GEOMETRY

Through Algebra

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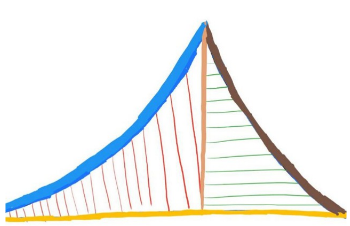

Contents

Introduction

This book shows how to solve problems in geometry using trigonometry and coordinate geometry.

Chapter 1

Triangle

Consider a triangle with vertices

$$\mathbf{A} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -4 \\ 6 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -3 \\ -5 \end{pmatrix} \tag{1.1}$$

1.1. Vectors

1.1.1. The direction vector of AB is defined as

$$\mathbf{B} - \mathbf{A} \tag{1.1.1.1}$$

Find the direction vectors of AB, BC and CA.

Solution:

(a) The Direction vector of AB is

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} -4 \\ 6 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -4 - 1 \\ 6 - (-1) \end{pmatrix} = \begin{pmatrix} -5 \\ 7 \end{pmatrix} \tag{1.1.1.2}$$

(b) The Direction vector of BC is

$$\mathbf{C}-\mathbf{B}=\begin{pmatrix}-3\\-5\end{pmatrix}-\begin{pmatrix}-4\\6\end{pmatrix}=\begin{pmatrix}-3-(-4)\\-5-6\end{pmatrix}=\begin{pmatrix}1\\-11\end{pmatrix} \tag{1.1.1.3}$$

(c) The Direction vector of CA is

$$\mathbf{A}-\mathbf{C}=\begin{pmatrix}1\\-1\end{pmatrix}-\begin{pmatrix}-3\\-5\end{pmatrix}=\begin{pmatrix}1-(-3)\\-1-(-5)\end{pmatrix}=\begin{pmatrix}4\\4\end{pmatrix} \tag{1.1.1.4}$$

1.1.2. The length of side BC is

$$c=\left\|\mathbf{B}-\mathbf{A}\right\| \triangleq \sqrt{\left(\mathbf{B}-\mathbf{A}\right)^{\top}\left(\mathbf{B}-\mathbf{A}\right)} \tag{1.1.2.1}$$

where

$$\mathbf{A}^{\top} \triangleq \begin{pmatrix}1 & -1\end{pmatrix} \tag{1.1.2.2}$$

Similarly,

$$b=\left\|\mathbf{C}-\mathbf{B}\right\|,\ a=\left\|\mathbf{A}-\mathbf{C}\right\| \tag{1.1.2.3}$$

Find a, b, c.

(a) Since,

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 5 \\ -7 \end{pmatrix}, \tag{1.1.2.4}$$

$$c = \|\mathbf{A} - \mathbf{B}\| = \sqrt{\begin{pmatrix} 5 & -7 \end{pmatrix} \begin{pmatrix} 5 \\ -7 \end{pmatrix}} = \sqrt{(5)^2 + (7)^2} \tag{1.1.2.5}$$

$$= \sqrt{74} \tag{1.1.2.6}$$

(b) Similarly,

$$\mathbf{B} - \mathbf{C} = \begin{pmatrix} -1 \\ 11 \end{pmatrix} \tag{1.1.2.7}$$

$$\implies a = \|\mathbf{B} - \mathbf{C}\| = \sqrt{\begin{pmatrix} -1 & 11 \end{pmatrix} \begin{pmatrix} -1 \\ 11 \end{pmatrix}} = \sqrt{(1)^2 + (11)^2} \tag{1.1.2.8}$$

$$= \sqrt{122} \tag{1.1.2.9}$$

and

(c)

$$\mathbf{A}-\mathbf{C} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \tag{1.1.2.10}$$

$$\implies b = \left\lVert \mathbf{A}-\mathbf{C} \right\rVert = \sqrt{\begin{pmatrix} 4 & 4 \end{pmatrix}\begin{pmatrix} 4 \\ 4 \end{pmatrix}} = \sqrt{(4)^2 + (4)^2} \tag{1.1.2.11}$$

$$= \sqrt{32} \tag{1.1.2.12}$$

1.1.3. Points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are defined to be collinear if

$$\text{rank}\begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} = 2 \tag{1.1.3.1}$$

Are the given points in (1.1) collinear?

Solution: From (1.1),

$$\begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -4 & -3 \\ -1 & 6 & -5 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 + R_2} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -4 & -3 \\ 0 & 2 & -8 \end{pmatrix} \tag{1.1.3.2}$$

$$\xleftrightarrow{R_2 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 5 & 4 \\ 0 & 2 & -8 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 - \frac{2}{5} R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 5 & 4 \\ 0 & 0 & \frac{-48}{5} \end{pmatrix} \tag{1.1.3.3}$$

There are no zero rows. So,

$$\operatorname{rank}\begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} = 3 \tag{1.1.3.4}$$

Hence, the points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are not collinear. This is visible in Fig. 1.1.

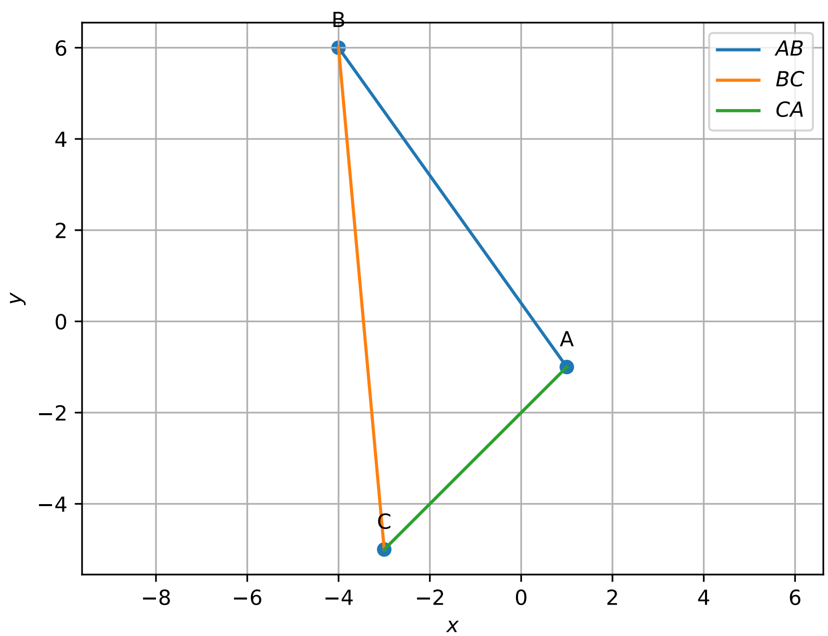

Figure 1.1: $\triangle ABC$

1.1.4. The parameteric form of the equation of AB is

$$\mathbf{x} = \mathbf{A} + k\mathbf{m} \tag{1.1.4.1}$$

where

$$\mathbf{m} = \mathbf{B} - \mathbf{A} \tag{1.1.4.2}$$

is the direction vector of AB. Find the parameteric equations of AB, BC and CA.

Solution: From (1.1.4.1) and (1.1.1.2), the parametric equation for AB is given by

$$AB : \mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + k \begin{pmatrix} -5 \\ 7 \end{pmatrix} \tag{1.1.4.3}$$

Similarly, from (1.1.1.3) and (1.1.1.4),

$$BC : \mathbf{x} = \begin{pmatrix} -4 \\ 6 \end{pmatrix} + k \begin{pmatrix} 1 \\ -11 \end{pmatrix} \tag{1.1.4.4}$$

$$CA : \mathbf{x} = \begin{pmatrix} -3 \\ -5 \end{pmatrix} + k \begin{pmatrix} 4 \\ 4 \end{pmatrix} \tag{1.1.4.5}$$

1.1.5. The normal form of the equation of AB is

$$\mathbf{n}^{\top} (\mathbf{x} - \mathbf{A}) = 0 \tag{1.1.5.1}$$

where

$$\mathbf{n}^\top \mathbf{m} = \mathbf{n}^\top (\mathbf{B} - \mathbf{A}) = 0 \tag{1.1.5.2}$$

$$\text{or, } \mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{m} \tag{1.1.5.3}$$

Find the normal form of the equations of AB, BC and CA.

Solution:

(a) From (1.1.1.3), the direction vector of side **BC** is

$$\mathbf{m} = \begin{pmatrix} 1 \\ -11 \end{pmatrix} \tag{1.1.5.4}$$

$$\implies \mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -11 \end{pmatrix} = \begin{pmatrix} -11 \\ -1 \end{pmatrix} \tag{1.1.5.5}$$

from (1.1.5.3). Hence, from (1.1.5.1), the normal equation of side BC is

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{B}) = 0 \tag{1.1.5.6}$$

$$\implies \begin{pmatrix} -11 & -1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -11 & -1 \end{pmatrix} \begin{pmatrix} -4 \\ 6 \end{pmatrix} \tag{1.1.5.7}$$

$$\implies BC: \quad \begin{pmatrix} 11 & 1 \end{pmatrix} \mathbf{x} = -38 \tag{1.1.5.8}$$

(b) Similarly, for AB, from (1.1.1.2),

$$\mathbf{m} = \begin{pmatrix} -5 \\ 7 \end{pmatrix} \tag{1.1.5.9}$$

$$\implies \mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -5 \\ 7 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix} \tag{1.1.5.10}$$

and

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{A}) = 0 \tag{1.1.5.11}$$

$$\implies AB: \quad \mathbf{n}^\top \mathbf{x} = \begin{pmatrix} 7 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{1.1.5.12}$$

$$\implies \begin{pmatrix} 7 & 5 \end{pmatrix} \mathbf{x} = 2 \tag{1.1.5.13}$$

(c) For CA, from (1.1.1.4),

$$\mathbf{m} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.1.5.14}$$

$$\implies \mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{1.1.5.15}$$

$$\tag{1.1.5.16}$$

$$\implies \mathbf{n}^\top (\mathbf{x} - \mathbf{C}) = 0 \tag{1.1.5.17}$$

$$\implies \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} -3 \\ -5 \end{pmatrix} = 2 \tag{1.1.5.18}$$

1.1.6. The area of $\triangle ABC$ is defined as

$$\frac{1}{2}\left\|(\mathbf{A}-\mathbf{B})\times(\mathbf{A}-\mathbf{C})\right\| \tag{1.1.6.1}$$

where

$$\mathbf{A}\times\mathbf{B} \triangleq \begin{vmatrix} 1 & -4 \\ -1 & 6 \end{vmatrix} \tag{1.1.6.2}$$

Find the area of $\triangle ABC$.

Solution: From (1.1.1.2) and (1.1.1.4),

$$\mathbf{A}-\mathbf{B} = \begin{pmatrix} 5 \\ -7 \end{pmatrix}, \mathbf{A}-\mathbf{C} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \tag{1.1.6.3}$$

$$\implies (\mathbf{A}-\mathbf{B})\times(\mathbf{A}-\mathbf{C}) = \begin{vmatrix} 5 & 4 \\ -7 & 4 \end{vmatrix} \tag{1.1.6.4}$$

$$= 5\times 4 - 4\times(-7) \tag{1.1.6.5}$$

$$= 48 \tag{1.1.6.6}$$

$$\implies \frac{1}{2}\left\|(\mathbf{A}-\mathbf{B})\times(\mathbf{A}-\mathbf{C})\right\| = \frac{48}{2} = 24 \tag{1.1.6.7}$$

which is the desired area.

1.1.7. Find the angles A, B, C if

$$\cos A \triangleq \frac{(\mathbf{B}-\mathbf{A})^{\top}\mathbf{C}-\mathbf{A}}{\left\|\mathbf{B}-\mathbf{A}\right\|\left\|\mathbf{C}-\mathbf{A}\right\|} \tag{1.1.7.1}$$

(a) From (1.1.1.2), (1.1.1.4), (1.1.2.6) and (1.1.2.12)

$$(\mathbf{B}-\mathbf{A})^\top(\mathbf{C}-\mathbf{A}) = \begin{pmatrix} -5 & 7 \end{pmatrix} \begin{pmatrix} -4 \\ -4 \end{pmatrix} \tag{1.1.7.2}$$

$$= -8 \tag{1.1.7.3}$$

$$\implies \cos A = \frac{-8}{\sqrt{74}\sqrt{32}} = \frac{-1}{\sqrt{37}} \tag{1.1.7.4}$$

$$\implies A = \cos^{-1}\frac{-1}{\sqrt{37}} \tag{1.1.7.5}$$

(b) From (1.1.1.2), (1.1.1.3), (1.1.2.6) and (1.1.2.9)

$$(\mathbf{C}-\mathbf{B})^\top(\mathbf{A}-\mathbf{B}) = \begin{pmatrix} 1 & -11 \end{pmatrix} \begin{pmatrix} 5 \\ -7 \end{pmatrix} \tag{1.1.7.6}$$

$$= 82 \tag{1.1.7.7}$$

$$\implies \cos B = \frac{82}{\sqrt{74}\sqrt{122}} = \frac{41}{\sqrt{2257}} \tag{1.1.7.8}$$

$$\implies B = \cos^{-1}\frac{41}{\sqrt{2257}} \tag{1.1.7.9}$$

(c) From (1.1.1.3), (1.1.1.4), (1.1.2.9) and (1.1.2.12)

$$(\mathbf{A}-\mathbf{C})^{\top}(\mathbf{B}-\mathbf{C}) = \begin{pmatrix} 4 & 4 \end{pmatrix}\begin{pmatrix} -1 \\ 11 \end{pmatrix} \tag{1.1.7.10}$$

$$= 40 \tag{1.1.7.11}$$

$$\implies \cos C = \frac{40}{\sqrt{32}\sqrt{122}} = \frac{5}{\sqrt{61}} \tag{1.1.7.12}$$

$$\implies C = \cos^{-1}\frac{5}{\sqrt{61}} \tag{1.1.7.13}$$

All codes for this section are available at

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codes/triangle/sides.py
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1.2. Median

1.2.1. If $\mathbf{D}$ divides BC in the ratio $k:1$,

$$\mathbf{D} = \frac{k\mathbf{C}+\mathbf{B}}{k+1} \tag{1.2.1.1}$$

Find the mid points $\mathbf{D}, \mathbf{E}, \mathbf{F}$ of the sides BC, CA and AB respectively.

Solution: Since $\mathbf{D}$ is the midpoint of BC,

$$k = 1, \tag{1.2.1.2}$$

$$\implies \mathbf{D} = \frac{\mathbf{C}+\mathbf{B}}{2} = \frac{1}{2}\begin{pmatrix} -7 \\ 1 \end{pmatrix} \tag{1.2.1.3}$$

Similarly,

$$\mathbf{E} = \frac{\mathbf{A} + \mathbf{C}}{2} = \begin{pmatrix} -1 \\ -3 \end{pmatrix} \tag{1.2.1.4}$$

$$\mathbf{F} = \frac{\mathbf{A} + \mathbf{B}}{2} = \frac{1}{2}\begin{pmatrix} -3 \\ 5 \end{pmatrix} \tag{1.2.1.5}$$

1.2.2. Find the equations of AD, BE and CF.

Solution: :

(a) The direction vector of AD is

$$\mathbf{m} = \mathbf{D} - \mathbf{A} = \begin{pmatrix} \frac{-7}{2} \\ \frac{1}{2} \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -9 \\ 3 \end{pmatrix} \equiv \begin{pmatrix} -3 \\ 1 \end{pmatrix} \tag{1.2.2.1}$$

$$\implies \mathbf{n} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{1.2.2.2}$$

Hence the normal equation of median AD is

$$\mathbf{n}^\top \left(\mathbf{x} - \mathbf{A}\right) = 0 \tag{1.2.2.3}$$

$$\implies \begin{pmatrix} 1 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -2 \tag{1.2.2.4}$$

(b) For BE,

$$\mathbf{m} = \mathbf{E} - \mathbf{B} = \begin{pmatrix} -1 \\ -3 \end{pmatrix} - \begin{pmatrix} -4 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 \\ -9 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ -3 \end{pmatrix} \tag{1.2.2.5}$$

$$\implies \mathbf{n} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{1.2.2.6}$$

Hence the normal equation of median BE is

$$\mathbf{n}^\top \left(\mathbf{x} - \mathbf{B}\right) = 0 \tag{1.2.2.7}$$

$$\implies \begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 3 & 1 \end{pmatrix} \begin{pmatrix} -4 \\ 6 \end{pmatrix} = -6 \tag{1.2.2.8}$$

(c) For median CF,

$$\mathbf{m} = \mathbf{F} - \mathbf{C} = \begin{pmatrix} \frac{-3}{2} \\ \frac{5}{2} \end{pmatrix} - \begin{pmatrix} -3 \\ -5 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ \frac{15}{2} \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 5 \end{pmatrix} \tag{1.2.2.9}$$

$$\implies \mathbf{n} = \begin{pmatrix} 5 \\ -1 \end{pmatrix} \tag{1.2.2.10}$$

Hence the normal equation of median CF is

$$\mathbf{n}^\top \left(\mathbf{x} - \mathbf{C}\right) = 0 \tag{1.2.2.11}$$

$$\implies \begin{pmatrix} 5 & -1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 5 & -1 \end{pmatrix} \begin{pmatrix} -3 \\ -5 \end{pmatrix} = -10 \tag{1.2.2.12}$$

1.2.3. Find the intersection $\mathbf{G}$ of BE and CF.

Solution: From (1.2.2.8) and (1.2.2.12), the equations of BE and CF are, respectively,

$$\begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -6 \end{pmatrix} \tag{1.2.3.1}$$

$$\begin{pmatrix} 5 & -1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -10 \end{pmatrix} \tag{1.2.3.2}$$

From (1.2.3.1) and (1.2.3.2) the augmented matrix is

$$\begin{pmatrix} 3 & 1 & -6 \\ 5 & -1 & -10 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 + R_2} \begin{pmatrix} 8 & 0 & -16 \\ 5 & -1 & -10 \end{pmatrix} \tag{1.2.3.3}$$

$$\xleftrightarrow{R_1 \leftarrow R_1/8} \begin{pmatrix} 1 & 0 & -2 \\ 5 & -1 & -10 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - 5R_1} \begin{pmatrix} 1 & 0 & -2 \\ 0 & -1 & 0 \end{pmatrix} \tag{1.2.3.4}$$

$$\xleftrightarrow{R_2 \leftarrow -R_2} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \end{pmatrix} \tag{1.2.3.5}$$

using Gauss elimination. Therefore,

$$\mathbf{G} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \tag{1.2.3.6}$$

1.2.4. Verify that

$$\frac{BG}{GE} = \frac{CG}{GF} = \frac{AG}{GD} = 2 \tag{1.2.4.1}$$

Solution:

(a) From (1.2.1.4) and (1.2.3.6),

$$\mathbf{G}-\mathbf{B}=\begin{pmatrix}2\\-6\end{pmatrix}, \mathbf{E}-\mathbf{G}=\begin{pmatrix}1\\-3\end{pmatrix} \tag{1.2.4.2}$$

$$\implies \mathbf{G}-\mathbf{B}=2\left(\mathbf{E}-\mathbf{G}\right) \tag{1.2.4.3}$$

$$\implies \left\|\mathbf{G}-\mathbf{B}\right\|=2\left\|\mathbf{E}-\mathbf{G}\right\| \tag{1.2.4.4}$$

$$\text{or, } \frac{BG}{GE}=2 \tag{1.2.4.5}$$

(b) From (1.2.1.5) and (1.2.3.6),

$$\mathbf{F}-\mathbf{G}=\frac{1}{2}\begin{pmatrix}1\\5\end{pmatrix}, \mathbf{G}-\mathbf{C}=\begin{pmatrix}1\\5\end{pmatrix} \tag{1.2.4.6}$$

$$\implies \mathbf{G}-\mathbf{C}=2\left(\mathbf{F}-\mathbf{G}\right) \tag{1.2.4.7}$$

$$\implies \left\|\mathbf{G}-\mathbf{C}\right\|=2\left\|\mathbf{F}-\mathbf{G}\right\| \tag{1.2.4.8}$$

$$\text{or, } \frac{CG}{GF}=2 \tag{1.2.4.9}$$

(c) From (1.2.1.3) and (1.2.3.6),

$$\mathbf{G}-\mathbf{A}=\begin{pmatrix}-3\\1\end{pmatrix}, \mathbf{D}-\mathbf{G}=\frac{1}{2}\begin{pmatrix}-3\\1\end{pmatrix} \tag{1.2.4.10}$$

$$\mathbf{G}-\mathbf{A}=2\left(\mathbf{D}-\mathbf{G}\right) \tag{1.2.4.11}$$

$$\implies \left\|\mathbf{G}-\mathbf{A}\right\|=2\left\|\mathbf{D}-\mathbf{G}\right\| \tag{1.2.4.12}$$

$$\text{or, } \frac{AG}{GD}=2 \tag{1.2.4.13}$$

From (1.2.4.5), (1.2.4.9), (1.2.4.13)

$$\frac{BG}{GE} = \frac{CG}{GF} = \frac{AG}{GD} = 2 \tag{1.2.4.14}$$

1.2.5. Show that $\mathbf{A}, \mathbf{G}$ and $\mathbf{D}$ are collinear.

Solution: Points $\mathbf{A}, \mathbf{D}, \mathbf{G}$ are defined to be collinear if

$$\text{rank}\begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{D} & \mathbf{G} \end{pmatrix} = 2 \tag{1.2.5.1}$$

$$\implies \begin{pmatrix} 1 & 1 & 1 \\ 1 & -\frac{7}{2} & -2 \\ -1 & \frac{1}{2} & 0 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 + R_2} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -\frac{7}{2} & -2 \\ 0 & -3 & -2 \end{pmatrix} \tag{1.2.5.2}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -\frac{9}{2} & -3 \\ 0 & -3 & -2 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 - \frac{2}{3}R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -\frac{9}{2} & -3 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.2.5.3}$$

Thus, the matrix (1.2.5.1) has rank 2 and the points are collinear. Thus, the medians of a triangle meet at the point $\mathbf{G}$. See Fig. 1.2.

1.2.6. Verify that

$$\mathbf{G} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \tag{1.2.6.1}$$

$\mathbf{G}$ is known as the centroid of $\triangle ABC$.

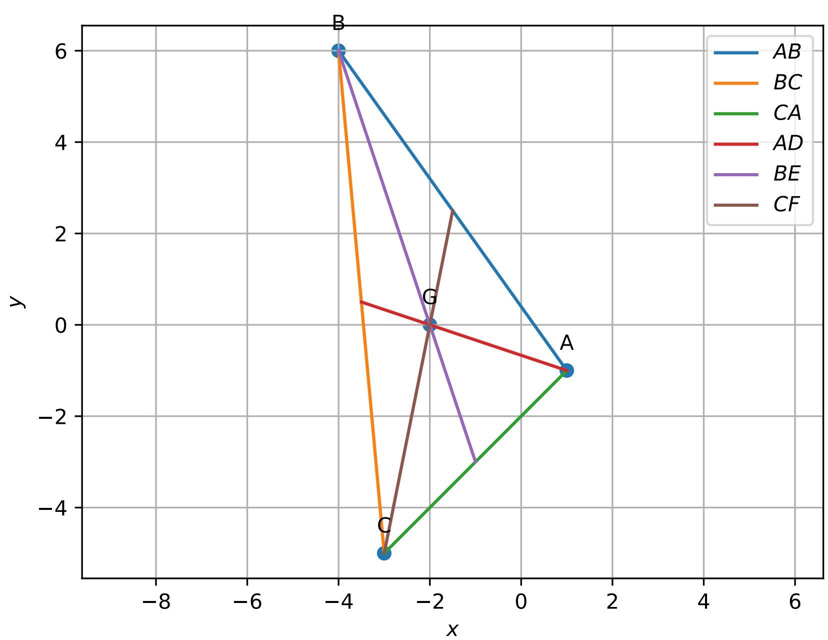

Figure 1.2: Medians of $\triangle ABC$ meet at $\mathbf{G}$.

Solution:

$$\mathbf{G} = \frac{\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -4 \\ 6 \end{pmatrix} + \begin{pmatrix} -3 \\ -5 \end{pmatrix}}{3} \tag{1.2.6.2}$$
$$= \begin{pmatrix} -2 \\ 0 \end{pmatrix}$$

1.2.7. Verify that

$$\mathbf{A} - \mathbf{F} = \mathbf{E} - \mathbf{D} \tag{1.2.7.1}$$

The quadrilateral $AFDE$ is defined to be a parallelogram.

Solution:

$$\mathbf{A} - \mathbf{F} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \begin{pmatrix} \frac{-3}{2} \\ \frac{5}{2} \end{pmatrix} = \begin{pmatrix} \frac{5}{2} \\ \frac{-7}{2} \end{pmatrix} \tag{1.2.7.2}$$

$$\mathbf{E} - \mathbf{D} = \begin{pmatrix} -1 \\ -3 \end{pmatrix} - \begin{pmatrix} \frac{-7}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{5}{2} \\ \frac{-7}{2} \end{pmatrix} \tag{1.2.7.3}$$

$$\implies \mathbf{A} - \mathbf{F} = \mathbf{E} - \mathbf{D} \tag{1.2.7.4}$$

See Fig. 1.3,

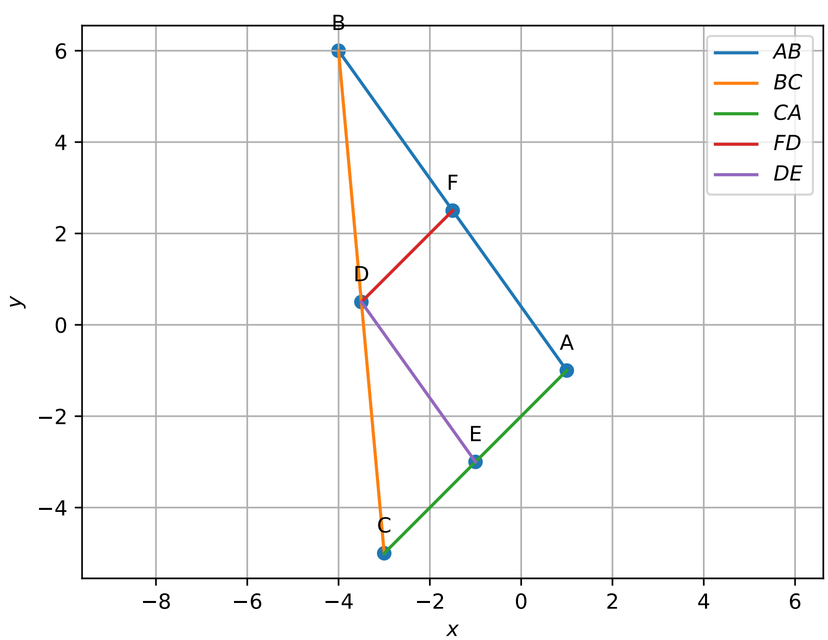

Figure 1.3: $AFDE$ forms a parallelogram in triangle ABC

1.3. Altitude

1.3.1. $\mathbf{D}_1$ is a point on BC such that

$$AD_1 \perp BC \tag{1.3.1.1}$$

and AD_1 is defined to be the altitude. Find the normal vector of AD_1.

Solution: The normal vector of AD_1 is the direction vector BC and is obtained from (1.1.1.3) as

$$\mathbf{n} = \begin{pmatrix} 1 \\ -11 \end{pmatrix} \tag{1.3.1.2}$$

1.3.2. Find the equation of AD_1.

Solution: The equation of AD_1 is

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{A}) = 0 \tag{1.3.2.1}$$

$$\implies \begin{pmatrix} -1 & 11 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -1 & 11 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -12 \tag{1.3.2.2}$$

1.3.3. Find the equations of the altitudes BE_1 and CF_1 to the sides AC and AB respectively.

Solution:

(a) From (1.1.1.4), the normal vector of CF_1 is

$$\mathbf{n} = \begin{pmatrix} -5 \\ 7 \end{pmatrix} \tag{1.3.3.1}$$

and the equation of CF_1 is

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{C}) = 0 \tag{1.3.3.2}$$

$$\implies \begin{pmatrix} -5 & 7 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} -3 \\ -5 \end{pmatrix} \right) = 0 \tag{1.3.3.3}$$

$$\implies \begin{pmatrix} 5 & -7 \end{pmatrix} \mathbf{x} = 20, \tag{1.3.3.4}$$

(b) Similarly, from (1.1.1.2), the normal vector of BE_1 is

$$\mathbf{n} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.3.3.5}$$

and the equation of BE_1 is

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{B}) = 0 \tag{1.3.3.6}$$

$$\implies \begin{pmatrix} 1 & 1 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} -4 \\ 6 \end{pmatrix} \right) = 0 \tag{1.3.3.7}$$

$$\implies \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 2, \tag{1.3.3.8}$$

1.3.4. Find the intersection $\mathbf{H}$ of BE_1 and CF_1.

Solution: The intersection of (1.3.3.8) and (1.3.3.4), is obtained from the matrix equation

$$\begin{pmatrix} 1 & 1 \\ 5 & -7 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 2 \\ 20 \end{pmatrix} \tag{1.3.4.1}$$

which can be solved as

$$\begin{pmatrix} 1 & 1 & 2 \\ 5 & -7 & 20 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - 5R_1} \begin{pmatrix} 1 & 1 & 2 \\ 0 & -12 & 10 \end{pmatrix} \tag{1.3.4.2}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{-12}} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & \frac{-5}{6} \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & 0 & \frac{17}{6} \\ 0 & 1 & \frac{-5}{6} \end{pmatrix} \tag{1.3.4.3}$$

yielding

$$\mathbf{H} = \frac{1}{6} \begin{pmatrix} 17 \\ -5 \end{pmatrix}, \tag{1.3.4.4}$$

See Fig. 1.4

1.3.5. Verify that

$$\left(\mathbf{A} - \mathbf{H}\right)^{\top} \left(\mathbf{B} - \mathbf{C}\right) = 0 \tag{1.3.5.1}$$

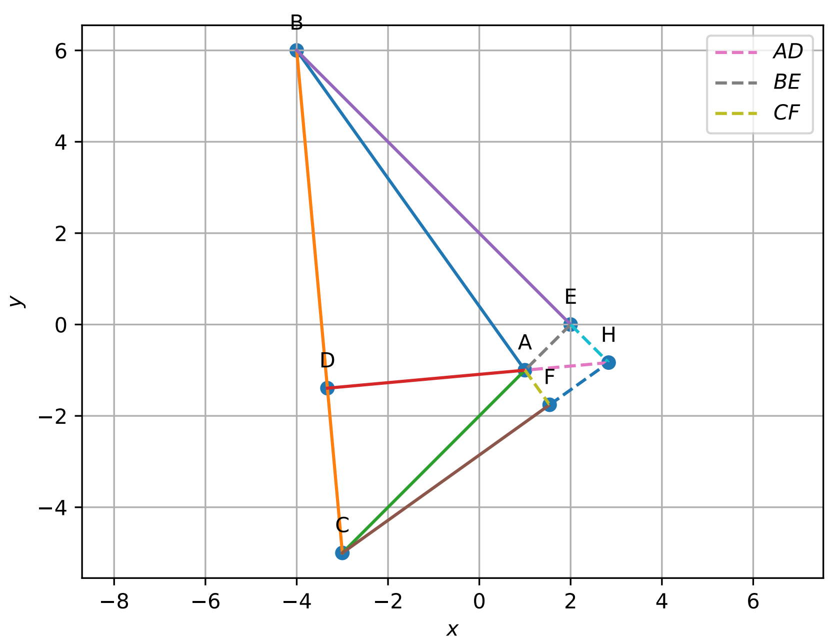

Figure 1.4: Altitudes BE_1 and CF_1 intersect at $\mathbf{H}$

Solution: From (1.3.4.4),

$$\mathbf{A}-\mathbf{H}=-\frac{1}{6}\begin{pmatrix}11\\1\end{pmatrix}, \mathbf{B}-\mathbf{C}=\begin{pmatrix}-1\\11\end{pmatrix} \tag{1.3.5.2}$$

$$\implies (\mathbf{A}-\mathbf{H})^{\top}(\mathbf{B}-\mathbf{C})=\frac{1}{6}\begin{pmatrix}11 & 1\end{pmatrix}\begin{pmatrix}-1\\11\end{pmatrix}=0 \tag{1.3.5.3}$$

1.4. Perpendicular Bisector

1.4.1. The equation of the perpendicular bisector of BC is

$$\left(\mathbf{x} - \frac{\mathbf{B}+\mathbf{C}}{2}\right)(\mathbf{B}-\mathbf{C}) = 0 \tag{1.4.1.1}$$

Substitute numerical values and find the equations of the perpendicular bisectors of AB, BC and CA.

Solution: From (1.1.1.2), (1.1.1.3), (1.1.1.4), (1.2.1.3), (1.2.1.4) and (1.2.1.5),

$$\frac{\mathbf{B}+\mathbf{C}}{\mathbf{2}} = \frac{1}{2}\begin{pmatrix} -7 \\ 1 \end{pmatrix}, \mathbf{B}-\mathbf{C} = \begin{pmatrix} -1 \\ 11 \end{pmatrix} \tag{1.4.1.2}$$

$$\frac{\mathbf{A}+\mathbf{B}}{\mathbf{2}} = \frac{1}{2}\begin{pmatrix} -3 \\ 5 \end{pmatrix}, \mathbf{A}-\mathbf{B} = \begin{pmatrix} 5 \\ -7 \end{pmatrix} \tag{1.4.1.3}$$

$$\frac{\mathbf{C}+\mathbf{A}}{\mathbf{2}} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}, \mathbf{C}-\mathbf{A} = \begin{pmatrix} -4 \\ -4 \end{pmatrix} \tag{1.4.1.4}$$

$$\tag{1.4.1.5}$$

yielding

$$(\mathbf{B}-\mathbf{C})^\top\left(\frac{\mathbf{B}+\mathbf{C}}{2}\right)=\begin{pmatrix}-1 & 11\end{pmatrix}\begin{pmatrix}-\frac{7}{2}\\ \frac{1}{2}\end{pmatrix}=9 \tag{1.4.1.6}$$

$$(\mathbf{A}-\mathbf{B})^\top\left(\frac{\mathbf{A}+\mathbf{B}}{2}\right)=\begin{pmatrix}5 & -7\end{pmatrix}\begin{pmatrix}-\frac{3}{2}\\ \frac{5}{2}\end{pmatrix}=-25 \tag{1.4.1.7}$$

$$(\mathbf{C}-\mathbf{A})^\top\left(\frac{\mathbf{C}+\mathbf{A}}{2}\right)=\begin{pmatrix}-4 & -4\end{pmatrix}\begin{pmatrix}-1\\ -3\end{pmatrix}=16 \tag{1.4.1.8}$$

Thus, the perpendicular bisectors are obtained from (1.4.1.1) as

$$BC: \quad \begin{pmatrix}-1 & 11\end{pmatrix}\mathbf{x}=9 \tag{1.4.1.9}$$

$$CA: \quad \begin{pmatrix}5 & -7\end{pmatrix}\mathbf{x}=-25 \tag{1.4.1.10}$$

$$AB: \quad \begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x}=-4 \tag{1.4.1.11}$$

1.4.2. Find the intersection $\mathbf{O}$ of the perpendicular bisectors of AB and AC.

Solution:

The intersection of (1.4.1.10) and (1.4.1.11), can be obtained as

$$\begin{pmatrix} 5 & -7 & -25 \\ 1 & 1 & -4 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow 5R_2 - R_1} \begin{pmatrix} 5 & -7 & -25 \\ 0 & 12 & 5 \end{pmatrix} \tag{1.4.2.1}$$

$$\xleftrightarrow{R_1 \leftarrow \frac{12}{7}R_1 + R_2} \begin{pmatrix} \frac{60}{7} & 0 & \frac{-265}{7} \\ 0 & 12 & 5 \end{pmatrix} \xleftrightarrow[R_1 \leftarrow \frac{7}{60}R_1]{R_2 \leftarrow \frac{1}{12}R_2} \begin{pmatrix} 1 & 0 & \frac{-53}{12} \\ 0 & 1 & \frac{5}{12} \end{pmatrix} \tag{1.4.2.2}$$

$$\implies \mathbf{O} = \begin{pmatrix} \frac{-53}{12} \\ \frac{5}{12} \end{pmatrix} \tag{1.4.2.3}$$

1.4.3. Verify that $\mathbf{O}$ satisfies (1.4.1.1). $\mathbf{O}$ is known as the circumcentre.

Solution: Substituing from (1.4.2.3) in (1.4.1.1), when substituted in the above equation,

$$\left(\mathbf{O} - \frac{\mathbf{B}+\mathbf{C}}{2}\right)^\top (\mathbf{B}-\mathbf{C})$$

$$= \left(\frac{1}{12}\begin{pmatrix} -53 \\ 5 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -7 \\ 1 \end{pmatrix}\right)^\top \begin{pmatrix} -1 \\ 11 \end{pmatrix}$$

$$= \frac{1}{12}\begin{pmatrix} -11 & -1 \end{pmatrix}\begin{pmatrix} -1 \\ 11 \end{pmatrix} = 0 \tag{1.4.3.1}$$

1.4.4. Verify that

$$OA = OB = OC \tag{1.4.4.1}$$

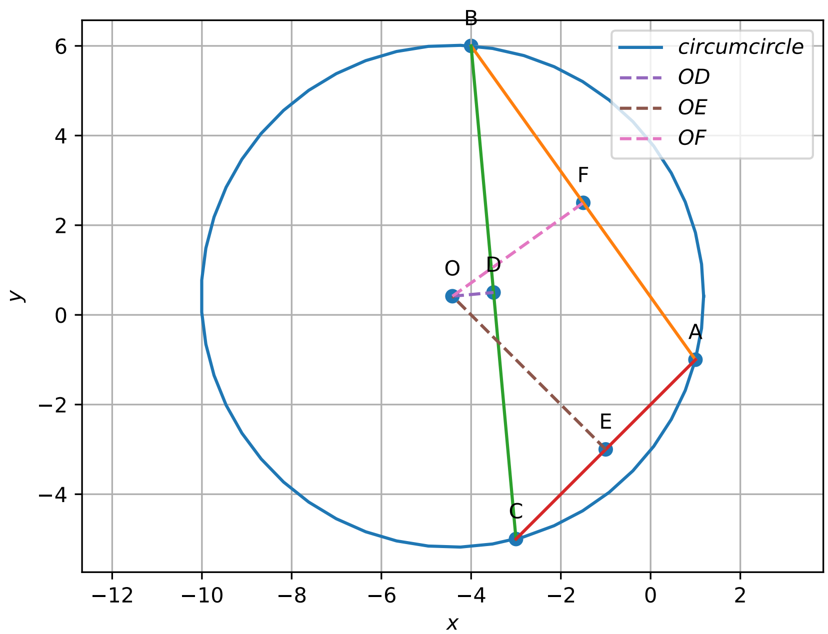

Figure 1.5: Circumcircle of $\triangle ABC$ with centre $\mathbf{O}$.

1.4.5. Draw the circle with centre at $\mathbf{O}$ and radius

$$R = OA \tag{1.4.5.1}$$

This is known as the circumradius.

Solution: See Fig. 1.5.

1.4.6. Verify that

$$\angle BOC = 2\angle BAC. \tag{1.4.6.1}$$

Solution:

(a) To find the value of $\angle BOC$:

$$\mathbf{B}-\mathbf{O}=\begin{pmatrix}\frac{5}{12}\\ \frac{67}{12}\end{pmatrix}, \mathbf{C}-\mathbf{O}=\begin{pmatrix}\frac{17}{12}\\ \frac{-65}{12}\end{pmatrix} \tag{1.4.6.2}$$

$$\implies (\mathbf{B}-\mathbf{O})^\top(\mathbf{C}-\mathbf{O})=\frac{-4270}{144} \tag{1.4.6.3}$$

$$\implies \|\mathbf{B}-\mathbf{O}\|=\frac{\sqrt{4514}}{12}, \|\mathbf{C}-\mathbf{O}\|=\frac{\sqrt{4514}}{12} \tag{1.4.6.4}$$

Thus,

$$\cos BOC=\frac{(\mathbf{B}-\mathbf{O})^\top(\mathbf{C}-\mathbf{O})}{\|\mathbf{B}-\mathbf{O}\|\|\mathbf{C}-\mathbf{O}\|}=\frac{-4270}{4514} \tag{1.4.6.5}$$

$$\implies \angle BOC=\cos^{-1}\left(\frac{-4270}{4514}\right) \tag{1.4.6.6}$$

$$=161.07536^\circ \text{ or } 198.92464^\circ \tag{1.4.6.7}$$

(b) To find the value of $\angle BAC$:

$$\mathbf{B}-\mathbf{A}=\begin{pmatrix}-5\\ 7\end{pmatrix}, \mathbf{C}-\mathbf{A}=\begin{pmatrix}-4\\ -4\end{pmatrix} \tag{1.4.6.8}$$

$$\implies (\mathbf{B}-\mathbf{A})^\top(\mathbf{C}-\mathbf{A})=-8 \tag{1.4.6.9}$$

$$\|\mathbf{B}-\mathbf{A}\|=\sqrt{74}\,\|\mathbf{C}-\mathbf{A}\|=4\sqrt{2} \tag{1.4.6.10}$$

Thus,

$$\cos BAC = \frac{(\mathbf{B}-\mathbf{A})^\top (\mathbf{C}-\mathbf{A})}{\|\mathbf{B}-\mathbf{A}\| \|\mathbf{C}-\mathbf{A}\|} = \frac{-8}{4\sqrt{148}} \tag{1.4.6.11}$$

$$\implies \angle BAC = \cos^{-1}\left(\frac{-8}{4\sqrt{148}}\right) \tag{1.4.6.12}$$

$$= 99.46232^\circ \tag{1.4.6.13}$$

From (1.4.6.13) and (1.4.6.7),

$$2 \times \angle BAC = \angle BOC \tag{1.4.6.14}$$

1.4.7. Let

$$\mathbf{P} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{1.4.7.1}$$

Find θ if

$$\mathbf{C}-\mathbf{O} = \mathbf{P}(\mathbf{A}-\mathbf{O}) \tag{1.4.7.2}$$

1.5. Angle Bisector

1.5.1. Let $\mathbf{D}_3, \mathbf{E}_3, \mathbf{F}_3$, be points on AB, BC and CA respectively such that

$$BD_3 = BF_3 = m, CD_3 = CE_3 = n, AE_3 = AF_3 = p. \tag{1.5.1.1}$$

Obtain m, n, p in terms of a, b, c obtained in Problem 1.1.2.

Solution: From the given information,

$$a = m + n, \tag{1.5.1.2}$$

$$b = n + p, \tag{1.5.1.3}$$

$$c = m + p \tag{1.5.1.4}$$

which can be expressed as

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} m \\ n \\ p \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{1.5.1.5}$$

$$\implies \begin{pmatrix} m \\ n \\ p \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{1.5.1.6}$$

Using row reduction,

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xleftrightarrow{R_3 \leftarrow R_3 - R_1} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right) \tag{1.5.1.7}$$

$$\xleftrightarrow[R_1 \leftarrow R_1 - R_2]{R_3 \leftarrow R_3 + R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right) \tag{1.5.1.8}$$

$$\xleftrightarrow[R_1 \leftarrow 2R_1+R_3]{R_2 \leftarrow 2R_2-R_3} \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & -1 & 1 \\ 0 & 2 & 0 & 1 & 1 & -1 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right) \tag{1.5.1.9}$$

yielding

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix} \tag{1.5.1.10}$$

Therefore,

$$\begin{aligned} p &= \frac{c+b-a}{2} = \frac{\sqrt{74}+\sqrt{32}-\sqrt{122}}{2} \\ m &= \frac{a+c-b}{2} = \frac{\sqrt{74}+\sqrt{122}-\sqrt{32}}{2} \\ n &= \frac{a+b-c}{2} = \frac{\sqrt{122}+\sqrt{32}-\sqrt{74}}{2} \end{aligned} \tag{1.5.1.11}$$

upon substituting from (1.1.2.6), (1.1.2.9) and (1.1.2.12).

1.5.2. Using section formula, find

$$\mathbf{D}_3 = \frac{m\mathbf{C}+n\mathbf{B}}{m+n}, \mathbf{E}_3 = \frac{n\mathbf{A}+p\mathbf{C}}{n+p}, \mathbf{F}_3 = \frac{p\mathbf{B}+m\mathbf{A}}{p+m} \tag{1.5.2.1}$$

1.5.3. Find the circumcentre and circumradius of $\triangle D_3E_3F_3$. These are the incentre and inradius of $\triangle ABC$.

1.5.4. Draw the circumcircle of $\triangle D_3E_3F_3$. This is known as the incircle of $\triangle ABC$.

Solution: See Fig. 1.6

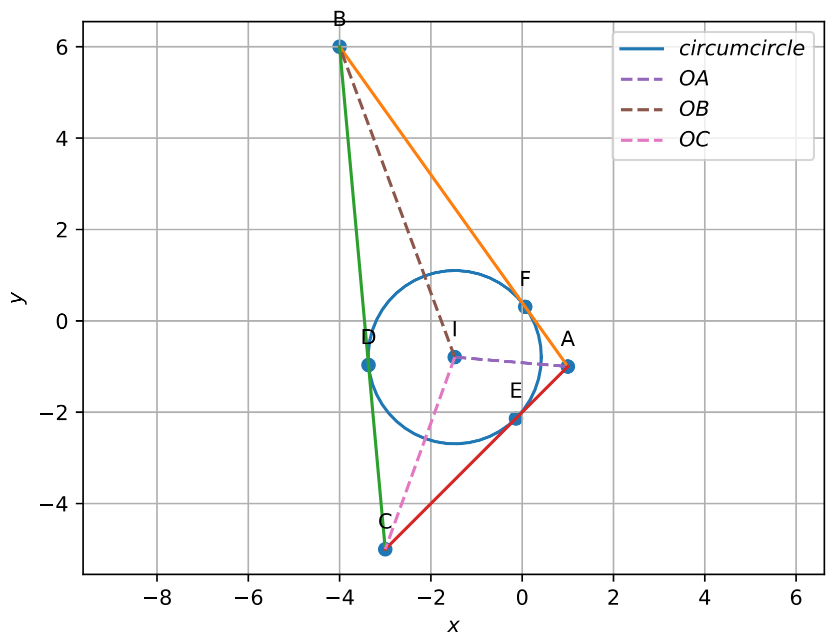

Figure 1.6: Incircle of $\triangle ABC$

1.5.5. Using (1.1.7.1) verify that

$$\angle BAI = \angle CAI. \tag{1.5.5.1}$$

AI is the bisector of $\angle A$.

1.5.6. Verify that BI, CI are also the angle bisectors of $\triangle ABC$.

1.6. Eigenvalues and Eigenvectors

The equation of the incircle is given by

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{6.1}$$

where

$$\mathbf{V} = \mathbf{I}, \mathbf{u} = -\mathbf{O}, f = \|\mathbf{O}\| - r^2, \tag{6.2}$$

$\mathbf{O}$ being the incentre and r the inradius. Here $\mathbf{I}$ is the identity matrix.

1.6.1. Compute

$$\mathbf{\Sigma} = (\mathbf{V}\mathbf{h} + \mathbf{u})(\mathbf{V}\mathbf{h} + \mathbf{u})^\top - g(\mathbf{h})\mathbf{V} \tag{1.6.1.1}$$

for $\mathbf{h} = \mathbf{A}$.

1.6.2. Find the roots of the equation

$$\left|\lambda\mathbf{I} - \mathbf{\Sigma}\right| = 0 \tag{1.6.2.1}$$

These are known as the eigenvalues of $\mathbf{\Sigma}$.

1.6.3. Find $\mathbf{p}$ such that

$$\mathbf{\Sigma}\mathbf{p} = \lambda\mathbf{p} \tag{1.6.3.1}$$

using row reduction. These are known as the eigenvectors of Σ.

1.6.4. Define

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \tag{1.6.4.1}$$

$$\mathbf{P} = \begin{pmatrix} \frac{\mathbf{p}_1}{\|\mathbf{p}_1\|} & \frac{\mathbf{p}_2}{\|\mathbf{p}_2\|} \end{pmatrix} \tag{1.6.4.2}$$

1.6.5. Verify that

$$\mathbf{P}^\top = \mathbf{P}^{-1}. \tag{1.6.5.1}$$

$\mathbf{P}$ is defined to be an orthogonal matrix.

1.6.6. Verify that

$$\mathbf{P}^\top \mathbf{\Sigma} \mathbf{P} = \mathbf{D}, \tag{1.6.6.1}$$

This is known as the spectral (eigenvalue) decomposition of a symmetric matrix

1.6.7. The direction vectors of the tangents from a point $\mathbf{h}$ to the circle in (6.1) are given by

$$\mathbf{m} = \mathbf{P} \begin{pmatrix} \sqrt{|\lambda_2|} \\ \pm\sqrt{|\lambda_1|} \end{pmatrix} \tag{1.6.7.1}$$

1.6.8. The points of contact of the pair of tangents to the circle in (6.1) from a point $\mathbf{h}$ are given by

$$\mathbf{x} = \mathbf{h} + \mu \mathbf{m} \tag{1.6.8.1}$$

where

$$\mu = -\frac{\mathbf{m}^\top \left(\mathbf{V}\mathbf{h} + \mathbf{u} \right)}{\mathbf{m}^\top \mathbf{V}\mathbf{m}} \tag{1.6.8.2}$$

for $\mathbf{m}$ in (1.6.7.1). Compute the points of contact. You should get the same points that you obtained in the previous section.

1.7. Matrices

The matrix of the veritices of the triangle is defined as

$$\mathbf{P} = \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \tag{8.3}$$

1.7.1. Vectors

1.7.1.1. Obtain the direction matrix of the sides of $\triangle ABC$ defined as

$$\mathbf{M} = \begin{pmatrix} \mathbf{A} - \mathbf{B} & \mathbf{B} - \mathbf{C} & \mathbf{C} - \mathbf{A} \end{pmatrix} \tag{1.7.1.1.1}$$

Solution:

$$\mathbf{M} = \begin{pmatrix} \mathbf{A}-\mathbf{B} & \mathbf{B}-\mathbf{C} & \mathbf{C}-\mathbf{A} \end{pmatrix} \tag{1.7.1.1.2}$$

$$= \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \tag{1.7.1.1.3}$$

where the second matrix above is known as a circulant matrix. Note that the 2nd and 3rd row of the above matrix are circular shifts of the 1st row.

1.7.1.2. Obtain the normal matrix of the sides of $\triangle ABC$

Solution: Considering the roation matrix

$$\mathbf{R} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{1.7.1.2.1}$$

the normal matrix is obtained as

$$\mathbf{N} = \mathbf{R}\mathbf{M} \tag{1.7.1.2.2}$$

1.7.1.3. Obtain a, b, c.

Solution: The sides vector is obtained as

$$\mathbf{d} = \sqrt{\text{diag}(\mathbf{M}^\top \mathbf{M})} \tag{1.7.1.3.1}$$

1.7.1.4. Obtain the constant terms in the equations of the sides of the triangle.

Solution: The constants for the lines can be expressed in vector form as

$$\mathbf{c} = \operatorname{diag}\left\{\left(\mathbf{N}^\top \mathbf{P}\right)\right\} \tag{1.7.1.4.1}$$

1.7.2. Median

1.7.2.1. Obtain the mid point matrix for the sides of the triangle

Solution:

$$\begin{pmatrix}\mathbf{D} & \mathbf{E} & \mathbf{F}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}\mathbf{A} & \mathbf{B} & \mathbf{C}\end{pmatrix}\begin{pmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{pmatrix} \tag{1.7.2.1.1}$$

1.7.2.2. Obtain the median direction matrix.

Solution: The median direction matrix is given by

$$\mathbf{M}_1 = \begin{pmatrix}\mathbf{A}-\mathbf{D} & \mathbf{B}-\mathbf{E} & \mathbf{C}-\mathbf{F}\end{pmatrix} \tag{1.7.2.2.1}$$

$$= \begin{pmatrix}\mathbf{A}-\frac{\mathbf{B}+\mathbf{C}}{2} & \mathbf{B}-\frac{\mathbf{C}+\mathbf{A}}{2} & \mathbf{C}-\frac{\mathbf{A}+\mathbf{B}}{2}\end{pmatrix} \tag{1.7.2.2.2}$$

$$= \begin{pmatrix}\mathbf{A} & \mathbf{B} & \mathbf{C}\end{pmatrix}\begin{pmatrix}1 & -\frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & 1 & -\frac{1}{2}\\ -\frac{1}{2} & -\frac{1}{2} & 1\end{pmatrix} \tag{1.7.2.2.3}$$

1.7.2.3. Obtain the median normal matrix.

1.7.2.4. Obtian the median equation constants.

1.7.2.5. Obtain the centroid by finding the intersection of the medians.

1.7.3. Altitude

1.7.3.1. Find the normal matrix for the altitudes

Solution: The desired matrix is

$$\mathbf{M}_2 = \begin{pmatrix} \mathbf{B} - \mathbf{C} & \mathbf{C} - \mathbf{A} & \mathbf{A} - \mathbf{B} \end{pmatrix} \tag{1.7.3.1.1}$$

$$= \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \tag{1.7.3.1.2}$$

1.7.3.2. Find the constants vector for the altitudes.

Solution: The desired vector is

$$\mathbf{c}_2 = \operatorname{diag}\left\{\left(\mathbf{M}^\top \mathbf{P}\right)\right\} \tag{1.7.3.2.1}$$

1.7.4. Perpendicular Bisector

1.7.4.1. Find the normal matrix for the perpendicular bisectors

Solution: The normal matrix is $\mathbf{M}_2$

1.7.4.2. Find the constants vector for the perpendicular bisectors.

Solution: The desired vector is

$$\mathbf{c}_3 = \operatorname{diag}\left\{\mathbf{M}_2^\top \begin{pmatrix} \mathbf{D} & \mathbf{E} & \mathbf{F} \end{pmatrix}\right\} \tag{1.7.4.2.1}$$

1.7.5. Angle Bisector

1.7.5.1. Find the points of contact.

Solution: The points of contact are given by

$$\begin{pmatrix} \frac{m\mathbf{C}+n\mathbf{B}}{m+n} & \frac{n\mathbf{A}+p\mathbf{C}}{n+p} & \frac{p\mathbf{B}+m\mathbf{A}}{p+m} \end{pmatrix} = \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \begin{pmatrix} 0 & \frac{n}{b} & \frac{m}{c} \\ \frac{n}{a} & 0 & \frac{p}{c} \\ \frac{m}{a} & \frac{p}{b} & 0 \end{pmatrix} \tag{1.7.5.1.1}$$

Appendix A

Trigonometry

A.1. Ratios

A right angled triangle looks like Fig. A.1. with angles $\angle A, \angle B$ and $\angle C$ and

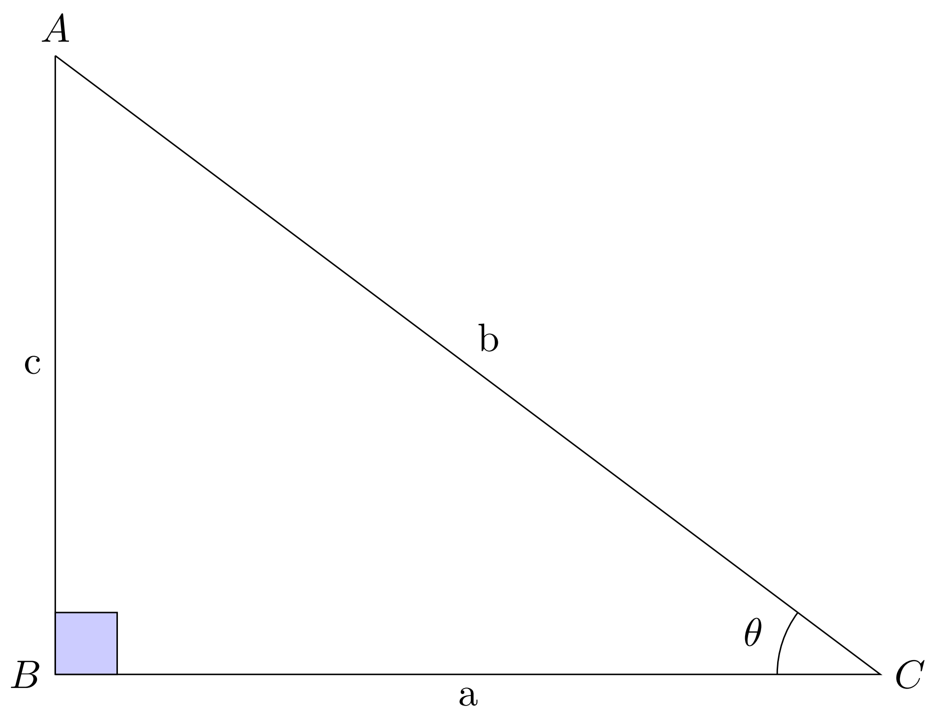

Figure A.1: Right Angled Triangle

sides a, b and c. The unique feature of this triangle is $\angle B$ which is defined to be $90°$.

A.1.1. For simplicity, let the greek letter $\theta = \angle C$. We have the following definitions.

$$\begin{aligned} \sin\theta &= \tfrac{c}{b} & \cos\theta &= \tfrac{a}{b} \\ \tan\theta &= \tfrac{c}{a} & \cot\theta &= \tfrac{1}{\tan\theta} \\ \csc\theta &= \tfrac{1}{\sin\theta} & \sec\theta &= \tfrac{1}{\cos\theta} \end{aligned} \tag{A.1.1.1}$$

A.1.2. Show that

$$\cos\theta = \sin(90° - \theta) \tag{A.1.2.1}$$

Solution: From (A.1.1.1),

$$\cos\angle BAC = \cos\alpha = \cos(90° - \theta) = \frac{c}{b} = \sin\angle ABC = \sin\theta \tag{A.1.2.2}$$

A.2. The Baudhayana Theorem

Use Fig. A.2 for all problems in this section.

A.2.1. Show that

$$b = a\cos\theta + c\sin\theta \tag{A.2.1.1}$$

Solution: We observe that

$$BD = a\cos\theta \tag{A.2.1.2}$$

$$AD = c\cos\alpha = c\sin\theta \quad \text{(From} \quad \text{(A.1.2.2))} \tag{A.2.1.3}$$

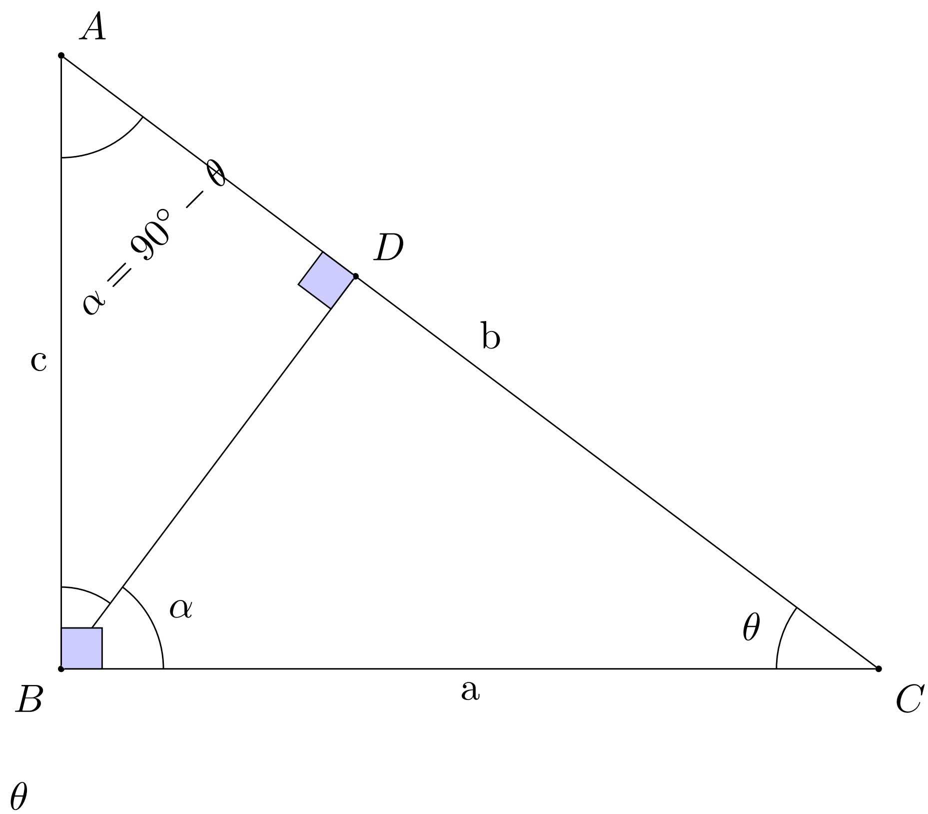

Figure A.2: Baudhayana Theorem

Thus,

$$BD + AD = b = a\cos\theta + c\sin\theta \tag{A.2.1.4}$$

A.2.2. From (A.2.1.1), show that

$$\sin^2\theta + \cos^2\theta = 1 \tag{A.2.2.1}$$

Solution: Dividing both sides of (A.2.1.1) by b,

$$1 = \frac{a}{b}\cos\theta + \frac{c}{b}\sin\theta \tag{A.2.2.2}$$

$$\Rightarrow \sin^2\theta + \cos^2\theta = 1 \quad \text{(from} \quad \text{(A.1.1.1))} \tag{A.2.2.3}$$

A.2.3. In a right angled triangle, the hypotenuse is the longest side.

Solution: From (A.2.2.1),

$$0 \le \sin\theta, \cos\theta \le 1 \tag{A.2.3.1}$$

Hence,

$$b\sin\theta \le b \implies c \le b \tag{A.2.3.2}$$

Similalry,

$$a \le b \tag{A.2.3.3}$$

A.2.4. Using (A.2.1.1), show that

$$b^2 = a^2 + c^2 \tag{A.2.4.1}$$

(A.2.4.1) is known as the Baudhayana theorem. It is also known as the Pythagoras theorem.

Solution: From (A.2.1.1),

$$b = a\frac{a}{b} + c\frac{c}{b} \quad \text{(from} \quad \text{(A.1.1.1))} \tag{A.2.4.2}$$

$$\implies b^2 = a^2 + c^2 \tag{A.2.4.3}$$

A.3. Area of a Triangle

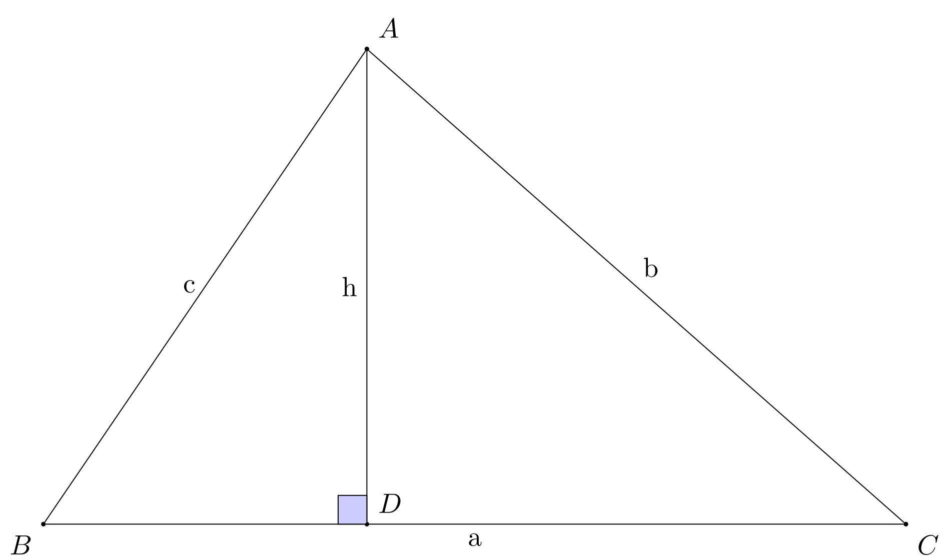

Figure A.3: Area of a Triangle

A.3.1. Show that the area of ΔABC in Fig. A.3 is $\frac{1}{2}ab\sin C$.

Solution: We have

$$ar\left(\Delta ABC\right) = \frac{1}{2}ah = \frac{1}{2}ab\sin C \quad \left(\because \quad h = b\sin C\right). \tag{A.3.1.1}$$

A.3.2. Show that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \tag{A.3.2.1}$$

Solution: Fig. A.3 can be suitably modified to obtain

$$ar\left(\Delta ABC\right) = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B \tag{A.3.2.2}$$

Dividing the above by abc, we obtain

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \tag{A.3.2.3}$$

This is known as the sine formula.

A.3.3. Show that

$$\alpha > \beta \implies \sin\alpha > \sin\beta \tag{A.3.3.1}$$

Solution: In Fig. A.4,

$$ar\left(\triangle ABD\right) < ar\left(\triangle ABC\right) \tag{A.3.3.2}$$

$$\implies \frac{1}{2}lc\sin\theta_1 < \frac{1}{2}ac\sin\left(\theta_1+\theta_2\right) \tag{A.3.3.3}$$

$$\implies \frac{l}{a} < \frac{\sin\left(\theta_1+\theta_2\right)}{\sin\theta_1} \tag{A.3.3.4}$$

$$\text{or, } 1 < \frac{l}{a} < \frac{\sin\left(\theta_1+\theta_2\right)}{\sin\theta_1} \tag{A.3.3.5}$$

$$\implies \frac{\sin\left(\theta_1+\theta_2\right)}{\sin\theta_1} > 1 \tag{A.3.3.6}$$

from Theorem A.2.3. This proves (A.3.3.1).

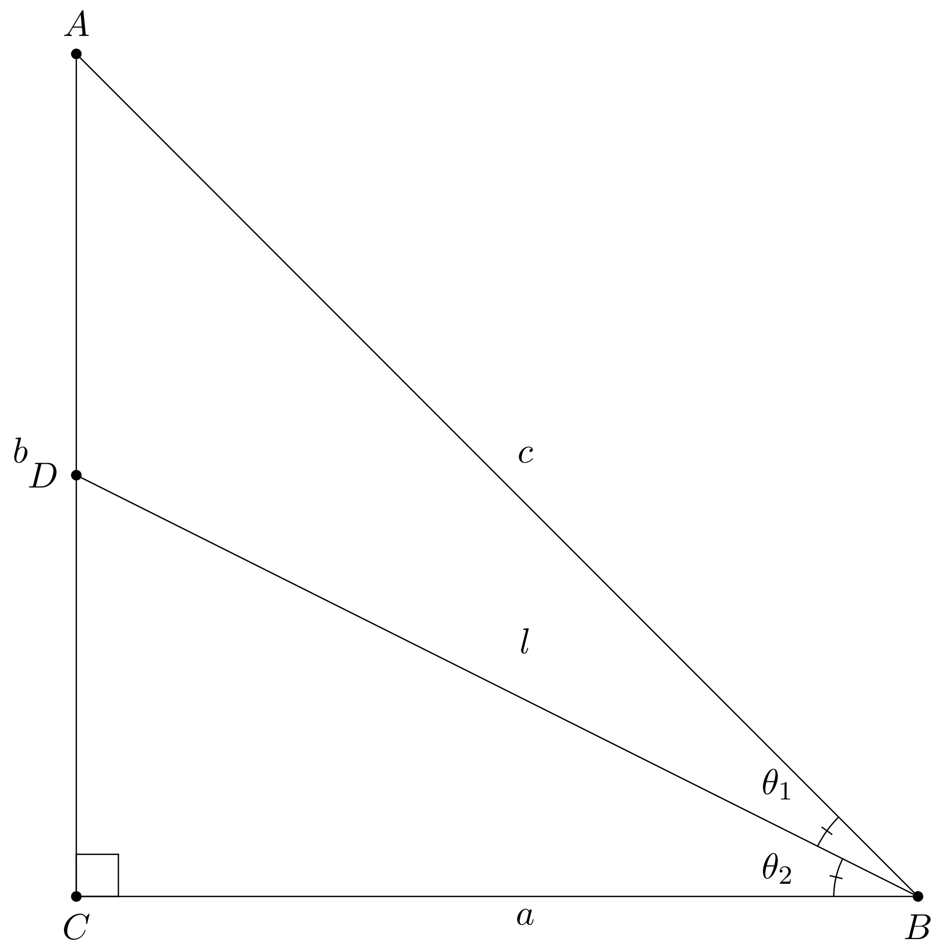

Figure A.4:

A.3.4. Using Fig. A.4, show that

$$\sin\theta_1 = \sin(\theta_1+\theta_2)\cos\theta_2 - \cos(\theta_1+\theta_2)\sin\theta_2 \qquad \text{(A.3.4.1)}$$

Solution: The following equations can be obtained from the figure using the forumula for the area of a triangle

$$ar(\Delta ABC) = \frac{1}{2}ac\sin(\theta_1 + \theta_2) \tag{A.3.4.2}$$

$$= ar(\Delta BDC) + ar(\Delta ADB) \tag{A.3.4.3}$$

$$= \frac{1}{2}cl\sin\theta_1 + \frac{1}{2}al\sin\theta_2 \tag{A.3.4.4}$$

$$= \frac{1}{2}ac\sin\theta_1\sec\theta_2 + \frac{1}{2}a^2\tan\theta_2 \tag{A.3.4.5}$$

($\because l = a\sec\theta_2$). From the above,

$$\sin(\theta_1 + \theta_2) = \sin\theta_1\sec\theta_2 + \frac{a}{c}\tan\theta_2 \tag{A.3.4.6}$$

$$= \sin\theta_1\sec\theta_2 + \cos(\theta_1 + \theta_2)\tan\theta_2 \tag{A.3.4.7}$$

Multiplying both sides by $\cos\theta_2$,

$$\sin(\theta_1 + \theta_2)\cos\theta_2 = \sin\theta_1 + \cos(\theta_1 + \theta_2)\sin\theta_2 \tag{A.3.4.8}$$

resulting in (A.3.4.1).

A.3.5. Find Hero's formula for the area of a triangle.

Solution: From (A.3.1), the area of $\triangle ABC$ is

$$\frac{1}{2}ab\sin C = \frac{1}{2}ab\sqrt{1-\cos^2 C} \quad \text{(from (A.2.2.1))} \tag{A.3.5.1}$$

$$= \frac{1}{2}ab\sqrt{1-\left(\frac{a^2+b^2-c^2}{2ab}\right)^2} \text{(from (B.3.3.1))} \tag{A.3.5.2}$$

$$= \frac{1}{4}\sqrt{(2ab)^2-(a^2+b^2-c^2)} \tag{A.3.5.3}$$

$$= \frac{1}{4}\sqrt{(2ab+a^2+b^2-c^2)(2ab-a^2-b^2+c^2)} \tag{A.3.5.4}$$

$$= \frac{1}{4}\sqrt{\left\{(a+b)^2-c^2\right\}\left\{c^2-(a-b)^2\right\}} \tag{A.3.5.5}$$

$$= \frac{1}{4}\sqrt{(a+b+c)(a+b-c)(a+c-b)(b+c-a)} \tag{A.3.5.6}$$

Substituting

$$s = \frac{a+b+c}{2} \tag{A.3.5.7}$$

in (A.3.5.6), the area of $\triangle ABC$ is

$$\sqrt{s(s-a)(s-b)(s-c)} \tag{A.3.5.8}$$

This is known as Hero's formula.

A.4. Angle Bisectors

A.4.1. In Fig. A.4.1.1, the bisectors of $\angle B$ and $\angle C$ meet at **I**. Show that IA bisects $\angle A$.

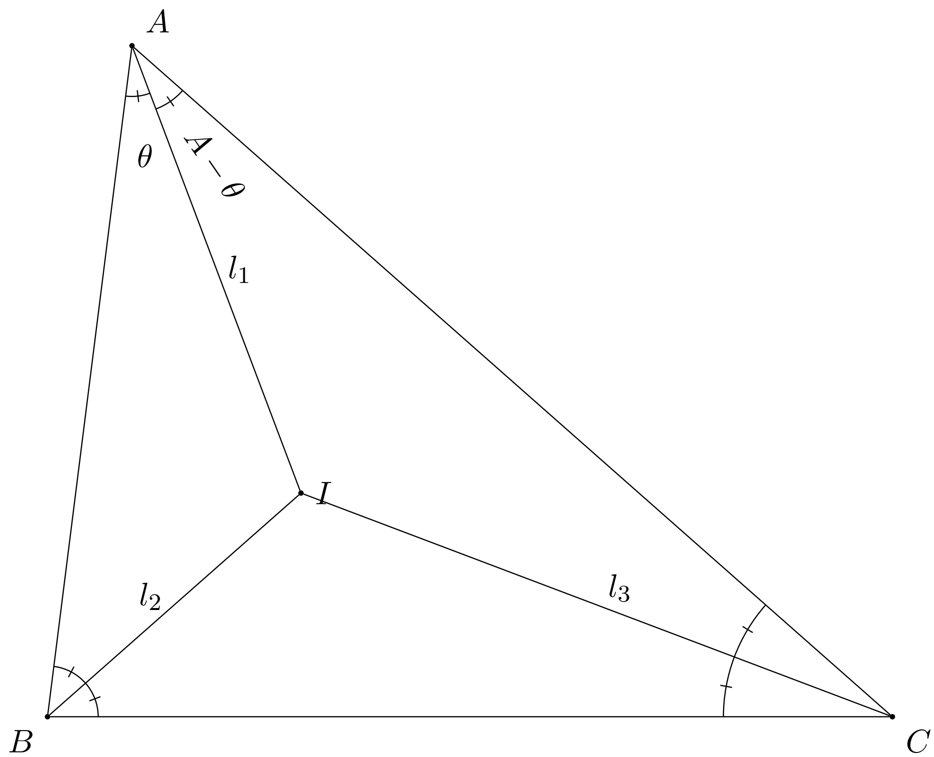

Figure A.4.1.1: Incentre I of $\triangle ABC$

Solution: Using sine formula in (A.3.2.3)

$$\frac{l_1}{\sin\frac{C}{2}} = \frac{l_3}{\sin(A-\theta)} \tag{A.4.1.1}$$

$$\frac{l_3}{\sin\frac{B}{2}} = \frac{l_2}{\sin\frac{C}{2}} \tag{A.4.1.2}$$

$$\frac{l_1}{\sin\frac{B}{2}} = \frac{l_2}{\sin\theta} \tag{A.4.1.3}$$

Multiplying the above equations,

$$\sin\theta = \sin(A-\theta) \implies \theta = \frac{A}{2} \tag{A.4.1.4}$$

A.4.2. In Fig. A.4.2.1,

$$ID \perp BC,\ IE \perp AC,\ IF \perp AB. \tag{A.4.2.1}$$

Show that

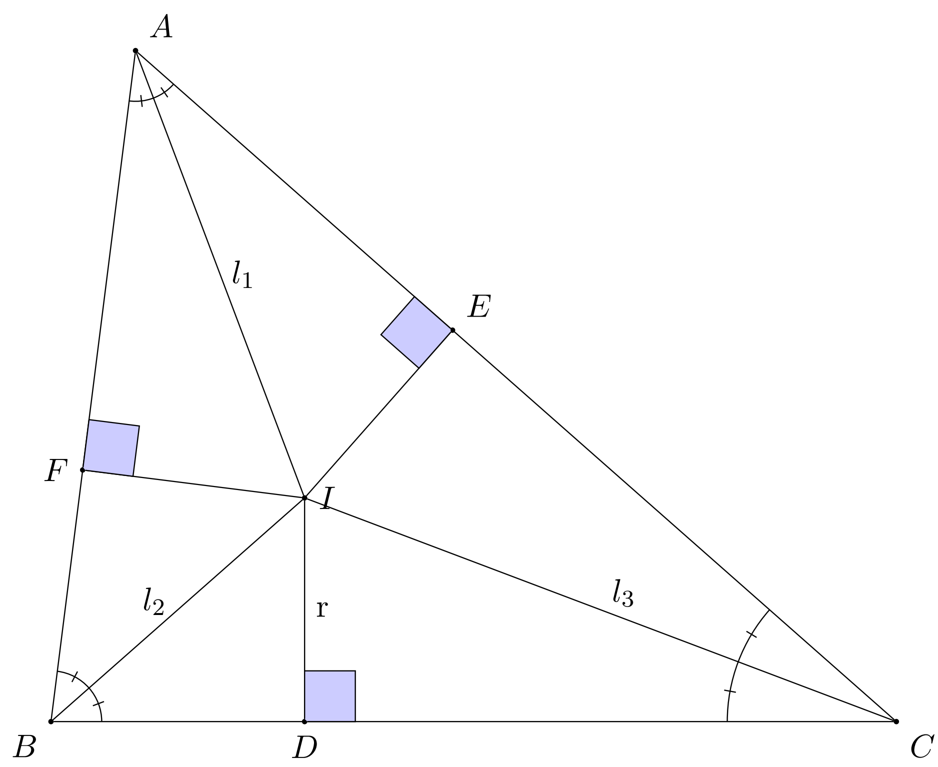

Figure A.4.2.1: Inradius r of $\triangle ABC$

$$ID = IE = IF = r \tag{A.4.2.2}$$

Solution: In $\triangle$s IDC and IEC,

$$ID = IE = \frac{l_3}{\sin\frac{C}{2}} \tag{A.4.2.3}$$

Similarly, in $\triangle$s IEA and IFA,

$$IF = IE = \frac{l_1}{\sin\frac{A}{2}} \tag{A.4.2.4}$$

yielding (A.4.2.2)

A.4.3. In Fig. A.4.2.1, show that

$$BD = BF,\ AE = AF,\ CD = CE \tag{A.4.3.1}$$

Solution: From Fig. A.4.2.1, in $\triangle$s IBD and IBF,

$$x = BD = BF = r\cot\frac{B}{2} \tag{A.4.3.2}$$

Similarly, other results can be obtained.

A.4.4. The circle with centre **I** and radius r in Fig. A.4.4.1 is known as the incircle. Find the radius r.

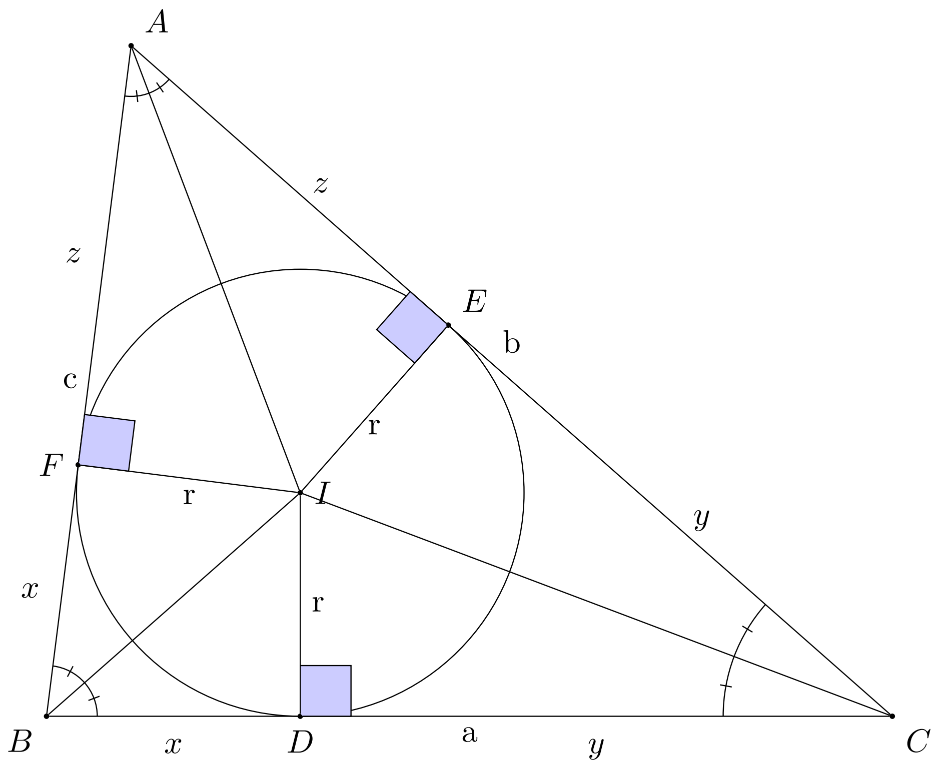

Figure A.4.4.1: Incircle of $\triangle ABC$

Solution: In $\triangle IBC$,

$$a = x + y = r\cot\frac{B}{2} + r\cot\frac{C}{2} \tag{A.4.4.1}$$

$$\implies r = \frac{a}{\cot\frac{B}{2} + \cot\frac{C}{2}} \tag{A.4.4.2}$$

A.5. Circumradius

A.5.1. In Fig. A.5.1.1,

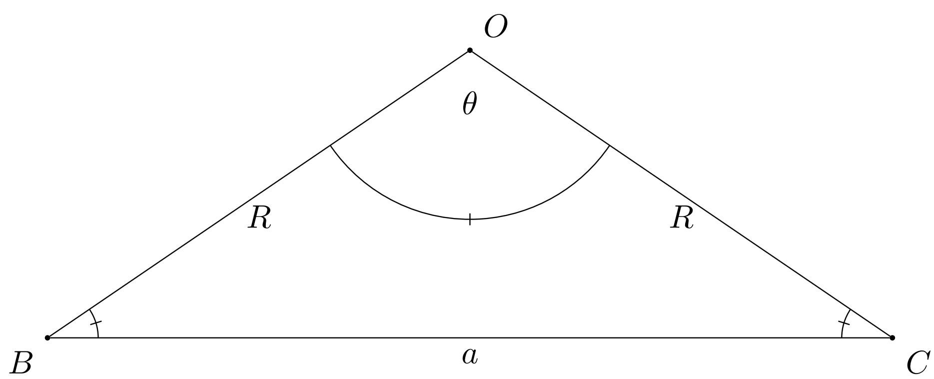

Figure A.5.1.1: Isosceles Triangle

$$OB = OC = R \tag{A.5.1.1}$$

Such a triangle is known as an isosceles triangle. Show that

$$\angle B = \angle C \tag{A.5.1.2}$$

Solution: Using (A.3.2.3),

$$\frac{\sin B}{R} = \frac{\sin C}{R} \tag{A.5.1.3}$$

$$\implies \sin B = \sin C \tag{A.5.1.4}$$

$$\text{or, } \angle B = \angle C. \tag{A.5.1.5}$$

A.5.2. In Fig. A.5.1.1, show that

$$a = 2R \sin \frac{\theta}{2} \tag{A.5.2.1}$$

Solution: In $\triangle OBC$, using the cosine formula from (B.3.3.1),

$$\cos\theta = \frac{R^2 + R^2 - a^2}{2R^2} = 1 - \frac{a^2}{2R^2} \tag{A.5.2.2}$$

$$\implies \frac{a^2}{2R^2} = 2\sin^2\frac{\theta}{2} \tag{A.5.2.3}$$

yielding (A.5.2.1).

A.5.3. In Fig. B.7.2.1, show that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R. \tag{A.5.3.1}$$

Solution: From (B.7.6.1) and (A.5.2.1)

$$a = 2R\sin A \tag{A.5.3.2}$$

A.6. Tangent

A.6.1. In Fig. B.8.2.1, show that $PA.PB = PC^2$.

Solution: In $\triangle$s APC and BPC, using (B.8.2.1),

$$\frac{AP}{\sin\theta} = \frac{AC}{\sin P} \tag{A.6.1.1}$$

$$\frac{PC}{\sin\theta} = \frac{BC}{\sin P} \tag{A.6.1.2}$$

$$\implies \frac{PC}{AP} = \frac{BC}{AC}\left(= \frac{BP}{CP}\right) \tag{A.6.1.3}$$

which gives the desired result. $\triangle$s APC and BPC are said to be similar.

A.7. Identities

A.7.1. Show that

$$\cos 90^\circ = 0 \tag{A.7.1.1}$$

Solution: Using (B.3.3.1) in Fig. A.1,

$$\cos 90^\circ = \frac{a^2 + c^2 - b^2}{2ac} = 0 \tag{A.7.1.2}$$

upon substituting from (A.2.4.1).

A.7.2. Show that

$$\sin 90^\circ = 1 \tag{A.7.2.1}$$

Solution: Trivial from (A.1.2.1).

A.7.3. Prove the following identities

(a)

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta. \tag{A.7.3.1}$$

(b)

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta. \tag{A.7.3.2}$$

Solution: In (A.3.4.1), let

$$\begin{aligned} \theta_1 + \theta_2 &= \alpha \\ \theta_2 &= \beta \end{aligned} \tag{A.7.3.3}$$

This gives (A.7.3.1). In (A.7.3.1), replace α by $90^\circ - \alpha$. This results in

$$\sin(90^\circ - \alpha - \beta) = \sin(90^\circ - \alpha)\cos\beta - \cos(90^\circ - \alpha)\sin\beta \tag{A.7.3.4}$$

$$\implies \cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \tag{A.7.3.5}$$

A.7.4. Using (A.3.4.1) and (A.7.3.2), show that

$$\sin(\theta_1 + \theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 \tag{A.7.4.1}$$

$$\cos(\theta_1 - \theta_2) = \cos\theta_1\cos\theta_2\sin\theta_1\sin\theta_2 \tag{A.7.4.2}$$

Solution: From (A.3.4.1),

$$\sin(\theta_1 + \theta_2)\cos\theta_2 = \sin\theta_1 + \cos(\theta_1 + \theta_2)\sin\theta_2 \tag{A.7.4.3}$$

Using (A.7.3.2) in the above,

$$\begin{aligned} \sin(\theta_1 + \theta_2)\cos\theta_2 = \sin\theta_1 + (&\cos\theta_1\cos\theta_2 \\ &- \sin\theta_1\sin\theta_2)\sin\theta_2 \end{aligned} \tag{A.7.4.4}$$

which can be expressed as

$$\sin(\theta_1+\theta_2)\cos\theta_2 = \sin\theta_1 \\ +\cos\theta_1\cos\theta_2\sin\theta_2 - \sin\theta_1\sin^2\theta_2 \quad \text{(A.7.4.5)}$$

Since

$$\sin^2\theta_2 = 1-\cos^2\theta_2, \quad \text{(A.7.4.6)}$$

we obtain

$$\sin(\theta_1+\theta_2)\cos\theta_2 \;=\; \cos\theta_1\cos\theta_2\sin\theta_2 \;+\; \sin\theta_1\cos^2\theta_2 \quad \text{(A.7.4.7)}$$

resulting in

$$\sin(\theta_1+\theta_2) = \cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2 \quad \text{(A.7.4.8)}$$

after factoring out $\cos\theta_2$. Using a similar approach, (A.7.4.2) can also be proved.

A.7.5. Show that

$$\sin\theta_1+\sin\theta_2 = 2\sin\left(\frac{\theta_1+\theta_2}{2}\right)\cos\left(\frac{\theta_1-\theta_2}{2}\right) \quad \text{(A.7.5.1)}$$

$$\cos\theta_1+\cos\theta_2 = 2\cos\left(\frac{\theta_1+\theta_2}{2}\right)\cos\left(\frac{\theta_1-\theta_2}{2}\right) \quad \text{(A.7.5.2)}$$

$$\sin\theta_1-\sin\theta_2 = 2\sin\left(\frac{\theta_1-\theta_2}{2}\right)\cos\left(\frac{\theta_1+\theta_2}{2}\right) \quad \text{(A.7.5.3)}$$

$$\cos\theta_1-\cos\theta_2 = 2\sin\left(\frac{\theta_1+\theta_2}{2}\right)\cos\left(\frac{\theta_2-\theta_1}{2}\right) \quad \text{(A.7.5.4)}$$

Solution: Let

$$\begin{aligned} \theta_1 &= \alpha + \beta \\ \theta_2 &= \alpha - \beta \end{aligned} \tag{A.7.5.5}$$

From (A.7.4.1),

$$\sin\theta_1 + \sin\theta_2 = \sin(\alpha+\beta) + \sin(\alpha-\beta) \tag{A.7.5.6}$$

$$= \sin\alpha\cos\beta + \cos\alpha\sin\beta \tag{A.7.5.7}$$

$$+ \sin\alpha\cos\beta - \cos\alpha\sin\beta \tag{A.7.5.8}$$

$$= 2\sin\alpha\cos\beta \tag{A.7.5.9}$$

resulting in (A.7.5.1)

$$\because \alpha = \frac{\theta_1+\theta_2}{2} \tag{A.7.5.10}$$

$$\beta = \frac{\theta_1-\theta_2}{2} \tag{A.7.5.11}$$

from (A.7.5.5). Other identities may be proved similarly.

A.7.6. Show that

$$\sin 2\theta = 2\sin\theta\cos\theta \tag{A.7.6.1}$$

$$\cos 2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1 \tag{A.7.6.2}$$

$$= \cos^2\theta - \sin^2\theta \tag{A.7.6.3}$$

Appendix B

Analytic Geometry

B.1. Vectors

B.1.1. A matrix of the form

$$\mathbf{A} \triangleq \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \tag{B.1.1.1}$$

is defined be column vector, or simply, vector. In Fig. A.1 the point vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$ can be defined as

$$\mathbf{A} = \begin{pmatrix} 0 \\ c \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} a \\ 0 \end{pmatrix} \tag{B.1.1.2}$$

B.1.2.

$$\lambda\mathbf{A} \triangleq \begin{pmatrix} \lambda a_1 \\ \lambda a_2 \end{pmatrix} \tag{B.1.2.1}$$

B.1.3. For

$$\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \tag{B.1.3.1}$$

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \end{pmatrix} \tag{B.1.3.2}$$

B.1.4. The transpose of $\mathbf{A}$ is the <u>row vector</u> defined as

$$\mathbf{A}^\top = \begin{pmatrix} a_1 & a_2 \end{pmatrix} \tag{B.1.4.1}$$

B.1.5. The <u>inner product</u> or <u>dot product</u> is defined as

$$\mathbf{A}^\top\mathbf{B} \equiv \mathbf{A} \cdot \mathbf{B} = \begin{pmatrix} a_1 & a_2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = a_1 b_1 + a_2 b_2 \tag{B.1.5.1}$$

In Fig. A.1,

$$\mathbf{A}^\top\mathbf{C} = 0 \tag{B.1.5.2}$$

B.1.6. The <u>norm</u> of $\mathbf{A}$ is defined as

$$\|\mathbf{A}\| = \sqrt{\mathbf{A}^\top\mathbf{A}} = \sqrt{a_1^2 + a_2^2} \tag{B.1.6.1}$$

B.1.7. In Fig. A.1, it is easy to verify that

$$\|\mathbf{A} - \mathbf{C}\|^2 = \begin{pmatrix} -c & a \end{pmatrix} \begin{pmatrix} -c \\ a \end{pmatrix} = a^2 + c^2 = b^2 \tag{B.1.7.1}$$

from (A.2.4.1). Thus, the distance betwen any two points $\mathbf{A}$ and $\mathbf{B}$ is given by

$$\|\mathbf{A} - \mathbf{B}\| \tag{B.1.7.2}$$

B.1.8. Show that

$$\|\lambda \mathbf{A}\| = |\lambda| \, \|\mathbf{A}\| \tag{B.1.8.1}$$

B.2. Collinear Points

B.2.1. The direction vector of the line AB is

$$\mathbf{A} - \mathbf{B} \equiv \mathbf{B} - \mathbf{A} \equiv \kappa \begin{pmatrix} 1 \\ m \end{pmatrix}, \tag{B.2.1.1}$$

where m is defined to be the slope of AB. In Fig. A.1,

$$\mathbf{A} - \mathbf{C} = \begin{pmatrix} -c \\ a \end{pmatrix} \equiv \begin{pmatrix} 1 \\ -\frac{a}{c} \end{pmatrix} = \begin{pmatrix} 1 \\ -\tan\theta \end{pmatrix} \tag{B.2.1.2}$$

the slope of AC is $-\tan\theta$

B.2.2. Points $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are on a line if they have the same direction vector, i.e.

$$p\left(\mathbf{B}-\mathbf{A}\right)+q\left(\mathbf{C}-\mathbf{B}\right)=0 \implies p, q \neq 0. \tag{B.2.2.1}$$

$\left(\mathbf{A}-\mathbf{B}\right), \left(\mathbf{C}-\mathbf{B}\right)$ are then said to be linearly dependent.

B.2.3. If points $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are collinear,

$$\mathbf{B}=\frac{k\mathbf{A}+\mathbf{C}}{k+1} \tag{B.2.3.1}$$

Solution: From (B.2.2.1),

$$p\left(\mathbf{A}-\mathbf{B}\right)+q\left(\mathbf{A}-\mathbf{C}\right)=0 \implies \mathbf{B}=\frac{p\mathbf{A}+q\mathbf{C}}{p+q} \tag{B.2.3.2}$$

yielding (B.2.3.1) upon substituting

$$k=\frac{p}{q}. \tag{B.2.3.3}$$

This is known as section formula.

B.2.4. Consequently, points $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ form a triangle if

$$p\left(\mathbf{A}-\mathbf{B}\right)+q\left(\mathbf{C}-\mathbf{B}\right) \tag{B.2.4.1}$$

$$=\left(p+q\right)\mathbf{B}-p\mathbf{A}-q\mathbf{C}=0 \tag{B.2.4.2}$$

$$\implies p=0, q=0 \tag{B.2.4.3}$$

B.2.5. In Fig. B.2.5.1

$$AF = BF,\ AE = BE, \tag{B.2.5.1}$$

and the medians BE and CF meet at **G**. Show that

$$\frac{GB}{GE} = \frac{GC}{GF} = 2 \tag{B.2.5.2}$$

Solution: From (B.2.3.1),

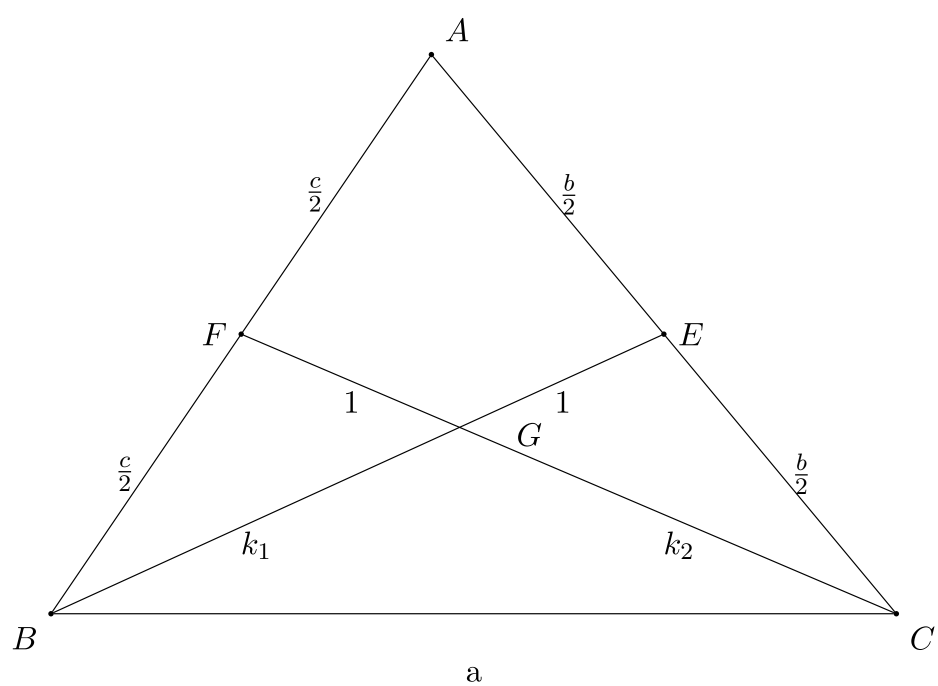

Figure B.2.5.1: $k_1 = k_2 = 2$.

$$\mathbf{G} = \frac{k_1\mathbf{E} + \mathbf{B}}{k_1 + 1} = \frac{k_2\mathbf{F} + \mathbf{C}}{k_2 + 1} \tag{B.2.5.3}$$

$$\implies \frac{k_1\left(\frac{\mathbf{A}+\mathbf{C}}{2}\right) + \mathbf{B}}{k_1 + 1} = \frac{k_2\left(\frac{\mathbf{A}+\mathbf{B}}{2}\right) + \mathbf{C}}{k_2 + 1} \tag{B.2.5.4}$$

$$\implies (k_2 + 1)\left\{k_1(\mathbf{A} + \mathbf{C}) + 2\mathbf{B}\right\} = (k_1 + 1)\left\{k_2(\mathbf{A} + \mathbf{B}) + 2\mathbf{C}\right\} \tag{B.2.5.5}$$

which can be expressed as

$$\left\{2 + k_2 - k_1k_2\right\}\mathbf{B} - (k_2 - k_1)\,\mathbf{A} - \left\{k_1 + 2 - k_1k_2\right\}\mathbf{C} = 0 \tag{B.2.5.6}$$

and is of the form (B.2.4.3) with

$$p = k_2 - k_1, q = k_1 + 2 - k_1k_2. \tag{B.2.5.7}$$

Thus, from (B.2.4.3)

$$k_2 - k_1 = 0, \tag{B.2.5.8}$$

$$k_1 + 2 - k_1k_2 = 0 \tag{B.2.5.9}$$

Thus, from (B.2.5.9)

$$k_1 = k_2 \tag{B.2.5.10}$$

and substituting the above in (B.2.5.9) results in the quadratic

$$k_1^2 - k_1 - 2 = 0 \tag{B.2.5.11}$$

$$\implies (k_1 - 2)(k_1 + 1) = 0 \tag{B.2.5.12}$$

admitting $k_1 = k_2 = 2$ as the only possible solution.

B.2.6. Substituting $k_1 = 2$ in (B.2.5.3)

$$\mathbf{G} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \tag{B.2.6.1}$$

B.2.7. In Fig. B.2.7.1, AG is extended to join BC at $\mathbf{D}$. Show that AD is also a median.

Solution: Considering the ratios in Fig. B.2.7.1,

$$\mathbf{G} = \frac{k_3\mathbf{D} + \mathbf{A}}{k_3 + 1} \tag{B.2.7.1}$$

$$\mathbf{D} = \frac{k_4\mathbf{C} + \mathbf{B}}{k_4 + 1} \tag{B.2.7.2}$$

Substituting from (B.2.6.1) in the above,

$$(k_3 + 1)\left(\frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3}\right) = k_3\left(\frac{k_4\mathbf{C} + \mathbf{B}}{k_4 + 1}\right) + \mathbf{A} \tag{B.2.7.3}$$

$$\implies (k_3 + 1)(k_4 + 1)(\mathbf{A} + \mathbf{B} + \mathbf{C}) = 3\left\{k_3(k_4\mathbf{C} + \mathbf{B}) + (k_4 + 1)\mathbf{A}\right\} \tag{B.2.7.4}$$

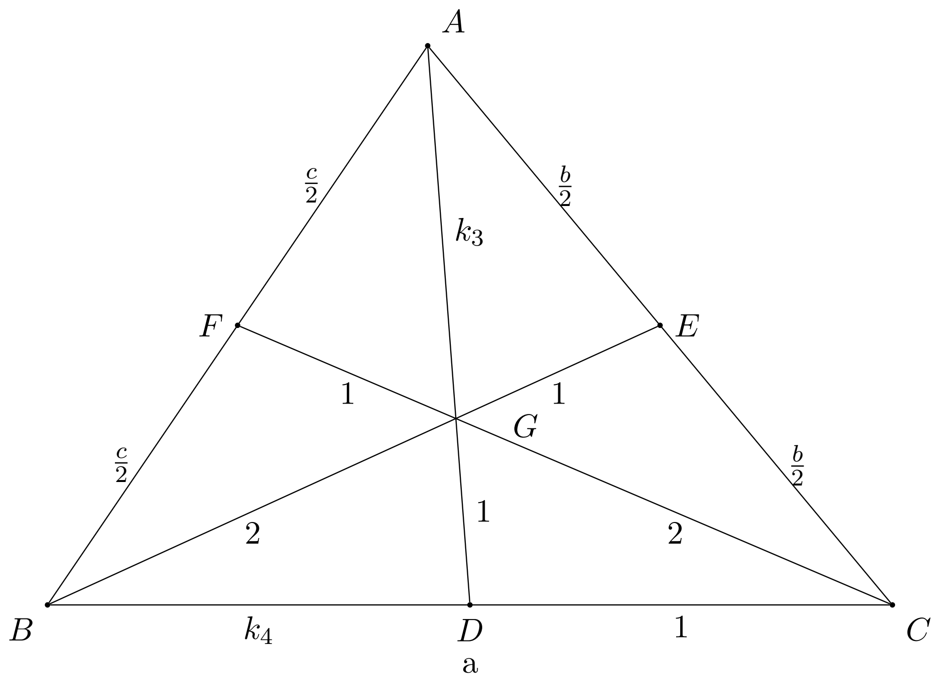

Figure B.2.7.1: $k_3 = 2, k_4 = 1$

which can be expressed as

$$\begin{aligned}&(k_3k_4 + k_3 - 2k_4 - 2)\,\mathbf{A}\\ &\qquad - (-k_3k_4 - k_4 + 2k_3 - 1)\,\mathbf{B}\\ &\qquad\qquad - (-k_3 - k_4 - 1 + 2k_3k_4)\,\mathbf{C} = \mathbf{0}\end{aligned} \quad \text{(B.2.7.5)}$$

Comparing the above with (B.2.4.3),

$$p = -k_3k_4 - k_4 + 2k_3 - 1, q = -k_3 - k_4 - 1 + 2k_3k_4 \qquad \text{(B.2.7.6)}$$

yielding

$$-k_3k_4 - k_4 + 2k_3 - 1 = 0 \tag{B.2.7.7}$$

$$-k_3 - k_4 - 1 + 2k_3k_4 = 0 \tag{B.2.7.8}$$

Subtracting (B.2.7.7) from (B.2.7.8),

$$3k_3\left(k_4 - 1\right) = 0 \tag{B.2.7.9}$$

$$\implies k_4 = 1 \tag{B.2.7.10}$$

which upon substituting in (B.2.7.7) yields

$$k_3 = 2 \tag{B.2.7.11}$$

B.3. Matrices: Cosine Formula

B.3.1. The determinant of the 2×2 matrix

$$\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \tag{B.3.1.1}$$

is defined as

$$\begin{vmatrix}\mathbf{M}\end{vmatrix} = \begin{vmatrix}\mathbf{A} & \mathbf{B}\end{vmatrix} \tag{B.3.1.2}$$

$$= \begin{vmatrix}a_1 & b_1 \\ a_2 & b_2\end{vmatrix} = a_1 b_2 - a_2 b_1 \tag{B.3.1.3}$$

B.3.2. In Fig. B.3.2.1, show that

$$\begin{pmatrix}0 & c & b \\ c & 0 & a \\ b & a & 0\end{pmatrix}\begin{pmatrix}\cos A \\ \cos B \\ \cos C\end{pmatrix} = \begin{pmatrix}a \\ b \\ c\end{pmatrix} \tag{B.3.2.1}$$

Solution: From Fig. B.3.2.1,

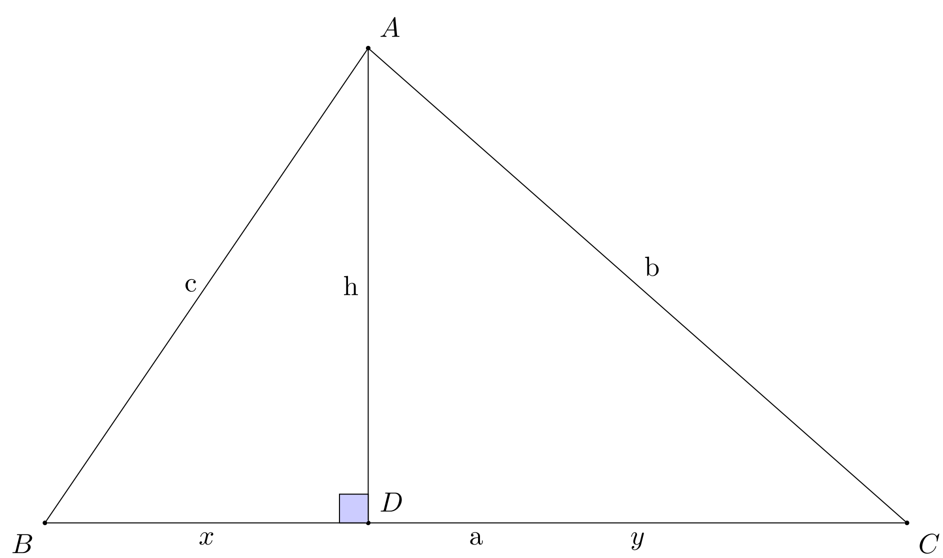

Figure B.3.2.1: The cosine formula

$$a = x + y = b\cos C + c\cos B = \begin{pmatrix} \cos C & \cos B \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix} \tag{B.3.2.2}$$

$$= \begin{pmatrix} 0 & b & c \end{pmatrix} \begin{pmatrix} \cos A \\ \cos C \\ \cos B \end{pmatrix} \tag{B.3.2.3}$$

Similarly,

$$b = c\cos A + a\cos C = \begin{pmatrix} c & 0 & a \end{pmatrix} \begin{pmatrix} \cos A \\ \cos C \\ \cos B \end{pmatrix} \tag{B.3.2.4}$$

$$c = b\cos A + a\cos B = \begin{pmatrix} b & a & 0 \end{pmatrix} \begin{pmatrix} \cos A \\ \cos C \\ \cos B \end{pmatrix} \tag{B.3.2.5}$$

The above equations can be expressed in matrix form as (B.3.2.1).

B.3.3. Show that

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \tag{B.3.3.1}$$

Solution: Using the properties of determinants,

$$\cos A = \frac{\begin{vmatrix} a & c & b \\ b & 0 & a \\ c & a & 0 \end{vmatrix}}{\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}} = \frac{ab^2 + ac^2 - a^3}{abc + abc} = \frac{b^2 + c^2 - a^2}{2abc} \tag{B.3.3.2}$$

B.4. Area of a Triangle: Cross Product

B.4.1. The cross product or vector product defined as $\mathbf{A} \times \mathbf{B}$ is given by (B.3.1.2) for 2×1 vectors.

B.4.2. The area of the triangle with vertices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ is given by

$$\frac{1}{2} \left\| (\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{C}) \right\| = \frac{1}{2} \left\| \mathbf{A} \times \mathbf{B} + \mathbf{B} \times \mathbf{C} + \mathbf{C} \times \mathbf{A} \right\| \tag{B.4.2.1}$$

B.4.3. If

$$\left\| \mathbf{A} \times \mathbf{B} \right\| = \left\| \mathbf{C} \times \mathbf{D} \right\|, \quad \text{then} \tag{B.4.3.1}$$

$$\mathbf{A} \times \mathbf{B} = \pm \left(\mathbf{C} \times \mathbf{D} \right) \tag{B.4.3.2}$$

where the sign depends on the orientation of the vectors.

B.5. Parallelogram

B.5.1. If $ABCD$ be a parallelogram,

$$\mathbf{B}-\mathbf{A}=\mathbf{C}-\mathbf{D} \tag{B.5.1.1}$$

B.5.2. The area of the parallelogram with vertices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{D}$ is given by

$$\left\|(\mathbf{A}-\mathbf{B}) \times (\mathbf{A}-\mathbf{D})\right\| = \left\|\mathbf{A}\times\mathbf{B}+\mathbf{B}\times\mathbf{C}+\mathbf{C}\times\mathbf{A}\right\| \tag{B.5.2.1}$$

B.6. Altitudes of a Triangle:Line Equation

B.6.1. Find the equation of the line BC.

Solution: Let $\mathbf{x}$ be any point on BC. Using section formula, for some k,

$$\mathbf{x}=\frac{k\mathbf{C}+\mathbf{B}}{k+1}=\frac{(k+1)\,\mathbf{C}+(\mathbf{B}-\mathbf{C})}{k+1} \tag{B.6.1.1}$$

$$\implies \mathbf{x}=\mathbf{C}+\lambda\mathbf{m} \tag{B.6.1.2}$$

where

$$\mathbf{m}=\frac{\mathbf{B}-\mathbf{C}}{k+1}\equiv\mathbf{B}-\mathbf{C} \tag{B.6.1.3}$$

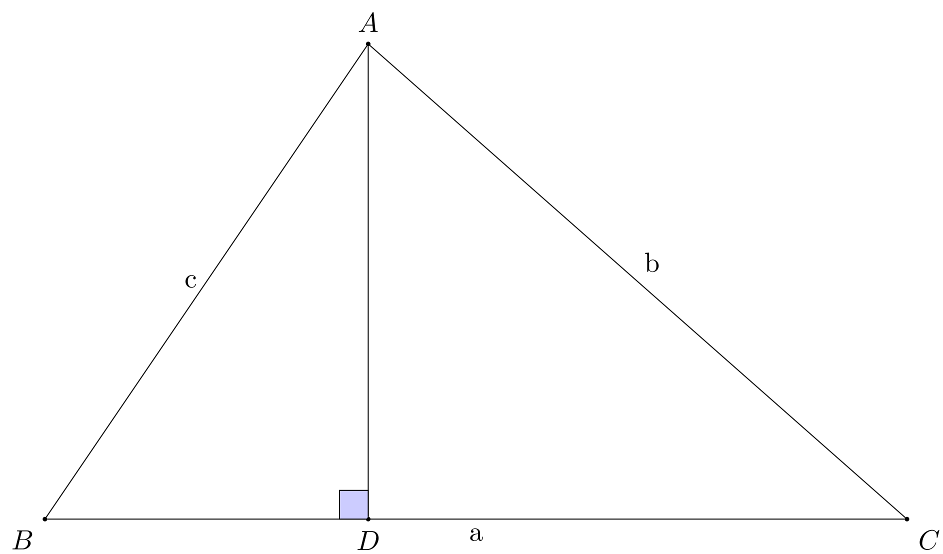

Figure B.6.1.1: Drawing the altitude

B.6.2. The normal vector to $\mathbf{m}$ is defined as

$$\mathbf{n}^\top \mathbf{m} = 0 \tag{B.6.2.1}$$

$$\mathbf{n} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{m} \tag{B.6.2.2}$$

B.6.3. From (B.6.2.1) and (B.6.1.2), it can be verified that

$$\mathbf{n}^\top \mathbf{x} = \mathbf{n}^\top \mathbf{C} + \lambda \mathbf{n}^\top \mathbf{m} \tag{B.6.3.1}$$

$$\implies \mathbf{n}^\top \mathbf{x} = \mathbf{n}^\top \mathbf{C} \tag{B.6.3.2}$$

(B.6.3.2) is defined to be the <u>normal form</u> of the line BC.

B.6.4. In Fig. B.6.5.1, $AD \perp BC$ and $BE \perp AC$ are defined to be the altitudes of $\triangle ABC$.

B.6.5. Let $\mathbf{H}$ be the intersection of the altitudes AD and BE as shown in Fig. B.6.5.1. CH is extended to meet AB at $\mathbf{F}$. Show that $CF \perp AB$.

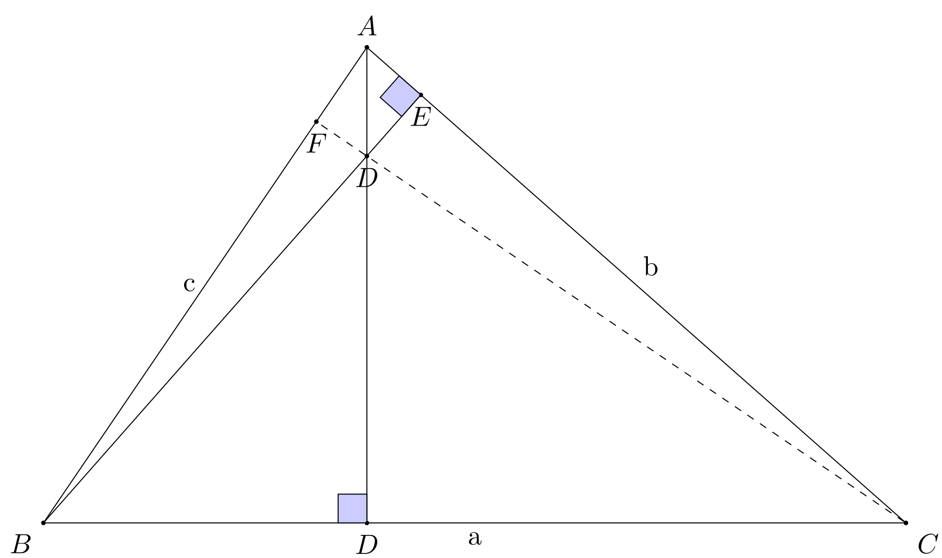

Figure B.6.5.1: Altitudes of a triangle meet at the orthocentre H

Solution: From (B.6.1.3) (B.6.2.1), (B.1.5.2) and (B.6.3.2), the equations of AD and BE are

$$(\mathbf{B}-\mathbf{C})^{\top}(\mathbf{x}-\mathbf{A}) = 0 \tag{B.6.5.1}$$

$$(\mathbf{C}-\mathbf{A})^{\top}(\mathbf{x}-\mathbf{B}) = 0 \tag{B.6.5.2}$$

$\because$ **H** lies on both AD and BE, it satisfies the above equations, and

$$(\mathbf{B} - \mathbf{C})^\top (\mathbf{H} - \mathbf{A}) = 0 \tag{B.6.5.3}$$

$$(\mathbf{C} - \mathbf{A})^\top (\mathbf{H} - \mathbf{B}) = 0 \tag{B.6.5.4}$$

Adding both the above and simplifying,

$$(\mathbf{B} - \mathbf{A})^\top (\mathbf{H} - \mathbf{C}) = 0 \tag{B.6.5.5}$$

$\implies CH \perp AB$ from (B.1.5.2), or $CF \perp AB$.

B.6.6. Altitudes of a $\triangle$ meet at the orthocentre H.

B.7. Circumcircle: Circle Equation

B.7.1. In Fig. B.7.1.1,

$$OB = OC = R, BD = DC. \tag{B.7.1.1}$$

Show that $OD \perp BC$.

Solution:

$$\|\mathbf{O} - \mathbf{C}\| = \|\mathbf{O} - \mathbf{B}\| = R \tag{B.7.1.2}$$

$$\implies \|\mathbf{O} - \mathbf{C}\|^2 = \|\mathbf{O} - \mathbf{B}\|^2 \tag{B.7.1.3}$$

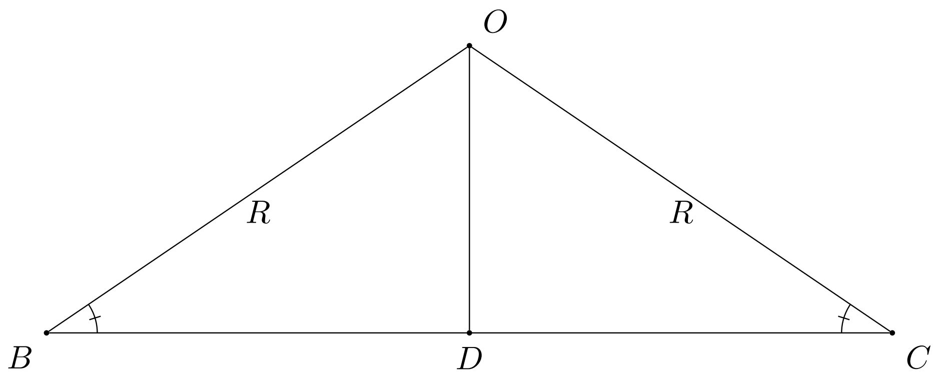

Figure B.7.1.1: Perpendicular bisector.

which can be expressed as

$$(\mathbf{O}-\mathbf{C})^\top(\mathbf{O}-\mathbf{C}) = (\mathbf{O}-\mathbf{B})^\top(\mathbf{O}-\mathbf{B}) \tag{B.7.1.4}$$

$$\|\mathbf{O}\|^2 - 2\mathbf{O}^\top\mathbf{C} + \|\mathbf{C}\|^2 = \|\mathbf{O}\|^2 - 2\mathbf{O}^\top\mathbf{B} + \|\mathbf{B}\|^2 \tag{B.7.1.5}$$

$$\implies (\mathbf{B}-\mathbf{C})^\top\mathbf{O} = \frac{\|\mathbf{B}\|^2 - \|\mathbf{C}\|^2}{2} \tag{B.7.1.6}$$

which can be simplified to obtain

$$(\mathbf{B}-\mathbf{C})^\top\left\{\mathbf{O}-\left(\frac{\mathbf{B}+\mathbf{C}}{2}\right)\right\} = 0 \tag{B.7.1.7}$$

$$\text{or, } (\mathbf{B}-\mathbf{C})^\top\{\mathbf{O}-\mathbf{D}\} = 0 \tag{B.7.1.8}$$

which proves the give result using (B.2.3.1) and (B.1.5.2).

B.7.2. The equation of the circle in Fig. B.7.2.1, is

$$\|\mathbf{x}-\mathbf{O}\| = R \tag{B.7.2.1}$$

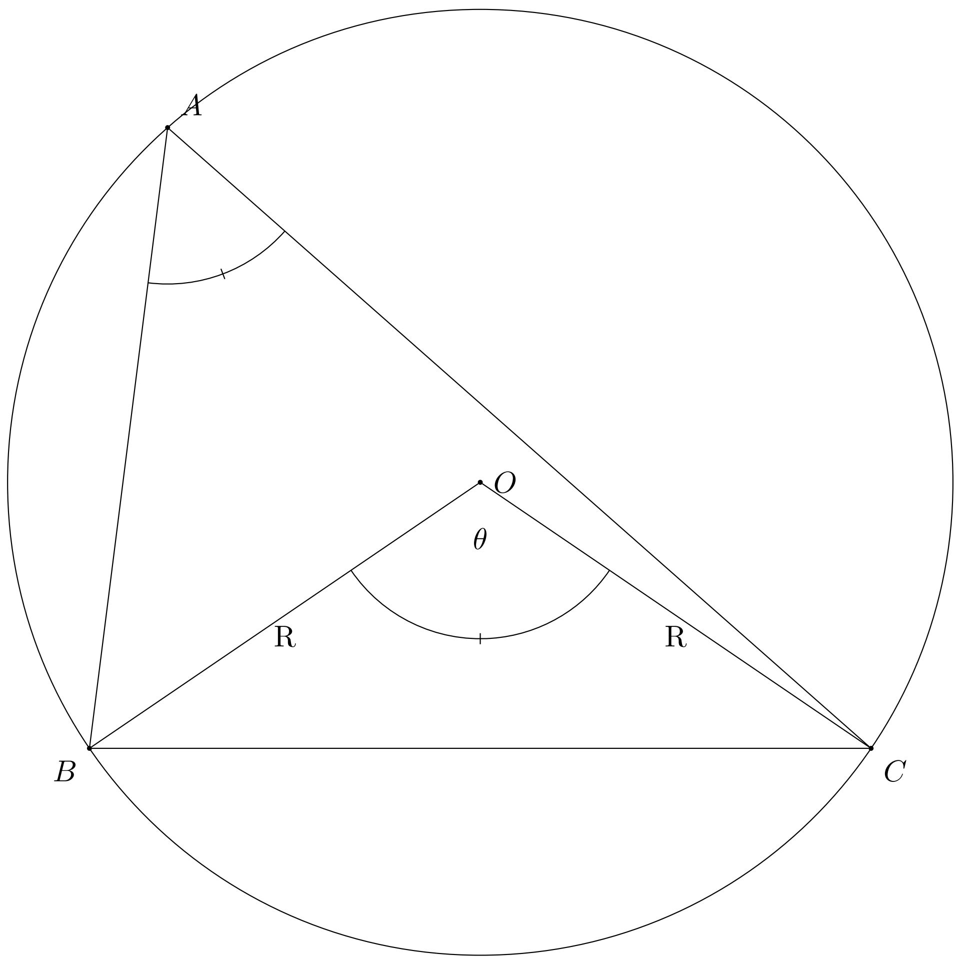

Figure B.7.2.1: Circumcircle of $\triangle ABC$

This is known as the circumcircle of $\triangle ABC$.

B.7.3. In Fig. B.3.2.1 show that

$$\cos A = \frac{(\mathbf{A}-\mathbf{B})^{\top}(\mathbf{A}-\mathbf{C})}{\|\mathbf{A}-\mathbf{B}\|\,\|\mathbf{A}-\mathbf{C}\|} \tag{B.7.3.1}$$

Solution: From (B.3.3.1), using (B.1.7.2),

$$\cos A = \frac{\|\mathbf{A} - \mathbf{B}\|^2 + \|\mathbf{A} - \mathbf{C}\|^2 - \|\mathbf{B} - \mathbf{C}\|^2}{2\|\mathbf{A} - \mathbf{B}\| \|\mathbf{A} - \mathbf{C}\|} \tag{B.7.3.2}$$

$$= \frac{\|\mathbf{A}\|^2 - \mathbf{A}^\top\mathbf{B} - \mathbf{A}^\top\mathbf{C} + \mathbf{B}^\top\mathbf{C}}{\|\mathbf{A} - \mathbf{B}\| \|\mathbf{A} - \mathbf{C}\|} \tag{B.7.3.3}$$

which can be expressed as (B.7.3.1).

B.7.4. Any point on the circle can be expressed as

$$\mathbf{x} = \mathbf{O} + R\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}, \quad 0 \in [0, 2\pi]. \tag{B.7.4.1}$$

B.7.5. Let

$$R = 1,\ \mathbf{O} = \mathbf{0},\ \mathbf{A} = \begin{pmatrix}\cos\theta_1 \\ \sin\theta_1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}\cos\theta_2 \\ \sin\theta_2\end{pmatrix}, \tag{B.7.5.1}$$

Show that

$$\|\mathbf{A} - \mathbf{B}\| = 2\sin\left(\frac{\theta_1 - \theta_2}{2}\right) \tag{B.7.5.2}$$

Solution: From (B.7.4.1).

$$\mathbf{A}-\mathbf{B} = \begin{pmatrix} \cos\theta_1 - \cos\theta_2 \\ \sin\theta_1 - \sin\theta_2 \end{pmatrix} \tag{B.7.5.3}$$

$$\implies \|\mathbf{A}-\mathbf{B}\|^2 = (\cos\theta_1 - \cos\theta_2)^2 + (\sin\theta_1 - \sin\theta_2)^2 \tag{B.7.5.4}$$

$$= 2\left\{1 - \cos(\theta_1-\theta_2)\right\} = 4\sin^2\left(\frac{\theta_1-\theta_2}{2}\right) \tag{B.7.5.5}$$

yielding (B.7.5.2) from (A.7.6.3).

B.7.6. In Fig. B.7.2.1, show that

$$\theta = 2A. \tag{B.7.6.1}$$

Solution: Let

$$\mathbf{C} = \begin{pmatrix} \cos\theta_3 \\ \sin\theta_3 \end{pmatrix} \tag{B.7.6.2}$$

Then, substituting from (B.7.5.2) in (B.7.3.2),

$$\cos A = \frac{4\sin^2\left(\frac{\theta_1-\theta_2}{2}\right) + 4\sin^2\left(\frac{\theta_1-\theta_3}{2}\right) - 4\sin^2\left(\frac{\theta_2-\theta_3}{2}\right)}{8\sin\left(\frac{\theta_1-\theta_2}{2}\right)\sin\left(\frac{\theta_1-\theta_3}{2}\right)} \tag{B.7.6.3}$$

$$= \frac{2\sin^2\left(\frac{\theta_1-\theta_2}{2}\right) + \cos(\theta_2-\theta_3) - \cos(\theta_1-\theta_3)}{4\sin\left(\frac{\theta_1-\theta_2}{2}\right)\sin\left(\frac{\theta_1-\theta_3}{2}\right)} \tag{B.7.6.4}$$

from (A.7.6.3). $\therefore$ from (A.7.5.4),

$$\cos A = \frac{2\sin^2\left(\frac{\theta_1-\theta_2}{2}\right) + 2\sin\left(\frac{\theta_1-\theta_2}{2}\right)\sin\left(\frac{\theta_1+\theta_2}{2} - \theta_3\right)}{4\sin\left(\frac{\theta_1-\theta_2}{2}\right)\sin\left(\frac{\theta_1-\theta_3}{2}\right)} \tag{B.7.6.5}$$

$$= \frac{\sin\left(\frac{\theta_1-\theta_2}{2}\right) + \sin\left(\frac{\theta_1+\theta_2}{2} - \theta_3\right)}{2\sin\left(\frac{\theta_1-\theta_3}{2}\right)} \tag{B.7.6.6}$$

From (A.7.5.1), the above equation can be expressed as

$$\cos A = \frac{2\sin\left(\frac{\theta_1-\theta_3}{2}\right)\cos\left(\frac{\theta_2-\theta_3}{2}\right)}{2\sin\left(\frac{\theta_1-\theta_3}{2}\right)} = \cos\left(\frac{\theta_2-\theta_3}{2}\right) \tag{B.7.6.7}$$

$$\implies 2A = \theta_2 - \theta_3 \tag{B.7.6.8}$$

Similarly,

$$\cos\theta = \frac{1 + 1 - 4\sin^2\left(\frac{\theta_2-\theta_3}{2}\right)}{2} = \cos\left(\theta_2 - \theta_3\right) = \cos 2A \tag{B.7.6.9}$$

B.8. Tangent

B.8.1. In Fig. B.8.1.1, OC is the radius and PC touches the circle at C. Show that

$$OC \perp PC. \tag{B.8.1.1}$$

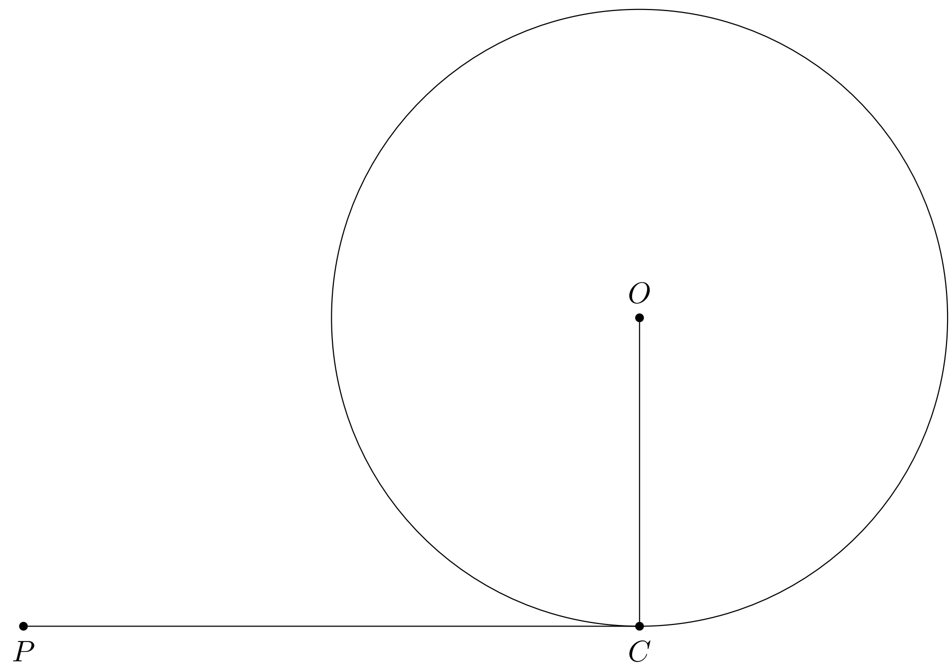

Figure B.8.1.1:

Solution: The equation of PC can be expressed as

$$\mathbf{x} = \mathbf{C} + \mu \mathbf{m} \tag{B.8.1.2}$$

and the equation of the circle is

$$\|\mathbf{x} - \mathbf{O}\| = R \tag{B.8.1.3}$$

Substituting (B.8.1.2) in (B.8.1.3),

$$\left\|\mathbf{C} + \mu\mathbf{m} - \mathbf{O}\right\|^2 = R^2 \quad \text{(B.8.1.4)}$$

$$\implies \mu^2 \left\|\mathbf{m}\right\|^2 + 2\mu\mathbf{m}^\top \left(\mathbf{C} - \mathbf{O}\right) + \left\|\mathbf{C} - \mathbf{O}\right\|^2 - R^2 = 0 \quad \text{(B.8.1.5)}$$

The above equation has only one root. Hence the discriminant of the above quadratic should be zero. So,

$$\left\{\mathbf{m}^\top \left(\mathbf{C} - \mathbf{O}\right)\right\}^2 - \left\|\mathbf{m}\right\|^2 \left\{\left\|\mathbf{C} - \mathbf{O}\right\|^2 - R^2\right\} = 0 \quad \text{(B.8.1.6)}$$

Since $\mathbf{C}$ is a point on the circle,

$$\left\|\mathbf{C} - \mathbf{O}\right\|^2 - R^2 = 0 \quad \text{(B.8.1.7)}$$

$$\implies \mathbf{m}^\top \left(\mathbf{C} - \mathbf{O}\right) = 0 \quad \text{(B.8.1.8)}$$

upon substituting in (B.8.1.6). Using the definition of the direction vector from (B.2.1.1)

$$\mathbf{m} = \mathbf{P} - \mathbf{C} \quad \text{(B.8.1.9)}$$

$$\implies \left(\mathbf{P} - \mathbf{C}\right)^\top \left(\mathbf{C} - \mathbf{O}\right) = 0 \quad \text{(B.8.1.10)}$$

which is equivalent to (B.8.1.1).

B.8.2. In Fig. B.8.2.1 show that

$$\theta = \alpha \quad \text{(B.8.2.1)}$$

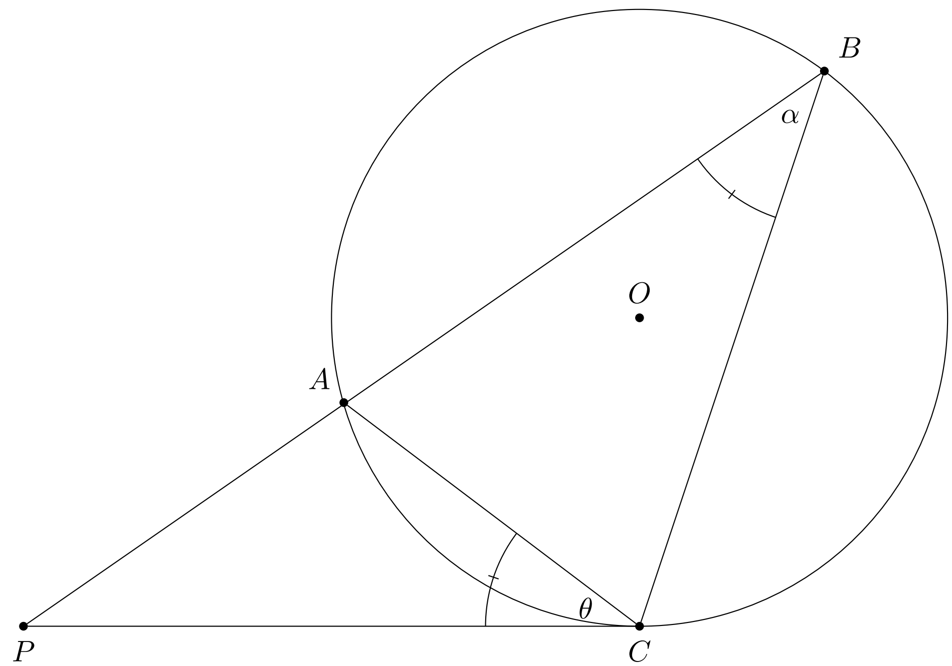

Figure B.8.2.1: $\theta = \alpha$.

Solution: Let Let

$$\mathbf{O} = \mathbf{0}\mathbf{A} = \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} \cos\theta_2 \\ \sin\theta_2 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} \cos\theta_3 \\ \sin\theta_3 \end{pmatrix} \tag{B.8.2.2}$$

Without loss of generality, let

$$\theta_3 = \frac{\pi}{2} \tag{B.8.2.3}$$

Then,

$$\mathbf{C} - \mathbf{O} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{B.8.2.4}$$

From from (B.8.1.10),

$$\mathbf{C} - \mathbf{P} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{B.8.2.5}$$

From (B.7.3.1) and (B.8.2.5),

$$\cos\theta = \frac{\begin{pmatrix} \cos\theta_3 - \cos\theta_1 & \sin\theta_3 - \sin\theta_1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}}{2\sin\left(\frac{\theta_1 - \theta_3}{2}\right)} \tag{B.8.2.6}$$

$$= \sin\left(\frac{\theta_1 + \theta_3}{2}\right) = \cos\left(\frac{\pi}{2} - \frac{\theta_1 + \theta_3}{2}\right) = \cos\left(\frac{\pi}{4} - \frac{\theta_1}{2}\right) \tag{B.8.2.7}$$

upon substituting from (B.8.2.3). Similarly, from (B.7.6.7),

$$\cos\alpha = \cos\left(\frac{\theta_1 - \theta_3}{2}\right) = \cos\left(\frac{\pi}{4} - \frac{\theta_1}{2}\right) = \cos\theta \tag{B.8.2.8}$$