrow{FS} C_n$$

(i) Linearity

If $x(t) \xleftrightarrow{FS} C_n$ and $y(t) \xrightarrow{FS} B_n$

Then $\alpha x(t) + \beta y(t) \xrightarrow{FS} \alpha C_n + \beta B_n$

(ii) Time-shifting

If $x(t) \xleftrightarrow{FS} C_n$

then $x(t-t_0) \xleftrightarrow{FS} e^{-jn\omega_0 t_0} C_n$

(iii) Frequency shifting

If $x(t) \xleftrightarrow{FS} C_n$

then, $e^{jm\omega_0 t} x(t) \xrightarrow{FS} C_{n-m}$.

(iv) Time scaling.

If $x(t) \xleftrightarrow{FS} C_n$ (period = T)

then $x(at) \xleftrightarrow{FS} C_n$ (period = T/a)

(v) Time reversal

If $x(t) \xleftrightarrow{FS} C_n$

then $x(-t) \xleftrightarrow{FS} C_{-n}$

(vi) Conjugation

If $x(t) \xleftrightarrow{FS} C_n$

then, $x^*(t) \xleftrightarrow{FS} C_{-n}^*$

(vii) Differentiation in time

$$\frac{d\,x(t)}{dt} \xleftrightarrow{FS} jn\omega_0 C_n$$

$$\frac{d^k x(t)}{dt^k} \xleftrightarrow{FS} (jn\omega_0)^k C_n.$$

(viii) Integration in time

$$\int_{-\infty}^{t} x(\tau)\, d\tau \xleftrightarrow{FS} \frac{C_n}{jn\omega_0}$$

(ix) Periodic Convolution.

$x(t) * y(t) \xleftrightarrow{FS} C_n \cdot T \cdot B_n$.

<u>Proof</u> :

The Fourier series expansion of $x(t) * y(t)$ is

$$\frac{1}{T}\int_0^T (x(t) * y(t))\, e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_0^T \left[\int_0^T x(\tau)\, y(t-\tau)\, d\tau\right] e^{-jn\omega_0 t}\, dt$$

Convolution over a period T is known as periodic convolution

$$= \frac{1}{T}\int_0^T \left[\int_0^T x(\tau)\, y(t-\tau)\, d\tau\right] e^{-jn\omega_0 t} \cdot \left[\frac{e^{jn\omega_0 \tau}}{e^{jn\omega_0 \tau}}\right] dt \, .$$

Since periodic convolution is commutative and associative

$$= \frac{1}{T}\int_0^T x(\tau)\, e^{-jn\omega_0 \tau}\, d\tau \cdot \int_0^T y(t-\tau)\, e^{-jn\omega_0 (t-\tau)}\, dt$$

Let $t-\tau = z \implies dt = dz$

$$= \frac{1}{T}\int_0^T x(\tau)\, e^{-jn\omega_0 \tau}\, d\tau \int_{-\tau}^{T} y(z)\, e^{-jn\omega_0 z}\, dz$$

$$= \left[\frac{1}{T}\int_0^T x(\tau)\, e^{-jn\omega_0 \tau}\, d\tau\right] T \left[\frac{1}{T}\int_{-\tau}^{T-\tau} y(z)\, e^{-jn\omega_0 z}\, dz\right]$$

$$= C_n \cdot T \cdot B_n \, .$$

(x) Multiplication

$$x(t)\cdot y(t) \xleftrightarrow{FS} \sum_{k=-\infty}^{\infty} C_k B_{n-k}$$

(xi) Parseval's Power Theorem: Parseval's Relation for continuous-time periodic signal

If $x(t) \longleftrightarrow C_n$

then,

$$\frac{1}{T}\int_0^T |x(t)|^2 dt = \sum_{n=-\infty}^{\infty} |C_n|^2$$

Proof:

By definition,

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

and

$$x^*(t) = \sum_{n=-\infty}^{\infty} C_n^* e^{-jn\omega_0 t}$$

Consider the integral

$$\frac{1}{T}\int_0^T |x(t)|^2 dt = \frac{1}{T}\int_0^T x(t).\, x^*(t)\, dt$$

$$= 1/T . \int_0^T x(t) . \left[\sum_{n=-\infty}^{\infty} C_n^* e^{-jn\omega_0 t}\right] dt.$$

$$= \sum_{n=-\infty}^{\infty} C_n^* \frac{1}{T}\int_0^T x(t)\, e^{-jn\omega_0 t} dt$$

$$= \sum_{n=-\infty}^{\infty} C_n^* C_n = \sum_{n=-\infty}^{\infty} |C_n|^2$$

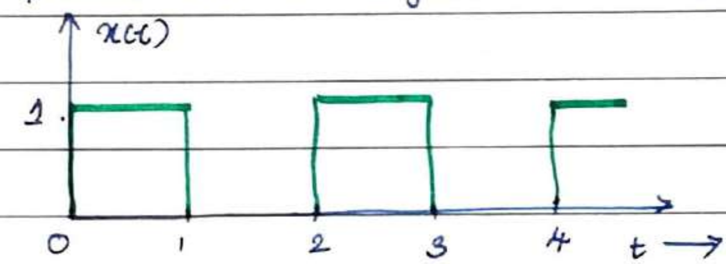
Thus,

$$\frac{1}{T}\int_0^T |x(t)|^2 dt = C_0^2 + 2\sum_{n=1}^{\infty} |C_n|^2 \; ; \text{ if } C_n^* = C_{-n}$$

Then, the power of the signal is defined as.

$$P = \sum_{n=-\infty}^{\infty} |C_n|^2$$

Q1. Find the power of the signal $x(t)$ upto second harmonic.



Ans: The fundamental period $T_0 = 2$

$$\Longrightarrow \omega_0 = 2\pi/2 = \pi \text{ rad/sec}$$

Defining $x(t)$ for its fundamental period is

$$x(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 0, & 1 \le t < 2 \end{cases}$$

Now $$C_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\omega_0 t}\, dt = \frac{1}{2}\left[\int_0^1 e^{-jn\pi t}\, dt + \int_1^2 0 \cdot dt\right]$$

$$= \frac{1}{2}\left[\frac{e^{-jn\pi t}}{-jn\pi}\right]_0^1 = \frac{1}{2}\left[\frac{1 - e^{-jn\pi}}{jn\pi}\right]$$

$$= \frac{e^{-jn\pi/2}\left[e^{jn\pi/2} - e^{-jn\pi/2}\right]}{2j(n\pi)}$$

$$\therefore C_n = \frac{e^{-jn\pi/2}}{n\pi}\left[\sin\frac{n\pi}{2}\right]$$

$$|C_n| = \frac{\sin n\pi/2}{n\pi}$$

$$|C_n| = \begin{cases} 0 ; & n \text{ is even} \\ 1/n\pi , & n \text{ is odd} \end{cases}$$

$$C_0 = \frac{1}{2}\int_0^1 dt = 1/2$$

Then, the power upto 2nd harmonic

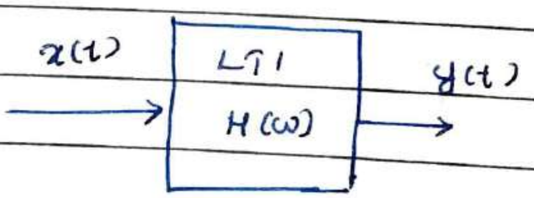
$$= |C_0|^2 + |C_1|^2 + |C_2|^2 + |C_{-1}|^2 + |C_{-2}|^2$$

$$= \underline{\underline{0.452\ W}}$$

Systems with Periodic Inputs

Complex exponential and sinusoidal are acting like eigen functions of an LTI system.
If the Fourier series coefficient of input $x(t)$ is C_n, then the Fourier series coefficient of output $y(t)$ will be $C_n H(n\omega_0)$

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$



Then output,

$$y(t) = \sum_{n=-\infty}^{\infty} C_n H(n\omega_0) e^{jn\omega_0 t}$$

Limitations of Fourier Series

(i) It can be used only for periodic inputs and thus not applicable for aperiodic one.

(ii) It cannot be used for unstable or even marginally stable systems.

Note:

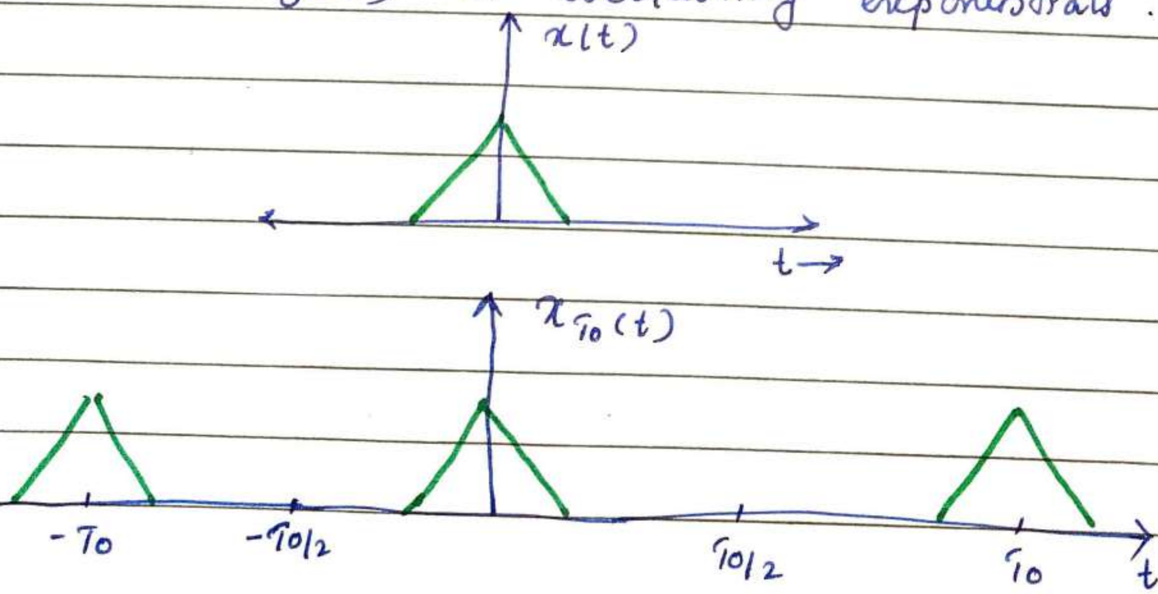
* Solution to first limitation is Fourier Transform, that is applicable for aperiodic signals also.
* Solution to the second limitation is Laplace transform.

CONTINOUS TIME FOURIER TRANSFORM (CTFT)

* Fourier Transform (FT) is applicable for both periodic and aperiodic signals.
* FT gives a frequency domain discription of time domain signal.
* Aperiodic signal is considered as periodic signal with an infinite period.
* As the period increases the fundamental frequency decreases which makes aperiodic signal infinitesimally close in frequency and the representation in terms of linear combination takes the form of an integral rather than sum.
* The resulting spectrum is called Fourier transform.
* Thus, FT is the extension of Fourier series for aperiodic signals.

Aperiodic Signal Representation by Fourier Integral

* Applying the limiting process, we can show that an aperiodic signal can be expressed as a continous sum (integral) of everlasting exponentials.



* Construction of a periodic signal $x_{T_0}(t)$ by periodic extension of $x(t)$; formed by repeating the signal $x(t)$ at intervals of T_0.

* If $T_0 \rightarrow \infty$, the pulse in the periodic signal repeat after an infinite interval and.

$$\lim_{T_0 \rightarrow \infty} x_{T_0}(t) = x(t)$$

Then, the exponential Fourier series expansion of $x_{T_0}(t)$ is

$$x_{T_0}(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$

$$\text{where} \quad D_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x_{T_0}(t)\, e^{-jn\omega_0 t}\, dt \quad ; \omega_0 = \frac{2\pi}{T_0}$$

Since $T_0 \longrightarrow \infty$; $(-T_0/2, T_0/2) \longrightarrow (-\infty, \infty)$

$$\therefore \quad D_n = \frac{1}{T_0} \int_{-\infty}^{\infty} x(t)\, e^{-jn\omega_0 t}\, dt$$

Now, let us define

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

; a continous function of ω.

Then, $$D_n = \frac{1}{T_0} X(n\omega_0)$$

Note: ie Fourier coefficients D_n are $1/T_0$ times the samples of $X(\omega)$ uniformly spaced at intervals of ω_0.

Then, when $T_0 \longrightarrow \infty$, $\omega_0 \longrightarrow 0$ and $D_n \rightarrow 0$

Then, $$x_{T_0}(t) = \sum_{n=-\infty}^{\infty} \frac{X(n\omega_0)}{T_0} e^{jn\omega_0 t}$$

As $T_0 \longrightarrow \infty$, ω_0 becomes infenitesimal and hence ω_0 is replaced with $\Delta\omega$.

$$\Delta\omega = 2\pi / T_0$$

$$\therefore \quad x_{T_0}(t) = \sum_{n=-\infty}^{\infty} \left[\frac{X(n\Delta\omega)\Delta\omega}{2\pi} \right] e^{jn\Delta\omega t}$$

Now, $$x(t) = \lim_{T_0 \to \infty} x_{T_0}(t)$$

$$= \lim_{T_0 \to \infty} \sum_{n=-\infty}^{\infty} \left[\frac{X(n\Delta\omega)\Delta\omega}{2\pi} \right] e^{jn\Delta\omega t}$$

$$\therefore \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} dt$$

Note: $X(\omega) \longrightarrow$ Fourier Transform of $x(t)$

$x(t) \longrightarrow$ Inverse Fourier Transform of $X(\omega)$

$X(\omega) = F(x(t))$ and $x(t) = F^{-1}(X(\omega))$

$$x(t) \xleftrightarrow{FT} X(\omega)$$

Note: In general, FT is of the nature of Fourier series with fundamental frequency $\Delta\omega$ (ω_0) approaching zero.

* $X(\omega)$ is the frequency-domain specification of $x(t)$.

$$X(\omega) = |X(\omega)|\, e^{j\angle X(\omega)}$$

* $|X(\omega)|$ – amplitude spectrum.
* $\angle X(\omega)$ – phase spectrum.

$$X(-\omega) = \int_{-\infty}^{\infty} x(t)\, e^{j\omega t}\, dt$$

and

$$X^*(-\omega) = \int_{-\infty}^{\infty} x^*(t)\, e^{-j\omega t}\, dt$$

$$\therefore \quad x^*(t) \Longleftrightarrow X^*(-\omega)$$

Note: If $x(t)$ is real function of 't', then $x(t) = x^*(t)$

$$X(-\omega) = X^*(\omega)$$

and

$$|X(-\omega)| = |X(\omega)| \rightarrow \text{even function of } \omega$$

$$\angle X(-\omega) = -\angle X(\omega) \rightarrow \text{odd function of } \omega.$$

* Spectrum of Fourier transform is continuous.
* FT is defined for both periodic and aperiodic signals.

- FT is defined for stable and energy signals, not for unstable one.
- FT is defined for power signal as approximation to energy signal.
- FT is not defined for "neither energy nor power signal" Eg: $x(t) = t\,u(t)$.

Conditions for existence of Fourier Transform.

Dirichlet conditions:

(i) Signal $x(t)$ should be absolutely integrable

ie. If $\int_{-\infty}^{\infty} x(t)\,dt < \infty$

then $|X(\omega)| < \infty$

(ii) Signal $x(t)$ should be deterministic over any finite interval.

(a) It should have finite number of maxima and minima over a finite interval.

(b) It should have finite number of discontinuities over finite intervals.

These conditions are sufficient not necessary.

Fourier Transform of Some Basic Signals.

(i) $x(t) = e^{-at}u(t)$, $a > 0$

$$X(\omega) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}\,dt = \int_{0}^{\infty} e^{-(a+j\omega)t}\,dt$$

$$X(\omega) = \frac{-1}{a+j\omega}\left[e^{-(a+j\omega)}\right]_0^{\infty} = \frac{1}{a+j\omega} \; ; \; a>0$$

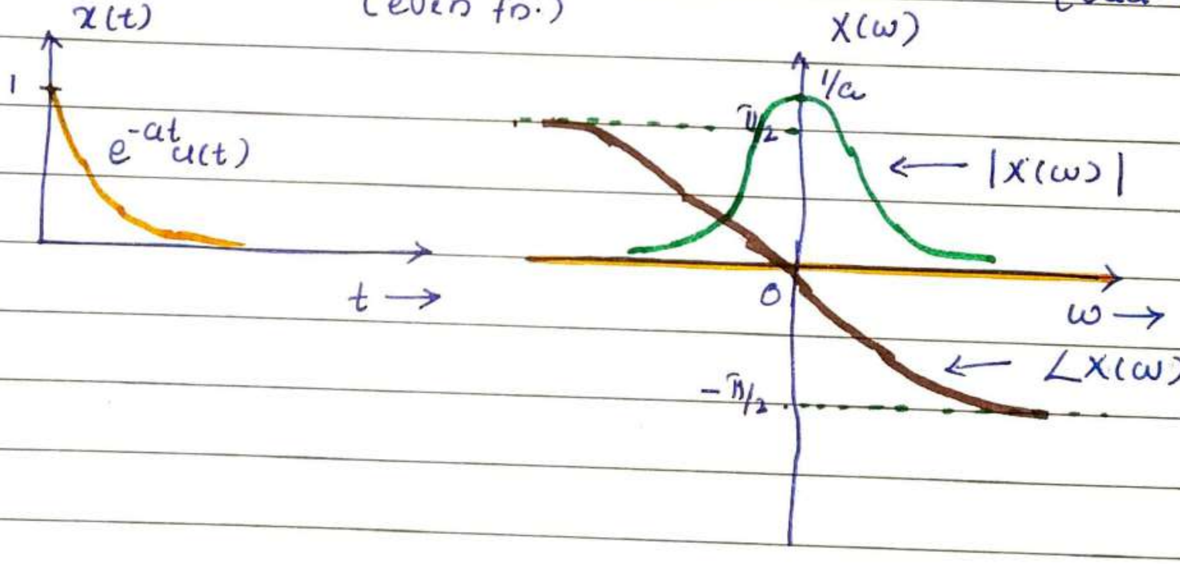
$$\therefore \quad e^{-at}u(t) \xleftrightarrow{FT} \frac{1}{a+j\omega} \, , \quad a>0$$

and in Polar form,

$$X(\omega) = \frac{1}{\sqrt{a^2+\omega^2}} e^{-j\tan^{-1}(\omega/a)}$$

$$|X(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}} \quad \text{and} \quad \angle X(\omega) = -\tan^{-1}(\omega/a)$$

(even fn.) (odd fn.)



(ii) $x(t) = e^{at}u(-t)$, $a>0$

$$X(\omega) = \int_{-\infty}^{\infty} e^{at}u(-t)\,e^{-j\omega t}\,dt = \int_{-\infty}^{0} e^{at}\,e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{0} e^{(a-j\omega)}\,dt = \frac{1}{a-j\omega} \; ; \quad a>0$$

$$e^{at}u(-t) \xleftrightarrow{FT} \frac{1}{a-j\omega}$$

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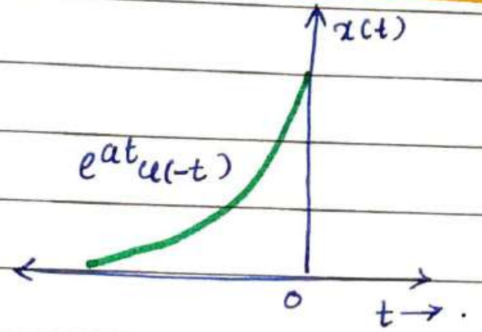
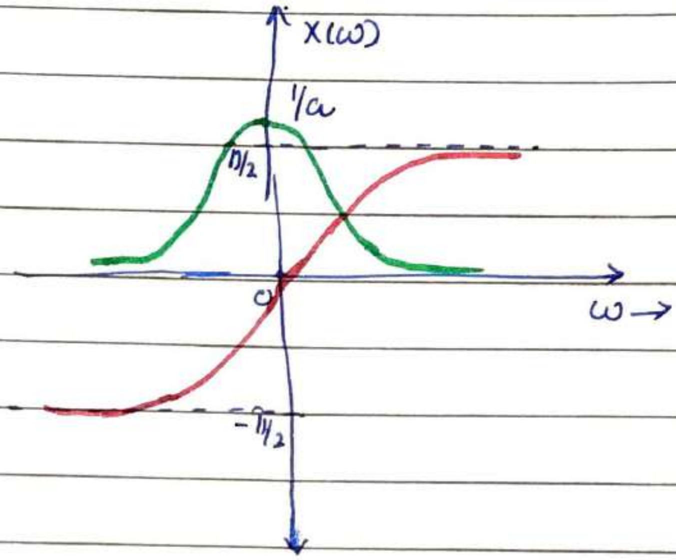
$$|X(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}} \qquad \angle H(\omega) = \tan^{-1}(\omega/a)$$





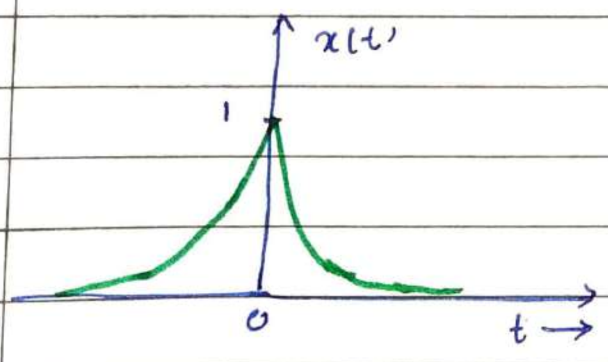
(iii) $x(t) = e^{-a|t|}u(t)$

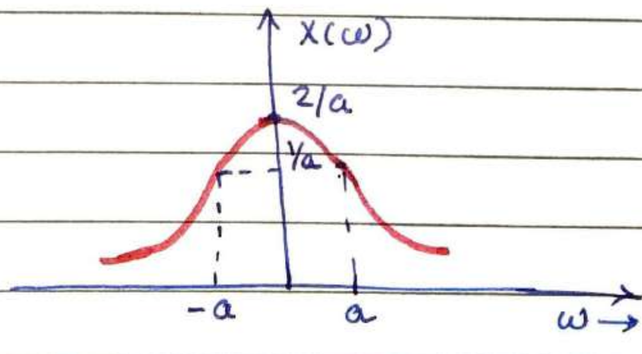
By defenition,

$$X(\omega) = \int_{-\infty}^{\infty} e^{-a|t|} e^{-j\omega t} dt = \int_{-\infty}^{0} e^{at} e^{-j\omega t} dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t} dt$$

$$= \frac{1}{a-j\omega} + \frac{1}{a+j\omega} = \frac{2a}{a^2+\omega^2}$$

$$e^{-a|t|}u(t) \Longleftrightarrow \frac{2a}{a^2+\omega^2}$$





Fourier Spectra.

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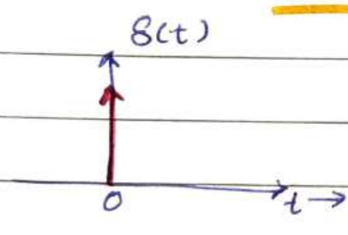
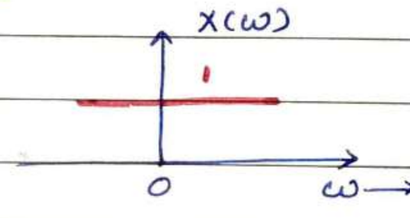
(iv) $x(t) = \delta(t)$

Then,

$$X(\omega) = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt = 1.$$

$$\therefore \quad \delta(t) \xLeftrightarrow{FT} 1.$$





(v) $x(t) = rect\,(t/\tau)$

By definition $\quad X(\omega) = \int_{-\infty}^{\infty} rect\,(t/\tau)\, e^{-\omega t}\, dt$

$$\because \; rect = \begin{cases} 1 ; & |t| < \tau/2 \\ 0 ; & \text{otherwise} \end{cases}$$

$$X(\omega) = \int_{-\tau/2}^{\tau/2} e^{-j\omega t}\, dt = -\frac{1}{j\omega}\left[e^{-j\omega t} dt \right]_{-\tau/2}^{\tau/2}$$

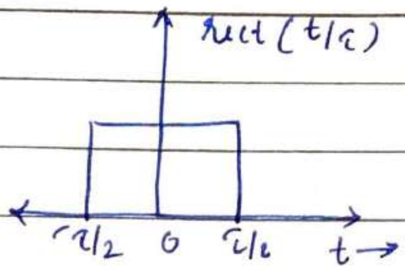
$$= -\frac{1}{j\omega}\left[e^{-j\omega\tau/2} - e^{j\omega\tau/2} \right]$$

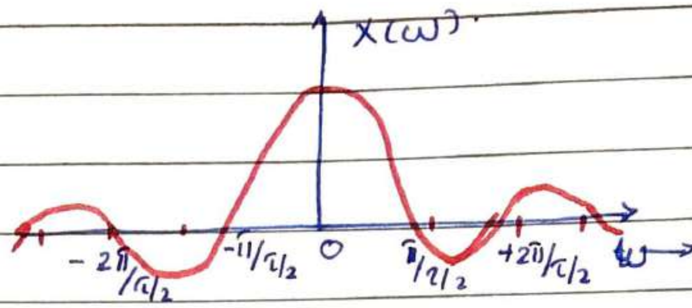
$$= \frac{2\sin(\omega\tau/2)}{\omega} ; \quad \omega \neq 0$$

The value of $X(\omega)$ for $\omega = 0$, using L'Hospital rule

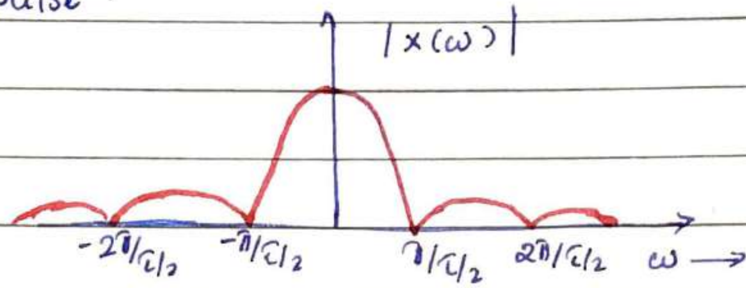
$$\lim_{\omega \to 0} \frac{2\sin(\omega\tau/2)}{\omega} = \tau$$

$$\therefore \quad rect\,(t/\tau) \Longleftrightarrow 2\, sinc\,(\omega\tau/2)$$





Gate pulse.



(vi) $x(t) = sgn(t)$

$$x(t) = sgn(t) = \begin{cases} 1 & , t > 0 \\ 0 & , t = 0 \\ -1 & t < 0 \end{cases}$$

also, $sgn(t) = u(t) - u(-t)$

This signal is not absolutely integrable, so we calculate Fourier transform of $sgn(t)$ as the limiting case of exponential $e^{-at}u(t) - e^{+at}u(-t)$ as $a \to 0$

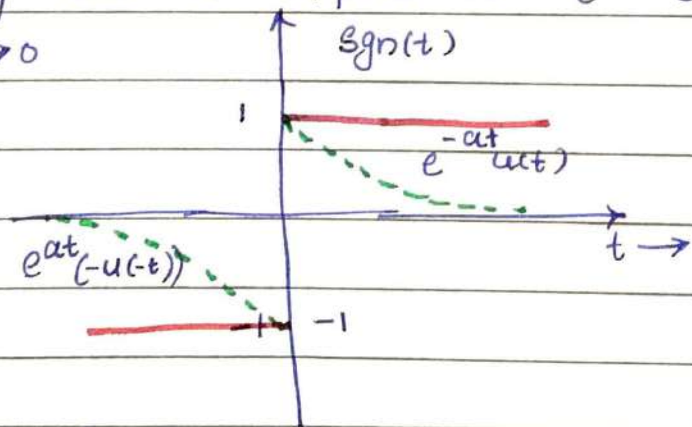


Fig: sgn function as limiting case of sum of exponentials

$$x(t) = sgn(t) = \lim_{a \to 0} \left[e^{-at}u(t) - e^{at}u(-t) \right]$$

Taking Fourier Transform,

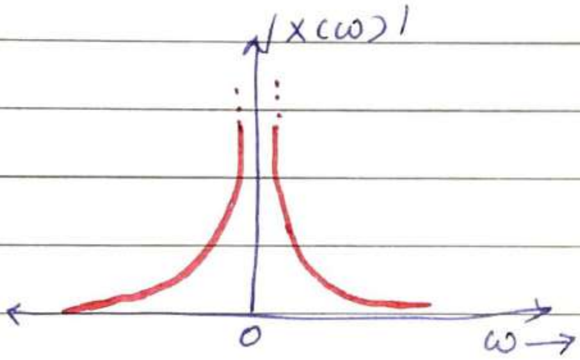
$$X(\omega) = \lim_{a \to 0} \left[\frac{1}{a+j\omega} - \frac{1}{a-j\omega} \right]$$

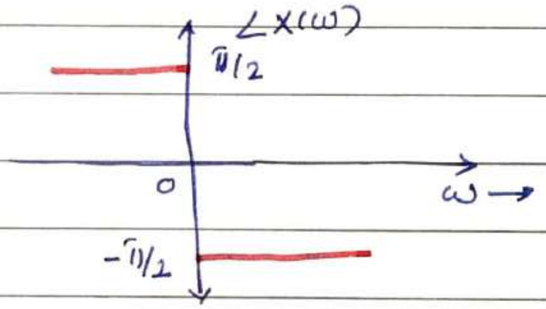
$$= \lim_{a \to 0} \left[\frac{-2j\omega}{a^2+\omega^2} \right] = \frac{-2j\omega}{\omega^2} = \frac{2}{j\omega}$$

$$\therefore \quad sgn(t) \overset{FT}{\longleftrightarrow} 2/j\omega$$

Fourier Spectra of $x(t) = sgn(t)$ is given below.

$$|X(\omega)| = \left| \frac{2}{\omega} \right| \quad \text{and} \quad \angle X(\omega) = \begin{cases} -\pi/2, & \omega > 0 \\ \pi/2, & \omega < 0 \end{cases}$$





(vii) $x(t) = u(t)$

The unit step function is not absolutely integrable so we calculate its fourier transform as limiting case of $e^{-at}u(t)$ as $a \to 0$.

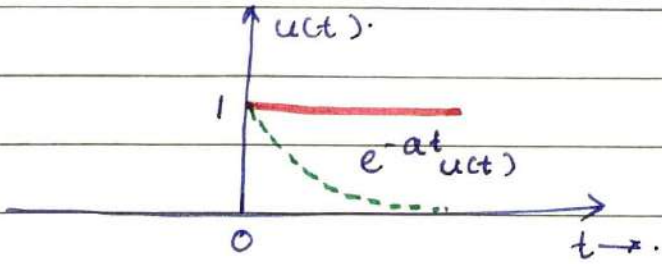


Fig: $u(t)$ as limiting case of $e^{-at}u(t)$ as $a \to 0$

$$\therefore \quad u(t) = \lim_{a \to 0} e^{-at}u(t)$$

Thus,

$$X(\omega) = \lim_{a \to 0} \left[\frac{1}{a+j\omega} \right]$$

$$= \lim_{a \to 0} \left[\frac{a}{a^2+\omega^2} - \frac{j\omega}{a^2+\omega^2} \right]$$

$$= \lim_{a \to 0} \left[\frac{a}{a^2+\omega^2} \right] + \frac{1}{j\omega}$$

The function $\left[\frac{a}{a^2+\omega^2} \right]$ has two interesting properties

(i) $\displaystyle\int_{-\infty}^{\infty} \frac{a}{a^2+\omega^2} \, d\omega = \left(\tan^{-1} \frac{\omega}{a} \right)_{-\infty}^{\infty} = \pi$

i.e. area under this function is π, regardless of the value of a.

(ii) When $a \to 0$, this function approaches zero for all $\omega \neq 0$ and its area is concentrated at a single point $\omega = 0$ i.e. an impulse at $\omega = 0$ with strength π.

$$\therefore \lim_{a \to 0} \left[\frac{a}{a^2+\omega^2} \right] = \pi \delta(\omega)$$

$$\therefore X(\omega) = \pi \delta(\omega) + \frac{1}{j\omega}$$

Alternatively,

$$u(t) = \frac{1 + \operatorname{sgn}(t)}{2}$$

$$\therefore FT(u(t)) = FT \left[\frac{1}{2} + \frac{1}{2} \operatorname{sgn}(t) \right] = \frac{1}{2} FT(1) + \frac{1}{2} FT[\operatorname{sgn}(t)]$$

$$= \frac{2\pi \delta(\omega)}{2} + \frac{1}{2} \times \frac{\cancel{2}}{\cancel{2}} \cdot \frac{2}{j\omega} = \pi \delta(\omega) + \frac{1}{j\omega}$$

$\therefore \quad u(t) \xleftrightarrow{FT} \pi\,\delta(\omega) + \frac{1}{j\omega}$

Inverse Fourier transform of some basic functions

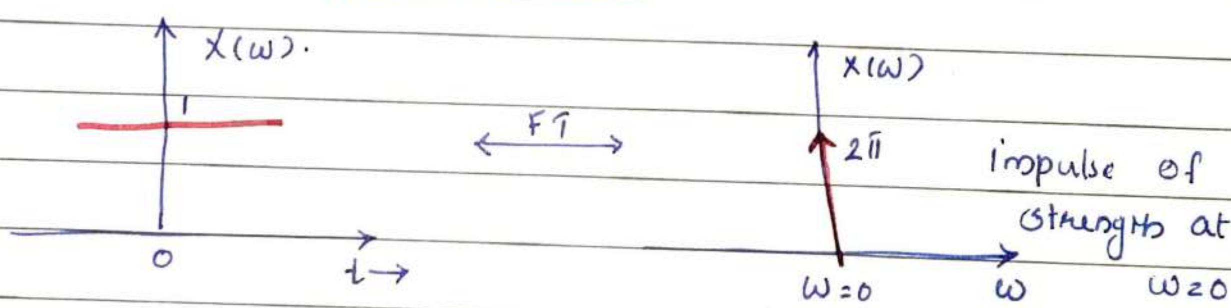
(i) $X(\omega) = \delta(\omega)$

By definition of IFT

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega)\, e^{j\omega t}\, d\omega = \frac{1}{2\pi}\, e^{j\omega t}\Big|_{\omega=0} = \frac{1}{2\pi}$$

$\therefore \quad \delta(\omega) \xleftrightarrow{IFT} 1/2\pi$

$\therefore \quad 1 \xleftrightarrow{FT} 2\pi\,\delta(\omega)$



(ii) $X(\omega) = \delta(\omega - \omega_0)$

$$FT^{-1}[X(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega-\omega_0)\, e^{j\omega t}\, dt = \frac{1}{2\pi}\, e^{j\omega t}\Big|_{\omega=\omega_0}$$

$$= \frac{1}{2\pi}\, e^{j\omega_0 t}$$

Thus, $\quad \delta(\omega-\omega_0) \xleftrightarrow{IFT} \frac{1}{2\pi}\, e^{j\omega_0 t}$

$\therefore \quad e^{j\omega_0 t} \xleftrightarrow{FT} 2\pi\,\delta(\omega-\omega_0)$

$FT(e^{j\omega_0 t})$ is an impulse shifted at $\omega = \omega_0$.

Q. HW

Find the FT of the following functions.

(i) $x(t) = \cos w_0 t$.

Hint : $\cos w_0 t = \frac{e^{jw_0 t} + e^{-jw_0 t}}{2}$

Ans : $\cos w_0 t \xleftrightarrow{FT} \pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]$

Plot the waveforms.

(ii) $x(t) = \sin w_0 t$

Hint : $\sin w_0 t = \frac{e^{jw_0 t} - e^{-jw_0 t}}{2j}$

Ans : $\sin w_0 t \xleftrightarrow{FT} \frac{\pi}{j}\left[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right]$

Plot the waveforms.

Note: The spectrum of $\sin w_0 t$ consists of two impulses at w_0 and $-w_0$.

Sl. No.	$x(t)$	$X(\omega)$
1.	$e^{-at}u(t)$	$\frac{1}{a+j\omega}$, $\quad a > 0$
2.	$e^{at}u(-t)$	$\frac{1}{a-j\omega}$, $\quad a > 0$
3.	$e^{-a\|t\|}$	$\frac{2a}{a^2+\omega^2}$, $\quad a > 0$
4.	$\delta(t)$	1
5.	$A \cdot \text{rect}\left(\frac{t}{\tau}\right)$	$\frac{2A\sin\left(\frac{\omega\tau}{2}\right)}{\omega}$, $\quad \omega \neq 0$
6.	$\text{sgn}(t)$	$\frac{2}{j\omega}$
7.	$u(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$
8.	1	$2\pi\,\delta(\omega)$
9.	$e^{j\omega_0 t}$	$2\pi\,\delta(\omega-\omega_0)$
10.	$\cos\omega_0 t$	$\pi\,[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)]$
11.	$\sin\omega_0 t$	$\frac{\pi}{j}\,[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$
12.	$\text{tri}\left(\frac{t}{\tau}\right)$	$\frac{\tau}{2}\text{sin}c^2\left(\frac{\omega\tau}{4}\right)$

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Properties of Fourier Transform

(i) Linearity

If $x_1(t) \xleftrightarrow{FT} X_1(\omega)$ and $x_2(t) \xleftrightarrow{FT} X_2(\omega)$

then $\alpha x_1(t) + \beta x_2(t) \xleftrightarrow{FT} \alpha X_1(\omega) + \beta X_2(\omega)$

(ii) Time-shifting

$$x(t-t_0) \xleftrightarrow{FT} e^{-j\omega t_0} X(\omega)$$

Proof :

$$FT\left[x(t-t_0)\right] = \int_{-\infty}^{\infty} x(t-t_0)\, e^{-j\omega t}\, dt$$

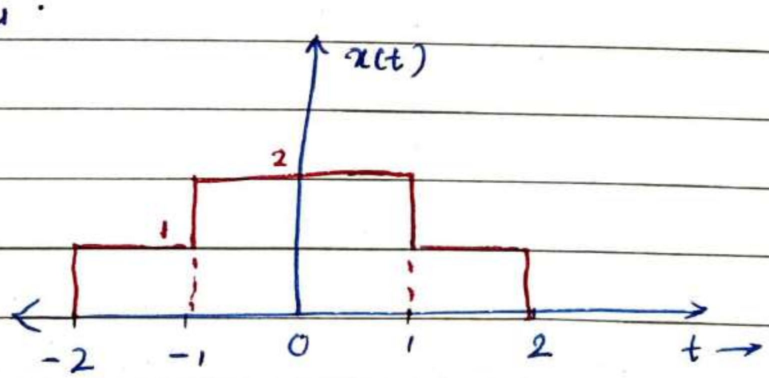
let $t - t_0 = \tau$ and $dt = d\tau$

$$= \int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega(t+\tau)}\, d\tau = e^{-j\omega t_0} X(\omega)$$

This shows that shifting a signal by t_0 doesnot

Note: ~~(iii)~~ change its amplitude spectrum, however changes in phase spectrum by $-\omega t_0$, which is a linear function of ω (i.e. linear phase).

Q. Determine the Fourier Transform of the signal shown in figure.



Ans: $x(t) = u(t+2) + u(t+1) - u(t-1) - u(t-2)$

and we have

$$u(t) \xleftrightarrow{FT} \frac{1}{j\omega} + \pi\,\delta(\omega) \quad \text{and.}$$

$$x(t-t_0) \xleftrightarrow{FT} e^{-j\omega t_0} X(\omega)$$

Then $F\{x(t)\} = X(\omega)$

$$= \left[e^{j2\omega} + e^{j\omega} - e^{-j\omega} - e^{-j2\omega}\right]\left[\frac{1}{j\omega} + \pi\delta(\omega)\right]$$

$$= \left[\frac{e^{j2\omega} - e^{-j2\omega}}{2j} + \frac{e^{j\omega} - e^{-j\omega}}{2j}\right]\left[\frac{2j}{j\omega} + j2\pi\delta(\omega)\right]$$

$$X(\omega) = \left[\sin 2\omega + \sin\omega\right]\left[\frac{2}{\omega} + j2\pi\delta(\omega)\right]$$

(iii) Frequency shifting

$$x(t)e^{j\omega_0 t} \xleftrightarrow{F} X(\omega - \omega_0)$$

Proof :

$$F\{x(t)e^{j\omega_0 t}\} = \int_{-\infty}^{\infty} x(t)e^{j\omega_0 t} \cdot e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{\infty} x(t)e^{-j(\omega-\omega_0)t}\,dt = X(\omega - \omega_0)$$

$\|^{ly}$,

$$F\{x(t)e^{-j\omega_0 t}\} = X(\omega + \omega_0)$$

Note:
* Multiplication of signal by exponential $e^{j\omega_0 t}$ shifts the spectrum of $x(t)$ by ω_0 in right direction.
* Frequency shifting in practice is obtained by multiplying $x(t)$ of sinusoids.

$$x(t)\cos\omega_0 t = \frac{1}{2}\left[x(t)e^{j\omega_0 t} + x(t)e^{-j\omega_0 t}\right]$$

$$x(t)\cos\omega_0 t \xleftrightarrow{F} \frac{1}{2}\left[X(\omega - \omega_0) + X(\omega + \omega_0)\right]$$

Q If FT of $x(t)$ is $2/\omega \sin(\pi\omega)$, then what is the FT of $e^{j5t}x(t)$?

Ans:

$$x(t) \xleftrightarrow{FT} 2/\omega \sin(\pi\omega)$$

$$e^{j5t}x(t) \xleftrightarrow{FT} \frac{2}{\omega-5}\sin\{\pi(\omega-5)\}$$

(iv) Time Scaling.

$$x(at) \xleftrightarrow{FT} \frac{1}{|a|}X(\omega/a) \text{ ; 'a' any real constant.}$$

Proof:

For $a>0$,

$$FT(x(at)) = \int_{-\infty}^{\infty} x(at)\, e^{-j\omega t}\,dt$$

Let $at=\tau$, thus $a\,dt = d\tau$

$$= \frac{1}{a}\int_{-\infty}^{\infty} x(\tau)\, e^{-j\left(\frac{\omega}{a}\right)\tau}\,d\tau = \frac{1}{a}X\left(\frac{\omega}{a}\right)$$

For $a<0$;

$$FT[x(-at)] = \int_{-\infty}^{\infty} x(-at)\, e^{-j\omega t}\,dt$$

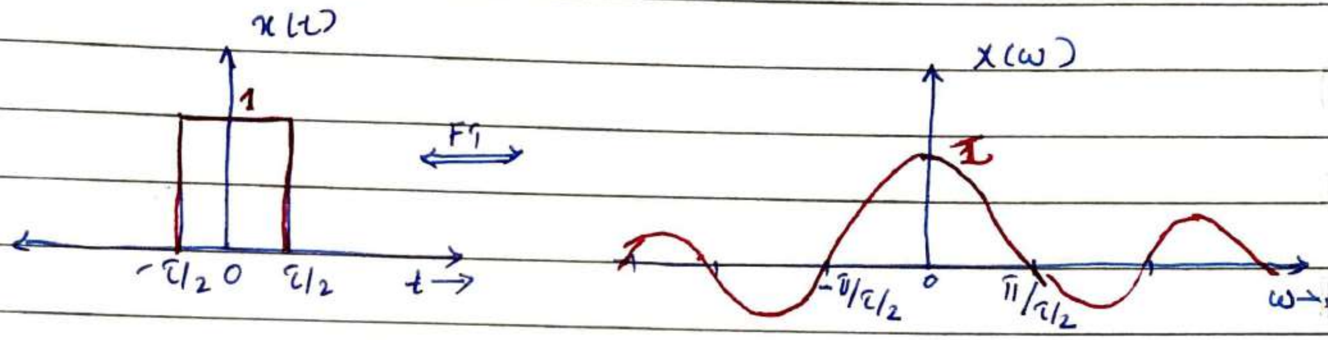
Let $-at=u$, $-a\,dt = du$

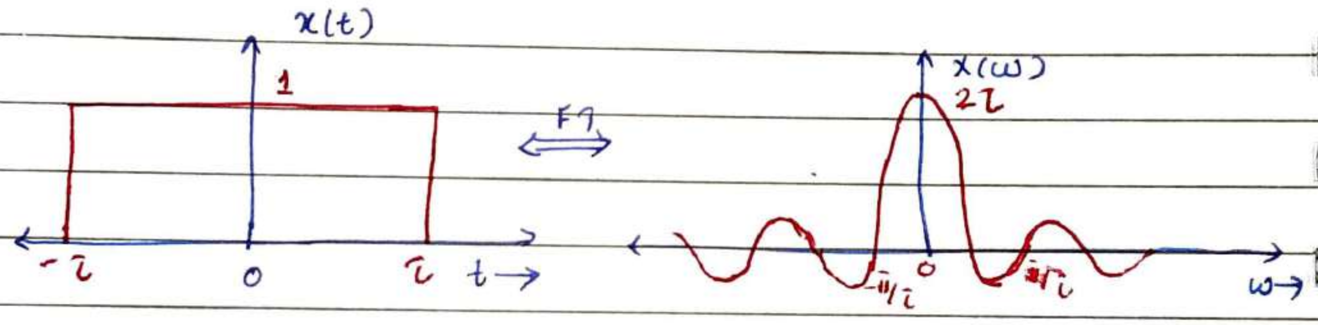
$$= -\frac{1}{a}\int_{+\infty}^{-\infty} x(u)\, e^{j\left(\frac{\omega}{a}\right)u}\,du$$

$$= \frac{1}{a}\int_{-\infty}^{\infty} x(u)\, e^{-j\left(-\frac{\omega}{a}\right)u}\,du = \frac{1}{a}X\left(-\frac{\omega}{a}\right)$$

$$\therefore FT[x(at)] = \frac{1}{|a|}X\left(\frac{\omega}{a}\right)$$

Note: The scaling property states that time compression of signal results in its spectral expansion and vice-versa.





Q FT of $e^{-\pi t^2}$ is $e^{-\pi f^2}$. Then find the FT of $e^{-\alpha t^2}$.

Ans:

$$e^{-\pi t^2} \xleftrightarrow{FT} e^{-\pi f^2}$$

$$e^{-\alpha t^2} = e^{-\pi\left(\sqrt{\frac{\alpha}{\pi}}\, t\right)^2}$$

Using the property,

$$x(at) \xleftrightarrow{FT} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

So, $$e^{-\pi\left(\sqrt{\frac{\alpha}{\pi}}\, t\right)^2} \xleftrightarrow{FT} \sqrt{\frac{\pi}{\alpha}}\, e^{-\pi f^2 \frac{\pi}{\alpha}} = \sqrt{\frac{\pi}{\alpha}}\, e^{\frac{-\pi^2 f^2}{\alpha}}$$

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(vi) Time Reversal

$$x(-t) \xleftrightarrow{FT} X(-\omega)$$

Note: This states that reversal of time also reverses its FT. This property is also called as inversion property of time and frequency.

(vii) Duality

$$X(t) \xleftrightarrow{FT} 2\pi x(-\omega)$$

Proof:

We have $\quad x(t) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$

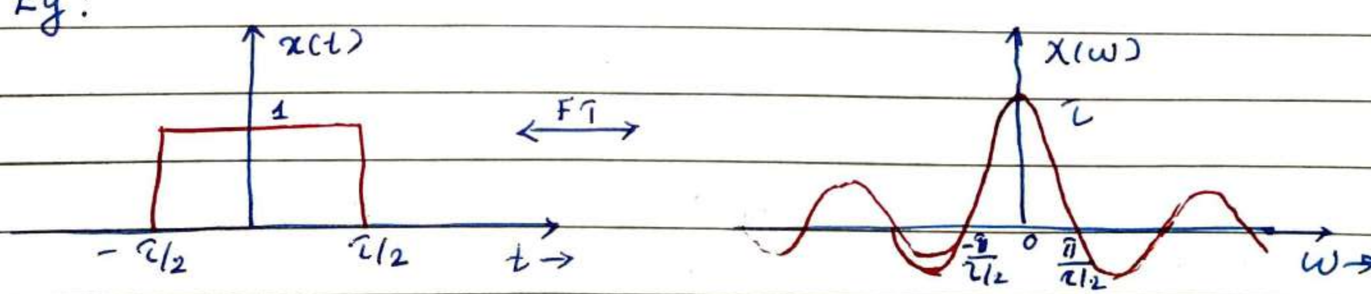
Changing 't' to '-t'

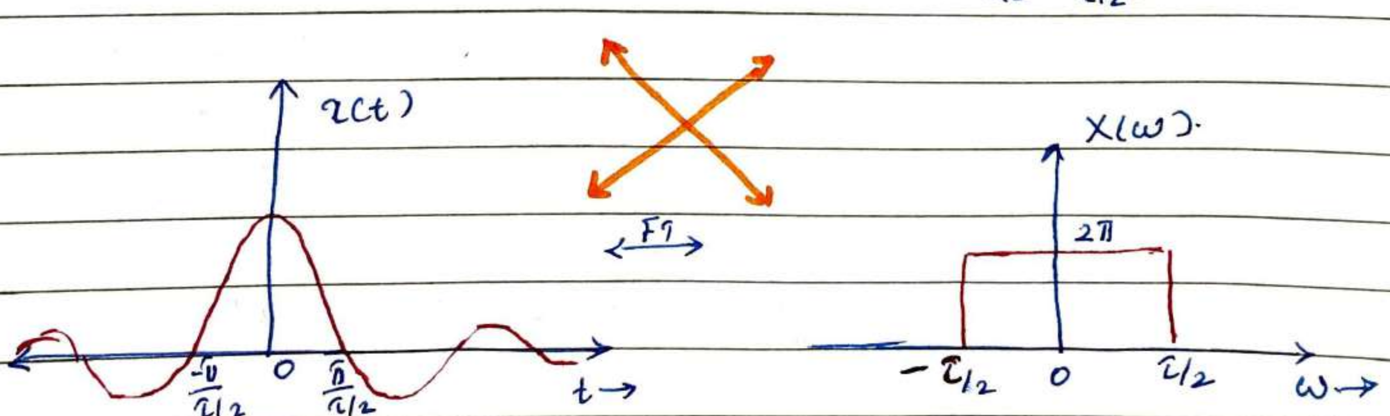
Hence $\quad 2\pi x(-t) = \displaystyle\int_{-\infty}^{\infty} X(\omega)\, e^{-j\omega t}\, d\omega$

Interchanging the variables 't' and 'ω' yields.

$$2\pi\, x(-\omega) = \int_{-\infty}^{\infty} X(t)\, e^{-j\omega t}\, dt = FT\{X(t)\}$$

Eg:





Q: a. Find the FT of $x(t) = \frac{1}{1+t^2}$.

Ans: We have

$$e^{-a|t|} \xleftrightarrow{FT} \frac{2a}{a^2+\omega^2}$$

$$\therefore \quad e^{-|t|} \xleftrightarrow{FT} \frac{2}{1+\omega^2} \; ; \; a=1$$

$$\underbrace{\frac{e^{-|t|}}{2}}_{x(t)} \xleftrightarrow{FT} \underbrace{\frac{1}{1+\omega^2}}_{X(\omega)}.$$

By applying the duality property.

$$X(t) \xleftrightarrow{FT} 2\pi x(-\omega)$$

$$\frac{1}{1+t^2} \xleftrightarrow{FT} 2\pi \frac{e^{-|\omega|}}{2} = \pi e^{-|\omega|}$$

b. $x(t) = 1/\pi t$

Ans: We have

$$\text{sgn } t \xleftrightarrow{FT} 2/j\omega$$

$$\Rightarrow \quad \frac{\text{sgn } t}{2} \xleftrightarrow{FT} 1/j\omega$$

From the application of duality,

$$X(t) \xleftrightarrow{FT} 2\pi x(-\omega)$$

$$\frac{1}{jt} \xleftrightarrow{FT} 2\pi \frac{\text{sgn}(-\omega)}{2}$$

$$\Rightarrow \quad \frac{1}{\pi t} \xleftrightarrow{FT} j\,\text{sgn}(-\omega)$$

since 'sgn' function is odd one, so $sgn(-\omega) = -sgn(\omega)$.

$$\therefore \quad FT\left[\frac{1}{\pi t}\right] = -j\, sgn\, \omega$$

(vii) Differentiation in time

$$\frac{d}{dt}x(t) \xleftrightarrow{FT} j\omega X(\omega)$$

and
$$\frac{d^n x(t)}{dt^n} \xleftrightarrow{FT} (j\omega)^n X(\omega)$$

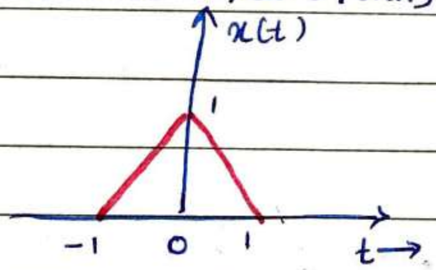
Note: This is important to note two things in differentiation property.

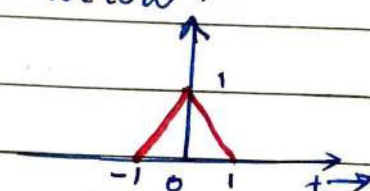
* $\frac{dx(t)}{dt}$ must satisfy Dirichlet's conditions

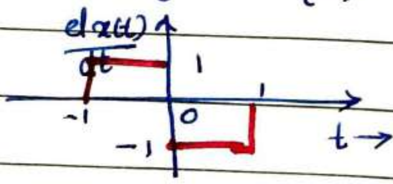
$$\int_{-\infty}^{\infty} \left| \frac{dx(t)}{dt} \right| dt < \infty$$

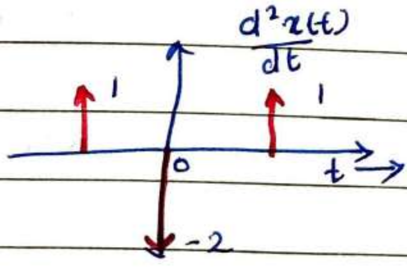
* Differentiation will destroy the DC component of the original spectrum and hence there is ~~there~~ no one to one relationship between $x(t)$ and $\frac{d}{dt}x(t)$.

Q. Using time differentiation property, find the Fourier transform of the signal below.









$$\therefore \frac{d^2 x(t)}{dt^2} = \delta(t+1) - 2\delta(t) + \delta(t-1)$$

Using the differentiation property of FT.

$$(j\omega)^2 X(\omega) = e^{j\omega} - 2 + e^{-j\omega} = -2 + 2\cos\omega$$

$$\therefore X(\omega) = \frac{2(1-\cos\omega)}{\omega^2} = \frac{2\left[1 - 1 + 2\sin^2\omega/2\right]}{\omega^2}$$

$$= \frac{4\sin^2\omega/2}{\omega^2} = \left[\frac{\sin\omega/2}{\omega/2}\right]^2$$

$$= \text{sinc}^2(\omega/2)$$

(viii) Integration in time

$$\int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{FT} \frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)$$

Proof:

Consider $x(t) * u(t) = \int_{-\infty}^{\infty} x(\tau) u(t-\tau)\,d\tau$

$$= \int_{-\infty}^{t} x(\tau)\,d\tau$$

Then, $F\left[\int_{-\infty}^{t} x(\tau)\,d\tau\right] = F\left[x(t) * u(t)\right]$

$$= F[x(t)] \cdot F[u(t)]$$

$$= X(\omega)\left[\frac{1}{j\omega} + \pi\delta(\omega)\right]$$

$$= \frac{X(\omega)}{j\omega} + \pi \delta(\omega) X(\omega)$$

$$= \frac{X(\omega)}{j\omega} + \pi \delta(\omega) X(0)$$ ($\because$ product property of impulses)

(ix) Convolution in time

The convolution in time domain is equivalent to multiplication in frequency domain.

If $x_1(t) \xleftrightarrow{FT} X_1(\omega)$ and $x_2(t) \xleftrightarrow{FT} X_2(\omega)$

~~Proof~~: then, $x_1(t) * x_2(t) \xleftrightarrow{FT} X_1(\omega) X_2(\omega)$

Proof:

By definition,

$$F\left[x_1(t) * x_2(t)\right] = \int_{-\infty}^{\infty} (x_1(t) * x_2(t))\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau\, e^{-j\omega t}\, dt$$

Let $t - \tau = \lambda \implies dt = d\tau$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1(\tau)\, x_2(\lambda)\, e^{-\omega(\tau+\lambda)}\, d\lambda\, d\tau$$

$$= \int_{-\infty}^{\infty} x_1(\tau) e^{-j\omega\tau}\, d\tau \int_{-\infty}^{\infty} x_2(\lambda)\, e^{-\omega\lambda}\, d\lambda$$

$$= X_1(\omega) \cdot X_2(\omega)$$

Note:

$$x(t) * h(t) \xleftrightarrow{FT} X(\omega)\cdot H(\omega) =$$

$$x(\alpha t) * h(\alpha t) \xleftrightarrow{FT} = \frac{1}{|\alpha|} y(\alpha t)$$

Q. If $g(t) * f(t) = h(t)$, then show that
$$g(t-T_1) * f(t-T_2) = h(t-T_1-T_2)$$

Ans:

$$g(t) * f(t) = h(t)$$

Taking FT on both sides,

$$G(\omega)\cdot F(\omega) = H(\omega)$$

Now, $x(t) = g(t-T_1) * f(t-T_2)$

Then

$$X(\omega) = e^{-j\omega T_1} G(\omega) \cdot e^{-j\omega T_2} F(\omega)$$

$$= e^{-j\omega(T_1+T_2)} G(\omega) F(\omega)$$

$$= e^{-j\omega(T_1+T_2)} H(\omega).$$

Taking the inverse FT on both sides, we get

$$x(t) = h(t-(T_1+T_2)) = \underline{h(t-T_1-T_2)}$$

(x) Frequency convolution (Multiplication or Modulation Property)

Note: Multiplication in time domain is equivalent to convolution in frequency domain.

If $x_1(t) \xleftrightarrow{FT} X_1(\omega)$ and $x_2(t) \xleftrightarrow{FT} X_2(\omega)$

then, $x_1(t)\cdot x_2(t) \xleftrightarrow{FT} \frac{1}{2\pi}\left[X_1(\omega) * X_2(\omega)\right]$

__/__/____

Proof:

Consider the IFT,

$$FT^{-1}\left[\frac{1}{2\pi} X_1(\omega) * X_2(\omega)\right]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1}{2\pi} X_1(\omega) * X_2(\omega)\right] e^{j\omega t}\, d\omega .$$

$$= \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X_1(\lambda)\, X_2(\omega-\lambda)\, e^{j\omega t} .\, d\lambda\, d\omega . \left[\frac{e^{j\lambda t}}{e^{j\lambda t}}\right]$$

$$= \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\lambda)\, e^{j\lambda t}\, d\lambda\right]\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X_2(\omega-\lambda)\, e^{j(\omega-\lambda)}\, d\omega\right]$$

$$= x_1(t) \cdot x_2(t)$$

$$\therefore \quad x_1(t) \cdot x_2(t) \xleftrightarrow{FT} \frac{1}{2\pi}\left[X_1(\omega) * X_2(\omega)\right]$$

Note: or $\quad x_1(t) \cdot x_2(t) \xleftrightarrow{FT} X_1(f) * X_2(f)$

(xi) Frequency Differentiation

If $\quad x(t) \xleftrightarrow{FT} X(\omega)$

then, $\quad t \cdot x(t) \xleftrightarrow{FT} j\,\frac{d}{d\omega} X(\omega)$

Proof:

By defenition,

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

Differentiating w.r.t. 'ω'

$$\frac{d\, X(\omega)}{d\omega} = \int_{-\infty}^{\infty} x(t)\,(jt)\, e^{-j\omega t}\, dt$$

$$\Longrightarrow \frac{1}{-j}\frac{dX(\omega)}{d\omega} = \int_{-\infty}^{\infty}[t\cdot x(t)]e^{-j\omega t}\,dt = F\{t\,x(t)\}$$

$$= j\frac{dX(\omega)}{d\omega}.$$

Q. Find the FT of Gaussian pulse $x(t) = e^{-\pi t^2}$.

Ans: $x(t) = e^{-\pi t^2}$

Given $\frac{d\,x(t)}{dt} \xleftrightarrow{FT} j\omega X(\omega)$

ie $\frac{d}{dt}(e^{-\pi t^2}) \xleftrightarrow{FT} j\omega X(\omega)$

ie. $-2\pi t e^{-\pi t^2} \xleftrightarrow{FT} j\omega X(\omega)$

$t\cdot e^{-\pi t^2} \xleftrightarrow{FT} \frac{j\omega X(\omega)}{-2\pi}$ ——(1)

From frequency differentiation property of FT

$t\cdot x(t) \xleftrightarrow{FT} j\frac{dX(\omega)}{d\omega}$

$\Longrightarrow$

$t\cdot e^{-\pi t^2} \xleftrightarrow{FT} j\frac{dX(\omega)}{d\omega}$ ——(2).

From Eq.(1) and (2)

$$\Longrightarrow j\frac{dX(\omega)}{d\omega} = \frac{j\omega X(\omega)}{-2\pi}$$

$$\Longrightarrow \frac{dX(\omega)}{X(\omega)} = -\frac{1}{2\pi}\omega\,d\omega$$

Integrating on both sides

$$\ln X(\omega) = -\frac{1}{2\pi}\left[\frac{\omega^2}{2}\right]$$

$$\Rightarrow \quad \ln X(\omega) = \frac{-\omega^2}{4\pi}$$

$$\Rightarrow \quad X(\omega) = e^{-\omega^2/4\pi} = e^{-\pi t^2}$$

(xii) Conjugate Property

If $\quad x(t) \xleftrightarrow{FT} X(\omega)$

then, $\quad x^*(t) \xleftrightarrow{FT} X^*(-\omega)$

Proof:

By defenition,

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

Conjugate on both sides,

$$X^*(\omega) = \int_{-\infty}^{\infty} x^*(t)\, e^{j\omega t}\, dt$$

Replacing ω by '-ω'.

$$X^*(-\omega) = \int_{-\infty}^{\infty} x^*(t)\, e^{-j\omega t}\, dt = FT\left[x^*(t)\right]$$

	Signal	Defenition	FT	Comment
1.	Real	$x^*(t) = x(t)$	$X^*(\omega) = X(\omega)$	Conjugate symmetric
2.	Real + Even	$x^*(t) = x(-t) = x(t)$	$X^*(\omega) = X(-\omega) = X(\omega)$	Real and Even
3.	Real + Odd	$x^*(t) = x(-t) = -x(t)$	$X^*(\omega) = X(-\omega) = -X(\omega)$	Imaginary and odd

(xiii) Parseval's Power Theorem.

The signal energy can be related to its Fourier spectrum $X(\omega)$ as,

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega .$$

Proof:

Consider the integral

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} x(t)\cdot x^*(t)\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega)\, e^{-j\omega t} d\omega\right] dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega) \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt \quad d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega)\, X(\omega)\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

Note: If $x(t) \xleftrightarrow{FT} X(\omega)$

then $x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) d\omega$

and $X(0) = \int_{-\infty}^{\infty} x(t)\, dt$

Applications of Fourier Transform.

Fourier Transform of Periodic Signal

A periodic signal $x(t)$ can be represented in Fourier series as,

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

Taking FT on both sides yields

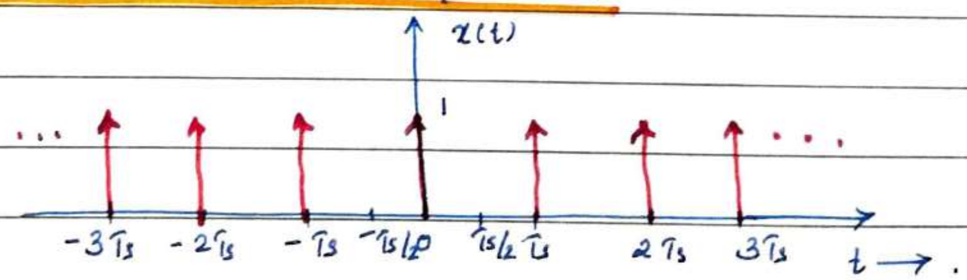
$$X(\omega) = FT\left[\sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}\right]$$

$$= \sum_{n=-\infty}^{\infty} C_n \left[2\pi\,\delta(\omega - n\omega_0)\right]$$

$$= 2\pi \sum_{n=-\infty}^{\infty} C_n\,\delta(\omega - n\omega_0)$$

Note: Thus, the Fourier transform of periodic signal is an impulse train having strength C_n and located at $\omega = n\omega_0$; $n = -\infty$ to $n = \infty$.

Fourier Transform of Impulse Train.



A periodic impulse train with fundamental period T_s is given as,

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \; ; \quad \omega_0 = 2\pi/T_0$$

Considering $x(t)$ over one period $-T_s/2$ to $T_s/2$, it is only an impulse located at $t = 0$

$$x(t) = \delta(t) \quad ; \quad -T_s/2 \le t \le T_s/2$$

Then, the Fourier series coefficient,

$$C_n = \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} x(t)\, e^{-jn\omega_0 t}\, dt = \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} \delta(t)\, e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{T_s} \times 1 = \frac{1}{T_s}$$

From the definition of FT of periodic signal

$$X(\omega) = 2\pi \sum_{n=-\infty}^{\infty} C_n\, \delta(\omega - n\omega_0)$$

$$= 2\pi \sum_{n=-\infty}^{\infty} C_n\, \delta(\omega - n\omega_0)$$

$$= 2\pi \sum_{n=-\infty}^{\infty} \frac{1}{T_s}\, \delta(\omega - n\omega_0)$$

$$= \frac{2\pi}{T_s} \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0) = \omega_0 \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0)$$

Note: FT of impulse train is also impulse train having strength ω_0.

Q. Given that $x(t)$ has a FT of $X(\omega)$. Then, find the FT of $x_1(t) = t \cdot \frac{d\,x(t)}{dt}$

Ans:

$$x_1(t) = t \cdot \frac{d\,x(t)}{dt}$$

Let $\phi(t) = \frac{d\,x(t)}{dt} \longrightarrow \phi(\omega) = j\omega\, X(\omega)$

$$\therefore \quad x_4(t) = t \cdot \phi(t)$$

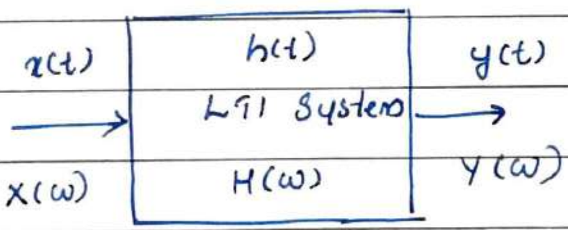
$$t \cdot \phi(t) \xleftrightarrow{FT} j \frac{d}{d\omega} \left[\phi(\omega) \right]$$

$$t \cdot \phi(t) \xleftrightarrow{FT} j \frac{d}{d\omega} \left[j\omega \, X(\omega) \right]$$

$$X_1(\omega) = j^2 \left[\omega \frac{d X(\omega)}{d\omega} + X(\omega) \cdot 1 \right]$$

$$= - \left[\omega \frac{d X(\omega)}{d\omega} + X(\omega) \right]$$

Transfer Function of an LTI Systems.



Let $\quad x(t) \xleftrightarrow{FT} X(\omega)$

$\quad h(t) \xleftrightarrow{FT} X(\omega)$

$\quad y(t) \xleftrightarrow{FT} Y(\omega)$

and $\quad y(t) = h(t) * x(t)$.

Then according to the time convolution property of FT.

$$Y(\omega) = H(\omega) \, X(\omega).$$

$\Longrightarrow \quad H(\omega) = \dfrac{Y(\omega)}{X(\omega)}$; is known as the Transfer Function (TF) of an LTI Systems.

Note: Transfer Function is the Fourier Transform of impulse response of the LTI Systems.

$H(\omega) = |H(\omega)|\, e^{j\angle H(\omega)}$ → phase spectrum

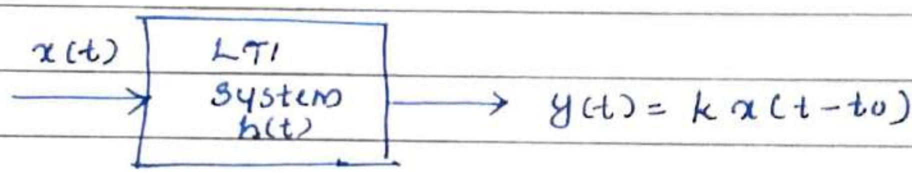
$|H(\omega)|$ → amplitude spectrum.

Frequency response of an LTI system → Amplitude spectrum: $|H(\omega)|$ vs ω; → Phase spectrum: $\angle H(\omega)$ vs ω.

Distortionless Transmission

Transmission through an LTI system is said to be distortionless if the output is the replica of input with scaling in amplitude and possible time shift.

Consider a continuous-time LTI system with impulse response $h(t)$.



Then, the distortionless system holds the following equation.

$$y(t) = k\, x(t - t_o) \quad \text{—— (1)}$$

where, 'k' accounts for scaling in amplitude and 'to' accounts for delay in transmission.

Taking the FT of Eq.(1), we will get

$$Y(\omega) = k\, e^{-j\omega t_o} X(\omega)$$

$$\therefore \frac{Y(\omega)}{X(\omega)} = H(\omega) = k\, e^{-j\omega t_o}$$

Thus, $H(\omega) = k e^{-j\omega t_0} = |H(\omega)| e^{j\angle H(\omega)}$

$\therefore$ The equivalent condition in frequency domain is,

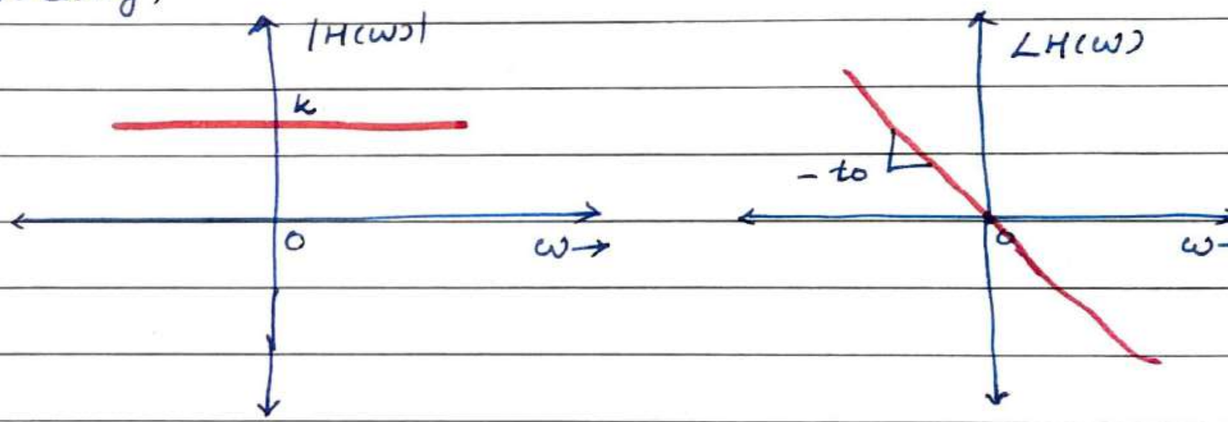
(i) the magnitude response $|H(\omega)|$ must be constant.

i.e. $|H(\omega)| = k =$ constant.

(ii) the response $\angle H(\omega)$ must be linear function of frequency.

$$\angle H(\omega) = -\omega t_0 \quad \text{or} \quad \angle H(\omega) \propto \omega.$$

Graphically,



(a) Magnitude Response (b) Phase response

Note:
* For distortionless transmission, we require linear phase characteristics
* The phase is not only linear function of ω, but it should also pass through $\omega = 0$.
* The two parameters that are associated with the frequency response $H(\omega)$ are phase delay and group delay.

Phase delay.

The time delay experienced by 'single frequency' signal when the signal passes through a system is referred to as phase delay and is given as,

$$\tau_p(\omega) = \frac{\angle H(\omega_0)}{\omega_0}.$$

Group delay.

The time delay experienced by 'group of frequency (or a small band centred at $\omega = \omega_0$) when an input signal that contains components with different frequencies (not harmonically related) passes through a system is referred as group delay and is given as,

$$\tau_g(\omega) = -\frac{d}{d\omega} \angle H(\omega)$$

Note:

* The group delay at each frequency equals to negative of slope of the phase at that frequency.
* If '$\tau_g(\omega)$' is constant, all the components are delayed by same interval.

Q. Comment about the τ_p and τ_g of a linear phase channel.

Ans: For linear phase channel

$$\theta(\omega) = -\omega t_0$$

Then,
$$\tau_p = \frac{-\theta(\omega)}{\omega} = \frac{-[-\omega t_0]}{\omega} = t_0$$

Then
$$\tau_g = \frac{-d\,\theta(\omega)}{d\omega} = t_0.$$

$\therefore \tau_p = \tau_g = t_0 =$ constant.

Note:
* If amplitude spectrum is not constant, then the distortion is known as amplitude distortion
* If phase spectrum is not linear function of frequency then it is known as phase distortion or delay distortion.
* For linear phase systems, both the phase delay and the group delay are constant.

Ideal Filters

As we have seen the magnitude spectrum of distortionless transmission which is constant over all frequency range which require infinite energy and that is not possible practically.

* So, we truncate frequency range from 0 to ω_c; this results in ideal filter.
* An ideal filter allow distortionless transmission of certain band of frequencies and suppress the remaining completely.
* The ideal low pass filter (LPF) is one that allow all components below $\omega = \omega_c$ to pass without distortion and suppress the components above $\omega = \omega_c$.
    - ω_c : cut-off frequency.
* $||^{ly}$ high pass filter (HPF)
* band pass filter (BPF)
* band suppression filter (BSF)
* Notch filter

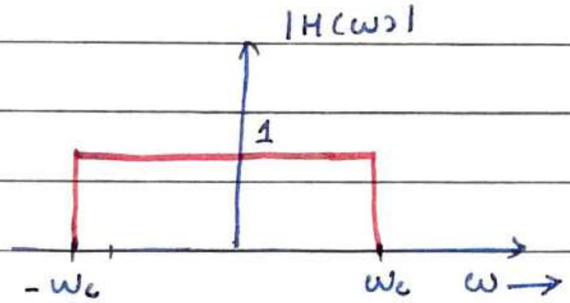
* The ideal LPF has linear phase of slope 't_0' which results in a time delay of 't_0' to all of its input components below 'ω_c' rad/sec.

Thus, $y(t) = x(t - t_0)$.

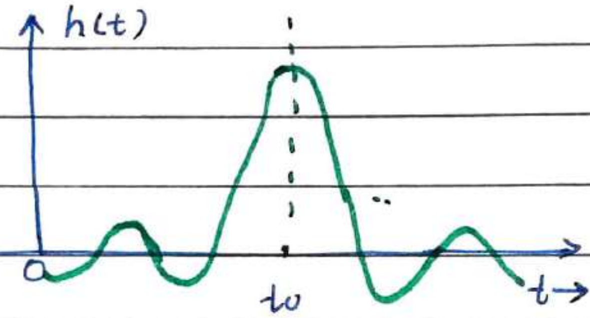
* $x(t)$ is transmitted without distortion, but with delay 't_0'.

* For this filter, the frequency response is given as,

$$|H(\omega)| = rec\left[\frac{\omega}{2\omega_c}\right] \quad \text{and} \quad \angle H(\omega) = e^{-j\omega t_0}$$



(a) Amplitude Spectrum of ideal LPF



(b) Impulse response of ideal LPF

Note: All ideal filters are non-causal and unstable and therefore physically unrealizable : why ?.

Bandwidth (BW) of a signal

The difference between the highest and lowest frequencies of spectral components of a signal is called the bandwidth of the signal.

BW : $\Delta\omega = \omega_2 - \omega_1$; $\Delta f = f_2 - f_1$

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Energy Spectral Density (ESD)

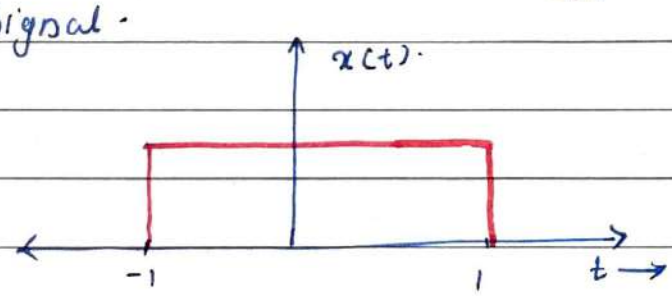
The energy per unit bandwidth is known as energy spectral density (ESD) and is denoted as $S_x(\omega)$.

ESD for an energy signal $x(t)$ is

$$S_x(\omega) = |X(\omega)|^2$$

It is defined for energy signals.

Q. Let $x(t) \xleftrightarrow{FT} X(\omega)$. Find $\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$ of the given signal.



Ans: $$\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega = \int_{-\infty}^{\infty} X(\omega) X^*(\omega)\, d\omega$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt \right] \cdot X^*(\omega)\, d\omega$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} X^*(\omega)\, e^{-j\omega t} d\omega \right] x(t)\, dt$$

$$= 2\pi \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \frac{1}{2\pi} \left\{ X(\omega)\, e^{-j\omega t} \right\}^* d\omega \right] x(t)\, dt$$

$$= 2\pi \int_{-\infty}^{\infty} x^*(t)\, x(t)\, dt = 2\pi \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$= 2\pi \int_{-1}^{1} 1\, dt = 2\pi \times 2 = \underline{\underline{4\pi}}$$

Power Spectral Density (PSD)

The power spectral density has the same relation to power signals as energy spectral density has to energy signal.

The PSD of a power signal $x(t)$ is,

$$S_x(\omega) = |X(\omega)|^2$$

Properties

(i) Area under PSD gives total power

i.e. $\int_{-\infty}^{\infty} S_x(\omega)\, d\omega$.

(ii) As power ≥ 0, so $S_x(\omega) \geq 0$ for all ω.

(iii) It is an even function of ω, $S_x(-\omega) = S_x(\omega)$

(iv) $S_x(\omega)$ is real.

Relation between input and output PSD of an LTI system.

Consider an LTI System

Let $S_x(\omega) \rightarrow$ PSD of input signal $x(t)$

$S_y(\omega) \longrightarrow$ PSD of output signal $y(t)$

$$y(t) = x(t) * h(t)$$

$$Y(\omega) = H(\omega) X(\omega)$$

$$\Longrightarrow \quad |Y(\omega)|^2 = |H(\omega)|^2 |X(\omega)|^2$$

$$S_y(\omega) = |H(\omega)|^2 S_x(\omega)$$

Correlation

* It is the similarity of coherance between two signals i.e. computing the correlation between the two signals is to measure the degree to which the two signals are similar. It is of two types.

(i) Auto - correlation function (ACF)

(ii) Cross -correlation function (CCF)

(i) Auto- correlation function

It gives correlation of function with its shifted version. Let $x(t)$ be an 'energy signal', its ACF is given as,

$$R_x(\tau) = \int_{-\infty}^{\infty} x(t)\, x(t-\tau)\, dt = \int_{-\infty}^{\infty} x(t+\tau)\, x(t)\, dt$$

If the signal is power signal, the ACF is given as,

$$R_x(\tau) = \lim_{T \to \infty} \int_{-T/2}^{T/2} x(t)\, x(t-\tau)\, dt.$$

where 'τ' is the shift parameter or lag parameter.

Properties of ACF

(i) ACF is an even function of 'τ'.

$$R_x(-\tau) = R_x(\tau)$$

Proof:

By definition

$$R_x(\tau) = \int_{-\infty}^{\infty} x(t)\, x(t-\tau)\, dt$$

Let $t-\tau = \lambda$, thus $dt = d\lambda$.

$$R_x(\tau) = \int_{-\infty}^{\infty} x(\lambda+\tau)\,x(\lambda)\,d\lambda = \int_{-\infty}^{\infty} x(\lambda)\,x(\lambda-(-\tau))\,d\lambda$$

$$= \underline{\underline{R_x(\tau) = R_x(-\tau)}}$$

(ii) ACF has its maximum magnitude at origin.

$$|R_x(0)| \geq |R_x(\tau)|$$

<u>Proof</u>:

Consider an integral

$$\int_{-\infty}^{\infty} \left[x(t) \pm x(t-\tau)\right]^2 dt \geq 0$$

$$\Rightarrow \int_{-\infty}^{\infty} x^2(t)\,dt + \int_{-\infty}^{\infty} x^2(t-\tau)\,dt \pm 2\int_{-\infty}^{\infty} x(t)\cdot x(t-\tau)\,dt \geq 0$$

$$\Longrightarrow R_x(0) + R_x(0) \pm 2R_x(\tau) \geq 0$$

$$2R_x(0) \pm 2R_x(\tau) \geq 0$$

$$R_x(0) \pm R_x(\tau) \geq 0$$

$$R_x(0) \geq \mp R_x(\tau)$$

It follows,

$$\underline{\underline{|R_x(0)| \geq |R_x(\tau)|}}$$

(iii) The convolution of signal with its reversal gives ACF.

$$R_x(\tau) = x(\tau) * x(-\tau)$$

Proof:

$$x(\tau) * x(-\tau) = \int_{-\infty}^{\infty} x(t)\, x(-(\tau-t))\,dt$$

$$= \int_{-\infty}^{\infty} x(t)\cdot x(t-\tau)\,dt = R_x(\tau)$$

(iv) Let $x(t)$ be energy signal

Then, energy in $x(t) = \int_{-\infty}^{\infty} |x(t)|^2 dt = R_x(0)$

Let $x(t)$ be power signal

Then, power in $x(t) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 dt = R_x(0)$

Proof:

$$R_x(\tau) = \int_{-\infty}^{\infty} x(t)\cdot x(t-\tau)\,dt$$

$$R_x(0) = \int_{-\infty}^{\infty} x(t)\,x(t)\,dt = \int_{-\infty}^{\infty} x^2(t)\,dt$$

$\|^{ly}$, it can be proved for power signal $x(t)$.

(v) ACF and PSD/ESD make Fourier transform pair

$$R_x(\tau) \xleftrightarrow{FT} S_x(\omega)$$

Proof:

We have $R_x(\tau) = x(\tau) * x(-\tau)$

Taking FT on both sides, yields

$$F[R_x(\tau)] = X(\omega)X(-\omega)$$

If $x(t)$ is real and even, $X(-\omega) = X^*(\omega)$,

Then,

$$F[R_x(\tau)] = X(\omega)X^*(\omega) = |X(\omega)|^2$$
$$= S_x(\omega)$$

$$R_x(\tau) \xleftrightarrow{FT} S_x(\omega)$$

Thus, $$S_x(\omega) = \int_{-\infty}^{\infty} R_x(\tau)\, e^{-j\omega\tau}\, d\tau$$

and $$R_x(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_x(\omega)\, e^{j\omega\tau}\, d\omega$$

This is known as "Weiner - ~~Khin~~ khinchine" relation

Then, $$R_x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_x(\omega)\, d\omega$$

$$= \frac{\text{Area under PSD/ESD curve}}{2\pi}$$

Cross correlation function (CCF)

It is the similarity between a signal and shifted version of other signal. Suppose $x(t)$ and $y(t)$ are two energy signals.

The cross correlation between $x(t)$ and $y(t)$, if both are real, is denoted as $R_{xy}(\tau)$ and is given as,

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t)\, y(t-\tau)\, dt = \int_{-\infty}^{\infty} x(t+\tau)\, y(t)\, dt$$

$$= x(\tau) * y(-\tau)$$

Similarly cross-correlation between $y(t)$ and $x(t)$ is denoted as $R_{yx}(\tau)$ and given as,

$$R_{yx}(\tau) = \int_{-\infty}^{\infty} y(t)\, x(t-\tau)\, dt = \int_{-\infty}^{\infty} x(t+\tau)\, x(t)\, dt$$

$$= y(\tau) * x(-\tau)$$

Properties of CCF

(i) It is even function i.e. $R_{xy}(\tau) = R_{yx}(-\tau)$

(ii) $|R_{xy}(\tau)| \leq \sqrt{R_x(0) R_y(0)}$

(iii) If $R_{xy} = 0$, then $x(t)$ and $y(t)$ are said to orthogonal.

Proof:

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t)\, y(t-\tau)\, dt$$

$$R_{xy}(0) = \int_{-\infty}^{\infty} x(t)\, y(t)\, dt$$

If $R_{xy}(0) = 0$, it yields

$$\int_{-\infty}^{\infty} x(t)\, y(t)\, dt = 0$$

i.e. $x(t)$ and $y(t)$ are ~~oth~~ orthogonal.

Q: If the PSD is $\eta/2$ W/Hz and auto correlation function defined by $R_x(\tau) = \frac{\eta}{2}\int_{-\infty}^{\infty} e^{j\omega\tau}\, df$. Then what is $x(t)$?

Ans:

$$R_x(\tau) \xleftrightarrow{FT} S_x(\omega)$$

$$\frac{\eta}{2}\int_{-\infty}^{\infty} e^{j\omega\tau}\, df \xleftrightarrow{FT} \eta/2$$

i.e. $$\int_{-\infty}^{\infty} e^{j\omega\tau}\, df \xleftrightarrow{FT} 1$$

Hence $x(t) = \delta(t)$

Comparison between time-domain and frequency-domain analysis

a) Signals : Time domain perspective

When we look at the signal, we consider its waveform, the signal width (duration) and the rate at which the waveform decays.

b) Systems : Time domain perspective

When we think of a system, we think of its impulse response, h(t). The width of h(t) indicates that the time constant (response time) i.e. how fast the system is capable of responding to an input and how much dispersion (spreading)

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it will perform.

c) Signals : Frequency- domain perspective

We think of the signal in terms of its frequency spectrum (i.e. in terms of its sinusoidal components and their amplitudes and phases); whether the spectrum is low pass, high pass, bandpass, etc.

d) Systems: Frequency- domain perspective

When we think about a system, we view it as a filter, which selectively transmits certain frequency components and suppress the other components. Knowing the input signal spectrum and frequency response of the system, we can create a mental image of the o/p signal spectrum.

Note: The Fourier transform is not defined for unstable signals therefore solution to this limitation is Laplace transform.

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Laplace Transform

* Fourier transform is not applicable for signals that are not absolutely integrable.

* This problem could be resolved by generalizing Fourier transform which leads to development of Laplace transform.

* Laplace transform can be applied to broad class of signals and systems. Eg: analysis of unstable systems.

* There are two varities of Laplace transform: bilateral and unilateral.

* Bilateral or two sided Laplace transform: is defined for $-\infty \leq t \leq \infty$ and can handle all causal and non-causal signals.
  It provides insights about system's characteristics such as stability, causality and frequency response.

* Unilateral or one-sided Laplace transform:
  - can handle only causal systems
  - mainly used to solve differential equations with initial conditions.

Definition

Consider a continous time signal $x(t)$. Its Laplace transform (bilateral) is defined as,

$$LT[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st} dt$$

where 's' is the complex frequency which is given as,

$$s = \sigma + j\omega$$

where,

σ - $Re\{s\}$ is damping factor

ω - $Im\{s\}$ is oscillation frequency in 'rad/s'.

$$x(t) \xleftrightarrow{LT} X(s)$$

and the inverse Laplace transform of $X(s)$ is given as,

$$ILT[X(s)] = L^{-1}[X(s)] = x(t)$$

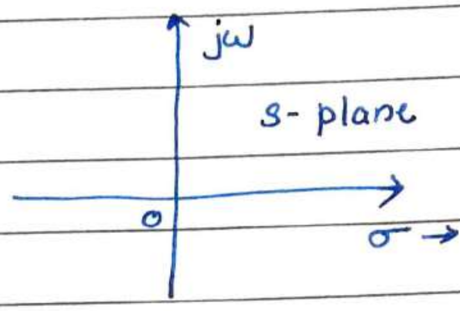
$$= \frac{1}{2\pi} \int_{c-j\infty}^{c+j\infty} X(s)\, e^{st}\, ds$$

where 'c' is constant and responsible for convergence of integral.

Relationship between Laplace Transform and Fourier Transform

For a signal $x(t)$, its Laplace transform $X(s)$ is given by,

$$L[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$



Put $s = \sigma + j\omega$, we get

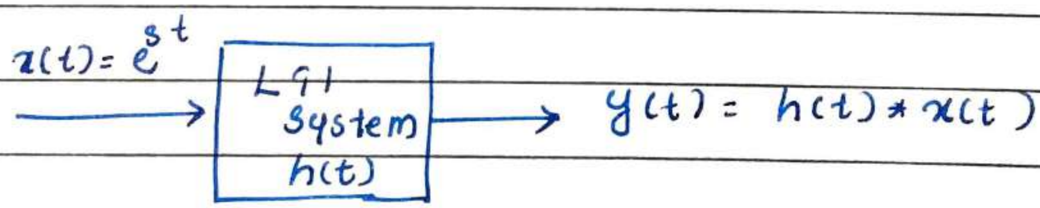
$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-(\sigma + j\omega)t}\, dt$$

$$= \int_{-\infty}^{\infty} \left[x(t)\, e^{-\sigma t}\right] e^{-j\omega t}\, dt$$

$$L[x(t)] = FT\left[x(t)\, e^{-\sigma t}\right]$$

At $\sigma = 0$, $s = j\omega \implies L[x(t)] = FT[x(t)]$

Eigen Value and Eigen Function



$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\, e^{s(t-\tau)}\, d\tau$$

$$= e^{st} \int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau = e^{st} \int_{-\infty}^{\infty} h(t)\, e^{-st}\, dt$$

$$y(t) = e^{st} H(s)$$

where $H(s) = L[h(t)]$ is known as transfer function of the LTI system

* For a signal, if system output is constant times the input, then the input is referred to as eigen function of system and amplitude factor is referred as eigen value of the system.

* Thus, 'e^{st}' is an eigen function and $H(s)$ is the corresponding eigen value of the system.

Region of Convergence (ROC) of Laplace Transform

* ROC is the set of values of 's' for which the Laplace transform of the signal converges.

* The LT is guaranteed to converge if $x(t)e^{-\sigma t}$ is absolutely integrable.

ie $$\int_{-\infty}^{\infty} |x(t) e^{-\sigma t}| dt < \infty$$

This guarantees that $X(s)$ will be within finite limit.

$$|X(s)| < \infty$$

* Thus, ROC consists of those values of 'σ' [Re(s)] for which the FT of $x(t) \cdot e^{-\sigma t}$ converges.

Properties of ROC

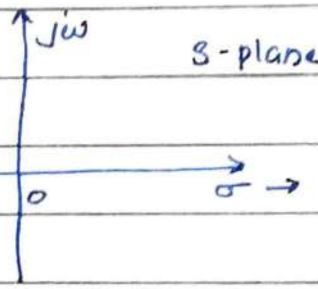
(i) ROC along with algebraic expression of $X(s)$ provides complete specification of LT.

(ii) In the case of identical algebraic expression of $X(s)$ the LT are distinguishable only by ROC.

(iii) The ROC of $X(s)$ consists of strips parallel to the $j\omega$-axis in the s-plane.

(iv) If signal $x(t)$ is of finite duration and absolutely integrable, then ROC is the entire s-plane.

The s-plane and poles and zeros



s-plane : with $s = \sigma + j\omega$, horizontal axis represents the $Re\{\sigma\} = \sigma$ and vertical axis represents the $Im\{\sigma\} = j\omega$.

Consider a ratio of two polynomials, say $F(s)$

$$F(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + b_0}$$

Factoring the polynomials in numerator and denominator.

$$F(s) = \frac{N(s)}{D(s)} = \frac{k(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)} \; ; \quad k = \frac{b_m}{a_n}$$

Zeros : Roots of $N(s) = 0$ are defined as zeros when the function $F(s)$ vanishes.

It is represented as 'z_i'.

i.e. $\lim_{s \to z_i} F(s) = 0$

Poles: Roots of $D(s) = 0$ are defined as poles where function $f(s)$ becomes infinity (unbounded).

It is represented as p_i'.

i.e. $\lim_{s \to p_i} F(s) = \infty$

* Locations of zeros in s-plane are denoted with 'o' symbol and locations of poles with 'x' symbol.

Laplace Transform to Some Basic Signals

(i) $x(t) = e^{-at} u(t)$; $a > 0$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt = \int_{-\infty}^{\infty} e^{-at} u(t) e^{-st} dt$$

$$= \int_{\cancel{-\infty}\ 0}^{\infty} e^{-(s+a)t} dt = \left. \frac{e^{-(s+a)t}}{-(s+a)} \right|_{0}^{\infty}$$

Substitute $s = \sigma + j\omega$,

$$X(s) = \left. \frac{e^{-(\sigma + j\omega + a)t}}{-(\sigma + j\omega + a)} \right|_{0}^{\infty}$$

Now, as $t \longrightarrow \infty$

$$\lim_{t \to \infty} e^{-(\sigma + a + j\omega)t} = \begin{cases} 0 & \sigma + a > 0 \\ \infty & \sigma + a < 0 \end{cases}$$

$\because |e^{j\omega t}| = 1$; regardless of ω.

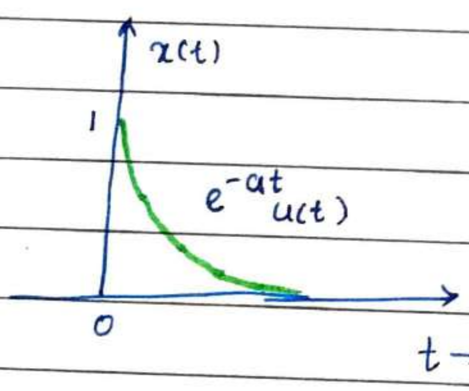
With the use of this result, $X(s)$ yields,

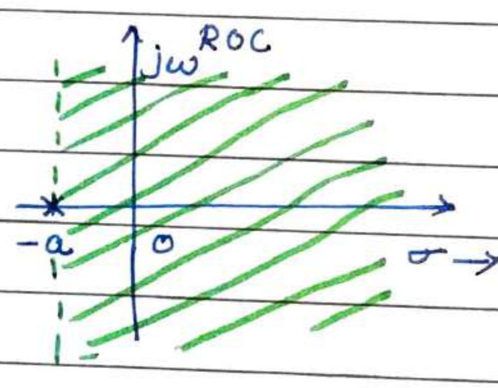
$$X(s) = \frac{1}{\sigma + a + j\omega} \quad ; \sigma + a > 0$$

$$= \frac{1}{s+a} \quad ; \sigma > -a$$

$$= \frac{1}{s+a} \quad ; Re(s) > -a$$

$\therefore$ $e^{-at}u(t) \xleftrightarrow{LT} \frac{1}{s+a}$; $Re(s) > -a$





(ii) $x(t) = -e^{-at}u(-t)$; $a > 0$

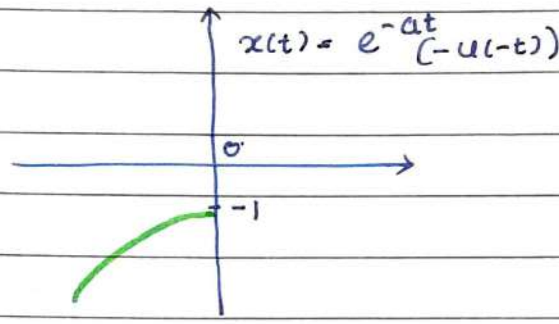
$$X(s) = \int_{-\infty}^{\infty} e^{-at}(-u(-t))e^{-st}\, dt$$

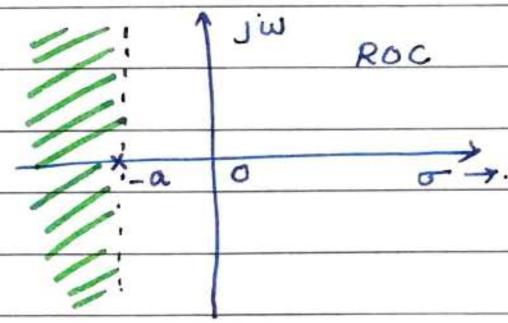
$$= -\int_{-\infty}^{0} e^{-(s+a)t}\, dt = -\left. \frac{e^{-(s+a)t}}{-(s+a)} \right|_{-\infty}^{0}$$

$$X(s) = \frac{1}{s+a} \; ; \quad \sigma + a < 0$$

$$= \frac{1}{s+a} \; ; \quad Re(s) < -a$$

$$\therefore \quad e^{-at}(-u(-t)) \xleftrightarrow{FT} \frac{1}{s+a} \; ; \quad Re(s) < -a$$





Comment: Since ROC does not contain $j\omega$-axis, the Fourier transform of $e^{-at}[-u(-t)]$ doesnot exist.

Note:

* Laplace transform of $e^{-at}u(t)$ and $e^{-at}[-u(-t)]$ has the same algebraic expression $\frac{1}{s+a}$; but different ROCs.

* Laplace transform has unique solution only when ROC is defined.

(iii) $x(t) = e^{at}u(t)$; $a > 0$

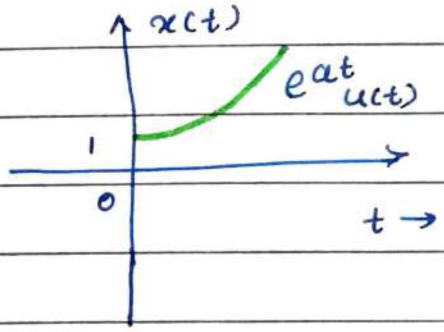
$$X(s) = \int_{-\infty}^{\infty} e^{at} u(t) e^{-st} \, dt$$

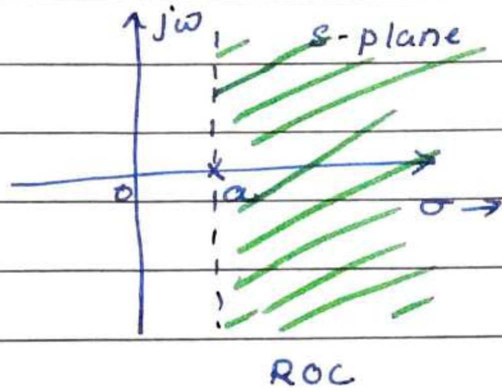
$$= \int_{0}^{\infty} e^{-(s-a)t} \, dt = \left. \frac{e^{-(s-a)t}}{-(s-a)} \right|_{0}^{\infty}$$

$$\Rightarrow X(s) = \frac{1}{s-a} \quad ; \quad \sigma - a > 0$$

$$= \frac{1}{s-a} \quad ; \quad Re(s) > a$$

$$\therefore \quad e^{at} u(t) \xleftrightarrow{LT} \frac{1}{s+a} \quad ; \quad Re(s) > a$$





Comment: Since ROC doesn't contain $j\omega$-axis, hence Fourier transform doesn't exist for $e^{at} u(t)$

(iv) $x(t) = -e^{at} u(-t) \quad ; \quad a > 0$

$$X(s) = \int_{-\infty}^{\infty} -e^{at} u(-t)\, e^{-st}\, dt$$

$$= -\int_{-\infty}^{0} e^{-(s-a)t}\, dt = -\left.\frac{e^{-(s-a)t}}{-(s-a)}\right|_{-\infty}^{0}$$

$$= \left.\frac{e^{-(\sigma - a + j\omega)t}}{(\sigma - a + j\omega)}\right|_{-\infty}^{0}$$

$$\therefore \quad X(s) = \frac{1}{s-a} \quad ; \quad \sigma - a < 0$$

$$= \frac{1}{s-a} \quad ; \quad Re[s] < a$$

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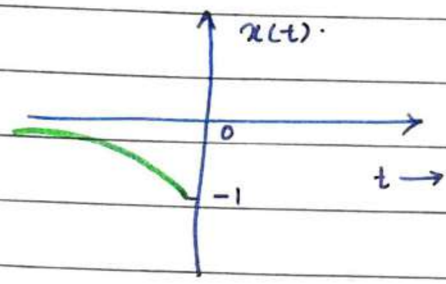
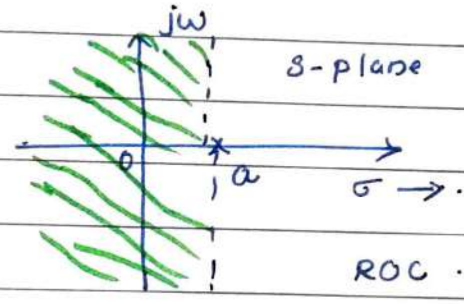
$$\therefore \quad e^{at}[-u(-t)] \xleftrightarrow{LT} \frac{1}{s-a} \; ; \; Re[s] < 0$$





(v) $x(t) = u(t)$

Since $e^{-at}u(t) \xleftrightarrow{LT} \frac{1}{s+a} \; ; \; Re(s) > -a$

Put $a = 0$

$$u(t) \xleftrightarrow{LT} \frac{1}{s} \; ; \; Re(s) > 0$$

(vi) $x(t) = u(-t)$

Since $e^{-at}[-u(-t)] \xleftrightarrow{LT} \frac{1}{s+a} \; ; \; Re[s] < -a$

Put $a = 0$

$$\Rightarrow \quad u(-t) \xleftrightarrow{LT} -1/s \; ; \; Re(s) < 0$$

(vii) $x(t) = t \cdot u(t)$

$$e^{-at}u(t) \xleftrightarrow{LT} \frac{1}{s+a} \; ; \; Re(s) > -a$$

Differentiating both sides

$$\Rightarrow (-t)\, e^{-at}u(t) \xleftrightarrow{LT} \frac{-1}{(s+a)^2} \; ; \; Re(s) > -a$$

when $a = 0$;

$$t\,u(t) \xleftrightarrow{LT} 1/s^2 \; ; \; Re(s) > 0$$

or $r(t)$ $\cancel{u(t)}$ $\xleftrightarrow{LT} 1/s^2$, $Re(s) > 0$

(viii) $x(t) = \sin \omega t\, u(t)$ and $\cos \omega t \cdot u(t)$

$$e^{-at} u(t) \xleftrightarrow{LT} \frac{1}{s+a} \; ; \quad Re(s) > -a$$

Put $a = j\omega$,

$$\Longrightarrow \quad e^{-j\omega t} u(t) \xleftrightarrow{LT} \frac{1}{s+j\omega} \; ; \quad Re(s) > 0$$

$$\Longrightarrow \quad [\cos \omega t - j \sin \omega t]\, u(t) \xleftrightarrow{LT} \frac{1}{(s+j\omega)} \cdot \frac{(s-j\omega)}{(s-j\omega)}$$

$$\Longrightarrow \quad \cos \omega t \cdot u(t) - j \sin \omega t\, u(t) \xleftrightarrow{LT} \frac{s}{s^2+\omega^2} - j \frac{\omega}{s^2+\omega^2}$$

on comparing the real and imaginary parts

$$\boxed{\begin{array}{lll} \cos \omega t \;\; u(t) \xleftrightarrow{LT} \dfrac{s}{s^2+\omega^2} & ; & Re(s) > 0 \\ \sin \omega t \cdot u(t) \xleftrightarrow{LT} \dfrac{\omega}{s^2+\omega^2} & ; & Re(s) > 0 \end{array}}$$

(ix) $x(t) = e^{-\alpha t} \sin \omega t \cdot u(t)$ and $e^{-\alpha t} \cos \omega t \cdot u(t)$

Since $e^{-at} u(t) \xleftrightarrow{LT} \dfrac{1}{s+a}$; $Re(s) > -a$

Put $a = \alpha + j\omega$,

$$e^{-(\alpha+j\omega)t} u(t) \xleftrightarrow{LT} \frac{1}{s+\alpha+j\omega} \; ; \; Re(s) > -\alpha$$

$$e^{-\alpha t}[\cos\omega t - j\sin\omega t]\,u(t) \xleftrightarrow{LT} \frac{1}{(s+\alpha)+j\omega} \times \frac{(s+\alpha)-j\omega}{(s+\alpha)-j\omega}$$

$$\Rightarrow e^{-\alpha t}[\cos\omega t - j\sin\omega t]\,u(t) \xleftrightarrow{LT} \frac{s+\alpha}{(s+\alpha)^2+\omega^2} - \frac{j\omega}{(s+\alpha)^2+\omega^2}$$

on comparing,

$$e^{-\alpha t}\cos\omega t \cdot u(t) \xleftrightarrow{LT} \frac{s+\alpha}{(s+\alpha)^2+\omega^2} \;;\; Re(s) > -\alpha$$

$$e^{-\alpha t}\sin\omega t \cdot u(t) \xleftrightarrow{LT} \frac{\omega}{(s+\alpha)^2+\omega^2} \;;\; Re(s) > -\alpha$$

(x) $x(t) = \sin(\omega t + \theta)$

$$L[\sin(\omega t+\theta)] = L[\sin\omega t \cdot \cos\theta + \cos\omega t \cdot \sin\theta]$$

$$= \frac{\omega}{s^2+\omega^2}\cdot\cos\theta + \frac{s}{s^2+\omega^2}\cdot\sin\theta$$

(xi) $x(t) = \cos(\omega t + \theta)$

$$L[\cos(\omega t+\theta)] \Longrightarrow L[\cos\omega t \cdot \cos\theta - \sin\omega t \cdot \sin\theta]$$

$$= \cos\theta \cdot \frac{s}{s^2+\omega^2} - \sin\theta \cdot \frac{\omega}{s^2+\omega^2}$$

(xii) $x(t) = \delta(t)$

$$X(s) = \int_{-\infty}^{\infty} \delta(t) e^{-st}\,dt = \int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

$\delta(t) \xleftrightarrow{LT} 1$; ROC : entire s-plane.

(xiii) $x(t) = t^n u(t)$

$$X(s) = \int_{-\infty}^{\infty} t^n u(t)\, e^{-st} dt = \int_{0}^{\infty} t^n e^{-st} dt$$

$$= \left[t^n \frac{e^{-st}}{-s} \right]_0^{\infty} - \int_0^{\infty} (n \cdot t^{n-1}) \left[\frac{e^{-st}}{-s} \right] dt$$

$$= \frac{n}{s} \int_0^{\infty} t^{n-1} e^{-st} dt$$

$$= \frac{n(n-1)(n-2)\cdots 3\cdot 2\cdot 1}{s^n} \cdot \frac{1}{s} = \frac{n!}{s^{n+1}}$$

$$\therefore \quad t^n u(t) \xleftrightarrow{LT} \frac{n!}{s^{n+1}}$$

(xiv) $x(t) = e^{-a|t|}$

$$x(t) = e^{-a|t|} = e^{-at} u(t) + e^{at} u(-t)$$

We have

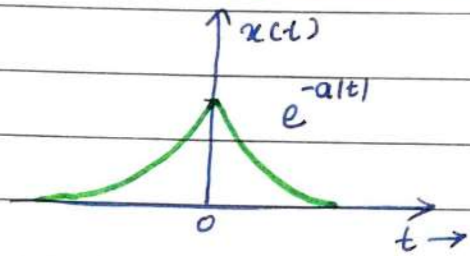
$$e^{-at} u(t) \xleftrightarrow{LT} \frac{1}{s+a} \quad ; \quad Re(s) > -a$$

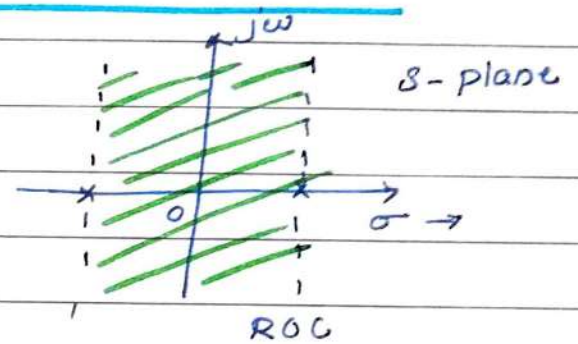
$$e^{at}[-u(-t)] \xleftrightarrow{LT} \frac{1}{s-a} \quad ; \quad Re(s) < a$$

$\therefore$ The ROC of resultant signal for which Laplace transform converges is $-a < Re(s) < a$

$$\therefore \quad X(s) = \frac{1}{s+a} + \frac{-1}{s-a} = \frac{-2a}{s^2-a^2}$$

$$e^{-a|t|} \xleftrightarrow{LT} \frac{-2a}{s^2-a^2} \quad ; \quad -a < Re(s) < a$$





Note: Laplace transform of 1 and sgn(t) doesn't exist since there is no common ROC

* $1 = u(t) + u(-t)$
* $sgn(t) = u(t) - u(-t)$

Q. $x(t) = e^{-t}u(t) + e^{-2t}u(t)$. Find the LT of $x(t)$.

Ans:

$$e^{-t}u(t) \xleftrightarrow{LT} \frac{1}{s+1} \quad ; \quad Re(s) > -1$$
$$e^{-2t}u(t) \xleftrightarrow{LT} \frac{1}{s+2} \quad , \quad Re(s) > -2$$

Hence common ROC exists and it is right of right most pole in s-plane and it is $Re(s) > -1$

$$\therefore \quad X(s) = \left[\frac{1}{s+1} + \frac{1}{s+2}\right] \quad ; \quad Re(s) > -1$$

$$= \frac{2s+3}{(s+1)(s+2)} \quad ; \quad Re(s) > -1$$

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Properties of Laplace Transform

(i) Linearity

If $x_1(t) \xleftrightarrow{LT} X_1(S)$ with ROC = R_1

and $x_2(t) \xleftrightarrow{LT} X_2(s)$ with ROC = R_2

then $\alpha x_1(t) + \beta x_2(t) \xleftrightarrow{LT} \alpha X_1(S) + \beta X_2(S)$

with ROC = $R_1 \cap R_2$

(ii) Time shifting

If $x(t) \xleftrightarrow{LT} X(S)$ with ROC = R

then $x(t-t_0) \xleftrightarrow{LT} e^{-st_0} X(S)$ with ROC = R

Proof:

$$L[x(t-t_0)] = \int_{-\infty}^{\infty} x(t-t_0)\, e^{-st}\, dt$$

Put $t - t_0 = \tau$; $dt = d\tau$

$$= \int_{-\infty}^{\infty} x(\tau)\, e^{-s(\tau+t_0)}\, d\tau = e^{-st_0} \int_{-\infty}^{\infty} x(\tau) e^{-s\tau}\, d\tau$$

$$= e^{-st_0} X(S)$$

(iii) Shift in domain

If $x(t) \xleftrightarrow{LT} X(S)$ with ROC = R

then, $e^{S_0 t} x(t) \xleftrightarrow{LT} X(s-s_0)$ with ROC = R + Re [s_0]

Proof:

$$LT\left[x(t) e^{S_0 t}\right] = \int_{-\infty}^{\infty} x(t)\, e^{-(s-s_0)t}\, dt = X(s-s_0)$$

Note that if $X(s)$ has a pole or zero at $s = a$, then $X(s-s_0)$ has pole or zero at $s - s_0 = a \Rightarrow s = a + s_0$.

(iv) Time ~~shifting~~ scaling

$$x(t) \xleftrightarrow{LT} X(s) \text{ with } ROC = R$$

then, $x(at) \xleftrightarrow{LT} \frac{1}{|a|} X(s/a)$ with $ROC = |a|R$

Proof:

$$L[x(at)] = \int_{-\infty}^{\infty} x(at) e^{-st} dt$$

Put $at = \tau$; $dt = d\tau/a$

$$= \int_{-\infty}^{\infty} x(\tau) e^{-\frac{s}{a}\tau} d\tau/a = \frac{1}{a} X(s/a)$$

as 'a' is real constant may be +ve or −ve, so generalizing it

$$L[x(at)] = \frac{1}{|a|} X(s/a)$$

(v) Time Reversal

If $x(t) \xleftrightarrow{LT} X(s)$ with $ROC = R$

then, $x(-t) \xleftrightarrow{LT} X(-s)$ with $ROC = -R$

(vi) Differentiation in time

If $x(t) \xleftrightarrow{LT} X(\omega)$

then, $\frac{dx(t)}{dt} \xleftrightarrow{LT} sX(s) - x(0^-)$

Repeated application of this property yields,

$$\frac{d^2x(t)}{dt^2} \xleftrightarrow{LT} s^2 X(s) - s x(0^-) - \dot{x}(0^-)$$

Ily,
$$\frac{d^n x(t)}{dt^n} \xleftrightarrow{LT} s^n X(s) - s^{n-1} x(0^-) - s^{n-2}\dot{x}(0^-) - \cdots - x^{(n-1)}(0^-)$$

$$= s^n X(s) - \sum_{k=1}^{n} s^{n-k} x^{(n-k)}(0^-)$$

where, $x^{(r)}(0^-) = \left.\frac{d^r x(t)}{dt^r}\right|_{t=0^-}$

(vii) Integration in time

If $x(t) \xleftrightarrow{LT} X(s)$

then,
$$\int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{LT} \frac{X(s)}{s} + \frac{\int_{-\infty}^{0^-} x(\tau)\,d\tau}{s}$$

(viii) Convolution in time domain.

If $x_1(t) \xleftrightarrow{LT} X_1(s)$ and $x_2(t) \xleftrightarrow{LT} X_2(s)$

ROC : R_1 and ROC : R_2

then $x_1(t) * x_2(t) \xleftrightarrow{LT} X_1(s) X_2(s)$ with ROC $R_1 \cap R_2$

Proof :

$$L[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} [x_1(t) * x_2(t)] e^{-st}\,dt$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1(\tau) x_2(t-\tau) e^{-st}\,d\tau \cdot dt \times \frac{e^{st}}{e^{st}}$$

On rearranging, we get

$$= \int_{-\infty}^{\infty} x_1(\tau) \cdot e^{-s\tau} d\tau \int_{-\infty}^{\infty} x_2(t-\tau) e^{-s(t-\tau)} dt = X_1(s) \cdot X_2(s)$$

(ix) Multiplication in time domain.

Multiplication in time domain corresponds to convolution in s-domain.

If $x_1(t) \xleftrightarrow{LT} X_1(s)$ with ROC = R_1

and $x_2(t) \xleftrightarrow{LT} X_2(s)$ with ROC : R_2.

then, $x_1(t) \cdot x_2(t) \xleftrightarrow{LT} \frac{1}{2\pi j} \cdot [X_1(s) * X_2(s)]$ with ROC containing $R_1 \cap R_2$

Proof:

$$LT[x_1(t) \cdot x_2(t)] = \int_{-\infty}^{\infty} x_1(t) x_2(t) e^{-st} dt$$

$$= \int_{-\infty}^{\infty} x_1(t) \left[\frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} X_2(s_0) e^{s_0 t} ds_0 \right] e^{-st} dt$$

On rearranging the integration,

$$= \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} X_2(s_0) \cdot \int_{-\infty}^{\infty} [x_1(t) e^{s_0 t}] e^{-st} dt \cdot ds_0$$

By using the frequency shifting property.

$$= \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} X_2(s_0) X_1(s-s_0) ds_0$$

$$= \frac{1}{2\pi j} \left[X_1(s) * X_2(s) \right]$$

(x) Differentiation in s-domain.

If $x(t) \xleftrightarrow{LT} X(s)$ with ROC = R

$-t \cdot x(t) \xleftrightarrow{LT} \frac{d}{ds} X(s)$ with ROC = R.

Proof:

We have $X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$

Differentiating w.r.t 's',

$$\frac{d X(s)}{ds} = \int_{-\infty}^{\infty} x(t)(-t) e^{-st} dt$$

$$= LT\left[-t x(t)\right]$$

$\therefore \quad -t x(t) \xleftrightarrow{LT} \frac{d X(s)}{ds}$

||ly,

$$(-t)^n x(t) \xleftrightarrow{LT} \frac{d^n X(s)}{ds^n}$$

(xi) Integration in s-domain.

If $x(t) \xleftrightarrow{LT} X(s)$ with ROC = S

then, $\frac{x(t)}{t} \xleftrightarrow{LT} \int_{s}^{\infty} X(s) \cdot ds$.

(xii) Conjugate property

If $x(t) \longleftrightarrow X(s)$

then $x^*(t) \longleftrightarrow X^*(s^*)$

Q1 Find the inverse LT of.

$$X(s) = \frac{s+3+5e^{-2s}}{(s+1)(s+2)}$$

Ans:

$$X(s) = \underbrace{\frac{s+3}{(s+1)(s+2)}}_{X_1(s)} + \underbrace{\frac{5e^{-2s}}{(s+1)(s+2)}}_{X_2(s)}$$

where,

$$X_1(s) = \frac{s+3}{(s+1)(s+2)} = \frac{2}{s+1} - \frac{1}{s+2}$$

$$X_2(s) = \frac{5}{(s+1)(s+2)} = \frac{5}{s+1} - \frac{5}{s+2}$$

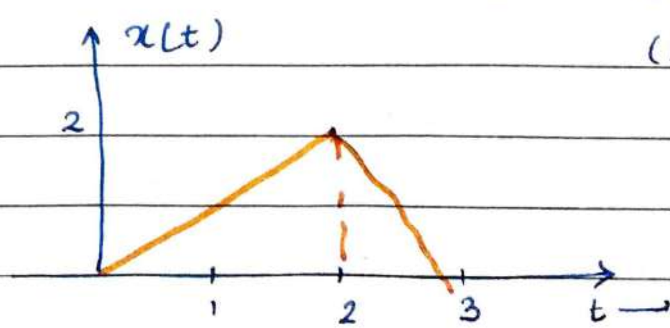
$$\therefore \quad x_1(t) = (2e^{-t} - e^{-2t})\,u(t)$$

$$x_2(t) = 5\,(e^{-t} - e^{-2t})\,u(t)$$

Since $X(s) = X_1(s) + X_2(s)\,e^{-2s}$

$$x(t) = (2e^{-t} - e^{-2t})\,u(t) + 5\,(e^{-t} - e^{-2t}]\,u(t-2)$$

Q2. (a) Find LT of the signal given in Figure.



(b) Find LT using time-differentiation and time shifting property.

Ans: $\frac{1}{s^2}\left[1 - 3e^{-2s} + 2e^{-3s}\right]$

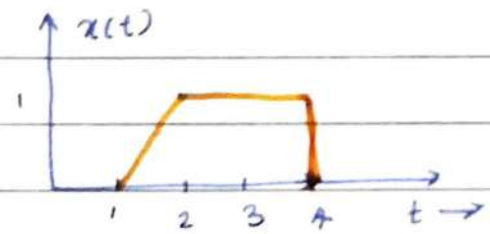
Q3. Find the LT of $c(t) = e^{at}u(t) * e^{bt}u(t)$ and then $c(t)$

Ans: By using the time-convolution property.

$$C(s) = \frac{1}{(s-a)(s-b)} = \frac{1}{a-b}\left[\frac{1}{s-a} - \frac{1}{s-b}\right]$$

Then $c(t) = \frac{1}{a-b}\left(e^{at} - e^{bt}\right)u(t)$

Q4. Find the LT of $x(t)$ shown in figure.



Ans: $x(t) = (t-1)\left[u(t-1) - u(t-2)\right] + \left[u(t-2) - u(t-4)\right]$

$= (t-1)u(t-1) - (t-1)u(t-2) + u(t-2) - u(t-4)$

Now $(t-1)u(t-2) = (t-2+1)u(t-2)$

$= (t-2)u(t-2) + u(t-2)$

$\therefore x(t) = (t-1)u(t-1) - (t-2)u(t-2) - u(t-4)$

Apply the time-differentiation property to

$$tu(t) \xleftrightarrow{LT} 1/s^2 \text{ yields,}$$

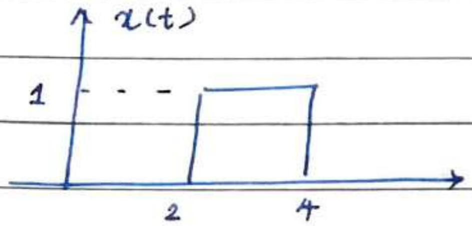
$$(t-1)u(t-1) \xleftrightarrow{LT} 1/s^2\, e^{-s} \quad \text{and}$$

$$(t-2)u(t-2) \xleftrightarrow{LT} 1/s^2\, e^{-2s}$$

$$u(t-4) \xleftrightarrow{LT} 1/s\, e^{-4s}$$

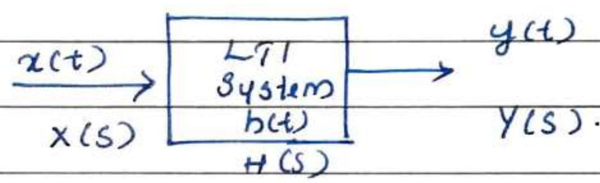
$$\therefore X(s) = \frac{+1}{s^2}e^{-s} - \frac{1}{s^2}e^{-2s} - \frac{1}{s}e^{-4s}$$

Q5 Find the LT and ROC for the following signal.



LTI System and Laplace Transform

Consider an LTI system with impulse response $h(t)$ whose input and output are $x(t)$ and $y(t)$ respectively.



In time-domain,

$$y(t) = h(t) * x(t)$$

Using the time-convolution property of Laplace transform

$$Y(S) = X(S) \cdot H(S)$$

$$H(S) = \frac{Y(S)}{X(S)}$$

H(S) is called Transfer Function.

Q. An LTI system has an impulse response of e^{2t}, $t>0$. If the initial conditions are zeros and input is e^{3t}, the output for $t>0$ is,

Ans:

$$h(t) = e^{2t}$$

$$\therefore \quad H(S) = \frac{1}{S-2}$$

input $x(t) = e^{3t}$

$$\therefore\ X(s) = \frac{1}{(s-3)}$$

$$\therefore\ Y(s) = H(s)X(s) = \frac{1}{(s-2)(s-3)} = \frac{1}{(s-3)} - \frac{1}{(s-2)}$$

$$\therefore\ y(t) = e^{3t} - e^{2t}$$

Impulse Response and Step Response

$$h(t) \xleftrightarrow{LT} H(s)$$

Step response $s(t) \xleftrightarrow{LT} S(s)$

As we know,

$$s(t) = h(t) * u(t)$$

$$\therefore\ S(s) = H(s)\cdot 1/s$$

$$\therefore\ H(s) = s\,S(s)$$

Q. The TF of a system is given by, $H(s) = \frac{1}{s^2(s-2)}$. The impulse response of the system is,

Ans: $h(t) = L^{-1}(H(s))$

$$= L^{-1}\left[\frac{1}{s^2(s-2)}\right] = L^{-1}\left[\frac{1}{s^2} \times \frac{1}{s-2}\right]$$

$$= \left[t * e^{2t}\right] u(t)$$

Steady-State Response of an LTI System to sinusoidal input $x(t) = A\cos(\omega_0 t + \theta)$

Consider an LTI system with Transfer function $H(s)$

$$\underset{A\cos(\omega_0 t+\theta)}{x(t)} \longrightarrow \boxed{\begin{array}{c}\text{LTI system}\\ H(s)\end{array}} \longrightarrow y(t) = \underline{k\cdot A\cos(\omega_0 t+\theta+\phi)}$$

where, $k\angle\phi = H(s)\big|_{s=j\omega_0}$.

Q. Given $H(s) = \dfrac{s-2}{s^2+4s+4}$. Find the steady state response for $x(t) = 8\cos 2t$.

Ans: $x(t) = 8\cos 2t$, $A = 8$ and $\omega_0 = 2$ rad/s

$$y(t) = H(s)\big|_{s=j2} \cdot 8\cos 2t = \left.\frac{s-2}{s^2+4s+4}\right|_{s=j2} \cdot (8\cos 2t)$$

$$= \frac{j2-2}{(j2)^2+4(j2)+4} \cdot (8\cos 2t)$$

$$= 8(0.353)\cos(2t-135^\circ)$$

Causality

- For a system to be causal if the impulse response $h(t)=0$ for $t<0$
- Thus, ~~thus~~ it is right-sided and corresponding ROC is right of the right most pole.
- However, it does not guarantee that the system is causal, rather it guarantee only that impulse response is right sided.

Stability

Let $H(s) = \frac{P(s)}{Q(s)} = b_0 \frac{(s-z_1)(s-z_2)\cdots(s-z_M)}{(s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_N)}$

Now, consider the denominator $Q(s)$. $= \sum_{i=1}^{N} \frac{A_i}{s-\lambda_i}$

* Since if $p(s)$ and $Q(s)$ have any common factors, they cancel each other and effective denominator of $H(s)$ is not necessarily equal to $Q(s)$.

* $H(s)$ and $h(t)$ are defined in terms of measurements at the external terminals.
* However, $Q(s)$ is an internal discription.
* We can define, the BIBO stability (external) from $H(s)$.

* If all the poles of $H(s)$ are in LHP (Left Hand Plane), all the terms in $h(t)$ are decaying exponentials. and $h(t)$ is abosolutely integrable. and the s/m is BIBO stable.

for Eg: $\frac{1}{s+a} \longrightarrow e^{-at}$

$$h(t) = L^{-1}(H(s)) = \sum_{i=1}^{N} A_i e^{\lambda_i t}$$

If $P(s)$ and $Q(s)$ have no common factors, the asymptotic stability criterion in terms of poles of TF are listed below.

1. An LTIC system is asymptotically stable if and only if all the poles of its transfer function $H(s)$ are in the LHP. The poles may be simple or repeated.

2. An LTIC system is marginally unstable if and only if either one or both of the following conditions exist. (i) atleast one pole of $H(s)$ is in the RHP (ii) there are repeated poles of $H(s)$ on the imajinary axis.

3. An LTIC system is marginally stable if and only if there are some unrepeated poles on the imaginary axis.

Note: Location of zeros of $H(s)$ has no role in determining the system stability.

Q. Solve the second order linear-differential equation

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = \frac{dx(t)}{dt} + x(t) \quad \text{---(1)}.$$

for the initial conditions $y(0^-) = 2$ and $\dot{y}(0^-) = 1$ and input $x(t) = e^{-4t}u(t)$.

Sol.

$$\frac{dy}{dt} \xleftrightarrow{LT} sY(s) - y(0^-) = sY(s) - 2$$

and

$$\frac{d^2y}{dt^2} \longleftrightarrow s^2Y(s) - s\,y(0^-) - \dot{y}(0^-) = s^2Y(s) - 2s - 1$$

For $x(t) = e^{-4t}u(t)$

$$X(s) = \frac{1}{s+4} \quad \text{and} \quad \frac{dx}{dt} \longleftrightarrow sX(s) - x(0^-)$$

$$= \frac{s}{s+4} - 0 = \frac{s}{s+4}$$

Take LT of Eq. (1)

$$\left[s^2 Y(s) - 2s - 1\right] + 5\left[sY(s) - 2\right] + 6Y(s) = \frac{s}{s+4} + \frac{1}{s+4}$$

On rearranging, we get

$$(s^2+5s+6)Y(s) - (2s+11) = \frac{s+1}{s+4}$$

$$(s^2+5s+6)Y(s) = (2s+11) + \frac{(s+1)}{(s+4)} = \frac{2s^2+20s+45}{s+4}$$

$$Y(s) = \frac{2s^2+20s+45}{(s+4)(s^2+5s+6)} = \frac{A}{(s+2)} + \frac{B}{(s+3)} + \frac{C}{(s+4)}$$

By applying partial fraction method

$$Y(s) = \frac{13/2}{s+2} - \frac{3}{s+3} - \frac{3/2}{s+4}$$

Taking the inverse LT,

$$y(t) = \left[\frac{13}{2}e^{-2t} - 3e^{-3t} - \frac{3}{2}e^{-4t}\right]u(t)$$

Q. An LTIC system is defined by,

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y(t) = \frac{dx}{dt} + x(t)$$

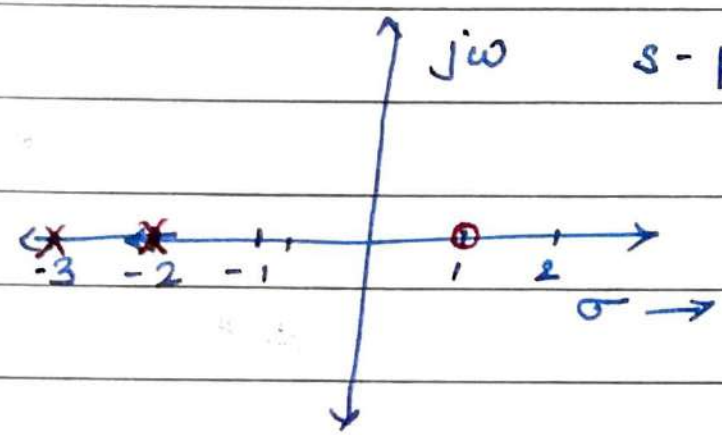
and the initial conditions are zero. Find the pole-zero plot and check the stability of the system.

Ans: The equation can be represented in terms of the differentiation operator as,

$(D^2 + 5D + 6)\, y(t) = (D+1)\, x(t)$.

∴ The transfer function is,

$$H(s) = \frac{P(s)}{Q(s)} = \frac{s+1}{s^2+5s+6} = \frac{(s+1)}{(s+3)(s+2)}$$



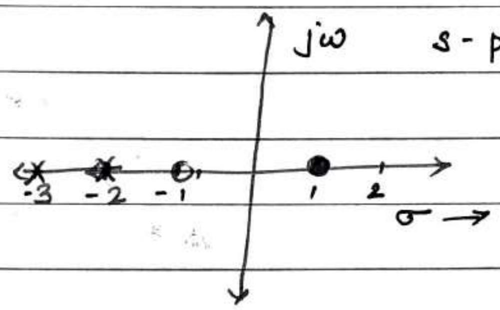
Pole - zero plot

Since all the poles of the TF are in the LHP of s-plane, the system is stable.

$(D^2 + 5D + 6)\, y(t) = (D+1)\, x(t)$.

$\therefore$ The transfer function is,

$$H(s) = \frac{P(s)}{Q(s)} = \frac{s+1}{s^2+5s+6} = \frac{(s+1)}{(s+3)(s+2)}$$



Since all the poles of the TF are in the LHP of s-plane, the system is stable.

Dependence of Frequency Response on Poles and Zeros of H(s)

Ref: Lathi 4.10

Frequency response: of a system is basically the information about the filtering capability of the system.

A system transfer function is given as,

$$H(s) = \frac{P(s)}{Q(s)} = b_0 \frac{(s-z_1)(s-z_2)\cdots(s-z_M)}{(s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_N)}$$

* Where $z_1, z_2 \ldots z_M$ are the zeros of the s/m. $\lambda_1, \lambda_2, \ldots \lambda_N$ represents the poles of the s/m.

Now the value of TF at some frequency $s = p$ is

$$H(s)\Big|_{s=p} = b_0 \frac{(p-z_1)(p-z_2)\cdots(p-z_M)}{(p-\lambda_1)(p-\lambda_2)\cdots(p-\lambda_N)}$$

* This equation consists of factors of the form $p - z_i$ and $p - \lambda_i$
* The factor $p - z_i$ is a complex number represented by a vector drawn from point z to the point p in the complex plane

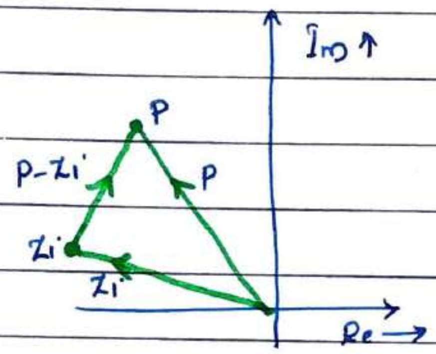


Fig (a)

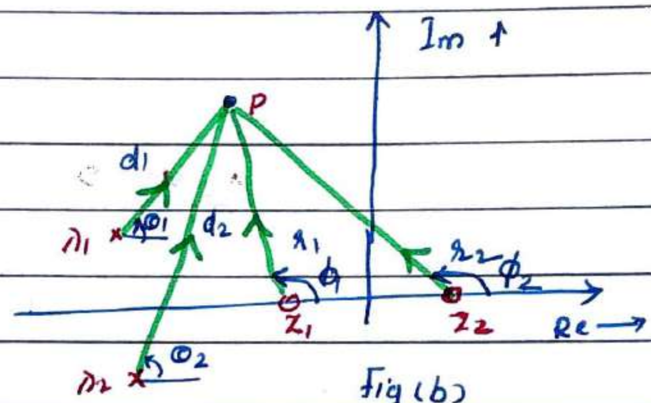


Fig (b)

$\therefore$ $p - z_i = r_i e^{j\phi_i}$, where r_i - length of $p - z_i$
and ϕ_i - angle with the horizontal axis to this line $p - z_i$.

$p - \lambda_i = d_i e^{j\theta_i}$ - where d_i - length of $p - \lambda_i$
and θ_i - angle with the horizontal axis to the line $p - \lambda_i$.

$$\therefore H(s)\Big|_{s=p} = b_0 \frac{(r_1 e^{j\phi_1})(r_2 e^{j\phi_2}) \cdots (r_M e^{j\phi_M})}{(d_1 e^{j\theta_1})(d_2 e^{j\theta_2}) \cdots (d_N e^{j\theta_N})}$$

$$= b_0 \frac{r_1 r_2 \cdots r_M}{d_1 d_2 \cdots d_N} e^{j[(\phi_1 + \phi_2 + \cdots \phi_M) - (\theta_1 + \theta_2 + \cdots \theta_N)]}$$

$$H(s)\Big|_{s=p} = b_0 \frac{r_1 r_2 \cdots r_M}{d_1 d_2 \cdots d_N}$$

$= b_0 \dfrac{\text{Product of the distances of zeros to P}}{\text{Product of the distances of poles to P}}$

and

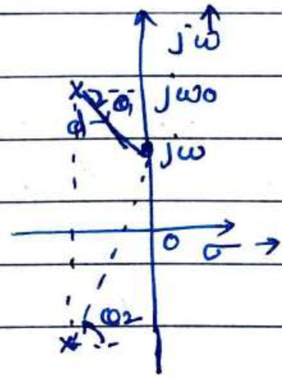
$$\angle H(s)\big|_{s=p} = (\phi_1 + \phi_2 + \cdots + \phi_N) - (\theta_1 + \theta_2 + \cdots + \theta_N)$$
$$= (\text{Sum of zero angles to } p) - (\text{sum of pole angles to } p)$$

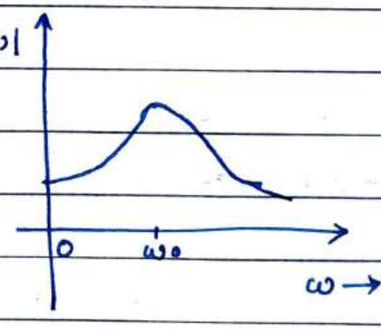
Note: If b_0 is negative, there is an additional phase π.

* To compute the frequency response $H(j\omega)$, we use $s = j\omega$ (a point on the imaginary axis), connect all poles and zeros to the point $j\omega$ and determine $|H(j\omega)|$ and $\angle H(j\omega)$.

* We can repeat this procedure for all values of 'ω' from 0 to ∞ to obtain the frequency response.

Gain Enchancement by a pole





Consider a single pole $-\alpha + j\omega_0$. To find the amplitude response $|H(j\omega)|$ for certain value of ω, we connect the pole to the point $j\omega$.

* If the length of this line is d, then $|H(j\omega)|$ is proportional to $1/d$.

$$\therefore \quad |H(j\omega)| = \frac{k}{d}$$

where, the exact value of 'k' is not important at this moment.

* As ω increases ~~beyond~~ from zero, 'd' decreases progressively until 'ω' reaches the value ω_0.
* As ω increases, beyond ω_0, d increases progressively.

* $\therefore$ A pole at $-\alpha + j\omega_0$ results in a frequency-selective behavior that enhances the gain at the frequency ω_0 (resonance).

* Also as the pole moves closer to the imaginary axis (as α is reduced), this enhancement (resonance) becomes more pronounced.

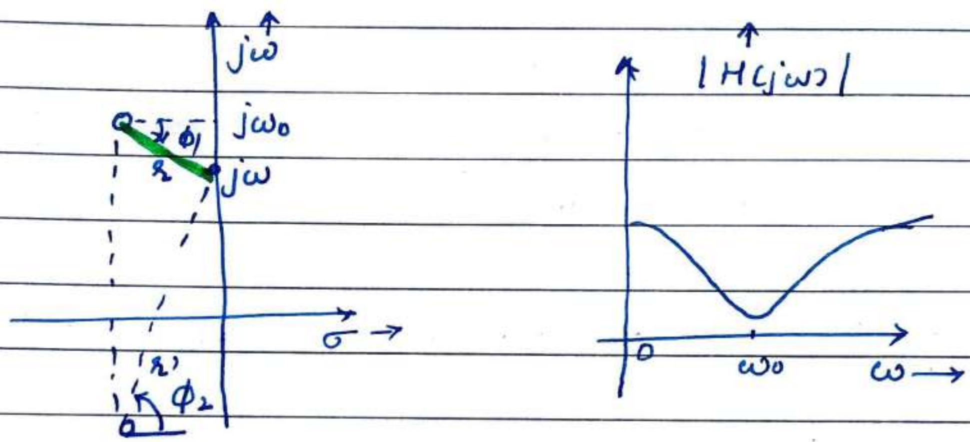
* In extreme case, when $\alpha = 0$ (pole on imaginary axis), the gain at ω_0 goes to infinity.

* Repeated poles further enhance the frequency-selective effect.

* For a real system, a complex pole $-\alpha + j\omega_0$ must accompany its conjugate $-\alpha - j\omega_0$

* Because, the conjugate pole is from $j\omega_0$, there is no dramatic change in the length d' as ω varies in the vicinity of ω_0.

Gain Suppression by a zero.



* Zeros at $-\alpha \pm j\omega_0$ will have excatly the opposite effect of suppressing the gain in the vicinity of ω_0

* A zero on the imaginary axis at $j\omega_0$ will totally suppress the gain (zero gain) at $\omega = \omega_0$.

* Repeated zeros will further enhance the effect.

Self Study

Influence of phase response with poles and zeros.

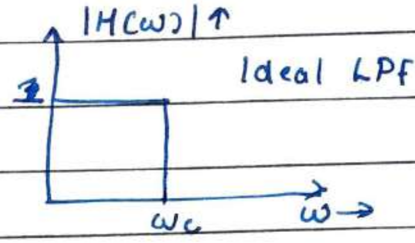
Low Pass Filters (LPF)

A typical LPF has a maximum gain at $\omega = 0$

* Because a pole enhances the gain at frequencies

in its vicinity, we need to place a pole (or poles) on the real axis opposite the origin ($j\omega = 0$).

For example,

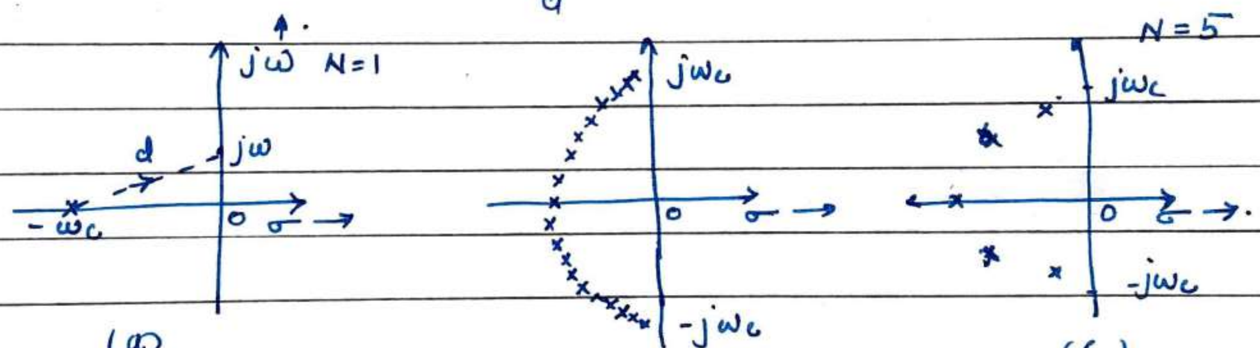


$$H(s) = \frac{\omega_c}{s + \omega_c}$$

The numerator is chosen ω_c to normalize the dc gain $H(0)$ to unity.

* If d is the distance from the pole $-\omega_c$ to a point $j\omega$, then

$$|H(\omega)| = \frac{\omega_c}{d} \quad \text{with} \quad H(0) = 1$$



(a) (b) (c)

* As ω increases, d increases and $|H(j\omega)|$ decreases monotonically with ω as given in Fig (a). with order $N = 1$.

Wall of Poles

* To attain the ideal low pass behaviour → (constant gain in the interval $0 : \omega_c$)

* To achive this we need to place a pole opposite every frequency in this band.

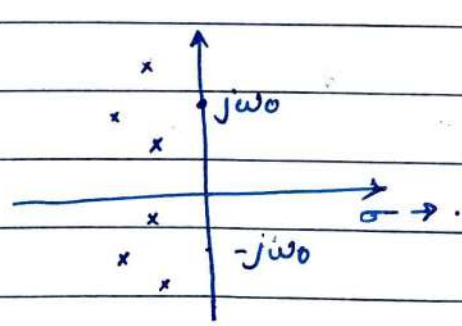
* Thus, we need to have a "continous wall of poles" facing the imaginary axis opposite the frequency band 0 to ω_c. as illustrated in fig.(b)

* In practice, we need to limit the no. of poles to a number N instead of infinite number of poles in this wall.
* N is called the order of the filter.
* As $N \to \infty$, the filter approaches ideal behaviour.

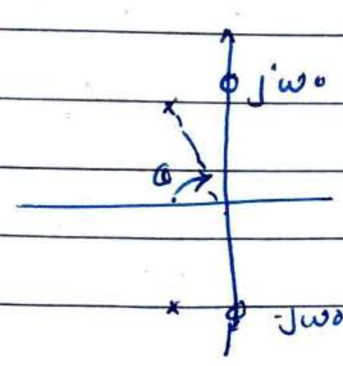
* Butterworth filters : wall of poles : semicircle.
* Chebyshev. filters : " : semiellipse.

	passband	stop band
Butterworth	Superior	Inferior
Chebyshev	Inferior (ripples)	Superior.

Pole-zero plots of BPF and Notch (BSF) filters.



BPF

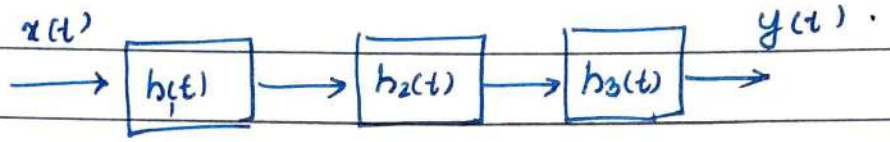


BSF

HW High pass filter (HPF)

Interconnection of LTI Systems

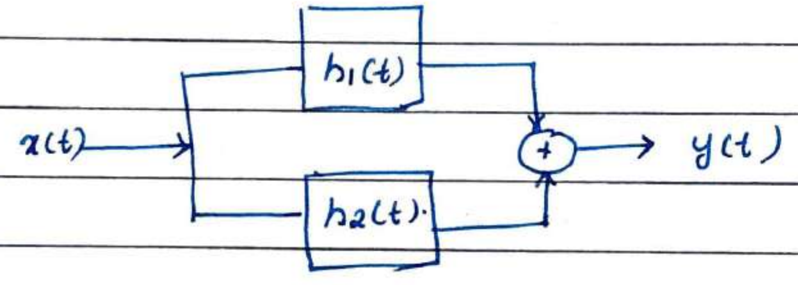
1. Series/ cascade connection.



Then, $h(t) = h_1(t) * h_2(t) * h_3(t)$.

$\therefore \quad H(s) = H_1(s)\, H_2(s)\, H_3(s)$.

2. Parallel Connection.

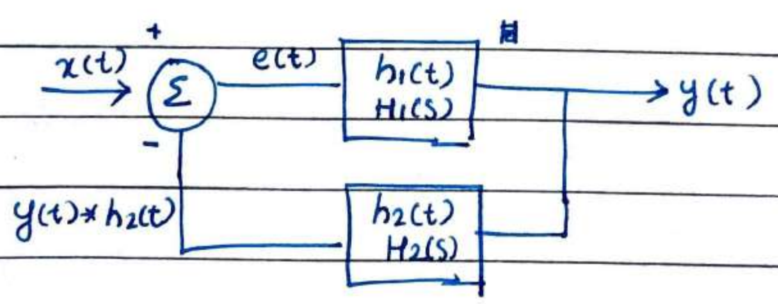


$$y(t) = x(t) * h_1(t) + x(t) * h_2(t).$$

$$\therefore \quad H(s) = H_1(s) + H_2(s).$$

$\cancel{e(t) = x(t) - y(t)}$

3. Feedback Connection.

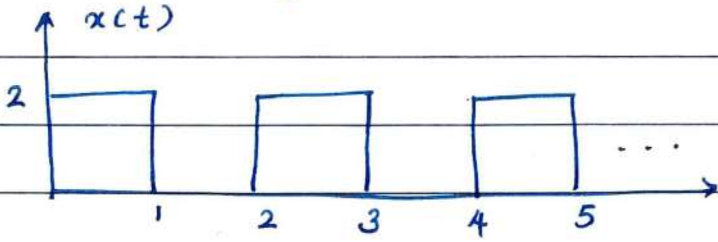


$$e(t) = x(t) - y(t) * h_2(t)$$

and $\quad y(t) = e(t) * h_1(t) \qquad H(s) = \dfrac{H_1(s)}{1 + H_1(s) H_2(s)}$

Laplace Transform of causal periodic signals

Consider a periodic signal as shown in figure.



Let $x_1(t), x_2(t), x_3(t), \ldots$ be the signal representing 1^{st}, 2^{nd}, $3^{rd}, \ldots$ cycles of $x(t)$ with fundamental time period T

$$\therefore \quad x_2(t) = x_1(t-T)$$

$$x_3(t) = x_2(t-2T)$$

$$\vdots$$

$$x(t) = x_1(t) + x_1(t-T) + x_1(t-2T) + \cdots$$

Taking the Laplace Transform, using the time shifting property

$$X(s) = X_1(s) + e^{-sT}X_1(s) + e^{-2sT}X_2(s) + \cdots$$

$$= \left(1 + e^{-sT} + e^{-2sT} + \cdots\right) X_1(s)$$

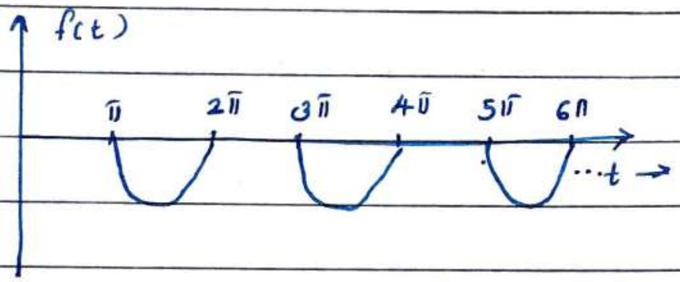
$$\therefore \quad X(s) = \frac{X_1(s)}{1 - e^{-sT}}$$

where $X_1(s) = \int_0^T x(t)\, e^{-st}\, dt$

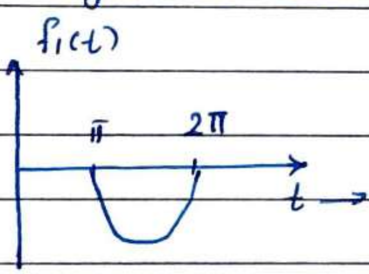
Q. Find the LT of the periodic signal given below.

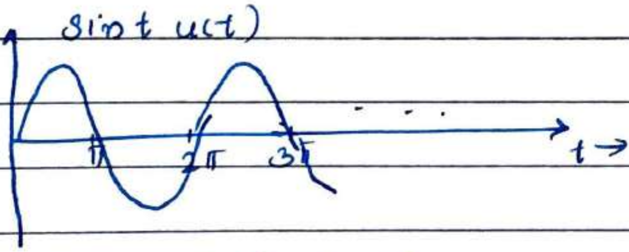
$$f(t) = \begin{cases} \sin t & ; \text{ if } (2n-1)\pi \le t \le 2n\pi \;;\; (n = 1, 2, 3, \cdots) \\ 0 & ; \text{ otherwise} \end{cases}$$

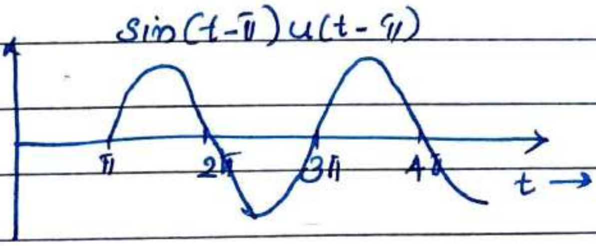
Ans:

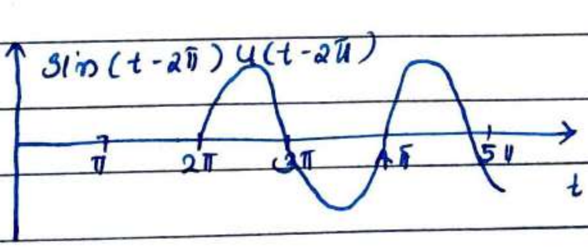


The given signal is causal periodic with period $T = 2\pi$









$$\therefore\; f_1(t) = -\left[\sin(t-\pi)\,u(t-\pi) + \sin(t-2\pi)\,u(t-2\pi)\right]$$

$$\sin t\, u(t) \xleftrightarrow{LT} \frac{1}{s^2+1}$$

$$\sin(t-\pi)\, u(t-\pi) \xleftrightarrow{LT} \frac{e^{-\pi s}}{s^2+1}$$

$$\sin(t-2\pi)\, u(t-2\pi) \xleftrightarrow{LT} \frac{e^{-\pi 2s}}{s^2+1}$$

$$F_1(s) = -\left[\frac{e^{-\pi s}}{s^2+1} + \frac{e^{-\pi 2s}}{s^2+1}\right]$$

$$= -e^{-\pi s}\left[\frac{1+e^{-\pi s}}{s^2+1}\right]$$

Then, $$F(s) = \frac{F_1(s)}{1-e^{-Ts}} = \frac{F_1(s)}{1-e^{-2\pi s}}$$

$$= -e^{-\pi s}\left[\frac{1+e^{-\pi s}}{s^2+1}\right] \times \frac{1}{[1-e^{-2\pi s}]}$$

$$= -e^{-\pi s}\left[\frac{1+e^{-\pi s}}{s^2+1}\right] \times \frac{1}{(1+e^{-\pi s})[1-e^{-\pi s}]}$$

$$= \frac{-e^{-\pi s}}{(s^2+1)(1-e^{-\pi s})}$$

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$$\sin t \, u(t) \xleftrightarrow{LT} \frac{1}{s^2+1}$$

$$\sin(t-\pi)\, u(t-\pi) \xleftrightarrow{LT} \frac{e^{-\pi s}}{s^2+1}$$

$$\sin(t-2\pi)\, u(t-2\pi) \xleftrightarrow{LT} \frac{e^{-\pi 2s}}{s^2+1}$$

$$F_1(s) = -\left[\frac{e^{-\pi s}}{s^2+1} + \frac{e^{-\pi 2s}}{s^2+1}\right]$$

$$= -e^{-\pi s}\left[\frac{1+e^{-\pi s}}{s^2+1}\right]$$

Then, $$F(s) = \frac{F_1(s)}{1-e^{-Ts}} = \frac{F_1(s)}{1-e^{-2\pi s}}$$

$$= -e^{-\pi s}\left[\frac{1+e^{-\pi s}}{s^2+1}\right] \times \frac{1}{\left[1-e^{-2\pi s}\right]}$$

$$= -e^{-\pi s}\left[\frac{1+e^{-\pi s}}{s^2+1}\right] \times \frac{1}{\left(1+e^{-\pi s}\right)\left[1-e^{-\pi s}\right]}$$

$$= \frac{-e^{-\pi s}}{(s^2+1)(1-e^{-\pi s})}$$

Inverse Laplace Transform

Consider $$F(s) = \frac{N(s)}{D(s)} \longleftrightarrow f(t)$$

We use partial fraction expansion method to break $F(s)$ down into simple terms whose inverse transform is obtained by the use of basic Laplace transform pairs.

We need to consider 3-possible cases of roots of $F(s)$
(i) simple poles (ii) Repeated poles (iii) Complex poles.

(i) Simple poles.

If $F(s)$ has simple poles, then $D(s)$ becomes product of factors, as

$$F(s) = \frac{N(s)}{(s+a_1)(s+a_2)\cdots(s+a_n)}$$

$F(s)$ can be decomposed as,

$$F(s) = \frac{A}{s+a_1} + \frac{B}{s+a_2} + \cdots + \frac{A_n}{s+a_n}.$$

where $A, B, \cdots$ are known as residues of $F(s)$.

$$A = (s+a_1)\,F(s)\Big|_{s=-a_1}$$

$$B = (s+a_2)\,F(s)\Big|_{s=-a_2}$$

This is known as 'Heavyside' theorem.

Since $\mathcal{L}^{-1}\left[\frac{1}{s+a_n}\right] = e^{-a_n t}$

$$f(t) = A e^{-a_1 t} + B e^{-a_2 t} + \cdots$$

(ii) Repeated poles.

$$F(s) = \frac{A}{(s+a)^n} + \frac{B}{(s+a)^{n-1}} + \frac{C}{(s+a)^{n-2}} + \cdots + \frac{Z}{(s+a)} + F_1(s)$$

where $F_1(s)$ = part of $F(s)$ that doesn't have a pole at $s = -a$.

$$A = (s+a)^n F(s) \Big|_{s=-a}$$

$$B = \frac{1}{\lfloor 1} \frac{d}{ds} \left[(s+a)^n F(s)\right] \Big|_{s=-a}$$

$$C = \frac{1}{\lfloor 2} \frac{d^2}{ds^2} \left[(s+a)^n F(s)\right] \Big|_{s=-a}$$

using $L^{-1}\left[\frac{1}{(s+a)^n}\right] = \frac{t^{n-1} e^{-at}}{\lfloor n-1} u(t)$, we can find out the inverse LT.

(iii) Complex poles.

$$F(s) = \frac{As+B}{s^2+as+b} + F_1(s)$$

let $s^2+as+b = (s+\alpha)^2 + \beta^2$

and $As+B = A_1(s+\alpha) + B_1\beta$

then, $$F(s) = \frac{A_1(s+\alpha)}{(s+\alpha)^2+\beta^2} + \frac{B_1\beta}{(s+\alpha)^2+\beta^2} + F_1(s)$$

Taking the inverse LT,

$$f(t) = A_1 e^{-\alpha t}\cos\beta t + B_1 e^{-\alpha t}\sin\beta t + f_1(t)$$

Q1. Find the inverse LT of $X(s) = \dfrac{8s+10}{(s+1)(s+2)^3}$

Ans:

$$\frac{8s+10}{(s+1)(s+2)^3} = \frac{A}{(s+1)} + \frac{B}{(s+2)^3} + \frac{C}{(s+2)^2} + \frac{D}{(s+2)}$$

Calculation of A and B: residue method.

$$A = (s+1)X(s)\Big|_{s=-1} = \frac{8s+10}{(s+2)^3}\Big|_{s=-1} = 2$$

$$B = (s+2)^3 X(s)\Big|_{s=-2} = \frac{8s+10}{s+1}\Big|_{s=-2} = 6$$

Calculation of C and D: differentiation formula.

$$C = \frac{d}{ds}\left[(s+2)^3 X(s)\right]\Big|_{s=-2} = \frac{d}{ds}\left[\frac{8s+10}{s+1}\right]\Big|_{s=-2}$$

$$= -2$$

$$D = \frac{1}{\lfloor 2}\frac{d^2}{ds^2}\left[(s+2)^3 X(s)\right]\Big|_{s=-2} = \frac{1}{2}\cdot\frac{d^2}{ds^2}\left[\frac{8s+10}{s+1}\right]\Big|_{s=-2}$$

$$= -2$$

$$\therefore\ X(s) = \frac{2}{s+1} + \frac{6}{(s+2)^3} + \frac{-2}{(s+2)^2} + \frac{-2}{(s+2)}$$

$$\therefore\ x(t) = \left[2e^{-t} + \underline{\underline{(3t^2-2t-2)e^{-2t}}}\right]u(t)$$

Q2. Find the inverse LT of $X(s) = \dfrac{20}{(s+3)(s^2+8s+25)}$.

Ans: Using partial fraction expression,

$$\frac{20}{(s+3)(s^2+8s+25)} = \frac{A}{(s+3)} + \frac{Bs+C}{(s^2+8s+25)} \quad \text{——} (1)$$

Calculation of A: residue method

$$A = (s+3)X(s)\Big|_{s=-3} = \frac{20}{(s^2+8s+25)}\Big|_{s=-3} = 2$$

Calculation of B and C:

Substitute $s = 0, 1$ (possibly other than the poles of $X(s)$)

When $s = 0$, Eq (1) $\Longrightarrow$

$$\frac{20}{3 \times 25} = \frac{A}{3} + \frac{C}{25} \Longrightarrow 20 = 25A + 3C$$

with $A = 2$, $C = -10$

When $s = 1$, Eq. (1) $\Longrightarrow$

$$\frac{20}{4 \times 34} = \frac{A}{4} + \frac{B+C}{34}$$

With $A = 2$, $C = -10$, then $B = -2$

$$\therefore X(s) = \frac{2}{s+3} + \frac{-2s-10}{s^2+8s+25}$$

Thus, $X(s) = \frac{2}{s+3} - \frac{2(s+4)}{(s^2+8s+25)}$

$$= \frac{2}{s+3} - \frac{2(s+4)}{[(s+4)^2+9]} - \frac{2\times3}{3[(s+4)^2+9]}$$

$$= 2\cdot\frac{1}{s+3} - 2\cdot\frac{s+4}{[(s+4)^2+9]} - \frac{2}{3}\times\frac{3}{[(s+4)^2+9]}$$

Taking the inverse LT,

$$x(t) = \left(2e^{-3t} - 2e^{-4t}\cos 3t - \frac{2}{3}e^{-4t}\sin 3t\right]u(t)$$

Unilateral Laplace Transform.

The unilateral LT of $x(t)$ is given by,

$$X(s) = \int_{0^-}^{\infty} x(t)e^{-st}\,dt$$

$t = 0^-$, indicates a time just before $t = 0$.

$$x(t) \xleftrightarrow{ULT} X(s)$$

Comparison between unilateral and bilateral LT

	Bilateral LT	Unilateral LT
1.	$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$	$X(s) = \int_{0^-}^{\infty} x(t)e^{-st}\,dt$.

2.	Limits of integration : $-\infty$ to $+\infty$	0^- to $+\infty$
3.	ROC is mandatory.	No need to specify ROC (ROC must always be RHS of s-plane
4.	BLT is unique if ROC is specified.	ULT is unique.
5.	Handles both causal and non-causal systems.	Handles only causal systems.
6	Application: Gives insight about system characteristics such as stability, causality, and frequency response.	Application: In solving linear differential equations with non-zero initial conditions.

Relationship between BLT and ULT

Any arbitary signal $x(t)$ can be split into its causal and anticausal components.

$$\therefore \quad x(t) = \underbrace{x(t)\,u(t)}_{\text{causal}} + \underbrace{x(t)\,u(-t)}_{\text{anticausal}}$$

$$= \underbrace{x(t)\,u(t)}_{\text{causal}} + \underbrace{x(-t)\,u(t)}_{\text{causal}}$$

Taking LT on both sides.

$$X(s) = ULT\left[x(t)\,u(t)\right] + ULT\left[x(-t)\,u(t)\right]$$

Initial Value Theorem-

Under the conditions that $x(t)=0$ for $t<0$ and there is no impulse or higher order singularities at $t=0$, the initial value of signal $x(t)$ ($x(0^+)$) i.e. the value of $x(t)$ as $t\to 0$ can be calculated as,

$$x(0^+) = \lim_{s\to\infty} s\,X(s) = \lim_{t\to 0} x(t)$$

Proof:

$$ULT\left[\frac{dx(t)}{dt}\right] = \int_{0^-}^{\infty} \frac{dx(t)}{dt} e^{-st}\,dt$$

$$s\,X(s) - x(0^-) = \int_{0^-}^{0^+} \frac{dx(t)}{dt} e^{-st}\,dt + \int_{0^+}^{\infty} \frac{dx(t)}{dt} e^{-st}\,dt$$

$$s\,X(s) - x(0^-) = \Big[x(t)\Big]_{0^-}^{0^+} + \int_{0^+}^{\infty} \frac{dx(t)}{dt} e^{-st}\,dt$$

$$s\,X(s) - x(0^-) = x(0^+) - x(0^-) + \int_{0^+}^{\infty} \frac{dx(t)}{dt} e^{-st}\,dt$$

As $t\to 0 \Rightarrow s\to\infty$, taking limits on both sides of the equation.

$$\lim_{s\to\infty} s\,X(s) = x(0^+) + \lim_{s\to\infty}\int_{0^+}^{\infty} \frac{dx(t)}{dt} e^{-st}\,dt$$

$$\therefore \lim_{s\to\infty} s\,X(s) = x(0^+)$$

Note: * The initial value theorem applies only if $X(s)$ is strictly proper (i.e. power of $N(s)$ < power of $D(s)$) $\because X(s) = \frac{N(s)}{D(s)}$

* For improper $X(s)$, using long division method $X(s)$ is expressed as polynomial in 's' plus a strict proper function. In this case, the value of remainder fraction gives initial value.

Q. Calculate the initial value of the following functions.

(a) $X(s) = \frac{4s+5}{2s+1}$

Ans: Since the given function is improper, apply long division method.

$$X(s) = 2 + \frac{3}{2s+1}$$

By considering the remainder function, $\left[\frac{3}{2s+1}\right]$

$$x(0^+) = \lim_{s\to\infty} s\left[\frac{3}{2s+1}\right] = \lim_{s\to\infty}\left[\frac{3}{2+1/s}\right] = 3/2.$$

(b) $\frac{s^3+3s^2+s+1}{s^2+2s+1} = X(s)$

Ans: Since the given function is improper,

$$X(s) = (s+1) - \frac{2s}{s^2+2s+1}$$

The remainder is $\left[\frac{-2s}{s^2+2s+1}\right]$

$$\therefore x(0^+) = \lim_{s\to\infty} s \cdot \left[\frac{-2s}{s^2+2s+1}\right] = \underline{\underline{-2}}$$

Final Value Theorem.

It states that $x(\infty) = \lim_{s\to 0} s\,X(s)$

Proof:

$$\text{ULT}\left[\frac{dx(t)}{dt}\right] = \int_{0^-}^{\infty} \frac{dx(t)}{dt} \cdot e^{-st}\,dt$$

$$s\,X(s) - x(0^-) = \int_{0^-}^{\infty} \frac{dx(t)}{dt} \cdot e^{-st}\,dt$$

Taking limits on both sides of the equation as $s \to 0$

$$\lim_{s\to 0}\left[s\,X(s) - x(0^-)\right] = \lim_{s\to 0} \int_{0^-}^{\infty} \frac{dx(t)}{dt} \cdot e^{-st}\,dt$$

$$= \left[x(t)\right]_{0^-}^{\infty}$$

$$= x(\infty) - x(0^-)$$

$$\therefore x(\infty) = \lim_{s\to 0} s\,X(s)$$

Note: The final value theorem is applicable only if poles of $X(s)$ lies in left half of s-plane (including $s=0$)

Q. Let $f(t) \xleftrightarrow{LT} F(s) = \frac{\omega_0}{s^2+\omega_0^2}$; $Re(s) > 0$. Find the final value of ~~f(t)~~ . $f(t)$.

Ans: $L^{-1}[F(s)] = \sin \omega_0 t$

$\therefore f(t) = \sin \omega_0 t$.

$\therefore -1 \leq f(\infty) \leq 1$

Q. $y(t) \xleftrightarrow{LT} Y(s) = \frac{1}{s(s-1)}$. Find the final value of $y(t)$

Ans: Since $s=1$ which is a pole at RHP of s-plane, the final value of $y(t)$ is unbounded.

Zero-input response and Zero-state response

Total response

Zero-input response/	Zero-state response/
Natural response/	Forced response/
Transient response/	Steady state response/
complementary function	Particular integral.

Zero-input response

The input is considered as zero and the response is due to the initial conditions.

Zero-state response

The initial conditions are considered as zero and the response is due to the applied input.

Q. The unit step response $y(t)$ of a linear system is $y(t) = (1 - 3e^{-t} + 3e^{-2t})\,u(t)$. Find the frequency at which the forced response becomes zero.

Ans:
$$y(t) = (1 - 3e^{-t} + 3e^{-2t})\,u(t)$$

$\therefore$

$$Y(s) = \frac{1}{s} - \frac{3}{(s+1)} + \frac{3}{s+2}$$

$$= \frac{s^2+2}{s(s+1)(s+2)}$$

The forced response becomes zero, when

$$s^2+2 = 0$$

$$\Rightarrow s = \pm j\sqrt{2}$$

$$\therefore \omega = \sqrt{2} \text{ rad/sec}$$

Sampling.

- The process of sampling is a bridge between continous-time and discrete-time domains.

* It is the process of continous-time signal to discrete-time signal and under certain conditions the continous-time signal can be completely recovered from its sampled sequence.

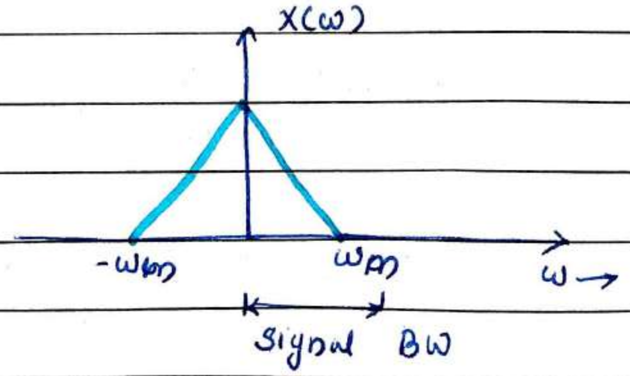
Sampling Theorem.

According to Sampling theorm, " A band limited signal of finite energy can be completely reconstructed from its samples taken uniformly at a rate $\omega_s \geq 2\omega_m$ samples rad/sec) or $f_s \geq 2f_m$ samples/sec.

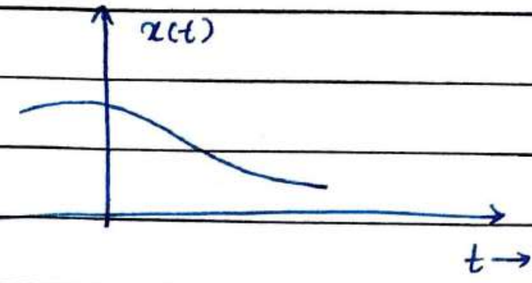
* In other words, the minimum sampling rate is $f_s = 2f_m$ Hz.

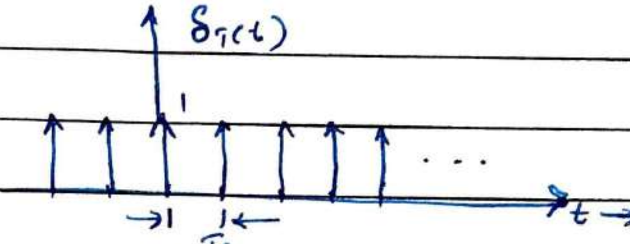
* Let $x(t)$ be a continous time finite energy signal whose spectrum is limited to ω_m rad/sec.

i.e. $X(\omega) = 0$ for $|\omega| > \omega_m$.

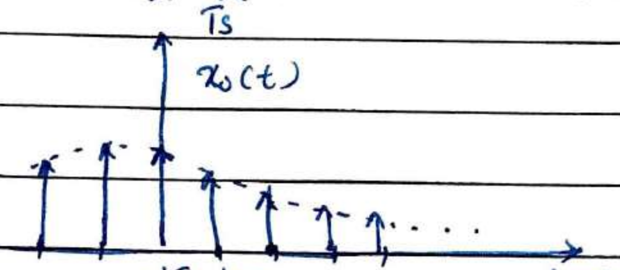


* The input $x(t)$ is sampled at the rate of f_s Hz by multiplying $x(t)$ by a periodic impulse train $\delta_T(t)$ with period $T_s = 1/f_s$.





* The sampled signal '$x_s(t)$' consists of impulses spaced every T_s second whose strength (area) is equal to instantaneous value of input signal $x(t)$.



$$\therefore \quad x_s(t) = x(t)\cdot\delta_T(t)$$

$$= x(t)\cdot\sum_{n=-\infty}^{\infty}\delta(t-nT_s)$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(t-nT_s)$$

∵ Sampling property of impulse

* Taking FT on both sides

$$X_s(\omega) = \frac{1}{2\pi}\left[X(\omega) * \frac{2\pi}{T_s}\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_s)\right]$$

$$= \frac{1}{T}\left[X(\omega) * \sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_s)\right]$$

Using convolution property of unit impulse

$$X_s(\omega) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X(\omega - n\omega_s)$$

in form of 'f' Hz

$$X_s(f) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X(f - nf_s)$$

Note: This result means that the spectrum $X_s(\omega)$ consists of $X(\omega)$ repeating periodically with period $T_s = \frac{2\pi}{\omega_s}$ sec

* We have assumed spectrum of $x(t)$ is band limited to ω_m.

The sampling frequency can take three possible values with respect to the spectrum width $(2\omega_m)$.

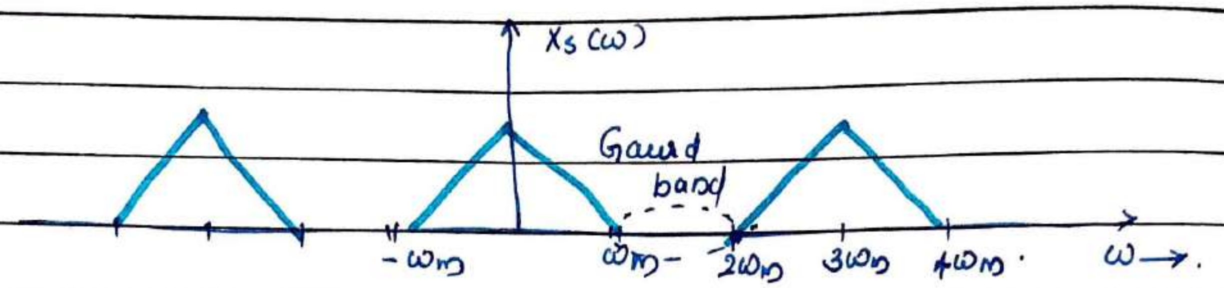
(i) $\omega_s > 2\omega_s$ (ii) $\omega_s = 2\omega_m$ (iii) $\omega_s < 2\omega_m$.

Case I : $\omega_s > 2\omega_m$.

Let $\omega_s = 3\omega_m$

The spectrum of sampled signal will be

$$X_s(\omega) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X(\omega - 3n\omega_s)$$



* For $\omega_s > 2\omega_m$, there is no overlap between the shifted spectrum of $X(\omega)$.

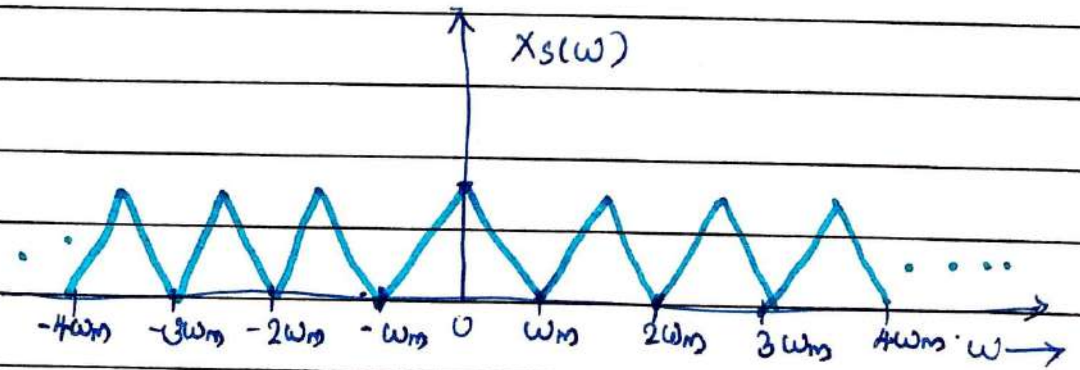
* As long as the sampling frequency ω_s is greater than twice the signal spectral bandwidth ($2\omega_m$), ~~between ω_m and~~ $x(t)$ can be recovered by passing the sampled signal $x_s(t)$ through ideal or practical LPF having BW between ω_m and $(\omega_s - \omega_m)$ rad/sec.

Case II : $\omega_s = 2\omega_m$

The spectrum of sampled signal will be

$$X_s(\omega) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X(\omega - 2n\omega_m)$$

Then, the spectrum of $X_s(\omega)$ is



* For $\omega_s = 2\omega_m$, there is no overlap between shifted spectrum of $X(\omega)$

* $x(t)$ can be recovered from its sampled signal by means of an ideal low pass filter (LPF).

* The minimum sampling rate $\omega_s = 2\omega_m$ (or $f_s = 2f_m$) required to recover $x(t)$ is called Nyquist rate.

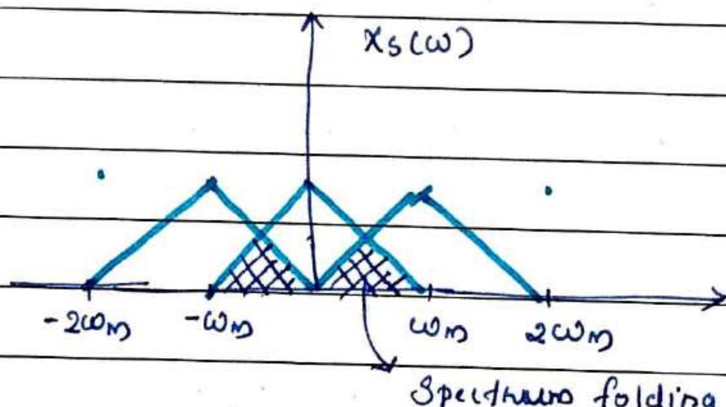
Case : III $\omega_s < 2\omega_m$

The spectrum of sampled signal will be

$$X_s(\omega) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X(\omega - n\omega_m)$$

Let $\omega_s = \omega_m$.

Then, the spectrum of $X_s(\omega)$ is



* We see for $\omega_s < 2\omega_m$, there is overlap between shifted spectrum of $X(\omega)$.
* Consequently, the signal $x(t)$ cannot be exactly recovered from its sampled signal.

Aliasing or Spectrum folding.

The overlap in shifted spectrum of original signal is termed as "aliasing".

* Aliasing is an irreversible process once aliasing has occurred, then signal cannot be recovered back.

Q. a) Find the Nyquist rate (NR) of

$$x(t) = \cos(2\pi \times 10^3 t) + \cos(6\pi \times 10^3 t)$$

Ans: $$x(t) = \cos(2\pi \times 10^3 t) + \cos(2\pi \times 3 \times 10^3 t)$$

$$\downarrow \qquad\qquad \downarrow$$

$$f_{m_1} \qquad\qquad f_{m_2}$$

$$f_{m_1} = 10^3 \text{ Hz} \quad \text{and} \quad f_{m_2} = 3 \times 10^3 \text{ Hz}.$$

$$f_m = \max(f_{m_1}, f_{m_2}) = 3 \text{ kHz}$$

$$\therefore \text{Nyquist rate} = 2 f_m = 2 \times 3\text{kHz} = \underline{\underline{6 \text{ kHz}}}$$

b) $$x(t) = \mathrm{sinc}(300t) + \mathrm{sinc}^2(300t)$$

Ans: $$x(t) = \frac{\sin(300\pi t)}{300\pi t} + \frac{\sin^2(300\pi t)}{(300\pi t)^2}$$

$$= \frac{\sin(300\pi t)}{300\pi t} + \frac{(1 - \cos(2 \times 300\pi t))}{2 \times (300\pi t)^2}$$

$\therefore$ The max. frequency component $\omega_m = 600\pi$ rad/sec.

$\therefore$ The Nyquist rate $\omega_s = 2\omega_m$

$= \underline{\underline{1200\pi}}$ rad/sec.

$\therefore$ $f_s = \underline{\underline{600 Hz}}$

Important Results

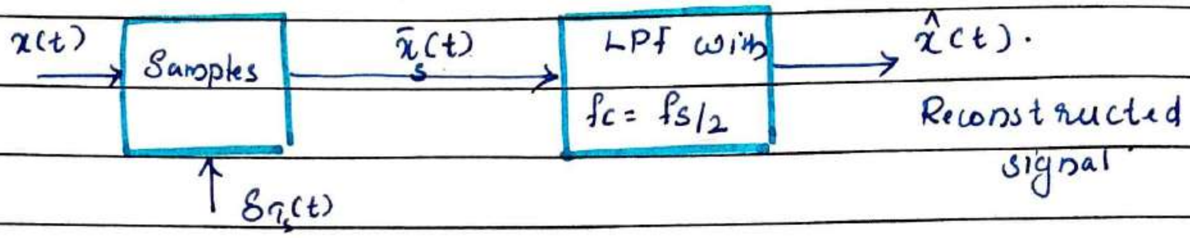
Signal, $x(t)$	Nyquist rate, ω_0 (NR)
$x^2(t)$	$2\omega_0$
$x(t+1)$	ω_0
$x(2t)$	$\cancel{2}\omega_0/2$
$x(t/2)$	$\cancel{\omega_0/2}$ $2\omega_0$
$\frac{dx(t)}{dt}$	ω_0
$\int_{-\infty}^{t} x(\tau)d\tau$	ω_0
$x(t) * x(t)$	$2\omega_0$

Q. Let $x(t)$ be band limited to F Hz. Then, what is the NR of $y(t) = x(0.5t) + x(t) - x(2t)$?

Ans: Expansion in time domain is compression in frequency domain and vice-versa.

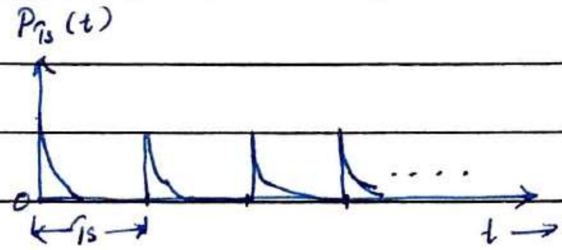
$\therefore$ Max. frequency component in $y(t)$ is $2F$.

Then, $NR = 2 \times 2F = \underline{\underline{4F \text{ Hz}}}$



Practical Sampling

* In ideal case, we multiply a signal $x(t)$ by an impulse train that is physically unrealizable.
* In practice, we multiply a signal $x(t)$ by a train of pulses of finite width.

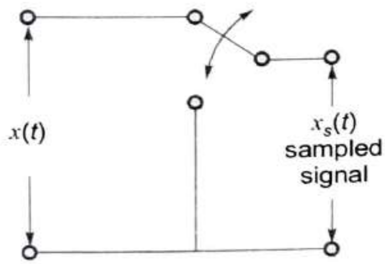
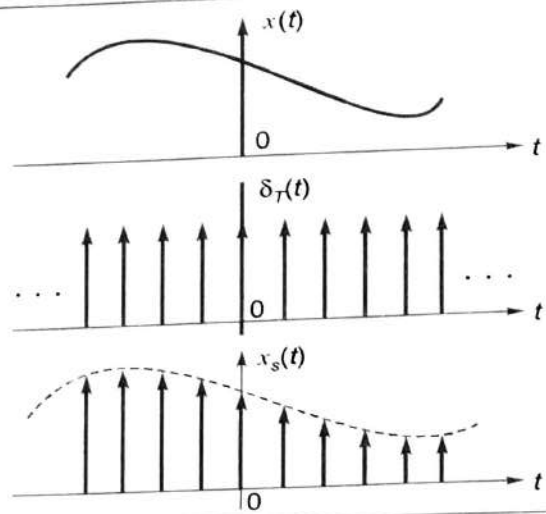
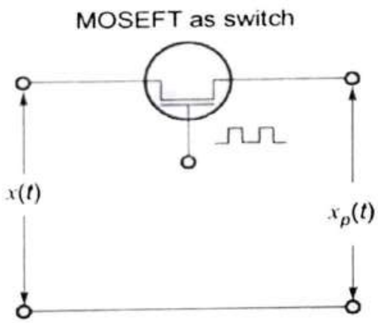
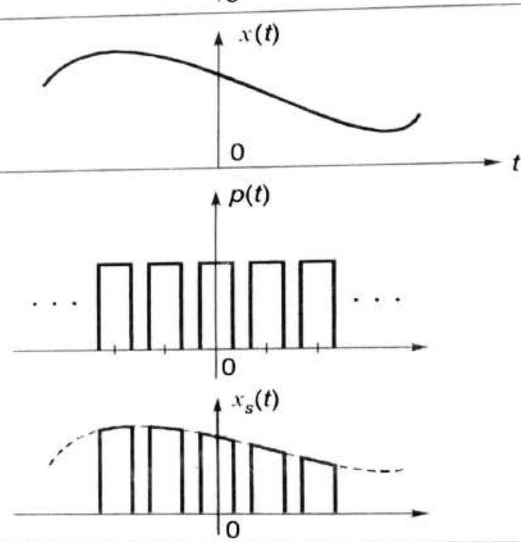
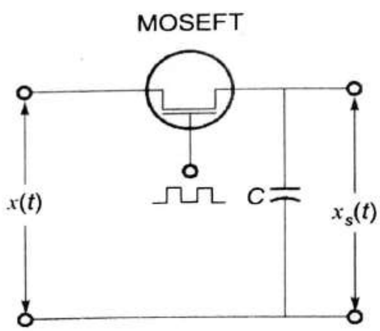
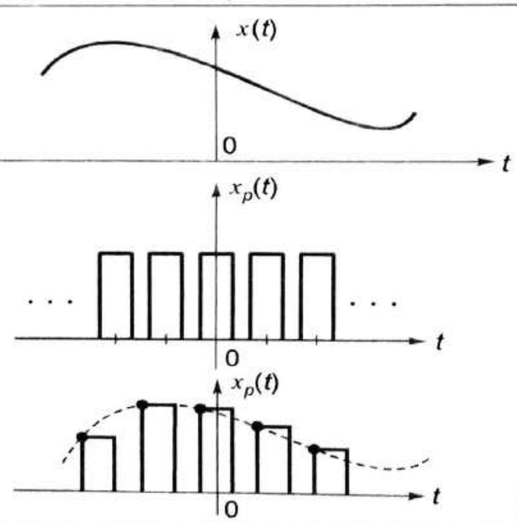


SIGNAL RECONSTRUCTION

* The process of reconstructing a continuous-time signal $x(t)$ from its samples is known as interpolation.

* The reconstruction is accomplished by passing the sampled signal $x_s(t)$ through an ideal LPF with cut-off frequency $f_c = f_m = f_s/2$, gain $1/T$ and BW 'f_m'.
* Since ideal LPF is not practically realizable, we use practical LPF

.2 Sampling Techniques

<table>
<tr><th>Sampling Technique</th><th></th><th>Sampler Circuit</th><th>Sampling Process</th></tr>
<tr><td>Ideal Sampling</td><td>Instantaneous or Impulse Train Sampling
• Sampling signal is periodic impulse train.
• The area of each impulse in sampled signal is equal to instantaneous value of input signal $x(t)$.</td><td>
Fig. Impulse train sampling</td><td></td></tr>
<tr><td rowspan="2">Practical Sampling</td><td>Natural Sampling
• Sampling signal is pulse train.
• The top of each pulse in sampled signal $x_p(t)$ retains shape of input signal $x(t)$ during pulse interval.</td><td>
Fig. Natural sampling using MOSFET as switches</td><td></td></tr>
<tr><td>Flat-Top Sampling
• Sampling signal is pulse train.
• The top of each pulse in sampled signal $x_p(t)$ remain constant and equal to instantaneous value of $x(t)$ at the start of samples.</td><td>
Fig. Sample and hold circuit</td><td></td></tr>
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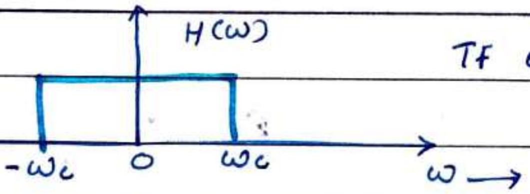
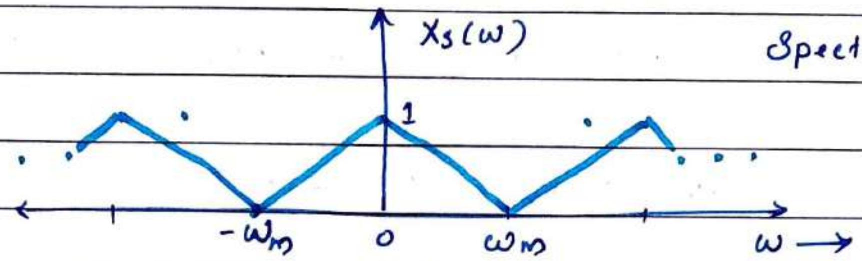
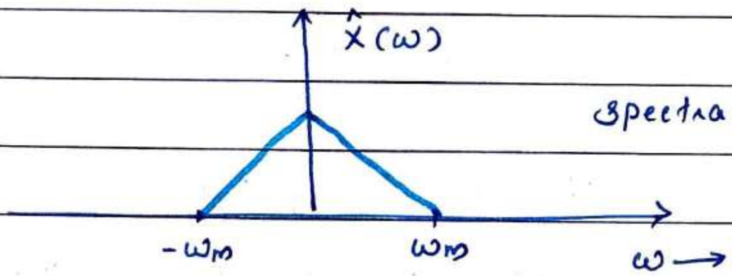


TF of ideal LPF



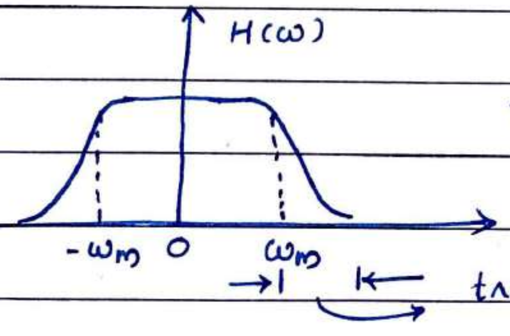
Spectra of sampled signal



spectra of reconstructed signal.

* For practical LPF, we cannot use critical sampling ($\omega_s = 2\omega_m$) and we have to use over sampling and seperation in over sampling is decided by the transition band of LPF.
* This seperation band is called "guard band".

* Thus, the BW of LPF must be in between f_m and $f_s - f_m$.



TF of practical LPF.

transition band.

* With the choice of cut off frequency and gain T, the ideal LPF required for signal reconstruction is,

$$H(\omega) = T_s\, rect\left[\frac{\omega}{2\pi f_s}\right] = T_s\, rect\left[\frac{\omega T_s}{2\pi}\right]$$

Ideal Interpolation : Time domain view.

The impulse response of LPF is,

$$h(t) = sinc\left(\frac{\pi t}{T_s}\right)$$

For the Nyquist sampling $T = 1/2f_m$.

and $h(t) = sinc(2\pi f_m t)$

* When $x_s(t)$ is applied to the input of this filter, the o/p is $\hat{x}(t)$

* Each sample in $x_s(t)$ being an impulse, generates a 'sinc' pulse of height ~~is~~ equal to the strength of the sample.

* The nth sample of $x_s(t)$ is $x(nT_s)\,\delta(t - nT_s)$ and the filter output is $x(nT_s)\,h(t - nT_s)$

* Then the filter o/p is

$$\hat{x}(t) = \sum_n x(nT_s)\, h(t - nT_s)$$

$$= \sum_n x(nT_s)\, sinc\left(\frac{\pi}{T_s}(t - nT_s)\right)$$

At $T_s = 1/2f_m$,

$$x(t) = \sum_n x(nT_s)\, \mathrm{sinc}\,(2\pi f_m t - n\pi)$$

→ Interpolation formula

Note: ⇒ which yields values of $x(t)$ between samples as weighted sum of all the sample values.

The Treachery of Aliasing.

* Another practical difficulty in reconstructing signals from its samples.
* Sampling theorem was developed based on the assumption that the signal is bandlimited.
* However, all practical signals are time-limited.
* But, a signal cannot be bandlimited and time-limited simultaneously.

Ref. Lathi Problem 8-23 ←

* However, it can be non-band limited and non-time-limited simultaneously.

* Thus, due to the infinite BW of time-limited signals, the spectral overlap in the sampled signal is unavoidable.

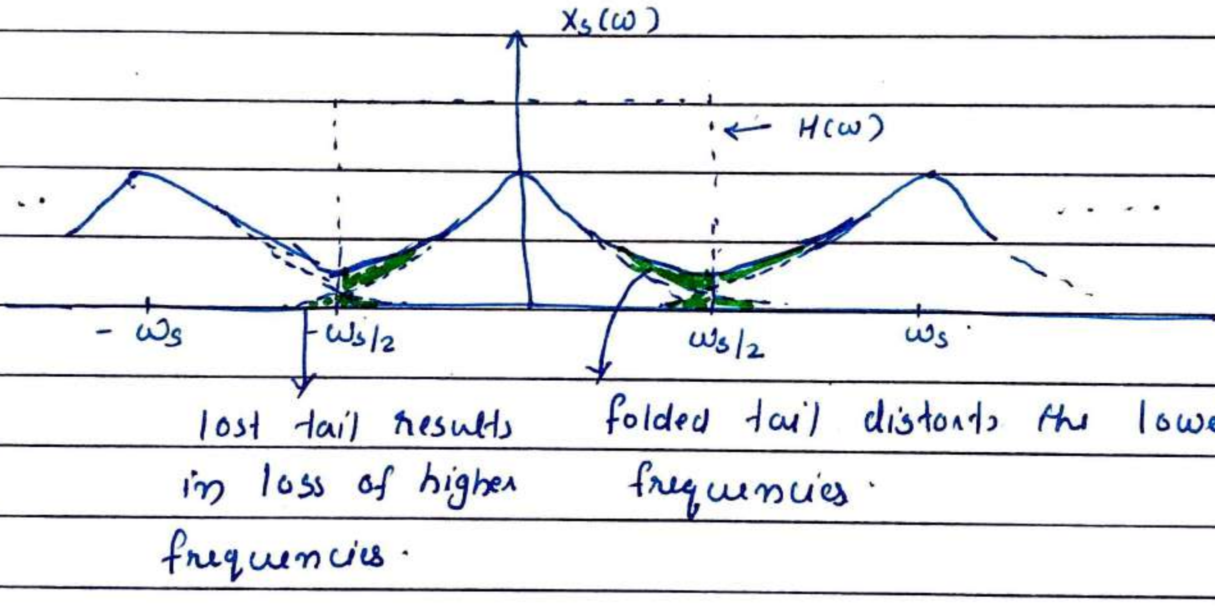
* Hence if we use an ideal LPF with cutoff frequency $f_s/2$, the following points are resulting.

(i) The loss of tail of $X(\omega)$ beyond $|f| > f_s/2$ Hz.

(ii) The reappearance of this tail inverted or folded onto the spectrum. Note that the spectra cross at $f = f_s/2 = 1/2T_s$ Hz.

* This frequency is called folding frequency.

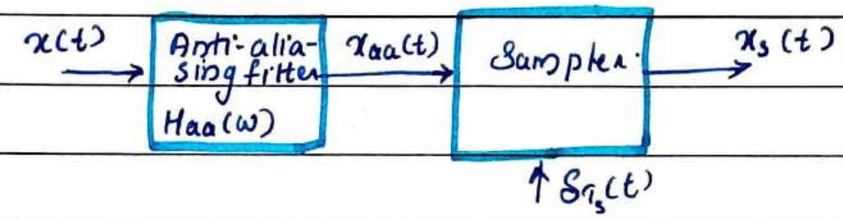
* Thus, the components of frequencies above $f_s/2$ reappear as the components below $f_s/2$. This tail inversion is known as spectral folding or aliasing.
* Thus aliasing destroys the integrity of the frequency component below below $f_s/2$



Anti-aliasing Filter

We must eliminate (suppress) the frequency components beyond the folding frequency $f_s/2$ from $x(t)$ before sampling of $x(t)$

* Such suppression can be accomplished by an ideal LPF of cut-off frequency $f_s/2$ Hz. This is known as anti-aliasing filter.



Thus, an anti-aliasing filter will band limit the signal $x(t)$ to $f_s/2$ $\longrightarrow$ reduces the noise / distortion.

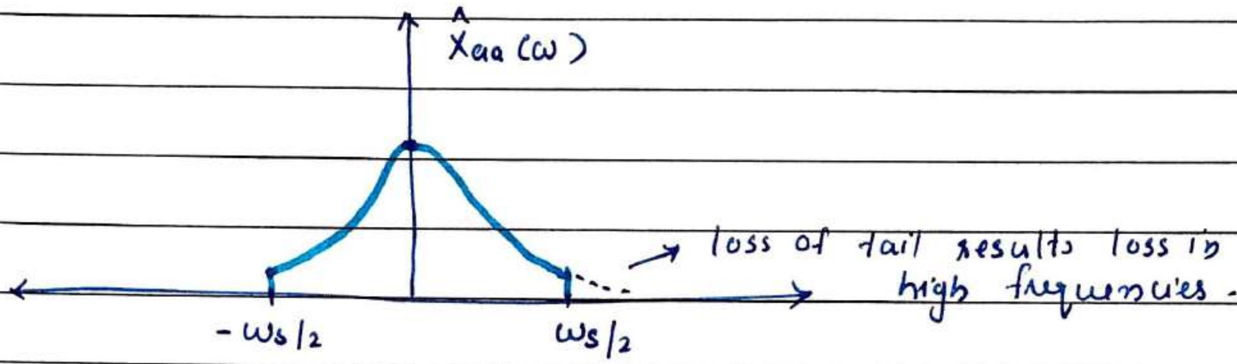


Fig: Reconstructed signal when anti-aliasing filter is used.

Dual of ~~Spectral~~ time sampling : Spectral sampling

* Applicable for time-limited signals.

Statement:

The spectral sampling theorem states that the spectrum $X(\omega)$ of a signal $x(t)$ time-limited to a duration 'τ' seconds can be reconstructed from the samples of $X(\omega)$ taken at a rate 'R' samples/Hz, when $R > \tau$ in seconds.

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{0}^{\tau} x(t) e^{-j\omega t} dt.$$

When $R = \frac{1}{f_0} > \tau$ samples/Hz.

Fourier Analysis of Discrete-Time Signals

Discrete-time periodic signals.

* A continous-time sinusoid $\cos \omega t$ is a periodic signal regardless of the value of ω.

* However, a discrete-time sinusoid $\cos \Omega n$ (or $e^{j\Omega n}$) is periodic only if $\Omega/2\pi$ is a rational number. If the sinusoid is N_0 periodic, then

$$\cos \Omega (n + N_0) = \cos \Omega n$$

This is possible only if

$$\Omega N_0 = 2\pi m \; ; \; m \text{ is an integer}$$

Here, both m and N_0 are integers. Hence

$$\Omega / 2\pi = m / N_0 \text{ is a rational number}$$

$\therefore$ $\cos \Omega n$ or $e^{j\Omega n}$ is periodic only if

$$\boxed{\frac{\Omega}{2\pi} = \frac{m}{N_0}}.$$

$$\therefore N_0 = m\left(\frac{2\pi}{\Omega}\right)$$

We must choose the smallest value of m that will make $m(2\pi/\Omega)$ an integer.

Eg: (i) $\cos\left(\frac{4\pi}{17} n\right)$.

$$m(2\pi/\Omega) \longrightarrow \text{integer} = m(17/2) \Longrightarrow m = 2.$$

$$\therefore \quad N_0 = m\left(\frac{2\pi}{\Omega}\right) = 2\left(\frac{17}{2}\right) = \underline{\underline{17}}$$

(ii) $\cos(0.8n)$

* It is not periodic, since $0.8/2\pi$ is not a rational number.

Periodic Signal Representation by Discrete - Time Fourier Series.

In continuous-time Fourier series, periodic signals can represented as sums of infinite complex exponentials /sinusoids

* However, the discrete time fourier representation is of finite series

* Because of the compactness and ease of mathematical manupulations, exponential form is more preferable than trigonometric form.

* The exponential fourier series consists of exponentials $\cancel{e^{j0\Omega_0}}$, $\cancel{e^{\pm}}$ e^{j0n}, $e^{\pm j\Omega_0 n}$, $e^{\pm j2\Omega_0 n}$, ...

However,

$$e^{j(\Omega \pm 2\pi m)n} = e^{j\Omega n} \cdot e^{\pm 2\pi m n} = e^{j\Omega n}, \quad m\text{-integer}$$

$\therefore$

* The consequence of this result is that the r^{th} harmonic is identical to the $(r+N_0)^{th}$ harmonic. Let g_n denote the n^{th} harmonic $e^{jn\Omega_0 n}$.

Then, $g_{r+N_0} = e^{j(r+N_0)\Omega_0 n} = e^{j(r\Omega_0 n + 2\pi n)}$

$$= e^{jr\Omega_0 n} = g_r$$

and

$$g_r = g_{r+N_0} = g_{r+2N_0} = \cdots = g_{r+mN_0}, \quad m\text{ - integer.}$$

∴ The first harmonic is identical to the (N_0+1) harmonic and second harmonic is identical to the (N_0+2) harmonic and so on.

Note: ∴ In other words, there are only N_0 independent harmonics and their frequencies range over an interval 2π (because the harmonics are seperated by $\Omega_0 = 2\pi / N_0$).

* Thus, unlike the continous-time counterpart, that the discrete-time Fourier series has only a finite number (N_0) of terms.

Ref: Lathi Section 5.5.1

* All discrete-time signals are band-limited to a band from $-\pi$ to π.
This band can be taken from 0 to 2π or any other contiguous band of width 2π.

Thus, for the exponentials $e^{jr\Omega_0 n}$ for $r = 0, 1, 2, \cdots N_0-1$, Fourier series for an N_0-periodic signal $x[n]$ consists of only these N_0 harmonics and can be expressed as,

$$\boxed{x[n] = \sum_{r=0}^{N_0-1} D_r e^{jr\Omega_0 n} \;; \quad \Omega_0 = \frac{2\pi}{N_0}}$$

$$\therefore \quad x[n] = \sum_{r=0}^{N_0-1} D_r \, e^{jr\frac{2\pi n}{N_0}}$$

To compute coefficients D_r. multiply both sides by $e^{-jm\Omega_0 n}$ and sum over n from $n = 0$ to $(N_0 - 1)$.

$$\sum_{n=0}^{N_0-1} x[n] \, e^{-jm\Omega_0 n} = \sum_{n=0}^{N_0-1} \sum_{r=0}^{N_0-1} D_r \, e^{j(r-m)\Omega_0 n}$$

$$= \sum_{r=0}^{N_0-1} D_r \left[\sum_{n=0}^{N_0-1} e^{j(r-m)\Omega_0 n} \right].$$

Ref: Lathi Eq. 8.28

The inner sum is zero for all values of $r \neq m$.
~~It is non-zero with a value N_0 only when $r = m$.~~
It is non-zero with a value N_0 only when $r = m$.
$\therefore$ This fact means the outside sum has only one term $D_m N_0$ (corresponding to $r = m$).

$$\therefore \quad \sum_{n=0}^{N_0-1} x[n] \, e^{-jm\Omega_0 n} = D_m N_0$$

and

$$D_m = \frac{1}{N_0} \sum_{n=0}^{N_0-1} x[n] \, e^{-jm\Omega_0 n}$$

$\therefore$ We have a discrete-time Fourier Series (DTFS) representation of an N_0 - Periodic signal $x[n]$ as,

$$\boxed{x[n] = \sum_{r=0}^{N_0-1} D_r \, e^{jr\Omega_0 n}.}$$

$$\boxed{\text{where} \quad D_r = \frac{1}{N_0} \sum_{n=0}^{N_0-1} x[n] \, e^{-jr\Omega_0 n} \; ; \; \Omega_0 = \frac{2\pi}{N_0}}$$

Fourier Spectra of a Periodic Signal $x[n]$

The Fourier series consists of N_0 component
$$D_0,\ D_1 e^{j\Omega_0 n},\ D_2 e^{j2\Omega_0 n},\ \dots D_{N_0-1} e^{j(N_0-1)\Omega_0 n}$$

Since fourier coefficients D_r are complex,
$$D_r = |D_r| e^{j\angle D_r}.$$

The plot of $|D_r|$ vs Ω $\rightarrow$ amplitude spectrum.
The plot of $\angle D_r$ vs Ω $\longrightarrow$ Phase spectrum.
$\rightarrow$ together known as Fourier spectra.

Also, $$\sum_{r=0}^{N_0-1} \phi(r) = \sum_{r=\langle N_0\rangle} \phi(r)$$ $\phi(r) \rightarrow$ N_0-periodic signal.

where $r = \langle N_0 \rangle$ indicates summation over any N_0 consecutive values of r.

$$\therefore \quad D_r = \frac{1}{N_0} \sum_{n=\langle N_0\rangle} x[n]\, e^{-jr\Omega_0 n}. \qquad 0 \le r \le N_0 - 1$$

If $x[n]$ is real, D_r is complex in general and D_{-r} is the conjugate of D_r.

$$\therefore \quad \underbrace{|D_r| = |D_{-r}|}_{\text{even fn}} \quad \text{and} \quad \underbrace{\angle D_r = -\angle D_{-r}}_{\text{odd fn.}}$$

Q. Find the DTFS for $x[n] = \sin 0.1\pi n$. Sketch the amplitude and phase spectra.

Ans: $\sin 0.1\pi n$ is periodic because $\Omega/2\pi = 1/20$ is a rational number. The period is

$$N_0 = m\left(\frac{2\pi}{\Omega}\right) = m\left(\frac{2\pi}{0.1\pi}\right) = 20m$$

The smallest value of 'm' that makes $2m$ an integer is $m=1$. $\therefore N_0 = 20$

$$x[n] = \sum_{r=\langle 20 \rangle} D_r e^{j0.1\pi n r}$$

Take the range $-10 \le r < 10$ (-10 to 9)

$$\therefore x[n] = \sum_{r=-10}^{9} D_r e^{j0.1\pi n r}$$

and $$D_r = \frac{1}{20}\sum_{n=-10}^{9} \sin 0.1\pi n \, e^{-j0.1\pi r n}$$

$$= \frac{1}{20}\sum_{n=-10}^{9} \frac{1}{2j}\left[e^{j0.1\pi n} - e^{-j0.1\pi n}\right] e^{-j0.1\pi n r}$$

$$= \frac{1}{40j}\left[\sum_{n=-10}^{9} e^{j0.1\pi n(1-r)} - \sum_{n=-10}^{9} e^{-j0.1\pi n(1+r)}\right]$$

r takes all values between -10 and 9. Here the first sum on the RHS will be zero except $r=1$. $||^{ly}$ the second sum on the RHS will be zero except $r=-1$; when its sum is equal to $N_0 = 20$

$\therefore \quad D_1 = \frac{1}{2j}$ and $D_{-1} = \frac{-1}{2j}$

All other coefficients are zeros.

$$\therefore \quad x[n] = \sin 0.1\pi n = \frac{1}{2j}\left(e^{j0.1\pi n} - e^{-j0.1\pi n}\right)$$

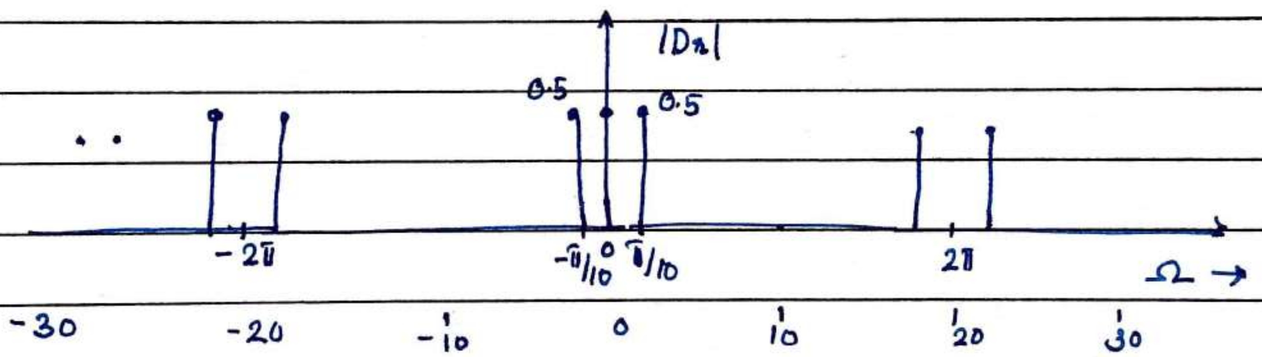
and $\quad D_1 = \frac{1}{2j} = \frac{1}{2} e^{-j\pi/2}$

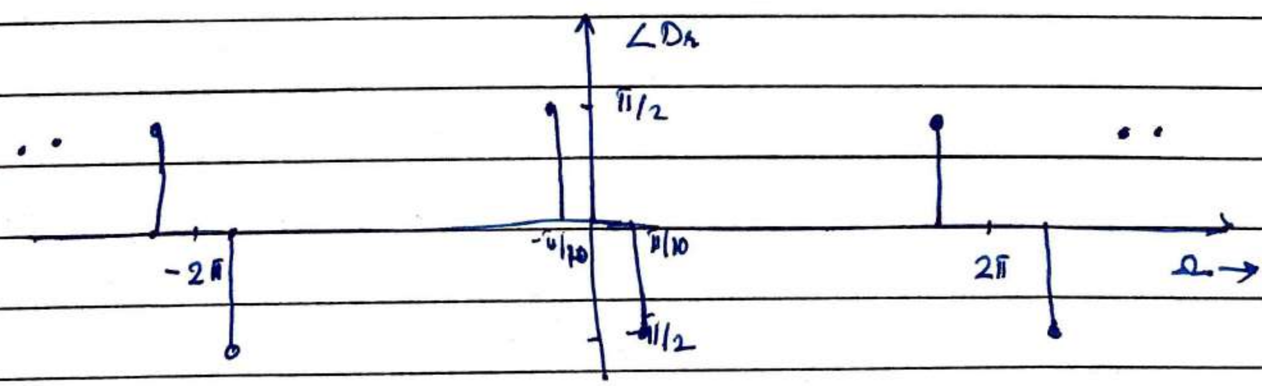
and

$$D_{-1} = \frac{-1}{2j} = \frac{1}{2} e^{j\pi/2}$$

$\therefore \quad |D_1| = |D_{-1}| = 1/2$

and $\angle D_1 = -\pi/2$ and $\angle D_{-1} = \pi/2$





Aperiodic Signal Representation by Fourier Integral

Self Study

* The procedure is identical conceptually for continous-time signals.
* Applying the limiting case, we can show that an aperiodic signal $x[n]$ can be expressed as a continous sum (integral) of everlasting exponentials.
* The periodic extension of $x[n]$ is termed as $x_{N_0}[n]$ when $N_0 \to \infty$ $(N_0 \geq 2N+1)$, where N is the width of the pulse.

* We call $X(\Omega)$ the discrete-time Fourier transform (DTFT) of $x[n]$ and $x[n]$ is the inverse discrete-time fourier transform (IDTFT) of $X(\Omega)$

$$\boxed{\begin{aligned} X(\Omega) &= \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n} \\ \text{and} \quad x[n] &= \frac{1}{2\pi} \int_{2\pi} X(\Omega)\, e^{jn\Omega}\, d\Omega \end{aligned}}$$

$$x[n] \overset{DTFT}{\Longleftrightarrow} X(\Omega) \rightarrow \text{Continous fn of } \Omega.$$

Periodic Nature of DTFT.

$$X(\Omega+2\pi) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j(\Omega+2\pi)n}$$

$$= \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n} \cdot e^{-j2\pi n} = \underline{X(\Omega)}$$

Note: Thus $X(\Omega)$ is a continous, periodic function of Ω with period 2π

Conjugate Symmetry of $X(\Omega)$

$$DTFT\left[x^*[n]\right] = \sum_{n=-\infty}^{\infty} x^*[n]\, e^{-j\Omega n} = X^*(-\Omega)$$

For real $x[n]$,

$$X(\Omega) = X^*(-\Omega)$$

$\because$ for real $x[n]$, $X(\Omega)$ and $X(-\Omega)$ are conjugates.

Then $X(\Omega) = |X(\Omega)|\, e^{j\angle X(\Omega)}$

Because of the conjugate symmetry of $X(\Omega)$, for real $x[n]$,

$$\boxed{\begin{aligned} |X(\Omega)| &= |X(-\Omega)| \quad \text{and} \\ \angle X(\Omega) &= -\angle X(-\Omega) \end{aligned}}$$

$\rightarrow$ even fn. of Ω

$\rightarrow$ odd fn. of Ω.

Existence of the DTFT

As we know, $X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}$

Because $|e^{-j\Omega n}| = 1$, the existence of $X(\Omega)$ is guaranteed if $x[n]$ is absolutely summable, that is

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

$\therefore$ The condition of absolute summability is a sufficient condition for the existence of the DTFT representation.

__/__/____

The inequality,

$$\left[\sum_{n=-\infty}^{\infty} |x[n]|\right]^2 \geq \sum_{n=-\infty}^{\infty} |x[n]|^2$$ shows that

the energy of an absolutely summable sequence is finite. However, not all finite-energy signals are absolutely summable. (Eg: $x[n] = \text{sinc}[n]$. For such signals, DTFT converges, not uniformly, but in the mean.

To summarize, $X(\Omega)$ exists under a weaker condition

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

* The DTFT under this condition is guaranteed to converge in the mean.

* The DTFT of $x[n] = r^n u[n]$; $r>1$ does not exist why?

DTFT of basic signals

(i) $x[n] = a^n u[n]$, $|a|<1$

Then $X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} a^n u[n] e^{-j\Omega n}$

$$= \sum_{n=0}^{\infty} (a e^{-j\Omega})^n$$

$$= \frac{1}{(1 - a e^{-j\Omega})} \quad ; |a|<1$$ → periodic with Period 2π

If $|a|>1$, $X(\Omega)$ does not converge (i.e. DTFT is not defined).

Amplitude and Phase Spectra

$$X(\Omega) = \frac{1}{(1 - a\cos\Omega + ja\sin\Omega)}$$

$$\therefore |X(\Omega)| = \frac{1}{\sqrt{(1-a\cos\Omega)^2 + (a\sin\Omega)^2}} = \frac{1}{\sqrt{1+a^2-2a\cos\Omega}}$$

and $\angle X(\Omega) = -\tan^{-1}\left(\frac{a\sin\Omega}{1-a\cos\Omega}\right)$

HW Draw Amplitude and Phase Spectra.

(ii) $x[n] = a^n u[-n-1]$; $a > 1$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} a^n u[-n-1]\, e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{-1} a^n e^{-j\Omega n}$$

Put $n = -m$, yields

$$X(\Omega) = \sum_{m=1}^{\infty} (a^{-1} e^{j\Omega})^m$$

$$= \frac{1/a\, e^{j\Omega}}{1 - 1/a\, e^{j\Omega}} = \frac{1}{ae^{-j\Omega} - 1}, \quad |a| > 1$$

(iii) $x[n] = a^{-n} u[-n-1]$; $|a| < 1$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} a^{-n} u[-n-1]\, e^{-j\Omega n} = \sum_{n=-\infty}^{-1} (a e^{j\Omega})^{-n}$$

Setting $n = -m$, yields

$$X(\Omega) = \sum_{m=1}^{\infty} (a e^{j\Omega})^m = a e^{j\Omega}\left(\frac{1}{1 - a e^{j\Omega}}\right) ; \; |a|<1$$

(iv) $x[n] = a^{|n|} \; ; \; |a|<1$

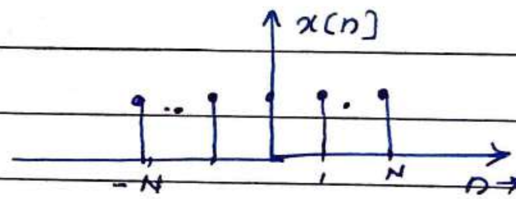
$$= a^n u[n] + a^{-n} u[-n-1]$$

Then $$X(\Omega) = \frac{1}{1 - a e^{-j\Omega}} + \frac{a e^{j\Omega}}{1 - a e^{j\Omega}}$$

$$= \frac{1 - a^2}{1 - 2a\cos\Omega + a^2} \; ; \; |a|<1$$

(v) Discrete-time Gate Pulse

$$x[n] = \begin{cases} 1 ; & -N \le n \le N \\ 0 ; & \text{otherwise} \end{cases}$$



Put $n + N = m \Rightarrow n = m - N$

$$X(\Omega) = \sum_{n=-N}^{N} 1 \cdot e^{-j\Omega n} = \sum_{m=0}^{2N} (e^{-j\Omega})^{m-N}$$

$$= \cancel{\sum} e^{j\Omega N} \sum_{m=0}^{2N} (e^{-j\Omega})^m$$

$$= e^{j\Omega N}\left[\frac{1 - (e^{-j\Omega})^{2N+1}}{1 - e^{-j\Omega}}\right]$$

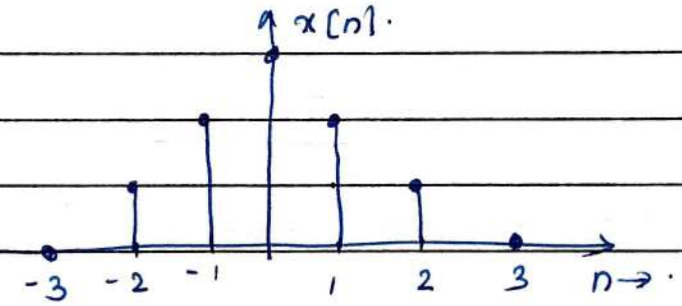
$$= \frac{e^{j\Omega N} \cdot e^{-j\Omega\left(\frac{2N+1}{2}\right)}}{e^{-j\Omega/2}}\left[\frac{e^{j\Omega\left(\frac{2N+1}{2}\right)} - e^{-j\Omega\left(\frac{2N+1}{2}\right)}}{e^{j\Omega/2} - e^{-j\Omega/2}}\right]$$

$$= \frac{\sin \Omega \left[\frac{2N+1}{2}\right]}{\sin(\Omega/2)}$$

(vi) $x[n] = \delta[n]$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} \delta[n]\, e^{-j\Omega n} = e^{-\Omega n}\Big|_{n=0} = \underline{1}$$

(vii) Discrete-time triangular pulse



$$x[n] = \delta[n+2] + 2\delta[n+1] + 3\delta[n] + 2\delta[n-1] + \delta[n-2]$$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} \left[\delta[n+2] + 2\delta[n+1] + 3\delta[n] + 2\delta[n-1] + \delta[n-2]\right] e^{-j\Omega n}$$

$$= e^{-j\Omega(-2)} + 2e^{-j\Omega(-1)} + 3e^{-j\Omega(0)} + 2e^{-j\Omega(1)} + e^{-j\Omega(2)}$$

$$= 2\cos 2\Omega + 4\cos\Omega + 3$$

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Sudhish N George, NIT Calicut

Properties of DTFT

(i) Linearity

If $x_1[n] \longleftrightarrow X_1(\Omega)$ and $x_2[n] \longleftrightarrow X_2(\Omega)$

then $\alpha x_1[n] + \beta x_2[n] \longleftrightarrow \alpha X_1(\Omega) + \beta X_2(\Omega)$

(ii) Time shifting

$$x[n-n_0] \longleftrightarrow e^{-j\Omega n_0} X(\Omega)$$

Proof:

$$DTFT\left[x[n-n_0]\right] = \sum_{n=-\infty}^{\infty} x[n-n_0] e^{-j\Omega n}$$

Substitute $n - n_0 = m$

$$= \sum_{m=-\infty}^{\infty} x[m] e^{-j\Omega(n_0+m)}$$

$$= e^{-j\Omega n_0} \sum_{m=-\infty}^{\infty} x[m] e^{-j\Omega m} = e^{-j\Omega n_0} X(\Omega)$$

(iii) Frequency Shifting

$$e^{j\Omega_0 n} x[n] \xleftrightarrow{DTFT} X(\Omega - \Omega_0)$$

Proof:

$$DTFT\left[e^{j\Omega_0 n} x[n]\right] = \sum_{n=-\infty}^{\infty} x[n] e^{j\Omega_0 n} \cdot e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} x[n] e^{-j(\Omega-\Omega_0)n} = X(\Omega - \Omega_0)$$

Note: $x[n] e^{-j\Omega_0 n} \xleftrightarrow{DTFT} X(\Omega+\Omega_0)$

(iv) Modulation Property

$$x[n]\cos\Omega_0 n \xleftrightarrow{DTFT} \frac{1}{2}\left[X(\Omega-\Omega_0)+X(\Omega+\Omega_0)\right]$$

Proof: From the frequency shifting property, we have

$$x[n]\,e^{j\Omega_0 n} \xleftrightarrow{DTFT} X(\Omega-\Omega_0)$$
$$x[n]\,e^{-j\Omega_0 n} \xleftrightarrow{DTFT} X(\Omega+\Omega_0)$$

Adding pair to pair, we obtain,

$$x[n]\left[e^{j\Omega_0 n}+e^{-j\Omega_0 n}\right] \longleftrightarrow X(\Omega-\Omega_0)+X(\Omega+\Omega_0)$$

$$\Rightarrow x[n]\cos\Omega_0 n \xleftrightarrow{DTFT} \frac{1}{2}\left[X(\Omega-\Omega_0)+X(\Omega+\Omega_0)\right]$$

(v) Time Scaling

$$x[n/k] \longleftrightarrow X(k\Omega)$$

Proof:

$$DTFT\left[x[n/k]\right] = \sum_{n=-\infty}^{\infty} x[n/k]\,e^{-j\Omega n}$$

Put $n/k = m$

$$= \sum_{m=-\infty}^{\infty} x[m]\,e^{-j\Omega mk} = X(k\Omega)$$

(vi) Time Reversal

$$x[-n] \longleftrightarrow X(-\Omega)$$

Proof: $$DTFT\left[x[-n]\right] = \sum_{n=-\infty}^{\infty} x[-n]\,e^{-j\Omega n}$$

$$= \sum_{m=-\infty}^{\infty} x[m]\, e^{j\Omega m} = X(-\Omega)$$

(viii) Conjugate Symmetry.

$$x^*[n] \longleftrightarrow X^*(-\Omega)$$

Proof:

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}$$

Replacing Ω by $-\Omega$

$$X(-\Omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{j\Omega n}.$$

Conjugating on both sides,

$$X^*(-\Omega) = \sum_{n=-\infty}^{\infty} x^*[n]\, e^{-j\Omega n} = DTFT[x^*[n]]$$

Case I : $x[n]$ is real ; $X(-\Omega) = X^*(\Omega)$

Case II : $x[n]$ is complex :

$$X(\Omega) = |X(\Omega)|\, e^{j\angle X(\Omega)}$$

$$|X(\Omega)| = |X(-\Omega)|$$

and

$$\angle X(\Omega) = -\angle X(-\Omega)$$

This is known as conjugate symmetry property.

(viii) Frequency differentiation: Multiplication by n.

$$n \cdot x[n] \longleftrightarrow j \frac{d}{d\Omega} X(\Omega)$$

Proof:

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$

Differentiating both sides w.r.t. Ω,

$$\frac{dX(\Omega)}{d\Omega} = -j \sum_{n=-\infty}^{\infty} n x[n] e^{-j\Omega n}$$

$$\Longrightarrow \quad j \frac{dX(\Omega)}{d\Omega} = \sum_{n=-\infty}^{\infty} \left[n x[n] \right] e^{-j\Omega n}$$

(ix) Convolution in

$$x_1[n] * x_2[n] \longleftrightarrow X_1(\Omega) \cdot X_2(\Omega)$$

Proof:

$$\text{DTFT}\left[x_1[n] * x_2[n] \right] = \sum_{n=-\infty}^{\infty} \left[x_1[n] * x_2[n] \right] e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x_1[m] \, x_2[n-m] \, e^{-j\Omega n}$$

Put $n-m = p$,

$$= \sum_{p=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x_1[m] \, x_2[p] \cdot e^{-j\Omega(p+m)}$$

$$= \sum_{m=-\infty}^{\infty} x_1[m]\, e^{-j\Omega m} \sum_{p=-\infty}^{\infty} x_2[p]\, e^{-j\Omega p} = X_1(\Omega) X_2(\Omega)$$

(x) Multiplication (Modulation or Windowing) Property

$$x_1[n]\, x_2[n] \xleftrightarrow{DTFT} \frac{1}{2\pi}\left[X_1(\Omega) \cdot X_2(\Omega) \right]$$

Proof:

$$DTFT\left[x_1[n]\, x_2[n] \right] = \sum_{n=-\infty}^{\infty} \left[x_1[n]\, x_2[n] \right] e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{2\pi} X_1(\theta)\, e^{j\Omega\theta}\, d\theta \right] x_2[n]\, e^{-j\Omega n}$$

$$= \frac{1}{2\pi} \int_{2\pi} X_1(\theta) \left[\sum_{n=-\infty}^{\infty} x_2[n]\, e^{-j(\Omega-\theta)n} \right] d\theta .$$

$$= \frac{1}{2\pi} \int_{2\pi} X_1(\theta)\, X_2(\Omega-\theta)\, d\theta = \frac{1}{2\pi}\left[X(\Omega) * X_2(\Omega) \right]$$

(xi) First Difference

The first difference in discrete time is equivalent to first derivative in continous-time.

$$x[n] - x[n-1] \longleftrightarrow (1 - e^{-j\Omega})\, X(\Omega$$

(xii) Parseval's Theorem.

According to Parseval's theorem, the energy E_x of $x[n]$ is given by,

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{2\pi} |X(\Omega)|^2 d\Omega$$

Proof:

By defenition,

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j\Omega n} d\Omega$$

Taking conjugate of the equation,

$$x^*[n] = \frac{1}{2\pi} \int_{2\pi} X^*(\Omega) e^{-j\Omega n} d\Omega$$

Now consider $\sum_{n=-\infty}^{\infty} \left| x[k] \right|^2 = \sum_{n=-\infty}^{\infty} x[n] x^*[n]$

$$= \sum_{n=-\infty}^{\infty} x[n] \cdot \left[\frac{1}{2\pi} \int_{2\pi} X^*(\Omega) e^{-j\Omega n} d\Omega \right]$$

$$= \frac{1}{2\pi} \int_{2\pi} X^*(\Omega) \left[\sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} \right] d\Omega$$

$$= \frac{1}{2\pi} \int_{2\pi} X^*(\Omega) X(\Omega) d\Omega$$

$$= \frac{1}{2\pi} \int_{2\pi} |X(\Omega)|^2 d\Omega$$

Q1 Find the DTFT of $x[n] = (n+1)a^n u[n]$.

Ans: $x(n) = n a^n u[n] + a^n u(n)$.

$$a^n u[n] \longleftrightarrow \frac{1}{[1-ae^{-j\Omega}]}$$

$$n a^n u(n) \longleftrightarrow j\frac{d}{d\Omega}\left[\frac{1}{1-ae^{-j\Omega}}\right] = \frac{ae^{-j\Omega}}{[1-ae^{-j\Omega}]^2}$$

$$\therefore \quad x[n] \longleftrightarrow \frac{ae^{-j\Omega}}{[1-ae^{-j\Omega}]^2} + \frac{1}{[1-ae^{-j\Omega}]}$$

$$= \frac{1}{[1-ae^{-j\Omega}]^2}$$

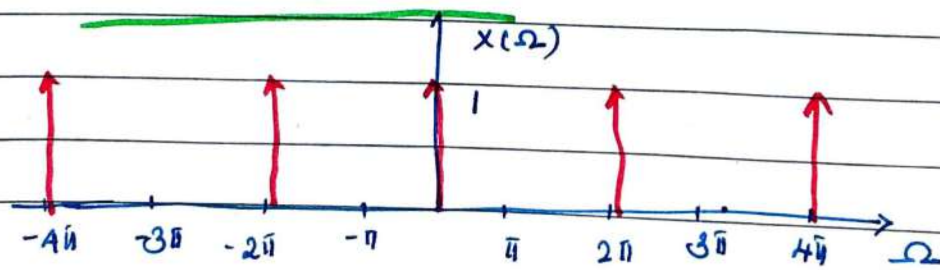
Q2. $x(n) = 4\cdot(0.5)^{n+3} u(n]$.

$$= 4\,(0.5)^n (0.5)^3 u[n]$$

$$\therefore X(\Omega) = 4\,(0.5)^3\left[\frac{1}{1-0.5e^{-j\Omega}}\right] = \frac{0.5}{1-0.5e^{-j\Omega}}$$

Fourier Transform Pairs using inverse DTFT

(i) $X(\Omega) = \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k)$



We may write,

$$X(\Omega) = \delta(\Omega) \quad ; \quad -\pi \le \Omega \le \pi$$

Then

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)\, e^{j\Omega n}\, d\Omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} \delta(\Omega)\, e^{j\Omega n}\, d\Omega$$

$$= \frac{1}{2\pi} e^{j\Omega n}\Big|_{\Omega=0} = \frac{1}{2\pi}$$

$$\therefore\ x[n] \xleftrightarrow{DTFT} X(\Omega)$$

$$\therefore\ \frac{1}{2\pi} \xleftrightarrow{DTFT} \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k)$$

$$\therefore\ 1 \xleftrightarrow{DTFT} 2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k)$$

(ii) $x[n] = e^{j\Omega_0 n}$.

$$1 \xleftrightarrow{DTFT} 2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k)$$

$$\therefore\ 1 \cdot e^{j\Omega_0 n} \xleftrightarrow{DTFT} 2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - \Omega_0 - 2\pi k)$$

$\because$ Frequency shifting.

(iii) $x[n] = \cos \Omega_0 n$.

$$DTFT[\cos \Omega_0 n] = \frac{1}{2} DTFT\left[e^{j\Omega_0 n} + e^{-j\Omega_0 n}\right]$$

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$$= \frac{1}{2}\left[2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - \Omega_0 - 2\pi k) + 2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega + \Omega_0 - 2\pi k)\right]$$

$$= \pi \sum_{k=-\infty}^{\infty} \left[\delta(\Omega - \Omega_0 - 2\pi k) + \delta(\Omega + \Omega_0 - 2\pi k)\right]$$

(iv) $x(n) = \sin \Omega_0 n$.

$$\longleftrightarrow \frac{\pi}{j} \sum_{k=-\infty}^{\infty} \left[\delta(\Omega - \Omega_0 - 2\pi k) - \delta(\Omega + \Omega_0 - 2\pi k)\right]$$

Fourier Transform of Periodic Signals

A periodic signal $x[n]$ can be represented in DTFS as.

$$x[n] = \sum_{k=0}^{N_0-1} C_k \, e^{jk\left[\frac{2\pi}{N_0}\right]n}$$

or equivalently $x[n] = \sum_{k=0}^{N_0-1} C_k \, e^{jk\Omega_0 n}$.

$\frac{2\pi}{N_0} = \Omega_0$

Taking FT on both sides,

$$X(\Omega) = \sum_{k=0}^{N_0-1} C_k \left[DTFT(e^{jk\Omega_0 n})\right]$$

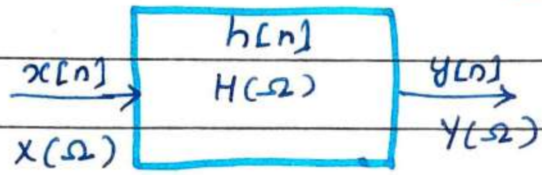
$$= \sum_{k=0}^{\infty} C_k \left[2\pi \delta(\Omega - k\Omega_0)\right]$$

$$= 2\pi \sum_{k=0}^{\infty} C_k \cdot \delta(\Omega - k\Omega_0)$$

It is an impulse train located at $\Omega = k\Omega_0$ with strength $2\pi C_k$

LTI System Analysis and DTFT

Let $x[n]$ and $y[n]$ are the input and output of an LTI system with impulse response $h[n]$.



LTI System.

$$y[n] = x[n] * h[n]$$

Using time convolution property of DTFT.

$$Y(\Omega) = X(\Omega)\, H(\Omega)$$

$$\therefore \quad |Y(\Omega)| = |H(\Omega)|\,|X(\Omega)|$$

and

$$\angle Y(\Omega) = \angle H(\Omega) + \angle X(\Omega)$$

or

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)}$$

where $H(\Omega)$ is known as frequency response of the system.

Special case:

Let $x[n] = e^{j\Omega_0 n}$

Then, $y[n] = h[n] * x[n]$

$$= \sum_{k=-\infty}^{\infty} h[k]\, x[n-k] = \sum_{k=-\infty}^{\infty} h[k]\, e^{j\Omega_0 (n-k)}$$

$$= \sum_{k=-\infty}^{\infty} h[k]\, e^{-j\Omega_0 k} \cdot e^{j\Omega_0 n}$$

$$= e^{j\Omega_0 n} \cdot H(\Omega_0)$$

$$\therefore \; y[n] = H(\Omega)\Big|_{\Omega=\Omega_0} \cdot e^{j\Omega_0 n}$$

$\therefore$ The o/p of an LTI system is a complex exponential of same frequency.

$$\therefore \quad e^{j\Omega_0 n} \Longrightarrow H(\Omega_0)\, e^{j\Omega_0 n}$$

Applications of DTFT

Distortionless Transmission

A transmission is said to be distortionless if input $x[n]$ and output $y[n]$ satisfy the condition.

$$y[n] = k\,x[n-n_0]$$

where, 'k' is a constant and accounts for scaling in amplitude and n_0 accounts for the delay and is an integer (in samples).

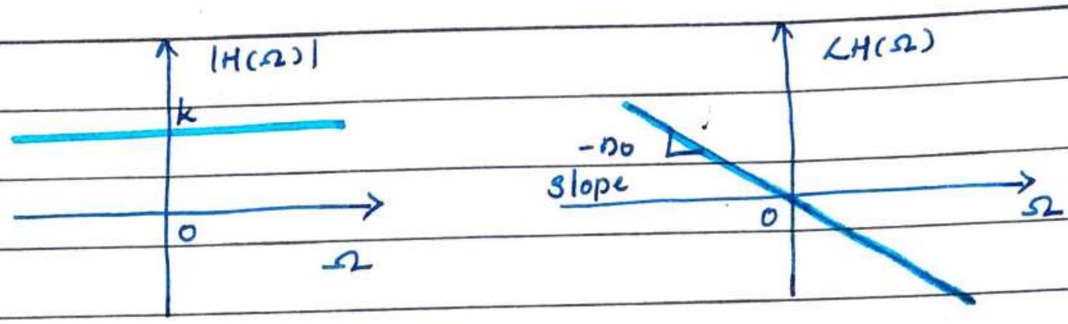
Taking the DTFT of the previous equation,

$$Y(\Omega) = k\,X(\Omega)\,e^{-j\Omega n_0}$$

$$\therefore \; H(\Omega) = k\,e^{-j\Omega n_0} = |H(\Omega)|\,e^{j\angle H(\Omega)}$$

$\therefore \; |H(\Omega)| = k$; a constant and

$\angle H(\Omega) = -\Omega n_0$; n_0 delay in number of samples.



(a) Magnitude response (b) Phase response

* Linear phase characteristics and must pass through $\Omega = 0$.
* In practice many systems have phase response that may be approximately linear, thus the slope varies with Ω.
* This variation is measured in terms of phase and group velocities.

Phase Delay

The time delay experienced by 'single frequency' signal when the signal passes through a system is referred to as phase delay and is given as,

$$\tau_p(\Omega) = \frac{-\angle H(\Omega_0)}{\Omega_0}$$

Group Delay.

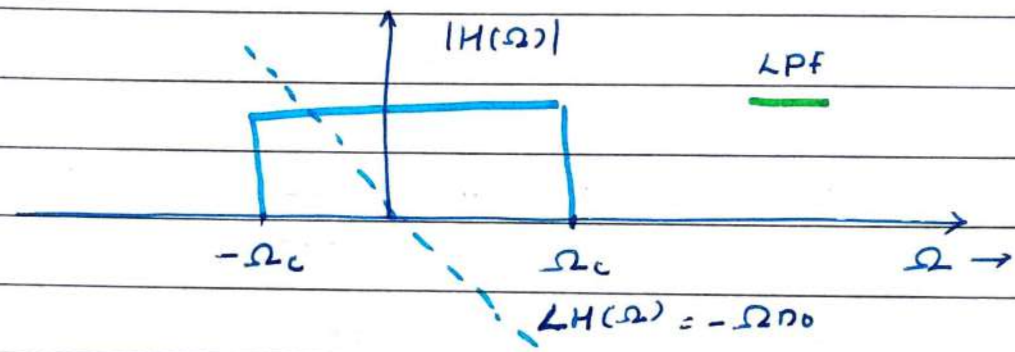
The time delay experienced by group of frequencies when an input signal that contains components with different frequencies (not harmonically related) passes through a system is referred as group delay and is given as,

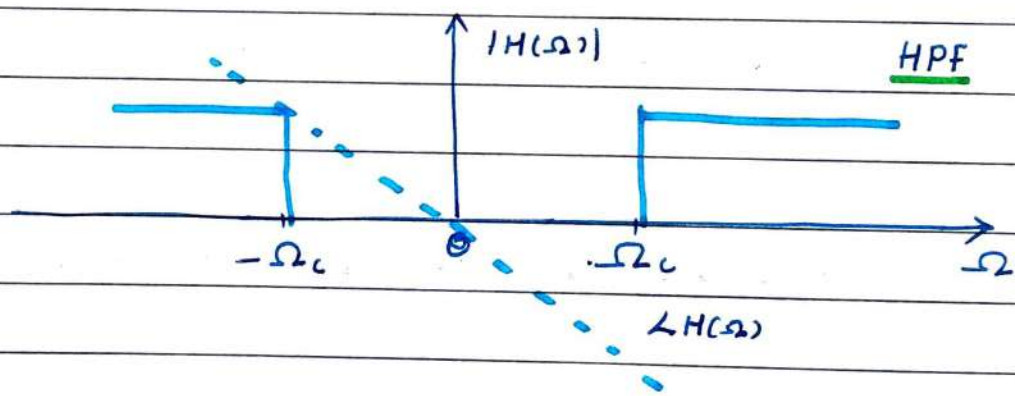
$$\tau_g(\Omega) = \frac{-d}{d\Omega} \angle H(\Omega)$$

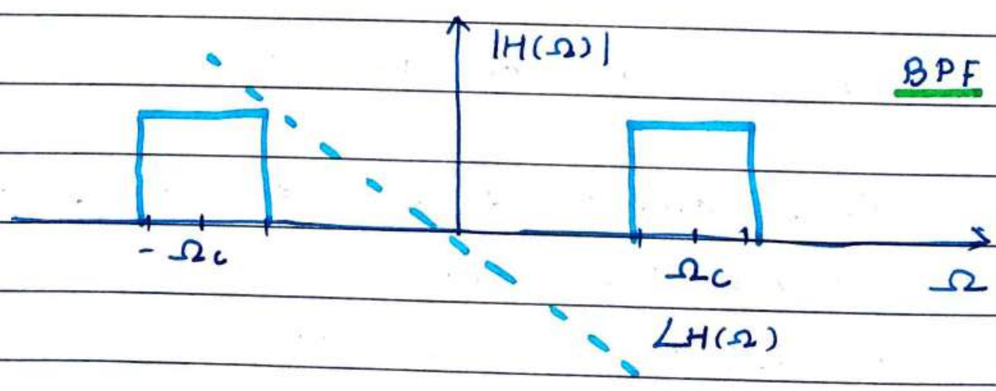
Note: For linear phase systems, both the phase ~~velocity~~ delay and group delay are constant.

Ideal and Practical Filters

Ideal filters allow distortionless transmission of certain band of frequencies and suppress all the remaining frequencies.







* The ideal LPF has linear phase of slope n_0 which results in a delay of n_0 to all it's input components below Ω_c rad/sec.

i.e. $y[n] = x[n-n_0]$

The signal $x[n]$ is transmitted by the system without distortion but with delay of n_0 samples.

For this filter, the frequency response is given by,

$$|H(\Omega)| = \text{rect}\left[\frac{\Omega}{2\Omega_c}\right]$$

$$\angle H(\Omega) = -\Omega n_0$$

$$\therefore \quad H(\Omega) = \text{rect}\left[\frac{\Omega}{2\Omega_c}\right] e^{-j\Omega n_0}$$

Taking the inverse DTFT,

$$h[n] = \frac{\sin(\Omega_c(n-n_0))}{\pi(n-n_0)}$$

Note: All ideal filters are non-causal and therefore physically unrealizable.

Response of LTI Discrete System

It is similar to the CTFT in analysis of LTI system. in continous-time domain.

* The DTFT is used to solve linear coefficient difference equations.

Q. An LTI system is specified by the equation

$$y[n] - \frac{1}{2} y[n-1] = x[n].$$

Find the frequency response and impulse response.

Ans:
$$y[n] - 0.5\, y[n-1] = x[n]$$

Taking DTFT on both sides.

$$Y(\Omega) - 0.5\, e^{-j\Omega} Y(\Omega) = X(\Omega).$$

$$\therefore \quad Y(\Omega)\left[1 - 0.5\, e^{-j\Omega}\right] = X(\Omega).$$

$$\therefore \text{ Frequency response } \quad H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{1}{\left[1 - 0.5\, e^{-j\Omega}\right]}$$

Taking the inverse DTFT

$$h[n] = (1/2)^n u[n]$$

Q. Determine the type of filter for the following difference equations.

(i) $y[n] = x[n] + x[n-1]$ and (ii) $h[n] = 0.8\,\delta[n] + 0.36(-0.8)^{n-1} u(n-1)$

Ans: (i) $y[n] = x[n] + x[n-1]$.

Taking DTFT on both sides,

$$Y(\Omega) = X(\Omega) + e^{-j\Omega} X(\Omega).$$

$$\therefore \quad H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = 1 + e^{-j\Omega}$$

$$= e^{-j\Omega/2}\left[e^{j\Omega/2} + e^{-j\Omega/2}\right] \times \frac{2}{2}$$

$$\therefore \; H(\Omega) = e^{-j\Omega/2} \cdot \cos(\Omega/2)$$

$$|H(\Omega)| = |2\cos(\Omega/2)|$$

$\therefore$ It is a LPF

(b) $h[n] = 0.8\,\delta[n] + 0.36(-0.8)^{n-1}u(n-1)$

Taking DTFT,

$$H(\Omega) = 0.8 + 0.36\left[\frac{e^{-j\Omega}}{1+0.8e^{-j\Omega}}\right]$$

$$= \frac{0.8 + 0.64e^{-j\Omega} + 0.36e^{-j\Omega}}{[1+0.8e^{-j\Omega}]}$$

$$= \frac{0.8 + e^{-j\Omega}}{1+0.8e^{-j\Omega}}$$

Here $|H(\Omega)| = 1$

Hence it is an all-pass filter. Here poles and zeros are reciprocal to each other.

Q. Given $x[n] = \{-1, 0, 1, \underset{\uparrow}{2}, 1, 0, 1, 2, 1, 0, -1\}$

and $x[n] \xleftrightarrow{DTFT} X(\Omega)$. Perform the following operations without finding DTFT.

(i) $\angle X(\Omega)$ (ii) $\int_{-\pi}^{\pi} X(\Omega)\,d\Omega$ (iii) $\int_{-\pi}^{\pi} |X(\Omega)|^2\,d\Omega$.

Ans: (i) Since $x[n]$ is symmetric about 2, $\angle X(\Omega) = \underline{-2\Omega}$

(ii) Consider

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)\, e^{j\Omega n}\, d\Omega$$

At $n=0$,

$$x[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)\, e^{0}\, d\Omega.$$

$$\Longrightarrow \int_{-\pi}^{\pi} X(\Omega)\, d\Omega = 2\pi\, x[0] = 2\pi \times 2 = \underline{4\pi}.$$

(iii) As per Parseval's theorem,

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\Omega)|^2\, d\Omega.$$

$$\Longrightarrow \int_{-\pi}^{\pi} |X(\Omega)|^2\, d\Omega = 2\pi \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$= 2\pi\left[1+0+1+4+1+0+1+4+1+0+1\right]$$

$$= 2\pi \times 14 = \underline{28\pi}$$

Relationship between CTFT and DTFT

Consider a continous time signal $x_c(t)$ with FT $X_c(\omega)$ which is band limited to B Hz. It is sampled at Nyquist rate $\left(T \le \frac{1}{2B}\right)$

The sampled signal $\overline{x_c(t)}$ is expressed as,

$$\overline{x_c(t)} = \sum_{n=-\infty}^{\infty} x_c(nT)\, \delta(t - nT)$$

Taking CTFT,

$$\overline{X_c(\omega)} = \sum_{n=-\infty}^{\infty} x_c(nT)\, e^{-jnT\omega} \quad \text{---(1)}$$

Here, the signal $x_c(nT)$ is constructed to make discrete time signal $x[n]$ such that

$$x[n] = x_c[nT]$$

Now the DTFT of $x[n]$ is given as,

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} x_c[nT]\, e^{-j\Omega n} \quad \text{---(2)}$$

On comparing Eq.(1) and (2), gives.

$$X(\Omega) = \overline{X_c(\omega)}\Big|_{\omega T = \Omega}$$

Or $\quad X(\Omega) = \overline{X_c}\,[\Omega/T]$

∴ The relationship between continous-time frequency 'ω' and discrete-time frequency 'Ω'.

$$\boxed{\omega = \frac{\Omega}{T}}$$

Energy Spectral Density (ESD)

According to Parsival's theorem, total energy E_x in a signal $x[n]$ is given as,

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{\pi}^{\pi} |X(\Omega)|^2 \, d\Omega$$

It relates energy per unit time $(|x[n]|^2)$ to energy per unit bandwidth $(|X(\Omega)|^2)$.

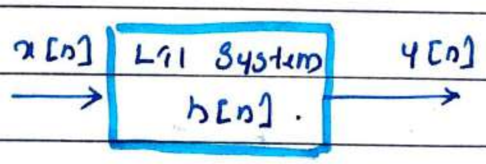
The energy per unit bandwidth is known as Energy Spectral Density (ESD) and is denoted as $S_x(\Omega)$

$$\boxed{S_x(\Omega) = |X(\Omega)|^2}$$

Power Spectral Density (PSD)

The PSD has the same relation to power signals as ESD has to energy signals.

$$\text{PSD}, \quad S_x(\Omega) = |X(\Omega)|^2$$



$$y[n] = h[n] * x[n]$$

$$Y(\Omega) = H(\Omega)\, X(\Omega)$$

$$|Y(\Omega)|^2 = |X(\Omega)|^2 \, |H(\Omega)|^2$$

$$\therefore \; S_Y(\Omega) = |H(\Omega)|^2 \, S_X(\Omega)$$

Auto-correlation Function (ACF)

It gives correlation of a sequence with its shifted version.

If $x[n]$ is an energy sequence, the ACF of discrete time signal $x[n]$ is given as,

$$R_x(k) = \sum_{n=-\infty}^{\infty} x[n]\,x[n-k] = \sum_{n=-\infty}^{\infty} x[n+k]\,x[n]$$

Properties of ACF

(i) ACF is an even function of time 'k'; $R_x(k) = R_x(-k)$

(ii) $R_x(0) = E_x = \sum_{n=-\infty}^{\infty} x^2[n]$; energy of the signal

(iii) $R_x(0) = P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x^2[n]$; power signal

(iv) $R_x(k) = x[n] * x[-n]$

(v) $R_x(k) \xleftrightarrow{DTFT} S_x(\Omega)$; auto-correlation function and PSD makes FT pairs.

Cross Correlation Function (CCF)

It gives correlation of a sequence with shifted version of another sequence.

The CCF of any two energy signal $x[n]$ and $y[n]$ is defined as,

$$R_{xy}[k] = \sum_{n=-\infty}^{\infty} x[n]\, y[n-k] = \sum_{n=-\infty}^{\infty} x[n+k]\, y[n].$$

It is an even function of 'k'

$$\underline{\underline{R_{xy}[k] = R_{yx}(-k)}}.$$

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Z - Transform

While studying DTFT, we saw two limitations with DTFT, these are.

(i) DTFT exists only for absolutely summable signals. The DTFT does not exist for exponentially or even linearly growing signals

(ii) DTFT can be applied only for BIBO stable systems. It cannot be used for unstable or even marginally stable systems.

* These problems can be solved by generalizing exponentials $e^{j\Omega n}$ to $e^{(\sigma + j\Omega)n}$.

* This generalization leads to z-transform and can be used for the analysis of many unstable systems in discrete-time domain.

* The discrete-time counterpart of Laplace transform is z-transform.

* The frequency domain analysis of discrete-time system allows us to represent any arbitary signal $x[n]$ as the sum of exponentials of the form z^n.

* There are two varients for z-transform: bilateral / two-sided and unilateral / one-sided.

* Bi-lateral z-transform can handle causal and non-causal signals. It provides insights about system's characteristics stability, causality and frequency response.

* Uni-lateral z-transform handles only causal signals and mainly used to solve difference equations with initial conditions.

The Definition

Consider a discrete-time signal $x[n]$, its Z-transform is defined as,

$$\boxed{ZT\left[x[n]\right] = X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}}$$

where 'z' is a complex variable which is given as,

$$z = r\, e^{j\Omega}$$

r - distance from origin in z-plane (magnitude of z)

Ω - an angle from positive real axis in z-plane (angle of z)

$$\boxed{x[n] \xleftrightarrow{ZT} X(z)} \Longrightarrow \text{z-transform pairs.}$$

The inverse z-transform $X(z)$ is given as,

$$x[n] = ZT^{-1}\left[X(z)\right] = \frac{1}{2\pi j}\oint X(z)\, z^{n-1}\, dz$$

The symbol $\oint$ indicates an integration in counter clockwise direction around a closed path in the complex plane.

Relationship between Z-transform and DTFT

For a signal $x[n]$, its z-transform $X(z)$ is given by,

$$ZT\left[x[n]\right] = X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

Put $z = r e^{j\Omega}$, we get

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] \cdot (r e^{j\Omega})^{-n} = \sum_{n=-\infty}^{\infty} \left[x[n]\, r^{-n}\right] e^{-j\Omega n}$$

$= \text{DTFT}\left[x[n]\, r^{-n}\right]$; $|z| = 1$

Region of Convergence of Z-transform

The z-transform is guaranteed to converge if $x[n] \cdot r^{-n}$ is absolutely summable.

$$\therefore \quad \sum_{n=-\infty}^{\infty} \left| x[n]\, r^{-n} \right| < \infty$$

This condition shows that $X(z)$ will be finite, if

$$|X(z)| < \infty$$

$\therefore$ ROC consists of those values of $|z| = |r\, e^{j\Omega}| = r$ for which the Z-transform converges.

Properties of ROC

(i) The ROC of $X(z)$ consists of a ring in the z-plane centered about the origin.

(ii) The ROC of $X(z)$ does not contain any poles.

(iii) If $x[n]$ is of finite duration, the ROC is entire Z-plane except possibly $z = 0$ and/or $z = \infty$

(iv) If $x[n]$ is a right-sided sequence, and if the circle $|z| = r_0$ is in the ROC, then all finite values of z for which $|z| > r_0$ will also be in the ROC.

(v) If $x[n]$ is a left-sided sequence, and if the circle $|z| = r_0$ is in the ROC, then all values of z for which $0 < |z| < r_0$ will also be in the ROC.

(vi) If $x[n]$ is two sided, and if the circle $|z| = r_0$ is in the ROC, then the ROC will consist of a ring in the z-plane that includes the circle $|z| = r_0$

(vii) If the Z-transform $X(z)$ of $x[n]$ is rational, then its ROC is bounded by poles or extended to infinity.

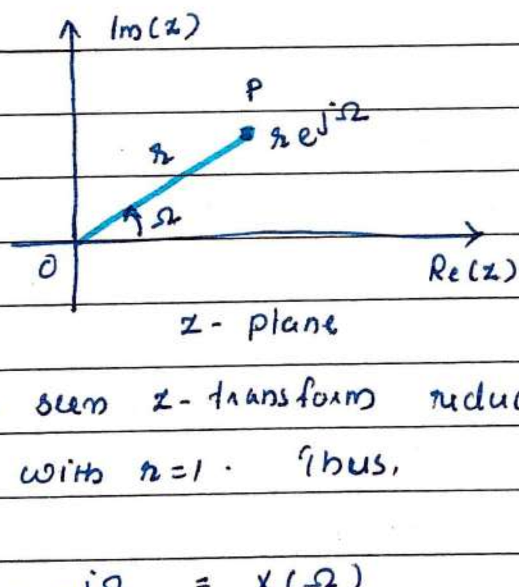
(viii) If the z-transform $X(z)$ of $x[n]$ is rational and if $x[n]$ is right sided, then the ROC is the region in the z-plane outside the outer-most pole. i.e. outside the circle of radius equal to the largest magnitude of the poles of $X(z)$.

Furthermore, if $x[n]$ is causal, then the ROC also includes $z = \infty$.

ix) If the z-transform $X(z)$ of $x[n]$ is rational and if $x[n]$ is left sided, then the ROC is in the region in the z-plane inside the inner-most non-zero pole. In particular, if $x[n]$ is anticausal, the ROC also includes $z = 0$.

The z-plane and poles and zeros

The graphical representation of complex number $z = re^{j\Omega}$ in terms of the complex plane is called as z-plane.

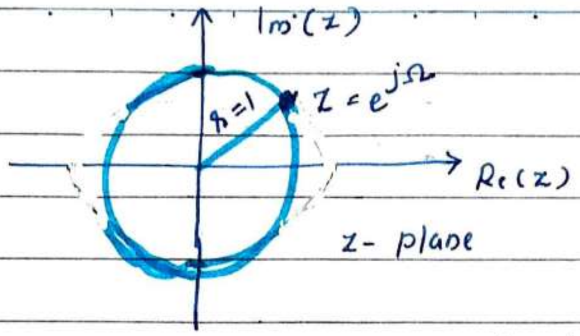


z- plane

As we have seen z-transform reduces to DTFT on the contour in z-plane with $r = 1$. Thus,

$$X(z)\Big|_{z = e^{j\Omega}} = X(\Omega)$$

$z = e^{j\Omega}$ represents a unit circle in the z-plane



Note: It is concluded that DTFT corresponds to z-transform evaluated on unit circle.

Poles and zeros

Consider a ratio of two polynomials, say F(z),

$$F(z) = \frac{N(z)}{D(z)} = \frac{b_m z^m + b_{m-1} z^{m-1} + \dots + b_1 z + b_0}{a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0}$$

Zeros : Roots of $N(z) = 0$, when $F(z)$ vanishes

i.e. $\lim_{z \to z_i} F(z) = 0$

Poles : Roots of $D(z) = 0$, when $F(z)$ becomes infinity

$$\lim_{z \to p_i} F(z) = \infty$$

Z- Transform of Some Basic Signals

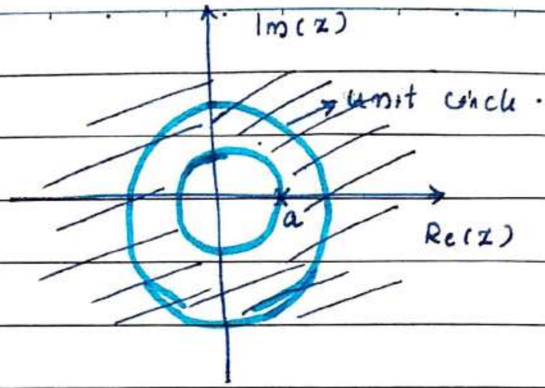
1. $x[n] = a^n u[n]$, $0 < |a| < 1$

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1-az^{-1}}$$

Then X(z) is converged if $|az^{-1}| < 1$ or
$|z| > |a|$

$$\therefore X(z) = \frac{1}{1-az^{-1}}, \quad |z| > |a|$$

ROC



(ii) $x[n] = -a^n u(-n-1]$

Then $X(z) = \sum_{n=-\infty}^{\infty} -a^n u[-n-1] z^{-n} = \sum_{n=-\infty}^{-1} (a^{-1}z)^{-n}$

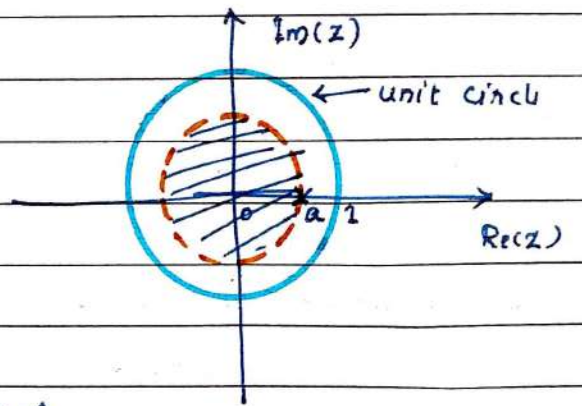
Setting $n = -m$.

$$X(z) = -\sum_{m=1}^{\infty} (a^{-1}z)^m = -a^{-1}z\left[\frac{1}{1-a^{-1}z}\right]$$

$X(z)$ is to be converged if $|a^{-1}z| < 1$ or $\underbrace{|z| < |a|}_{ROC}$

$$\therefore X(z) = \frac{a^{-1}z}{a^{-1}z-1}, \quad |z| < |a|$$

$$= \frac{1}{1-az^{-1}}; \quad |z| < |a|$$



(iii) $x[n] = u[n]$

Since $a^n u[n] \xleftrightarrow{ZT} \frac{1}{1-az^{-1}}; \quad |z| > |a|$

Setting $a = 1$

$$u[n] \xleftrightarrow{ZT} \frac{1}{1-z^{-1}}; |z| > 1$$

(iv) $x[n] = -u[-n-1]$

We have $-a^n u[-n-1] \xleftrightarrow{ZT} \dfrac{1}{1-az^{-1}}$; $|z| < |a|$

Put $a=1 \Rightarrow$ $-u[-n-1] \xleftrightarrow{ZT} \dfrac{1}{1-az^{-1}}$; $|z| < 1$

(v) $x[n] = \delta[n]$

$$X(z) = \sum_{n=-\infty}^{\infty} \delta[n] z^{-n} = 1 \quad ; \quad \text{ROC : entire z-plane}$$

(vi) $x[n] = u[-n]$

$$X(z) = \sum_{n=-\infty}^{\infty} u[-n] z^{-n} = \sum_{n=-\infty}^{0} z^{-n} = \sum_{n=0}^{\infty} z^{n}$$

$$= \frac{1}{1-z} \; ; \; |z|<1 = \frac{-z^{-1}}{1-z^{-1}} \; ; \; |z|<1$$

vii) $x[n] = a^{|n|}$

Case I : when $0 < a < 1$

$$x[n] = \begin{cases} a^n ; & n \geq 0 \\ a^{-n} ; & n < 0 \end{cases}$$

$$= a^n u[n] + a^{-n} u[-n-1]$$

$$a^n u[n] \xleftrightarrow{ZT} \frac{1}{1-az^{-1}} \; ; \; |z| > |a|$$

and

$$a^{-n} u[-n-1] \xleftrightarrow{ZT} \frac{-1}{1-\frac{1}{a}z^{-1}} \; ; \; |z| < \frac{1}{|a|}$$

$$\therefore X(z) = \left[\frac{1}{1-az^{-1}} - \frac{1}{1-\frac{1}{a}z^{-1}} \right] \; ; \; |a| < |z| < \frac{1}{|a|}$$

ROC will be a ring.

Case 2 : When $1 < a < \infty$

$$x[n] = a^n u[n] + a^{-n} u[-n-1]$$

We have

$$a^n u[n] \xleftrightarrow{ZT} \frac{1}{1-az^{-1}} \quad ; \quad |z| > |a|$$

and

$$a^{-n} u[-n-1] \xleftrightarrow{ZT} \frac{1}{1-a^{-1}z^{-1}} \quad : \quad |z| < \frac{1}{|a|}$$

Since $a > 1$, there is no common ROC and Z-transform of a^n for $1 < a < \infty$ does not exist.

(viii) $x[n] = \sin(\Omega_0 n)\, u[n]$

$$\sin[\Omega_0 n]\, u[n] = \frac{1}{2j}\left[e^{j\Omega_0 n} - e^{-j\Omega_0 n}\right] u[n].$$

We have

$$e^{j\Omega_0 n} u[n] \xleftrightarrow{ZT} \frac{1}{1-e^{j\Omega_0}z^{-1}} \quad ; \quad |z| > |e^{j\Omega_0}| \text{ or } |z| > 1$$

and

$$e^{-j\Omega_0 n} u[n] \xleftrightarrow{ZT} \frac{1}{1-e^{-j\Omega_0}z^{-1}} \quad ; \quad |z| > |e^{-j\Omega_0}| \text{ or } |z| > 1$$

$$\therefore \quad \sin[\Omega_0 n]\, u[n] \xleftrightarrow{ZT} \frac{1}{2j}\left[\frac{1}{1-e^{j\Omega_0}z^{-1}} + \frac{1}{1-e^{-j\Omega_0}z^{-1}}\right] ; \quad |z| > 1$$

$$= \frac{z^{-1}\sin\Omega_0}{1-2z^{-1}\cos\Omega_0 + z^{-2}} \quad ; \quad |z| > 1$$

(ix) $x[n] = \cos[\Omega_0 n]\, u[n]$

$$\cos[\Omega_0 n]\, u[n] = \frac{1}{2}\left[e^{j\Omega_0 n} + e^{-j\Omega_0 n}\right] u[n]$$

$$\therefore \quad \cos(\Omega_0 n) u[n] \xleftrightarrow{ZT} \frac{1}{2}\left[\frac{1}{1-e^{j\Omega_0}z^{-1}} + \frac{1}{1-e^{-j\Omega_0}z^{-1}}\right] : \quad |z| > 1$$

$$\therefore \quad X(z) = \frac{1 - z^{-1}\cos\Omega_0}{1 - 2z^{-1}\cos\Omega_0 + z^{-2}} \quad ; \quad |z| > 1$$

Q) Find the Z-transform of the following signals and comment on the ROC :

(i) $x_1[n] = \{\underset{\uparrow}{1}, 1, 1\}$ (ii) $x_2[n] = \{1, 1, \underset{\uparrow}{1}\}$

(iii) $x_3[n] = \{1, \underset{\uparrow}{1}, 1\}$

Ans: (i) $x_1[n] = \{\underset{\uparrow}{1}, 1, 1\}$ right sided signal

$$X_1(z) = \sum_{n=0}^{2} x_1[n] z^{-n} = 1 + z^{-1} + z^{-2}$$

Since $x_1[n]$ is finite duration signal and hence ROC is the entire Z-plane, but $X_1(z)$ becomes unbounded as $z \to 0$ so, ROC will not include $z = 0$.

(ii) $x_2[n] = \{1, 1, \underset{\uparrow}{1}\}$ is a left-sided signal

$$X_2(z) = \sum_{n=-2}^{0} x[n] z^{-n} = z^2 + z + 1$$

Since $x_2[n]$ is the finite duration signal and hence ROC is the entire Z-plane. However $X_2(z)$ will become unbounded as $z \to \infty$

$\therefore$ ROC is the entire Z-plane, except $z = \infty$

(iii) $x_3[n] = \{1, \underset{\uparrow}{1}, 1\}$ is a two sided signal

$$X_3(z) = z^{-1} + 1 + z$$

Since $x_3[n]$ is a finite duration signal, ROC is the entire Z-plane. However, $X_3(z)$ will be unbounded at $z = 0$ and $z = \infty$

$\therefore$ ROC will not include $z = 0$ and $z = \infty$.

Properties of Z- transform

(i) Linearity

If $x_1[n] \xleftrightarrow{ZT} X_1(z)$ with ROC = R_1 and

$x_2[n] \xleftrightarrow{ZT} X_2(z)$ with ROC = R_2

then, $\alpha x_1[n] + \beta x_2[n] \xleftrightarrow{ZT} \alpha X_1(z) + \beta X_2(z)$ with

ROC = $R_1 \cap R_2$

(ii) Time- shifting

If $x[n] \xleftrightarrow{ZT} X(z)$; ROC = R

then, $x[n-n_0] \xleftrightarrow{ZT} z^{-n_0} X(z)$ with ROC = R

except for the possible addition or deletion of $z=0$ or ∞.

Eg: $\delta[n] \xleftrightarrow{ZT} 1$; ROC : entire z-plane.

$\delta[n-1] \xleftrightarrow{ZT} z^{-1}$; ROC : entire z-plane except $z=0$

$\delta[n+1] \xleftrightarrow{ZT} z^{+1}$; ROC : entire z-plane except $z=\infty$

$\delta[n-1] + \delta[n] + \delta[n+1] \longleftrightarrow [z^{-1}+1+z]$

ROC: entire z-plane except $z=0$ and ∞

(iii) Exponential multiplication (scaling in z-domain).

$a^n x[n] \xleftrightarrow{ZT} X(z/a)$ with ROC = $|a| R$

(iv) Time Reversal

$x[-n] \xleftrightarrow{ZT} X(z^{-1})$ with ROC = $1/R$

Reversal or Reflection in time domain corresponds to inversion in the z-domain.

(v) Differentiation in z-domain

$$n\,x[n] \xleftrightarrow{ZT} z \cdot \frac{dX(z)}{dz} \quad \text{with ROC} = R.$$

On generalizing, we will get

$$(n)^k\,x[n] \xleftrightarrow{ZT} (-z)^k \frac{d^k X(z)}{dz^k}$$

(vi) Convolution in time domain

$$x_1[n] * x_2[n] \xleftrightarrow{ZT} X_1(z)\,X_2(z)$$

Multiplication in z-domain

Eg: $x_1[n] * x_2[-n] \xleftrightarrow{ZT} X_1(z) \cdot X_2(1/z)$

(vii) Time expansion

The time expansion of a signal is applicable only for +ve integer 'm', because discrete-time index is defined only for integer values.

$$x_m[n] = \begin{cases} x\left[\frac{n}{m}\right] & ; \text{ if } n \text{ is multiple of } m \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\therefore x\left[\frac{n}{m}\right] \xleftrightarrow{ZT} X(z^m) \;;\; ROC = R^{1/m} \;;\; n = k \cdot m.$$

(viii) Conjugate Symmetry

$$x^*[n] \xleftrightarrow{ZT} X^*(z^*) \quad \text{with ROC} = R.$$

(ix) Correlation property

$$x_1[m] * x_2[-m] \xleftrightarrow{ZT} X_1(z) \cdot X_2(1/z)$$

(x) First difference

$$x(n) - x(n-1) \xleftrightarrow{ZT} [1 - z^{-1}] X(z)$$

Q. Find the Z-transform of

(i) $x[n] = a^n \cos(\Omega_0 n) u[n]$

$\because a^n x(n) \xleftrightarrow{ZT} X(z/a)$

$$X(z) = \frac{1 - (z/a)^{-1} \cos\Omega_0}{1 - 2(z/a)^{-1} + (z/a)^{-2}} \quad ; \quad |z| > |a|$$

(ii) $x[n] = a^n u[n]$, then $ZT[x(n) * x[n]]$

$$ZT[x[n] * x[n]] = \frac{1}{[1 - az^{-1}]^2}$$

(iii) $x[n] = u[(n-1)/2]$

$$u[n] \xleftrightarrow{ZT} \frac{1}{1 - z^{-1}}$$

$$u[n/2] \xleftrightarrow{ZT} \left[\frac{1}{1 - z^{-2}}\right]$$

$$u\left[\frac{1}{2}(n-1)\right] \xleftrightarrow{ZT} \frac{z^{-1}}{(1 - z^{-2})}$$

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Inverse Z - Transform

The methods that are used to evaluate the inverse Z-transform are.

(i) Partial fraction expansion

(ii) Power series expansion.

(i) Partial fraction expansion

Three possibilities (i) simple poles (ii) Repeated poles and (iii) complex poles.

Q. Consider $X(z) = \frac{z^2}{(z-1)(z-0.5)}$ evaluate $x[n]$ if ROCs are

(i) ROC: $|z| > 1$ (ii) ROC: $|z| < 0.5$ (iii) ROC: $0.5 < |z| < 1$

Ans: (i) $X(z) = \frac{z^2}{(z-1)(z-0.5)}$

$X(z)$ consists of simple poles and partial fraction method can be applied.

$$\therefore \frac{X(z)}{z} = \frac{z}{(z-1)(z-0.5)} = \frac{A}{(z-1)} + \frac{B}{(z-0.5)}$$

Then $A = (z-1)\frac{X(z)}{z}\Big|_{z=1} = 2$

$B = (z-0.5)\frac{X(z)}{z}\Big|_{z=0.5} = -1$

$$\therefore \frac{X(z)}{z} = \frac{2}{z-1} + \frac{-1}{z-0.5}$$

$$X(z) = 2 \cdot \frac{z}{z-1} - \frac{z}{z-0.5} = \frac{2}{1-z^{-1}} - \frac{1}{1-0.5z^{-1}}$$

(i) ROC : $|z| > 1$; suggest ~~the~~ outsides the outermost pole, and hence the corresponding inverse signal are causal one Then,

$$\frac{2}{1-z^{-1}} \longleftrightarrow 2u[n]$$

and
$$\frac{1}{1-0.5z^{-1}} \longleftrightarrow (0.5)^n u[n]$$

$$\therefore x[n] = 2u[n] - (0.5)^n u[n]$$

(ii) ROC : $|z| < 0.5$, suggests ROC insides the inner most pole; and the signal is anticausal one.

$$\therefore \frac{2}{1-z^{-1}} \longleftrightarrow 2\left(-u(-n-1)\right) ; |z| < 1$$

$$\frac{1}{1-0.5z^{-1}} \longleftrightarrow (0.5)^n \left[-u(-n-1)\right] ; |z| < 0.5$$

$$\therefore x[n] = -2u[-n-1] - (0.5)^n \left[-u[-n-1]\right]$$

$$= \left[-2 + (0.5)^n\right] u(-n-1)$$

(iii) ROC : $0.5 < |z| < 1$, suggests a ring in z-plane. The inverse corresponding to $0.5 < |z|$ is causal one and for $|z| < 1$ is anticausal.

$$\therefore \frac{2}{1-z^{-1}} \xleftrightarrow{IZT} 2\left[-u(-n-1)\right] ; |z| < 1$$

$$\frac{1}{1-0.5z^{-1}} \xleftrightarrow{IZT} 0.5\, u[n], |z| > 0.5$$

$$x[n] = -2u[-n-1] - (0.5)^n u[n]$$

Q $$X(z) = \frac{1}{(1+z^{-1})(1-z^{-1})^2} \quad ; \quad |z| > 1$$

If $x[n]$ is a causal one, then determine the z-transform of $X(z)$.

Ans :

$$X(z) = \frac{z^3}{(z+1)(z-1)^2}$$

Then $$\frac{X(z)}{z} = \frac{z^2}{(z+1)(z-1)^2} = \frac{A}{(z+1)} + \frac{B}{(z-1)^2} + \frac{C}{z-1}$$

Where, $$A = (z+1)\left.\frac{X(z)}{z}\right|_{z=-1} = \frac{1}{4}$$

$$B = (z-1)^2\left.\frac{X(z)}{z}\right|_{z=1} = \frac{1}{2}$$

$$C = \frac{d}{dz}\left[(z-1)^2 \frac{d\,X(z)}{dz}\right]_{z=1} = \left.\frac{d}{dz}\left[\frac{z^2}{z+1}\right]\right|_{z=1} = 3/4$$

Thus, $$\frac{X(z)}{z} = \frac{1/4}{z+1} + \frac{1/2}{(z-1)^2} + \frac{3/4}{z-1}$$

$$\therefore \quad X(z) = \frac{1/4\,z}{(z+1)} + \frac{1/2\,z}{(z-1)^2} + \frac{3/4\,z}{(z-1)}$$

$$= \frac{1}{4}\left[\frac{1}{1-(-1)z^{-1}}\right] + \frac{1}{2}\,\frac{z^{-1}}{(1-z^{-1})^2} + \frac{3}{4}\cdot\frac{1}{1-z^{-1}}$$

Since $|z| > 1$, i.e. $x[n]$ is right sided one, hence causal signal

$$x[n] = \frac{1}{4}(-1)^n u[n] + \frac{1}{2} n\,u[n] + \frac{3}{4} u[n]$$

Q. Consider $x(z) = \dfrac{2z(3z+17)}{(z-1)(z^2-6z+25)}$. If $x[n]$ is a causal, then determine inverse z-transform.

Ans: $\dfrac{X(z)}{z} = \dfrac{2(3z+17)}{(z-1)(z^2-6z+25)} = \dfrac{A}{(z-1)} + \dfrac{Bz+C}{z^2-6z+25}$

where,

$$A = (z-1)\frac{X(z)}{z}\Big|_{z=1} = \underline{\underline{2}}$$

Now,

$$\frac{2(3z+17)}{(z-1)(z^2-6z+25)} = \frac{2}{(z-1)} + \frac{Bz+C}{z^2-6z+25}$$

Put $z=0 \implies$

$$\frac{-34}{25} = -2 + \frac{C}{25}$$

$$C = \underline{\underline{16}}$$

Then, $B = \underline{\underline{-2}}$

$$\therefore \quad \frac{X(z)}{z} = \frac{2}{z-1} + \frac{-2z+16}{z^2-6z+25}$$

$$\therefore \quad X(z) = \frac{2z}{z-1} + \frac{(-2z+16)z}{z^2-6z+25}$$

$$= \frac{2}{1-z^{-1}} + \frac{-2(1-8z^{-1})}{1-6z^{-1}+25z^{-2}}$$

$$= \frac{2}{1-z^{-1}} + \frac{(-2)(1-3z^{-1})}{1-6z^{-1}+25z^{-2}} + \frac{-2\times 5z^{-1}}{1-5z^{-1}+25z^{-2}}$$

$$= \frac{2}{1-z^{-1}} + \frac{(-2)(1-3z^{-1})}{1-6z^{-1}+25z^{-2}} + \frac{(5/2)\,4z^{-1}}{1-6z^{-1}+5^2z^{-2}}$$

We have

$$\frac{1 - a\cos\Omega_0 z^{-1}}{1 - 2a\cos\Omega_0 z^{-1} + a^2 z^{-2}} \longleftrightarrow a^n \cos(\Omega_0 n)\, u[n]$$

and

$$\frac{a \sin\Omega_0 z^{-1}}{1 - 2a\cos\Omega_0 z^{-1} + a^2 z^{-2}} \longleftrightarrow a^n \sin(\Omega_0 n)\, u[n]$$

Then, $a\cos\Omega_0 = 3$ and $a \sin\Omega_0 = 4$

$$a^2 = 25 \implies a = 5$$

$$\Omega_0 = 0.925$$

$$\therefore\ X(z) = \frac{2}{1-z^{-1}} + \frac{(-2)\left[1 - 5\cos(0.927) z^{-1}\right]}{1 - 2\times 5\times \cos(0.927) z^{-1} + 5^2 z^{-2}}$$

$$+ \left(\frac{5}{2}\right) \frac{5 \cdot \sin(0.927) z^{-1}}{1 - 2\times 5 \times \cos(0.927) z^{-1} + 5^2 z^{-2}}$$

Taking the inverse z-transform.

$$x[n] = 2u[n] - 2(5)^n \cos(0.927n)\, u[n]$$
$$+ \frac{5}{2}(5)^n \sin(0.927n)\, u[n]$$

Power series expansion (Long division method)

By defenition

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \dots + x[-1] z + x[0] + x[1] z^{-1} + x[2] z^{-2} + \dots$$

This result is a power series in z^{-1} or z. The coefficients of this power series can be identified as, $\dots x(-1), x(0), x[1], \dots$

The power series expansion method particularly is useful.

(i) whenever we want to know only the first few terms of the sequence $x[n]$.

(ii) for non-rational z-transform ; eg: $\log(1+az^{-1})$

Q. Using long division method, the inverse z-transform of $X(z) = \frac{1}{1-az^{-1}}$ If the ROCs (i) $|z|>a$ and (ii) $|z|<|a|$

Ans: (i) ROC : $|z|>|a|$

This expression can be expanded in power series by long division

$$
\begin{array}{r|l}
 & 1 + az^{-1} + a^2z^{-2} + \cdots \\
\hline
1 - az^{-1} & 1 \\
 & 1 - az^{-1} \\
\hline
 & az^{-1} \\
 & az^{-1} - a^2z^{-2} \\
\hline
 & a^2z^{-2} \\
 & a^2z^{-2} - a^3z^{-3} \\
\hline
 & a^3z^{-3}
\end{array}
$$

$\therefore X(z) = 1 + az^{-1} + a^2z^{-2} + \cdots \quad |z|>|a|$

$\therefore x[0] = 1, \quad x[1] = a, \quad x[2] = a^2, \cdots$

$x[n] = 0 ; n<0$

$\therefore x[n] = a^n u[n]$

(ii) ROC : $|z|<|a|$

$$
\begin{array}{r|l}
 & -a^{-1}z - a^{-2}z^2 + \cdots \\
\hline
-az^{-1}+1 & 1 \\
 & 1 - a^{-1}z \\
\hline
 & a^{-1}z
\end{array}
$$

$\therefore X(z) = -a^{-1}z - a^{-2}z^{-2} - \cdots \quad |z|<|a|$

$$\therefore \; x[-1] = -a^{-1} \; ; \; x[-2] = -a^{-2}, \ldots$$

$$x[n] = -a^n u[-n-1]$$

Discrete-time LTI Systems and Z-Transform

$x[n]$, $X(z)$ $\longrightarrow$ $h[n]$ LTI system $H(z)$ $\longrightarrow$

$$y[n] = x[n] * h[n]$$

$$Y(z) = H(z)\,X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} \Longrightarrow \text{Transfer function}$$

For $z = e^{j\Omega}$; $H(z) = H(e^{j\Omega})$ is known as frequence response

Impulse Response and Step Response

Impulse response $h[n] \xleftrightarrow{ZT} H(z)$, transfer fn.

$$\therefore \; h[n] = IZT\,[H(z)]$$

As we know, the step response

$$s[n] = h[n] * u[n]$$

$$= \sum_{k=-\infty}^{\infty} h[k]\,u[n-k]$$

$$= \sum_{k=-\infty}^{n} h[k]$$

Taking z-transform

$$S(z) = \frac{H(z)}{1-z^{-1}}$$ (Accumulation property

$$\therefore \; H(z) = (1 - z^{-1})\, S(z)$$

$$H(z) = S(z) - z^{-1} S(z)$$

Taking the inverse z-transform,

$$h(n) = s[n] - s[n-1]$$

Note: First difference of step response gives impulse response.

Q. Consider the transfer function of a causal LTI system

$$H(z) = \frac{z^{-1} + \frac{1}{2} z^{-2}}{1 - \frac{3}{5} z^{-1} + \frac{2}{25} z^{-2}}$$

Find impulse and step response.

Ans:

$$H(z) = \frac{z^{-1} + 1/2\, z^{-2}}{1 - 3/5\, z^{-1} + 2/25\, z^{-2}} = \frac{z^{-1} + 1/2\, z^{-2}}{\left[1 - 1/5\, z^{-1}\right]\left[1 - 2/5\, z^{-1}\right]}$$

Using partial fraction method,

$$\frac{H(z)}{z} = \frac{1 + \frac{1}{2} z^{-1}}{\left(z - 1/5\right)\left[z - \frac{2}{5}\right]} = \frac{A}{\left[z - 1/5\right]} + \frac{B}{\left[z - 2/5\right]}$$

$$A = \left[z - \frac{1}{5}\right] \frac{H(z)}{z} \Bigg|_{z = 1/5} = \frac{-35}{2}$$

$$B = \left[z - \frac{2}{5}\right] \frac{H(z)}{z} \Bigg|_{z = 2/5} = \frac{45}{4}$$

$$\frac{H(z)}{z} = \frac{-35/2}{z - 1/5} + \frac{45/4}{z - 2/5}$$

$$\therefore H(z) = -\frac{35}{2}\left[\frac{z}{z - 1/5}\right] + \frac{45}{4}\left[\frac{z}{z - 2/5}\right]$$

$$\therefore h(n) = IZT[H(z)] = \left[-\frac{35}{2}\left(\frac{4}{5}\right)^n + \frac{45}{4}\left(\frac{2}{5}\right)^n\right] u(n)$$

As we know,

$$S(z) = H(z) \cdot \frac{1}{1 - z^{-1}} = H(z)\left[\frac{z}{1 - z}\right]$$

Homework

Step response $s(n) = \frac{35}{8}\left(\frac{1}{5}\right)^n u(n) - \frac{15}{2}\left(\frac{2}{5}\right)^n u(n) + \frac{25}{8} u(n)$

Causality and Stability

Causality

* For a discrete-time LTI system to be causal, if $h(n) = 0$ for $n < 0$ and is right-sided.
* $\therefore$ The corresponding ROC is in the region in z-plane exterior of a circle outside the outermost pole
* For a rational system function $H(z)$ expressed as ratio of polynomial in z; in order to be causal, the degree of numerator cannot be greater than the degree of denominator.

Stability

A discrete time LTI system to be stable if ROC includes unit circle. i.e. all poles of rational transfer function $H(z)$ are within the unit circle.

A system is to be unstable if and only if.

(i) atleast one pole of $H(z)$ is outside the unit circle or

(ii) there are repeated poles of $H(z)$ on the unit circle.

A system is said to be marginally stable if and only if some poles are on the unit circle and no poles of $H(z)$ lies outside the unit circle.

Stability to Causality

A causal system is one whose ROC is outside the outermost pole. For a causal system to be stable, the outermost poles of $H(z)$ must be inside the unit circle.

Q) Check the TFs of LTI system are causal or not

(i) $$H_1(z) = \frac{z^3 - 2z^2 + z}{z^2 + \frac{1}{4}z + 1/8}$$

Since numerator of $H_1(z)$ is of higher than the denominator. Hence the system is not causal.

(ii) $$H_2(z) = \left[\frac{1}{1-0.5z^{-1}} + \frac{2}{1-3z^{-1}}\right] \quad ; \quad |z| > 3$$

Since the ROC for this system is exterior of a circle outside the outermost pole. Hence $H_2(z)$ is causal System.

(iii)

$$H(z) = \frac{1}{\left[1-\frac{1}{2}z^{-1}\right]\left[1+\frac{1}{4}z^{-1}\right]}$$

* This system has poles at $z = 1/2$ and $z = -1/4$. Since its the ROC is outside the outermost pole, hence system is causal.
* Also noting that ROC includes the unit circle, consequently system is stable.

Z-Transform of Causal Periodic Signals

Let $x_1[n]$, $x_2[n]$, $x_3[n]$, ... be the signal representing 1st, 2nd, 3rd, ... cycles of a periodic signal with fundamental period N.

$$\therefore \quad x_2[n] = x_1[n-N]$$
$$x_3[n] = x_1[n-2N].$$
$$\vdots$$

$$\therefore \quad x[n] = x_1[n] + x_1[n-N] + x_1[n-2N] + \cdots$$

Taking the z-transform,

$$X(z) = X_1(z) + X_1(z)z^{-N} + X_1(z)z^{-2N} + \cdots$$

$$= X_1(z) \cdot \frac{1}{1-z^{-N}} \quad \because |z^{-N}| < 1$$

$$\text{where} \quad X_1(z) = \sum_{n=0}^{N-1} x_1[n] z^{-n}.$$

Relation between Laplace Transform and Z-Transform

Consider the sampled signal $x_s(t)$ obtained by sampling a continuous-time signal $x(t)$ with sampling interval T.

$\therefore$

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\delta(t-nT)$$

Taking LT on both sides

$$X_s(s) = \int_{-\infty}^{\infty} x_s(t)\,e^{-st}\,dt = \int_{-\infty}^{\infty}\left[\sum_{n=-\infty}^{\infty} x(nT)\,\delta(t-nT)\right]e^{-st}\,dt$$

$$= \sum_{n=-\infty}^{\infty} x(nT)\int_{-\infty}^{\infty}\delta(t-nT)\,e^{-st}\,dt = \sum_{n=-\infty}^{\infty} x(nT)\left(e^{sT}\right)^{-n}$$

Under the conditions of sampling theorem a continuous-time signal $x(t)$ is exactly represented by a sequence of instantaneous sample values $x(nT)$; i.e. the discrete time sequence $x[n]$ is related to $x(nT)$ as,

$$x[n] = x(nT)$$

$$\therefore \quad X_s(s) = \sum_{n=-\infty}^{\infty} x[n]\left(e^{sT}\right)^{-n}$$

Introducing $z = e^{sT}$

$$X_s(s) = \sum_{n=-\infty}^{\infty} x[n]\,z^{-n}$$

$$\therefore \quad X_s(s)\Big|_{z=e^{sT}} = X(z)$$

$$\therefore \quad s = \frac{1}{T}\ln(z)$$

$$s = \sigma + j\omega$$

$$z = e^{(\sigma+j\omega)T} = e^{\sigma T}\cdot e^{j\omega T}$$

$$|z| = e^{\sigma T}$$

Q) Find the z-transform of the signal, obtained by sampling of

$$x(t) = e^{-at} u(t)$$

Ans: Let the sampling interval be T

$$x(t)\Big|_{t=nT} = x(nT) = e^{-anT} u(nT) = (e^{-aT})^n u(n)$$

$$x[n] = x(nT) \quad \text{(from sampling conditions)}$$

$$\therefore x[n] = (e^{-aT})^n u[n] .$$

$$\therefore X(z) = \sum_{n=-\infty}^{\infty} (e^{-aT})^n u[n] z^{-n} = \sum_{n=0}^{\infty} (e^{-aT} z^{-1})^n$$

$$= \frac{1}{1 - e^{-aT} z^{-1}} \quad : |z| > |e^{-aT}|$$

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Unilateral Z- Transform

* The unilateral z-transform can handle causal signals and systems.
* The UZT of $x[n]$ is given as

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

$$\therefore \quad x[n] \xleftrightarrow{UZT} X(z)$$

Properties of Z- transform

The UZT and BZT have many properties in common. The following properties are valid only for UZT.

(i) Right shift (Delay)

If $x[n] \xleftrightarrow{UZT} X(z)$

Then $x[n-1] \xleftrightarrow{UZT} z^{-1}X(z) + x[-1]$

Proof:

$$UZT\left[x[n-1]\right] = \sum_{n=0}^{\infty} x[n-1] z^{-n}$$

$$= x[-1] + x[0]z^{-1} + x[1]z^{-2} + \cdots$$

$$= x[-1] + z^{-1}\left[x[0] + x[1]z^{-1} + \cdots\right]$$

$$= x[-1] + z^{-1}X(z)$$

Then $UZT\left[x[n-2]\right] = z^{-2}X(z) + z^{-1}x[-1] + x[-2]$

On generalizing,

$$x[n-m] \xleftrightarrow{UZT} z^{-m}X(z) + z^{-m}\sum_{n=1}^{m} x[-n] z^{n}$$

(ii) Left Shift (Advance)

$$x[n+1] \xleftrightarrow{UZT} zX(z) - z\,x[0]$$

Proof :

$$UZT\left[x[n+1]\right] = \sum_{n=0}^{\infty} x[n+1]\,z^{-n}$$

$$= x[1] + x[2]z^{-1} + x[3]\,z^{-2} + \cdots$$

Multiplying with z on the expression of $X(z)$

$$zX(z) = z\,x[0] + x[1] + x[2]z^{-1} + \cdots$$

$$\therefore\; zX(z) - z\,x[0] = x[1] + x[2]z^{-1} + \cdots$$

$$= UZT\left[x[n+1]\right]$$

$||^{ly}$, $$x[n+2] \xleftrightarrow{UZT} z^2X(z) - z^2x[0] - z\,x[1].$$

On generalizing

$$x[n+m] \xleftrightarrow{UZT} z^m X(z) - z^m \sum_{n=0}^{m-1} x[n]\,z^{-n}$$

(iii) Initial Value Theorem

It states that

$$x[0] = \lim_{z\to\infty} X(z)$$

Proof :

$$X(z) = \sum_{n=0}^{\infty} x[n]\,z^{-n}$$

$$= x[0] + x[1]\,z^{-1} + x[2]\,z^{-2} + \cdots$$

$$\lim_{z\to\infty} X(z) = x[0]$$

(iv) Final Value Theorem

It states that

$$x[\infty] = \lim_{z \to 1} (1 - z^{-1})\, X(z)$$

Pole of $(1-z^{-1})X(z)$ must be inside the unit circle.

Note: For a ratio polynomial function, if the order of the numerator polynomial is greater than the denominator polynomial, then the initial value theorem is not applicable.

Q) Evaluate the initial value of the signal whose z-transform is,

$$X(z) = \frac{1+z^{-1}}{(1-z^{-1})(1+0.5z^{-1})}$$

Ans:

$$x[0] = \lim_{z \to \infty} X(z)$$

$$= \lim_{z \to \infty} \left[\frac{1+z^{-1}}{[1-z^{-1}][1+0.5z^{-1}]} \right] = \frac{1+0}{(1-0)(1+0)} = \underline{\underline{1}}$$

Q) Evaluate the final value of the signal whose z-transform is

$$X(z) = \frac{2z^{-1}}{1-1.8z^{-1}+0.8z^{-2}}$$

Ans:

$$x[\infty] = \lim_{z \to 1} (1-z^{-1})\, X(z) = \lim_{z \to 1} (1-z^{-1}) \frac{2z^{-1}}{1-1.8z^{-1}+0.8z^{-2}}$$

$$= \lim_{z \to 1} \frac{(1-z^{-1})\,2z^{-1}}{(1-z^{-1})(1-0.8z^{-1})} = \lim_{z \to 1} \frac{2z^{-1}}{(1-0.8z^{-1})} = \underline{\underline{10}}$$

Z-transform Solution of Linear Difference Equations with Initial Conditions

Q Solve the difference equations

$$y[n] + 3y[n-1] = x[n]$$

with initial conditions $y[-1]=1$ and determine $y[n]$ for the input $x[n] = 8\,u[n]$.

Ans:

$$y[n] + 3y[n-1] = x[n] \quad \text{with} \quad y[-1]=1$$

Applying unilateral z-transform,

$$Y(z) + 3\left[z^{-1}Y(z) + y[-1]\right] = \frac{8}{1-z^{-1}}$$

$$\therefore \; Y(z)\left[1+3z^{-1}\right] + 3 = \frac{8}{1-z^{-1}}$$

$$\therefore \; Y(z) = \frac{-3}{1+3z^{-1}} + \frac{8}{\left(1-z^{-1}\right)\left[1+3z^{-1}\right]}$$

$$= \frac{-3}{1+3z^{-1}} + \frac{A}{1+z^{-1}} + \frac{B}{1+3z^{-1}}$$

Using partial fractions method, $A = 2$ and $B = 6$

$$\therefore \; Y(z) = \frac{-3}{1+3z^{-1}} + \frac{2}{1+z^{-1}} + \frac{6}{1+3z^{-1}}$$

$$= \frac{3}{1+3z^{-1}} + \frac{2}{1+z^{-1}}$$

Taking inverse UZT,

$$y[n] = 3(-3)^n u[n] + 2\,u[n]$$

Q) Solve the difference equation

$y[n+2] - 5y[n+1] + 6y[n] = 3x[n+1] + 5x[n]$ with initial conditions are $y[-1] = \frac{11}{6}$, $y[-2] = \frac{37}{36}$ and the input $x[n] = (1/2)^n u[n]$.

Ans: $y[n+2] - 5y[n+1] + 6y[n] = 3x[n+1] + 5x[n]$.

On application of UZT, it can be seen that shift property requires the knowledge of auxilary conditions $y(0), y[1], \ldots$ rather than $y[-1], y[-2], \ldots$

This problem is overcome by replacing n with n-2.

$\therefore\ y[n] - 5y[n-1] + 6y[n-2] = 3x[n-1] + 5x[n-2]$

Applying UZT,

$$Y(z) - 5\left[z^{-1}Y(z) + y[-1]\right] + 6\left[z^{-2}Y(z) + z^{-1}y[-1] + y[-2]\right]$$

$$= 3\left[z^{-1}X(z) + x[-1]\right] + 5\left[z^{-2}X(z) + z^{-1}x[-1] + x[-2]\right]$$

Since $x[n] = (1/2)^n u[n]$ is a causal signal, $x[-1] = x[-2] = 0$

and $X(z) = \dfrac{1}{1 - \frac{1}{2}z^{-1}}$

Then,

$$Y(z) - 5\left[z^{-1}Y(z) + \frac{11}{6}\right] + 6\left[z^{-2}Y(z) + \frac{11}{6}z^{-1} + \frac{37}{36}\right]$$

$$= \frac{3z^{-1}}{1 - \frac{1}{2}z^{-1}} + \frac{5z^{-2}}{1 - \frac{1}{2}z^{-1}}$$

Then $$Y(z) = \frac{10.5\,z^{-2} - 9.5z^{-1} + 3}{\left[1 - \frac{1}{2}z^{-1}\right]\left[1 - 5z^{-1} + 6z^{-2}\right]}$$

Then $$\frac{Y(z)}{z} = \frac{3z^2 - 9.5z + 10.5}{(z-0.5)(z-2)(z-3)}$$

$$= \frac{A}{(z-0.5)} + \frac{B}{(z-2)} + \frac{C}{(z-3}$$

$$= \frac{26/15}{z-0.5} - \frac{7/3}{z-2} + \frac{18/5}{z-3}$$

$$\therefore\ Y(z) = \frac{26}{15}\left[\frac{z}{z-0.5}\right] - \frac{7}{3}\left[\frac{z}{z-2}\right] + \frac{18}{5}\left[\frac{z}{z-3}\right]$$

$$\therefore\ y[n] = \left[\frac{26}{15}(0.5)^n - \frac{7}{3}(2)^n + \frac{18}{5}(3)^n\right]u[n]$$

Zero input response and zero-state response

Zero input response (ZIR)

The system response due to the initial conditions when the input is considered to be zero is known as ZIR

Zero state response (ZSR)

The system response to input $x[n]$ when all initial conditions are zero; i.e. zero states.

HW Evaluate ZIR and ZSR for the given difference equation $y[n] - 5y[n-1] + 6y[n-2] = 3x[n-1] + 5x[n-2]$. with initial conditions are $y[-1] = 11/6$, $y[-2] = 37/36$ and the input $x[n] = (1/2)^n u[n]$.

Ans : $y_{ZIR}[n] = \left[5(2)^n - 2(3)^n\right] u[n]$

$$y_{ZSR}[n] = \left[\frac{26}{5}(0.5)^n - \frac{22}{3}(2)^n + \frac{28}{5}(3)^n\right] u[n]$$

$$y_{total}[n] = \left[\frac{26}{15}(0.5)^n + \frac{7}{3}(2)^n + \frac{18}{5}(3)^n\right] u[n]$$

The transformed Representation of LTID System.

$$x[n] \longrightarrow \boxed{ZT} \xrightarrow{X(z)} \boxed{H(z)} \xrightarrow{Y = H(z)X(z)} \boxed{IZT} \longrightarrow y[n]$$

System Realization

→ Block diagram representation using basic operations such as adder, a scalar multiplication, unit delay and pick off points. They are classified into $1/z$

(i) Direct - form Realization I (DF I)

(ii) Canonic ~~for~~ direct-form Realization (DF II)

(iii) transpose form of DF II

unit delay

$$\longrightarrow \boxed{z^{-1}} \longrightarrow$$

Consider a general N^{th} order LTID system which is given as,

$$H(z) = \frac{b_0 z^N + b_1 z^{N-1} + \cdots + b_{N-1} z + b_N}{z^N + a_1 z^{N-1} + \cdots + a_{N-1} z + a_N} = \frac{Y(z)}{X(z)}$$

$$= \frac{b_0 + b_1 z^{-1} + \cdots + b_{N-1} z^{-(N-1)} + b_N z^{-N}}{1 + a z^{-1} + \cdots + a_{N-1} z^{-(N-1)} + a_N z^{-N}}$$

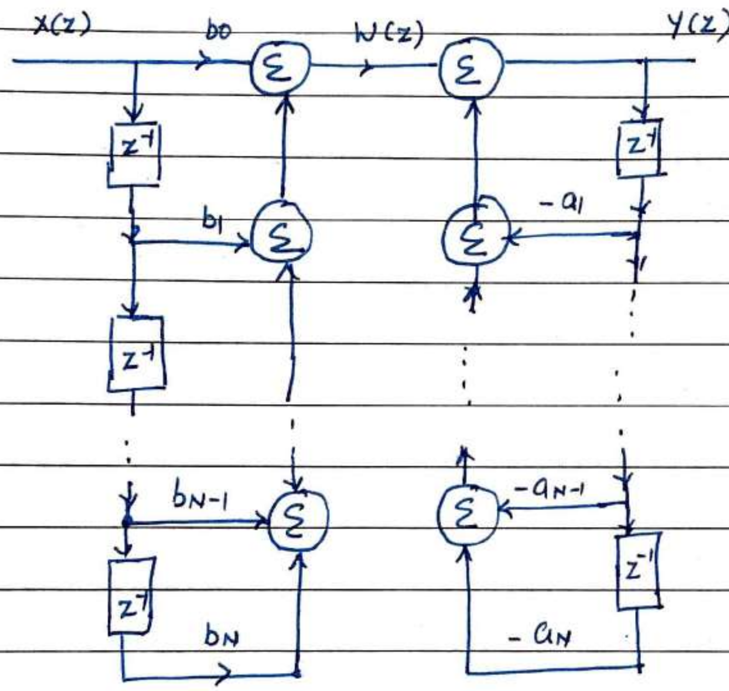


Fig : (a) DF I

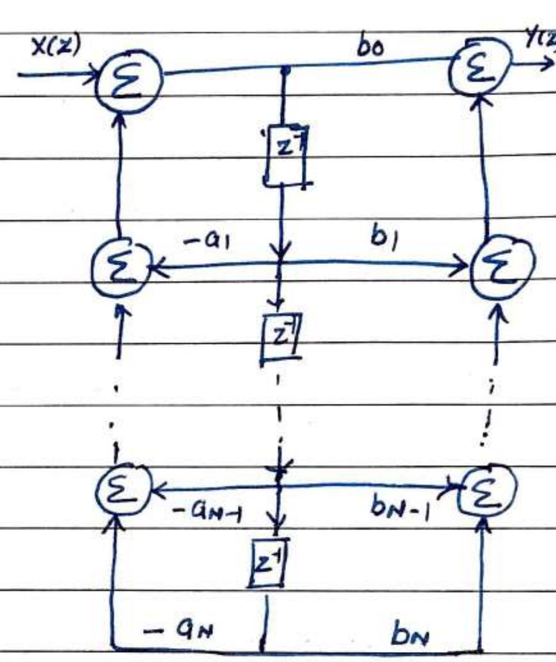
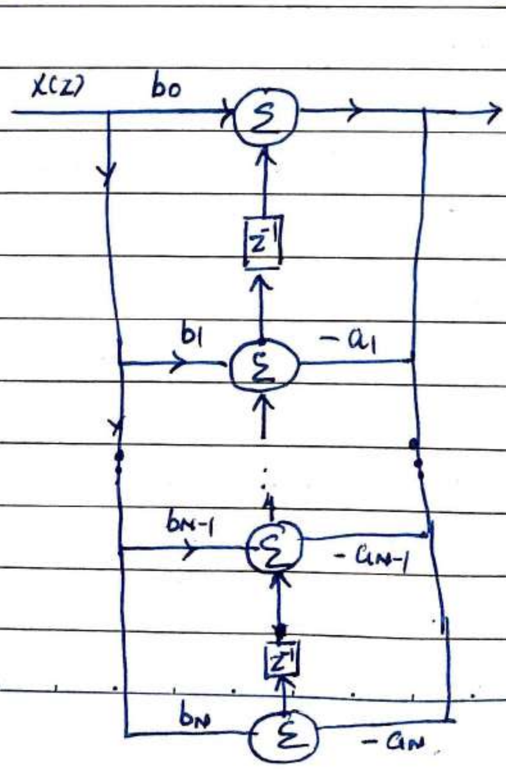


Fig : (b) Canonic direct (DF II)

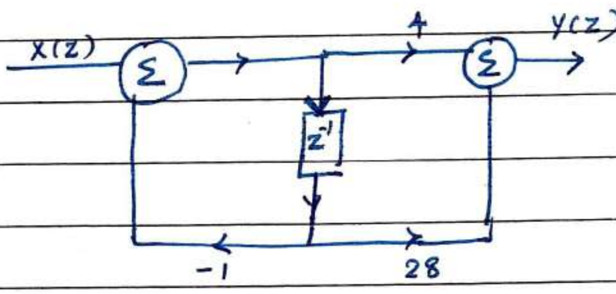


Fig(c) : transpose form of DF II

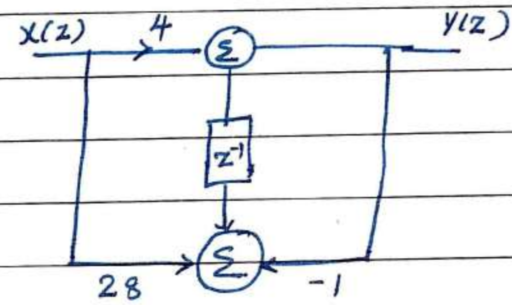
Q Find the canonic direct and transposed canonic direct realization of

$$H(z) = \frac{4z+28}{z+1}$$

Ans: $a_1 = 1$, $b_0 = 4$ and $b_1 = 28$



(a) canonic direct form



(b) its transpose

Frequency Response / System Response

Q For the system,

$$y[n+1] - 0.8\,y[n] = x[n+1]$$

find the system response to the input $x[n] = 1^n = 1$

Ans: $$y[n+1] - 0.8\,y[n] = x[n+1]$$

Taking z-transform on both sides

$$z\,Y(z) - 0.8\,Y(z) = z\,X(z).$$

$$\therefore\ H(z) = \frac{Y(z)}{X(z)} = \frac{z}{z-0.8} = \frac{1}{1-0.8z^{-1}}$$

Then, the frequency response

$$H(e^{j\Omega}) = \frac{1}{1-0.8e^{-j\Omega}} = \frac{1}{1-0.8\cos\Omega + j0.8\sin\Omega}$$

$$\therefore \quad |H(e^{j\Omega})| = \frac{1}{\sqrt{(1-0.8\cos\Omega)^2 + (0.8\sin\Omega)^2}}$$

$$= \frac{1}{\sqrt{1.64 - 1.6\cos\Omega}}$$

$$\angle H(e^{j\Omega}) = -\tan^{-1}\left[\frac{0.8\sin\Omega}{1-0.8\cos\Omega}\right]$$

At $\Omega = 0$, $\quad |H(e^{j\Omega})| = 5$ and $\angle H(e^{j\Omega}) = 0$

Ref. Eg: 5.10 (Lathi) for plots.

Note: Periodic nature of frequency response.

$H(e^{j\Omega}) = H(e^{j(\Omega + 2\pi m)})$; m - integer.

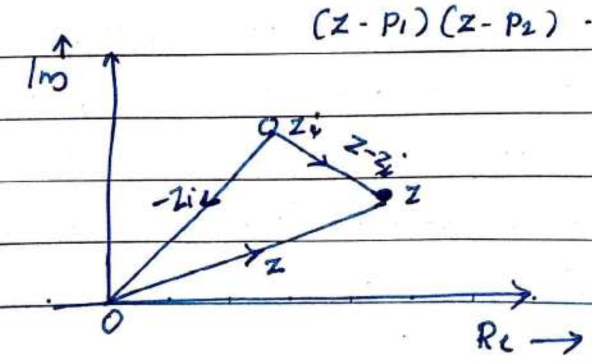
with period 2π.

Frequency Response from Pole-Zero Location

* The frequency response of a system is determined by pole-zero locations of $H(z)$.

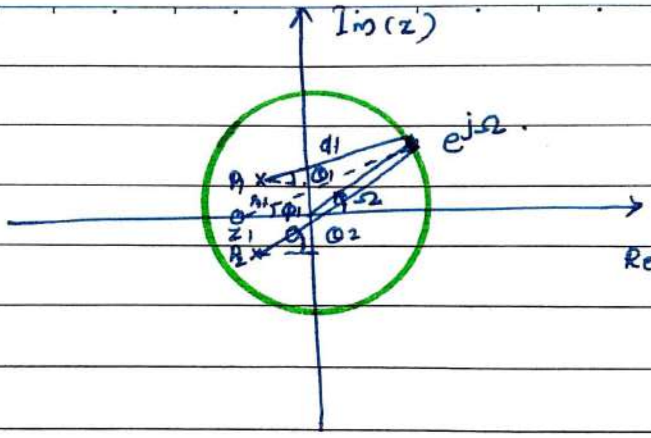
* The N^{th} order TF $H(z)$ is given by,

$$H(z) = b_0 \frac{(z-z_1)(z-z_2)\cdots(z-z_N)}{(z-p_1)(z-p_2)\cdots(z-p_N)}$$



To compute $H(e^{j\Omega})$

Put $z = e^{j\Omega}$



Then $H(z)\Big|_{z=e^{j\Omega}} = H(e^{j\Omega}) = b_0 \dfrac{(r_1 e^{j\phi_1}) \dots (r_N e^{j\phi_N})}{(d_1 e^{j\theta_1}) \dots (d_N e^{j\theta_N})}$

$$= b_0 \frac{(r_1 r_2 \dots r_N)}{(d_1 d_2 \dots d_N)} e^{j[(\phi_1 + \phi_2 + \dots + \phi_N] - (\theta_1 + \dots + \theta_N)}$$

$\therefore \; |H(z)|_{z=e^{j\Omega}} = b_0 \dfrac{r_1 r_2 \dots r_N}{d_1 d_2 \dots d_N}$

$$= b_0 \frac{\text{Product of zeros distance to } e^{j\Omega}}{\text{Product of distances of poles to } e^{j\Omega}}$$

and

$$\angle H(z)\Big|_{z=e^{j\Omega}} = (\phi_1 + \phi_2 + \dots + \phi_N) - (\theta_1 + \theta_2 + \dots + \theta_N)$$

$= (\text{Sum of zero angles to } e^{j\Omega}) - (\text{sum of pole angles to } e^{j\Omega})$

Controlling gain by placement of poles and zeros

* The nearer the pole (or zero) is to a point $e^{j\Omega}$ (on the unit circle), the more influence of that pole (or zero) wields on the amplitude response at that frequency because the length of the vector joining that pole (or zero) to the point $e^{j\Omega}$ is small.

* The proximity of a pole (or zero) has the similar effect on the phase response.
* To enhance the amplitude response at a frequency Ω, we should place a pole as close as possible to the point $e^{j\Omega}$ (on the unit circle).
* Similarly, to suppress the amplitude response at a frequency Ω, we should place a zero as close as possible to the point $e^{j\Omega}$.
* Pole must be inside the unit circle for a stable system, whereas, zeros may lie anywhere.

Ref: Fig. 5.19 (Lathi)

Eg: 5.14 (BPF)
: 5.15 (Notch / BSF)

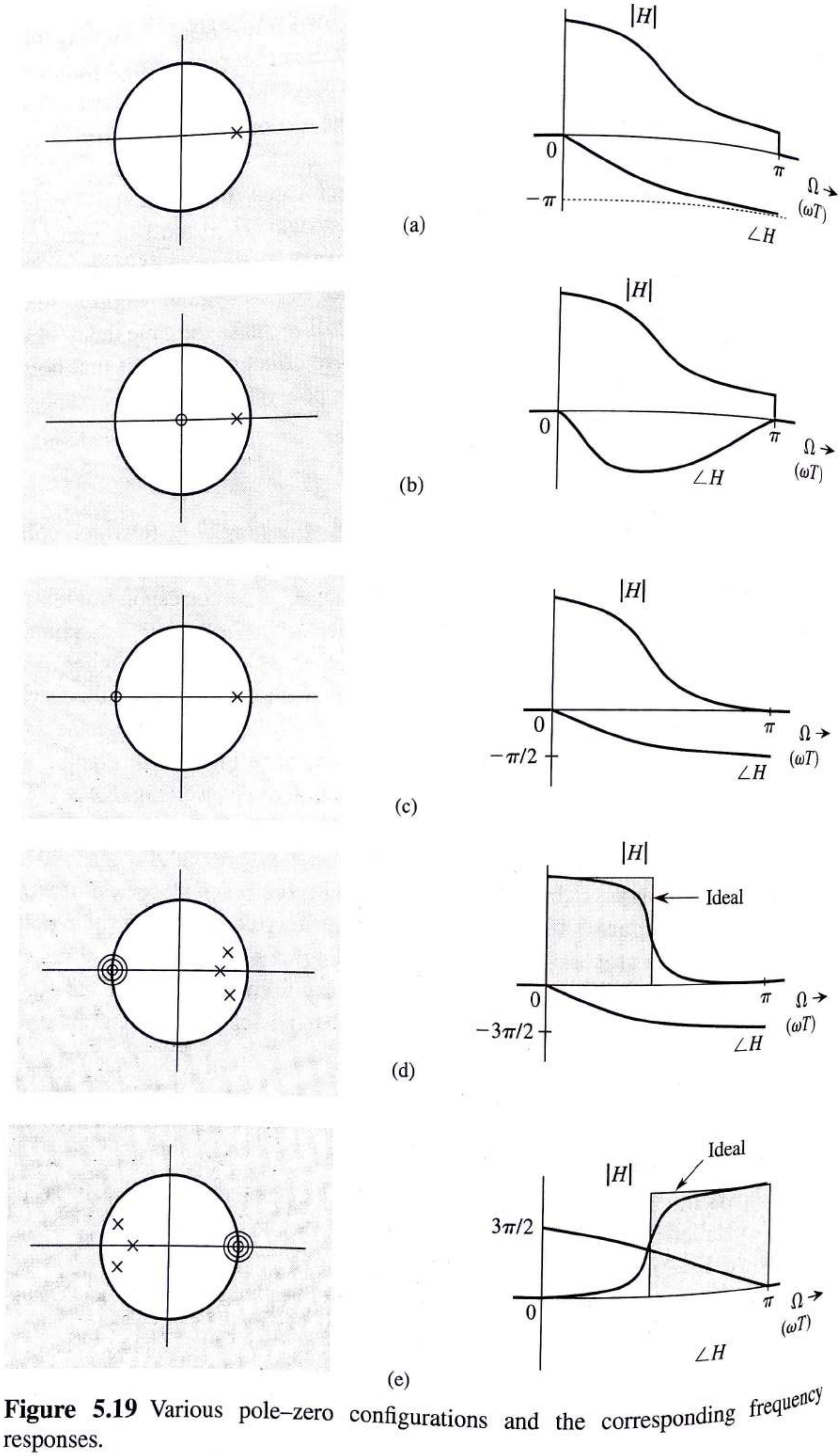
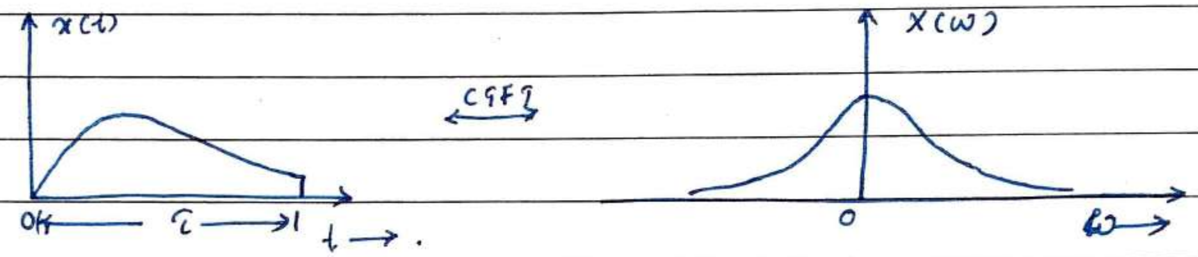


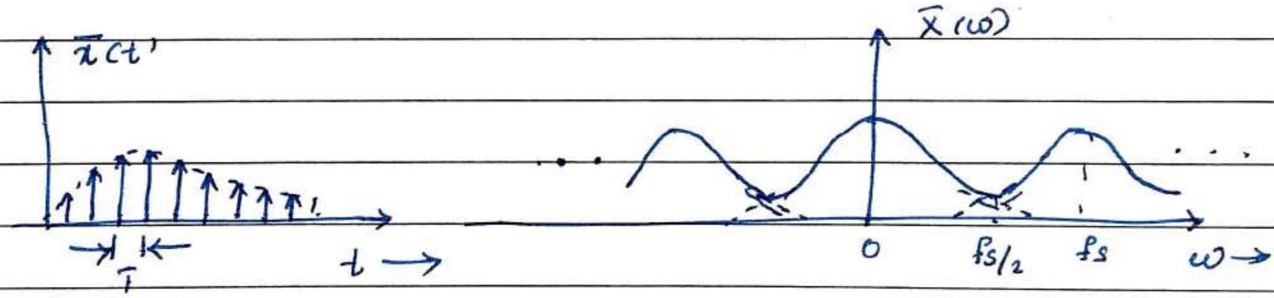
Figure 5.19 Various pole–zero configurations and the corresponding frequency responses.

Discrete Fourier Transform (DFT)

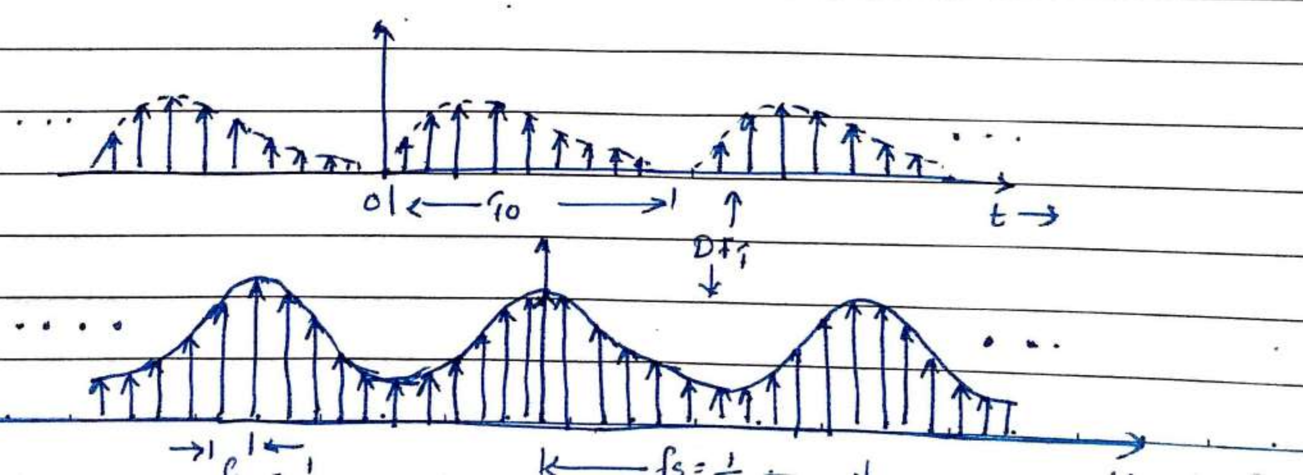
* Digital processing of fourier transform of continuous time signal $x(t)$ requires sample values of $x(t)$ and also a computer can compute $X(\omega)$ only at some discrete values of 'ω'.
* $\therefore$ We need to relate the samples of $x(t)$ to samples of $X(\omega)$.



* On sampling, the spectrum $\bar{X}(\omega)$ of sampled signal $\bar{x}(t)$ consists of $X(\omega)$ repeating every f_s Hz with $f_s = 1/T$



* The sampled signal is repeated periodically every T_0 second.
* According to spectral sampling theorem, such an operation results in sampling of spectrum at a rate of T_0 samples/Hz with $f_0 = 1/T_0$ Hz.



* The overall operation can viewed as when a signal $x(t)$ is sampled and then periodically repeated, the corresponding spectrum is also sampled and periodically repeated.

* The discrete Fourier transform (DFT) is the Fourier transform of "sampled signal repeated periodically".
* The number of points in DFT is known as N-point DFT.

The Definition

Consider a sampled and periodic signal $x[n]$. Its DFT is given as,

$$DFT[x[n]] = X(k) = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k}{N}n} \qquad k = 0, 1, \dots N-1$$

The signal $x[n]$ and $X[k]$ make DFT pairs

$$\underbrace{x[n]}_{\substack{\text{Discrete} \\ \text{periodic}}} \xleftrightarrow{DFT} \underbrace{X[k]}_{\substack{\text{Discrete} \\ \text{periodic}}}$$

and the inverse DFT is given as,

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi k}{N}n} \quad ; \quad \Omega = \frac{2\pi}{N}.$$

when N is the period of the signal $x[n]$.

* DFT relates the samples of time domain to samples of frequency domain.

(Q) Find DFT of signal $x[n] = \{\underset{\uparrow}{1}, 2, 3, 4\}$.

Ans: Here $N = 4$, $k = 0, 1, 2, 3$

$$X[k] = \sum_{n=0}^{3} x[n]\, e^{-j\frac{2\pi}{N} k \cdot n}$$

$$= \sum_{n=0}^{3} x[n] \left[e^{-j\pi/2} \right]^{kn} = \sum_{n=0}^{3} x[n] (-j)^{kn}$$

$$x[n] = \{\underset{\uparrow}{1}, 2, 3, 4\}$$

$\therefore$

$$X[k] = 1 + 2(-j)^k + 3(-j)^{2k} + 4(-j)^{3k}$$

$\therefore$ $X[0] = 1+2+3+4 = 10 \rightarrow$ sum of samples.

$X[1] = 1 - 2j - 3 + 4j = -2 + j2$

$X[2] = 1 - 2 + 3 - 4 = -2 \rightarrow$ alternate sum and difference of samples.

$X[3] = 1 + 2j - 3 - 4j = -2 - 2j \rightarrow$ conjugate of $X[1]$

The 4-point DFT of $x[n]$ is given as,

$$X[k] = \{10, -2+j2, -2, 2-j2\}.$$

Matrix Interpretation of the DFT

* The DFT maps N - numbers - $x[0], x[1], \ldots, x[N-1]$ into new numbers - $X[0], X[1], \ldots X[N-1]$.

* Interpret this operation as the product of an $N \times N$ DFT matrix and an $N \times 1$ vector of signal values.
  * useful way to view DFT
  * power of matrix (linear) algebra in signal processing.

Example $(N = 4)$

$$X[k] = \sum_{n=0}^{3} x[n]\, e^{-jk\frac{2\pi}{4}n} \quad ; \quad \boxed{W_N = e^{-j\frac{2\pi}{N}}}$$

twiddle factor

$$\therefore \quad X[k] = x[0]\,W_4^{k\cdot 0} + x[1]\,W_4^{k\cdot 1} + x[2]\,W_4^{k\cdot 2} + x[3]\,W_4^{3k}$$

$\therefore$

$$\begin{array}{c} k=0 \\ k=1 \\ k=2 \\ k=3 \end{array} \begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix} = \underbrace{\begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix}}_{\text{DFT matrix for } N=4} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}$$

In general,

Let $\bar{x} = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}$ and $\bar{X} = \begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{bmatrix}$

$$\Longrightarrow \quad \bar{X} = W\bar{x}.$$

where $W = \begin{bmatrix} W_N^0 & W_N^0 & \cdots & W_N^0 \\ W_N^0 & W_N^1 & \cdots & W_N^{1(N-1)} \\ W_N^0 & W_N^2 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \cdots & \vdots \\ W_N^0 & W_N^{N-1} & \cdots & W_N^{(N-1)^2} \end{bmatrix}$ N-point DFT matrix

So that $X[k] = W_N^0\,x[0] + W_N^k\,x[1] + W_N^{2k}\,x[2] + \cdots + W_N^{k(N-1)}\,x[N-1]$

Inverse DFT

If $\bar{X} = W\bar{x}$ is the DFT, then $\bar{x} = W^{-1}X$ is the inverse DFT.

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}kn} = \frac{1}{N}\Big[X[0]\, W_N^{-0} + X[1]\, W_N^{-n} + \dots$$
$$+ X[2]\, W_N^{-2n} + \dots + X[N-1]\, W_N^{-(N-1)n}\Big]$$

$$\therefore \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ \vdots \\ x[N-1] \end{bmatrix} = \frac{1}{N}\begin{bmatrix} W_N^{-0} & W_N^{-0} & \dots & W_N^{-0} \\ W_N^{-0} & W_N^{-1} & \dots & W_N^{-(N-1)} \\ W_N^{-0} & W_N^{-2} & \dots & W_N^{-2(N-1)} \\ - & - & - & - \\ \vdots & & & \vdots \\ W_N^{-0} & W_N^{-(N-1)} & \dots & W_N^{-(N-1)^2} \end{bmatrix}\begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ \vdots \\ X[N-1] \end{bmatrix}$$

$$\bar{x} = W^{-1}\bar{X}$$

Note : $W^{-1} = \frac{1}{N} W^{H}$
$\downarrow$
Complex conjugate transpose.

Check $W W^{-1} = W^{-1} W = I$.

Q. The 4-point DFT of a discrete time sequence $\{1, 0, 2, 3\}$ is

Ans :

$$\bar{X} = W\bar{x}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ -1+3j \\ 0 \\ -1-3j \end{bmatrix} \qquad W_4 = e^{-j\frac{2\pi}{4}}$$

Properties of DFT

(i) Linearity

If $x_1[n] \xleftrightarrow{DFT} X_1[k]$ and $x_2[n] \xleftrightarrow{DFT} X_2[k]$

then $\alpha x_1[n] + \beta x_2[n] \xleftrightarrow{DFT} \alpha X_1[k] + \beta X_2[k]$

(ii) Time shifting

If $x[n] \xleftrightarrow{DFT} X[k]$

then $x[n-n_0] \xleftrightarrow{DFT} e^{-j\frac{2\pi}{N}kn_0} X[k]$

Proof:

$$DFT\left[x(n-n_0)\right] = \sum_{n=0}^{N-1} x[n-n_0]\, e^{-j\frac{2\pi}{N}k\cdot n}$$

Put $n-n_0 = m$.

$$= \sum_{m=-n_0}^{N-1-n_0} x[m]\cdot e^{-j\frac{2\pi}{N}k(m+n_0)}$$

$$= e^{-j\frac{2\pi}{N}kn_0} \sum_{m=-n_0}^{N-1-n_0} x[m]\, e^{-j\frac{2\pi}{N}k\cdot m} = e^{-j\frac{2\pi}{N}kn_0} X[k]$$

(iii) Shift in k-domain.

$$x[n]\, e^{j\frac{2\pi}{N}k_0 n} \xleftrightarrow{DFT} X[k-k_0]$$

Proof:

$$DFT\left[x[n]\, e^{j\frac{2\pi}{N}k_0 n}\right] = \sum_{n=0}^{N-1} x[n]\, e^{j\frac{2\pi}{N}k_0 n}\cdot e^{-j\frac{2\pi}{N}k\cdot n}$$

$$= \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}(k-k_0)n} = X\left[k-k_0\right]$$

(iv) Conjugate Symmetry

$x^*[n] \xleftrightarrow{DFT} X^*[-k]$

Proof:

By definition,

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}k\cdot n}$$

Replacing k with $-k$

$$X[-k] = \sum_{n=0}^{N-1} x[n]\, e^{j\frac{2\pi}{N}k\cdot n}$$

Taking conjugate on both sides as,

$$X^*[-k] = \sum_{n=0}^{N-1} x^*[n]\, e^{-j\frac{2\pi}{N}k\cdot n} = \underline{\underline{DFT[x^*[n]]}}$$

(v) Note: a) For real-valued $x[n]$: $X^*[k] = X[-k]$

$X^*[-k] = X[k]$

This is known as even conjugate.

b) Since $X[k]$ is periodic in N, hence

$X^*[-k] = X^*[N-k]$

This is known as conjugate symmetric.

(v) Time Reversal

$x[-n] \xleftrightarrow{DFT} X[-k]$ and

$x[N-n] \xleftrightarrow{DFT} X[N-k]$

(vi) Linear Convolution*

$x_1[n] \xleftrightarrow{DFT} X_1[k]$ with period N_1

$x_2[n] \xleftrightarrow{DFT} X_2[k]$ with period N_2

then $x_1[n] * x_2[n] \xleftrightarrow{DFT} X_1[k]\, X_2[k]$ with period N_0.

(vii) Periodic Convolution*

If $x_1[n] \xleftrightarrow{DFT} X_1[k]$ with period N_0 and

$x_2[n] \xleftrightarrow{DFT} X_2[k]$ with period N_0.

then,

$$x_1[n] \circledN x_2[n] \xleftrightarrow{DFT} X_1[k]\, X_2[k] \text{ with period } N_0.$$

(viii) Parseval's theorem.

$$\text{energy, } E = \sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2$$

Proof :

By definition

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{+j\frac{2\pi}{N}kn}$$

Taking conjugate, we have

$$x^*[n] = \frac{1}{N}\sum_{k=0}^{N-1} X^*[k]\, e^{-j\frac{2\pi}{N}kn}$$

Now consider,

$$\sum_{n=0}^{N-1} |x[n]|^2 = \sum_{n=0}^{N-1} x[n]\cdot x^*[n]$$

$$= \sum_{n=0}^{N-1} x[n]\, \frac{1}{N}\sum_{k=0}^{N-1} X^*[k]\, e^{-j\frac{2\pi}{N}kn}$$

Adjusting the above equation,

$$= \frac{1}{N}\sum_{k=0}^{N-1} X^*[k] \left[\sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}\right]$$

$$= \frac{1}{N}\sum_{k=0} X^*[k]\, X[k] = \frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2$$

* To be discussed in DSP.

Q) Consider $x[n] = \{2, 1, 1, 0, 3, 2, 0, 3, 4, 6\}$ whose DFT is $X[k]$. Evaluate the following

(i) $X[0]$ (ii) $X[5]$ (iii) $\sum_{k=0}^{9} X[k]$ (iv) $|X[k]|^2$

Ans :

$N = 10$

By definition $X[k] = \sum_{k=0}^{9} x[n] e^{-j\frac{2\pi}{10}k \cdot n}$

(i) $X[0] = \sum_{n=0}^{9} x[n] = 2+1+1+0+3+2+0+3+4+6 = \underline{\underline{22}}$

(ii) $X[5] = X[\frac{N}{2}] = X[\frac{10}{2}]$

$\therefore X[5] = \sum_{n=0}^{9} x[n] e^{-j\frac{2\pi}{N}\cdot\frac{N}{2}\cdot n} = \sum_{n=0}^{9} x[n](-1)^n$

$= 2-1+1-0+3-2+0-3+4-6 = \underline{\underline{-2}}$

(iii) $\sum_{k=0}^{9} X[k]$

We have $x[0] = \frac{1}{10}\sum_{k=0}^{9} X[k]$

$\therefore \sum_{k=0}^{9} X[k] = 10 \times x[0] = \underline{\underline{20}}$

(iv) $|X[k]|^2$

According to Parseval's theorem.

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2$$

$$\therefore \sum_{k=0}^{N-1} |X[k]|^2 = N\sum_{n=0}^{N-1} |x[n]|^2 = 10[2^2+1^2+1^2+0+3^2+2^2+0+3^2+4^2+6^2]$$

$$= \underline{\underline{800}}$$

Introduction to Fast Fourier Transform (FFT) *

The N-point DFT is given as,

$$X[k] = \sum_{n=0}^{N-1} x[n] \left[e^{-j\frac{2\pi}{N}}\right]^{kn} = \sum_{n=0}^{N-1} x[n] \, W_N^{kn}$$

* To compute one point (one sample) DFT, it is required 'N' multiplications and N-1 additions.

* To compute N-point DFT, it is required 'N^2' multiplications and N (N-1) additions.

*

* In general, we can say that as N increases these computation become time consuming even for high-speed computer.

* The use of Fast Fourier Transform (FFT) reduces the number of computations from N^2 to N log N.
(multiplications)