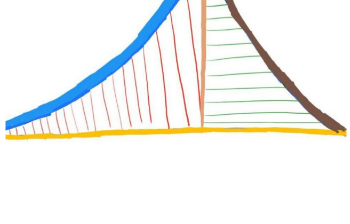
GEOMETRY

Through Algebra

G. V. V. Sharma



Contents

Introduction

This book shows how to solve problems in geometry using trigonometry and coordinate geometry.

Chapter 1

Triangle

Consider a triangle with vertices

$$\mathbf{A} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -4 \\ 6 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -3 \\ -5 \end{pmatrix} \tag{1.1}$$

1.1. Vectors

1.1.1. The direction vector of AB is defined as

$$\mathbf{B} - \mathbf{A} \tag{1.1.1.1}$$

Find the direction vectors of AB, BC and CA.

Solution:

(a) The Direction vector of AB is

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} -4 \\ 6 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -4 - 1 \\ 6 - (-1) \end{pmatrix} = \begin{pmatrix} -5 \\ 7 \end{pmatrix} \tag{1.1.1.2}$$

(b) The Direction vector of BC is

$$\mathbf{C}-\mathbf{B}=\begin{pmatrix}-3\\-5\end{pmatrix}-\begin{pmatrix}-4\\6\end{pmatrix}=\begin{pmatrix}-3-(-4)\\-5-6\end{pmatrix}=\begin{pmatrix}1\\-11\end{pmatrix} \tag{1.1.1.3}$$

(c) The Direction vector of CA is

$$\mathbf{A}-\mathbf{C}=\begin{pmatrix}1\\-1\end{pmatrix}-\begin{pmatrix}-3\\-5\end{pmatrix}=\begin{pmatrix}1-(-3)\\-1-(-5)\end{pmatrix}=\begin{pmatrix}4\\4\end{pmatrix} \tag{1.1.1.4}$$

1.1.2. The length of side BC is

$$c=\left\|\mathbf{B}-\mathbf{A}\right\|\triangleq\sqrt{\left(\mathbf{B}-\mathbf{A}\right)^{\top}\left(\mathbf{B}-\mathbf{A}\right)} \tag{1.1.2.1}$$

where

$$\mathbf{A}^{\top}\triangleq\begin{pmatrix}1 & -1\end{pmatrix} \tag{1.1.2.2}$$

Similarly,

$$b=\left\|\mathbf{C}-\mathbf{B}\right\|,\ a=\left\|\mathbf{A}-\mathbf{C}\right\| \tag{1.1.2.3}$$

Find a, b, c.

(a) Since,

$$\mathbf{A}-\mathbf{B}=\begin{pmatrix}5\\-7\end{pmatrix}, \tag{1.1.2.4}$$

$$c=\left\|\mathbf{A}-\mathbf{B}\right\|=\sqrt{\begin{pmatrix}5 & -7\end{pmatrix}\begin{pmatrix}5\\-7\end{pmatrix}}=\sqrt{(5)^2+(7)^2} \tag{1.1.2.5}$$

$$=\sqrt{74} \tag{1.1.2.6}$$

(b) Similarly,

$$\mathbf{B}-\mathbf{C}=\begin{pmatrix}-1\\11\end{pmatrix} \tag{1.1.2.7}$$

$$\implies a=\left\|\mathbf{B}-\mathbf{C}\right\|=\sqrt{\begin{pmatrix}-1 & 11\end{pmatrix}\begin{pmatrix}-1\\11\end{pmatrix}}=\sqrt{(1)^2+(11)^2} \tag{1.1.2.8}$$

$$=\sqrt{122} \tag{1.1.2.9}$$

and

(c)

$$\mathbf{A}-\mathbf{C} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \tag{1.1.2.10}$$

$$\implies b = \left\lVert \mathbf{A}-\mathbf{C} \right\rVert = \sqrt{\begin{pmatrix} 4 & 4 \end{pmatrix}\begin{pmatrix} 4 \\ 4 \end{pmatrix}} = \sqrt{(4)^2+(4)^2} \tag{1.1.2.11}$$

$$= \sqrt{32} \tag{1.1.2.12}$$

1.1.3. Points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are defined to be collinear if

$$\text{rank}\begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} = 2 \tag{1.1.3.1}$$

Are the given points in (1.1) collinear?

Solution: From (1.1),

$$\begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -4 & -3 \\ -1 & 6 & -5 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3+R_2} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -4 & -3 \\ 0 & 2 & -8 \end{pmatrix} \tag{1.1.3.2}$$

$$\xleftrightarrow{R_2 \leftarrow R_1-R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 5 & 4 \\ 0 & 2 & -8 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3-\frac{2}{5}R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 5 & 4 \\ 0 & 0 & \frac{-48}{5} \end{pmatrix} \tag{1.1.3.3}$$

There are no zero rows. So,

$$\text{rank}\begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} = 3 \tag{1.1.3.4}$$

Hence, the points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are not collinear. This is visible in Fig. 1.1.

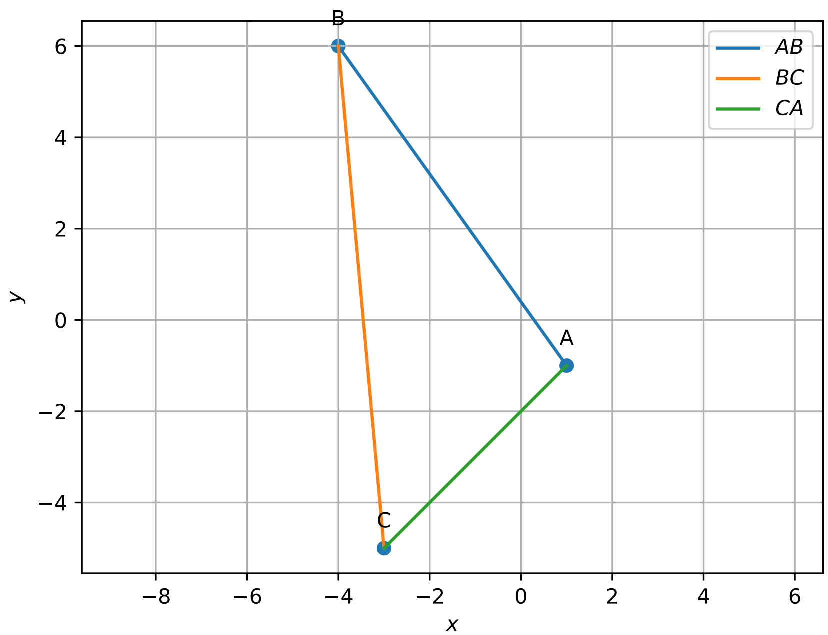


Figure 1.1: $\triangle ABC$

1.1.4. The parameteric form of the equation of AB is

$$\mathbf{x} = \mathbf{A} + k\mathbf{m} \tag{1.1.4.1}$$

where

$$\mathbf{m} = \mathbf{B} - \mathbf{A} \tag{1.1.4.2}$$

is the direction vector of AB. Find the parameteric equations of AB, BC and CA.

Solution: From (1.1.4.1) and (1.1.1.2), the parametric equation for AB is given by

$$AB : \mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + k \begin{pmatrix} -5 \\ 7 \end{pmatrix} \tag{1.1.4.3}$$

Similarly, from (1.1.1.3) and (1.1.1.4),

$$BC : \mathbf{x} = \begin{pmatrix} -4 \\ 6 \end{pmatrix} + k \begin{pmatrix} 1 \\ -11 \end{pmatrix} \tag{1.1.4.4}$$

$$CA : \mathbf{x} = \begin{pmatrix} -3 \\ -5 \end{pmatrix} + k \begin{pmatrix} 4 \\ 4 \end{pmatrix} \tag{1.1.4.5}$$

1.1.5. The normal form of the equation of AB is

$$\mathbf{n}^{\top} (\mathbf{x} - \mathbf{A}) = 0 \tag{1.1.5.1}$$

where

$$\mathbf{n}^\top \mathbf{m} = \mathbf{n}^\top (\mathbf{B} - \mathbf{A}) = 0 \tag{1.1.5.2}$$

$$\text{or, } \mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{m} \tag{1.1.5.3}$$

Find the normal form of the equations of AB, BC and CA.

Solution:

(a) From (1.1.1.3), the direction vector of side $\mathbf{BC}$ is

$$\mathbf{m} = \begin{pmatrix} 1 \\ -11 \end{pmatrix} \tag{1.1.5.4}$$

$$\implies \mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -11 \end{pmatrix} = \begin{pmatrix} -11 \\ -1 \end{pmatrix} \tag{1.1.5.5}$$

from (1.1.5.3). Hence, from (1.1.5.1), the normal equation of side BC is

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{B}) = 0 \tag{1.1.5.6}$$

$$\implies \begin{pmatrix} -11 & -1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -11 & -1 \end{pmatrix} \begin{pmatrix} -4 \\ 6 \end{pmatrix} \tag{1.1.5.7}$$

$$\implies BC : \quad \begin{pmatrix} 11 & 1 \end{pmatrix} \mathbf{x} = -38 \tag{1.1.5.8}$$

(b) Similarly, for AB, from (1.1.1.2),

$$\mathbf{m} = \begin{pmatrix} -5 \\ 7 \end{pmatrix} \tag{1.1.5.9}$$

$$\implies \mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -5 \\ 7 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix} \tag{1.1.5.10}$$

and

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{A}) = 0 \tag{1.1.5.11}$$

$$\implies AB: \quad \mathbf{n}^\top \mathbf{x} = \begin{pmatrix} 7 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{1.1.5.12}$$

$$\implies \begin{pmatrix} 7 & 5 \end{pmatrix} \mathbf{x} = 2 \tag{1.1.5.13}$$

(c) For CA, from (1.1.1.4),

$$\mathbf{m} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.1.5.14}$$

$$\implies \mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{1.1.5.15}$$

$$\tag{1.1.5.16}$$

$$\implies \mathbf{n}^\top (\mathbf{x} - \mathbf{C}) = 0 \tag{1.1.5.17}$$

$$\implies \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} -3 \\ -5 \end{pmatrix} = 2 \tag{1.1.5.18}$$

1.1.6. The area of $\triangle ABC$ is defined as

$$\frac{1}{2}\left\|(\mathbf{A}-\mathbf{B})\times(\mathbf{A}-\mathbf{C})\right\| \tag{1.1.6.1}$$

where

$$\mathbf{A}\times\mathbf{B} \triangleq \begin{vmatrix} 1 & -4 \\ -1 & 6 \end{vmatrix} \tag{1.1.6.2}$$

Find the area of $\triangle ABC$.

Solution: From (1.1.1.2) and (1.1.1.4),

$$\mathbf{A}-\mathbf{B} = \begin{pmatrix} 5 \\ -7 \end{pmatrix}, \mathbf{A}-\mathbf{C} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \tag{1.1.6.3}$$

$$\implies (\mathbf{A}-\mathbf{B})\times(\mathbf{A}-\mathbf{C}) = \begin{vmatrix} 5 & 4 \\ -7 & 4 \end{vmatrix} \tag{1.1.6.4}$$

$$= 5\times 4 - 4\times(-7) \tag{1.1.6.5}$$

$$= 48 \tag{1.1.6.6}$$

$$\implies \frac{1}{2}\left\|(\mathbf{A}-\mathbf{B})\times(\mathbf{A}-\mathbf{C})\right\| = \frac{48}{2} = 24 \tag{1.1.6.7}$$

which is the desired area.

1.1.7. Find the angles A, B, C if

$$\cos A \triangleq \frac{(\mathbf{B}-\mathbf{A})^{\top}\mathbf{C}-\mathbf{A}}{\left\|\mathbf{B}-\mathbf{A}\right\|\left\|\mathbf{C}-\mathbf{A}\right\|} \tag{1.1.7.1}$$

(a) From (1.1.1.2), (1.1.1.4), (1.1.2.6) and (1.1.2.12)

$$(\mathbf{B}-\mathbf{A})^{\top}(\mathbf{C}-\mathbf{A}) = \begin{pmatrix} -5 & 7 \end{pmatrix} \begin{pmatrix} -4 \\ -4 \end{pmatrix} \tag{1.1.7.2}$$

$$= -8 \tag{1.1.7.3}$$

$$\implies \cos A = \frac{-8}{\sqrt{74}\sqrt{32}} = \frac{-1}{\sqrt{37}} \tag{1.1.7.4}$$

$$\implies A = \cos^{-1}\frac{-1}{\sqrt{37}} \tag{1.1.7.5}$$

(b) From (1.1.1.2), (1.1.1.3), (1.1.2.6) and (1.1.2.9)

$$(\mathbf{C}-\mathbf{B})^{\top}(\mathbf{A}-\mathbf{B}) = \begin{pmatrix} 1 & -11 \end{pmatrix} \begin{pmatrix} 5 \\ -7 \end{pmatrix} \tag{1.1.7.6}$$

$$= 82 \tag{1.1.7.7}$$

$$\implies \cos B = \frac{82}{\sqrt{74}\sqrt{122}} = \frac{41}{\sqrt{2257}} \tag{1.1.7.8}$$

$$\implies B = \cos^{-1}\frac{41}{\sqrt{2257}} \tag{1.1.7.9}$$

(c) From (1.1.1.3), (1.1.1.4), (1.1.2.9) and (1.1.2.12)

$$(\mathbf{A}-\mathbf{C})^\top(\mathbf{B}-\mathbf{C}) = \begin{pmatrix} 4 & 4 \end{pmatrix} \begin{pmatrix} -1 \\ 11 \end{pmatrix} \tag{1.1.7.10}$$

$$= 40 \tag{1.1.7.11}$$

$$\implies \cos C = \frac{40}{\sqrt{32}\sqrt{122}} = \frac{5}{\sqrt{61}} \tag{1.1.7.12}$$

$$\implies C = \cos^{-1}\frac{5}{\sqrt{61}} \tag{1.1.7.13}$$

All codes for this section are available at

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codes/triangle/sides.py
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1.2. Median

1.2.1. If $\mathbf{D}$ divides BC in the ratio $k:1$,

$$\mathbf{D} = \frac{k\mathbf{C}+\mathbf{B}}{k+1} \tag{1.2.1.1}$$

Find the mid points $\mathbf{D}, \mathbf{E}, \mathbf{F}$ of the sides BC, CA and AB respectively.

Solution: Since $\mathbf{D}$ is the midpoint of BC,

$$k = 1, \tag{1.2.1.2}$$

$$\implies \mathbf{D} = \frac{\mathbf{C}+\mathbf{B}}{2} = \frac{1}{2}\begin{pmatrix} -7 \\ 1 \end{pmatrix} \tag{1.2.1.3}$$

Similarly,

$$\mathbf{E} = \frac{\mathbf{A}+\mathbf{C}}{2} = \begin{pmatrix} -1 \\ -3 \end{pmatrix} \tag{1.2.1.4}$$

$$\mathbf{F} = \frac{\mathbf{A}+\mathbf{B}}{2} = \frac{1}{2}\begin{pmatrix} -3 \\ 5 \end{pmatrix} \tag{1.2.1.5}$$

1.2.2. Find the equations of AD, BE and CF.

Solution: :

(a) The direction vector of AD is

$$\mathbf{m} = \mathbf{D} - \mathbf{A} = \begin{pmatrix} \frac{-7}{2} \\ \frac{1}{2} \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -9 \\ 3 \end{pmatrix} \equiv \begin{pmatrix} -3 \\ 1 \end{pmatrix} \tag{1.2.2.1}$$

$$\implies \mathbf{n} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{1.2.2.2}$$

Hence the normal equation of median AD is

$$\mathbf{n}^\top \left(\mathbf{x} - \mathbf{A}\right) = 0 \tag{1.2.2.3}$$

$$\implies \begin{pmatrix} 1 & 3 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = -2 \tag{1.2.2.4}$$

(b) For BE,

$$\mathbf{m} = \mathbf{E} - \mathbf{B} = \begin{pmatrix} -1 \\ -3 \end{pmatrix} - \begin{pmatrix} -4 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 \\ -9 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ -3 \end{pmatrix} \tag{1.2.2.5}$$

$$\implies \mathbf{n} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{1.2.2.6}$$

Hence the normal equation of median BE is

$$\mathbf{n}^\top \left(\mathbf{x} - \mathbf{B} \right) = 0 \tag{1.2.2.7}$$

$$\implies \begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 3 & 1 \end{pmatrix} \begin{pmatrix} -4 \\ 6 \end{pmatrix} = -6 \tag{1.2.2.8}$$

(c) For median CF,

$$\mathbf{m} = \mathbf{F} - \mathbf{C} = \begin{pmatrix} \frac{-3}{2} \\ \frac{5}{2} \end{pmatrix} - \begin{pmatrix} -3 \\ -5 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ \frac{15}{2} \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 5 \end{pmatrix} \tag{1.2.2.9}$$

$$\implies \mathbf{n} = \begin{pmatrix} 5 \\ -1 \end{pmatrix} \tag{1.2.2.10}$$

Hence the normal equation of median CF is

$$\mathbf{n}^\top \left(\mathbf{x} - \mathbf{C} \right) = 0 \tag{1.2.2.11}$$

$$\implies \begin{pmatrix} 5 & -1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 5 & -1 \end{pmatrix} \begin{pmatrix} -3 \\ -5 \end{pmatrix} = -10 \tag{1.2.2.12}$$

1.2.3. Find the intersection $\mathbf{G}$ of BE and CF.

Solution: From (1.2.2.8) and (1.2.2.12), the equations of BE and CF are, respectively,

$$\begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -6 \end{pmatrix} \tag{1.2.3.1}$$

$$\begin{pmatrix} 5 & -1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -10 \end{pmatrix} \tag{1.2.3.2}$$

From (1.2.3.1) and (1.2.3.2) the augmented matrix is

$$\begin{pmatrix} 3 & 1 & -6 \\ 5 & -1 & -10 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 + R_2} \begin{pmatrix} 8 & 0 & -16 \\ 5 & -1 & -10 \end{pmatrix} \tag{1.2.3.3}$$

$$\xleftrightarrow{R_1 \leftarrow R_1/8} \begin{pmatrix} 1 & 0 & -2 \\ 5 & -1 & -10 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - 5R_1} \begin{pmatrix} 1 & 0 & -2 \\ 0 & -1 & 0 \end{pmatrix} \tag{1.2.3.4}$$

$$\xleftrightarrow{R_2 \leftarrow -R_2} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \end{pmatrix} \tag{1.2.3.5}$$

using Gauss elimination. Therefore,

$$\mathbf{G} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \tag{1.2.3.6}$$

1.2.4. Verify that

$$\frac{BG}{GE} = \frac{CG}{GF} = \frac{AG}{GD} = 2 \tag{1.2.4.1}$$

Solution:

(a) From (1.2.1.4) and (1.2.3.6),

$$\mathbf{G}-\mathbf{B} = \begin{pmatrix} 2 \\ -6 \end{pmatrix}, \mathbf{E}-\mathbf{G} = \begin{pmatrix} 1 \\ -3 \end{pmatrix} \tag{1.2.4.2}$$

$$\implies \mathbf{G}-\mathbf{B} = 2\left(\mathbf{E}-\mathbf{G}\right) \tag{1.2.4.3}$$

$$\implies \left\|\mathbf{G}-\mathbf{B}\right\| = 2\left\|\mathbf{E}-\mathbf{G}\right\| \tag{1.2.4.4}$$

$$\text{or, } \frac{BG}{GE} = 2 \tag{1.2.4.5}$$

(b) From (1.2.1.5) and (1.2.3.6),

$$\mathbf{F}-\mathbf{G} = \frac{1}{2}\begin{pmatrix} 1 \\ 5 \end{pmatrix}, \mathbf{G}-\mathbf{C} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \tag{1.2.4.6}$$

$$\implies \mathbf{G}-\mathbf{C} = 2\left(\mathbf{F}-\mathbf{G}\right) \tag{1.2.4.7}$$

$$\implies \left\|\mathbf{G}-\mathbf{C}\right\| = 2\left\|\mathbf{F}-\mathbf{G}\right\| \tag{1.2.4.8}$$

$$\text{or, } \frac{CG}{GF} = 2 \tag{1.2.4.9}$$

(c) From (1.2.1.3) and (1.2.3.6),

$$\mathbf{G}-\mathbf{A} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}, \mathbf{D}-\mathbf{G} = \frac{1}{2}\begin{pmatrix} -3 \\ 1 \end{pmatrix} \tag{1.2.4.10}$$

$$\mathbf{G}-\mathbf{A} = 2\left(\mathbf{D}-\mathbf{G}\right) \tag{1.2.4.11}$$

$$\implies \left\|\mathbf{G}-\mathbf{A}\right\| = 2\left\|\mathbf{D}-\mathbf{G}\right\| \tag{1.2.4.12}$$

$$\text{or, } \frac{AG}{GD} = 2 \tag{1.2.4.13}$$

From (1.2.4.5), (1.2.4.9), (1.2.4.13)

$$\frac{BG}{GE} = \frac{CG}{GF} = \frac{AG}{GD} = 2 \tag{1.2.4.14}$$

1.2.5. Show that $\mathbf{A}, \mathbf{G}$ and $\mathbf{D}$ are collinear.

Solution: Points $\mathbf{A}, \mathbf{D}, \mathbf{G}$ are defined to be collinear if

$$\text{rank} \begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{D} & \mathbf{G} \end{pmatrix} = 2 \tag{1.2.5.1}$$

$$\implies \begin{pmatrix} 1 & 1 & 1 \\ 1 & -\frac{7}{2} & -2 \\ -1 & \frac{1}{2} & 0 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 + R_2} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -\frac{7}{2} & -2 \\ 0 & -3 & -2 \end{pmatrix} \tag{1.2.5.2}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -\frac{9}{2} & -3 \\ 0 & -3 & -2 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 - \frac{2}{3} R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -\frac{9}{2} & -3 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.2.5.3}$$

Thus, the matrix (1.2.5.1) has rank 2 and the points are collinear. Thus, the medians of a triangle meet at the point $\mathbf{G}$. See Fig. 1.2.

1.2.6. Verify that

$$\mathbf{G} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \tag{1.2.6.1}$$

$\mathbf{G}$ is known as the <u>centroid</u> of $\triangle ABC$.

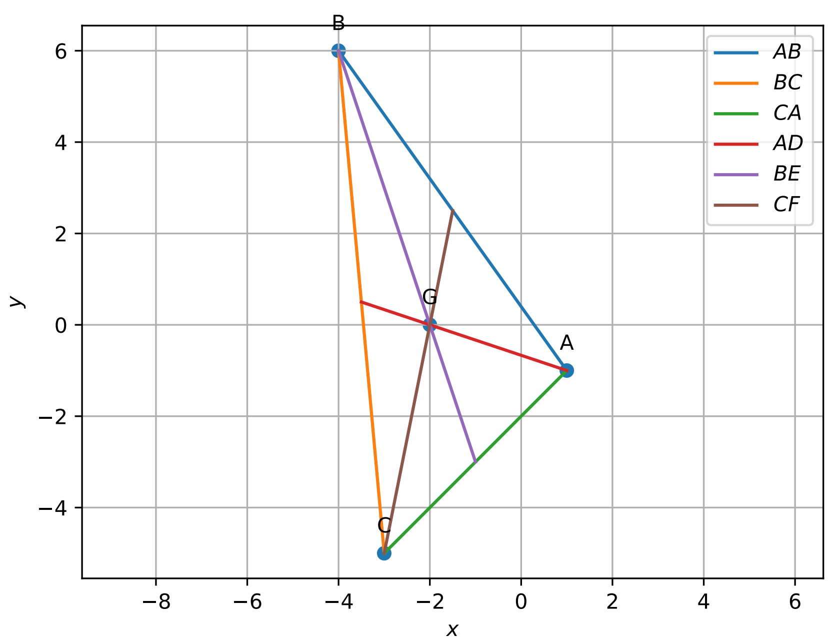


Figure 1.2: Medians of $\triangle ABC$ meet at $\mathbf{G}$.

Solution:

$$\mathbf{G} = \frac{\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -4 \\ 6 \end{pmatrix} + \begin{pmatrix} -3 \\ -5 \end{pmatrix}}{3} \tag{1.2.6.2}$$
$$= \begin{pmatrix} -2 \\ 0 \end{pmatrix}$$

1.2.7. Verify that

$$\mathbf{A} - \mathbf{F} = \mathbf{E} - \mathbf{D} \tag{1.2.7.1}$$

The quadrilateral $AFDE$ is defined to be a parallelogram.

Solution:

$$\mathbf{A} - \mathbf{F} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \begin{pmatrix} \frac{-3}{2} \\ \frac{5}{2} \end{pmatrix} = \begin{pmatrix} \frac{5}{2} \\ \frac{-7}{2} \end{pmatrix} \tag{1.2.7.2}$$

$$\mathbf{E} - \mathbf{D} = \begin{pmatrix} -1 \\ -3 \end{pmatrix} - \begin{pmatrix} \frac{-7}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{5}{2} \\ \frac{-7}{2} \end{pmatrix} \tag{1.2.7.3}$$

$$\implies \mathbf{A} - \mathbf{F} = \mathbf{E} - \mathbf{D} \tag{1.2.7.4}$$

See Fig. 1.3,

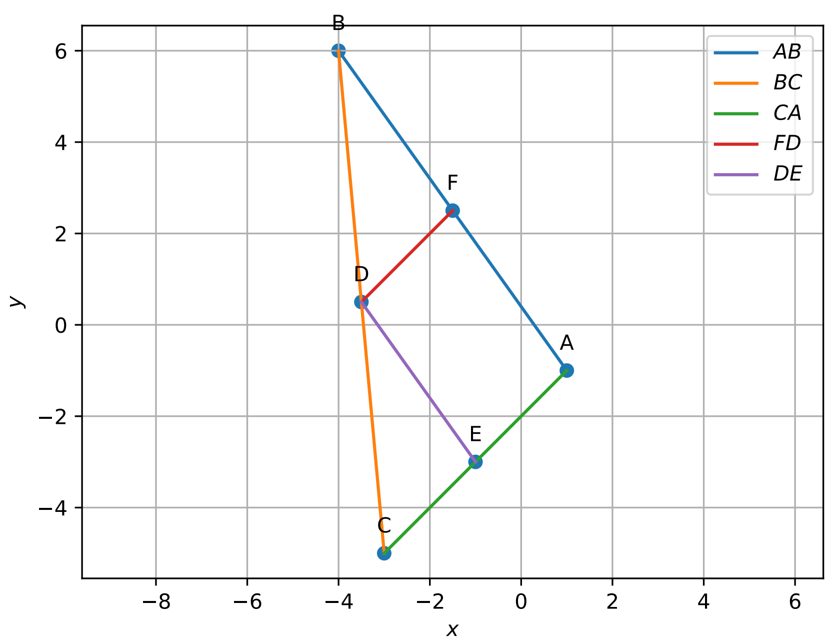


Figure 1.3: $AFDE$ forms a parallelogram in triangle ABC

1.3. Altitude

1.3.1. $\mathbf{D}_1$ is a point on BC such that

$$AD_1 \perp BC \tag{1.3.1.1}$$

and AD_1 is defined to be the altitude. Find the normal vector of AD_1.

Solution: The normal vector of AD_1 is the direction vector BC and is obtained from (1.1.1.3) as

$$\mathbf{n} = \begin{pmatrix} 1 \\ -11 \end{pmatrix} \tag{1.3.1.2}$$

1.3.2. Find the equation of AD_1.

Solution: The equation of AD_1 is

$$\mathbf{n}^\top(\mathbf{x} - \mathbf{A}) = 0 \tag{1.3.2.1}$$

$$\implies \begin{pmatrix} -1 & 11 \end{pmatrix}\mathbf{x} = \begin{pmatrix} -1 & 11 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = -12 \tag{1.3.2.2}$$

1.3.3. Find the equations of the altitudes BE_1 and CF_1 to the sides AC and AB respectively.

Solution:

(a) From (1.1.1.4), the normal vector of CF_1 is

$$\mathbf{n} = \begin{pmatrix} -5 \\ 7 \end{pmatrix} \tag{1.3.3.1}$$

and the equation of CF_1 is

$$\mathbf{n}^\top \left(\mathbf{x} - \mathbf{C}\right) = 0 \tag{1.3.3.2}$$

$$\implies \begin{pmatrix} -5 & 7 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} -3 \\ -5 \end{pmatrix}\right) = 0 \tag{1.3.3.3}$$

$$\implies \begin{pmatrix} 5 & -7 \end{pmatrix} \mathbf{x} = 20, \tag{1.3.3.4}$$

(b) Similarly, from (1.1.1.2), the normal vector of BE_1 is

$$\mathbf{n} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.3.3.5}$$

and the equation of BE_1 is

$$\mathbf{n}^\top \left(\mathbf{x} - \mathbf{B}\right) = 0 \tag{1.3.3.6}$$

$$\implies \begin{pmatrix} 1 & 1 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} -4 \\ 6 \end{pmatrix}\right) = 0 \tag{1.3.3.7}$$

$$\implies \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 2, \tag{1.3.3.8}$$

1.3.4. Find the intersection $\mathbf{H}$ of BE_1 and CF_1.

Solution: The intersection of (1.3.3.8) and (1.3.3.4), is obtained from the matrix equation

$$\begin{pmatrix} 1 & 1 \\ 5 & -7 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 2 \\ 20 \end{pmatrix} \tag{1.3.4.1}$$

which can be solved as

$$\begin{pmatrix} 1 & 1 & 2 \\ 5 & -7 & 20 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - 5R_1} \begin{pmatrix} 1 & 1 & 2 \\ 0 & -12 & 10 \end{pmatrix} \tag{1.3.4.2}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{-12}} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & \frac{-5}{6} \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & 0 & \frac{17}{6} \\ 0 & 1 & \frac{-5}{6} \end{pmatrix} \tag{1.3.4.3}$$

yielding

$$\mathbf{H} = \frac{1}{6} \begin{pmatrix} 17 \\ -5 \end{pmatrix}, \tag{1.3.4.4}$$

See Fig. 1.4

1.3.5. Verify that

$$(\mathbf{A} - \mathbf{H})^\top (\mathbf{B} - \mathbf{C}) = 0 \tag{1.3.5.1}$$

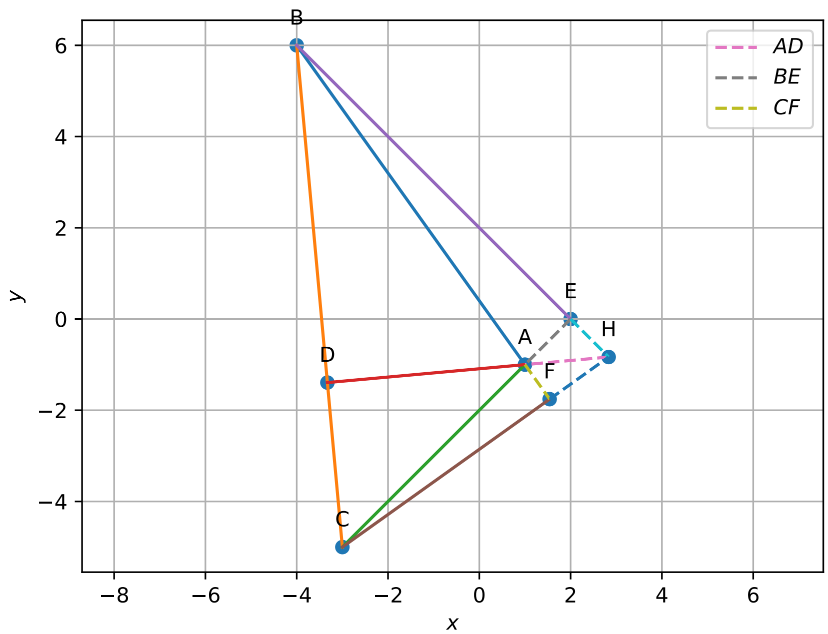


Figure 1.4: Altitudes BE_1 and CF_1 intersect at $\mathbf{H}$

Solution: From (1.3.4.4),

$$\mathbf{A} - \mathbf{H} = -\frac{1}{6}\begin{pmatrix} 11 \\ 1 \end{pmatrix}, \mathbf{B} - \mathbf{C} = \begin{pmatrix} -1 \\ 11 \end{pmatrix} \tag{1.3.5.2}$$

$$\implies (\mathbf{A} - \mathbf{H})^\top (\mathbf{B} - \mathbf{C}) = \frac{1}{6}\begin{pmatrix} 11 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 11 \end{pmatrix} = 0 \tag{1.3.5.3}$$

1.4. Perpendicular Bisector

1.4.1. The equation of the perpendicular bisector of BC is

$$\left(\mathbf{x} - \frac{\mathbf{B}+\mathbf{C}}{2}\right)(\mathbf{B}-\mathbf{C}) = 0 \tag{1.4.1.1}$$

Substitute numerical values and find the equations of the perpendicular bisectors of AB, BC and CA.

Solution: From (1.1.1.2), (1.1.1.3), (1.1.1.4), (1.2.1.3), (1.2.1.4) and (1.2.1.5),

$$\frac{\mathbf{B}+\mathbf{C}}{\mathbf{2}} = \frac{1}{2}\begin{pmatrix}-7\\1\end{pmatrix}, \mathbf{B}-\mathbf{C} = \begin{pmatrix}-1\\11\end{pmatrix} \tag{1.4.1.2}$$

$$\frac{\mathbf{A}+\mathbf{B}}{\mathbf{2}} = \frac{1}{2}\begin{pmatrix}-3\\5\end{pmatrix}, \mathbf{A}-\mathbf{B} = \begin{pmatrix}5\\-7\end{pmatrix} \tag{1.4.1.3}$$

$$\frac{\mathbf{C}+\mathbf{A}}{\mathbf{2}} = \begin{pmatrix}-1\\-3\end{pmatrix}, \mathbf{C}-\mathbf{A} = \begin{pmatrix}-4\\-4\end{pmatrix} \tag{1.4.1.4}$$

$$\tag{1.4.1.5}$$

yielding

$$(\mathbf{B}-\mathbf{C})^\top\left(\frac{\mathbf{B}+\mathbf{C}}{2}\right) = \begin{pmatrix}-1 & 11\end{pmatrix}\begin{pmatrix}-\frac{7}{2}\\ \frac{1}{2}\end{pmatrix} = 9 \tag{1.4.1.6}$$

$$(\mathbf{A}-\mathbf{B})^\top\left(\frac{\mathbf{A}+\mathbf{B}}{2}\right) = \begin{pmatrix}5 & -7\end{pmatrix}\begin{pmatrix}-\frac{3}{2}\\ \frac{5}{2}\end{pmatrix} = -25 \tag{1.4.1.7}$$

$$(\mathbf{C}-\mathbf{A})^\top\left(\frac{\mathbf{C}+\mathbf{A}}{2}\right) = \begin{pmatrix}-4 & -4\end{pmatrix}\begin{pmatrix}-1\\ -3\end{pmatrix} = 16 \tag{1.4.1.8}$$

Thus, the perpendicular bisectors are obtained from (1.4.1.1) as

$$BC: \quad \begin{pmatrix}-1 & 11\end{pmatrix}\mathbf{x} = 9 \tag{1.4.1.9}$$

$$CA: \quad \begin{pmatrix}5 & -7\end{pmatrix}\mathbf{x} = -25 \tag{1.4.1.10}$$

$$AB: \quad \begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = -4 \tag{1.4.1.11}$$

1.4.2. Find the intersection $\mathbf{O}$ of the perpendicular bisectors of AB and AC.

Solution:

The intersection of (1.4.1.10) and (1.4.1.11), can be obtained as

$$\begin{pmatrix} 5 & -7 & -25 \\ 1 & 1 & -4 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow 5R_2 - R_1} \begin{pmatrix} 5 & -7 & -25 \\ 0 & 12 & 5 \end{pmatrix} \tag{1.4.2.1}$$

$$\xleftrightarrow{R_1 \leftarrow \frac{12}{7}R_1 + R_2} \begin{pmatrix} \frac{60}{7} & 0 & \frac{-265}{7} \\ 0 & 12 & 5 \end{pmatrix} \xleftrightarrow[R_1 \leftarrow \frac{7}{60}R_1]{R_2 \leftarrow \frac{1}{12}R_2} \begin{pmatrix} 1 & 0 & \frac{-53}{12} \\ 0 & 1 & \frac{5}{12} \end{pmatrix} \tag{1.4.2.2}$$

$$\implies \mathbf{O} = \begin{pmatrix} \frac{-53}{12} \\ \frac{5}{12} \end{pmatrix} \tag{1.4.2.3}$$

1.4.3. Verify that $\mathbf{O}$ satisfies (1.4.1.1). $\mathbf{O}$ is known as the circumcentre.

Solution: Substituing from (1.4.2.3) in (1.4.1.1), when substituted in the above equation,

$$\begin{aligned} &\left(\mathbf{O} - \frac{\mathbf{B}+\mathbf{C}}{2}\right)^\top (\mathbf{B} - \mathbf{C}) \\ &\quad = \left(\frac{1}{12}\begin{pmatrix} -53 \\ 5 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -7 \\ 1 \end{pmatrix}\right)^\top \begin{pmatrix} -1 \\ 11 \end{pmatrix} \\ &\quad = \frac{1}{12}\begin{pmatrix} -11 & -1 \end{pmatrix}\begin{pmatrix} -1 \\ 11 \end{pmatrix} = 0 \end{aligned} \tag{1.4.3.1}$$

1.4.4. Verify that

$$OA = OB = OC \tag{1.4.4.1}$$

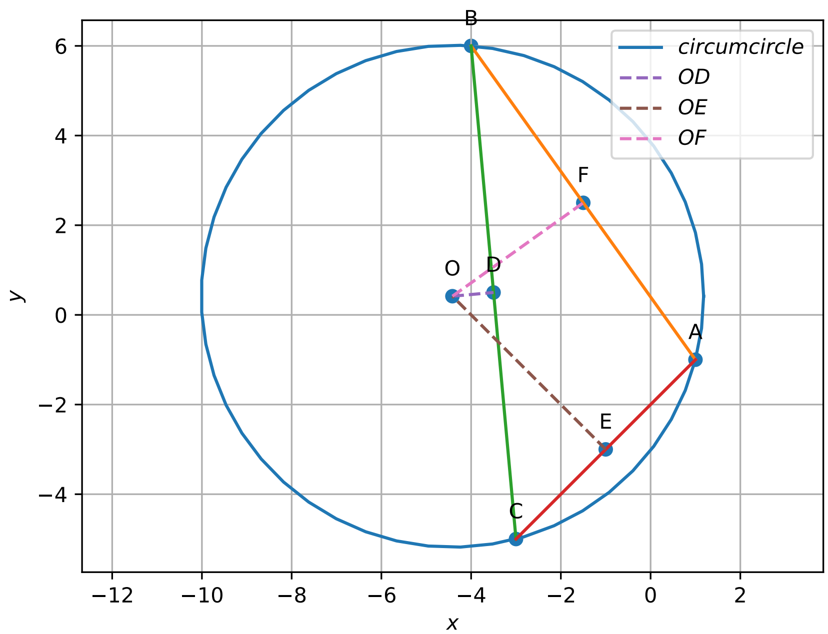


Figure 1.5: Circumcircle of $\triangle ABC$ with centre $\mathbf{O}$.

1.4.5. Draw the circle with centre at $\mathbf{O}$ and radius

$$R = OA \tag{1.4.5.1}$$

This is known as the circumradius.

Solution: See Fig. 1.5.

1.4.6. Verify that

$$\angle BOC = 2\angle BAC. \tag{1.4.6.1}$$

Solution:

(a) To find the value of $\angle BOC$:

$$\mathbf{B}-\mathbf{O}=\begin{pmatrix}\frac{5}{12}\\ \frac{67}{12}\end{pmatrix}, \mathbf{C}-\mathbf{O}=\begin{pmatrix}\frac{17}{12}\\ \frac{-65}{12}\end{pmatrix} \tag{1.4.6.2}$$

$$\implies (\mathbf{B}-\mathbf{O})^\top(\mathbf{C}-\mathbf{O})=\frac{-4270}{144} \tag{1.4.6.3}$$

$$\implies \|\mathbf{B}-\mathbf{O}\|=\frac{\sqrt{4514}}{12}, \|\mathbf{C}-\mathbf{O}\|=\frac{\sqrt{4514}}{12} \tag{1.4.6.4}$$

Thus,

$$\cos BOC=\frac{(\mathbf{B}-\mathbf{O})^\top(\mathbf{C}-\mathbf{O})}{\|\mathbf{B}-\mathbf{O}\|\,\|\mathbf{C}-\mathbf{O}\|}=\frac{-4270}{4514} \tag{1.4.6.5}$$

$$\implies \angle BOC=\cos^{-1}\left(\frac{-4270}{4514}\right) \tag{1.4.6.6}$$

$$=161.07536^\circ \text{ or } 198.92464^\circ \tag{1.4.6.7}$$

(b) To find the value of $\angle BAC$:

$$\mathbf{B}-\mathbf{A}=\begin{pmatrix}-5\\ 7\end{pmatrix}, \mathbf{C}-\mathbf{A}=\begin{pmatrix}-4\\ -4\end{pmatrix} \tag{1.4.6.8}$$

$$\implies (\mathbf{B}-\mathbf{A})^\top(\mathbf{C}-\mathbf{A})=-8 \tag{1.4.6.9}$$

$$\|\mathbf{B}-\mathbf{A}\|=\sqrt{74}\,\|\mathbf{C}-\mathbf{A}\|=4\sqrt{2} \tag{1.4.6.10}$$

Thus,

$$\cos BAC = \frac{(\mathbf{B}-\mathbf{A})^\top(\mathbf{C}-\mathbf{A})}{\|\mathbf{B}-\mathbf{A}\|\,\|\mathbf{C}-\mathbf{A}\|} = \frac{-8}{4\sqrt{148}} \tag{1.4.6.11}$$

$$\implies \angle BAC = \cos^{-1}\left(\frac{-8}{4\sqrt{148}}\right) \tag{1.4.6.12}$$

$$= 99.46232^\circ \tag{1.4.6.13}$$

From (1.4.6.13) and (1.4.6.7),

$$2 \times \angle BAC = \angle BOC \tag{1.4.6.14}$$

1.4.7. Let

$$\mathbf{P} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{1.4.7.1}$$

Find θ if

$$\mathbf{C}-\mathbf{O} = \mathbf{P}(\mathbf{A}-\mathbf{O}) \tag{1.4.7.2}$$

1.5. Angle Bisector

1.5.1. Let $\mathbf{D}_3, \mathbf{E}_3, \mathbf{F}_3$, be points on AB, BC and CA respectively such that

$$BD_3 = BF_3 = m, CD_3 = CE_3 = n, AE_3 = AF_3 = p. \tag{1.5.1.1}$$

Obtain m, n, p in terms of a, b, c obtained in Problem 1.1.2.

Solution: From the given information,

$$a = m + n, \tag{1.5.1.2}$$

$$b = n + p, \tag{1.5.1.3}$$

$$c = m + p \tag{1.5.1.4}$$

which can be expressed as

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} m \\ n \\ p \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{1.5.1.5}$$

$$\implies \begin{pmatrix} m \\ n \\ p \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{1.5.1.6}$$

Using row reduction,

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xleftrightarrow{R_3 \leftarrow R_3 - R_1} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right) \tag{1.5.1.7}$$

$$\xleftrightarrow[R_1 \leftarrow R_1 - R_2]{R_3 \leftarrow R_3 + R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right) \tag{1.5.1.8}$$

$$\xleftrightarrow[R_1 \leftarrow 2R_1+R_3]{R_2 \leftarrow 2R_2-R_3} \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & -1 & 1 \\ 0 & 2 & 0 & 1 & 1 & -1 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right) \tag{1.5.1.9}$$

yielding

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix} \tag{1.5.1.10}$$

Therefore,

$$\begin{aligned} p &= \frac{c+b-a}{2} = \frac{\sqrt{74}+\sqrt{32}-\sqrt{122}}{2} \\ m &= \frac{a+c-b}{2} = \frac{\sqrt{74}+\sqrt{122}-\sqrt{32}}{2} \\ n &= \frac{a+b-c}{2} = \frac{\sqrt{122}+\sqrt{32}-\sqrt{74}}{2} \end{aligned} \tag{1.5.1.11}$$

upon substituting from (1.1.2.6), (1.1.2.9) and (1.1.2.12).

1.5.2. Using section formula, find

$$\mathbf{D}_3 = \frac{m\mathbf{C}+n\mathbf{B}}{m+n}, \mathbf{E}_3 = \frac{n\mathbf{A}+p\mathbf{C}}{n+p}, \mathbf{F}_3 = \frac{p\mathbf{B}+m\mathbf{A}}{p+m} \tag{1.5.2.1}$$

1.5.3. Find the circumcentre and circumradius of $\triangle D_3E_3F_3$. These are the incentre and inradius of $\triangle ABC$.

1.5.4. Draw the circumcircle of $\triangle D_3E_3F_3$. This is known as the incircle of $\triangle ABC$.

Solution: See Fig. 1.6

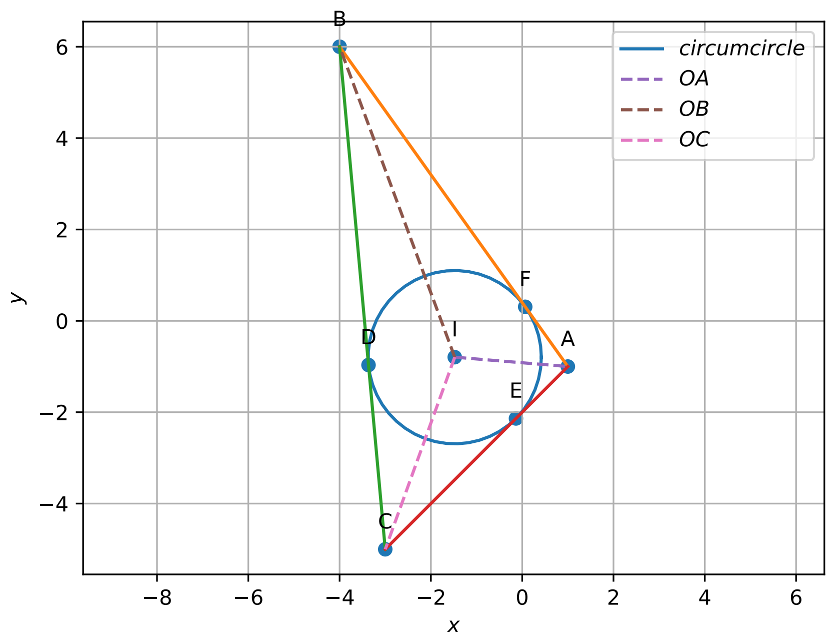


Figure 1.6: Incircle of $\triangle ABC$

1.5.5. Using (1.1.7.1) verify that

$$\angle BAI = \angle CAI. \tag{1.5.5.1}$$

AI is the bisector of $\angle A$.

1.5.6. Verify that BI, CI are also the angle bisectors of $\triangle ABC$.

1.6. Matrices

The matrix of the veritices of the triangle is defined as

$$\mathbf{P} = \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \tag{6.1}$$

1.6.1. Vectors

1.6.1.1. Obtain the direction matrix of the sides of $\triangle ABC$ defined as

$$\mathbf{M} = \begin{pmatrix} \mathbf{A}-\mathbf{B} & \mathbf{B}-\mathbf{C} & \mathbf{C}-\mathbf{A} \end{pmatrix} \tag{1.6.1.1.1}$$

Solution:

$$\mathbf{M} = \begin{pmatrix} \mathbf{A}-\mathbf{B} & \mathbf{B}-\mathbf{C} & \mathbf{C}-\mathbf{A} \end{pmatrix} \tag{1.6.1.1.2}$$

$$= \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \tag{1.6.1.1.3}$$

where the second matrix above is known as a circulant matrix. Note that the 2nd and 3rd row of the above matrix are circular shifts of the 1st row.

1.6.1.2. Obtain the normal matrix of the sides of $\triangle ABC$

Solution: Considering the roation matrix

$$\mathbf{R} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{1.6.1.2.1}$$

the normal matrix is obtained as

$$\mathbf{N} = \mathbf{R}\mathbf{M} \tag{1.6.1.2.2}$$

1.6.1.3. Obtain a, b, c.

Solution: The sides vector is obtained as

$$\mathbf{d} = \sqrt{\text{diag}(\mathbf{M}^\top \mathbf{M})} \tag{1.6.1.3.1}$$

1.6.1.4. Obtain the constant terms in the equations of the sides of the triangle.

Solution: The constants for the lines can be expressed in vector form as

$$\mathbf{c} = \text{diag}\left\{\left(\mathbf{N}^\top \mathbf{P}\right)\right\} \tag{1.6.1.4.1}$$

1.6.2. Median

1.6.2.1. Obtain the mid point matrix for the sides of the triangle

Solution:

$$\begin{pmatrix} \mathbf{D} & \mathbf{E} & \mathbf{F} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \tag{1.6.2.1.1}$$

1.6.2.2. Obtain the median direction matrix.

Solution: The median direction matrix is given by

$$\mathbf{M}_1 = \begin{pmatrix} \mathbf{A}-\mathbf{D} & \mathbf{B}-\mathbf{E} & \mathbf{C}-\mathbf{F} \end{pmatrix} \tag{1.6.2.2.1}$$

$$= \begin{pmatrix} \mathbf{A}-\frac{\mathbf{B}+\mathbf{C}}{2} & \mathbf{B}-\frac{\mathbf{C}+\mathbf{A}}{2} & \mathbf{C}-\frac{\mathbf{A}+\mathbf{B}}{2} \end{pmatrix} \tag{1.6.2.2.2}$$

$$= \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix}\begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix} \tag{1.6.2.2.3}$$

1.6.2.3. Obtain the median normal matrix.

1.6.2.4. Obtian the median equation constants.

1.6.2.5. Obtain the centroid by finding the intersection of the medians.

1.6.3. Altitude

1.6.3.1. Find the normal matrix for the altitudes

Solution: The desired matrix is

$$\mathbf{M}_2 = \begin{pmatrix} \mathbf{B}-\mathbf{C} & \mathbf{C}-\mathbf{A} & \mathbf{A}-\mathbf{B} \end{pmatrix} \tag{1.6.3.1.1}$$

$$= \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \tag{1.6.3.1.2}$$

1.6.3.2. Find the constants vector for the altitudes.

Solution: The desired vector is

$$\mathbf{c}_2 = \operatorname{diag}\left\{\left(\mathbf{M}^\top \mathbf{P}\right)\right\} \tag{1.6.3.2.1}$$

1.6.4. Perpendicular Bisector

1.6.4.1. Find the normal matrix for the perpendicular bisectors

Solution: The normal matrix is $\mathbf{M}_2$

1.6.4.2. Find the constants vector for the perpendicular bisectors.

Solution: The desired vector is

$$\mathbf{c}_3 = \operatorname{diag}\left\{\mathbf{M}_2^\top \begin{pmatrix} \mathbf{D} & \mathbf{E} & \mathbf{F} \end{pmatrix}\right\} \tag{1.6.4.2.1}$$

1.6.5. Angle Bisector

1.6.5.1. Find the points of contact.

Solution: The points of contact are given by

$$\begin{pmatrix} \frac{m\mathbf{C}+n\mathbf{B}}{m+n} & \frac{n\mathbf{A}+p\mathbf{C}}{n+p} & \frac{p\mathbf{B}+m\mathbf{A}}{p+m} \end{pmatrix} = \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \begin{pmatrix} 0 & \frac{n}{b} & \frac{m}{c} \\ \frac{n}{a} & 0 & \frac{p}{c} \\ \frac{m}{a} & \frac{p}{b} & 0 \end{pmatrix} \tag{1.6.5.1.1}$$

Chapter 2

Linear Equations

2.1. 9

2.1.1. 9.3.3

1. In which quadrant or on which axis do each of the points(-2,4),(3,-1),(-1,0),(1,2) and (-3,-5) lie? Verify your answer by locating them on the Cartesian plane.

2. Plot the points (x, y) given in the following table on the plane,choosing the suitable units of distance on the axes.

x	-2	-1	0	1	3
y	8	7	-1.25	3	-1

Table 2.1: table of values

2.1.2. 9.4.1

1. The cost of a notebook is twice the cost of a pen.Write a linear equation in two variables to represent this statement. (Take the Cost of a

notebook to be x and that of a pen to be y).

2. Express the following linear equation in the form $ax + by + c = 0$ and indicate the values of a, b and c in each case:

   (i) $2x + 3y = 9.3\overline{5}$

   (ii) $x - \frac{y}{5} - 10 = 10$

   (iii) $-2x + 3y = 6$

   (iv) $x = 3y$

   (v) $2x = -5y$

   (vi) $3x + 2 = 0$

   (vii) $y - 2 = 0$

   (viii) $5 = 2x$

2.1.3. 9.4.2

1. Which one of the following options is true, and why? $y = 3x + 5$ has

   (i) a unique solution

   (ii) only two solutions

   (iii) infinitely many solutions

2. Write four solutions for each of the following equations

   (i) $2x + y = 7$

   (ii) $\pi x + y = 9$

(iii) $x = 4y$

3. Check which of the following are solutions of the equation $x - 2y = 4$ and which are not

   (i) $(0, 2)$

   (ii) $(2, 0)$

   (iii) $(4, 0)$

   (iv) $(\sqrt{2}, 4\sqrt{2})$

   (v) $(1, 1)$

4. Find the value of k if $x = 2$, $y = 1$ is a solution of the equation of the equation $2x + 3y = 4$

2.1.4. 9.4.3

1. Draw the graph of each of the following linear equations in two variables:

   (i) $x + y = 4$

   (ii) $x - y = 2$

   (iii) $y = 3x$

   (iv) $3 = 2x + y$

2. Give the equations of two lines passing through (2,14).How many more such lines are there and why?

3. If the point(3,4) lies on the graph of the equation $3y = ax + 7$ find the value of a

4. The taxi fare in the city is as follows:for te first kilometre,the fare is ₹ 8 and for the subsquent distance is ₹ 5 per km.Taking the distance Covered as x km and total fare as ₹ y.Write a linear equation for this information,and draw its graph.

5. From the choices given below,choose the equation whose graphs are given Fig 4.6 Fig 4.7

   For fig-4.6

   (i) $y = x$

   (ii) $x + y = 0$

   (iii) $y = 2x$

   (iv) $2 + 3y = 7x$

   For fig-4.7

   (i) $y = x + 2$

   (ii) $y = x - 2$

   (iii) $y = -x + 2$

   (iv) $x + 2y = 6$

6. If the work done by a body of a constant force is directly proportional to the distance travelled by the body,express this in the form of an equation and draw the graph of the same by taking the varables and

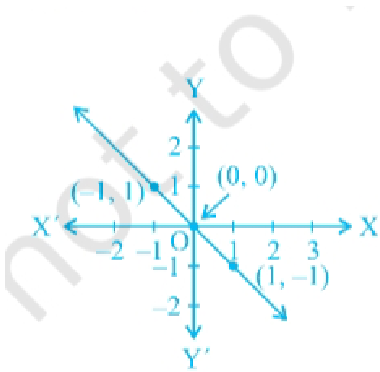


Fig. 4.6

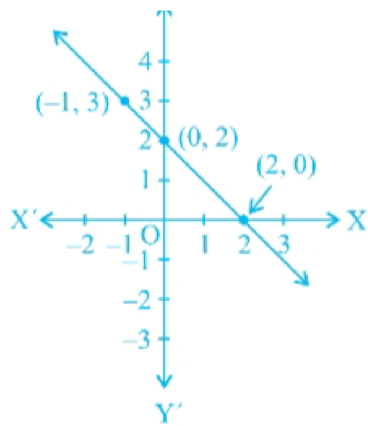


Fig. 4.7

Figure 2.1: Graph

draw the graph of the same by taking the constant force as $5 units$.Also read from the graph the work done when the distance travelled by the body is

(i) 2Units

(ii) 0Unit

7. Yamini and Fatima,two students of class IX of a school,together contributed ₹ 100 towards the prime minister's reief fund to help the earhquake victims.Write a linear equation which satisfies this data. (you may take their contributions ₹ x and ₹ y.Draw the graph of the same.

8. In Countries like USA and Canada temperature is measured in Celsius. Here is a linear equation that converts Farenheit to celsius: $F = \frac{9}{5}C + 32$

(i) Draw the graph of the linear equation above using Celsius for x axis and Farenheit for y axis

(ii) If the temperature is $30°C$,what is the temperature in farenheight?

(iii) If the temperature is $95°F$,what is the temperature in celsius?

(iv) If the temperature is $0°C$.What is the temperature in Farenheit and if the temperature in celsius?

(v) Is there a temperature Which is numerically same in both Farenheit and Celsius? If yes find it.

2.2. 10

2.2.1. Examples:-1-19 (10.3)

1. Let us take the example given in Section 3.1. Akhila goes to a fair with ₹ 20 and wants to have rides on the Giant Wheel and play Hoopla. Represent this situation algebraically and graphically (geometrically).

2. Romila went to a stationary shop and purchased 2 pencils and 3 erasers for ₹ 9. Her friend Sonali saw the new variety of pencils and erasers with Romila, and she also bought 4 pencils and 6 erasers of the same kind for ₹ 18. Represent this situation algebraically and graphically.

3. Two rails are represented by the equations $x + 2y - 4 = 0$ and $2x + 4y - 12 = 0$. Represent this situation geometrically.

4. Check graphically whether the pair of equations.

$$x + 3y = 6 \tag{4.1}$$

$$\text{and } 2x - 3y = 12 \tag{4.2}$$

is consistent. If so, Solve them graphically.

5. Graphically, find whether the following pair of equatons has no solution, unique solution or infinitely many solutions.

$$5x - 8y + 1 = 0 \tag{5.1}$$

$$3x - \frac{24}{5}y + \frac{3}{5} = 0 \tag{5.2}$$

6. Champa went to a "Sale" to purchase some pants and skirts. When her friends asked her how many of each she had boughtc she answered, "The number of skirts is two less than twice the number of pants purchased. Also, the number of skirts is four less than four times the number of pants purchased". Help her friends to find how many pants and skirts Champa bought.

7. Solve the following pair of equations by substitution method:

$$7x - 15y = 2 \tag{7.1}$$

$$x + 2y = 3 \tag{7.2}$$

8. Solve Q.1 of Exercise 3.1 by the method of substitution.

Aftab tells his daughter, “Seven years ago, I was seven times as old as you were then. Also, three years from now, I shall be three times as old as you will be.” (Isn’t this interesting ?) Represent this situation algebraically and graphically.

9. Let us consider Example 2 in Section 3.3 i.e., the cost of 2 pencils and 3 erasers is ₹ 9 and the cost of 4 pencils and 6 erasers is ₹ 18. Find the cost of each pencil and each eraser.

10. Let us consider the Example 3 of Section 3.2. Will the rails cross each other? Two rails are represented by the equations $x + 2y - 4 = 0$ and $2x + 4y - 12 = 0$. Represent this situation geometrically.

11. The ratio of incomes of two persons is 9:7 and the ratio of their expenditures is 4 : 3. If each of them manages to save ₹ 2000 per month, find their monthly incomes.

12. Use elimination method to find all possible solutions of the following pair of linear equations:-

$$2x + 3y = 8 \tag{12.1}$$

$$4x + 6y = 7 \tag{12.2}$$

13. The sum of a two digit number and the number obtained by reversing the digits is 66. If the digits of the number differ by 2, find the number. How many such numberes are there?

14. From a bus stand in Bangalore, if we buy 2 tickets to Malleshwaram

and 2 tickets to yeshwanthpur the total cost is ₹ 74. Find the fare from the bus stand to Malleshwaram, and to Yeshwanthpur.

15. For which values p does the pair of equations given below has unique solution.

$$4x + py + 8 = 0 \tag{15.1}$$
$$2x + 2y + 2 = 0 \tag{15.2}$$

16. For what values of k will the following pair of linear equations have infinitely many solutions.

$$kx + 3y - (k - 3) = 0 \tag{16.1}$$
$$12x + ky - k = 0 \tag{16.2}$$

17. Solve the pair of equations:

$$\frac{2}{x} + \frac{3}{y} = 13 \tag{17.1}$$
$$\frac{5}{x} + \frac{4}{y} = -2 \tag{17.2}$$

18. Solve the following pair of linear equations by reducing them to a pair of linear equations

$$\frac{5}{x-1} + \frac{1}{y-2} = 2 \tag{18.1}$$
$$\frac{6}{x-1} - \frac{3}{y-2} = 1 \tag{18.2}$$

19. A boat goes 30 km upstream and 44 km downstream in 10 hours. In 13 hours, it can go 40 km upstream and 55 km down-stream. Determine the speed of the stream and that of the boat in still water.

2.2.2. 10.3.1

1. Aftab tells his daughter, "Seven years ago, I was seven times as old as you were then. Also, 3 years from now, I shall be 3 times as old as you will be. "(Isn't it interesting? Represent this situation algebraically and graphically.

2. The coach of a cricket team buys 3 bats and 6 balls for $Rs.3900$. Later, she buys another bat and 3 more balls of the same kind for $Rs.3300$. Represent this situation algebraically and geometrically.

3. The cost of $2kg$ of apples and $1kg$ of grapes on a day was found to be $Rs.160$. After a month, the cost of $4kg$ of apples and $2kg$ of grapes is $Rs.300$. Represent this situation algebraically and geometrically.

2.2.3. 10.3.2

1. Form the pair of linear equations in the following problems and find their solutions graphically:

   (i) 10 students of Class X took part in a Mathematics quiz. If the number of girls is 4 more than the number of boys, find the number of boys and girls who took part in the quiz.

(ii) 5 pencils and 7 pens together cost $Rs.50$ whereas 7 pencils and 5 pens together cost $Rs.46$. Find the cost of one pencil and that of one pen.

2. On comparing the ratios $\frac{a_1}{a_2}, \frac{b_1}{b_2}$ and $\frac{c_1}{c_2}$, find out whether the lines representing the following pairs of linear equations intersect at a point, are parallel or coincident:

(i)

$$5x - 4y + 8 = 0 \tag{2.1}$$

$$7x + 6y - 9 = 0 \tag{2.2}$$

(ii)

$$9x + 3y + 12 = 0 \tag{2.3}$$

$$18x + 6y + 24 = 0 \tag{2.4}$$

(iii)

$$6x - 3y + 10 = 0 \tag{2.5}$$

$$2x - y + 9 = 0 \tag{2.6}$$

3. On comparing the ratios $\frac{a_1}{a_2}, \frac{b_1}{b_2}$ and $\frac{c_1}{c_2}$, find out whether the following equations are consistent, or inconsistent:

(i)

$$3x + 2y = 5; \tag{3.1}$$

$$2x - 3y = 7 \tag{3.2}$$

(ii)

$$2x - 3y = 8; \tag{3.3}$$

$$4x - 6y = 9 \tag{3.4}$$

(iii)

$$\frac{3}{2}x + \frac{5}{3}y = 7; \tag{3.5}$$

$$9x - 10y = 14 \tag{3.6}$$

(iv)

$$5x - 3y = 11; \tag{3.7}$$

$$-10x + 6y = -22 \tag{3.8}$$

(v)

$$\frac{4}{3}x + 2y = 8; \tag{3.9}$$

$$2x + 3y = 12 \tag{3.10}$$

4. Which of the following pairs of linear equations are consistent/inconsistent? If consistent, obtain solution graphically:

(i)

$$x + y = 5; \tag{4.1}$$
$$2x + y = 10 \tag{4.2}$$

(ii)

$$x - y = 8; \tag{4.3}$$
$$3x - 3y = 16 \tag{4.4}$$

(iii)

$$2x + y - 6 = 0; \tag{4.5}$$
$$4x - 2y + 4 = 0 \tag{4.6}$$

(iv)

$$2x - 2y - 2 = 0; \tag{4.7}$$
$$4x - 4y - 5 = 0 \tag{4.8}$$

5. Half the perimeter of a rectangular garden,whose length is $4m$, more than its width,is $36m$. Find the dimensions of the garden.

6. Given the linear equation $2x+3y-8 = 0$, write another linear equation

in two variables such that geometrical representation of the pair so formed is:

(i) intersecting lines

(ii) parallel lines

(iii) coincident lines

7. Draw the graphs of the equations $x - y + 1 = 0$ and $3x + 2y - 12 = 0$. Determine the coordinates of the vertices of the triangle formed by these lines and the axis and shade the triangular region.

2.2.4. 10.3.3

1. Solve the following pair of linear equations by the substitution method.

(i)

$$x + y = 14 \tag{1.1}$$

$$x - y = 4 \tag{1.2}$$

(ii)

$$s - t = 3 \tag{1.3}$$

(iii)

$$3x - y = 3 \tag{1.4}$$

$$9x - 3y = 9 \tag{1.5}$$

$$0.2x + 0.3y = 1.3 \tag{1.6}$$

$$0.4x + 0.5y = 23 \tag{1.7}$$

(iv)

$$\sqrt{2x} + \sqrt{3y} = 0 \tag{1.8}$$

$$\sqrt{3x} - \sqrt{8y} = 0 \tag{1.9}$$

(v)

$$\frac{3x}{2} - \frac{5y}{2} = -2 \tag{1.10}$$

$$\frac{x}{3} + \frac{y}{2} = \frac{13}{6} \tag{1.11}$$

2. Solve $2x + 3y = 11$ and $2x + 4y = -24$ and hence find the value of m for which $y = mx + 3$

3. Form the pair of linear equations for the following problems and find their solutions by the substitution method

   (I) The difference between two numbers is 26 and one number is three times the other. Find them.

(II) The larger of two supplementary angles exceeds the smaller by 18 degrees. Find them.

(III) The coach of a cricket team buys 7 balls and 6 balls for ₹ 3800. Later, she buys 3 bats and 5 balls for ₹ 1750. Find the cost of each bat and each ball.

(IV) The taxi charges in a city consist of a fixed charge together with the charges for the distance covered. For a distance of 10 km, the charge paid is ₹ 105 and for a distance of 15 km, the charge paid is ₹ 155. What are the fixed charges and the charge per km? How much does a person have to pay for travelling a distance of 25 km ?

(V) A fraction becomes $\frac{9}{11}$ if 2 is added to both the numerator and the denominator. If 3 is added to both the numerator and the denominator, it becomes $\frac{5}{6}$. Find the fraction.

(VI) Five years hence, the age of Jacob will be three times that of his son. Five years ago, Jacob's age was seven times that of his son. What are their present ages?

2.2.5. 10.3.4

1. Solve the following pair of linear equations by the elimination method and the substitution method:

   (i) $x + y = 5$ and $2x - 3y = 4$

   (ii) $3x + 4y = 10$ and $2x - 2y = 2$

(iii) $3x - 5y - 4 = 0$ and $9x = 2y + 7$

(iv) $\frac{x}{2} + \frac{2y}{3} = -1$ and $x - \frac{y}{3} = 0$

2. Form the pair of linear equations in the following problem, and find their solutions(if they exist) by the elimination method:

   (i) If we add 1 to the numerator and subtract 1 from the denominator, a fraction reduces to 1. It becomes $\frac{1}{2}$ if we only add 1 to the denominator. What is the fraction?

   (ii) Five years ago, Nuri was thrice as old as sonu. Ten years later, Nuri will be twice as old as sonu. How old are Nuri and sonu?

   (iii) The Sum of the digits of a two-digit number is 9. Also, nine times this number is twice the number obtained by reversing the order of the digits. Find the number.

   (iv) Meena Went to a bank to withdraw ₹ 2000. She asked the cashier to give her ₹ 50 and ₹ 100 notes only. Meena got 25 notes in all. Find how many notes ₹ 50 and ₹ 100 she received.

   (v) A lending library has a fixed charge for the first three days and an additional charge for each day thereafter. Sarita paid ₹ 27 for seven days, While susy paid ₹ 21 for the book she paid for five days. Find the fixed charge and the charge for each extra day.

2.2.6. 10.3.5

1. Which of the following pairs of linear equations has unique solution, no solution or infinitely many solutions.In case there is a unique solu-

tion,find it by using cross multiplication method:

(i)

$$x - 3y - 3 = 0 \tag{1.1}$$

$$3x - 9y - 2 = 0 \tag{1.2}$$

(ii)

$$2x + y = 5 \tag{1.3}$$

$$3x + 2y = 8 \tag{1.4}$$

(iii)

$$3x - 5y = 20 \tag{1.5}$$

$$6x - 10y = 40 \tag{1.6}$$

(iv)

$$x - 3y - 7 = 0 \tag{1.7}$$

$$3x - 3y - 15 = 0 \tag{1.8}$$

2. (i) For which values of a and b does the following pair of linear

equations have an infinite number of solutions?

$$2x + 3y = 7 \tag{2.1}$$

$$(a-b)x + (a-b)y = 3a + b - 2 \tag{2.2}$$

(ii) For which value of k will the following pair of linear equation have no solution?

$$3x + y = 1 \tag{2.3}$$

$$(2k-1)x + (k-1)y = 2k + 1 \tag{2.4}$$

3. Solve the following pair of linear equations by the substituions and cross multiplication method:

$$8x + 5y = 9 \tag{3.1}$$

$$3x + 2y = 4 \tag{3.2}$$

4. Form the pair of linear equations in the following problems and find their solutions by any algebraic method:

   (i) A part of monthly hostel charges is fixed and the remaining depends on the number of days one has taken food in the mess. When a student A takes food for 20 days she has to pay Rs.1000 as hostel charges whereas a student B who takes food for 26 days, pays Rs.1180 as hostel charges. Find the fixed charges and the cost of food per day.

(ii) A fraction becomes $\frac{1}{3}$ when 1 is subtracted from the numerator and it becomes $\frac{1}{4}$ when 8 is added to the denominator. Find the fraction.

(iii) Yash scored 40 marks in a test, getting 3 marks for each right answer and losing 1 mark for each wrong answer. Had 4 marks been awarded for each correct answer and 2 marks been deducted for each incorrect answer, then Yash would have scored 50 marks. How many questions were there in the test?

(iv) Places A and B are $100km$ apart on a highway. One car starts from A and another from B at the same time. If the car travel in the same direction at different speeds, they meet in 5hrs. If they travel towards each other, they meet in 1hr. What are the speeds of the two cars?

(v) The area of rectangle gets reduced by 9 square units, if its length is reduced by 5 units and breadth is increased by 3 units. If we increase the length by 3 units and the breadth by 2 units, the area increases by 67 square units. Find the dimensions of the rectangle.

2.2.7. 10.3.6

1. Solve the following pair of equations by reducing them to a pair of linear equations:

(i)

$$\frac{1}{2x} + \frac{1}{3y} = 2 \tag{1.1}$$

$$\frac{1}{3x} + \frac{1}{2y} = \frac{13}{6} \tag{1.2}$$

(ii)

$$\frac{2}{\sqrt{x}} + \frac{3}{\sqrt{y}} = 2 \tag{1.3}$$

$$\frac{4}{\sqrt{x}} - \frac{9}{\sqrt{y}} = -1 \tag{1.4}$$

(iii)

$$\frac{4}{x} + 3y = 14 \tag{1.5}$$

$$\frac{3}{x} - 4y = 23 \tag{1.6}$$

(iv)

$$\frac{5}{x-1} + \frac{1}{y-2} = 2 \tag{1.7}$$

$$\frac{6}{x-1} - \frac{3}{y-2} = 1 \tag{1.8}$$

(v)

$$\frac{7x-2y}{xy} = 5 \tag{1.9}$$

$$\frac{8x+7y}{xy} = 15 \tag{1.10}$$

(vi)

$$6x + 3y = 6xy \tag{1.11}$$

$$2x + 4y = 5xy \tag{1.12}$$

(vii)

$$\frac{10}{x+y} + \frac{2}{x-y} = 4 \tag{1.13}$$

$$\frac{15}{x+y} - \frac{5}{x-y} = -2 \tag{1.14}$$

(viii)

$$\frac{1}{3x+y} + \frac{1}{3x-y} = \frac{3}{4} \tag{1.15}$$

$$\frac{1}{2(3x+y)} - \frac{1}{2(3x-y)} = \frac{-1}{8} \tag{1.16}$$

2. Formulate the following problems as a pair of equations,and hence find their solutions:

   (i) Ritu can row downstream $20km$ in 2 hours, and upstream $4km$ in 2 hours. Find her speed of rowing in still water and the speed of the current.

   (ii) 2 women and 5 men can together finish an embroidery work in 4 days, while 3 women and 6 men can finish it in 3 days. Find the time taken by 1 women along to finish the work, and also that taken by 1 men alone.

(iii) Roohi travels $300km$ to her home partly by train and partly by bus. She takes 4 hours if she travels $60km$ by train and the remaining by bus. If she travels $100km$ by train and the remaining by bus, she takes 10 minutes longer. Find the speed of the train and the bus seperately.

2.2.8. 10.3.7

1. The ages of two friends ani and Biju differ by 3 years. Ani's father dharam is twice as old as Ani and Biju is twice as old as sister cathy. The ages of cathy and dharam differ by 30 years. Find the ages of Ani and Biju.

2. One says, "Give me a hundred, Friend ! I shall then become twice as rich as you ". The other "if you give me ten, i shall be six times as rich as you ". Tell me What is the amount of their (respective) capital? [From the bijaganita of bhaskara II]
   $[Hint : X + 100 = 2(y - 100), y + 10 = 6(x - 10)]$

3. A train Covered a certain distance at a uniform speed. If the train would have been 10 km/h faster, it would have taken 2 hours less than the scheduled time. And, if the train were slower by 10 km/h; it would have taken 3 hours more than the scheduled time. Find the distance covered by the train.

4. The students of a class are made to stand in rows. If 3 Students are extra in a row, there would be 1 row less. If 3 students are less in a

row, there would be 2 rows more. Find the number of stuents in the class.

5. In a $\triangle ABC, \angle C = 3\angle B = 2(\angle A + \angle B)$. Find the three angles

6. Draw the graphs of the equations $5x - y = 5$ and $3x - y = 3$. Determine the Co-ordinates of the vertices of the triangle formed by these lines and the y axis.

7. Solve the following pair of linear equations;

   (a)

$$px + qy = p - q \tag{7.1}$$

$$qx - py = p + q \tag{7.2}$$

   (b)

$$ax + by = c \tag{7.3}$$

$$bx + ay = 1 + c \tag{7.4}$$

   (c)

$$\frac{x}{a} - \frac{y}{b} = 0 \tag{7.5}$$

$$ax + by = a^2 + b^2 \tag{7.6}$$

(d)

$$(a-b)x+(a+b)y=a^2-2ab-b^2 \tag{7.7}$$

$$(a+b)(x+y)=a^2+b^2 \tag{7.8}$$

(e)

$$152x-378y=-74 \tag{7.9}$$

$$-378x+152y=-604 \tag{7.10}$$

8. $ABCD$ is a cyclic quadrilateral [see Fig. 2.2]. Find the angles of the cyclic quadrilateral.

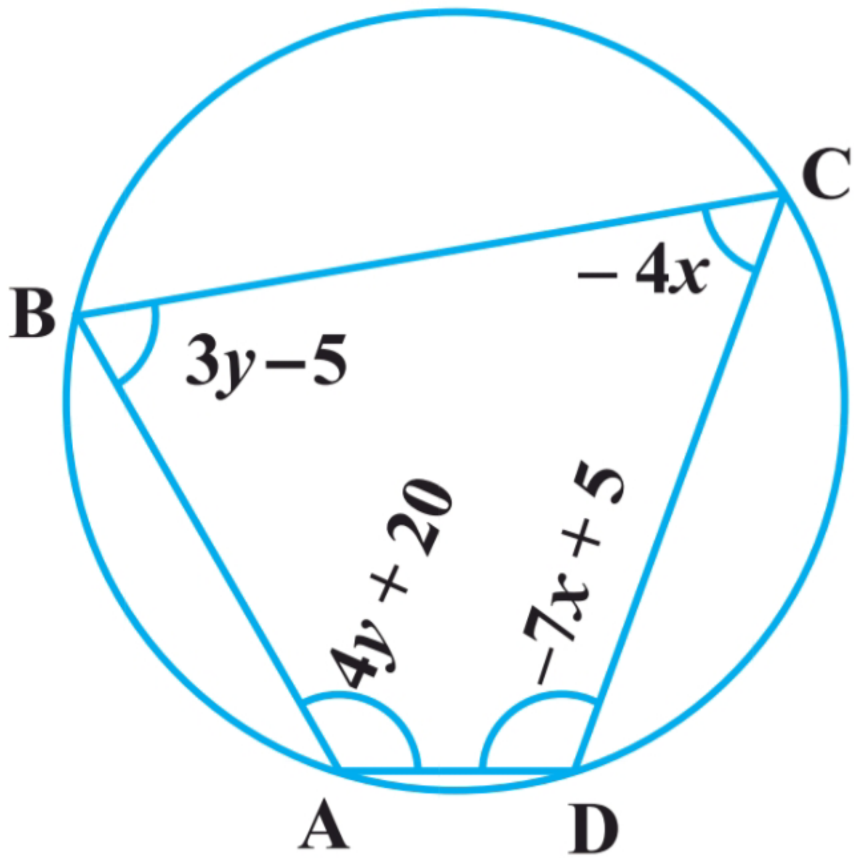


Figure 2.2: 3.7

Chapter 3

Quadratic Equations

3.1. 10

3.1.1. Examples:-1-18 (10.4)

1. Represent the following situation mathematically:

   (i) John and Jevanti together have 45 marbles. Both of them lost 5 marbles each, and the product of the number of marbles they now have is 124. We would like to find out how many marbles they had to start with.

   (ii) A cottage industry produces a certain number of toys a day. The cost of production of each toy (in Rupees) was found to be 55 minus the number of toys produced in a day. On a particular day, the total cost of production was ₹ 750. We woud like to find out the number of toys produced on that day.

2. Check whether the following are quadratic equations:

   (i) $(x-2)^2 + 1 = 2x - 3$

(ii) $x(x+1)+8=(x+2)(x-2)$

(iii) $x(2x+3)=X^2+1$

(iv) $(x+2)^3=x^3-4$

3. Find the roots of the equation $2x^2-5x+3=0$ by factorisation.

4. Find the roots of the quadratic equation $6x^2-x-2=0$.

5. Find the roots of the quadratic equation $3x^2-2\sqrt{6}x+2=0$

6. Find the dimensions of the prayer hall discussed in Section 4.1. A charity trust decides to build a prayer hall having a carpet area of 300 square metres with its length one metre more than twice its breath. What should be the length and breadth of the hall?

7. Solve the equation given in Example 3 by the method of completing the square.

8. Find the roots of the equation $5x^2-6x-2=0$ by the method of completing the square.

9. Find the roots of $4x^2+3x+5=0$ by the method of completing the square.

10. Solve Q.2(i) of exercise 4.1 by using the quadratic formula.

    (i) The area of a rectangle plot is $528m^2$. The length of the plot (in metres) is one more than twice itsbreadth. We need to find the length and breadth of the plot.

11. Find two consecutive odd positive integers, sum of whose squares is 290.

12. A rectangular park is to be designed whose breadth is 3 m less than its length. Its are is to be 4 square metres than the area of a park that has already been made in the shape of a isoceles triangle with its base as the breadth of the rectangular park and of altitude 12 m (see Fig. 3.1). Find its length and breadth.

13. Find the roots of the following quadratic equations, if they exist, using the quadratic formula.

    (i) $3x^2 - 5x + 2 = 0$

    (ii) $x^2 + 4x + 5 = 0$

    (iii) $2x^2 - 2\sqrt{2}x + 1 = 0$

14. Find the roots of the following equations:

    (i) $x + \frac{1}{x} = 3, x \neq 0$

    (ii) $\frac{1}{x} - \frac{1}{x-2} = 3, x \neq 0, 2$

15. A motor boat whose speed is 18 km/h in still water takes 1 hour more to go 24 km upstream than to return downstream to the same spot. Find the speed of the stream.

16. Find the discriminant of the quadratic equation $2x^2 - 4x + 3 = 0$, and hence find the nature of its roots.

17. A pole has to be erected at a point on the boundary of a circular park of diameter 1.3 metres in such a way that the difference of its distances

from two diametrically opposite fixed gates A and B on the boundary is 7 metres. Is it possible to do so? If yes, at what distances from the two gatees should the pole be erected ?

18. Find the discriminant of the equation $3x^2 - 2x + \frac{1}{3} = 0$ and hence find the nature of its roots. Find them, if they are real.

3.1.2. 10.4.1

1. Check whether the following are quadratic equations:

   (i) $(x+1)^2 = 2(x-3)$

   (ii) $x^2 - 2x = (-2)(3-x)$

   (iii) $(x-2)(x+1) = (x-1)(x+3)$

   (iv) $(x-3)(2x-1) = x(x+5)$

   (v) $(2x-1)(x-3) = (x+5)(x-1)$

   (vi) $x^2 + 3x + 1 = (x-2)^2$

   (vii) $(x+2)^3 = 2x(x^2-1)$

   (viii) $x^3 - 4x^2 - x - 1 = (x-2)^3$

2. Represent the folloing situations in form of quadratic equations:

   (i) The area of rectangulr plot is $528m^2$. The length of the plot (in metres) is one more than twice its breadth. We need to find the length and breadth of the plot.

   (ii) The product of two consecutive positive integers is 306. We need to find the integers.

(iii) Rohan's mother is 26 years older than him. The product of their ages(in years) 3 years from now will be 360. We would like to find Rohan's present age.

(iv) A train travels a distance of $480km$ at a unifom speed. If the speed had been $8km/h$ less, then it would have taken $3hours$ more to cover the same distance. We need to find the speed of the train.

3.1.3. 10.4.2

1. Find the roots of the following quadratic equations by factorisation:

    (i) $x^2 - 3x - 10 = 0$

    (ii) $2x^2 + x - 6 = 0$

    (iii) $\sqrt{2}x^2 + 7x + 5\sqrt{2} = 0$

    (iv) $2x^2 - x + \frac{1}{8} = 0$

    (v) $100x^2 - 20x + 1 = 0$

2. Represent the following situations mathematically;

    (i) John and Jivanti together have 45 marbles. Both of them lost 5 marbles each, and the product of the number of marbles they have is 124. We would like to find out how many marbles they had to start with.

    (ii) A cottage industry produces a certain number of toys in a day. The cost of production of each toy (in rupees) was found to be 55 minus the number of toys produced in a day. On a particular day, the total cost of production was ₹ 750. We would like to find out the number of toys produced on that day.

3. Find two numbers whose sum is 27 and product is 182.

4. Find two consecutive positive integers, sum of whose squares is 365.

5. The altitude of a right triangle is 7 cm less than its base. If the hypotenuse is 13 cm, find the other two sides.

6. A cottage industry produces a certain number of pottery articles in a day. It was observed on a particular day that the cost of production of each article (in rupees) was 3 more than twice the number of articles produced on that day. If the total cost of production on that day was ₹ 90, find the number of articles produced and the cost of each article.

3.1.4. 10.4.3

1. Find the roots of the following quadratic equations, if they exist, by the method of completing the square:

   (i) $2x^2 - 7x + 3 = 0$

   (ii) $2x^2 + x - 4 = 0$

   (iii) $4x^2 + 4\sqrt{3}x + 3 = 0$

   (iv) $2x^2 + x + 4 = 0$

2. Find the roots of the quadratic equations given in Q.1 above by applying the quadratic formula.

3. Find the roots of the following equations:

   (i)

$$x - \frac{1}{x} = 3, x \neq 0 \tag{3.1}$$

   (ii)

$$\frac{1}{x+4} - \frac{1}{x-7} = \frac{11}{30}, x \neq -4, 7 \tag{3.2}$$

4. The sum of the reciprocals of Rehman's ages, (in years) 3 years ago and 5 years from now is $\frac{1}{3}$. Find his present age.

5. In a class test, the sum of shefali's marks in Mathematics and english is 30. Had she got 2 marks more in Mathematics and 3 marks less in English, the product of their marks would have been 210. Find her marks in the two subjects.

6. The diagonal of a rectangular field is 60 metres more than the Shorter side. If the longer side is 30 metres more than the shorter side, find the sides of the field.

7. The difference of squares of two numbers is 180. The square of the smaller number is 8 times the larger number. Find the two numbers.

8. A train travels 360 km at a uniform speed. If the speed had been 5 km/hr more, it would have taken 1 hour less for the same journey. Find the speed of the train.

9. Two Water taps together can fill a tank in $9\frac{3}{8}$ hours.The tap of larger diameter takes 10 hours.The tap of larger diameter takes 10 hours less than the smaller one to fill the tank seperately. Find the time in which each tap can seperately fill the tank.

10. An express train takes 1 hour less than a passenger train to travel 132 km between mysore and bangalore (without taking into consideration the time they stop at intermediate statioons). If the average speed of the express train is 11 Km/h more than that of the passenger train, find the average speed of the two trains.

11. Sum of the areas of two square is $468m^2$. If the difference of their perimeter is 24m, find the sides of the two squares.

3.1.5. 10.4.4

1. Find the nature of the roots of the following quadratic equations. If real roots exist, find them:

   (i) $2x^2 - 3x + 5 = 0$

   (ii) $3x^2 - 4sqrt3x + 4 = 0$

   (iii) $2x^2 - 6x + 3 = 0$

2. Find the values of k for each of the following quadratic equations, so that they hav equal roots:

   (i) $2x^2 = kx - 3 = 0$

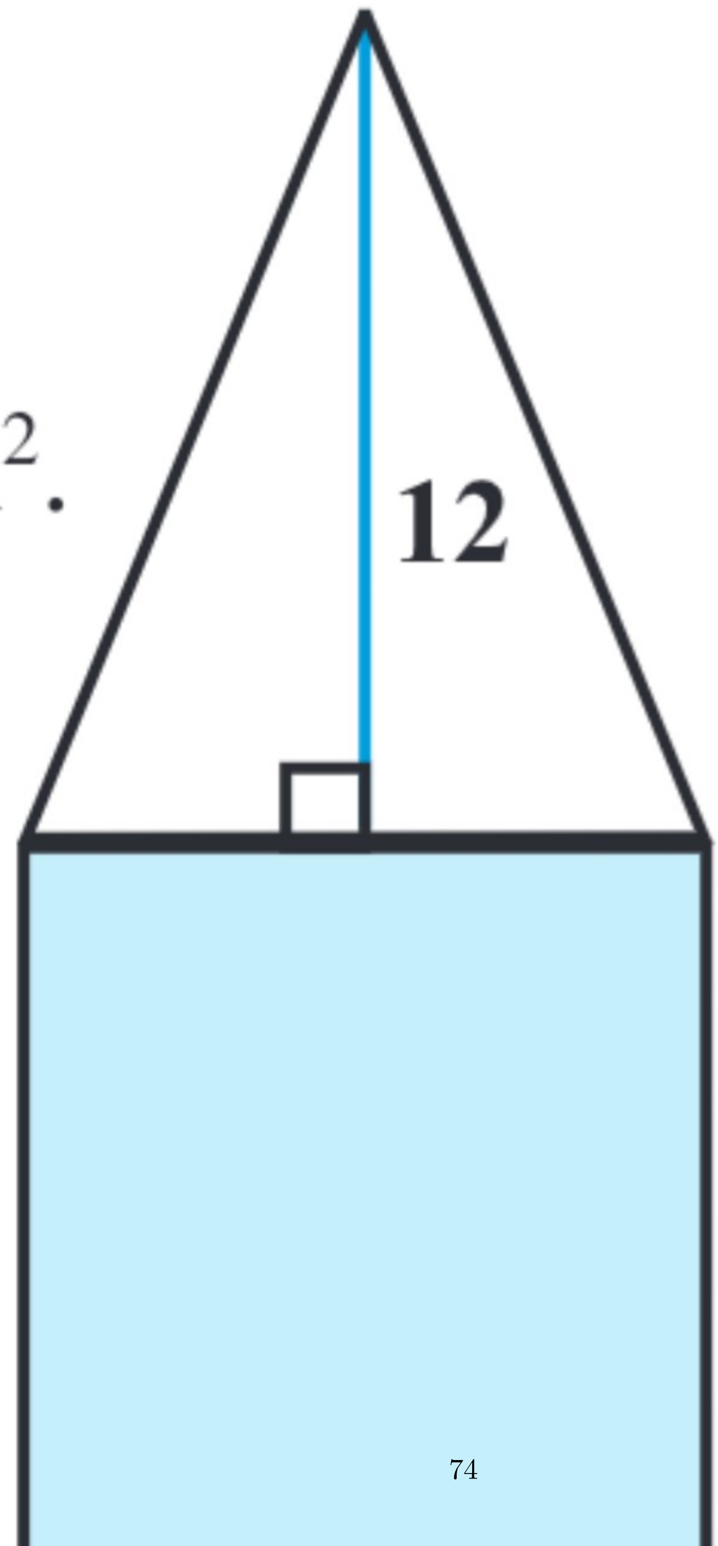
   (ii) $kx(x - 2) + 6 = 0$

3. Is it possible to design a rectangular mango grove whose length is twice its breadth, and the area is $800m^2$? If so, find its length and breadth.

4. Is the following situation possible? If so, determine their present ages. The sum of the ages of the two friends is 20 years. Four years ago, the product of their ages in years was 48.

5. Is it possible to design a rectangular park of perimeter 80m and area of $400m^2$. If so, find its length and breadth.

l^2.



Chapter 4

Coordinate Geometry

4.1. 10

4.1.1. Examples:-1-15 (10.7)

1. Do the points $(3,2), (-2,-3)$ and $(2,3)$ form a triangle? If so, name the type of triangle formed.

2. Show that the points $(1,7), (4,2), (-1,-1)$ and $(-4,4)$ are the vertices of a square.

3. Fig. 4.1 shows the arrangement of desks in a classroom. Ashima, Bharti and Camella are seated at $A(3,1), B(6,4)$ and $C(8,6)$ respectively. Do you think they are seated in a line? Give reasons for your answer.

4. Find a relation between x and y such that the point (x,y) is equidistant from the points $(7,1)$ and $(3,5)$.

5. Find a point on the Y-axis which is equidistant from the points $A(6,5)$ and $B(-4,3)$.

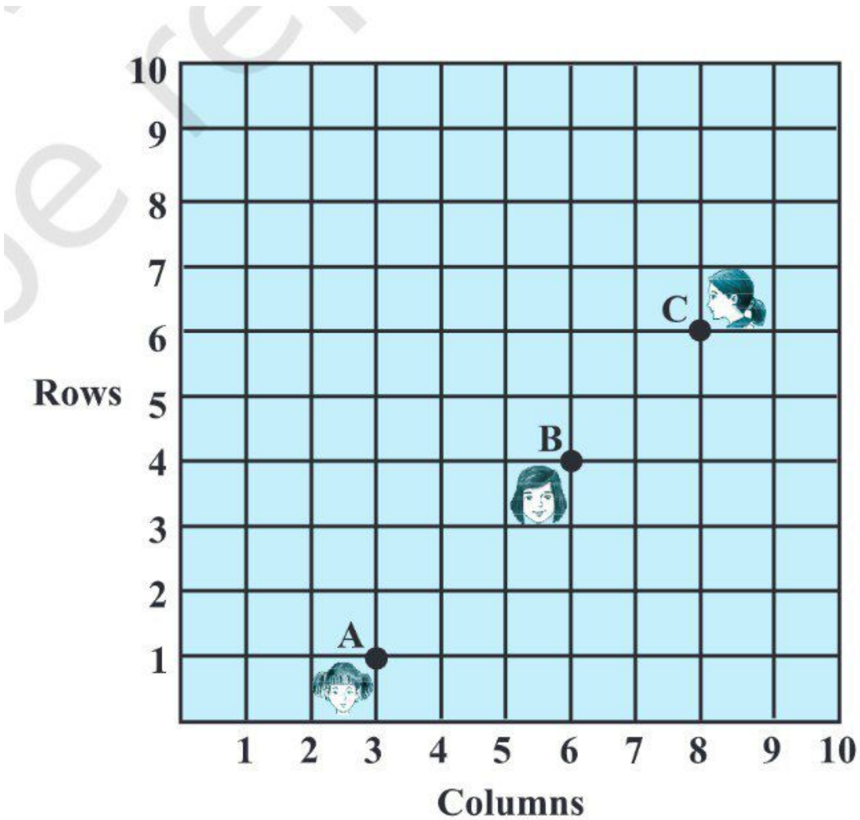


Figure 4.1: 7.6

6. Find the coordinates of the point which divides the line segment joining the points $(4,-3)$ and $(8,5)$ in the ratio $3:1$ internally.

7. In what ratio does the point $(-4,6)$ divide the line segment joining the points $A(-6,0)$ and $B(3,-8)$?

8. Find the coordinates of the points of trisection (i.e. points dividing to three equal parts) of the line segment joining the points $A(2,-2)$ and $B(-7,4)$.

9. Find the ratio in which the Y-axis divides the line segment joining the

points $(5, -6)$ and $(-1, -4)$. Also find the point of intersection.

10. If the points $A(6, 1), B(8, 2), C(9, 4)$ and $D(p, 3)$ are the vertices of a parallelogram, taken in order, find the value of p.

11. Find the area of the triangle whose vertices are $(1, -1), (-4, 6)$ and $(-3, 5)$.

12. Find the area of a triangle formed by the points $A(5, 2), B(4, 7)$ and $(7, -4)$.

13. Find the area of the triangle formed by the points $P(-1.5, 3), Q(6, -2)$ and $R(-3, 4)$.

14. Find the values of k if the points $A(2, 3), B(4, k)$ and $C(6, -3)$ are collinear.

15. If $A(-5, 7), B(-4, -5), C(-1, -6)$ and $D(4, 5)$ are the vertices of a quadrilateral, find the area of quadrilateral ABCD.

4.1.2. 10.7.1

1. Find the distance between the following pairs of points:

   (i) $(2, 3), (4, 1)$

   (ii) $(-5, 7), (-1, 3)$

   (iii) $(a, b), (-a, b)$

2. Find the distance between the points $(0, 0)$ and $(36, 15)$. Can you now find the two town A and B discussed in section 7.2.

3. Determine if the points $(1, 5), (2, 3)$ and $(-2, 11)$ are collinear.

4. Check whether $(5, 2), (6, 4)$ and $(7, 2)$ are the vertices of an isoceles triangle.

5. In a classroom, 4 friends are seated at the points A, B, C and D as shown Fig. 4.2 in Champa and chameli walk into the class and after observing for a fwe minutes champa asks chameli, "Don't you think ABCD is a square?" Chameli disagrees Using distance formula, find which of them is correct.

6. Name the type of quadrilateral formed, if any, by the following points, and give reasons for your answer:

   (i) $(-1, 2), (1, 0), (-1, 2), (3, 0)$

   (ii) $(-3, 5), (3, 1), (0, 3), (-1, -4)$

   (iii) $(4, 5), (7, 6), (4, 3), (1, 2)$

7. Find the point on the x axis which is equidistant from $(2, 5)$ and $(2, 9)$.

8. Find the values of y for which the distance between the points $P(2, -3)$ and $Q(10, y)$ is 10 units.

9. $Q(0, 1)$ is equidistant from $P(5, -3)$ and $R(x, 6)$,find the values of x. Also find the distances QR and PR.

10. Find a relation between x and y such that (x, y) is equidistant from the point $(3, 6)$ and $(-3, 4)$.

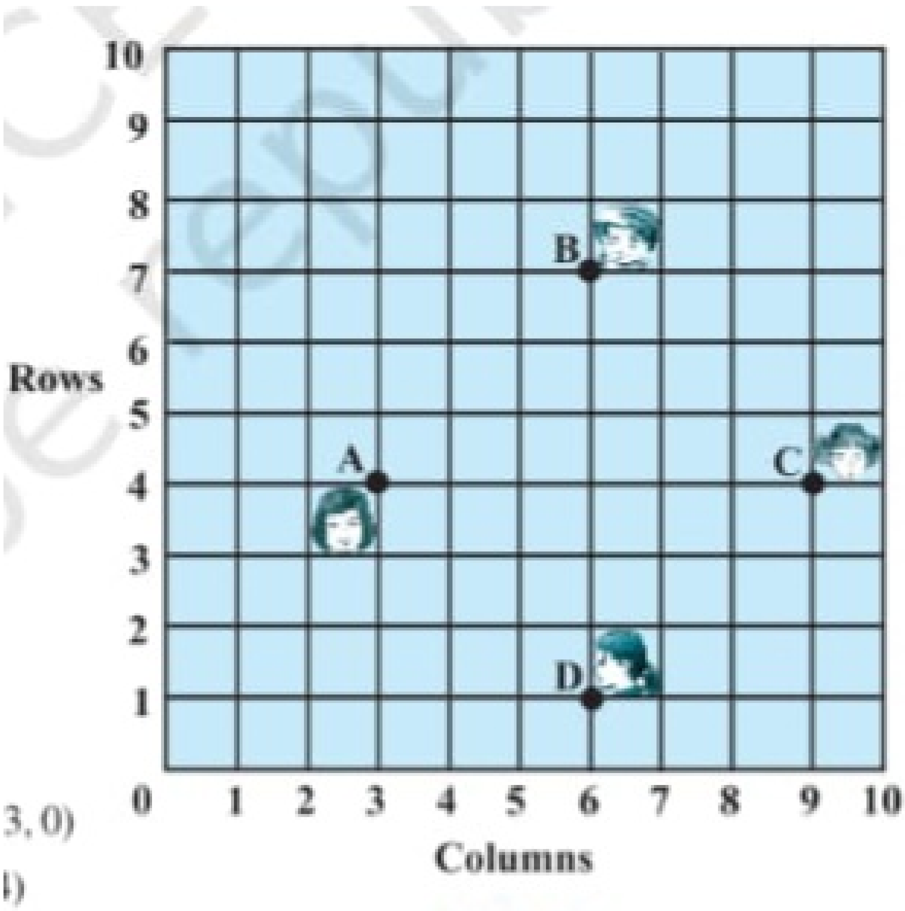


Figure 4.2: 7.8

4.1.3. 10.7.2

1. Find the coordinates of the point which divides the join of $(-1,7)$ and $(4,-3)$ in the ratio 2:3.

2. Find the coordinates of the points of trisection of the line segment joining $(4,-1)$ and $(-2,3)$.

3. To conduct Sports Day activities, in your rectangular shaped school

ground ABCD, lines have been drawn with chalk powder at a distance of 1m each. 100 flower pots have been placed at a distance of 1m from each other along AD, as shown in Fig. 4.3. Niharika runs $\frac{1}{4}th$ distance AD on the 2nd line and posts a green flag. Preet runs $\frac{1}{5}th$ the distance AD on the eighth line and posts a red flag. What is the distance between both the flags? If Rashmi has to post a blue flag exactly halfway between the line segment joining the two flags, where should she post her flag?

4. Find the ratio in which the line segment joining the points $(-3, 10)$ and $(6, -8)$ is divided by $(1, -6)$.

5. Find the ratio in which the line segment joining $A(1, -5)$ and $B(-4, 5)$ is divided by the x-axis. Also find the coordinates of the point of division.

6. If $(1, 2), (4, y), (x, 6)$ and $(3, 5)$ are the vertices of parallelogram taken in order, find x and y.

7. Find the coordinates of a point A, where AB is the diameter of a circle whose centre is $(2, -3)$ and B is $(1, 4)$.

8. If A and B are $(-2, -2)$ and $(2, -4)$ respectively, find the coordinates of P such that AP= $\frac{3}{7}$ AB and P lies on the line segment AB.

9. Find the coordinates of the points which divide the line segment joining A $(2, -2)$ and B $(2, 8)$ into four equal parts.

10. Find the area of a rhombus if its vertices are $(3,0),(4,5),(1,-4)$ and $(-2,-1)$ taken in order.

4.1.4. 10.7.3

1. Find the area of the triangles whose vertices are:

   (i) $(2,3),(-1,0),(2,-4)$

   (ii) $(-5,1),(3,-5),(5,2)$

2. In each of the following value of 'K', for which the points are collinear.

3. Find the area of the triangle by joining the mid-points of the sides of the triangle whose vertices are $(0,1),(2,1)$ and $(0,3)$. Find the ratio of this area to the area of the given triangle.

4. Find the area of the quadrilateral whose vertices, taken in order, are $(-4,-2),(-3,-5),(3,-2)$ and $(2,3)$.

5. You have studied in class IX (chapter 9, Example 3),that a median of a triangle divides in two triangles of equal areas. Verify this result for $\triangle ABC$ whose vertices are $A(4,-6),B(3,-2)$ and $C(5,2)$.

4.1.5. 10.7.4

1. Determine the ratio in which the line $2x+y-4=0$ divides the line segment joining the points $A(2,-2)$ and $B(3,7)$.

2. Find a relation between x and y if the points $(x, y), (1, 2)$ and $(7, 0)$ are collinear.

3. Find the centre of a circle passing through the points $(6, -6), (3, -7)$ and $(3, 3)$.

4. The two opposite vertices of a square are $(-1, 2)$ and $(3, 2)$. Find the coordinates of the two other vertices.

5. The Class X students of a secondary school in Krishinagar have been allotted a rectangular plot of land for their gardening activity. Sapling of Gulmohar are planted on the boundary at a distance of 1m from each other. there is a triangular grassy lawn in the plot as shown in Fig. 4.4. The students are to sow seeds of flowering plants on the remaining area of the plot.

   (i) Taking A as origin, find the coordinates of the vertices of the triangle.

   (ii) What will be tthe coordinates of the vertices of $\triangle PQR$ if C is the origin Also calculate the areas of the triangles in these cases. What do you observe?

6. The vertices of a $\triangle ABC$ are $A(4, 6), B(1, 5)$ and $C(7, 2)$. A line is drawn to intersect sides AB and AC at D and E respectively, such that $\frac{AD}{AB} = \frac{AE}{AC} = \frac{1}{4}$. Calculate the area of the $\triangle AD$ and compare it with he area of $\triangle ABC$.

7. Let $A(4, 2), B(6, 5)$ and $C(1, 4)$ be the vertices of $\triangle ABC$.

(i) The median from A meets BC at D. Find the coordinates of the points D.

(ii) Find the coordinates of the point P on AD such that $AP : PD = 2 : 1$.

(iii) Find the coordinates of points Q and R on medians BE and CF respectively such that $BQ : QE = 2 : 1$ and $CR : RF = 2 : 1$.

(iv) What do you observe?

(v) If $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$ are the vertices of $\triangle ABC$, find the coordinates of the triangle.

8. $ABCD$ is a rectangle formed by the points $A(-1, -1), B(-1, 4), C(5, 4)$ and $D(5, -1)$. P, Q, R and S are the mid points of AB, BC, CD and DA respectively. Is the quadrilateral $PQRS$ a square? a rectangle? or a rhombus? Justify your answer.

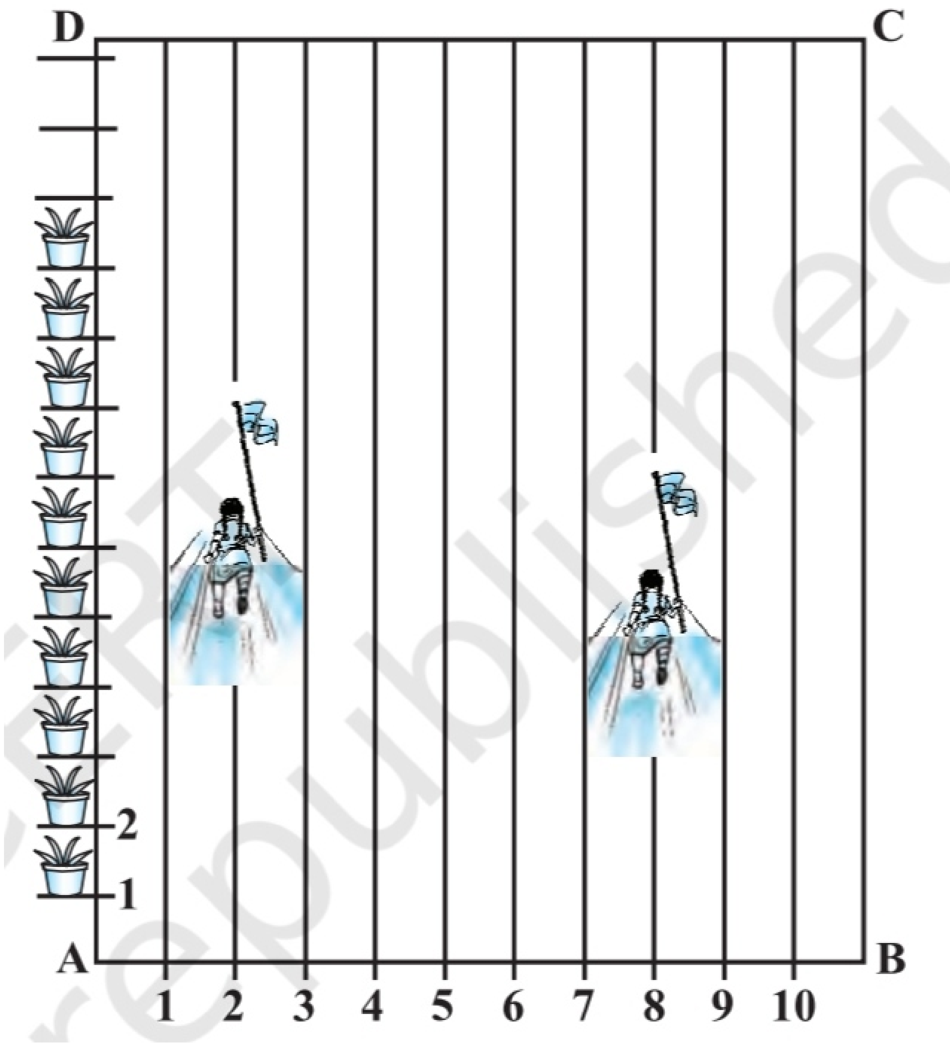


Figure 4.3: 7.12

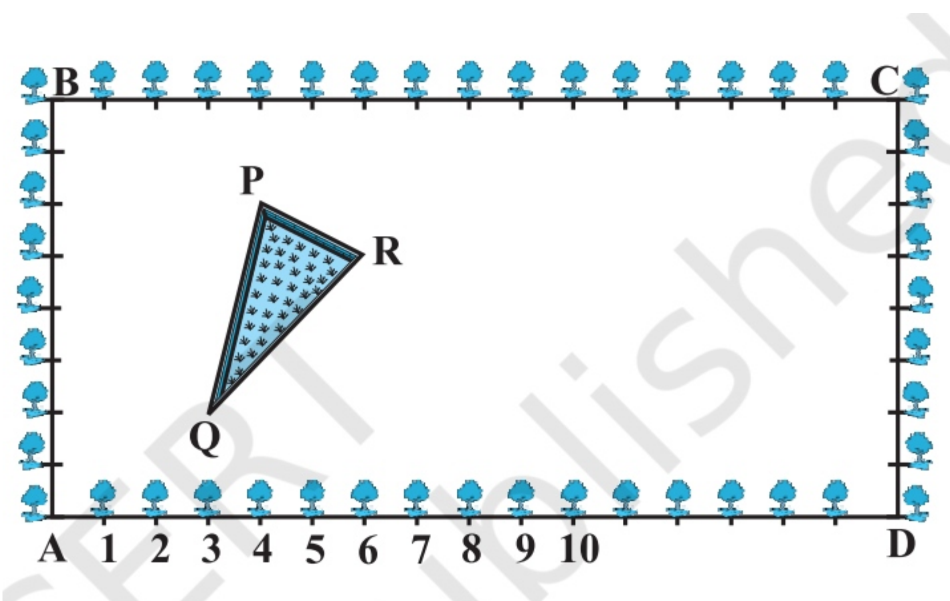


Figure 4.4: 7.14

Chapter 5

Straight Lines

5.1. 11

5.1.1. Examples:-1-25 (11.10)

1. Find the slope of lines:

   (a) Passing through the points $(3, -2)$ and $(-1, 4)$

   (b) Passing through the points $(3, -2)$ and $(7, -2)$

   (c) passing through the points $(3, -2)$ and $(3, 4)$

   (d) Making inclination of $60°$ with the positive direction of x-axis.

2. If the angle between two lines is $\frac{\pi}{4}$ and slope of one of the lines is $\frac{1}{2}$, find the slope of the other line.

3. Line through the points $(-2, 6)$ and $(4, 8)$ is perpendicular to the line through the points $(8, 12)$ and $(x, 24)$. Find the value of x.

4. Three points $(h, k), Q(x_1, y_1)$ and $R(x_2, y_2)$ lie on a line. Show that $(h - x_1)(y_2 - y_1) = (k - y_1)(x_2 - x_1)$.

5. In Fig. 5.1, time and distance graph of a linear motion is given. Two positions of line and distance are recorded as, when $T = 0, D = 2$ and when $T = 3, D = 8$. Use the concept of slope, find law of motion i.e, how distance depends upon time.

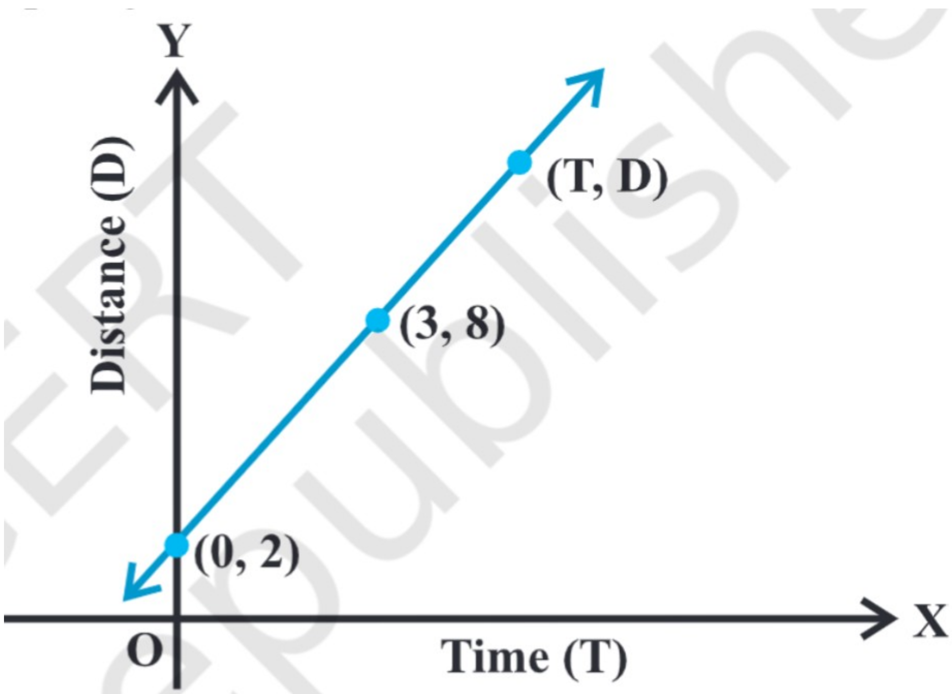


Figure 5.1: 10.9

6. Find the equations of the lines parallel to axes and passing through $(2, 3)$.

7. Find the equation of the line through $(-2, 3)$ with slope -4

8. Write the equation of the line through the points $(1, -1)$ and $(3, 5)$.

9. Write the equation of the lines for which $\tan\theta = \frac{1}{2}$, where θ is the

inclination of the line and

(i) y-intercepts is $\frac{-3}{2}$

(ii) x-intercept is 4.

10. Find the equation of the lines which makes intercepts -3 and 2 on the x- and y-axes respectively.

11. Find the equation of the line whose perpendicular distance from the origin is 4 units and the angle which the normal makes with positive direction of x-axis is 15°.

12. The Fahrenheit temperature F and absolute temperature K satisfy a linear equation. Given that $K = 273$ when $F = 32$ and that $K = 373$ when $F = 212$. Express K in terms of F and find the value of F, when $K = 0$.

13. Equation of a line is $3x - 4y + 10 = 0$, Find its

(i) Slope

(ii) x and y-intercepts.

14. Reduce the equation $\sqrt{3}x + y - 8 = 0$ into normal form. Find the values of p and ω.

15. Find the angle between the lines $y - \sqrt{3}x - 5 = 0$ and $\sqrt{3}y - x + 6 = 0$.

16. Show that two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ where $b_1b - 2 \neq 0$ are:

(a) parallel if $\frac{a_1}{b_1} = \frac{a_2}{b_2}$ and

(b) Perpendicular if $a_1a_2 - b_1b_2 = 0$.

17. Find the equation of a line perpendicular to the line $x + 2y + 3 = 0$ and passing through the point $(1, -2)$.

18. Find the distance of the point $(3, -5)$ from the line $3x - 4y - 26 = 0$.

19. Find the distaance between the parallel lines $3x - 4y + 7 = 0$ and $3x - 4y + 5 = 0$.

20. If the lines $2x + y - 3 = 0, 5x + ky - 3 = 0$ and $3x - y - 2 = 0$ are concurrent, find the value of k.

21. Find the distance of the line $4x - y - 0$ from the point $p(4, 1)$ measured along the line making an angle of $135°$ with the positive x-axis.

22. Assuming that straight lines work as the plane mirror for a point, find the image of the point $(1, 2)$ in the line $x - 3y + 4 = 0$.

23. Show that the area of the triangle formed by the lines $y = m_1x + c_1, y = m_2x + c_2$ and $x = 0$ is $\frac{c_1 - c_2^2}{2\left|m_1 - m_2\right|}$

24. A line is such that its segment between the lines $5x - y + 4 = 0$ and $3x + 4y - 4 = 0$ is bisected at the point $(1, 5)$. Obtain its equation.

25. Show that the path of a moving point such that its distances from two lines $3x - 2y = 5$ and $3x + 2y = 5$ are equal is a straight line.

5.1.2. 11.10.1

1. Draw a quadrilateral in the Cartesian plane, whose vertices are $(-4, 5), (0, 7), (5, -5)$ and $(-4, -2)$. Also, find its area.

2. The base of an equilateral triangle with side $2a$ lies along the y-axis such that the mid-point of the base is at the origin. Find vertices of the triangle.

3. Find the distance between $P(x_1, y_1), Q(x_2, y_2)$ when:

   (i) PQ is parallel to the y-axis.

   (ii) PQ is parellel to the x-axis.

4. Find the point x-axis, which is equidistant from the points $(7, 6)$ and $(3, 4)$.

5. Find the slope of a line, which passes through the origin, and the mid-point of the line segment joining the points $P(0, -4)$ and $B(8, 0)$.

6. Without using the Pythagoras thorem, show that the points $(4, 4), (3, 5)$ and $(-1, -1)$ are the vertices of a right angled triangle.

7. Find the slope of the line, which makes an angle of $30°$ with the positive direction of y-axis measured anticlockwise.

8. Find the value of x for which the points $(x, -1), (2, 1)$ and $(4, 5)$ are collinear.

9. Without using distance formula, show that points $(-2, -1), (4, 0), (3, 3)$ and $(-3, 2)$ are the vertices of the parallelogram.

10. Find the angle between the x-axis and the line joining the points $(3, -1)$ and $(4, -2)$.

11. The slope of a line is double of the slope of another line. If tangent of the angle between them is $\frac{!}{3}$, find the slopes of the lines.

12. A line passes through (x_1, y_1) and (h, k). If slope of the line is m, show that:

13. $k - y_1 = m(h - x_1)$

14. If three points $(h, 0)$, (a, b) and $(0, k)$ lie on a line, show that $\frac{a}{h} + \frac{b}{k} = 1$.

15. Consider the following population and year graph Fig. 5.2, find the slope of the line AB and using it, find what will be the population in the year 2010?

5.1.3. 11.10.2

In excercises 1 to 8, find the equation of the line which satisfy the given conditions:

1. Write the equations for x and y axes.

2. Passing through the point $(-4, 3)$ with slope $\frac{1}{2}$.

3. Passing through $(0, 0)$ with slope m.

4. Passing through $(2, \sqrt{3})$ and inclined with x axis at an angle of $75°$.

5. Intersecting the x axis at a distance of 3 units to the left of the origin with slope -2.

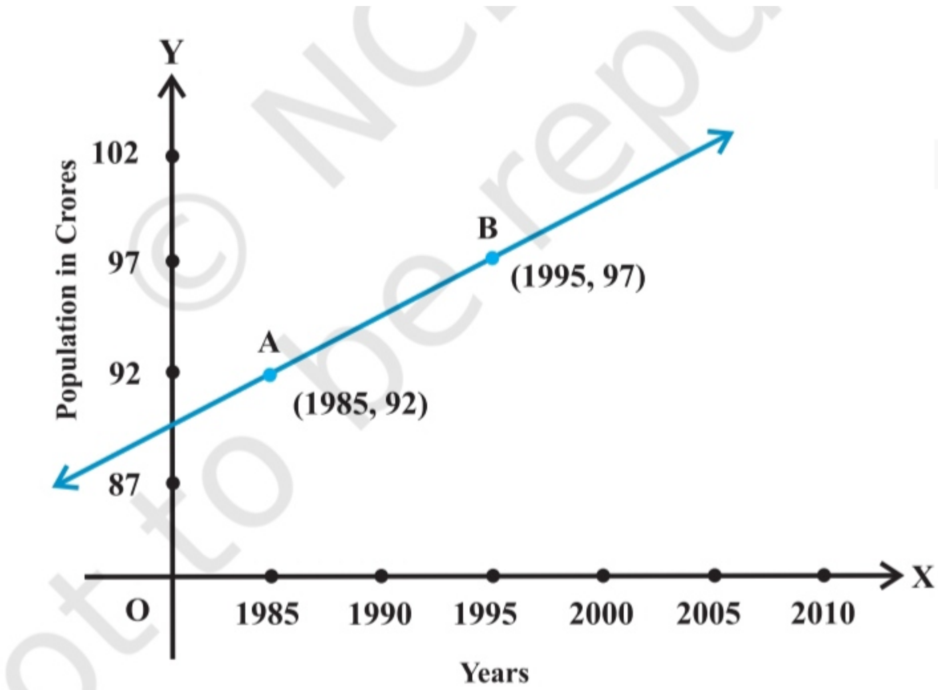


Figure 5.2: 10.10

6. Intersecting the y axis at a distance of 2 units above the origin and making an angle of 30 with positive direction of the x axis.

7. Passing through the points $(-1, 1)$ and $(2, -4)$.

8. Perpendicular distance from the origin is 5 units and the angle made by the perpendicular with the positive x axis is 30.

9. The vertices of $\triangle PQR$ are $P(2, 1), Q(-2, 3)$ and $R(4, 5)$. Find equation of the median through the vertex R.

10. Find the equation of the line passing through $(-3, 5)$ and perpendicular to the line through the points $(2, 5)$ and $(-3, 6)$.

11. A line perpendicular to the line segment joining the points $(1, 0)$ and $(2, 3)$ divides it in the ratio $1 : n$. Find the equation of the line.

12. Find the equation of the line that cuts off equal axes and passes through the point $(2, 3)$.

13. Find equation of the line passing through the point $(2, 2)$ and cutting off intercepts on the axes whose sum is 9.

14. Find equation of the line through the point $(0, 2)$ making an angle $\frac{2\pi}{3}$ with the positive x axis. Also, find the equation of the parallel to it and crossing the y axis at a distance of 2 units below the origin.

15. The perpendicular from the origin to a line meets it at the point $(-2, 9)$, find the equation of the line.

16. The length L [in centimetre of a copper rod is a linear function of its celsius temperature C]. In an experiment, if $L = 124.942$. When $C = 20$ and $L = 125.134$ When $C = 110$, express L in terms of C.

17. The owner of a milk store finds that, he can sell 980 litres of milk each week at ₹ 14/litre and 1220 litres of milk each week at ₹ 16/litre. Assuming a linear relationship between selling price and demand, how many litres could he sell weekly at ₹ 17/ litre?

18. $P(a, b)$ is the mid-point of a line segment between axes. Show that equation of the line is $\frac{x}{a} + \frac{y}{b} = 2$

19. Point $R(h, k)$ divides a line segment betwween the axes in the ratio $1 : 2$. find equation of the line.

20. By Using the concept of equation of a line, prove that the three points (3,0), (-2,-2) and (8,2) are collinear.

5.1.4. 11.10.3

1. Reduce the following equations into slope-intercept form and find their slopes and the y-intercepts:

   (i) $x + 7y = 0$

   (ii) $6x + 3y - 5 = 0$

   (iii) $y = 0$

2. Reduce the following equations into intercept form and find their intercepts on the axes:

   (i) $3x + 2y - 12 = 0$

   (ii) $4x - 3y = 6$

   (iii) $3y + 2 = 0$

3. Reduce the folloeing equations into normal form. Find their perpendicular distances from the origin and angle between perpendicular and the positive x-axis:

   (i) $x - \sqrt{3}y + 8 = 0$

   (ii) $y - 2 = 0$

   (iii) $x - y = 4$

4. Find the distance of the point $(-1, 1)$ from the line $12(x+6) = 5(y-2)$.

5. Find the points on the x-axis, whose distances from the line $\frac{x}{3} + \frac{y}{4} = 1$ are 4 units.

6. Find the distance between parallel lines:

   (i)

$$15x + 8y - 34 = 0 \text{ and } 15x + 8y + 31 = 0 \tag{6.1}$$

   (ii)

$$l(x + y) + p = 0 \text{ and } l(x + y) - r = 0 \tag{6.2}$$

7. Find equation of the line parallel to the line $3x - 4y + 2 = 0$ and passing through the point $(-2, 3)$.

8. Find equation of the line perpendicular to the line $x - 7y + 5 = 0$ and having x intercept 3.

9. Find angles between the lines $\sqrt{3}x + y = 1$ and $x + \sqrt{3}y = 1$.

10. The line through the points $(h, 3)$ and $(4, 1)$ intersects the line $7x - 9y - 19 = 0$ at right angle. Find the value of h.

11. Prove that the line through the point (x, y) and parallel to the line $Ax + By + C = 0$ is
    $A(x - x_1) + B(y - y_1) = 0$.

12. Two lines passing through the point $(2, 3)$ intersects each other at an angle at $60°$. If the shape of one line is 2, find equation of the other

line.

13. Find the equation of the right bisector of the line segment joining the point $(3, 4)$ and $(-1, 2)$.

14. Find the coordinates of the foot of perpendicular4 from the point $(-1, 3)$ to the line $3x + 4y - 16 = 0$

15. The perpendicular from the origin to the line $y = mx + c$ meets it at the point $(-1, 2)$. Find the values of m and c.

16. If p and q are the lengths of perpendiculars from the origin to the lines $x \cos\theta - y \sin\theta = k \cos 2\theta$ and $x \sec\theta + y \operatorname{cosec}\theta = k$ respectively, prove that $p^2 + 4q^2 = k^2$.

17. In the triangle ABC ith vertices $A(2, 3), B(4, -1)$ and $C(1, 2)$, find the equation and length of altitude from the vertex A.

18. If p is the length of perpendicular from the origin to the line whose intercepts on the axes are a and b, then show that $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$.

5.1.5. 11.10.4

1. Find the values of k for which the line $(k-3), x-(4-k^2)y+k^2-7k+6 = 0$ is

    (a) Parallel to the x axis.

    (b) Parallel to the y axis.

    (c) Passing through the origin.

2. Find the values of θ and p, if the equation $x\cos\theta + y\sin\theta = p$ is the normal form of the line $\sqrt{3}x + y + 2 = 0$.

3. Find the equations of the lines, which cut-off intercepts on the axes whose sum and product are 1 and -6, respectively.

4. What are the points on the y axis whose distance from the line $\frac{x}{3} + \frac{y}{4} = 1$ is 4 units.

5. Find perpendicular distance from the origin to the line joining the points $(\cos\theta, \sin\theta)$ and $(\cos\phi, \sin\phi)$.

6. Find the equation of the line parallel to y axis and drawn through the point of intersection of the lines $x - 7y + 5 = 0$ and $3x + y = 0$.

7. Find equation of a line drawn perpendicular to the line $\frac{x}{4} + \frac{y}{6} = 1$ through the point, where it meets the y axis.

8. Find the area of the triangle formed by the lines $y - x = 0, x + y = 0$ and $x - k = 0$.

9. Find the value of p so that the three lines $3x+y-2 = 0, px+2y-3 = 0$ and $2x - y - 3 = 0, px + 2y - 3 = 0$ and $2x - y - 3 = 0$ may intersect at one point.

10. If three lines when equation are $y = m_1x + c_1y = m_2x + c_2$ and $y = m_1x + c_1$ are concurrent, then show that $m_1(c_2 - c_3) + m_2(c_1 - c_2) = 0$

11. Find the equation of the lines through the point (3,2) which make an angle of $45°$ with the line $x - 2y = 3$

12. Find the equation of the line passing through the point of intersection of the lines $4x + 7y - 3 = 0$ and $2x - 3y + 1 = 0$ that has equal interceptson the axes.

13. Show that the equation of the line passing through the origin and making an angle θ with the line $y = mx + c$ is $\frac{y}{x} = \frac{m \pm \tan\theta,}{1 \mp m \tan\theta,}$.

14. In what ratio, the line joining $(-1, 1)$ and $(5, 7)$ is divided by the line $x + y = 4$?

15. Find the distance of the line $4x = 7y + 5 = 0$ from the point $(1, 2)$ along the line $2x - y = 0$.

16. Find the direction in which a straight line must be drawn through the point $(-1, 2)$ so that the point of intersection with the line $x + y = 4$ may be at a distance of 3 units from this point.

17. The hypothesis of a right angled triangle has its ends at the points $(1, 3)$ and $(-4, 1)$. Find an equation of the legs (perpendicular sides of the triangle.

18. Find the image of the point $(3, 8)$ with respect to the line $x + 3y = 7$ assuming the line to be a plane mirror.

19. If the lines $y = 3x + 1$ and $2y = x + 3$ are equally inclined to the line $y = mx + 4$. Find the value of m.

20. If sum of the perpendicular distance of a variable point $P(x, y)$ from the lines $x + y - 5 = 0$ and $3x - 2y + 7 = 0$ is always 10. Show that P must move on a line.

21. Find equation of the line which is equidistant from parallel lines $9x + 6y = -7$ and $3x + 2y + 6 = 0$.

22. A ray of the light passing through the point $(1, 2)$ reflects on the x axis at point A and the reflected ray passes through the point $(5, 3)$. Find the coordinates of A.

23. Prove that the product of the lengths of the perpendiculars drawn from the points $(\sqrt{a^2 - b^2}, 0)$ and $(-\sqrt{a^2 - b^2}, 0)$ to the line $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$ is b^2

24. A person standing at the junction (crossing) of two straight paths represented by the equations $2x - 3y + 4 = 0$ and $3x + 4y - 5 = 0$ and $3x+4y-5 = 0$ wants to reach the path whose equation is $6x-7y+8 = 0$ in the least time. Find the equation of the path that he should follow.

Chapter 6

Circles

6.1. 11

6.1.1. 11.11.1

In each of the following exercise 6.1.1.1 to 6.1.1.5, find the equation of the circle with:

1. centre $(0, 2)$ and radius 2
2. centre $(-2, 3)$ and radius 4
3. centre $\frac{(}{1}2, \frac{1}{4})$ and radius $\frac{1}{!2}$
4. centre $(1, 1)$ and radius 2
5. centre $(-a, -b)$ and radius $\sqrt{a^2 - b^2}$

In each of the following exercise 6 to 9, find the centre and radius of the circles

6. $(x - 5)^2 + (y - 3)^2 = 36$
7. $x^2 + y^2 - 4x - 8y - 45 = 0$

8. $x^2 + y^2 - 8x + 10y - 12 = 0$

9. $2x^2 + 2y^2 - x = 0$

10. Find the equation of the circle passing through the points $(4, 1)$ and $(6, 5)$ and whose centre is on the line $4x + y = 16$.

11. Find the equation of the circle passing through the points $(2, 3)$ and $(-1, 1)$ and whose centre is on the line $x - 3y - 11 = 0$.

12. Find the equation of the circle with radius 5 whose centre lies on x-axis and passes through the point $(2, 3)$.

13. Find the equation of the circle passing through $(0, 0)$ and making intercepts a and b on the coordinate axes.

14. Find the equation of a circle with centre $(2, 2)$ and passes through the point $(4, 5)$.

15. Does the point $(-2.5, 3.5)$ lie inside, outside or on the circle $x^2 + y^2 = 25$.

Chapter 7

3D Geometry

7.1. 11

7.1.1. Examples:-1-13 (11.12)

1. In Fig. 7.1, if P is $(2, 4, 5)$, find the coordinates of F.

2. Find the octant in which the points $(-3, 1, 2)$ and $(-3, 1, -2)$ lie.

3. Find the distance between the origin O and any point $Q(x_2, y_2, z_2)$.

4. Show that the points $P(-2, 3, 5), Q(1, 2, 3)$ and $R(7, 0, -1)$ are collinear.

5. Are the points $A(3, 6, 9), B(10, 20, 30)$ and $C(24, -41, 5)$ the vertices of a right angled triangle?

6. Find the equation of set of points P such that $PA^2 + PB^2 = 2k^2$, where A and B are the points $(3, 4, 5)$ and $(-1, 3, -7)$, respectively.

7. Find the coordinates of the point which divides the line segment joining the points $(1, -2, 3)$ and $(3, 4, -5)$ in the ratio $2 : 3$

   (i) internally, and

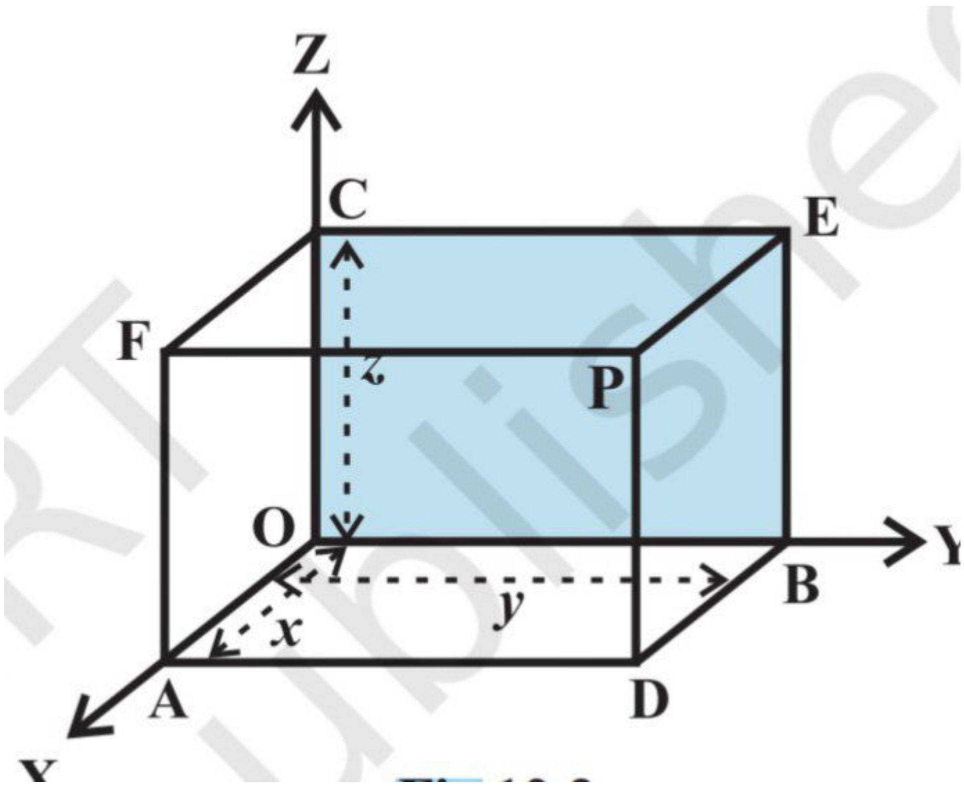


Figure 7.1: 12.3

(ii) externally

8. Using section formula, prove that the three points $(-4, 6, 10), (2, 4, 6)$ and $(14, 0, -2)$ are collinear.

9. Find the coordinates of the centroid of the triangle whose vertices are $(x_1, y_1, z_1), (x_2, y_2, z_2)$ and (x_3, y_3, z_3).

10. Find the ratio in which the line segment joining the points $(4, 8, 10)$ and $(6, 10, -8)$ is divided by the YZ- plane.

11. Show that the points $A(1, 2, 3), B(-1, -2, -1), C(2, 3, 2)$ and $D(4, 7, 6)$

are the vertices of a parallelogram $ABCD$, but it is not a rectangle.

12. Find the equation of the set of the points P such that its distances from the points $A(3, 4, -5)$ and $B(-2, 1, 4)$ are equal.

13. The centroid of a triangle ABC is at the point $(1, 1, 1)$. If the coordinates of A and B are $(3, -5, 7)$ and $(-1, 7, -6)$, respectively find the coordinates of the point C.

7.1.2. 11.12.1

1. A points is on the axis. What are its y coordinate and z coordinates?

2. A point in the XZ-plane. What can you say about its y coordinates?

3. Name the octants in which the following points lie:

$$(1, 2, 3), (4, -2, 3), (4, -2, -5), (4, 2, -5), (-4, 2, -5), (-4, 2, -5), (-3, -1, 6), (-2, -4, -7) \tag{3.1}$$

4. Fill in the blanks:-

   (i) The x axis and y axis taken together determine a plane known as ______ .

   (ii) The coordinates of points in the XY. plane are of the form ______.

   (iii) Coordinate plane divide the space into ______ octants.

7.1.3. 11.12.2

1. Find the distance between the following pairs of points:

   (i) $(2, 3, 5)$ and $(4, 3, 1)$

   (ii) $(-3, 7, 2)$ and $(2, 4, -1)$

   (iii) $(-1, 3, -4)$ and $(1, -3, 4)$

   (iv) $(2, -1, 3)$ and $(-2, 1, 3)$

2. Show that the points $(-2, 3, 5), (1, 2, 3)$ and $(7, 0, -1)$ are collinear.

3. Verify the following:

   (i) $(0, 7, -10), (1, 6, -6)$ and $(4, 9, -6)$ are the vertices of an isoceles triangle.

   (ii) $(0, 7, 10), (-1, 6, 6)$ and $(-4, 9, 6)$ are the vertices of a right angled triangle.

   (iii) $(-1, 2, 1), (1, -2, 5), (4, -7, 8)$ and $(2, -3, 4)$ are the vertices of a parallelogram.

4. Find the equation of the set of points which are equidistant from the points $(1, 2, 3)$ and $(3, 2, -1)$.

5. Find the equation of the set of points P, the sum of whose distances from $A(4, 0, 0)$ and $B(-4, 0, 0)$ is equal to 10.

7.1.4. 11.12.3

1. Find the coordinates of the point which divides the line segment joining the points which divides the line segment joining the points $(-2, 3, 5)$ and $(1, -4, 6)$ in the ratio

   (a) $2 : 3$ internally,

   (b) $2 : 3$ externally

2. Given that $P(3, 2, -4), Q(5, 4, -6)$ and $R(9, 8, -10)$ are Collinear. Find the ratio in which Q divides PR.

3. Find the ratio in which the yz plane divides the line segment formed by joining the points $(-2, 4, 7)$ and $(3, -5, 8)$.

4. Using section formula, show that the points $A(2, -3, 4), B(-1, 2, 1)$ and $C(0, \frac{1}{3}, 2)$ are collinear.

5. Find the coordinates of the points which triset the line segment joining the points $P(4, 2, -6)$ and $Q(10, -16, 6)$.

7.1.5. 11.12.4

1. Three vertices of a parallelogram $ABCD$ are $A(3, -1, 2), B(1, -2, 4)$ and $C(-1, 1, 2)$. Find the coordinates of the fourth vertex.

2. Find the lengths of the medians of the triangle with vertice $A(0, 0, 6), B(0, 4, 0)$ and $(6, 0, 0)$.

3. If the origin is the centroid of the triangle PQR with vertices $P(2a, 2, 6), Q(-4, 3b, -10)$ and $R(8, 14, 2c)$, then find the values of a, b and c.

4. Find the coordinates of a point on y-axis which are at a distance of $5\sqrt{2}$ from the point $P(3, -2, 5)$.

5. A point R with x-coordinate 4 lies on the line segment joining the points $P(2, -3, 4)$ and $Q(, 0, 10)$. Find the coordinates of the point R.

6. If A and B be the points $(3, 4, 5)$ and $(-1, 3, -7)$ respectively, find the equation of the set of the points P such that $PA^2 + PB^2 = K^2$ where K is a constant.

Chapter 8

Matrices

8.1. 12

8.1.1. Examples:-1-28 (12.3)

1. Consider the following information regarding the number of men ad women workers in three factories I, II and III

	Men Workers	Women Workers
I	30	27
II	25	31
III	27	26

Table 8.1:

   Represent the above information in the form of a 3×2 matrix. What does the entry in the third row and second column represent ?

2. If a matrix has 8 elements, what are the possible orders it can have ?

3. Construct a 3×2 matrix whose elements are given by $a_{ij} = \frac{1}{2} |1 - 3j|$

4. If $\begin{pmatrix} x+3 & z+4 & 2y-7 \\ -6 & a-1 & 0 \\ b-3 & -21 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 6 & 3y-2 \\ -6 & -3 & 2c+2 \\ 2b+4 & -21 & 0 \end{pmatrix}$. Find the values of a, b, c, x, y and z.

5. Find the values of a, b, c and d from the following equation :

$$\begin{pmatrix} 2a+b & a-2b \\ 5c-d & 4c+d \end{pmatrix} = \begin{pmatrix} 4 & -3 \\ 11 & 24 \end{pmatrix} \tag{5.1}$$

6. Given $A = \begin{pmatrix} \sqrt{3} & 1 & -1 \\ 2 & 3 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & \sqrt{5} & 1 \\ -2 & 3 & \frac{1}{2} \end{pmatrix}$, find $A+B$.

7. If $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{pmatrix}$, then find $2A - B$.

8. If $A = \begin{pmatrix} 8 & 0 \\ 4 & -2 \\ 3 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -2 \\ 4 & 2 \\ -5 & 1 \end{pmatrix}$ then find that X, such that $2A + 3X = 5B$

9. Find X and Y, if $X + Y = \begin{pmatrix} 5 & 2 \\ 0 & 9 \end{pmatrix}$ and $X - Y = \begin{pmatrix} 3 & 6 \\ 0 & -1 \end{pmatrix}$

10. Find the values of x and y from the following equations:

$$2\begin{pmatrix} x & 5 \\ 7 & y-3 \end{pmatrix} + \begin{pmatrix} 3 & -4 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 7 & 6 \\ 15 & 14 \end{pmatrix} \tag{10.1}$$

11. Two farmers Ramkishan and Gurucharan Singh cultivate only three varities of rice namely Basmati, Permal and Naura. The sale (in Rupees) of these three varities of rice by both the farmers in the month of September and October are given by the following matrices A and B.

September Sales (in Rupees)

$$A = \begin{array}{c} \begin{pmatrix} \text{Basmati} & \text{Permal} & \text{Naura} \\ 10000 & 20000 & 30000 \\ 50000 & 30000 & 10000 \end{pmatrix} \end{array} \begin{array}{c} \text{Ramkishan} \\ \text{Gurucharan Singh} \end{array} \tag{11.1}$$

October Sales (in Rupees)

$$B = \begin{array}{c} \begin{pmatrix} \text{Basmati} & \text{Permal} & \text{Naura} \\ 5000 & 10000 & 6000 \\ 20000 & 10000 & 10000 \end{pmatrix} \end{array} \begin{array}{c} \text{Ramakishan} \\ \text{Gurucharan Singh} \end{array} \tag{11.2}$$

(i) Find the combined sales in Sepember and October for each farmer in each variety.

(ii) Find the decrease in sales from September to October.

(iii) If both farmers receive 2% profit on gross sales, compute the profit for each farmer and for each variety sold in October.

12. Find AB, if $A = \begin{pmatrix} 6 & 9 \\ 2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 6 & 0 \\ 7 & 9 & 8 \end{pmatrix}$.

13. If $A = \begin{pmatrix} 1 & -2 & -3 \\ -4 & 2 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{pmatrix}$, then find AB, BA. Show that $AB \neq BA$.

14. If $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then $AB = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $BA = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ clearly $AB \neq BA$. Thus matrix multiplication is not commutative.

15. Find AB, if $A = \begin{pmatrix} 0 & -1 \\ 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 5 \\ 0 & 0 \end{pmatrix}$.

16. If $A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ 0 & 2 \\ -1 & 4 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 2 & 3 & -4 \\ 2 & 0 & -2 & 1 \end{pmatrix}$ find $A(BC), (AB)C$ and show that $(AB)C = A(BC)$.

17. If $A = \begin{pmatrix} 0 & 6 & 7 \\ -6 & 0 & 8 \\ 7 & -8 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 3 \\ -2 \\ 3 \end{pmatrix}$ Calculate AC, BC and $(A+B)C$. Also, verify that $(A+B)C = AC + BC$.

18. If $A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{pmatrix}$, then show that $A^3 - 23A - 40I = 0$.

19. In a legislative assembly election, a political group hired a public relations firm to promote its candidate in three ways: telephone, housecalls

and letters. The cost per contact (in paise) is given in matrix A as cost per contact

$$A = \begin{pmatrix} 40 \\ 100 \\ 50 \end{pmatrix} \begin{array}{c} \text{Telephone} \\ \text{Housecall} \\ \text{Letter} \end{array} \tag{19.1}$$

.

The number of contacts of each type made in two cities X and Y is given by

$$B = \begin{pmatrix} \text{Telephone} & \text{Housecall} & \text{Letter} \\ 1000 & 500 & 5000 \\ 3000 & 1000 & 10000 \end{pmatrix} \begin{array}{c} X \\ Y \end{array} \tag{19.2}$$

.

Find the total amount spent by the group in the two cities X and Y.

20. If $A = \begin{pmatrix} 3 & \sqrt{3} & 2 \\ 4 & 2 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -1 & 2 \\ 1 & 2 & 4 \end{pmatrix}$, verify that

    (a) $(A')' = A$

    (b) $(A+B)' = A' + B'$

    (c) $(kB)' = kB'$, where k is any constant.

21. If $A = \begin{pmatrix} -2 \\ 4 \\ 5 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 3 & -6 \end{pmatrix}$, verify that $(AB)' = B'A'$.

22. Express the matrix $B = \begin{pmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix}$ as the sum of symmetric and a skew symmetric matrix.

23. By using elementary operations, find the inverse of the matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

24. Obtain the inverse of the following matrix using elementary operations.

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix} \tag{24.1}$$

.

25. Find P^{-1}, if it exists, given $P = \begin{pmatrix} 10 & -2 \\ -5 & 1 \end{pmatrix}$.

26. If $A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, then prove that $A^n = \begin{pmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{pmatrix}$, $n \in N$.

27. If A and B are symmetric matrices of the same order, then show that AB is symmetric if and only if A and B commute, that is $AB = BA$.

28. Let $A = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 5 & 2 \\ 7 & 4 \end{pmatrix}$, $C = \begin{pmatrix} 2 & 5 \\ 3 & 8 \end{pmatrix}$. Find a matrix D such that $CD - AB = 0$.

8.1.2. 12.3.1

1. In the matrix $A = \begin{pmatrix} 2 & 5 & 19 & -7 \\ 25 & -2 & \frac{5}{2} & 12 \\ \sqrt{3} & 1 & -5 & 17 \end{pmatrix}$, write:

   (i) The order of the matrix

   (ii) The number of elements

   (iii) Write the elements $a_{13}, a_{21}, a_{33}, a_{24}, a_{23}$

2. If a matrix has 24 elements, what are the possible order it can have? What if, it has 13 elements?

3. If a matrix has 18 elements, what are the possible orders it can have? What, if it has 5 elements?

4. Construct a 2×2 matrix, $A = [a_{ij}]$, whose elements are given by:

   (i) $[a_{ij}] = \frac{(i+j)^2}{2}$

   (ii) $[a_{ij}] = \frac{i}{j}$

   (iii) $[a_{ij}] = \frac{(i+2j)^2}{2}$

5. Construct a 3×4 matrix, whose elements are given by:

   (i) $[a_{ij}] = \frac{1}{2}\left|-3i + j\right|$

   (ii) $[a_{ij}] = 2i - j$

6. Find the values of x, y and z from the following equations:

   (i) $\begin{pmatrix} 4 & 3 \\ x & 5 \end{pmatrix} = \begin{pmatrix} y & z \\ 1 & 5 \end{pmatrix}$

(ii) $\begin{pmatrix} x+y & 2 \\ 5+z & xy \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 5 & 8 \end{pmatrix}$

(iii) $\begin{pmatrix} x+y+z \\ x+z \\ y+z \end{pmatrix} = \begin{pmatrix} 9 \\ 5 \\ 7 \end{pmatrix}$

7. Find the value of a, b, c and d from the equation:

$$\begin{pmatrix} a-b & 2a-c \\ 2a-b & 3c+d \end{pmatrix} = \begin{pmatrix} -1 & 5 \\ 0 & 13 \end{pmatrix} \tag{7.1}$$

8. $A = [a_{ij}]_{m \times n}$ is a square matrix, if:

   (a) $m \leq n$

   (b) $m \geq n$

   (c) $m = n$

   (d) None of these

9. Which of the given values of x and y make the following pair of matrices equal:

$$\begin{pmatrix} 3x+7 & 5 \\ y+1 & 2-3x \end{pmatrix}, \begin{pmatrix} 0 & y-2 \\ 8 & 4 \end{pmatrix} \tag{9.1}$$

   (a) $x = \frac{1}{3}, y = 7$

   (b) Not possible to find

   (c) $y = 7, x = \frac{2}{3}$

(d) $x = \frac{1}{3}, y = \frac{2}{3}$

10. The number of all possible matrices of order 3×3 with each entry 0 or 1 is:

    (a) 27

    (b) 81

    (c) 18

    (d) 512

8.1.3. 12.3.2

1. Let $A = \begin{pmatrix} 2 & 4 \\ 3 & 2 \end{pmatrix}, B = \begin{pmatrix} 1 & 3 \\ -2 & 5 \end{pmatrix}, C = \begin{pmatrix} -2 & 5 \\ 3 & 4 \end{pmatrix}$. Find each of the following:

    (i) $A + B$

    (ii) $A - B$

    (iii) $3A - C$

    (iv) AB

    (v) BA

2. Compute the following:

    (i) $\begin{pmatrix} a & b \\ -b & a \end{pmatrix} + \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$

    (ii) $\begin{pmatrix} a^2 + b^2 & b^2 + c^2 \\ a^2 + c^2 & a^2 + b^2 \end{pmatrix} + \begin{pmatrix} 2ab & 2ac \\ -2ac & -2ab \end{pmatrix}$

(iii) $\begin{pmatrix} -1 & 4 & -6 \\ 8 & 5 & 16 \\ 2 & 8 & 5 \end{pmatrix} + \begin{pmatrix} 12 & 7 & 6 \\ 8 & 0 & 5 \\ 3 & 2 & 4 \end{pmatrix}$

(iv) $\begin{pmatrix} \cos^2 x & \sin^2 x \\ \sin^2 x & \cos^2 x \end{pmatrix} + \begin{pmatrix} \sin^2 x & \cos^2 x \\ \cos^2 x & \sin^2 x \end{pmatrix}$

3. Compute the following products:

(i) $\begin{pmatrix} a & b \\ b & -a \end{pmatrix} \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

(ii) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} 2 & 3 & 4 \end{pmatrix}$

(iii) $\begin{pmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & -3 & 5 \\ 0 & 2 & 4 \\ 5 & 0 & 5 \end{pmatrix}$

(iv) $\begin{pmatrix} 2 & 1 \\ 3 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \end{pmatrix}$

(v) $\begin{pmatrix} 3 & -1 & 3 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 1 & 0 \\ 3 & 1 \end{pmatrix}$

4. If $A = \begin{pmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{pmatrix}$ and $C = \begin{pmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & -2 & 3 \end{pmatrix}$,

then compute $(A+B)$ and $(B+C)$. Also, verify that $A+(B-C) = (A+B)-C$.

5. If $A = \begin{pmatrix} \frac{2}{3} & 1 & \frac{5}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{4}{3} \\ \frac{7}{3} & 2 & \frac{2}{3} \end{pmatrix}$ and $B = \begin{pmatrix} \frac{2}{5} & \frac{3}{5} & 1 \\ \frac{1}{5} & \frac{2}{5} & \frac{4}{5} \\ \frac{7}{5} & \frac{6}{5} & \frac{2}{5} \end{pmatrix}$, then compute $3A - 5B$.

6. Simplify $\cos\theta \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} + \sin\theta \begin{pmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{pmatrix}$.

7. Find X and Y, if:

   (i) $X+Y = \begin{pmatrix} 7 & 0 \\ 2 & 5 \end{pmatrix}$ and $X-Y = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$.

   (ii) $2X+3Y = \begin{pmatrix} 2 & 3 \\ 4 & 0 \end{pmatrix}$ and $3X+2Y = \begin{pmatrix} 2 & -2 \\ -1 & 5 \end{pmatrix}$

8. Find X, if $Y = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$ and $2X+Y = \begin{pmatrix} 1 & 0 \\ -3 & 2 \end{pmatrix}$.

9. Find x and y, if $2\begin{pmatrix} 1 & 3 \\ 0 & x \end{pmatrix} + \begin{pmatrix} y & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 6 \\ 1 & 8 \end{pmatrix}$.

10. Solve the equation for x, y, z and t, if $2\begin{pmatrix} x & y \\ z & t \end{pmatrix} + 3\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} = 3\begin{pmatrix} 3 & 5 \\ 4 & 6 \end{pmatrix}$.

11. If $x\begin{pmatrix}2\\3\end{pmatrix}+y\begin{pmatrix}-1\\1\end{pmatrix}=\begin{pmatrix}10\\5\end{pmatrix}$, find the values of x and y.

12. Given $3\begin{pmatrix}x & y\\z & w\end{pmatrix}=\begin{pmatrix}x & 6\\-1 & 2w\end{pmatrix}+\begin{pmatrix}4 & x+y\\z+w & 3\end{pmatrix}$, find the values of x, y, z and w.

13. If $F(x)=\begin{pmatrix}\cos x & -\sin x & 0\\\sin x & \cos x & 0\\0 & 0 & 1\end{pmatrix}$, show that $F(x)+F(y)=F(x+y)$.

14. Show that:

(i) $\begin{pmatrix}5 & -1\\6 & 7\end{pmatrix}\begin{pmatrix}2 & 1\\3 & 4\end{pmatrix}\neq\begin{pmatrix}2 & 1\\3 & 4\end{pmatrix}\begin{pmatrix}5 & -1\\6 & 7\end{pmatrix}$.

(ii) $\begin{pmatrix}1 & 2 & 3\\0 & 1 & 0\\1 & 1 & 0\end{pmatrix}\begin{pmatrix}-1 & 1 & 0\\0 & -1 & 1\\2 & 3 & 4\end{pmatrix}\neq\begin{pmatrix}-1 & 1 & 0\\0 & -1 & 1\\2 & 3 & 4\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\0 & 1 & 0\\1 & 1 & 0\end{pmatrix}$

15. Find $A^2-5A+6I$, if $A=\begin{pmatrix}2 & 0 & 1\\2 & 1 & 3\\1 & -1 & 0\end{pmatrix}$.

16. If $A=\begin{pmatrix}1 & 0 & 2\\0 & 2 & 1\\2 & 0 & 3\end{pmatrix}$, prove that $A^3-6A^2+7A+2I=0$.

17. If $A=\begin{pmatrix}3 & -2\\4 & -2\end{pmatrix}$ and $I=\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$, find k so that $A^2=kA-2I$.

18. If $A = \begin{pmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{pmatrix}$ and I is the identity matrix of order 2, show that $I + A = (I - A)\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$.

19. A trust fund has ₹ 30000 that must be invested in two different types of bonds. The first bomd pays 5% interest per year, and the secon bond pays 7% interest per year. Using matrix multiplication, detemine how to divide ₹ 30000 among the 2 types of bonds. If the trust fund must obtain an annual total interest of:

    (a) ₹ 1800

    (b) ₹ 2000

20. The bookshop of a particular school has 10 dozen chemistry books, 8 dozen phyaics books, 10 dozen economics books. Their selling prices are ₹ 80, ₹ 60 and ₹ 40 each respectively. Find the total amount the bookshop will receive from selling all the books using matrix algebra. Assume X, Y, Z, W and P are matrices of order $2 \times n, 3 \times k, 2 \times p, n \times 3$ and $p \times k$, respectively. Choose the correct answer in 21 and 22.

21. The restriction on n, k and p so that $PY + WY$ will be defined are:

    (a) $k = 3, p = n$

    (b) k is arbitrary, $p = 2$.

    (c) p is arbitrary, $k = 3$

    (d) $k = 2, p = 3$

22. If $n = p$, then order of the matrix $7X - 5Z$ is:

(a) $p \times 2$

(b) $2 \times n$

(c) $n \times 3$

(d) $p \times n$

8.1.4. 12.3.3

1. Find the transpose of eaach of the following matrices:

(i) $\begin{pmatrix} 5 \\ \frac{1}{2} \\ -1 \end{pmatrix}$

(ii) $\begin{pmatrix} 1 & -1 \\ 2 & -3 \end{pmatrix}$

(iii) $\begin{pmatrix} -1 & 5 & 6 \\ \sqrt{3} & 5 & 6 \\ 2 & 3 & 1 \end{pmatrix}$

2. If $A = \begin{pmatrix} -1 & 2 & 3 \\ 5 & 7 & 9 \\ -2 & 3 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -4 & 1 & -5 \\ 1 & 2 & 0 \\ 1 & 3 & 1 \end{pmatrix}$, then verify that

(a) $(A + B) = A' + B'$

(b) $(A - B)' = A' - B'$

3. If $A = \begin{pmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix}$, then find $(A + 2B)'$

4. If $A = \begin{pmatrix} -2 & 3 \\ 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix}$, then find the $(A + 2B)'$

5. For the matrices A and B, Verify that $(AB)' = B'A'$, where

   (i) $A = \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix}, B = \begin{pmatrix} -1.2 & 1 \end{pmatrix}$

   (ii) $A = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, B = \begin{pmatrix} 1 & 5 & 7 \end{pmatrix}$

6. If

   (i) $A = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$, then verify that $A + A' = I$

   (ii) $A = \begin{pmatrix} \sin\alpha & \cos\alpha \\ -\cos\alpha & \sin\alpha \end{pmatrix}$, then verify that $A + A' = I$

7. (i) Show that the matrix $A = \begin{pmatrix} 1 & -1 & 5 \\ -1 & 2 & 1 \\ 5 & 1 & 3 \end{pmatrix}$ is a symmetrical matrix.

(ii) Show that the matrix $A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$ is a skew symmetric matrix.

8. For the matrix $A = \begin{pmatrix} 1 & 5 \\ 5 & 7 \end{pmatrix}$, verify that

   (i) $(A + A)$ is a symmetric matrix.

   (ii) $(A - A)$ ia a skew symmetric matrix.

9. Find $\frac{1}{2}(A + A')$ and $\frac{1}{2}(A - A')$, when $A = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$

10. Express the following matrices as the sum of a symmetric and a skew symmetric matrix:

    (i) $\begin{pmatrix} 3 & 5 \\ 1 & -1 \end{pmatrix}$

    (ii) $\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$

    (iii) $\begin{pmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{pmatrix}$

    (iv) $\begin{pmatrix} 1 & 5 \\ -1 & 2 \end{pmatrix}$

11. If A, B are symmetric matrices of same order, then $AB - BA$ is a

    (a) Skew symmetric matrix

    (b) Symmetric matrix

    (c) Zero matrix

    (d) Identity matrix

12. If $A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$ and $A + A' = 1$ then the value of α is

    (a) $\frac{\pi}{6}$

    (b) $\frac{\pi}{3}$

    (c) π

    (d) $\frac{3\pi}{2}$

8.1.5. 12.3.5

1. Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, show that $(aI + bA)^n = a^n I + na^{n-1}bA$, where I is the identity matrix of order 2 and $n \in N$.

2. If $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, Prove that $A^n = \begin{pmatrix} 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \end{pmatrix}$, $n \in N$.

3. If $A = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$, then prove that $A^n = \begin{pmatrix} 1+2n & -4n \\ n & 1-2n \end{pmatrix}$,Where n is any positive integer.

Market	Product X	Product Y	Product Z
I	10,000	2,000	18,000
II	6,000	20,000	8,000

Table 8.2:

4. If A and B are symmetric matrices prove that $AB - BA$ is a skew symmetric matrix.

5. Show that the matrix $B'AB$ is a symmetric or skew symmetric according as A is symmetric or skew symmetric.

6. Find the value of x, y, z if the matrix $A = \begin{pmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{pmatrix}$ satisfy the equation $A'A = I$.

7. For what values of x : $\begin{pmatrix} 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ x \end{pmatrix} = 0$

8. If $A = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$, show that $A^2 - 5A + 7I = 0$.

9. Find x, if $\begin{pmatrix} x & -5 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ 4 \\ 1 \end{pmatrix} = 0$.

10. A manufacturer produces three products x, y, z which he sells in two markets. Annual Sales are indicated below:

    (a) If unit sale Prices of x, y and z are ₹ 2.50, ₹ 1.50 and ₹ 1.00, respectively. Find the total revenue in each market with the help of matrix algebra.

(b) If the unit costs of the above three commodities are ₹ 2.00, ₹ 1.00 and 50 paise respectively. Find the gross profit.

11. Find the matrix X so that $X\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{pmatrix}$

12. If A and B are square matrices of the same order such that $AB = BA$, then prove by induction that $(AB)^n = B^n A^n$. Further prove that $(AB)^n = A^n B^n$ for all $n \in N$. Choose the correct answer in the following questions:

13. If $A = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$ is such that $A^2 = I$, then

(a) $1 + \alpha^2 + \beta\gamma = 0$

(b) $1 - \alpha^2 + \beta\gamma = 0$

(c) $1 - \alpha^2 - \beta\gamma = 0$

(d) $1 + \alpha^2 - \beta\gamma = 0$

14. If the matrix A is both symmetric and skew symmetric, then

(a) A is a diagonal matrix

(b) A is a Zero matrix

(c) A is a Square matrix

(d) None of these

15. If A is square matrix such that $A^2 = A$, then $(I + A)^3 - 7A$ is equal to

(a) A

(b) $I - A$

(c) I

(d) $3A$

8.2. Determinants

8.2.1. Examples:-1-34 (12.4)

1. Evaluate $\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix}$

2. Evaluate $\begin{vmatrix} x & x+1 \\ x-1 & x \end{vmatrix}$

3. Evaluate the determinant $\Delta = \begin{vmatrix} 1 & 2 & 4 \\ -1 & 3 & 0 \\ 4 & 1 & 0 \end{vmatrix}$

4. Evaluate $\Delta = \begin{vmatrix} 0 & \sin\alpha & -\cos\alpha \\ -\sin\alpha & 0 & \sin\beta \\ \cos\alpha & -\sin\beta & 0 \end{vmatrix}$

5. Find the values of x for which $\begin{vmatrix} 3 & x \\ x & 1 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix}$.

6. Verify Property 1 for $\Delta = \begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$

7. Verify Property 2 for $\Delta = \begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$

8. Evaluate $\Delta = \begin{vmatrix} 3 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 2 & 3 \end{vmatrix}$

9. Evaluate $\begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$.

10. Show that $\begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix} = 0$

11. Prove that $\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix} = a^3$

12. Without expanding, prove that $\Delta = \begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix} = 0$

13. Evaluate $\Delta \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$

14. Prove that $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc.$

15. If x, y, z are different and $\Delta = \begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$, then show that $1 + xyz = 0$.

16. Show that $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc \left| 1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right| = abc + bc + ca + ab.$

17. Find the area of the triangle whose vertices are $(3, 8)$, $(-4, 2)$ and $(5, 1)$.

18. Find the equation of the line joining $A(1, 3)$ and $B(0, 0)$ using determinants and find k if $D(k, 0)$ is a point such that area of triangle ABD is 3 sq. units.

19. Find the minor of element 6 in the determinant $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$.

20. Find minors and cofactors of all the elements of the determinant

$$\begin{vmatrix} 1 & -2 \\ 4 & 3 \end{vmatrix}.$$

21. Find minors and cofactors of the elements a_{11}, a_{21} in the determinant
$$\delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

22. Find minors and cofactors of the elements of the determinant $\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$ and verify that $a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33} = 0$.

23. Find adj A for $A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$.

24. If $A = \begin{pmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix}$, then verify that A adj $A = |A|$. Also find A^{-1}.

25. If $A = \begin{pmatrix} 2 & 3 \\ 1 & -4 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$, then verify that $(AB)^{-1} = B^{-1}A^{-1}$.

26. Show that the matrix $A = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$ satisfies the equation $A^2 - 4A + I = O$, where I is 2×2 identity matrix and O is 2×2 zero matrix. Using this equation, find A^{-1}.

27. Solve the system of equations

$$2x + 5Y = 1, \tag{27.1}$$

$$3x + 2y = 7 \tag{27.2}$$

.

28. Solve the following system of equations by matrix method.

$$3x - 2y + 3z = 8 \tag{28.1}$$

$$2x + y - z = 1 \tag{28.2}$$

$$4x - 3y + 2z = 4 \tag{28.3}$$

29. The sum of three numbers is 6. If we multiply third number by 3 and add second number to it, we get 11. By adding first and third numbers, we get double of the second number. Represent it algebraically and find the numbers using matrix method.

30. If a, b, c are positive and unequal, show that value of the determinant $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ is negative.

31. If a, b, c are in $A.P$ find the value of $\begin{vmatrix} 2y+4 & 5y+7 & 8y+a \\ 3y+5 & 6y+8 & 9y+b \\ 4y+6 & 7y+ & 10y+c \end{vmatrix}$.

32. Show that $\Delta = \begin{vmatrix} (y+z)^2 & xy & zx \\ xy & (x+z)^2 & yz \\ xz & yz & (x+y)^2 \end{vmatrix} = 2xyz(x+y+z)^3$.

33. Use product $\begin{vmatrix} 1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 2 & 4 \end{vmatrix} \begin{vmatrix} 2 & 0 & 1 \\ 9 & 2 & 3 \\ 6 & 1 & 2 \end{vmatrix}$ to solve the system of equations.

$$x - y + 2z = 1 \tag{33.1}$$

$$2z - 3z = 1 \tag{33.2}$$

$$3x - 2y + 4z = 2 \tag{33.3}$$

34. Prove that $\Delta = \begin{vmatrix} a+bx & c+dx+p+qx & \\ ax+b & cx+d+px+q & \\ u & v & w \end{vmatrix} = (1-x^2) \begin{vmatrix} a & c & p \\ b & d & q \\ u & v & w \end{vmatrix}$.

Chapter 9

Vector Algebra

Appendix A

Trigonometry

A.1. Ratios

A right angled triangle looks like Fig. A.1. with angles $\angle A, \angle B$ and $\angle C$ and

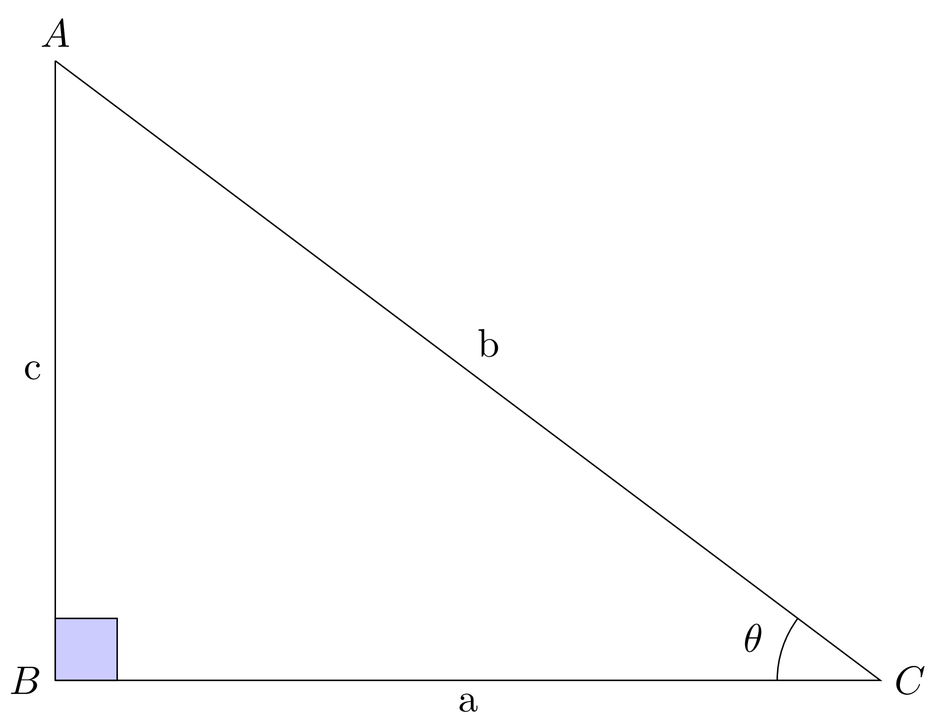


Figure A.1: Right Angled Triangle

sides a, b and c. The unique feature of this triangle is $\angle B$ which is defined to be $90°$.

A.1.1. For simplicity, let the greek letter $\theta = \angle C$. We have the following definitions.

$$\begin{aligned} \sin\theta &= \tfrac{c}{b} & \cos\theta &= \tfrac{a}{b} \\ \tan\theta &= \tfrac{c}{a} & \cot\theta &= \tfrac{1}{\tan\theta} \\ \csc\theta &= \tfrac{1}{\sin\theta} & \sec\theta &= \tfrac{1}{\cos\theta} \end{aligned} \tag{A.1.1.1}$$

A.1.2. Show that

$$\cos\theta = \sin(90° - \theta) \tag{A.1.2.1}$$

Solution: From (A.1.1.1),

$$\cos\angle BAC = \cos\alpha = \cos(90° - \theta) = \frac{c}{b} = \sin\angle ABC = \sin\theta \tag{A.1.2.2}$$

A.2. The Baudhayana Theorem

Use Fig. A.2 for all problems in this section.

A.2.1. Show that

$$b = a\cos\theta + c\sin\theta \tag{A.2.1.1}$$

Solution: We observe that

$$BD = a\cos\theta \tag{A.2.1.2}$$

$$AD = c\cos\alpha = c\sin\theta \quad \text{(From} \quad \text{(A.1.2.2))} \tag{A.2.1.3}$$

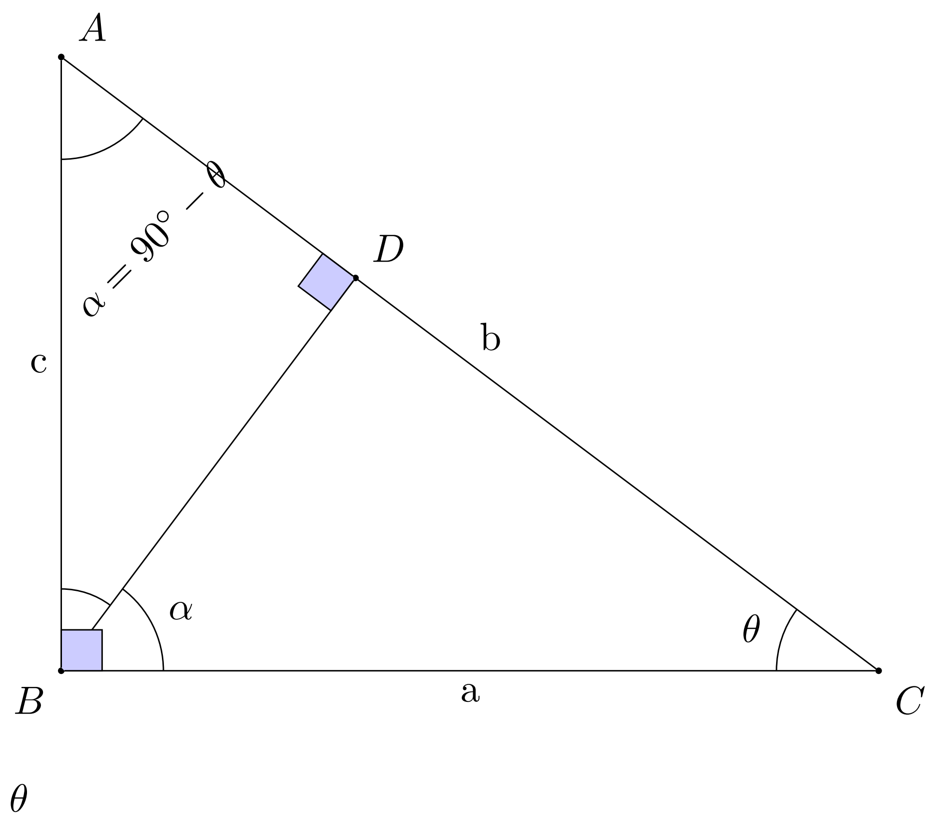


Figure A.2: Baudhayana Theorem

Thus,

$$BD + AD = b = a\cos\theta + c\sin\theta \tag{A.2.1.4}$$

A.2.2. From (A.2.1.1), show that

$$\sin^2\theta + \cos^2\theta = 1 \tag{A.2.2.1}$$

Solution: Dividing both sides of (A.2.1.1) by b,

$$1 = \frac{a}{b}\cos\theta + \frac{c}{b}\sin\theta \tag{A.2.2.2}$$

$$\Rightarrow \sin^2\theta + \cos^2\theta = 1 \quad \text{(from} \quad \text{(A.1.1.1))} \tag{A.2.2.3}$$

A.2.3. In a right angled triangle, the hypotenuse is the longest side.

Solution: From (A.2.2.1),

$$0 \le \sin\theta, \cos\theta \le 1 \tag{A.2.3.1}$$

Hence,

$$b\sin\theta \le b \implies c \le b \tag{A.2.3.2}$$

Similalry,

$$a \le b \tag{A.2.3.3}$$

A.2.4. Using (A.2.1.1), show that

$$b^2 = a^2 + c^2 \tag{A.2.4.1}$$

(A.2.4.1) is known as the Baudhayana theorem. It is also known as the Pythagoras theorem.

Solution: From (A.2.1.1),

$$b = a\frac{a}{b} + c\frac{c}{b} \quad (\text{from} \quad (\text{A.1.1.1})) \tag{A.2.4.2}$$

$$\implies b^2 = a^2 + c^2 \tag{A.2.4.3}$$

A.3. Area of a Triangle

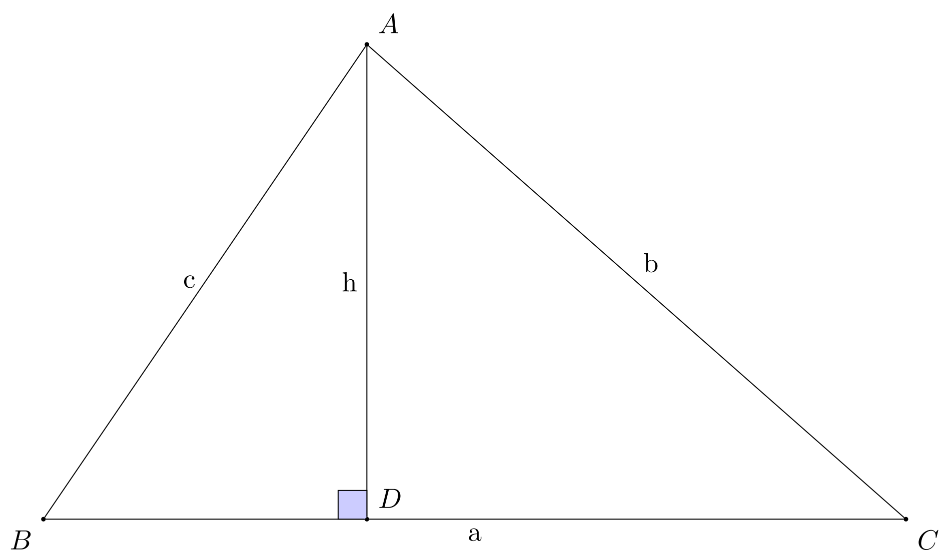


Figure A.3: Area of a Triangle

A.3.1. Show that the area of ΔABC in Fig. A.3 is $\frac{1}{2}ab\sin C$.

Solution: We have

$$ar\left(\Delta ABC\right) = \frac{1}{2}ah = \frac{1}{2}ab\sin C \quad \left(\because \quad h = b\sin C\right). \tag{A.3.1.1}$$

A.3.2. Show that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \tag{A.3.2.1}$$

Solution: Fig. A.3 can be suitably modified to obtain

$$ar\left(\Delta ABC\right) = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B \tag{A.3.2.2}$$

Dividing the above by abc, we obtain

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \tag{A.3.2.3}$$

This is known as the sine formula.

A.3.3. Show that

$$\alpha > \beta \implies \sin\alpha > \sin\beta \tag{A.3.3.1}$$

Solution: In Fig. A.4,

$$ar\left(\triangle ABD\right) < ar\left(\triangle ABC\right) \tag{A.3.3.2}$$

$$\implies \frac{1}{2}lc\sin\theta_1 < \frac{1}{2}ac\sin\left(\theta_1 + \theta_2\right) \tag{A.3.3.3}$$

$$\implies \frac{l}{a} < \frac{\sin\left(\theta_1 + \theta_2\right)}{\sin\theta_1} \tag{A.3.3.4}$$

$$\text{or, } 1 < \frac{l}{a} < \frac{\sin\left(\theta_1 + \theta_2\right)}{\sin\theta_1} \tag{A.3.3.5}$$

$$\implies \frac{\sin\left(\theta_1 + \theta_2\right)}{\sin\theta_1} > 1 \tag{A.3.3.6}$$

from Theorem A.2.3. This proves (A.3.3.1).

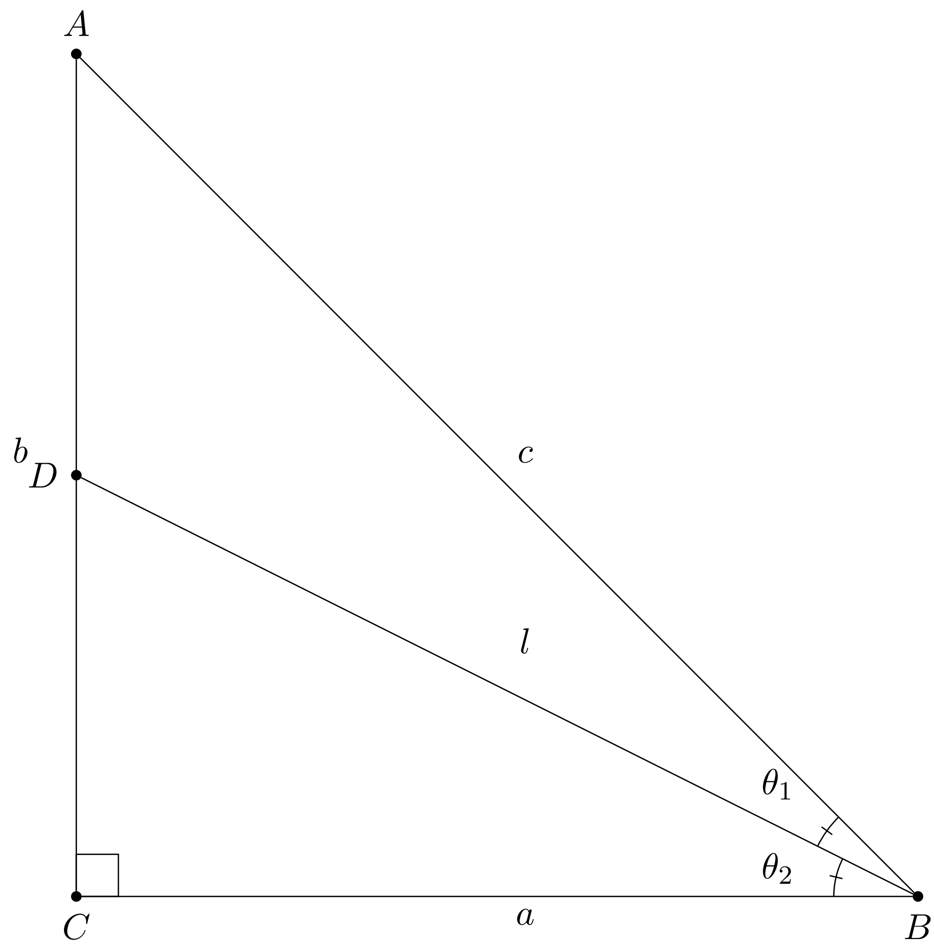


Figure A.4:

A.3.4. Using Fig. A.4, show that

$$\sin\theta_1 = \sin(\theta_1+\theta_2)\cos\theta_2 - \cos(\theta_1+\theta_2)\sin\theta_2 \qquad \text{(A.3.4.1)}$$

Solution: The following equations can be obtained from the figure using the forumula for the area of a triangle

$$
\begin{aligned}
ar\left(\Delta ABC\right) &= \frac{1}{2}ac\sin\left(\theta_1+\theta_2\right) && \text{(A.3.4.2)}\\
&= ar\left(\Delta BDC\right)+ar\left(\Delta ADB\right) && \text{(A.3.4.3)}\\
&= \frac{1}{2}cl\sin\theta_1+\frac{1}{2}al\sin\theta_2 && \text{(A.3.4.4)}\\
&= \frac{1}{2}ac\sin\theta_1\sec\theta_2+\frac{1}{2}a^2\tan\theta_2 && \text{(A.3.4.5)}
\end{aligned}
$$

($\because l = a\sec\theta_2$). From the above,

$$
\begin{aligned}
\sin\left(\theta_1+\theta_2\right) &= \sin\theta_1\sec\theta_2+\frac{a}{c}\tan\theta_2 && \text{(A.3.4.6)}\\
&= \sin\theta_1\sec\theta_2+\cos\left(\theta_1+\theta_2\right)\tan\theta_2 && \text{(A.3.4.7)}
\end{aligned}
$$

Multiplying both sides by $\cos\theta_2$,

$$
\sin\left(\theta_1+\theta_2\right)\cos\theta_2 = \sin\theta_1+\cos\left(\theta_1+\theta_2\right)\sin\theta_2 \qquad \text{(A.3.4.8)}
$$

resulting in (A.3.4.1).

A.3.5. Find Hero's formula for the area of a triangle.

Solution: From (A.3.1), the area of $\triangle ABC$ is

$$\begin{aligned}
\frac{1}{2}ab\sin C &= \frac{1}{2}ab\sqrt{1-\cos^2 C} \quad \text{(from (A.2.2.1))} && \text{(A.3.5.1)}\\
&= \frac{1}{2}ab\sqrt{1-\left(\frac{a^2+b^2-c^2}{2ab}\right)^2} \text{(from (B.3.3.1))} && \text{(A.3.5.2)}\\
&= \frac{1}{4}\sqrt{(2ab)^2-(a^2+b^2-c^2)} && \text{(A.3.5.3)}\\
&= \frac{1}{4}\sqrt{(2ab+a^2+b^2-c^2)(2ab-a^2-b^2+c^2)} && \text{(A.3.5.4)}\\
&= \frac{1}{4}\sqrt{\left\{(a+b)^2-c^2\right\}\left\{c^2-(a-b)^2\right\}} && \text{(A.3.5.5)}\\
&= \frac{1}{4}\sqrt{(a+b+c)(a+b-c)(a+c-b)(b+c-a)} && \text{(A.3.5.6)}
\end{aligned}$$

Substituting

$$s = \frac{a+b+c}{2} \tag{A.3.5.7}$$

in (A.3.5.6), the area of $\triangle ABC$ is

$$\sqrt{s(s-a)(s-b)(s-c)} \tag{A.3.5.8}$$

This is known as Hero's formula.

A.4. Angle Bisectors

A.4.1. In Fig. A.4.1.1, the bisectors of $\angle B$ and $\angle C$ meet at **I**. Show that IA bisects $\angle A$.

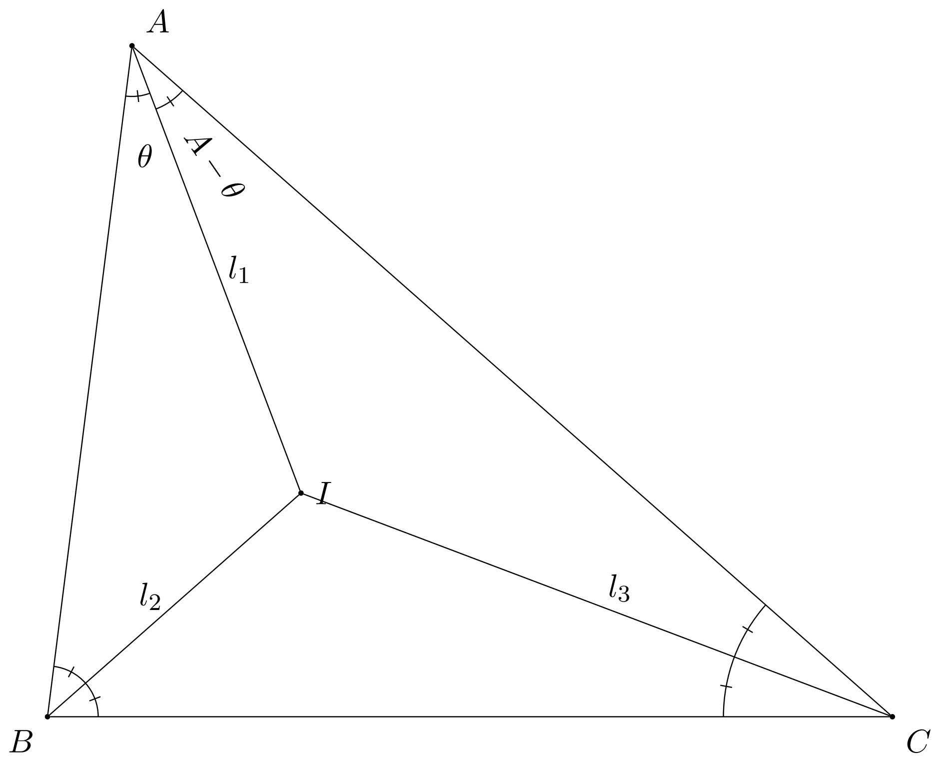


Figure A.4.1.1: Incentre I of $\triangle ABC$

Solution: Using sine formula in (A.3.2.3)

$$\frac{l_1}{\sin\frac{C}{2}} = \frac{l_3}{\sin(A-\theta)} \tag{A.4.1.1}$$

$$\frac{l_3}{\sin\frac{B}{2}} = \frac{l_2}{\sin\frac{C}{2}} \tag{A.4.1.2}$$

$$\frac{l_1}{\sin\frac{B}{2}} = \frac{l_2}{\sin\theta} \tag{A.4.1.3}$$

Multiplying the above equations,

$$\sin\theta = \sin(A-\theta) \implies \theta = \frac{A}{2} \tag{A.4.1.4}$$

A.4.2. In Fig. A.4.2.1,

$$ID \perp BC,\, IE \perp AC,\, IF \perp AB. \tag{A.4.2.1}$$

Show that

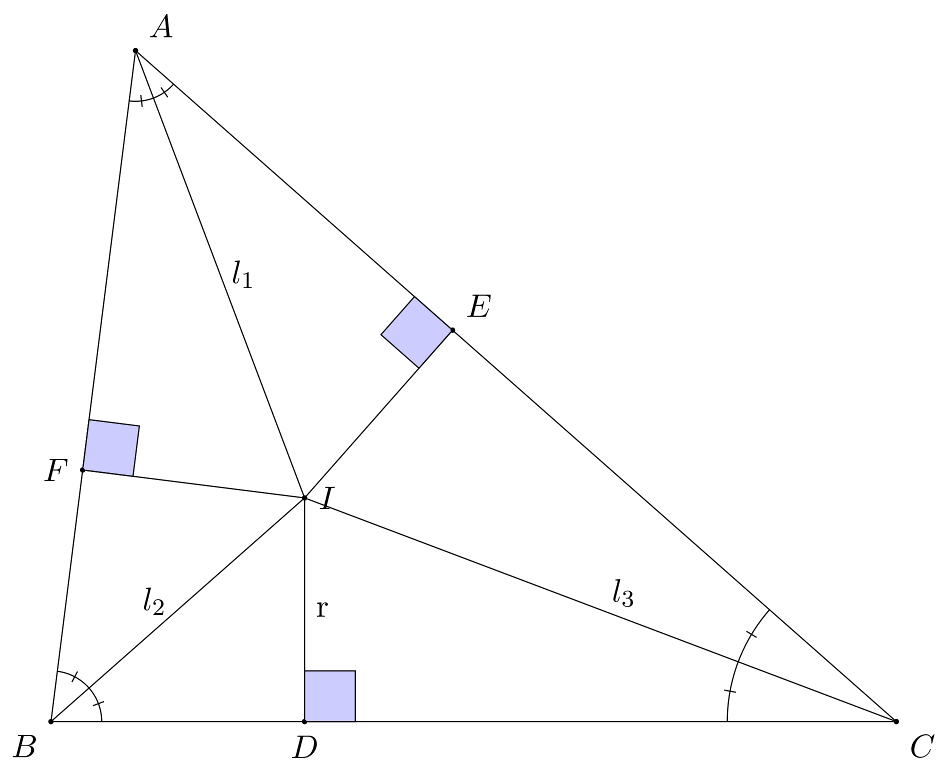


Figure A.4.2.1: Inradius r of $\triangle ABC$

$$ID = IE = IF = r \tag{A.4.2.2}$$

Solution: In $\triangle$s IDC and IEC,

$$ID = IE = \frac{l_3}{\sin\frac{C}{2}} \tag{A.4.2.3}$$

Similarly, in $\triangle$s IEA and IFA,

$$IF = IE = \frac{l_1}{\sin\frac{A}{2}} \tag{A.4.2.4}$$

yielding (A.4.2.2)

A.4.3. In Fig. A.4.2.1, show that

$$BD = BF,\ AE = AF,\ CD = CE \tag{A.4.3.1}$$

Solution: From Fig. A.4.2.1, in $\triangle$s IBD and IBF,

$$x = BD = BF = r\cot\frac{B}{2} \tag{A.4.3.2}$$

Similarly, other results can be obtained.

A.4.4. The circle with centre **I** and radius r in Fig. A.4.4.1 is known as the incircle. Find the radius r.

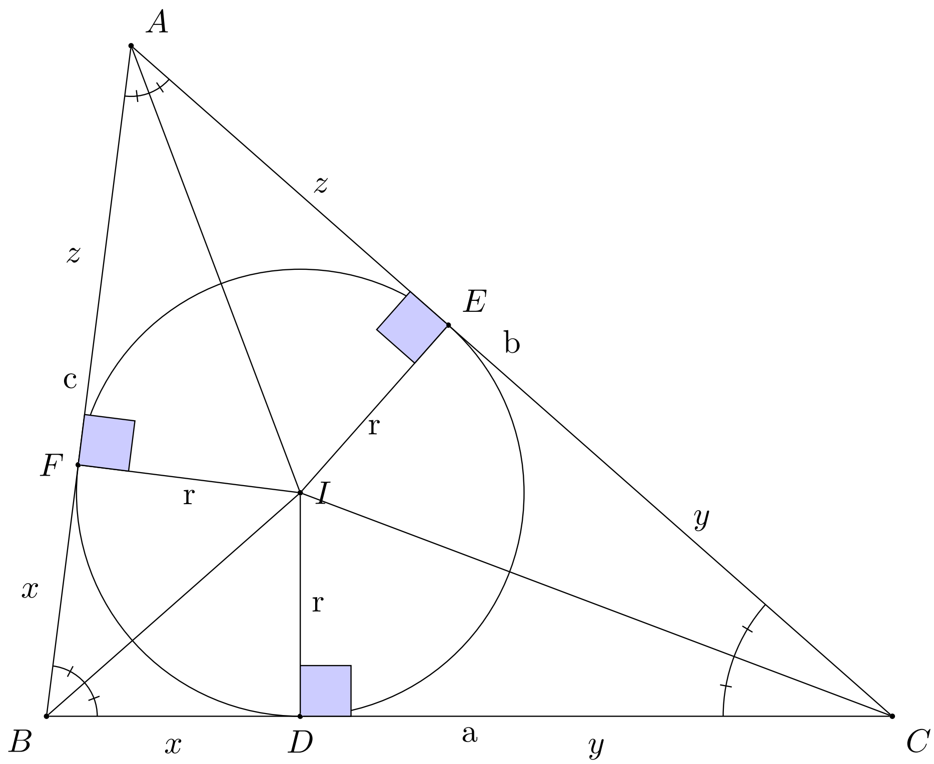


Figure A.4.4.1: Incircle of $\triangle ABC$

Solution: In $\triangle IBC$,

$$a = x + y = r\cot\frac{B}{2} + r\cot\frac{C}{2} \tag{A.4.4.1}$$

$$\implies r = \frac{a}{\cot\frac{B}{2} + \cot\frac{C}{2}} \tag{A.4.4.2}$$

A.5. Circumradius

A.5.1. In Fig. A.5.1.1,

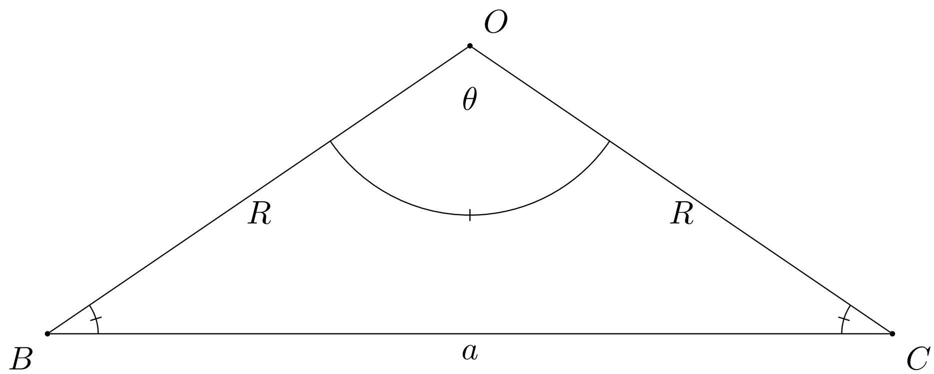


Figure A.5.1.1: Isosceles Triangle

$$OB = OC = R \tag{A.5.1.1}$$

Such a triangle is known as an isosceles triangle. Show that

$$\angle B = \angle C \tag{A.5.1.2}$$

Solution: Using (A.3.2.3),

$$\frac{\sin B}{R} = \frac{\sin C}{R} \tag{A.5.1.3}$$

$$\implies \sin B = \sin C \tag{A.5.1.4}$$

$$\text{or, } \angle B = \angle C. \tag{A.5.1.5}$$

A.5.2. In Fig. A.5.1.1, show that

$$a = 2R\sin\frac{\theta}{2} \tag{A.5.2.1}$$

Solution: In $\triangle OBC$, using the cosine formula from (B.3.3.1),

$$\cos\theta = \frac{R^2 + R^2 - a^2}{2R^2} = 1 - \frac{a^2}{2R^2} \tag{A.5.2.2}$$

$$\implies \frac{a^2}{2R^2} = 2\sin^2\frac{\theta}{2} \tag{A.5.2.3}$$

yielding (A.5.2.1).

A.5.3. In Fig. B.7.2.1, show that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R. \tag{A.5.3.1}$$

Solution: From (B.7.6.1) and (A.5.2.1)

$$a = 2R\sin A \tag{A.5.3.2}$$

A.6. Tangent

A.6.1. In Fig. B.8.2.1, show that $PA.PB = PC^2$.

Solution: In $\triangle$s APC and BPC, using (B.8.2.1),

$$\frac{AP}{\sin\theta} = \frac{AC}{\sin P} \tag{A.6.1.1}$$

$$\frac{PC}{\sin\theta} = \frac{BC}{\sin P} \tag{A.6.1.2}$$

$$\implies \frac{PC}{AP} = \frac{BC}{AC}\left(= \frac{BP}{CP}\right) \tag{A.6.1.3}$$

which gives the desired result. $\triangle$s APC and BPC are said to be <u>similar</u>.

A.7. Identities

A.7.1. Show that

$$\cos 90^\circ = 0 \tag{A.7.1.1}$$

Solution: Using (B.3.3.1) in Fig. A.1,

$$\cos 90^\circ = \frac{a^2 + c^2 - b^2}{2ac} = 0 \tag{A.7.1.2}$$

upon substituting from (A.2.4.1).

A.7.2. Show that

$$\sin 90^\circ = 1 \tag{A.7.2.1}$$

Solution: Trivial from (A.1.2.1).

A.7.3. Prove the following identities

(a)

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta. \tag{A.7.3.1}$$

(b)

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta. \tag{A.7.3.2}$$

Solution: In (A.3.4.1), let

$$\begin{aligned}\theta_1 + \theta_2 &= \alpha \\ \theta_2 &= \beta\end{aligned} \tag{A.7.3.3}$$

This gives (A.7.3.1). In (A.7.3.1), replace α by $90^\circ - \alpha$. This results in

$$\sin(90^\circ - \alpha - \beta) = \sin(90^\circ - \alpha)\cos\beta - \cos(90^\circ - \alpha)\sin\beta \tag{A.7.3.4}$$

$$\implies \cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \tag{A.7.3.5}$$

A.7.4. Using (A.3.4.1) and (A.7.3.2), show that

$$\sin(\theta_1 + \theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 \tag{A.7.4.1}$$

$$\cos(\theta_1 - \theta_2) = \cos\theta_1\cos\theta_2\sin\theta_1\sin\theta_2 \tag{A.7.4.2}$$

Solution: From (A.3.4.1),

$$\sin(\theta_1 + \theta_2)\cos\theta_2 = \sin\theta_1 + \cos(\theta_1 + \theta_2)\sin\theta_2 \tag{A.7.4.3}$$

Using (A.7.3.2) in the above,

$$\begin{aligned}\sin(\theta_1 + \theta_2)\cos\theta_2 = \sin\theta_1 + (&\cos\theta_1\cos\theta_2 \\ &- \sin\theta_1\sin\theta_2)\sin\theta_2\end{aligned} \tag{A.7.4.4}$$

which can be expressed as

$$\sin(\theta_1+\theta_2)\cos\theta_2 = \sin\theta_1 + \cos\theta_1\cos\theta_2\sin\theta_2 - \sin\theta_1\sin^2\theta_2 \quad \text{(A.7.4.5)}$$

Since

$$\sin^2\theta_2 = 1-\cos^2\theta_2, \quad \text{(A.7.4.6)}$$

we obtain

$$\sin(\theta_1+\theta_2)\cos\theta_2 = \cos\theta_1\cos\theta_2\sin\theta_2 + \sin\theta_1\cos^2\theta_2 \quad \text{(A.7.4.7)}$$

resulting in

$$\sin(\theta_1+\theta_2) = \cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2 \quad \text{(A.7.4.8)}$$

after factoring out $\cos\theta_2$. Using a similar approach, (A.7.4.2) can also be proved.

A.7.5. Show that

$$\sin\theta_1+\sin\theta_2 = 2\sin\left(\frac{\theta_1+\theta_2}{2}\right)\cos\left(\frac{\theta_1-\theta_2}{2}\right) \quad \text{(A.7.5.1)}$$

$$\cos\theta_1+\cos\theta_2 = 2\cos\left(\frac{\theta_1+\theta_2}{2}\right)\cos\left(\frac{\theta_1-\theta_2}{2}\right) \quad \text{(A.7.5.2)}$$

$$\sin\theta_1-\sin\theta_2 = 2\sin\left(\frac{\theta_1-\theta_2}{2}\right)\cos\left(\frac{\theta_1+\theta_2}{2}\right) \quad \text{(A.7.5.3)}$$

$$\cos\theta_1-\cos\theta_2 = 2\sin\left(\frac{\theta_1+\theta_2}{2}\right)\cos\left(\frac{\theta_2-\theta_1}{2}\right) \quad \text{(A.7.5.4)}$$

Solution: Let

$$\begin{aligned}\theta_1 &= \alpha + \beta \\ \theta_2 &= \alpha - \beta\end{aligned} \tag{A.7.5.5}$$

From (A.7.4.1),

$$\sin\theta_1 + \sin\theta_2 = \sin(\alpha+\beta) + \sin(\alpha-\beta) \tag{A.7.5.6}$$

$$= \sin\alpha\cos\beta + \cos\alpha\sin\beta \tag{A.7.5.7}$$

$$+ \sin\alpha\cos\beta - \cos\alpha\sin\beta \tag{A.7.5.8}$$

$$= 2\sin\alpha\cos\beta \tag{A.7.5.9}$$

resulting in (A.7.5.1)

$$\because \alpha = \frac{\theta_1+\theta_2}{2} \tag{A.7.5.10}$$

$$\beta = \frac{\theta_1-\theta_2}{2} \tag{A.7.5.11}$$

from (A.7.5.5). Other identities may be proved similarly.

A.7.6. Show that

$$\sin 2\theta = 2\sin\theta\cos\theta \tag{A.7.6.1}$$

$$\cos 2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1 \tag{A.7.6.2}$$

$$= \cos^2\theta - \sin^2\theta \tag{A.7.6.3}$$

Appendix B

Analytic Geometry

B.1. Vectors

B.1.1. A matrix of the form

$$\mathbf{A} \triangleq \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \tag{B.1.1.1}$$

is defined be column vector, or simply, vector. In Fig. A.1 the point vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$ can be defined as

$$\mathbf{A} = \begin{pmatrix} 0 \\ c \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} a \\ 0 \end{pmatrix} \tag{B.1.1.2}$$

B.1.2.

$$\lambda\mathbf{A} \triangleq \begin{pmatrix} \lambda a_1 \\ \lambda a_2 \end{pmatrix} \tag{B.1.2.1}$$

B.1.3. For

$$\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \tag{B.1.3.1}$$

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \end{pmatrix} \tag{B.1.3.2}$$

B.1.4. The transpose of **A** is the <u>row vector</u> defined as

$$\mathbf{A}^\top = \begin{pmatrix} a_1 & a_2 \end{pmatrix} \tag{B.1.4.1}$$

B.1.5. The <u>inner product</u> or <u>dot product</u> is defined as

$$\mathbf{A}^\top \mathbf{B} \equiv \mathbf{A} \cdot \mathbf{B} = \begin{pmatrix} a_1 & a_2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = a_1 b_1 + a_2 b_2 \tag{B.1.5.1}$$

In Fig. A.1,

$$\mathbf{A}^\top \mathbf{C} = 0 \tag{B.1.5.2}$$

B.1.6. The <u>norm</u> of **A** is defined as

$$\|\mathbf{A}\| = \sqrt{\mathbf{A}^\top \mathbf{A}} = \sqrt{a_1^2 + a_2^2} \tag{B.1.6.1}$$

B.1.7. In Fig. A.1, it is easy to verify that

$$\|\mathbf{A} - \mathbf{C}\|^2 = \begin{pmatrix} -c & a \end{pmatrix} \begin{pmatrix} -c \\ a \end{pmatrix} = a^2 + c^2 = b^2 \tag{B.1.7.1}$$

from (A.2.4.1). Thus, the distance betwen any two points **A** and **B** is given by

$$\|\mathbf{A} - \mathbf{B}\| \tag{B.1.7.2}$$

B.1.8. Show that

$$\|\lambda \mathbf{A}\| = |\lambda| \, \|\mathbf{A}\| \tag{B.1.8.1}$$

B.2. Collinear Points

B.2.1. The direction vector of the line AB is

$$\mathbf{A} - \mathbf{B} \equiv \mathbf{B} - \mathbf{A} \equiv \kappa \begin{pmatrix} 1 \\ m \end{pmatrix}, \tag{B.2.1.1}$$

where m is defined to be the slope of AB. In Fig. A.1,

$$\mathbf{A} - \mathbf{C} = \begin{pmatrix} -c \\ a \end{pmatrix} \equiv \begin{pmatrix} 1 \\ -\frac{a}{c} \end{pmatrix} = \begin{pmatrix} 1 \\ -\tan\theta \end{pmatrix} \tag{B.2.1.2}$$

the slope of AC is $-\tan\theta$

B.2.2. Points $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are on a line if they have the same direction vector, i.e.

$$p\left(\mathbf{B}-\mathbf{A}\right)+q\left(\mathbf{C}-\mathbf{B}\right)=0 \implies p, q \neq 0. \tag{B.2.2.1}$$

$\left(\mathbf{A}-\mathbf{B}\right), \left(\mathbf{C}-\mathbf{B}\right)$ are then said to be <u>linearly dependent</u>.

B.2.3. If points $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are collinear,

$$\mathbf{B}=\frac{k\mathbf{A}+\mathbf{C}}{k+1} \tag{B.2.3.1}$$

Solution: From (B.2.2.1),

$$p\left(\mathbf{A}-\mathbf{B}\right)+q\left(\mathbf{A}-\mathbf{C}\right)=0 \implies \mathbf{B}=\frac{p\mathbf{A}+q\mathbf{C}}{p+q} \tag{B.2.3.2}$$

yielding (B.2.3.1) upon substituting

$$k=\frac{p}{q}. \tag{B.2.3.3}$$

This is known as <u>section formula</u>.

B.2.4. Consequently, points $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ form a triangle if

$$p\left(\mathbf{A}-\mathbf{B}\right)+q\left(\mathbf{C}-\mathbf{B}\right) \tag{B.2.4.1}$$

$$=\left(p+q\right)\mathbf{B}-p\mathbf{A}-q\mathbf{C}=0 \tag{B.2.4.2}$$

$$\implies p=0, q=0 \tag{B.2.4.3}$$

B.2.5. In Fig. B.2.5.1

$$AF = BF,\ AE = BE, \tag{B.2.5.1}$$

and the medians BE and CF meet at **G**. Show that

$$\frac{GB}{GE} = \frac{GC}{GF} = 2 \tag{B.2.5.2}$$

Solution: From (B.2.3.1),

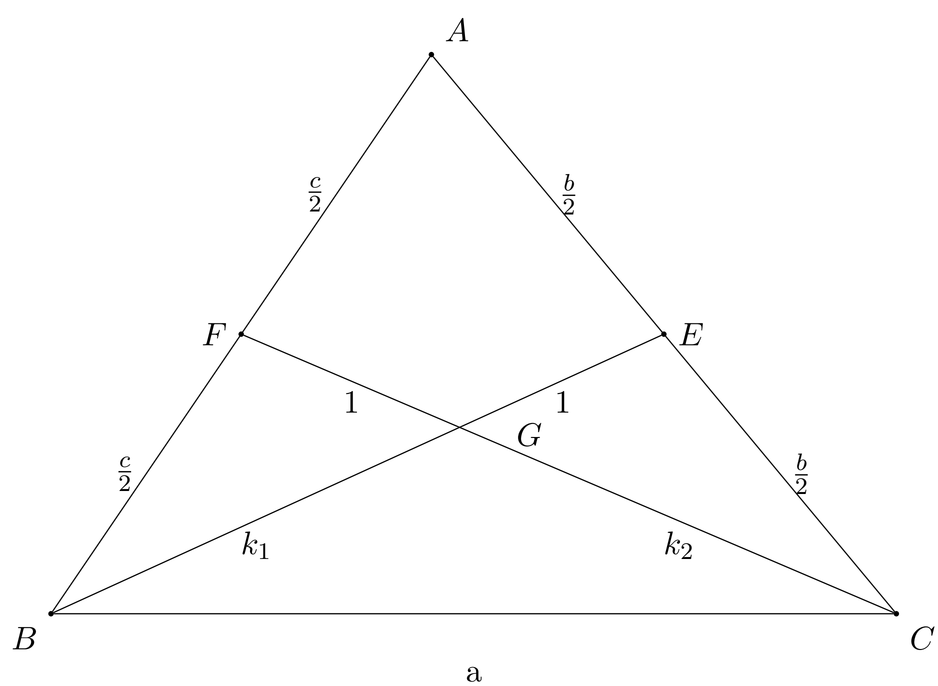


Figure B.2.5.1: $k_1 = k_2 = 2$.

$$\mathbf{G} = \frac{k_1\mathbf{E} + \mathbf{B}}{k_1 + 1} = \frac{k_2\mathbf{F} + \mathbf{C}}{k_2 + 1} \tag{B.2.5.3}$$

$$\implies \frac{k_1\left(\frac{\mathbf{A}+\mathbf{C}}{2}\right) + \mathbf{B}}{k_1 + 1} = \frac{k_2\left(\frac{\mathbf{A}+\mathbf{B}}{2}\right) + \mathbf{C}}{k_2 + 1} \tag{B.2.5.4}$$

$$\implies (k_2 + 1)\left\{k_1(\mathbf{A} + \mathbf{C}) + 2\mathbf{B}\right\} = (k_1 + 1)\left\{k_2(\mathbf{A} + \mathbf{B}) + 2\mathbf{C}\right\} \tag{B.2.5.5}$$

which can be expressed as

$$\left\{2 + k_2 - k_1k_2\right\}\mathbf{B} - (k_2 - k_1)\mathbf{A} - \left\{k_1 + 2 - k_1k_2\right\}\mathbf{C} = 0 \tag{B.2.5.6}$$

and is of the form (B.2.4.3) with

$$p = k_2 - k_1, q = k_1 + 2 - k_1k_2. \tag{B.2.5.7}$$

Thus, from (B.2.4.3)

$$k_2 - k_1 = 0, \tag{B.2.5.8}$$

$$k_1 + 2 - k_1k_2 = 0 \tag{B.2.5.9}$$

Thus, from (B.2.5.9)

$$k_1 = k_2 \tag{B.2.5.10}$$

and substituting the above in (B.2.5.9) results in the quadratic

$$k_1^2 - k_1 - 2 = 0 \tag{B.2.5.11}$$

$$\implies (k_1 - 2)(k_1 + 1) = 0 \tag{B.2.5.12}$$

admitting $k_1 = k_2 = 2$ as the only possible solution.

B.2.6. Substituting $k_1 = 2$ in (B.2.5.3)

$$\mathbf{G} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \tag{B.2.6.1}$$

B.2.7. In Fig. B.2.7.1, AG is extended to join BC at $\mathbf{D}$. Show that AD is also a median.

Solution: Considering the ratios in Fig. B.2.7.1,

$$\mathbf{G} = \frac{k_3\mathbf{D} + \mathbf{A}}{k_3 + 1} \tag{B.2.7.1}$$

$$\mathbf{D} = \frac{k_4\mathbf{C} + \mathbf{B}}{k_4 + 1} \tag{B.2.7.2}$$

Substituting from (B.2.6.1) in the above,

$$(k_3 + 1)\left(\frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3}\right) = k_3\left(\frac{k_4\mathbf{C} + \mathbf{B}}{k_4 + 1}\right) + \mathbf{A} \tag{B.2.7.3}$$

$$\implies (k_3 + 1)(k_4 + 1)(\mathbf{A} + \mathbf{B} + \mathbf{C}) = 3\left\{k_3(k_4\mathbf{C} + \mathbf{B}) + (k_4 + 1)\mathbf{A}\right\} \tag{B.2.7.4}$$

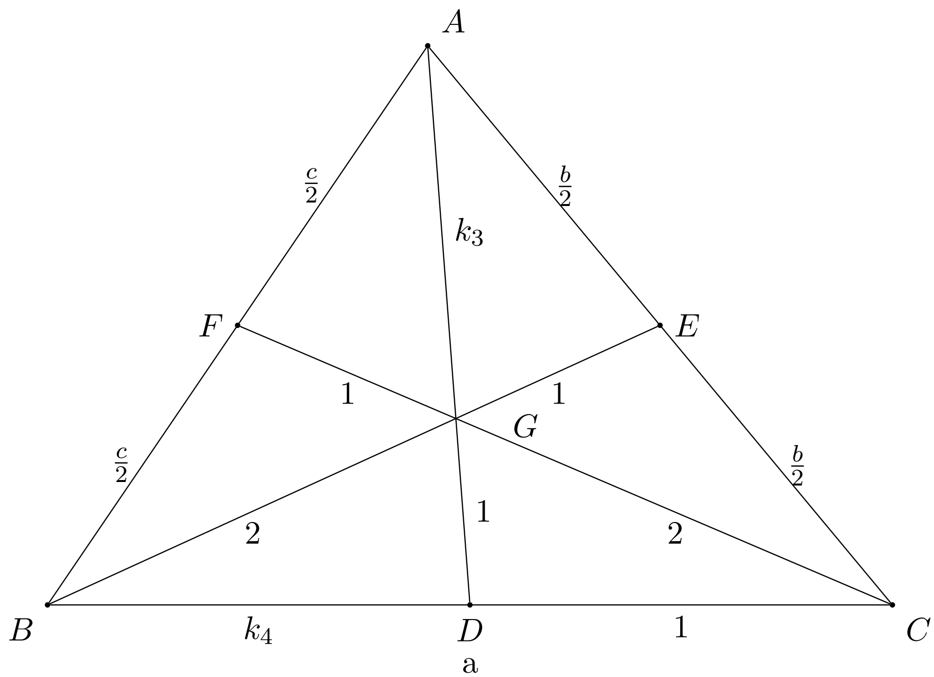


Figure B.2.7.1: $k_3 = 2, k_4 = 1$

which can be expressed as

$$
\begin{aligned}
&(k_3k_4 + k_3 - 2k_4 - 2)\,\mathbf{A} \\
&\qquad - (-k_3k_4 - k_4 + 2k_3 - 1)\,\mathbf{B} \\
&\qquad\qquad - (-k_3 - k_4 - 1 + 2k_3k_4)\,\mathbf{C} = \mathbf{0} \quad \text{(B.2.7.5)}
\end{aligned}
$$

Comparing the above with (B.2.4.3),

$$p = -k_3k_4 - k_4 + 2k_3 - 1, q = -k_3 - k_4 - 1 + 2k_3k_4 \qquad \text{(B.2.7.6)}$$

yielding

$$-k_3k_4 - k_4 + 2k_3 - 1 = 0 \tag{B.2.7.7}$$

$$-k_3 - k_4 - 1 + 2k_3k_4 = 0 \tag{B.2.7.8}$$

Subtracting (B.2.7.7) from (B.2.7.8),

$$3k_3\left(k_4 - 1\right) = 0 \tag{B.2.7.9}$$

$$\implies k_4 = 1 \tag{B.2.7.10}$$

which upon substituting in (B.2.7.7) yields

$$k_3 = 2 \tag{B.2.7.11}$$

B.3. Matrices: Cosine Formula

B.3.1. The determinant of the 2×2 matrix

$$\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \tag{B.3.1.1}$$

is defined as

$$\begin{vmatrix}\mathbf{M}\end{vmatrix} = \begin{vmatrix}\mathbf{A} & \mathbf{B}\end{vmatrix} \tag{B.3.1.2}$$

$$= \begin{vmatrix}a_1 & b_1 \\ a_2 & b_2\end{vmatrix} = a_1b_2 - a_2b_1 \tag{B.3.1.3}$$

B.3.2. In Fig. B.3.2.1, show that

$$\begin{pmatrix}0 & c & b \\ c & 0 & a \\ b & a & 0\end{pmatrix}\begin{pmatrix}\cos A \\ \cos B \\ \cos C\end{pmatrix} = \begin{pmatrix}a \\ b \\ c\end{pmatrix} \tag{B.3.2.1}$$

Solution: From Fig. B.3.2.1,

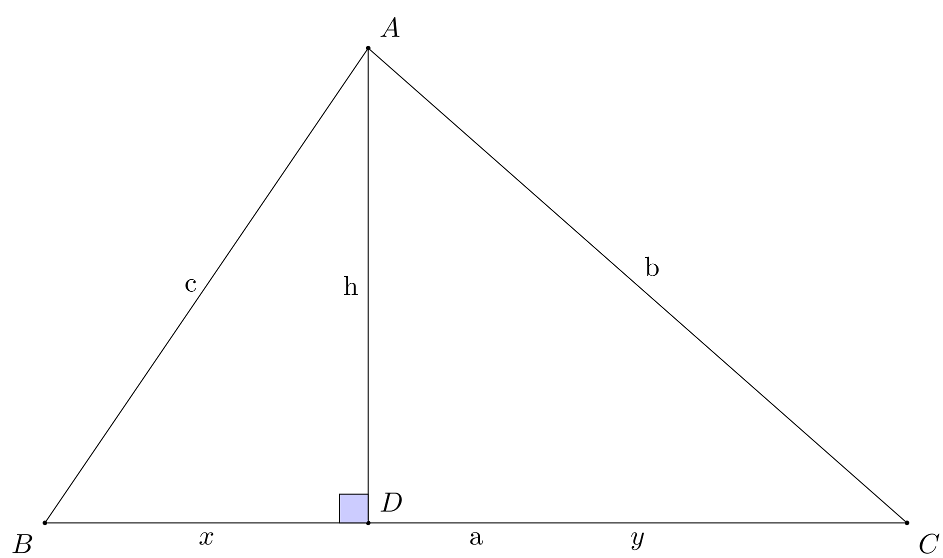


Figure B.3.2.1: The cosine formula

$$a = x + y = b\cos C + c\cos B = \begin{pmatrix} \cos C & \cos B \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix} \tag{B.3.2.2}$$

$$= \begin{pmatrix} 0 & b & c \end{pmatrix} \begin{pmatrix} \cos A \\ \cos C \\ \cos B \end{pmatrix} \tag{B.3.2.3}$$

Similarly,

$$b = c\cos A + a\cos C = \begin{pmatrix} c & 0 & a \end{pmatrix} \begin{pmatrix} \cos A \\ \cos C \\ \cos B \end{pmatrix} \tag{B.3.2.4}$$

$$c = b\cos A + a\cos B = \begin{pmatrix} b & a & 0 \end{pmatrix} \begin{pmatrix} \cos A \\ \cos C \\ \cos B \end{pmatrix} \tag{B.3.2.5}$$

The above equations can be expressed in matrix form as (B.3.2.1).

B.3.3. Show that

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \tag{B.3.3.1}$$

Solution: Using the properties of determinants,

$$\cos A = \frac{\begin{vmatrix} a & c & b \\ b & 0 & a \\ c & a & 0 \end{vmatrix}}{\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}} = \frac{ab^2 + ac^2 - a^3}{abc + abc} = \frac{b^2 + c^2 - a^2}{2abc} \tag{B.3.3.2}$$

B.4. Area of a Triangle: Cross Product

B.4.1. The cross product or vector product defined as $\mathbf{A} \times \mathbf{B}$ is given by (B.3.1.2) for 2×1 vectors.

B.4.2. The area of the triangle with vertices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ is given by

$$\frac{1}{2}\left\|(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{C})\right\| = \frac{1}{2}\left\|\mathbf{A} \times \mathbf{B} + \mathbf{B} \times \mathbf{C} + \mathbf{C} \times \mathbf{A}\right\| \tag{B.4.2.1}$$

B.4.3. If

$$\left\|\mathbf{A} \times \mathbf{B}\right\| = \left\|\mathbf{C} \times \mathbf{D}\right\|, \quad \text{then} \tag{B.4.3.1}$$

$$\mathbf{A} \times \mathbf{B} = \pm\left(\mathbf{C} \times \mathbf{D}\right) \tag{B.4.3.2}$$

where the sign depends on the orientation of the vectors.

B.5. Parallelogram

B.5.1. If $ABCD$ be a parallelogram,

$$\mathbf{B} - \mathbf{A} = \mathbf{C} - \mathbf{D} \tag{B.5.1.1}$$

B.5.2. The area of the parallelogram with vertices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{D}$ is given by

$$\left\|(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{D})\right\| = \left\|\mathbf{A} \times \mathbf{B} + \mathbf{B} \times \mathbf{C} + \mathbf{C} \times \mathbf{A}\right\| \tag{B.5.2.1}$$

B.6. Altitudes of a Triangle:Line Equation

B.6.1. Find the equation of the line BC.

Solution: Let $\mathbf{x}$ be any point on BC. Using section formula, for some k,

$$\mathbf{x} = \frac{k\mathbf{C} + \mathbf{B}}{k+1} = \frac{(k+1)\,\mathbf{C} + (\mathbf{B} - \mathbf{C})}{k+1} \tag{B.6.1.1}$$

$$\implies \mathbf{x} = \mathbf{C} + \lambda \mathbf{m} \tag{B.6.1.2}$$

where

$$\mathbf{m} = \frac{\mathbf{B} - \mathbf{C}}{k+1} \equiv \mathbf{B} - \mathbf{C} \tag{B.6.1.3}$$

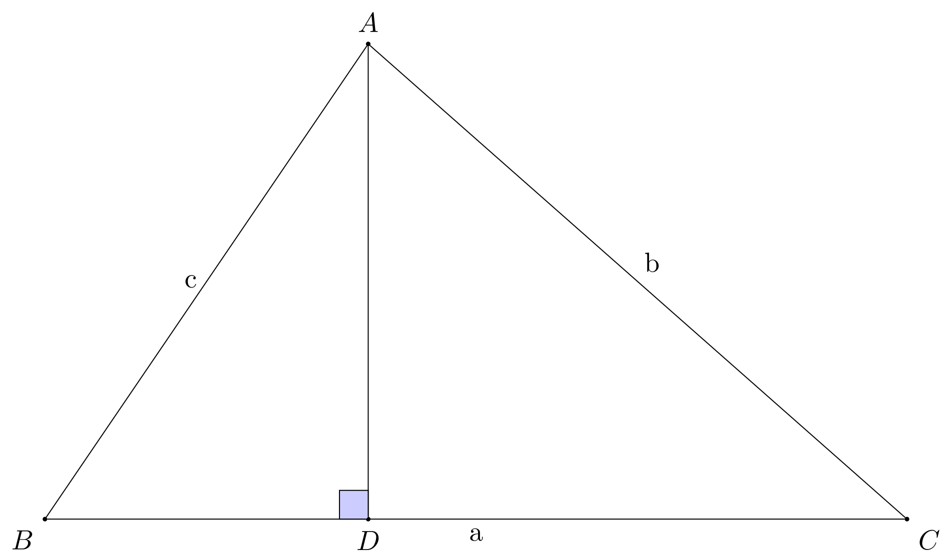


Figure B.6.1.1: Drawing the altitude

B.6.2. The normal vector to $\mathbf{m}$ is defined as

$$\mathbf{n}^\top \mathbf{m} = 0 \tag{B.6.2.1}$$

$$\mathbf{n} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{m} \tag{B.6.2.2}$$

B.6.3. From (B.6.2.1) and (B.6.1.2), it can be verified that

$$\mathbf{n}^\top \mathbf{x} = \mathbf{n}^\top \mathbf{C} + \lambda \mathbf{n}^\top \mathbf{m} \tag{B.6.3.1}$$

$$\implies \mathbf{n}^\top \mathbf{x} = \mathbf{n}^\top \mathbf{C} \tag{B.6.3.2}$$

(B.6.3.2) is defined to be the normal form of the line BC.

B.6.4. In Fig. B.6.5.1, $AD \perp BC$ and $BE \perp AC$ are defined to be the altitudes of $\triangle ABC$.

B.6.5. Let $\mathbf{H}$ be the intersection of the altitudes AD and BE as shown in Fig. B.6.5.1. CH is extended to meet AB at $\mathbf{F}$. Show that $CF \perp AB$.

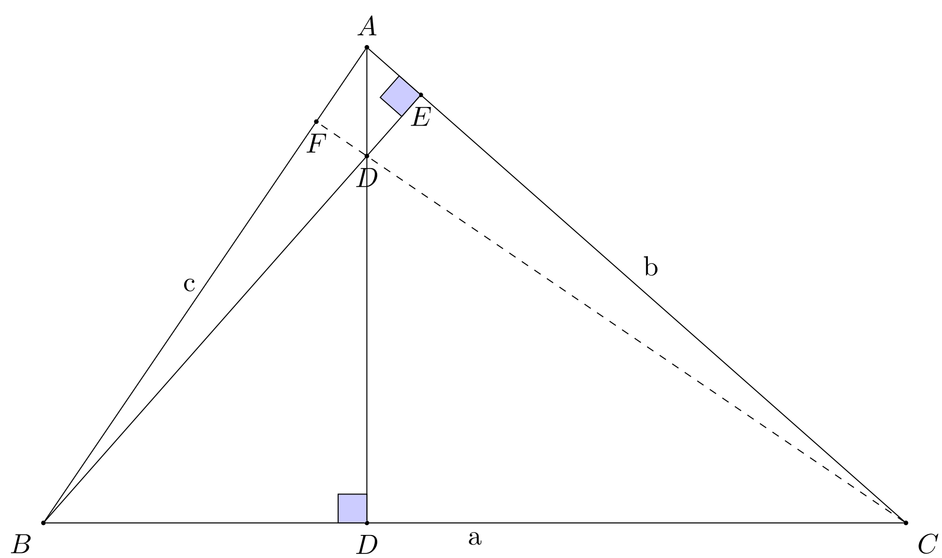


Figure B.6.5.1: Altitudes of a triangle meet at the orthocentre H

Solution: From (B.6.1.3) (B.6.2.1), (B.1.5.2) and (B.6.3.2), the equations of AD and BE are

$$(\mathbf{B}-\mathbf{C})^\top (\mathbf{x}-\mathbf{A}) = 0 \tag{B.6.5.1}$$

$$(\mathbf{C}-\mathbf{A})^\top (\mathbf{x}-\mathbf{B}) = 0 \tag{B.6.5.2}$$

$\because$ **H** lies on both AD and BE, it satisfies the above equations, and

$$(\mathbf{B}-\mathbf{C})^\top (\mathbf{H}-\mathbf{A}) = 0 \tag{B.6.5.3}$$

$$(\mathbf{C}-\mathbf{A})^\top (\mathbf{H}-\mathbf{B}) = 0 \tag{B.6.5.4}$$

Adding both the above and simplifying,

$$(\mathbf{B}-\mathbf{A})^\top (\mathbf{H}-\mathbf{C}) = 0 \tag{B.6.5.5}$$

$\implies CH \perp AB$ from (B.1.5.2), or $CF \perp AB$.

B.6.6. Altitudes of a $\triangle$ meet at the orthocentre H.

B.7. Circumcircle: Circle Equation

B.7.1. In Fig. B.7.1.1,

$$OB = OC = R, BD = DC. \tag{B.7.1.1}$$

Show that $OD \perp BC$.

Solution:

$$\|\mathbf{O}-\mathbf{C}\| = \|\mathbf{O}-\mathbf{B}\| = R \tag{B.7.1.2}$$

$$\implies \|\mathbf{O}-\mathbf{C}\|^2 = \|\mathbf{O}-\mathbf{B}\|^2 \tag{B.7.1.3}$$

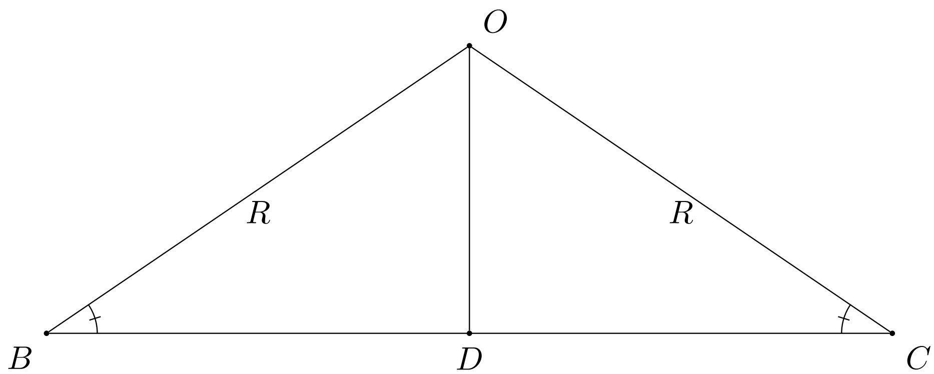


Figure B.7.1.1: Perpendicular bisector.

which can be expressed as

$$(\mathbf{O}-\mathbf{C})^{\top}(\mathbf{O}-\mathbf{C}) = (\mathbf{O}-\mathbf{B})^{\top}(\mathbf{O}-\mathbf{B}) \tag{B.7.1.4}$$

$$\|\mathbf{O}\|^2 - 2\mathbf{O}^{\top}\mathbf{C} + \|\mathbf{C}\|^2 = \|\mathbf{O}\|^2 - 2\mathbf{O}^{\top}\mathbf{B} + \|\mathbf{B}\|^2 \tag{B.7.1.5}$$

$$\implies (\mathbf{B}-\mathbf{C})^{\top}\mathbf{O} = \frac{\|\mathbf{B}\|^2 - \|\mathbf{C}\|^2}{2} \tag{B.7.1.6}$$

which can be simplified to obtain

$$(\mathbf{B}-\mathbf{C})^{\top}\left\{\mathbf{O} - \left(\frac{\mathbf{B}+\mathbf{C}}{2}\right)\right\} = 0 \tag{B.7.1.7}$$

$$\text{or, } (\mathbf{B}-\mathbf{C})^{\top}\{\mathbf{O}-\mathbf{D}\} = 0 \tag{B.7.1.8}$$

which proves the give result using (B.2.3.1) and (B.1.5.2).

B.7.2. The equation of the circle in Fig. B.7.2.1, is

$$\|\mathbf{x}-\mathbf{O}\| = R \tag{B.7.2.1}$$

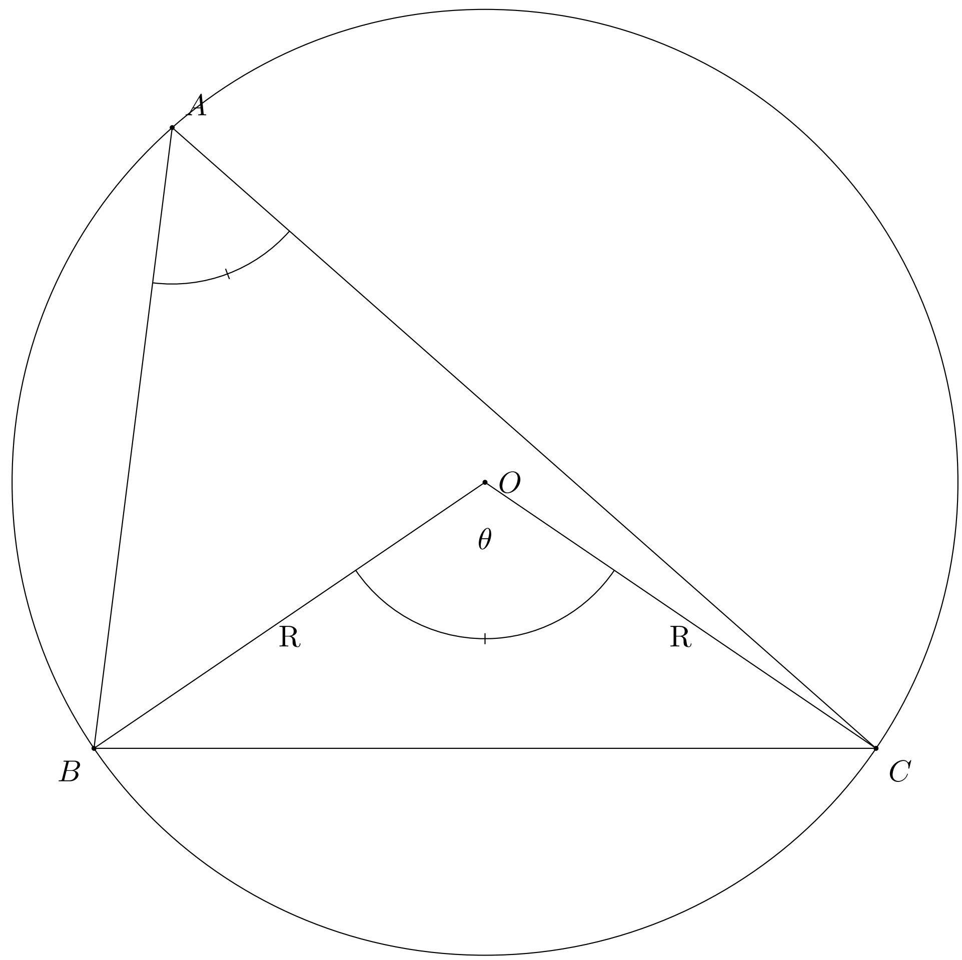


Figure B.7.2.1: Circumcircle of $\triangle ABC$

This is known as the circumcircle of $\triangle ABC$.

B.7.3. In Fig. B.3.2.1 show that

$$\cos A = \frac{(\mathbf{A}-\mathbf{B})^{\top}(\mathbf{A}-\mathbf{C})}{\|\mathbf{A}-\mathbf{B}\|\,\|\mathbf{A}-\mathbf{C}\|} \tag{B.7.3.1}$$

Solution: From (B.3.3.1), using (B.1.7.2),

$$\cos A = \frac{\|\mathbf{A}-\mathbf{B}\|^2 + \|\mathbf{A}-\mathbf{C}\|^2 - \|\mathbf{B}-\mathbf{C}\|^2}{2\,\|\mathbf{A}-\mathbf{B}\|\,\|\mathbf{A}-\mathbf{C}\|} \tag{B.7.3.2}$$

$$= \frac{\|\mathbf{A}\|^2 - \mathbf{A}^\top\mathbf{B} - \mathbf{A}^\top\mathbf{C} + \mathbf{B}^\top\mathbf{C}}{\|\mathbf{A}-\mathbf{B}\|\,\|\mathbf{A}-\mathbf{C}\|} \tag{B.7.3.3}$$

which can be expressed as (B.7.3.1).

B.7.4. Any point on the circle can be expressed as

$$\mathbf{x} = \mathbf{O} + R\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}, \quad 0 \in [0, 2\pi]\,. \tag{B.7.4.1}$$

B.7.5. Let

$$R = 1,\ \mathbf{O} = \mathbf{0},\ \mathbf{A} = \begin{pmatrix}\cos\theta_1 \\ \sin\theta_1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}\cos\theta_2 \\ \sin\theta_2\end{pmatrix}, \tag{B.7.5.1}$$

Show that

$$\|\mathbf{A}-\mathbf{B}\| = 2\sin\left(\frac{\theta_1-\theta_2}{2}\right) \tag{B.7.5.2}$$

Solution: From (B.7.4.1).

$$\mathbf{A}-\mathbf{B} = \begin{pmatrix} \cos\theta_1 - \cos\theta_2 \\ \sin\theta_1 - \sin\theta_2 \end{pmatrix} \tag{B.7.5.3}$$

$$\implies \|\mathbf{A}-\mathbf{B}\|^2 = (\cos\theta_1 - \cos\theta_2)^2 + (\sin\theta_1 - \sin\theta_2)^2 \tag{B.7.5.4}$$

$$= 2\left\{1 - \cos(\theta_1 - \theta_2)\right\} = 4\sin^2\left(\frac{\theta_1 - \theta_2}{2}\right) \tag{B.7.5.5}$$

yielding (B.7.5.2) from (A.7.6.3).

B.7.6. In Fig. B.7.2.1, show that

$$\theta = 2A. \tag{B.7.6.1}$$

Solution: Let

$$\mathbf{C} = \begin{pmatrix} \cos\theta_3 \\ \sin\theta_3 \end{pmatrix} \tag{B.7.6.2}$$

Then, substituting from (B.7.5.2) in (B.7.3.2),

$$\cos A = \frac{4\sin^2\left(\frac{\theta_1-\theta_2}{2}\right) + 4\sin^2\left(\frac{\theta_1-\theta_3}{2}\right) - 4\sin^2\left(\frac{\theta_2-\theta_3}{2}\right)}{8\sin\left(\frac{\theta_1-\theta_2}{2}\right)\sin\left(\frac{\theta_1-\theta_3}{2}\right)} \tag{B.7.6.3}$$

$$= \frac{2\sin^2\left(\frac{\theta_1-\theta_2}{2}\right) + \cos(\theta_2-\theta_3) - \cos(\theta_1-\theta_3)}{4\sin\left(\frac{\theta_1-\theta_2}{2}\right)\sin\left(\frac{\theta_1-\theta_3}{2}\right)} \tag{B.7.6.4}$$

from (A.7.6.3). $\therefore$ from (A.7.5.4),

$$\cos A = \frac{2\sin^2\left(\frac{\theta_1-\theta_2}{2}\right) + 2\sin\left(\frac{\theta_1-\theta_2}{2}\right)\sin\left(\frac{\theta_1+\theta_2}{2} - \theta_3\right)}{4\sin\left(\frac{\theta_1-\theta_2}{2}\right)\sin\left(\frac{\theta_1-\theta_3}{2}\right)} \tag{B.7.6.5}$$

$$= \frac{\sin\left(\frac{\theta_1-\theta_2}{2}\right) + \sin\left(\frac{\theta_1+\theta_2}{2} - \theta_3\right)}{2\sin\left(\frac{\theta_1-\theta_3}{2}\right)} \tag{B.7.6.6}$$

From (A.7.5.1), the above equation can be expressed as

$$\cos A = \frac{2\sin\left(\frac{\theta_1-\theta_3}{2}\right)\cos\left(\frac{\theta_2-\theta_3}{2}\right)}{2\sin\left(\frac{\theta_1-\theta_3}{2}\right)} = \cos\left(\frac{\theta_2-\theta_3}{2}\right) \tag{B.7.6.7}$$

$$\implies 2A = \theta_2 - \theta_3 \tag{B.7.6.8}$$

Similarly,

$$\cos\theta = \frac{1+1-4\sin^2\left(\frac{\theta_2-\theta_3}{2}\right)}{2} = \cos\left(\theta_2-\theta_3\right) = \cos 2A \tag{B.7.6.9}$$

B.8. Tangent

B.8.1. In Fig. B.8.1.1, OC is the radius and PC touches the circle at C. Show that

$$OC \perp PC. \tag{B.8.1.1}$$

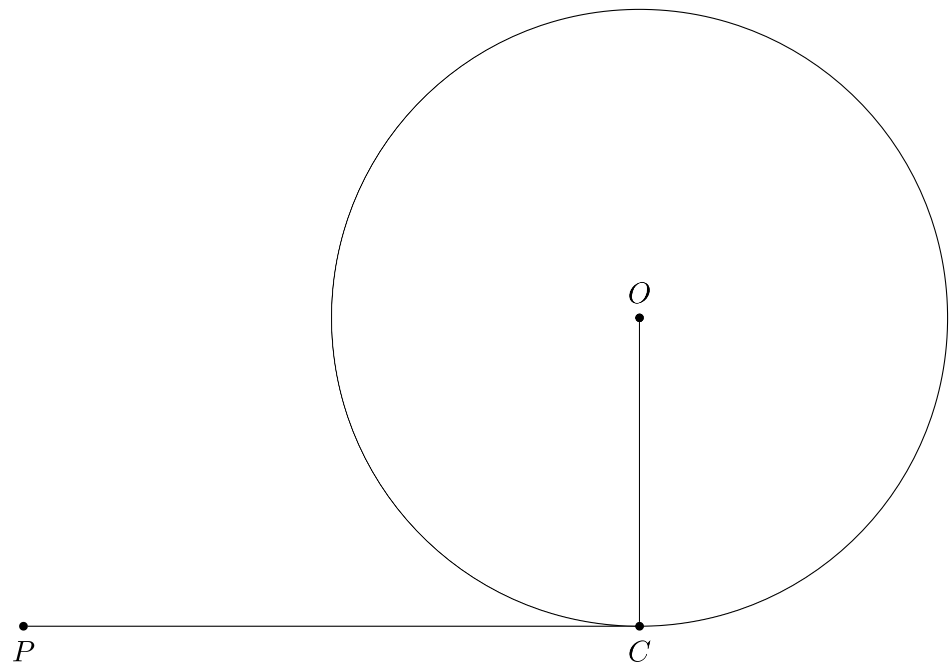


Figure B.8.1.1:

Solution: The equation of PC can be expressed as

$$\mathbf{x} = \mathbf{C} + \mu\mathbf{m} \tag{B.8.1.2}$$

and the equation of the circle is

$$\|\mathbf{x} - \mathbf{O}\| = R \tag{B.8.1.3}$$

Substituting (B.8.1.2) in (B.8.1.3),

$$\left\|\mathbf{C} + \mu\mathbf{m} - \mathbf{O}\right\|^2 = R^2 \tag{B.8.1.4}$$

$$\implies \mu^2 \left\|\mathbf{m}\right\|^2 + 2\mu\mathbf{m}^\top \left(\mathbf{C} - \mathbf{O}\right) + \left\|\mathbf{C} - \mathbf{O}\right\|^2 - R^2 = 0 \tag{B.8.1.5}$$

The above equation has only one root. Hence the discriminant of the above quadratic should be zero. So,

$$\left\{\mathbf{m}^\top \left(\mathbf{C} - \mathbf{O}\right)\right\}^2 - \left\|\mathbf{m}\right\|^2 \left\{\left\|\mathbf{C} - \mathbf{O}\right\|^2 - R^2\right\} = 0 \tag{B.8.1.6}$$

Since $\mathbf{C}$ is a point on the circle,

$$\left\|\mathbf{C} - \mathbf{O}\right\|^2 - R^2 = 0 \tag{B.8.1.7}$$

$$\implies \mathbf{m}^\top \left(\mathbf{C} - \mathbf{O}\right) = 0 \tag{B.8.1.8}$$

upon substituting in (B.8.1.6). Using the definition of the direction vector from (B.2.1.1)

$$\mathbf{m} = \mathbf{P} - \mathbf{C} \tag{B.8.1.9}$$

$$\implies \left(\mathbf{P} - \mathbf{C}\right)^\top \left(\mathbf{C} - \mathbf{O}\right) = 0 \tag{B.8.1.10}$$

which is equivalent to (B.8.1.1).

B.8.2. In Fig. B.8.2.1 show that

$$\theta = \alpha \tag{B.8.2.1}$$

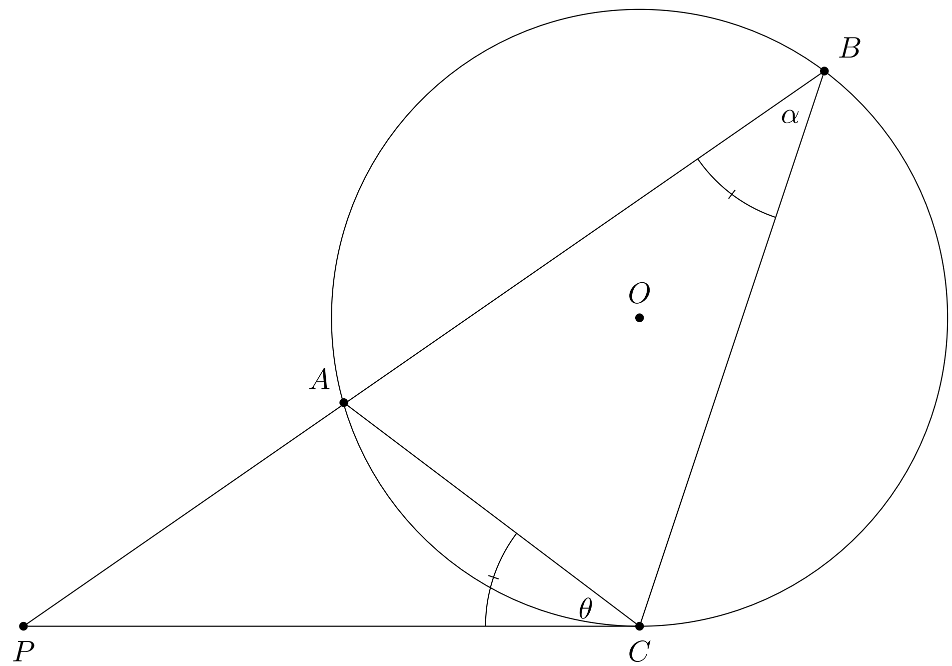


Figure B.8.2.1: $\theta = \alpha$.

Solution: Let Let

$$\mathbf{O} = \mathbf{0}\mathbf{A} = \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} \cos\theta_2 \\ \sin\theta_2 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} \cos\theta_3 \\ \sin\theta_3 \end{pmatrix} \tag{B.8.2.2}$$

Without loss of generality, let

$$\theta_3 = \frac{\pi}{2} \tag{B.8.2.3}$$

Then,

$$\mathbf{C} - \mathbf{O} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{B.8.2.4}$$

From from (B.8.1.10),

$$\mathbf{C} - \mathbf{P} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{B.8.2.5}$$

From (B.7.3.1) and (B.8.2.5),

$$\cos\theta = \frac{\begin{pmatrix} \cos\theta_3 - \cos\theta_1 & \sin\theta_3 - \sin\theta_1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}}{2\sin\left(\frac{\theta_1 - \theta_3}{2}\right)} \tag{B.8.2.6}$$

$$= \sin\left(\frac{\theta_1 + \theta_3}{2}\right) = \cos\left(\frac{\pi}{2} - \frac{\theta_1 + \theta_3}{2}\right) = \cos\left(\frac{\pi}{4} - \frac{\theta_1}{2}\right) \tag{B.8.2.7}$$

upon substituting from (B.8.2.3). Similarly, from (B.7.6.7),

$$\cos\alpha = \cos\left(\frac{\theta_1 - \theta_3}{2}\right) = \cos\left(\frac{\pi}{4} - \frac{\theta_1}{2}\right) = \cos\theta \tag{B.8.2.8}$$