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# GEOMETRY

## Through Algebra

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# Introduction

This book shows how to solve problems in geometry using trigonometry and coordinate geometry.



# Chapter 1

## Triangle

Consider a triangle with vertices

$$\mathbf{A} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -4 \\ 6 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -3 \\ -5 \end{pmatrix} \quad (1.1)$$

### 1.1. Vectors

1.1.1. The direction vector of  $AB$  is defined as

$$\mathbf{B} - \mathbf{A} \quad (1.1.1.1)$$

Find the direction vectors of  $AB$ ,  $BC$  and  $CA$ .

**Solution:**

(a) The Direction vector of  $AB$  is

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} -4 \\ 6 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -4 - 1 \\ 6 - (-1) \end{pmatrix} = \begin{pmatrix} -5 \\ 7 \end{pmatrix} \quad (1.1.1.2)$$



(b) The Direction vector of  $BC$  is

$$\mathbf{C} - \mathbf{B} = \begin{pmatrix} -3 \\ -5 \end{pmatrix} - \begin{pmatrix} -4 \\ 6 \end{pmatrix} = \begin{pmatrix} -3 - (-4) \\ -5 - 6 \end{pmatrix} = \begin{pmatrix} 1 \\ -11 \end{pmatrix} \quad (1.1.1.3)$$

(c) The Direction vector of  $CA$  is

$$\mathbf{A} - \mathbf{C} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \begin{pmatrix} -3 \\ -5 \end{pmatrix} = \begin{pmatrix} 1 - (-3) \\ -1 - (-5) \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \quad (1.1.1.4)$$

1.1.2. The length of side  $BC$  is

$$c = \|\mathbf{B} - \mathbf{A}\| \triangleq \sqrt{(\mathbf{B} - \mathbf{A})^\top (\mathbf{B} - \mathbf{A})} \quad (1.1.2.1)$$

where

$$\mathbf{A}^\top \triangleq \begin{pmatrix} 1 & -1 \end{pmatrix} \quad (1.1.2.2)$$

Similarly,

$$b = \|\mathbf{C} - \mathbf{B}\|, a = \|\mathbf{A} - \mathbf{C}\| \quad (1.1.2.3)$$

Find  $a, b, c$ .

(a) Since,

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 5 \\ -7 \end{pmatrix}, \quad (1.1.2.4)$$

$$c = \|\mathbf{A} - \mathbf{B}\| = \sqrt{\begin{pmatrix} 5 & -7 \end{pmatrix} \begin{pmatrix} 5 \\ -7 \end{pmatrix}} = \sqrt{(5)^2 + (7)^2} \quad (1.1.2.5)$$

$$= \sqrt{74} \quad (1.1.2.6)$$

(b) Similarly,

$$\mathbf{B} - \mathbf{C} = \begin{pmatrix} -1 \\ 11 \end{pmatrix} \quad (1.1.2.7)$$

$$\Rightarrow a = \|\mathbf{B} - \mathbf{C}\| = \sqrt{\begin{pmatrix} -1 & 11 \end{pmatrix} \begin{pmatrix} -1 \\ 11 \end{pmatrix}} = \sqrt{(1)^2 + (11)^2} \quad (1.1.2.8)$$

$$= \sqrt{122} \quad (1.1.2.9)$$

and

(c)

$$\mathbf{A} - \mathbf{C} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \quad (1.1.2.10)$$

$$\Rightarrow b = \|\mathbf{A} - \mathbf{C}\| = \sqrt{\begin{pmatrix} 4 & 4 \end{pmatrix} \begin{pmatrix} 4 \\ 4 \end{pmatrix}} = \sqrt{(4)^2 + (4)^2} \quad (1.1.2.11)$$

$$= \sqrt{32} \quad (1.1.2.12)$$

1.1.3. Points  $\mathbf{A}, \mathbf{B}, \mathbf{C}$  are defined to be collinear if

$$\text{rank} \begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} = 2 \quad (1.1.3.1)$$

Are the given points in (1.1) collinear?

**Solution:** From (1.1),

$$\begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -4 & -3 \\ -1 & 6 & -5 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 + R_2} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -4 & -3 \\ 0 & 2 & -8 \end{pmatrix} \quad (1.1.3.2)$$

$$\xleftrightarrow{R_2 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 5 & 4 \\ 0 & 2 & -8 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 - \frac{2}{5}R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 5 & 4 \\ 0 & 0 & \frac{-48}{5} \end{pmatrix} \quad (1.1.3.3)$$

There are no zero rows. So,

$$\text{rank} \begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} = 3 \quad (1.1.3.4)$$

Hence, the points  $\mathbf{A}, \mathbf{B}, \mathbf{C}$  are not collinear. This is visible in Fig. 1.1.

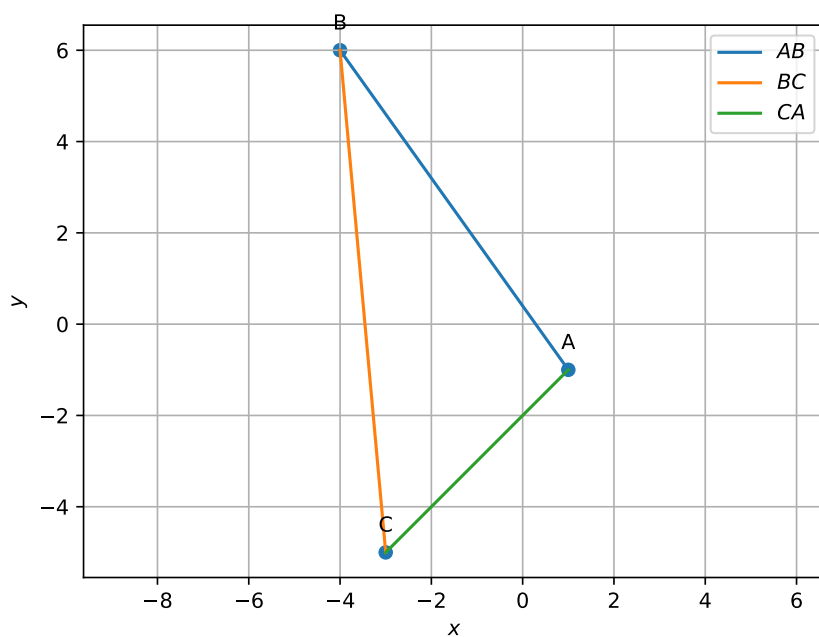


Figure 1.1:  $\triangle ABC$

1.1.4. The parametric form of the equation of  $AB$  is

$$\mathbf{x} = \mathbf{A} + k\mathbf{m} \quad (1.1.4.1)$$

where

$$\mathbf{m} = \mathbf{B} - \mathbf{A} \quad (1.1.4.2)$$

is the direction vector of  $AB$ . Find the parameteric equations of  $AB$ ,  $BC$  and  $CA$ .

**Solution:** From (1.1.4.1) and (1.1.1.2), the parametric equation for  $AB$  is given by

$$AB : \mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + k \begin{pmatrix} -5 \\ 7 \end{pmatrix} \quad (1.1.4.3)$$

Similarly, from (1.1.1.3) and (1.1.1.4),

$$BC : \mathbf{x} = \begin{pmatrix} -4 \\ 6 \end{pmatrix} + k \begin{pmatrix} 1 \\ -11 \end{pmatrix} \quad (1.1.4.4)$$

$$CA : \mathbf{x} = \begin{pmatrix} -3 \\ -5 \end{pmatrix} + k \begin{pmatrix} 4 \\ 4 \end{pmatrix} \quad (1.1.4.5)$$

1.1.5. The normal form of the equation of  $AB$  is

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{A}) = 0 \quad (1.1.5.1)$$

where

$$\mathbf{n}^\top \mathbf{m} = \mathbf{n}^\top (\mathbf{B} - \mathbf{A}) = 0 \quad (1.1.5.2)$$

$$\text{or, } \mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{m} \quad (1.1.5.3)$$

Find the normal form of the equations of  $AB$ ,  $BC$  and  $CA$ .

**Solution:**

(a) From (1.1.1.3), the direction vector of side  $\mathbf{BC}$  is

$$\mathbf{m} = \begin{pmatrix} 1 \\ -11 \end{pmatrix} \quad (1.1.5.4)$$

$$\Rightarrow \mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -11 \end{pmatrix} = \begin{pmatrix} -11 \\ -1 \end{pmatrix} \quad (1.1.5.5)$$

from (1.1.5.3). Hence, from (1.1.5.1), the normal equation of side  $BC$  is

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{B}) = 0 \quad (1.1.5.6)$$

$$\Rightarrow \begin{pmatrix} -11 & -1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -11 & -1 \end{pmatrix} \begin{pmatrix} -4 \\ 6 \end{pmatrix} \quad (1.1.5.7)$$

$$\Rightarrow BC : \begin{pmatrix} 11 & 1 \end{pmatrix} \mathbf{x} = -38 \quad (1.1.5.8)$$

(b) Similarly, for  $AB$ , from (1.1.1.2),

$$\mathbf{m} = \begin{pmatrix} -5 \\ 7 \end{pmatrix} \quad (1.1.5.9)$$

$$\Rightarrow \mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -5 \\ 7 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix} \quad (1.1.5.10)$$

and

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{A}) = 0 \quad (1.1.5.11)$$

$$\Rightarrow AB: \quad \mathbf{n}^\top \mathbf{x} = \begin{pmatrix} 7 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (1.1.5.12)$$

$$\Rightarrow \begin{pmatrix} 7 & 5 \end{pmatrix} \mathbf{x} = 2 \quad (1.1.5.13)$$

(c) For  $CA$ , from (1.1.1.4),

$$\mathbf{m} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (1.1.5.14)$$

$$\Rightarrow \mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (1.1.5.15)$$

$$(1.1.5.16)$$

$$\Rightarrow \mathbf{n}^\top (\mathbf{x} - \mathbf{C}) = 0 \quad (1.1.5.17)$$

$$\Rightarrow \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} -3 \\ -5 \end{pmatrix} = 2 \quad (1.1.5.18)$$

1.1.6. The area of  $\triangle ABC$  is defined as

$$\frac{1}{2} \|(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{C})\| \quad (1.1.6.1)$$

where

$$\mathbf{A} \times \mathbf{B} \triangleq \begin{vmatrix} 1 & -4 \\ -1 & 6 \end{vmatrix} \quad (1.1.6.2)$$

Find the area of  $\triangle ABC$ .

**Solution:** From (1.1.1.2) and (1.1.1.4),

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 5 \\ -7 \end{pmatrix}, \mathbf{A} - \mathbf{C} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \quad (1.1.6.3)$$

$$\Rightarrow (\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{C}) = \begin{vmatrix} 5 & 4 \\ -7 & 4 \end{vmatrix} \quad (1.1.6.4)$$

$$= 5 \times 4 - 4 \times (-7) \quad (1.1.6.5)$$

$$= 48 \quad (1.1.6.6)$$

$$\Rightarrow \frac{1}{2} \|(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{C})\| = \frac{48}{2} = 24 \quad (1.1.6.7)$$

which is the desired area.

1.1.7. Find the angles  $A, B, C$  if

$$\cos A \triangleq \frac{(\mathbf{B} - \mathbf{A})^\top \mathbf{C} - \mathbf{A}}{\|\mathbf{B} - \mathbf{A}\| \|\mathbf{C} - \mathbf{A}\|} \quad (1.1.7.1)$$



(a) From (1.1.1.2), (1.1.1.4), (1.1.2.6) and (1.1.2.12)

$$(\mathbf{B} - \mathbf{A})^\top (\mathbf{C} - \mathbf{A}) = \begin{pmatrix} -5 & 7 \end{pmatrix} \begin{pmatrix} -4 \\ -4 \end{pmatrix} \quad (1.1.7.2)$$

$$= -8 \quad (1.1.7.3)$$

$$\implies \cos A = \frac{-8}{\sqrt{74}\sqrt{32}} = \frac{-1}{\sqrt{37}} \quad (1.1.7.4)$$

$$\implies A = \cos^{-1} \frac{-1}{\sqrt{37}} \quad (1.1.7.5)$$

(b) From (1.1.1.2), (1.1.1.3), (1.1.2.6) and (1.1.2.9)

$$(\mathbf{C} - \mathbf{B})^\top (\mathbf{A} - \mathbf{B}) = \begin{pmatrix} 1 & -11 \end{pmatrix} \begin{pmatrix} 5 \\ -7 \end{pmatrix} \quad (1.1.7.6)$$

$$= 82 \quad (1.1.7.7)$$

$$\implies \cos B = \frac{82}{\sqrt{74}\sqrt{122}} = \frac{41}{\sqrt{2257}} \quad (1.1.7.8)$$

$$\implies B = \cos^{-1} \frac{41}{\sqrt{2257}} \quad (1.1.7.9)$$

(c) From (1.1.1.3), (1.1.1.4), (1.1.2.9) and (1.1.2.12)

$$(\mathbf{A} - \mathbf{C})^\top (\mathbf{B} - \mathbf{C}) = \begin{pmatrix} 4 & 4 \end{pmatrix} \begin{pmatrix} -1 \\ 11 \end{pmatrix} \quad (1.1.7.10)$$

$$= 40 \quad (1.1.7.11)$$

$$\implies \cos C = \frac{40}{\sqrt{32}\sqrt{122}} = \frac{5}{\sqrt{61}} \quad (1.1.7.12)$$

$$\implies C = \cos^{-1} \frac{5}{\sqrt{61}} \quad (1.1.7.13)$$

All codes for this section are available at

codes/triangle/sides.py

## 1.2. Median

1.2.1. If  $\mathbf{D}$  divides  $BC$  in the ratio  $k : 1$ ,

$$\mathbf{D} = \frac{k\mathbf{C} + \mathbf{B}}{k + 1} \quad (1.2.1.1)$$

Find the mid points  $\mathbf{D}, \mathbf{E}, \mathbf{F}$  of the sides  $BC, CA$  and  $AB$  respectively.

**Solution:** Since  $\mathbf{D}$  is the midpoint of  $BC$ ,

$$k = 1, \quad (1.2.1.2)$$

$$\implies \mathbf{D} = \frac{\mathbf{C} + \mathbf{B}}{2} = \frac{1}{2} \begin{pmatrix} -7 \\ 1 \end{pmatrix} \quad (1.2.1.3)$$

Similarly,

$$\mathbf{E} = \frac{\mathbf{A} + \mathbf{C}}{2} = \begin{pmatrix} -1 \\ -3 \end{pmatrix} \quad (1.2.1.4)$$

$$\mathbf{F} = \frac{\mathbf{A} + \mathbf{B}}{2} = \frac{1}{2} \begin{pmatrix} -3 \\ 5 \end{pmatrix} \quad (1.2.1.5)$$

1.2.2. Find the equations of  $AD$ ,  $BE$  and  $CF$ .

**Solution:** :

(a) The direction vector of  $AD$  is

$$\mathbf{m} = \mathbf{D} - \mathbf{A} = \begin{pmatrix} \frac{-7}{2} \\ \frac{1}{2} \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -9 \\ 3 \end{pmatrix} \equiv \begin{pmatrix} -3 \\ 1 \end{pmatrix} \quad (1.2.2.1)$$

$$\Rightarrow \mathbf{n} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad (1.2.2.2)$$

Hence the normal equation of median  $AD$  is

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{A}) = 0 \quad (1.2.2.3)$$

$$\Rightarrow \begin{pmatrix} 1 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -2 \quad (1.2.2.4)$$

(b) For  $BE$ ,

$$\mathbf{m} = \mathbf{E} - \mathbf{B} = \begin{pmatrix} -1 \\ -3 \end{pmatrix} - \begin{pmatrix} -4 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 \\ -9 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ -3 \end{pmatrix} \quad (1.2.2.5)$$

$$\Rightarrow \mathbf{n} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad (1.2.2.6)$$

Hence the normal equation of median  $BE$  is

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{B}) = 0 \quad (1.2.2.7)$$

$$\Rightarrow \begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 3 & 1 \end{pmatrix} \begin{pmatrix} -4 \\ 6 \end{pmatrix} = -6 \quad (1.2.2.8)$$

(c) For median  $CF$ ,

$$\mathbf{m} = \mathbf{F} - \mathbf{C} = \begin{pmatrix} \frac{-3}{2} \\ \frac{5}{2} \end{pmatrix} - \begin{pmatrix} -3 \\ -5 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ \frac{15}{2} \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 5 \end{pmatrix} \quad (1.2.2.9)$$

$$\Rightarrow \mathbf{n} = \begin{pmatrix} 5 \\ -1 \end{pmatrix} \quad (1.2.2.10)$$

Hence the normal equation of median  $CF$  is

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{C}) = 0 \quad (1.2.2.11)$$

$$\Rightarrow \begin{pmatrix} 5 & -1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 5 & -1 \end{pmatrix} \begin{pmatrix} -3 \\ -5 \end{pmatrix} = -10 \quad (1.2.2.12)$$

1.2.3. Find the intersection  $\mathbf{G}$  of  $BE$  and  $CF$ .

**Solution:** From (1.2.2.8) and (1.2.2.12), the equations of  $BE$  and  $CF$  are, respectively,

$$\begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -6 \end{pmatrix} \quad (1.2.3.1)$$

$$\begin{pmatrix} 5 & -1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -10 \end{pmatrix} \quad (1.2.3.2)$$

From (1.2.3.1) and (1.2.3.2) the augmented matrix is

$$\begin{pmatrix} 3 & 1 & -6 \\ 5 & -1 & -10 \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1 + R_2} \begin{pmatrix} 8 & 0 & -16 \\ 5 & -1 & -10 \end{pmatrix} \quad (1.2.3.3)$$

$$\xrightarrow{R_1 \leftarrow R_1/8} \begin{pmatrix} 1 & 0 & -2 \\ 5 & -1 & -10 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - 5R_1} \begin{pmatrix} 1 & 0 & -2 \\ 0 & -1 & 0 \end{pmatrix} \quad (1.2.3.4)$$

$$\xrightarrow{R_2 \leftarrow -R_2} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \end{pmatrix} \quad (1.2.3.5)$$

using Gauss elimination. Therefore,

$$\mathbf{G} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \quad (1.2.3.6)$$

1.2.4. Verify that

$$\frac{BG}{GE} = \frac{CG}{GF} = \frac{AG}{GD} = 2 \quad (1.2.4.1)$$

**Solution:**

(a) From (1.2.1.4) and (1.2.3.6),

$$\mathbf{G} - \mathbf{B} = \begin{pmatrix} 2 \\ -6 \end{pmatrix}, \mathbf{E} - \mathbf{G} = \begin{pmatrix} 1 \\ -3 \end{pmatrix} \quad (1.2.4.2)$$

$$\implies \mathbf{G} - \mathbf{B} = 2(\mathbf{E} - \mathbf{G}) \quad (1.2.4.3)$$

$$\implies \|\mathbf{G} - \mathbf{B}\| = 2\|\mathbf{E} - \mathbf{G}\| \quad (1.2.4.4)$$

$$\text{or, } \frac{BG}{GE} = 2 \quad (1.2.4.5)$$

(b) From (1.2.1.5) and (1.2.3.6),

$$\mathbf{F} - \mathbf{G} = \frac{1}{2} \begin{pmatrix} 1 \\ 5 \end{pmatrix}, \mathbf{G} - \mathbf{C} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \quad (1.2.4.6)$$

$$\implies \mathbf{G} - \mathbf{C} = 2(\mathbf{F} - \mathbf{G}) \quad (1.2.4.7)$$

$$\implies \|\mathbf{G} - \mathbf{C}\| = 2\|\mathbf{F} - \mathbf{G}\| \quad (1.2.4.8)$$

$$\text{or, } \frac{CG}{GF} = 2 \quad (1.2.4.9)$$

(c) From (1.2.1.3) and (1.2.3.6),

$$\mathbf{G} - \mathbf{A} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}, \mathbf{D} - \mathbf{G} = \frac{1}{2} \begin{pmatrix} -3 \\ 1 \end{pmatrix} \quad (1.2.4.10)$$

$$\mathbf{G} - \mathbf{A} = 2(\mathbf{D} - \mathbf{G}) \quad (1.2.4.11)$$

$$\implies \|\mathbf{G} - \mathbf{A}\| = 2\|\mathbf{D} - \mathbf{G}\| \quad (1.2.4.12)$$

$$\text{or, } \frac{AG}{GD} = 2 \quad (1.2.4.13)$$

From (1.2.4.5), (1.2.4.9), (1.2.4.13)

$$\frac{BG}{GE} = \frac{CG}{GF} = \frac{AG}{GD} = 2 \quad (1.2.4.14)$$

1.2.5. Show that **A**, **G** and **D** are collinear.

**Solution:** Points **A**, **D**, **G** are defined to be collinear if

$$\text{rank} \begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{D} & \mathbf{G} \end{pmatrix} = 2 \quad (1.2.5.1)$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 1 & -\frac{7}{2} & -2 \\ -1 & \frac{1}{2} & 0 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 + R_2} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -\frac{7}{2} & -2 \\ 0 & -3 & -2 \end{pmatrix} \quad (1.2.5.2)$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -\frac{9}{2} & -3 \\ 0 & -3 & -2 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 - \frac{2}{3}R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -\frac{9}{2} & -3 \\ 0 & 0 & 0 \end{pmatrix} \quad (1.2.5.3)$$

Thus, the matrix (1.2.5.1) has rank 2 and the points are collinear.

Thus, the medians of a triangle meet at the point **G**. See Fig. 1.2.

1.2.6. Verify that

$$\mathbf{G} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \quad (1.2.6.1)$$

**G** is known as the centroid of  $\triangle ABC$ .

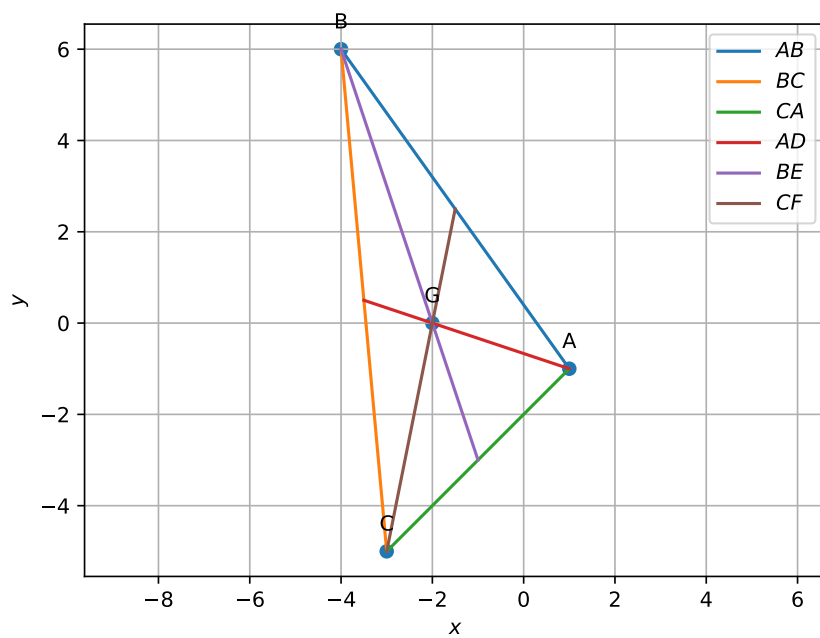


Figure 1.2: Medians of  $\triangle ABC$  meet at  $\mathbf{G}$ .



**Solution:**

$$\begin{aligned}\mathbf{G} &= \frac{\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -4 \\ 6 \end{pmatrix} + \begin{pmatrix} -3 \\ -5 \end{pmatrix}}{3} \\ &= \begin{pmatrix} -2 \\ 0 \end{pmatrix}\end{aligned}\tag{1.2.6.2}$$

1.2.7. Verify that

$$\mathbf{A} - \mathbf{F} = \mathbf{E} - \mathbf{D}\tag{1.2.7.1}$$

The quadrilateral  $AFDE$  is defined to be a parallelogram.

**Solution:**

$$\mathbf{A} - \mathbf{F} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \begin{pmatrix} \frac{-3}{2} \\ \frac{5}{2} \end{pmatrix} = \begin{pmatrix} \frac{5}{2} \\ \frac{-7}{2} \end{pmatrix}\tag{1.2.7.2}$$

$$\mathbf{E} - \mathbf{D} = \begin{pmatrix} -1 \\ -3 \end{pmatrix} - \begin{pmatrix} \frac{-7}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{5}{2} \\ \frac{-7}{2} \end{pmatrix}\tag{1.2.7.3}$$

$$\implies \mathbf{A} - \mathbf{F} = \mathbf{E} - \mathbf{D}\tag{1.2.7.4}$$

See Fig. 1.3,

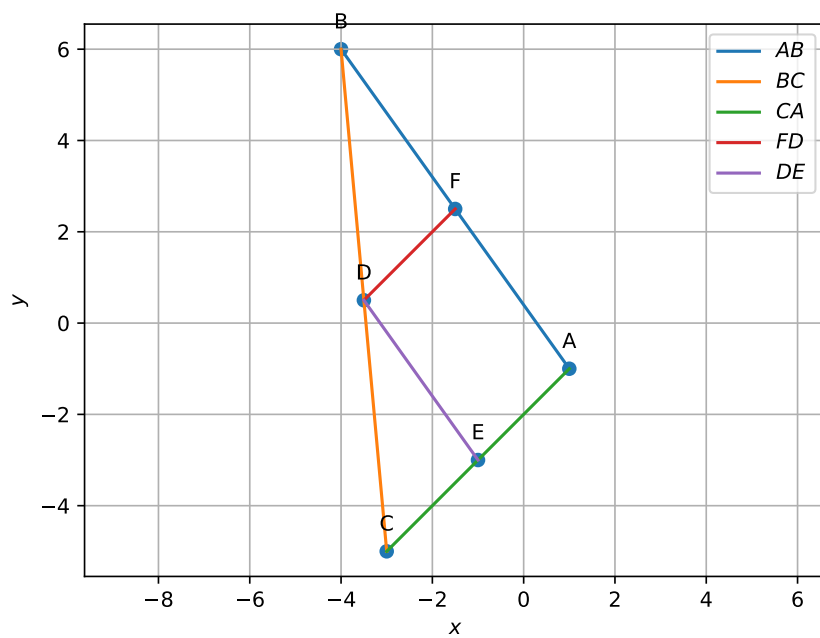


Figure 1.3:  $AFDE$  forms a parallelogram in triangle ABC

## 1.3. Altitude

1.3.1.  $D_1$  is a point on  $BC$  such that

$$AD_1 \perp BC \quad (1.3.1.1)$$

and  $AD_1$  is defined to be the altitude. Find the normal vector of  $AD_1$ .

**Solution:** The normal vector of  $AD_1$  is the direction vector  $BC$  and is obtained from (1.1.1.3) as

$$\mathbf{n} = \begin{pmatrix} 1 \\ -11 \end{pmatrix} \quad (1.3.1.2)$$

1.3.2. Find the equation of  $AD_1$ .

**Solution:** The equation of  $AD_1$  is

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{A}) = 0 \quad (1.3.2.1)$$

$$\Rightarrow \begin{pmatrix} -1 & 11 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -1 & 11 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -12 \quad (1.3.2.2)$$

1.3.3. Find the equations of the altitudes  $BE_1$  and  $CF_1$  to the sides  $AC$  and  $AB$  respectively.

**Solution:**

(a) From (1.1.1.4), the normal vector of  $CF_1$  is

$$\mathbf{n} = \begin{pmatrix} -5 \\ 7 \end{pmatrix} \quad (1.3.3.1)$$

and the equation of  $CF_1$  is

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{C}) = 0 \quad (1.3.3.2)$$

$$\Rightarrow \begin{pmatrix} -5 & 7 \end{pmatrix} \left( \mathbf{x} - \begin{pmatrix} -3 \\ -5 \end{pmatrix} \right) = 0 \quad (1.3.3.3)$$

$$\Rightarrow \begin{pmatrix} 5 & -7 \end{pmatrix} \mathbf{x} = 20, \quad (1.3.3.4)$$

(b) Similarly, from (1.1.1.2), the normal vector of  $BE_1$  is

$$\mathbf{n} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (1.3.3.5)$$

and the equation of  $BE_1$  is

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{B}) = 0 \quad (1.3.3.6)$$

$$\Rightarrow \begin{pmatrix} 1 & 1 \end{pmatrix} \left( \mathbf{x} - \begin{pmatrix} -4 \\ 6 \end{pmatrix} \right) = 0 \quad (1.3.3.7)$$

$$\Rightarrow \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 2, \quad (1.3.3.8)$$

1.3.4. Find the intersection  $\mathbf{H}$  of  $BE_1$  and  $CF_1$ .

**Solution:** The intersection of (1.3.3.8) and (1.3.3.4), is obtained from the matrix equation

$$\begin{pmatrix} 1 & 1 \\ 5 & -7 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 2 \\ 20 \end{pmatrix} \quad (1.3.4.1)$$

which can be solved as

$$\begin{pmatrix} 1 & 1 & 2 \\ 5 & -7 & 20 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - 5R_1} \begin{pmatrix} 1 & 1 & 2 \\ 0 & -12 & 10 \end{pmatrix} \quad (1.3.4.2)$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{-12}} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & \frac{-5}{6} \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & 0 & \frac{17}{6} \\ 0 & 1 & \frac{-5}{6} \end{pmatrix} \quad (1.3.4.3)$$

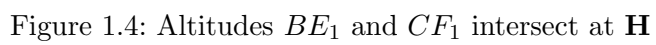
yielding

$$\mathbf{H} = \frac{1}{6} \begin{pmatrix} 17 \\ -5 \end{pmatrix}, \quad (1.3.4.4)$$

See Fig. 1.4

1.3.5. Verify that

$$(\mathbf{A} - \mathbf{H})^\top (\mathbf{B} - \mathbf{C}) = 0 \quad (1.3.5.1)$$


$$\mathbf{A} - \mathbf{H} = -\frac{1}{6} \begin{pmatrix} 11 \\ 1 \end{pmatrix}, \mathbf{B} - \mathbf{C} = \begin{pmatrix} -1 \\ 11 \end{pmatrix} \quad (1.3.5.2)$$

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## 1.4. Perpendicular Bisector

1.4.1. The equation of the perpendicular bisector of  $BC$  is

$$\left( \mathbf{x} - \frac{\mathbf{B} + \mathbf{C}}{2} \right) (\mathbf{B} - \mathbf{C}) = 0 \quad (1.4.1.1)$$

Substitute numerical values and find the equations of the perpendicular bisectors of  $AB$ ,  $BC$  and  $CA$ .

**Solution:** From (1.1.1.2), (1.1.1.3), (1.1.1.4), (1.2.1.3), (1.2.1.4) and (1.2.1.5),

$$\frac{\mathbf{B} + \mathbf{C}}{2} = \frac{1}{2} \begin{pmatrix} -7 \\ 1 \end{pmatrix}, \mathbf{B} - \mathbf{C} = \begin{pmatrix} -1 \\ 11 \end{pmatrix} \quad (1.4.1.2)$$

$$\frac{\mathbf{A} + \mathbf{B}}{2} = \frac{1}{2} \begin{pmatrix} -3 \\ 5 \end{pmatrix}, \mathbf{A} - \mathbf{B} = \begin{pmatrix} 5 \\ -7 \end{pmatrix} \quad (1.4.1.3)$$

$$\frac{\mathbf{C} + \mathbf{A}}{2} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}, \mathbf{C} - \mathbf{A} = \begin{pmatrix} -4 \\ -4 \end{pmatrix} \quad (1.4.1.4)$$

$$(1.4.1.5)$$

yielding

$$(\mathbf{B} - \mathbf{C})^\top \left( \frac{\mathbf{B} + \mathbf{C}}{2} \right) = \begin{pmatrix} -1 & 11 \end{pmatrix} \begin{pmatrix} -\frac{7}{2} \\ \frac{1}{2} \end{pmatrix} = 9 \quad (1.4.1.6)$$

$$(\mathbf{A} - \mathbf{B})^\top \left( \frac{\mathbf{A} + \mathbf{B}}{2} \right) = \begin{pmatrix} 5 & -7 \end{pmatrix} \begin{pmatrix} -\frac{3}{2} \\ \frac{5}{2} \end{pmatrix} = -25 \quad (1.4.1.7)$$

$$(\mathbf{C} - \mathbf{A})^\top \left( \frac{\mathbf{C} + \mathbf{A}}{2} \right) = \begin{pmatrix} -4 & -4 \end{pmatrix} \begin{pmatrix} -1 \\ -3 \end{pmatrix} = 16 \quad (1.4.1.8)$$

Thus, the perpendicular bisectors are obtained from (1.4.1.1) as

$$BC : \quad \begin{pmatrix} -1 & 11 \end{pmatrix} \mathbf{x} = 9 \quad (1.4.1.9)$$

$$CA : \quad \begin{pmatrix} 5 & -7 \end{pmatrix} \mathbf{x} = -25 \quad (1.4.1.10)$$

$$AB : \quad \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = -4 \quad (1.4.1.11)$$

1.4.2. Find the intersection  $\mathbf{O}$  of the perpendicular bisectors of  $AB$  and  $AC$ .

**Solution:**



The intersection of (1.4.1.10) and (1.4.1.11), can be obtained as

$$\begin{pmatrix} 5 & -7 & -25 \\ 1 & 1 & -4 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow 5R_2 - R_1} \begin{pmatrix} 5 & -7 & -25 \\ 0 & 12 & 5 \end{pmatrix} \quad (1.4.2.1)$$

$$\xleftrightarrow{R_1 \leftarrow \frac{12}{7}R_1 + R_2} \begin{pmatrix} \frac{60}{7} & 0 & -\frac{265}{7} \\ 0 & 12 & 5 \end{pmatrix} \xleftrightarrow[R_1 \leftarrow \frac{7}{60}R_1]{R_2 \leftarrow \frac{1}{12}R_2} \begin{pmatrix} 1 & 0 & -\frac{53}{12} \\ 0 & 1 & \frac{5}{12} \end{pmatrix} \quad (1.4.2.2)$$

$$\implies \mathbf{O} = \begin{pmatrix} -\frac{53}{12} \\ \frac{5}{12} \end{pmatrix} \quad (1.4.2.3)$$

1.4.3. Verify that  $\mathbf{O}$  satisfies (1.4.1.1).  $\mathbf{O}$  is known as the circumcentre.

**Solution:** Substituting from (1.4.2.3) in (1.4.1.1), when substituted in the above equation,

$$\begin{aligned} & \left( \mathbf{O} - \frac{\mathbf{B} + \mathbf{C}}{2} \right)^\top (\mathbf{B} - \mathbf{C}) \\ &= \left( \frac{1}{12} \begin{pmatrix} -53 \\ 5 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -7 \\ 1 \end{pmatrix} \right)^\top \begin{pmatrix} -1 \\ 11 \end{pmatrix} \\ &= \frac{1}{12} \begin{pmatrix} -11 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 11 \end{pmatrix} = 0 \quad (1.4.3.1) \end{aligned}$$

1.4.4. Verify that

$$OA = OB = OC \quad (1.4.4.1)$$

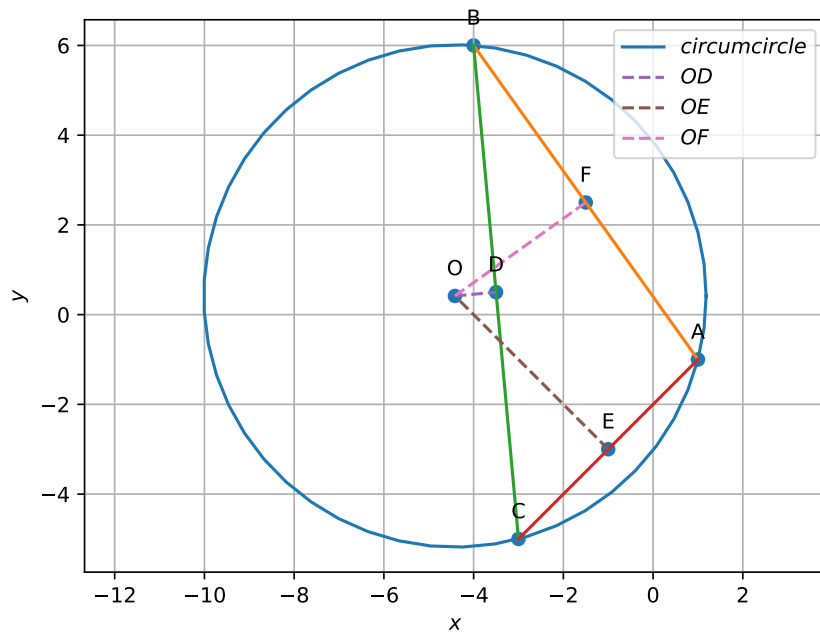


Figure 1.5: Circumcircle of  $\triangle ABC$  with centre  $\mathbf{O}$ .

1.4.5. Draw the circle with centre at  $\mathbf{O}$  and radius

$$R = OA \quad (1.4.5.1)$$

This is known as the circumradius.

**Solution:** See Fig. 1.5.

1.4.6. Verify that

$$\angle BOC = 2\angle BAC. \quad (1.4.6.1)$$

**Solution:**

(a) To find the value of  $\angle BOC$  :

$$\mathbf{B} - \mathbf{O} = \begin{pmatrix} \frac{5}{12} \\ \frac{67}{12} \end{pmatrix}, \mathbf{C} - \mathbf{O} = \begin{pmatrix} \frac{17}{12} \\ \frac{-65}{12} \end{pmatrix} \quad (1.4.6.2)$$

$$\Rightarrow (\mathbf{B} - \mathbf{O})^\top (\mathbf{C} - \mathbf{O}) = \frac{-4270}{144} \quad (1.4.6.3)$$

$$\Rightarrow \|\mathbf{B} - \mathbf{O}\| = \frac{\sqrt{4514}}{12}, \|\mathbf{C} - \mathbf{O}\| = \frac{\sqrt{4514}}{12} \quad (1.4.6.4)$$

Thus,

$$\cos BOC = \frac{(\mathbf{B} - \mathbf{O})^\top (\mathbf{C} - \mathbf{O})}{\|\mathbf{B} - \mathbf{O}\| \|\mathbf{C} - \mathbf{O}\|} = \frac{-4270}{4514} \quad (1.4.6.5)$$

$$\Rightarrow \angle BOC = \cos^{-1} \left( \frac{-4270}{4514} \right) \quad (1.4.6.6)$$

$$= 161.07536^\circ \text{ or } 198.92464^\circ \quad (1.4.6.7)$$

(b) To find the value of  $\angle BAC$  :

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} -5 \\ 7 \end{pmatrix}, \mathbf{C} - \mathbf{A} = \begin{pmatrix} -4 \\ -4 \end{pmatrix} \quad (1.4.6.8)$$

$$\Rightarrow (\mathbf{B} - \mathbf{A})^\top (\mathbf{C} - \mathbf{A}) = -8 \quad (1.4.6.9)$$

$$\|\mathbf{B} - \mathbf{A}\| = \sqrt{74}, \|\mathbf{C} - \mathbf{A}\| = 4\sqrt{2} \quad (1.4.6.10)$$

Thus,

$$\cos BAC = \frac{(\mathbf{B} - \mathbf{A})^\top (\mathbf{C} - \mathbf{A})}{\|\mathbf{B} - \mathbf{A}\| \|\mathbf{C} - \mathbf{A}\|} = \frac{-8}{4\sqrt{148}} \quad (1.4.6.11)$$

$$\implies \angle BAC = \cos^{-1} \left( \frac{-8}{4\sqrt{148}} \right) \quad (1.4.6.12)$$

$$= 99.46232^\circ \quad (1.4.6.13)$$

From (1.4.6.13) and (1.4.6.7),

$$2 \times \angle BAC = \angle BOC \quad (1.4.6.14)$$

1.4.7. Let

$$\mathbf{P} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \quad (1.4.7.1)$$

Find  $\theta$  if

$$\mathbf{C} - \mathbf{O} = \mathbf{P} (\mathbf{A} - \mathbf{O}) \quad (1.4.7.2)$$

## 1.5. Angle Bisector

1.5.1. Let  $\mathbf{D}_3, \mathbf{E}_3, \mathbf{F}_3$ , be points on  $AB, BC$  and  $CA$  respectively such that

$$BD_3 = BF_3 = m, CD_3 = CE_3 = n, AE_3 = AF_3 = p. \quad (1.5.1.1)$$

Obtain  $m, n, p$  in terms of  $a, b, c$  obtained in Problem 1.1.2.

**Solution:** From the given information,

$$a = m + n, \quad (1.5.1.2)$$

$$b = n + p, \quad (1.5.1.3)$$

$$c = m + p \quad (1.5.1.4)$$

which can be expressed as

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} m \\ n \\ p \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (1.5.1.5)$$

$$\Rightarrow \begin{pmatrix} m \\ n \\ p \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (1.5.1.6)$$

Using row reduction,

$$\left( \begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array} \right) \xleftrightarrow{R_3 \leftarrow R_3 - R_1} \left( \begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array} \right) \quad (1.5.1.7)$$

$$\xleftrightarrow[R_1 \leftarrow R_1 - R_2]{R_3 \leftarrow R_3 + R_2} \left( \begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array} \right) \quad (1.5.1.8)$$

$$\xleftrightarrow[R_1 \leftarrow 2R_1 + R_3]{R_2 \leftarrow 2R_2 - R_3} \left( \begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & -1 & 1 \\ 0 & 2 & 0 & 1 & 1 & -1 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array} \right) \quad (1.5.1.9)$$

yielding

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix} \quad (1.5.1.10)$$

Therefore,

$$\begin{aligned} p &= \frac{c+b-a}{2} = \frac{\sqrt{74} + \sqrt{32} - \sqrt{122}}{2} \\ m &= \frac{a+c-b}{2} = \frac{\sqrt{74} + \sqrt{122} - \sqrt{32}}{2} \\ n &= \frac{a+b-c}{2} = \frac{\sqrt{122} + \sqrt{32} - \sqrt{74}}{2} \end{aligned} \quad (1.5.1.11)$$

upon substituting from (1.1.2.6), (1.1.2.9) and (1.1.2.12).

1.5.2. Using section formula, find

$$\mathbf{D}_3 = \frac{m\mathbf{C} + n\mathbf{B}}{m+n}, \mathbf{E}_3 = \frac{n\mathbf{A} + p\mathbf{C}}{n+p}, \mathbf{F}_3 = \frac{p\mathbf{B} + m\mathbf{A}}{p+m} \quad (1.5.2.1)$$

1.5.3. Find the circumcentre and circumradius of  $\triangle D_3E_3F_3$ . These are the incentre and inradius of  $\triangle ABC$ .

1.5.4. Draw the circumcircle of  $\triangle D_3E_3F_3$ . This is known as the incircle of  $\triangle ABC$ .

**Solution:** See Fig. 1.6

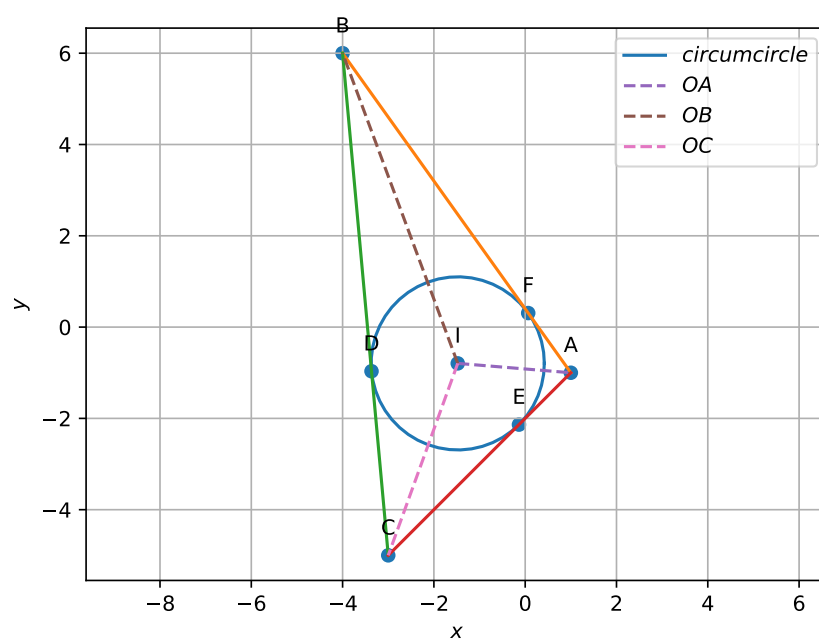


Figure 1.6: Incircle of  $\triangle ABC$

1.5.5. Using (1.1.7.1) verify that

$$\angle BAI = \angle CAI. \quad (1.5.5.1)$$

$AI$  is the bisector of  $\angle A$ .

1.5.6. Verify that  $BI, CI$  are also the angle bisectors of  $\triangle ABC$ .

## 1.6. Eigenvalues and Eigenvectors

The equation of the incircle is given by

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \quad (6.1)$$

where

$$\mathbf{V} = \mathbf{I}, \mathbf{u} = -\mathbf{O}, f = \|\mathbf{O}\|^2 - r^2, \quad (6.2)$$

$\mathbf{O}$  being the incentre and  $r$  the inradius. Here  $\mathbf{I}$  is the identity matrix.

1.6.1. Compute

$$\Sigma = (\mathbf{V}\mathbf{h} + \mathbf{u})(\mathbf{V}\mathbf{h} + \mathbf{u})^\top - g(\mathbf{h})\mathbf{V} \quad (1.6.1.1)$$

for  $\mathbf{h} = \mathbf{A}$ .

1.6.2. Find the roots of the equation

$$\left| \lambda \mathbf{I} - \Sigma \right| = 0 \quad (1.6.2.1)$$

These are known as the eigenvalues of  $\Sigma$ .

1.6.3. Find  $\mathbf{p}$  such that

$$\Sigma \mathbf{p} = \lambda \mathbf{p} \quad (1.6.3.1)$$

using row reduction. These are known as the eigenvectors of  $\Sigma$ .



1.6.4. Define

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad (1.6.4.1)$$

$$\mathbf{P} = \begin{pmatrix} \frac{\mathbf{p}_1}{\|\mathbf{p}_1\|} & \frac{\mathbf{p}_2}{\|\mathbf{p}_2\|} \end{pmatrix} \quad (1.6.4.2)$$

1.6.5. Verify that

$$\mathbf{P}^\top = \mathbf{P}^{-1}. \quad (1.6.5.1)$$

$\mathbf{P}$  is defined to be an orthogonal matrix.

1.6.6. Verify that

$$\mathbf{P}^\top \Sigma \mathbf{P} = \mathbf{D}, \quad (1.6.6.1)$$

This is known as the spectral (eigenvalue ) decomposition of a symmetric matrix

1.6.7. The direction vectors of the tangents from a point  $\mathbf{h}$  to the circle in (6.1) are given by

$$\mathbf{m} = \mathbf{P} \begin{pmatrix} \sqrt{|\lambda_2|} \\ \pm \sqrt{|\lambda_1|} \end{pmatrix} \quad (1.6.7.1)$$

1.6.8. The points of contact of the pair of tangents to the circle in (6.1) from

a point  $\mathbf{h}$  are given by

$$\mathbf{x} = \mathbf{h} + \mu \mathbf{m} \quad (1.6.8.1)$$

where

$$\mu = -\frac{\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u})}{\mathbf{m}^\top \mathbf{V}\mathbf{m}} \quad (1.6.8.2)$$

for  $\mathbf{m}$  in (1.6.7.1). Compute the points of contact. You should get the same points that you obtained in the previous section.

## 1.7. Matrices

The matrix of the vertices of the triangle is defined as

$$\mathbf{P} = \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \quad (8.3)$$

### 1.7.1. Vectors

1.7.1.1. Obtain the direction matrix of the sides of  $\triangle ABC$  defined as

$$\mathbf{M} = \begin{pmatrix} \mathbf{A} - \mathbf{B} & \mathbf{B} - \mathbf{C} & \mathbf{C} - \mathbf{A} \end{pmatrix} \quad (1.7.1.1.1)$$

**Solution:**

$$\mathbf{M} = \begin{pmatrix} \mathbf{A} - \mathbf{B} & \mathbf{B} - \mathbf{C} & \mathbf{C} - \mathbf{A} \end{pmatrix} \quad (1.7.1.1.2)$$

$$= \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \quad (1.7.1.1.3)$$

where the second matrix above is known as a circulant matrix. Note that the 2nd and 3rd row of the above matrix are circular shifts of the 1st row.

1.7.1.2. Obtain the normal matrix of the sides of  $\triangle ABC$

**Solution:** Considering the rotation matrix

$$\mathbf{R} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad (1.7.1.2.1)$$

the normal matrix is obtained as

$$\mathbf{N} = \mathbf{R}\mathbf{M} \quad (1.7.1.2.2)$$

1.7.1.3. Obtain  $a, b, c$ .

**Solution:** The sides vector is obtained as

$$\mathbf{d} = \sqrt{\text{diag}(\mathbf{M}^\top \mathbf{M})} \quad (1.7.1.3.1)$$

1.7.1.4. Obtain the constant terms in the equations of the sides of the triangle.

**Solution:** The constants for the lines can be expressed in vector form as

$$\mathbf{c} = \text{diag} \left\{ \left( \mathbf{N}^\top \mathbf{P} \right) \right\} \quad (1.7.1.4.1)$$

## 1.7.2. Median

1.7.2.1. Obtain the mid point matrix for the sides of the triangle

**Solution:**

$$\begin{pmatrix} \mathbf{D} & \mathbf{E} & \mathbf{F} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad (1.7.2.1.1)$$

1.7.2.2. Obtain the median direction matrix.

**Solution:** The median direction matrix is given by

$$\mathbf{M}_1 = \begin{pmatrix} \mathbf{A} - \mathbf{D} & \mathbf{B} - \mathbf{E} & \mathbf{C} - \mathbf{F} \end{pmatrix} \quad (1.7.2.2.1)$$

$$= \begin{pmatrix} \mathbf{A} - \frac{\mathbf{B} + \mathbf{C}}{2} & \mathbf{B} - \frac{\mathbf{C} + \mathbf{A}}{2} & \mathbf{C} - \frac{\mathbf{A} + \mathbf{B}}{2} \end{pmatrix} \quad (1.7.2.2.2)$$

$$= \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix} \quad (1.7.2.2.3)$$

1.7.2.3. Obtain the median normal matrix.

1.7.2.4. Obtain the median equation constants.

1.7.2.5. Obtain the centroid by finding the intersection of the medians.

### 1.7.3. Altitude

1.7.3.1. Find the normal matrix for the altitudes

**Solution:** The desired matrix is

$$\mathbf{M}_2 = \begin{pmatrix} \mathbf{B} - \mathbf{C} & \mathbf{C} - \mathbf{A} & \mathbf{A} - \mathbf{B} \end{pmatrix} \quad (1.7.3.1.1)$$

$$= \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \quad (1.7.3.1.2)$$

1.7.3.2. Find the constants vector for the altitudes.

**Solution:** The desired vector is

$$\mathbf{c}_2 = \text{diag} \left\{ \left( \mathbf{M}^\top \mathbf{P} \right) \right\} \quad (1.7.3.2.1)$$

### 1.7.4. Perpendicular Bisector

1.7.4.1. Find the normal matrix for the perpendicular bisectors

**Solution:** The normal matrix is  $\mathbf{M}_2$

1.7.4.2. Find the constants vector for the perpendicular bisectors.

**Solution:** The desired vector is

$$\mathbf{c}_3 = \text{diag} \left\{ \mathbf{M}_2^\top \begin{pmatrix} \mathbf{D} & \mathbf{E} & \mathbf{F} \end{pmatrix} \right\} \quad (1.7.4.2.1)$$

## 1.7.5. Angle Bisector

1.7.5.1. Find the points of contact.

**Solution:** The points of contact are given by

$$\begin{pmatrix} \frac{m\mathbf{C}+n\mathbf{B}}{m+n} & \frac{n\mathbf{A}+p\mathbf{C}}{n+p} & \frac{p\mathbf{B}+m\mathbf{A}}{p+m} \end{pmatrix} = \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \begin{pmatrix} 0 & \frac{n}{b} & \frac{m}{c} \\ \frac{n}{a} & 0 & \frac{p}{c} \\ \frac{m}{a} & \frac{p}{b} & 0 \end{pmatrix} \quad (1.7.5.1.1)$$



## Appendix A

# Trigonometry

### A.1. Ratios

A right angled triangle looks like Fig. A.1. with angles  $\angle A$ ,  $\angle B$  and  $\angle C$  and



Figure A.1: Right Angled Triangle



sides  $a, b$  and  $c$ . The unique feature of this triangle is  $\angle B$  which is defined to be  $90^\circ$ .

A.1.1. For simplicity, let the greek letter  $\theta = \angle C$ . We have the following definitions.

$$\begin{aligned}\sin \theta &= \frac{c}{b} & \cos \theta &= \frac{a}{b} \\ \tan \theta &= \frac{c}{a} & \cot \theta &= \frac{1}{\tan \theta} \\ \csc \theta &= \frac{1}{\sin \theta} & \sec \theta &= \frac{1}{\cos \theta}\end{aligned}\tag{A.1.1.1}$$

A.1.2. Show that

$$\cos \theta = \sin (90^\circ - \theta)\tag{A.1.2.1}$$

**Solution:** From (A.1.1.1),

$$\cos \angle BAC = \cos \alpha = \cos (90^\circ - \theta) = \frac{c}{b} = \sin \angle ABC = \sin \theta\tag{A.1.2.2}$$

## A.2. The Baudhayana Theorem

Use Fig. A.2 for all problems in this section.

A.2.1. Show that

$$b = a \cos \theta + c \sin \theta\tag{A.2.1.1}$$

**Solution:** We observe that

$$BD = a \cos \theta\tag{A.2.1.2}$$

$$AD = c \cos \alpha = c \sin \theta \quad (\text{From (A.1.2.2)})\tag{A.2.1.3}$$

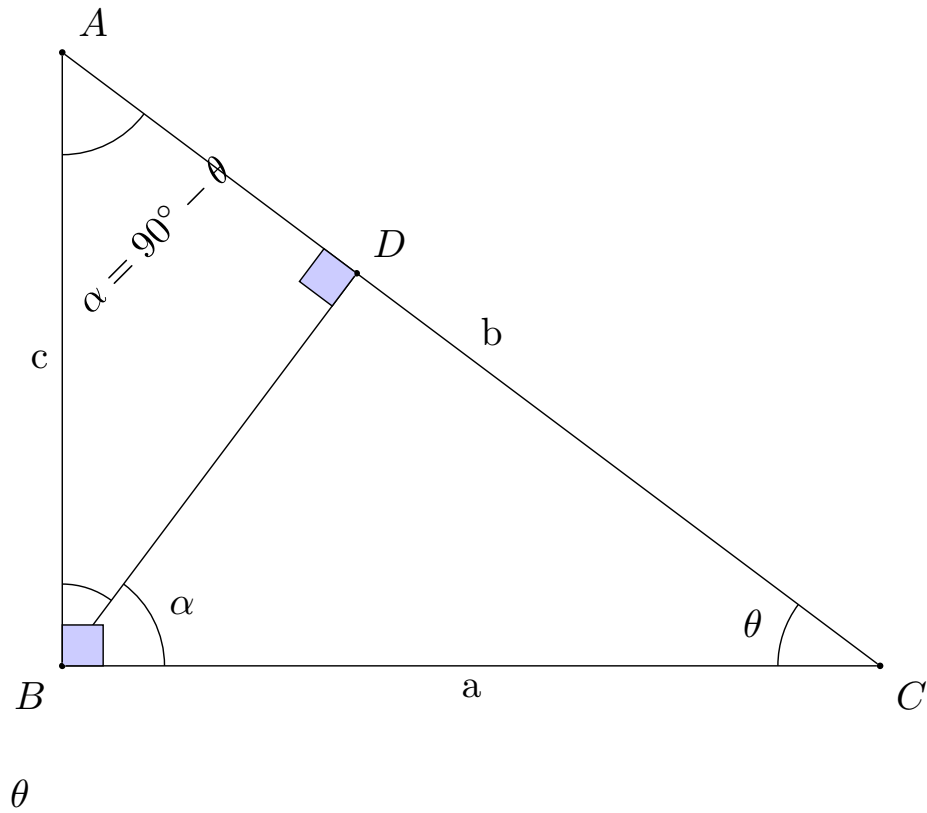


Figure A.2: Baudhayana Theorem

Thus,

$$BD + AD = b = a \cos \theta + c \sin \theta \quad (\text{A.2.1.4})$$

A.2.2. From (A.2.1.1), show that

$$\sin^2 \theta + \cos^2 \theta = 1 \quad (\text{A.2.2.1})$$

**Solution:** Dividing both sides of (A.2.1.1) by  $b$ ,

$$1 = \frac{a}{b} \cos \theta + \frac{c}{b} \sin \theta \quad (\text{A.2.2.2})$$

$$\Rightarrow \sin^2 \theta + \cos^2 \theta = 1 \quad (\text{from (A.1.1.1)}) \quad (\text{A.2.2.3})$$

A.2.3. In a right angled triangle, the hypotenuse is the longest side.

**Solution:** From (A.2.2.1),

$$0 \leq \sin \theta, \cos \theta \leq 1 \quad (\text{A.2.3.1})$$

Hence,

$$b \sin \theta \leq b \implies c \leq b \quad (\text{A.2.3.2})$$

Similalry,

$$a \leq b \quad (\text{A.2.3.3})$$

A.2.4. Using (A.2.1.1), show that

$$b^2 = a^2 + c^2 \quad (\text{A.2.4.1})$$

(A.2.4.1) is known as the Baudhayana theorem. It is also known as the Pythagoras theorem.

**Solution:** From (A.2.1.1),

$$b = a \frac{a}{b} + c \frac{c}{b} \quad (\text{from (A.1.1.1)}) \quad (\text{A.2.4.2})$$

$$\implies b^2 = a^2 + c^2 \quad (\text{A.2.4.3})$$

## A.3. Area of a Triangle

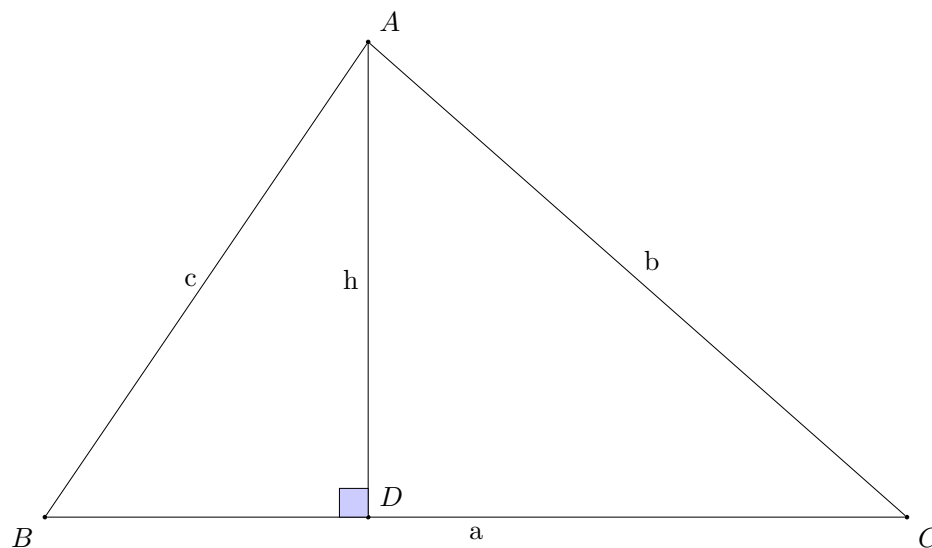


Figure A.3: Area of a Triangle

A.3.1. Show that the area of  $\triangle ABC$  in Fig. A.3 is  $\frac{1}{2}ab \sin C$ .

**Solution:** We have

$$\text{ar}(\triangle ABC) = \frac{1}{2}ah = \frac{1}{2}ab \sin C \quad (\because h = b \sin C). \quad (\text{A.3.1.1})$$

A.3.2. Show that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad (\text{A.3.2.1})$$

**Solution:** Fig. A.3 can be suitably modified to obtain

$$ar(\triangle ABC) = \frac{1}{2}ab \sin C = \frac{1}{2}bc \sin A = \frac{1}{2}ca \sin B \quad (\text{A.3.2.2})$$

Dividing the above by  $abc$ , we obtain

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad (\text{A.3.2.3})$$

This is known as the sine formula.

A.3.3. Show that

$$\alpha > \beta \implies \sin \alpha > \sin \beta \quad (\text{A.3.3.1})$$

**Solution:** In Fig. A.4,

$$ar(\triangle ABD) < ar(\triangle ABC) \quad (\text{A.3.3.2})$$

$$\implies \frac{1}{2}lc \sin \theta_1 < \frac{1}{2}ac \sin(\theta_1 + \theta_2) \quad (\text{A.3.3.3})$$

$$\implies \frac{l}{a} < \frac{\sin(\theta_1 + \theta_2)}{\sin \theta_1} \quad (\text{A.3.3.4})$$

$$\text{or, } 1 < \frac{l}{a} < \frac{\sin(\theta_1 + \theta_2)}{\sin \theta_1} \quad (\text{A.3.3.5})$$

$$\implies \frac{\sin(\theta_1 + \theta_2)}{\sin \theta_1} > 1 \quad (\text{A.3.3.6})$$

from Theorem A.2.3. This proves (A.3.3.1).

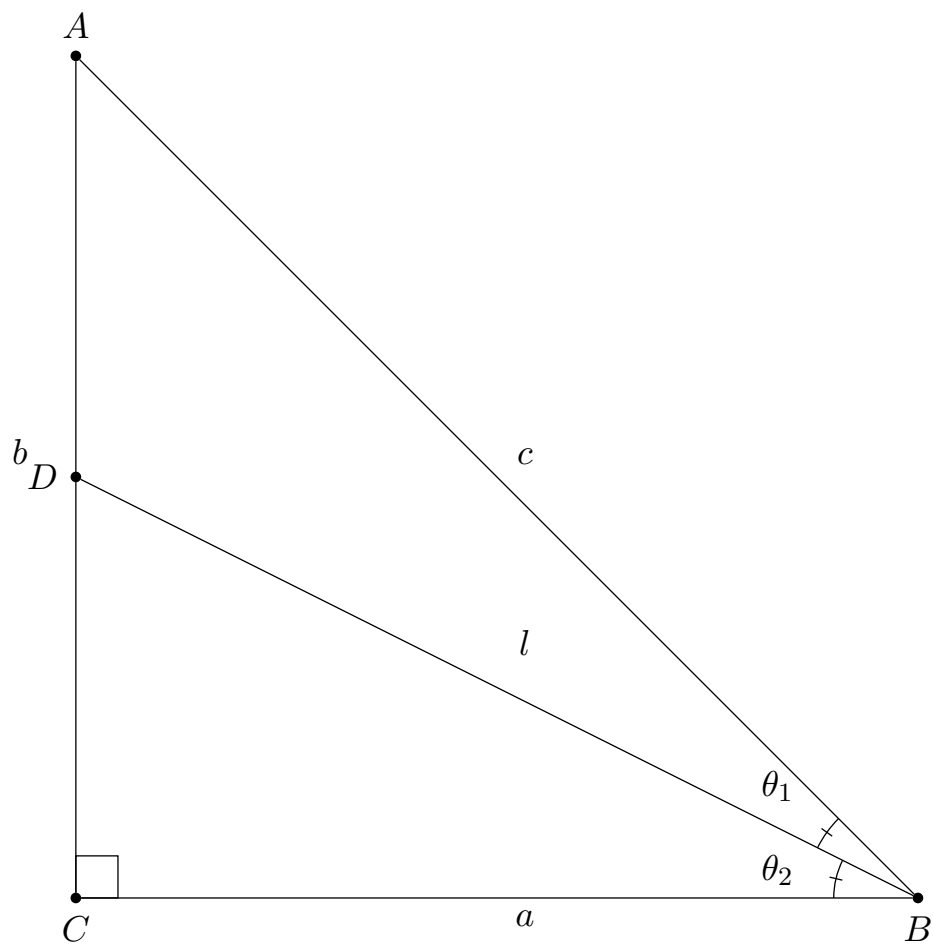


Figure A.4:

A.3.4. Using Fig. A.4, show that

$$\sin \theta_1 = \sin (\theta_1 + \theta_2) \cos \theta_2 - \cos (\theta_1 + \theta_2) \sin \theta_2 \quad (\text{A.3.4.1})$$

**Solution:** The following equations can be obtained from the figure

using the formula for the area of a triangle

$$ar(\Delta ABC) = \frac{1}{2}ac \sin(\theta_1 + \theta_2) \quad (\text{A.3.4.2})$$

$$= ar(\Delta BDC) + ar(\Delta ADB) \quad (\text{A.3.4.3})$$

$$= \frac{1}{2}cl \sin \theta_1 + \frac{1}{2}al \sin \theta_2 \quad (\text{A.3.4.4})$$

$$= \frac{1}{2}ac \sin \theta_1 \sec \theta_2 + \frac{1}{2}a^2 \tan \theta_2 \quad (\text{A.3.4.5})$$

( $\because l = a \sec \theta_2$ ). From the above,

$$\sin(\theta_1 + \theta_2) = \sin \theta_1 \sec \theta_2 + \frac{a}{c} \tan \theta_2 \quad (\text{A.3.4.6})$$

$$= \sin \theta_1 \sec \theta_2 + \cos(\theta_1 + \theta_2) \tan \theta_2 \quad (\text{A.3.4.7})$$

Multiplying both sides by  $\cos \theta_2$ ,

$$\sin(\theta_1 + \theta_2) \cos \theta_2 = \sin \theta_1 + \cos(\theta_1 + \theta_2) \sin \theta_2 \quad (\text{A.3.4.8})$$

resulting in (A.3.4.1).

A.3.5. Find Hero's formula for the area of a triangle.

**Solution:** From (A.3.1), the area of  $\triangle ABC$  is

$$\frac{1}{2}ab \sin C = \frac{1}{2}ab \sqrt{1 - \cos^2 C} \quad (\text{from (A.2.2.1)}) \quad (\text{A.3.5.1})$$

$$= \frac{1}{2}ab \sqrt{1 - \left( \frac{a^2 + b^2 - c^2}{2ab} \right)^2} \quad (\text{from (B.3.3.1)}) \quad (\text{A.3.5.2})$$

$$= \frac{1}{4} \sqrt{(2ab)^2 - (a^2 + b^2 - c^2)^2} \quad (\text{A.3.5.3})$$

$$= \frac{1}{4} \sqrt{(2ab + a^2 + b^2 - c^2)(2ab - a^2 - b^2 + c^2)} \quad (\text{A.3.5.4})$$

$$= \frac{1}{4} \sqrt{\{(a+b)^2 - c^2\} \{c^2 - (a-b)^2\}} \quad (\text{A.3.5.5})$$

$$= \frac{1}{4} \sqrt{(a+b+c)(a+b-c)(a+c-b)(b+c-a)} \quad (\text{A.3.5.6})$$

Substituting

$$s = \frac{a+b+c}{2} \quad (\text{A.3.5.7})$$

in (A.3.5.6), the area of  $\triangle ABC$  is

$$\sqrt{s(s-a)(s-b)(s-c)} \quad (\text{A.3.5.8})$$

This is known as Hero's formula.

## A.4. Angle Bisectors

A.4.1. In Fig. A.4.1.1, the bisectors of  $\angle B$  and  $\angle C$  meet at **I**. Show that  $IA$  bisects  $\angle A$ .



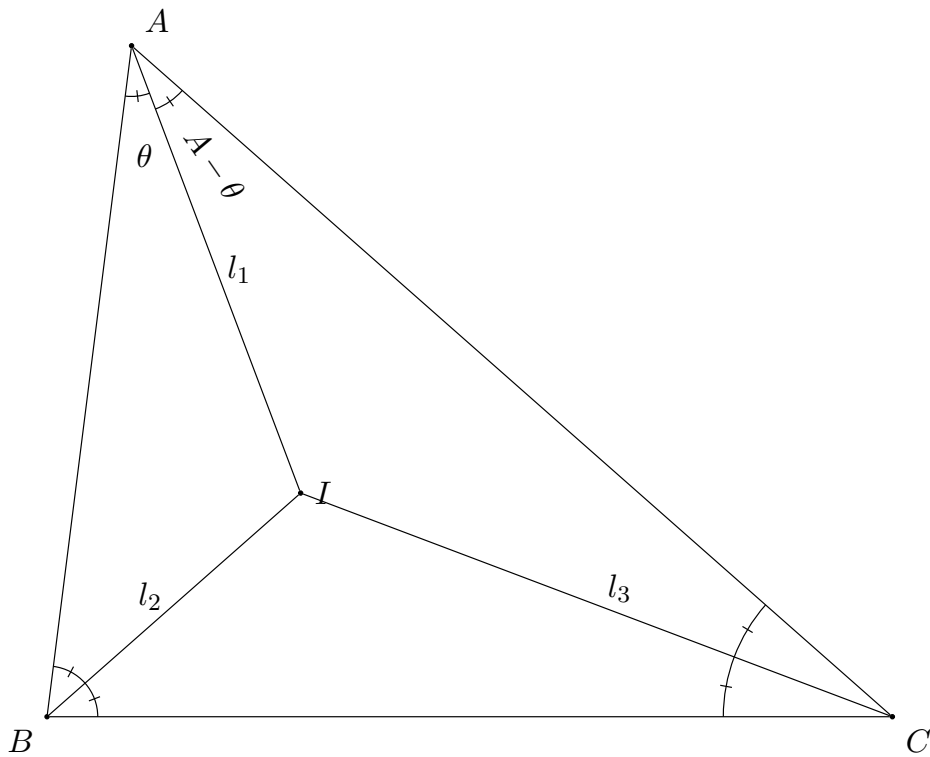


Figure A.4.1.1: Incentre  $I$  of  $\triangle ABC$

**Solution:** Using sine formula in (A.3.2.3)

$$\frac{l_1}{\sin \frac{C}{2}} = \frac{l_3}{\sin (A - \theta)} \quad (\text{A.4.1.1})$$

$$\frac{l_3}{\sin \frac{B}{2}} = \frac{l_2}{\sin \frac{C}{2}} \quad (\text{A.4.1.2})$$

$$\frac{l_1}{\sin \frac{B}{2}} = \frac{l_2}{\sin \theta} \quad (\text{A.4.1.3})$$

Multiplying the above equations,

$$\sin \theta = \sin (A - \theta) \implies \theta = \frac{A}{2} \quad (\text{A.4.1.4})$$

A.4.2. In Fig. A.4.2.1,

$$ID \perp BC, IE \perp AC, IF \perp AB. \quad (\text{A.4.2.1})$$

Show that

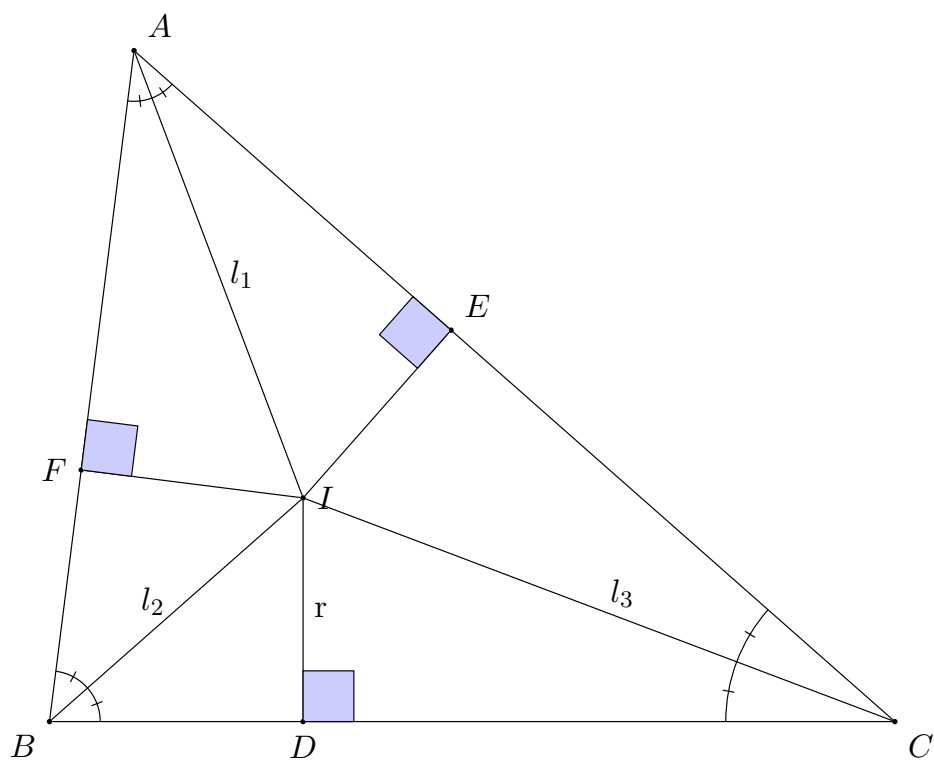


Figure A.4.2.1: Inradius  $r$  of  $\triangle ABC$

$$ID = IE = IF = r \quad (\text{A.4.2.2})$$

**Solution:** In  $\triangle IDC$  and  $IEC$ ,

$$ID = IE = \frac{l_3}{\sin \frac{C}{2}} \quad (\text{A.4.2.3})$$

Similarly, in  $\triangle IEA$  and  $IFA$ ,

$$IF = IE = \frac{l_1}{\sin \frac{A}{2}} \quad (\text{A.4.2.4})$$

yielding (A.4.2.2)

A.4.3. In Fig. A.4.2.1, show that

$$BD = BF, AE = AF, CD = CE \quad (\text{A.4.3.1})$$

**Solution:** From Fig. A.4.2.1, in  $\triangle IBD$  and  $IBF$ ,

$$x = BD = BF = r \cot \frac{B}{2} \quad (\text{A.4.3.2})$$

Similarly, other results can be obtained.

A.4.4. The circle with centre **I** and radius  $r$  in Fig. A.4.4.1 is known as the incircle. Find the radius  $r$ .

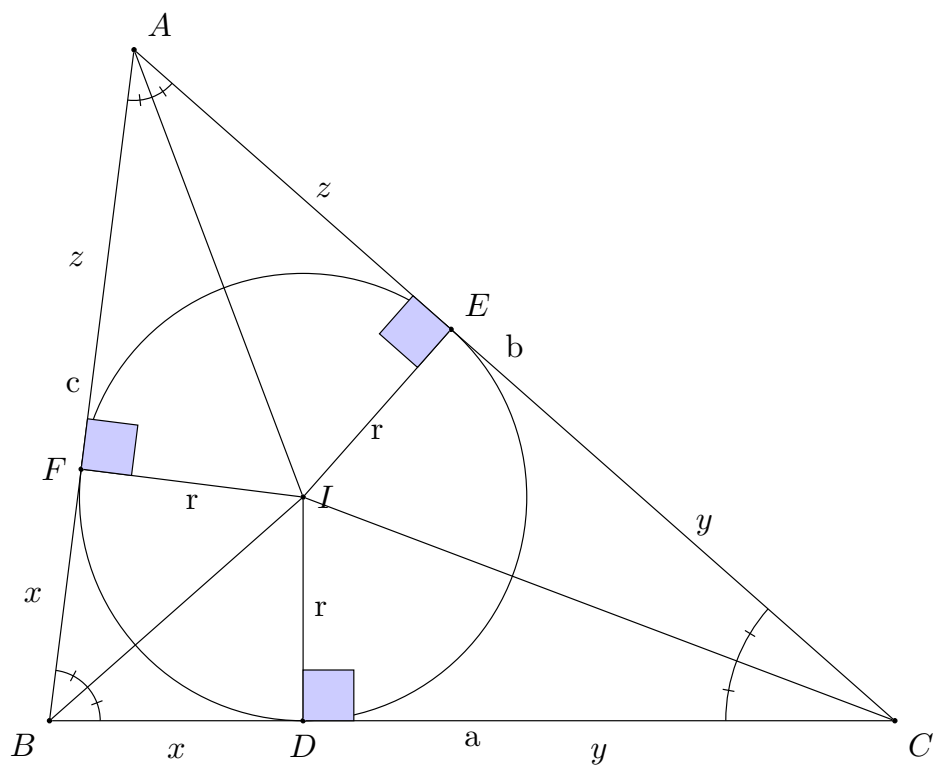


Figure A.4.4.1: Incircle of  $\triangle ABC$

**Solution:** In  $\triangle IBC$ ,

$$a = x + y = r \cot \frac{B}{2} + r \cot \frac{C}{2} \quad (\text{A.4.4.1})$$

$$\Rightarrow r = \frac{a}{\cot \frac{B}{2} + \cot \frac{C}{2}} \quad (\text{A.4.4.2})$$

## A.5. Circumradius

A.5.1. In Fig. A.5.1.1,

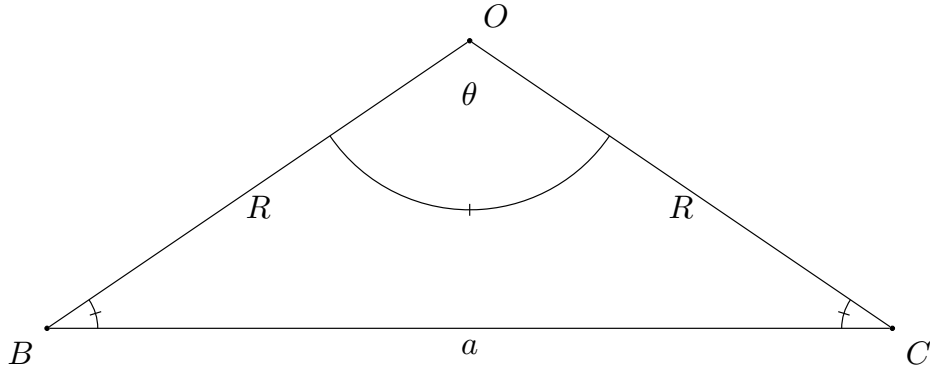


Figure A.5.1.1: Isosceles Triangle

$$OB = OC = R \quad (\text{A.5.1.1})$$

Such a triangle is known as an isosceles triangle. Show that

$$\angle B = \angle C \quad (\text{A.5.1.2})$$

**Solution:** Using (A.3.2.3),

$$\frac{\sin B}{R} = \frac{\sin C}{R} \quad (\text{A.5.1.3})$$

$$\implies \sin B = \sin C \quad (\text{A.5.1.4})$$

$$\text{or, } \angle B = \angle C. \quad (\text{A.5.1.5})$$

A.5.2. In Fig. A.5.1.1, show that

$$a = 2R \sin \frac{\theta}{2} \quad (\text{A.5.2.1})$$

**Solution:** In  $\triangle OBC$ , using the cosine formula from (B.3.3.1),

$$\cos \theta = \frac{R^2 + R^2 - a^2}{2R^2} = 1 - \frac{a^2}{2R^2} \quad (\text{A.5.2.2})$$

$$\implies \frac{a^2}{2R^2} = 2 \sin^2 \frac{\theta}{2} \quad (\text{A.5.2.3})$$

yielding (A.5.2.1).

A.5.3. In Fig. B.7.2.1, show that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R. \quad (\text{A.5.3.1})$$

**Solution:** From (B.7.6.1) and (A.5.2.1)

$$a = 2R \sin A \quad (\text{A.5.3.2})$$

## A.6. Tangent

A.6.1. In Fig. B.8.2.1, show that  $PA \cdot PB = PC^2$ .

**Solution:** In  $\triangle$ s  $APC$  and  $BPC$ , using (B.8.2.1),

$$\frac{AP}{\sin \theta} = \frac{AC}{\sin P} \quad (\text{A.6.1.1})$$

$$\frac{PC}{\sin \theta} = \frac{BC}{\sin P} \quad (\text{A.6.1.2})$$

$$\implies \frac{PC}{AP} = \frac{BC}{AC} \left( = \frac{BP}{CP} \right) \quad (\text{A.6.1.3})$$

which gives the desired result.  $\triangle$ s  $APC$  and  $BPC$  are said to be similar.

## A.7. Identities

A.7.1. Show that

$$\cos 90^\circ = 0 \quad (\text{A.7.1.1})$$

**Solution:** Using (B.3.3.1) in Fig. A.1,

$$\cos 90^\circ = \frac{a^2 + c^2 - b^2}{2ac} = 0 \quad (\text{A.7.1.2})$$

upon substituting from (A.2.4.1).

A.7.2. Show that

$$\sin 90^\circ = 1 \quad (\text{A.7.2.1})$$

**Solution:** Trivial from (A.1.2.1).

A.7.3. Prove the following identities

(a)

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta. \quad (\text{A.7.3.1})$$

(b)

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta. \quad (\text{A.7.3.2})$$

**Solution:** In (A.3.4.1), let

$$\begin{aligned}\theta_1 + \theta_2 &= \alpha \\ \theta_2 &= \beta\end{aligned}\tag{A.7.3.3}$$

This gives (A.7.3.1). In (A.7.3.1), replace  $\alpha$  by  $90^\circ - \alpha$ . This results in

$$\sin(90^\circ - \alpha - \beta) = \sin(90^\circ - \alpha) \cos \beta - \cos(90^\circ - \alpha) \sin \beta \tag{A.7.3.4}$$

$$\implies \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta \tag{A.7.3.5}$$

A.7.4. Using (A.3.4.1) and (A.7.3.2), show that

$$\sin(\theta_1 + \theta_2) = \sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2 \tag{A.7.4.1}$$

$$\cos(\theta_1 - \theta_2) = \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \tag{A.7.4.2}$$

**Solution:** From (A.3.4.1),

$$\sin(\theta_1 + \theta_2) \cos \theta_2 = \sin \theta_1 + \cos(\theta_1 + \theta_2) \sin \theta_2 \tag{A.7.4.3}$$

Using (A.7.3.2) in the above,

$$\begin{aligned}\sin(\theta_1 + \theta_2) \cos \theta_2 &= \sin \theta_1 + (\cos \theta_1 \cos \theta_2 \\ &\quad - \sin \theta_1 \sin \theta_2) \sin \theta_2\end{aligned}\tag{A.7.4.4}$$



which can be expressed as

$$\begin{aligned}\sin(\theta_1 + \theta_2) \cos \theta_2 &= \sin \theta_1 \\ &+ \cos \theta_1 \cos \theta_2 \sin \theta_2 - \sin \theta_1 \sin^2 \theta_2\end{aligned}\quad (\text{A.7.4.5})$$

Since

$$\sin^2 \theta_2 = 1 - \cos^2 \theta_2, \quad (\text{A.7.4.6})$$

we obtain

$$\sin(\theta_1 + \theta_2) \cos \theta_2 = \cos \theta_1 \cos \theta_2 \sin \theta_2 + \sin \theta_1 \cos^2 \theta_2 \quad (\text{A.7.4.7})$$

resulting in

$$\sin(\theta_1 + \theta_2) = \cos \theta_1 \sin \theta_2 + \sin \theta_1 \cos \theta_2 \quad (\text{A.7.4.8})$$

after factoring out  $\cos \theta_2$ . Using a similar approach, (A.7.4.2) can also be proved.

A.7.5. Show that

$$\sin \theta_1 + \sin \theta_2 = 2 \sin \left( \frac{\theta_1 + \theta_2}{2} \right) \cos \left( \frac{\theta_1 - \theta_2}{2} \right) \quad (\text{A.7.5.1})$$

$$\cos \theta_1 + \cos \theta_2 = 2 \cos \left( \frac{\theta_1 + \theta_2}{2} \right) \cos \left( \frac{\theta_1 - \theta_2}{2} \right) \quad (\text{A.7.5.2})$$

$$\sin \theta_1 - \sin \theta_2 = 2 \sin \left( \frac{\theta_1 - \theta_2}{2} \right) \cos \left( \frac{\theta_1 + \theta_2}{2} \right) \quad (\text{A.7.5.3})$$

$$\cos \theta_1 - \cos \theta_2 = 2 \sin \left( \frac{\theta_1 + \theta_2}{2} \right) \cos \left( \frac{\theta_2 - \theta_1}{2} \right) \quad (\text{A.7.5.4})$$

**Solution:** Let

$$\begin{aligned}\theta_1 &= \alpha + \beta \\ \theta_2 &= \alpha - \beta\end{aligned}\tag{A.7.5.5}$$

From (A.7.4.1),

$$\sin \theta_1 + \sin \theta_2 = \sin (\alpha + \beta) + \sin (\alpha - \beta) \tag{A.7.5.6}$$

$$= \sin \alpha \cos \beta + \cos \alpha \sin \beta \tag{A.7.5.7}$$

$$+ \sin \alpha \cos \beta - \cos \alpha \sin \beta \tag{A.7.5.8}$$

$$= 2 \sin \alpha \cos \beta \tag{A.7.5.9}$$

resulting in (A.7.5.1)

$$\therefore \alpha = \frac{\theta_1 + \theta_2}{2} \tag{A.7.5.10}$$

$$\beta = \frac{\theta_1 - \theta_2}{2} \tag{A.7.5.11}$$

from (A.7.5.5). Other identities may be proved similarly.

A.7.6. Show that

$$\sin 2\theta = 2 \sin \theta \cos \theta \tag{A.7.6.1}$$

$$\cos 2\theta = 1 - 2 \sin^2 \theta = 2 \cos^2 \theta - 1 \tag{A.7.6.2}$$

$$= \cos^2 \theta - \sin^2 \theta \tag{A.7.6.3}$$



## Appendix B

# Analytic Geometry

## B.1. Vectors

B.1.1. A matrix of the form

$$\mathbf{A} \triangleq \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \quad (\text{B.1.1.1})$$

is defined be column vector, or simply, vector. In Fig. A.1 the point vectors  $\mathbf{A}, \mathbf{B}, \mathbf{C}$  can be defined as

$$\mathbf{A} = \begin{pmatrix} 0 \\ c \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} a \\ 0 \end{pmatrix} \quad (\text{B.1.1.2})$$

B.1.2.

$$\lambda \mathbf{A} \triangleq \begin{pmatrix} \lambda a_1 \\ \lambda a_2 \end{pmatrix} \quad (\text{B.1.2.1})$$

B.1.3. For

$$\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \quad (\text{B.1.3.1})$$

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \end{pmatrix} \quad (\text{B.1.3.2})$$

B.1.4. The transpose of  $\mathbf{A}$  is the row vector defined as

$$\mathbf{A}^\top = \begin{pmatrix} a_1 & a_2 \end{pmatrix} \quad (\text{B.1.4.1})$$

B.1.5. The inner product or dot product is defined as

$$\mathbf{A}^\top \mathbf{B} \equiv \mathbf{A} \cdot \mathbf{B} = \begin{pmatrix} a_1 & a_2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = a_1 b_1 + a_2 b_2 \quad (\text{B.1.5.1})$$

In Fig. A.1,

$$\mathbf{A}^\top \mathbf{C} = 0 \quad (\text{B.1.5.2})$$

B.1.6. The norm of  $\mathbf{A}$  is defined as

$$\|\mathbf{A}\| = \sqrt{\mathbf{A}^\top \mathbf{A}} = \sqrt{a_1^2 + a_2^2} \quad (\text{B.1.6.1})$$

B.1.7. In Fig. A.1, it is easy to verify that

$$\|\mathbf{A} - \mathbf{C}\|^2 = \begin{pmatrix} -c & a \end{pmatrix} \begin{pmatrix} -c \\ a \end{pmatrix} = a^2 + c^2 = b^2 \quad (\text{B.1.7.1})$$

from (A.2.4.1). Thus, the distance between any two points  $\mathbf{A}$  and  $\mathbf{B}$  is given by

$$\|\mathbf{A} - \mathbf{B}\| \quad (\text{B.1.7.2})$$

B.1.8. Show that

$$\|\lambda \mathbf{A}\| = |\lambda| \|\mathbf{A}\| \quad (\text{B.1.8.1})$$

## B.2. Collinear Points

B.2.1. The direction vector of the line  $AB$  is

$$\mathbf{A} - \mathbf{B} \equiv \mathbf{B} - \mathbf{A} \equiv \kappa \begin{pmatrix} 1 \\ m \end{pmatrix}, \quad (\text{B.2.1.1})$$

where  $m$  is defined to be the slope of  $AB$ . In Fig. A.1,

$$\mathbf{A} - \mathbf{C} = \begin{pmatrix} -c \\ a \end{pmatrix} \equiv \begin{pmatrix} 1 \\ -\frac{a}{c} \end{pmatrix} = \begin{pmatrix} 1 \\ -\tan \theta \end{pmatrix} \quad (\text{B.2.1.2})$$

the slope of  $AC$  is  $-\tan \theta$

B.2.2. Points  $\mathbf{A}$ ,  $\mathbf{B}$  and  $\mathbf{C}$  are on a line if they have the same direction vector,  
i.e.

$$p(\mathbf{B} - \mathbf{A}) + q(\mathbf{C} - \mathbf{B}) = 0 \implies p, q \neq 0. \quad (\text{B.2.2.1})$$

$(\mathbf{A} - \mathbf{B}), (\mathbf{C} - \mathbf{B})$  are then said to be linearly dependent.

B.2.3. If points  $\mathbf{A}$ ,  $\mathbf{B}$  and  $\mathbf{C}$  are collinear,

$$\mathbf{B} = \frac{k\mathbf{A} + \mathbf{C}}{k + 1} \quad (\text{B.2.3.1})$$

**Solution:** From (B.2.2.1),

$$p(\mathbf{A} - \mathbf{B}) + q(\mathbf{A} - \mathbf{C}) = 0 \implies \mathbf{B} = \frac{p\mathbf{A} + q\mathbf{C}}{p + q} \quad (\text{B.2.3.2})$$

yielding (B.2.3.1) upon substituting

$$k = \frac{p}{q}. \quad (\text{B.2.3.3})$$

This is known as section formula.

B.2.4. Consequently, points  $\mathbf{A}$ ,  $\mathbf{B}$  and  $\mathbf{C}$  form a triangle if

$$p(\mathbf{A} - \mathbf{B}) + q(\mathbf{C} - \mathbf{B}) \quad (\text{B.2.4.1})$$

$$= (p + q)\mathbf{B} - p\mathbf{A} - q\mathbf{C} = 0 \quad (\text{B.2.4.2})$$

$$\implies p = 0, q = 0 \quad (\text{B.2.4.3})$$

B.2.5. In Fig. B.2.5.1

$$AF = BF, AE = BE, \quad (\text{B.2.5.1})$$

and the medians  $BE$  and  $CF$  meet at  $\mathbf{G}$ . Show that

$$\frac{GB}{GE} = \frac{GC}{GF} = 2 \quad (\text{B.2.5.2})$$

**Solution:** From (B.2.3.1),

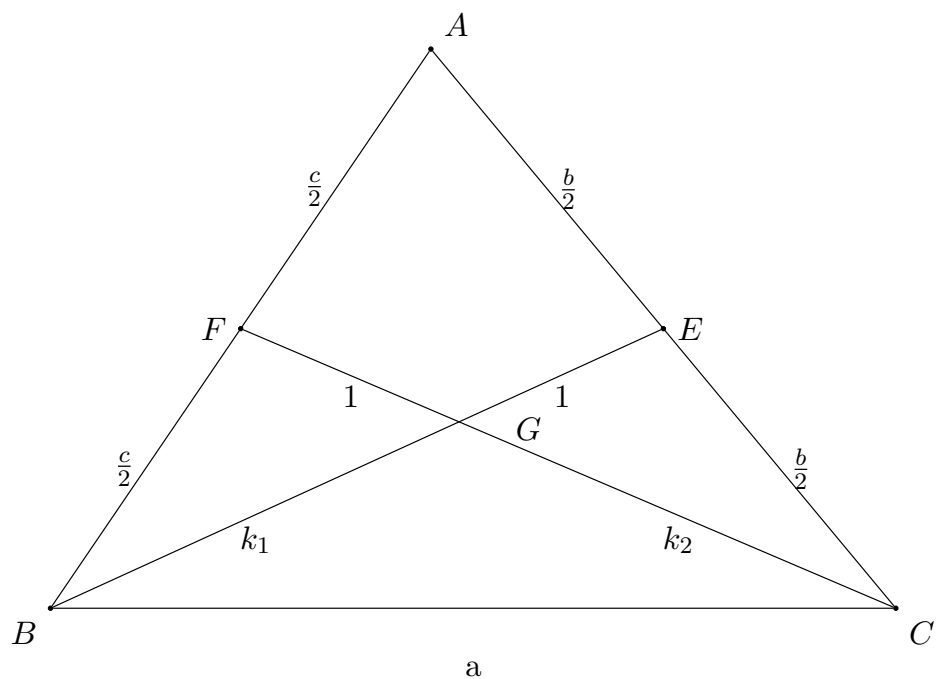


Figure B.2.5.1:  $k_1 = k_2 = 2$ .



$$\mathbf{G} = \frac{k_1 \mathbf{E} + \mathbf{B}}{k_1 + 1} = \frac{k_2 \mathbf{F} + \mathbf{C}}{k_2 + 1} \quad (\text{B.2.5.3})$$

$$\implies \frac{k_1 \left( \frac{\mathbf{A} + \mathbf{C}}{2} \right) + \mathbf{B}}{k_1 + 1} = \frac{k_2 \left( \frac{\mathbf{A} + \mathbf{B}}{2} \right) + \mathbf{C}}{k_2 + 1} \quad (\text{B.2.5.4})$$

$$\implies (k_2 + 1) \{k_1 (\mathbf{A} + \mathbf{C}) + 2\mathbf{B}\} = (k_1 + 1) \{k_2 (\mathbf{A} + \mathbf{B}) + 2\mathbf{C}\} \quad (\text{B.2.5.5})$$

which can be expressed as

$$\{2 + k_2 - k_1 k_2\} \mathbf{B} - (k_2 - k_1) \mathbf{A} - \{k_1 + 2 - k_1 k_2\} \mathbf{C} = 0 \quad (\text{B.2.5.6})$$

and is of the form (B.2.4.3) with

$$p = k_2 - k_1, q = k_1 + 2 - k_1 k_2. \quad (\text{B.2.5.7})$$

Thus, from (B.2.4.3)

$$k_2 - k_1 = 0, \quad (\text{B.2.5.8})$$

$$k_1 + 2 - k_1 k_2 = 0 \quad (\text{B.2.5.9})$$

Thus, from (B.2.5.9)

$$k_1 = k_2 \quad (\text{B.2.5.10})$$

and substituting the above in (B.2.5.9) results in the quadratic

$$k_1^2 - k_1 - 2 = 0 \quad (\text{B.2.5.11})$$

$$\implies (k_1 - 2)(k_1 + 1) = 0 \quad (\text{B.2.5.12})$$

admitting  $k_1 = k_2 = 2$  as the only possible solution.

B.2.6. Substituting  $k_1 = 2$  in (B.2.5.3)

$$\mathbf{G} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \quad (\text{B.2.6.1})$$

B.2.7. In Fig. B.2.7.1,  $AG$  is extended to join  $BC$  at  $\mathbf{D}$ . Show that  $AD$  is also a median.

**Solution:** Considering the ratios in Fig. B.2.7.1,

$$\mathbf{G} = \frac{k_3 \mathbf{D} + \mathbf{A}}{k_3 + 1} \quad (\text{B.2.7.1})$$

$$\mathbf{D} = \frac{k_4 \mathbf{C} + \mathbf{B}}{k_4 + 1} \quad (\text{B.2.7.2})$$

Substituting from (B.2.6.1) in the above,

$$(k_3 + 1) \left( \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \right) = k_3 \left( \frac{k_4 \mathbf{C} + \mathbf{B}}{k_4 + 1} \right) + \mathbf{A} \quad (\text{B.2.7.3})$$

$$\implies (k_3 + 1)(k_4 + 1)(\mathbf{A} + \mathbf{B} + \mathbf{C}) = 3\{k_3(k_4 \mathbf{C} + \mathbf{B}) + (k_4 + 1)\mathbf{A}\} \quad (\text{B.2.7.4})$$



Figure B.2.7.1:  $k_3 = 2, k_4 = 1$

which can be expressed as

$$\begin{aligned}
 & (k_3 k_4 + k_3 - 2k_4 - 2) \mathbf{A} \\
 & - (-k_3 k_4 - k_4 + 2k_3 - 1) \mathbf{B} \\
 & - (-k_3 - k_4 - 1 + 2k_3 k_4) \mathbf{C} = \mathbf{0} \quad (\text{B.2.7.5})
 \end{aligned}$$

Comparing the above with (B.2.4.3),

$$p = -k_3 k_4 - k_4 + 2k_3 - 1, q = -k_3 - k_4 - 1 + 2k_3 k_4 \quad (\text{B.2.7.6})$$

yielding

$$-k_3k_4 - k_4 + 2k_3 - 1 = 0 \quad (\text{B.2.7.7})$$

$$-k_3 - k_4 - 1 + 2k_3k_4 = 0 \quad (\text{B.2.7.8})$$

Subtracting (B.2.7.7) from (B.2.7.8),

$$3k_3(k_4 - 1) = 0 \quad (\text{B.2.7.9})$$

$$\implies k_4 = 1 \quad (\text{B.2.7.10})$$

which upon substituting in (B.2.7.7) yields

$$k_3 = 2 \quad (\text{B.2.7.11})$$

## B.3. Matrices: Cosine Formula

B.3.1. The determinant of the  $2 \times 2$  matrix

$$\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \quad (\text{B.3.1.1})$$

is defined as

$$|\mathbf{M}| = \begin{vmatrix} \mathbf{A} & \mathbf{B} \end{vmatrix} \quad (\text{B.3.1.2})$$

$$= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 \quad (\text{B.3.1.3})$$

B.3.2. In Fig. B.3.2.1, show that

$$\begin{pmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{pmatrix} \begin{pmatrix} \cos A \\ \cos B \\ \cos C \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (\text{B.3.2.1})$$

**Solution:** From Fig. B.3.2.1,

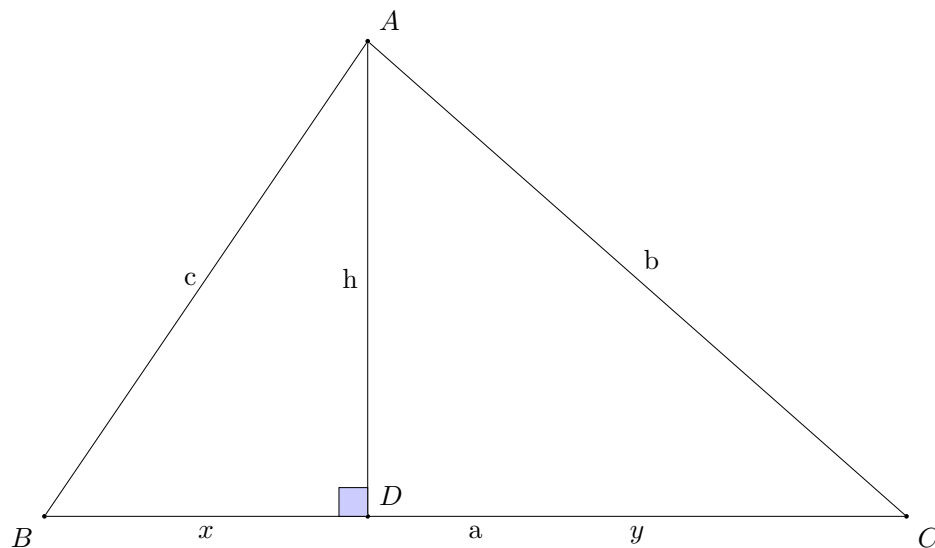


Figure B.3.2.1: The cosine formula

$$a = x + y = b \cos C + c \cos B = \begin{pmatrix} \cos C & \cos B \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix} \quad (\text{B.3.2.2})$$

$$= \begin{pmatrix} 0 & b & c \end{pmatrix} \begin{pmatrix} \cos A \\ \cos C \\ \cos B \end{pmatrix} \quad (\text{B.3.2.3})$$

Similarly,

$$b = c \cos A + a \cos C = \begin{pmatrix} c & 0 & a \end{pmatrix} \begin{pmatrix} \cos A \\ \cos C \\ \cos B \end{pmatrix} \quad (\text{B.3.2.4})$$

$$c = b \cos A + a \cos B = \begin{pmatrix} b & a & 0 \end{pmatrix} \begin{pmatrix} \cos A \\ \cos C \\ \cos B \end{pmatrix} \quad (\text{B.3.2.5})$$

The above equations can be expressed in matrix form as (B.3.2.1).

B.3.3. Show that

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \quad (\text{B.3.3.1})$$

**Solution:** Using the properties of determinants,

$$\cos A = \frac{\begin{vmatrix} a & c & b \\ b & 0 & a \\ c & a & 0 \end{vmatrix}}{\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}} = \frac{ab^2 + ac^2 - a^3}{abc + abc} = \frac{b^2 + c^2 - a^2}{2abc} \quad (\text{B.3.3.2})$$

## B.4. Area of a Triangle: Cross Product

B.4.1. The cross product or vector product defined as  $\mathbf{A} \times \mathbf{B}$  is given by (B.3.1.2) for  $2 \times 1$  vectors.

B.4.2. The area of the triangle with vertices  $\mathbf{A}, \mathbf{B}, \mathbf{C}$  is given by

$$\frac{1}{2} \|(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{C})\| = \frac{1}{2} \|\mathbf{A} \times \mathbf{B} + \mathbf{B} \times \mathbf{C} + \mathbf{C} \times \mathbf{A}\| \quad (\text{B.4.2.1})$$

B.4.3. If

$$\|\mathbf{A} \times \mathbf{B}\| = \|\mathbf{C} \times \mathbf{D}\|, \quad \text{then} \quad (\text{B.4.3.1})$$

$$\mathbf{A} \times \mathbf{B} = \pm (\mathbf{C} \times \mathbf{D}) \quad (\text{B.4.3.2})$$

where the sign depends on the orientation of the vectors.

## B.5. Parallelogram

B.5.1. If  $ABCD$  be a parallelogram,

$$\mathbf{B} - \mathbf{A} = \mathbf{C} - \mathbf{D} \quad (\text{B.5.1.1})$$

B.5.2. The area of the parallelogram with vertices  $\mathbf{A}, \mathbf{B}, \mathbf{C}$  and  $\mathbf{D}$  is given by

$$\|(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{D})\| = \|\mathbf{A} \times \mathbf{B} + \mathbf{B} \times \mathbf{C} + \mathbf{C} \times \mathbf{A}\| \quad (\text{B.5.2.1})$$

## B.6. Altitudes of a Triangle:Line Equation

B.6.1. Find the equation of the line  $BC$ .

**Solution:** Let  $\mathbf{x}$  be any point on  $BC$ . Using section formula, for some  $k$ ,

$$\mathbf{x} = \frac{k\mathbf{C} + \mathbf{B}}{k+1} = \frac{(k+1)\mathbf{C} + (\mathbf{B} - \mathbf{C})}{k+1} \quad (\text{B.6.1.1})$$

$$\implies \mathbf{x} = \mathbf{C} + \lambda \mathbf{m} \quad (\text{B.6.1.2})$$

where

$$\mathbf{m} = \frac{\mathbf{B} - \mathbf{C}}{k+1} \equiv \mathbf{B} - \mathbf{C} \quad (\text{B.6.1.3})$$



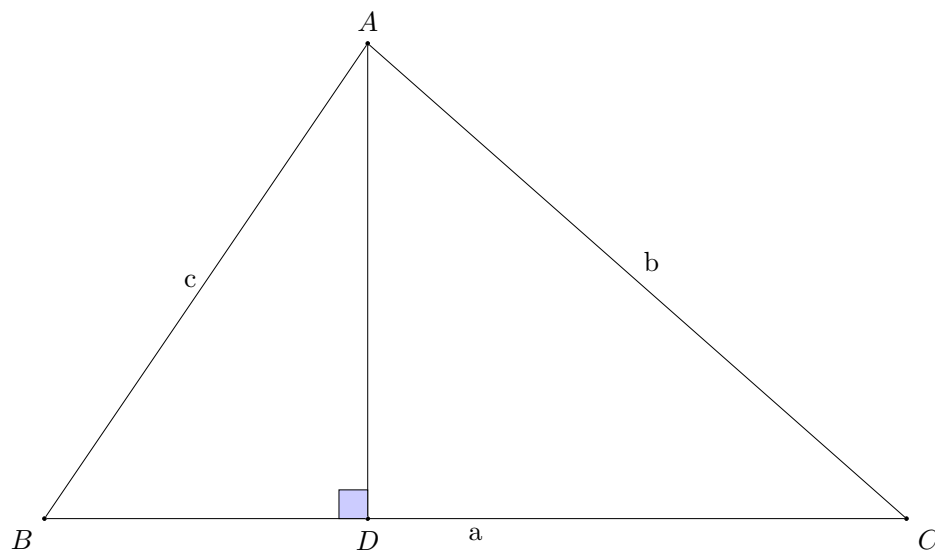


Figure B.6.1.1: Drawing the altitude

B.6.2. The normal vector to  $\mathbf{m}$  is defined as

$$\mathbf{n}^\top \mathbf{m} = 0 \quad (\text{B.6.2.1})$$

$$\mathbf{n} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{m} \quad (\text{B.6.2.2})$$

B.6.3. From (B.6.2.1) and (B.6.1.2), it can be verified that

$$\mathbf{n}^\top \mathbf{x} = \mathbf{n}^\top \mathbf{C} + \lambda \mathbf{n}^\top \mathbf{m} \quad (\text{B.6.3.1})$$

$$\implies \mathbf{n}^\top \mathbf{x} = \mathbf{n}^\top \mathbf{C} \quad (\text{B.6.3.2})$$

(B.6.3.2) is defined to be the normal form of the line  $BC$ .

B.6.4. In Fig. B.6.5.1,  $AD \perp BC$  and  $BE \perp AC$  are defined to be the altitudes of  $\triangle ABC$ .

B.6.5. Let  $\mathbf{H}$  be the intersection of the altitudes  $AD$  and  $BE$  as shown in Fig. B.6.5.1.  $CH$  is extended to meet  $AB$  at  $\mathbf{F}$ . Show that  $CF \perp AB$ .

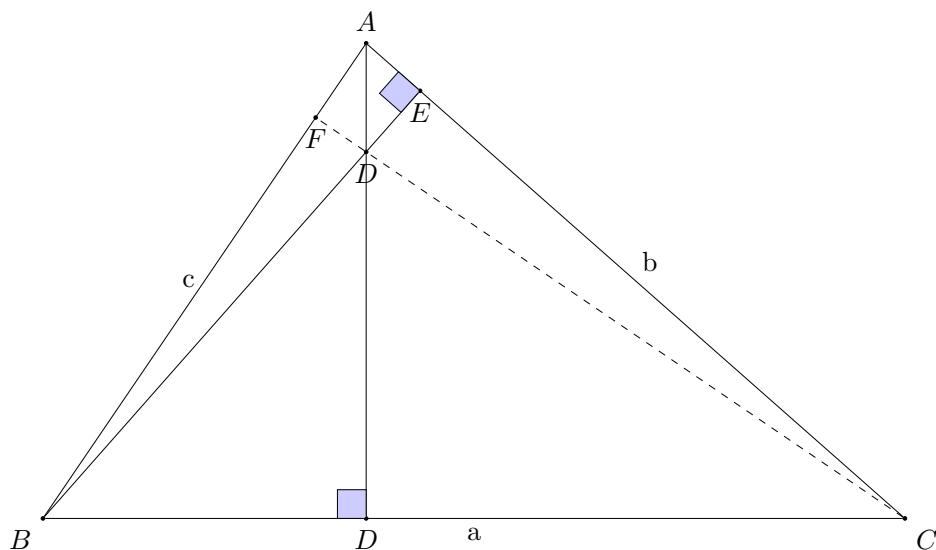


Figure B.6.5.1: Altitudes of a triangle meet at the orthocentre  $H$

**Solution:** From (B.6.1.3) (B.6.2.1), (B.1.5.2) and (B.6.3.2), the equations of  $AD$  and  $BE$  are

$$(\mathbf{B} - \mathbf{C})^\top (\mathbf{x} - \mathbf{A}) = 0 \quad (\text{B.6.5.1})$$

$$(\mathbf{C} - \mathbf{A})^\top (\mathbf{x} - \mathbf{B}) = 0 \quad (\text{B.6.5.2})$$

$\therefore \mathbf{H}$  lies on both  $AD$  and  $BE$ , it satisfies the above equations, and

$$(\mathbf{B} - \mathbf{C})^\top (\mathbf{H} - \mathbf{A}) = 0 \quad (\text{B.6.5.3})$$

$$(\mathbf{C} - \mathbf{A})^\top (\mathbf{H} - \mathbf{B}) = 0 \quad (\text{B.6.5.4})$$

Adding both the above and simplifying,

$$(\mathbf{B} - \mathbf{A})^\top (\mathbf{H} - \mathbf{C}) = 0 \quad (\text{B.6.5.5})$$

$\implies CH \perp AB$  from (B.1.5.2), or  $CF \perp AB$ .

B.6.6. Altitudes of a  $\triangle$  meet at the orthocentre  $H$ .

## B.7. Circumcircle: Circle Equation

B.7.1. In Fig. B.7.1.1,

$$OB = OC = R, BD = DC. \quad (\text{B.7.1.1})$$

Show that  $OD \perp BC$ .

**Solution:**

$$\|\mathbf{O} - \mathbf{C}\| = \|\mathbf{O} - \mathbf{B}\| = R \quad (\text{B.7.1.2})$$

$$\implies \|\mathbf{O} - \mathbf{C}\|^2 = \|\mathbf{O} - \mathbf{B}\|^2 \quad (\text{B.7.1.3})$$

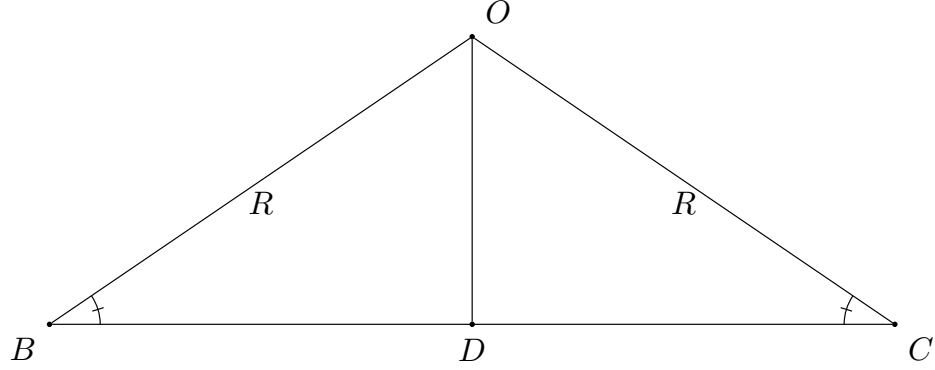


Figure B.7.1.1: Perpendicular bisector.

which can be expressed as

$$(\mathbf{O} - \mathbf{C})^\top (\mathbf{O} - \mathbf{C}) = (\mathbf{O} - \mathbf{B})^\top (\mathbf{O} - \mathbf{B}) \quad (\text{B.7.1.4})$$

$$\|\mathbf{O}\|^2 - 2\mathbf{O}^\top \mathbf{C} + \|\mathbf{C}\|^2 = \|\mathbf{O}\|^2 - 2\mathbf{O}^\top \mathbf{B} + \|\mathbf{B}\|^2 \quad (\text{B.7.1.5})$$

$$\implies (\mathbf{B} - \mathbf{C})^\top \mathbf{O} = \frac{\|\mathbf{B}\|^2 - \|\mathbf{C}\|^2}{2} \quad (\text{B.7.1.6})$$

which can be simplified to obtain

$$(\mathbf{B} - \mathbf{C})^\top \left\{ \mathbf{O} - \left( \frac{\mathbf{B} + \mathbf{C}}{2} \right) \right\} = 0 \quad (\text{B.7.1.7})$$

$$\text{or, } (\mathbf{B} - \mathbf{C})^\top \{\mathbf{O} - \mathbf{D}\} = 0 \quad (\text{B.7.1.8})$$

which proves the give result using (B.2.3.1) and (B.1.5.2).

B.7.2. The equation of the circle in Fig. B.7.2.1, is

$$\|\mathbf{x} - \mathbf{O}\| = R \quad (\text{B.7.2.1})$$

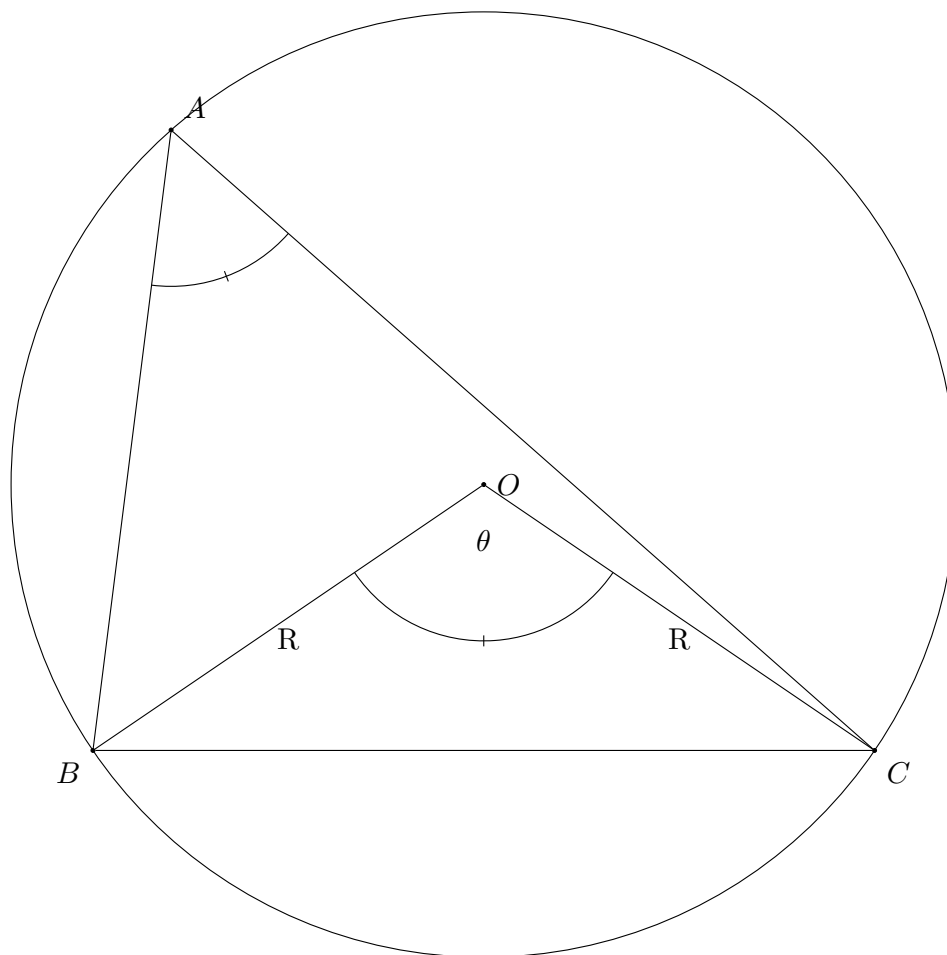


Figure B.7.2.1: Circumcircle of  $\triangle ABC$

This is known as the circumcircle of  $\triangle ABC$ .

B.7.3. In Fig. B.3.2.1 show that

$$\cos A = \frac{(\mathbf{A} - \mathbf{B})^\top (\mathbf{A} - \mathbf{C})}{\|\mathbf{A} - \mathbf{B}\| \|\mathbf{A} - \mathbf{C}\|} \quad (\text{B.7.3.1})$$

**Solution:** From (B.3.3.1), using (B.1.7.2),

$$\cos A = \frac{\|\mathbf{A} - \mathbf{B}\|^2 + \|\mathbf{A} - \mathbf{C}\|^2 - \|\mathbf{B} - \mathbf{C}\|^2}{2 \|\mathbf{A} - \mathbf{B}\| \|\mathbf{A} - \mathbf{C}\|} \quad (\text{B.7.3.2})$$

$$= \frac{\|\mathbf{A}\|^2 - \mathbf{A}^\top \mathbf{B} - \mathbf{A}^\top \mathbf{C} + \mathbf{B}^\top \mathbf{C}}{\|\mathbf{A} - \mathbf{B}\| \|\mathbf{A} - \mathbf{C}\|} \quad (\text{B.7.3.3})$$

which can be expressed as (B.7.3.1).

B.7.4. Any point on the circle can be expressed as

$$\mathbf{x} = \mathbf{O} + R \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \quad \theta \in [0, 2\pi]. \quad (\text{B.7.4.1})$$

B.7.5. Let

$$R = 1, \mathbf{O} = \mathbf{0}, \mathbf{A} = \begin{pmatrix} \cos \theta_1 \\ \sin \theta_1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} \cos \theta_2 \\ \sin \theta_2 \end{pmatrix}, \quad (\text{B.7.5.1})$$

Show that

$$\|\mathbf{A} - \mathbf{B}\| = 2 \sin \left( \frac{\theta_1 - \theta_2}{2} \right) \quad (\text{B.7.5.2})$$

**Solution:** From (B.7.4.1).

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} \cos \theta_1 - \cos \theta_2 \\ \sin \theta_1 - \sin \theta_2 \end{pmatrix} \quad (\text{B.7.5.3})$$

$$\implies \|\mathbf{A} - \mathbf{B}\|^2 = (\cos \theta_1 - \cos \theta_2)^2 + (\sin \theta_1 - \sin \theta_2)^2 \quad (\text{B.7.5.4})$$

$$= 2 \{1 - \cos(\theta_1 - \theta_2)\} = 4 \sin^2 \left( \frac{\theta_1 - \theta_2}{2} \right) \quad (\text{B.7.5.5})$$

yielding (B.7.5.2) from (A.7.6.3).

B.7.6. In Fig. B.7.2.1, show that

$$\theta = 2A. \quad (\text{B.7.6.1})$$

**Solution:** Let

$$\mathbf{C} = \begin{pmatrix} \cos \theta_3 \\ \sin \theta_3 \end{pmatrix} \quad (\text{B.7.6.2})$$

Then, substituting from (B.7.5.2) in (B.7.3.2),

$$\cos A = \frac{4 \sin^2 \left( \frac{\theta_1 - \theta_2}{2} \right) + 4 \sin^2 \left( \frac{\theta_1 - \theta_3}{2} \right) - 4 \sin^2 \left( \frac{\theta_2 - \theta_3}{2} \right)}{8 \sin \left( \frac{\theta_1 - \theta_2}{2} \right) \sin \left( \frac{\theta_1 - \theta_3}{2} \right)} \quad (\text{B.7.6.3})$$

$$= \frac{2 \sin^2 \left( \frac{\theta_1 - \theta_2}{2} \right) + \cos(\theta_2 - \theta_3) - \cos(\theta_1 - \theta_3)}{4 \sin \left( \frac{\theta_1 - \theta_2}{2} \right) \sin \left( \frac{\theta_1 - \theta_3}{2} \right)} \quad (\text{B.7.6.4})$$

from (A.7.6.3).  $\therefore$  from (A.7.5.4),

$$\cos A = \frac{2 \sin^2 \left( \frac{\theta_1 - \theta_2}{2} \right) + 2 \sin \left( \frac{\theta_1 - \theta_2}{2} \right) \sin \left( \frac{\theta_1 + \theta_2}{2} - \theta_3 \right)}{4 \sin \left( \frac{\theta_1 - \theta_2}{2} \right) \sin \left( \frac{\theta_1 - \theta_3}{2} \right)} \quad (\text{B.7.6.5})$$

$$= \frac{\sin \left( \frac{\theta_1 - \theta_2}{2} \right) + \sin \left( \frac{\theta_1 + \theta_2}{2} - \theta_3 \right)}{2 \sin \left( \frac{\theta_1 - \theta_3}{2} \right)} \quad (\text{B.7.6.6})$$

From (A.7.5.1), the above equation can be expressed as

$$\cos A = \frac{2 \sin \left( \frac{\theta_1 - \theta_3}{2} \right) \cos \left( \frac{\theta_2 - \theta_3}{2} \right)}{2 \sin \left( \frac{\theta_1 - \theta_3}{2} \right)} = \cos \left( \frac{\theta_2 - \theta_3}{2} \right) \quad (\text{B.7.6.7})$$

$$\implies 2A = \theta_2 - \theta_3 \quad (\text{B.7.6.8})$$

Similarly,

$$\cos \theta = \frac{1 + 1 - 4 \sin^2 \left( \frac{\theta_2 - \theta_3}{2} \right)}{2} = \cos (\theta_2 - \theta_3) = \cos 2A \quad (\text{B.7.6.9})$$

## B.8. Tangent

B.8.1. In Fig. B.8.1.1,  $OC$  is the radius and  $PC$  touches the circle at  $C$ . Show that

$$OC \perp PC. \quad (\text{B.8.1.1})$$





Figure B.8.1.1:

**Solution:** The equation of  $PC$  can be expressed as

$$\mathbf{x} = \mathbf{C} + \mu \mathbf{m} \quad (\text{B.8.1.2})$$

and the equation of the circle is

$$\|\mathbf{x} - \mathbf{O}\| = R \quad (\text{B.8.1.3})$$

Substituting (B.8.1.2) in (B.8.1.3),

$$\|\mathbf{C} + \mu\mathbf{m} - \mathbf{O}\|^2 = R^2 \quad (\text{B.8.1.4})$$

$$\implies \mu^2 \|\mathbf{m}\|^2 + 2\mu\mathbf{m}^\top (\mathbf{C} - \mathbf{O}) + \|\mathbf{C} - \mathbf{O}\|^2 - R^2 = 0 \quad (\text{B.8.1.5})$$

The above equation has only one root. Hence the discriminant of the above quadratic should be zero. So,

$$\left\{\mathbf{m}^\top (\mathbf{C} - \mathbf{O})\right\}^2 - \|\mathbf{m}\|^2 \left\{\|\mathbf{C} - \mathbf{O}\|^2 - R^2\right\} = 0 \quad (\text{B.8.1.6})$$

Since  $\mathbf{C}$  is a point on the circle,

$$\|\mathbf{C} - \mathbf{O}\|^2 - R^2 = 0 \quad (\text{B.8.1.7})$$

$$\implies \mathbf{m}^\top (\mathbf{C} - \mathbf{O}) = 0 \quad (\text{B.8.1.8})$$

upon substituting in (B.8.1.6). Using the definition of the direction vector from (B.2.1.1)

$$\mathbf{m} = \mathbf{P} - \mathbf{C} \quad (\text{B.8.1.9})$$

$$\implies (\mathbf{P} - \mathbf{C})^\top (\mathbf{C} - \mathbf{O}) = 0 \quad (\text{B.8.1.10})$$

which is equivalent to (B.8.1.1).

B.8.2. In Fig. B.8.2.1 show that

$$\theta = \alpha \quad (\text{B.8.2.1})$$

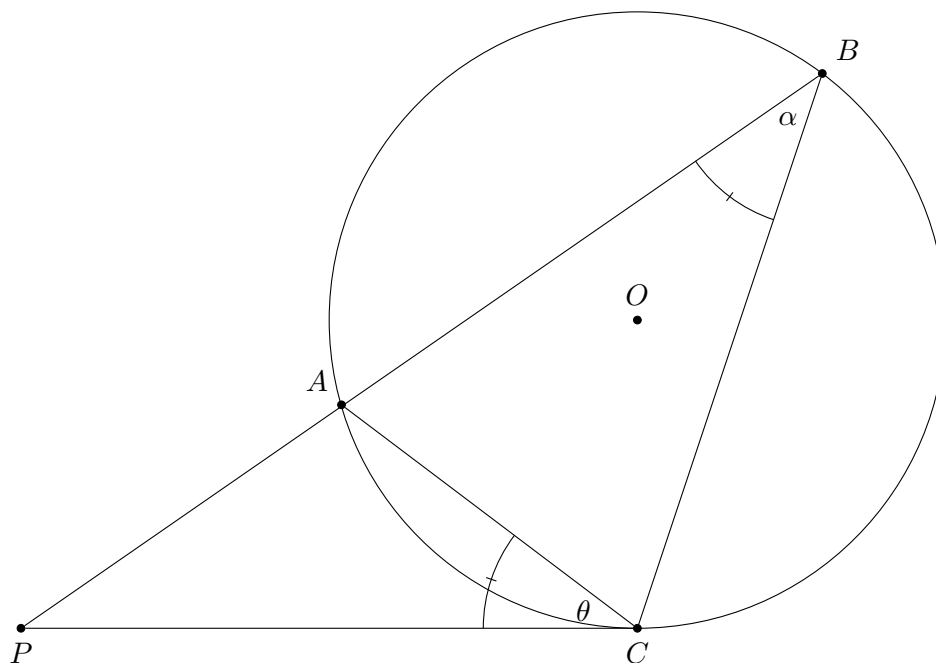


Figure B.8.2.1:  $\theta = \alpha$ .

**Solution:** Let Let

$$\mathbf{O} = \mathbf{OA} = \begin{pmatrix} \cos \theta_1 \\ \sin \theta_1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} \cos \theta_2 \\ \sin \theta_2 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} \cos \theta_3 \\ \sin \theta_3 \end{pmatrix} \quad (\text{B.8.2.2})$$

Without loss of generality, let

$$\theta_3 = \frac{\pi}{2} \quad (\text{B.8.2.3})$$

Then,

$$\mathbf{C} - \mathbf{O} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (\text{B.8.2.4})$$

From from (B.8.1.10),

$$\mathbf{C} - \mathbf{P} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (\text{B.8.2.5})$$

From (B.7.3.1) and (B.8.2.5),

$$\cos \theta = \frac{\begin{pmatrix} \cos \theta_3 - \cos \theta_1 & \sin \theta_3 - \sin \theta_1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}}{2 \sin \left( \frac{\theta_1 - \theta_3}{2} \right)} \quad (\text{B.8.2.6})$$

$$= \sin \left( \frac{\theta_1 + \theta_3}{2} \right) = \cos \left( \frac{\pi}{2} - \frac{\theta_1 + \theta_3}{2} \right) = \cos \left( \frac{\pi}{4} - \frac{\theta_1}{2} \right) \quad (\text{B.8.2.7})$$

upon substituting from (B.8.2.3). Similarly, from (B.7.6.7),

$$\cos \alpha = \cos \left( \frac{\theta_1 - \theta_3}{2} \right) = \cos \left( \frac{\pi}{4} - \frac{\theta_1}{2} \right) = \cos \theta \quad (\text{B.8.2.8})$$

