Module 7: Renewal Processes

Lecture 1 Renewal Function and Renewal Equation

Introduction

Renewal theory is the branch of probability theory that generalizes Poisson processes for arbitrary inter-arrival (holding) times. Let X_1 be the time to the first renewal and let $X_n (n = 2, 3, \ldots)$ be the time between $(n-1)$-th renewal and n-th renewal. Assume that X_n $(n = 1, 2, \ldots)$ are i.i.d. non-negative random variables with distribution function F. Let

$$\mu = E(X_n) = \int_0^\infty x dF(x)$$

which we assume to be positive and finite.

Definition 1. *Define the time of the n-th renewal by*

$$S_n = \sum_{i=1}^{n} X_i.$$

Let $N(t)$ be the number of renewals by time t so that

$$N(t) = max\{n : S_n \leq t\}.$$

Then, the counting process $\{N(t), t \geq 0\}$ will be a renewal process. If for some n, $S_n = t$, then a renewal is said to occur at t; S_n gives the time of the nth renewal and is called the nth renewal time.

In the above, we are assuming that 0 is a renewal time. Sometimes, more general processes can be considered where 0 is not a renewal time.

Remark 2.

1. *A counting process for which the inter-arrival times are i.i.d. with an arbitrary distribution is said to be a renewal process.*

2. *A Poisson process can be defined as a counting process for which the inter-arrival times are i.i.d. with an exponential distribution. A renewal process is more general counting process than Poisson process.*

3. *A continuous-time stochastic process in which the embedded jump chain (the discrete process registering what values the process takes) is a Markov chain,*

and where the holding times (time between jumps) are random variables with any distribution, whose distribution function may depend on the two states between which the move is made, we say it is called a semi-Markov process or Markov renewal process.

4. *When $P(X = c) = 1$ for some $c > 0$, $\{S_n = nc, n \geq 1\}$ is called a deterministic renewal process.*

Distribution

Consider a renewal process $\{N(t), t \geq 0\}$ with inter-arrival time distribution F. Note that, the event $\{N(t) \geq n\}$ is equivalent to the event $\{S_n \leq t\}$. Hence,

$$P\{N(t) \geq n\} = P(S_n \leq t\} = F^{(n)}(t)$$

where $F^{(n)}$ is n-fold convolution of F with $F^{(0)} = 1$.

$$\begin{aligned} P(N(t) = n) &= P(N(t) \geq n) - P(N(t) \geq n+1) \\ &= F^{(n)}(t) - F^{(n+1)}(t) \end{aligned}$$

For example, when inter-arrival time is uniform distribution between 0 and 1, the n-fold convolution of F is given by

$$F^{(n)}(t) = \frac{t^n}{n!}, \ n = 1, 2, \ldots; \quad 0 \leq t \leq 1$$

which can be shown from the mathematical induction.

$$\begin{aligned} F^{(n+1)}(t) &= P(S_{n+1} \leq t) \\ &= \int_0^t P(S_{n+1} \leq t \mid S_n = x) \frac{x^{n-1}}{(n-1)!} dx \\ &= \int_0^t P(X_{n+1} \leq t - x) \frac{x^{n-1}}{(n-1)!} dx \\ &= \frac{t^{n+1}}{(n+1)!}, \quad 0 \leq t \leq 1. \end{aligned}$$

Example 3. *Poisson process is a counting renewal process $\{N(t), t \geq 0\}$ whose inter-arrival time has a exponential distribution with parameter λ, i.e., $X \sim Exp(\lambda)$.*

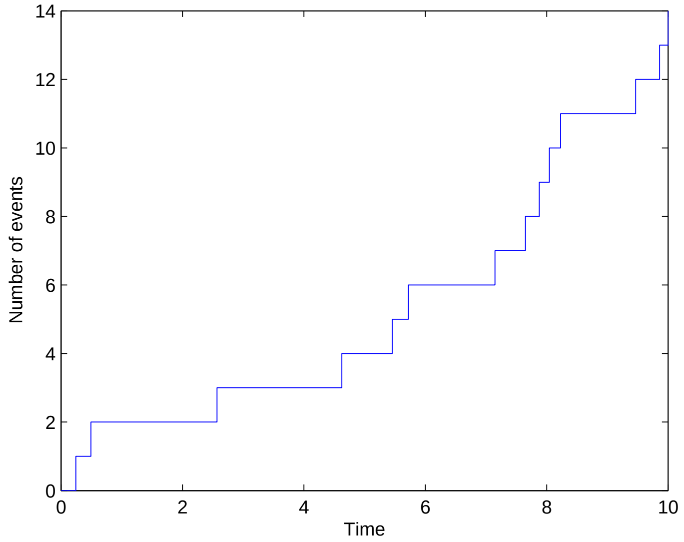

Figure 7.1: Sample path of Poisson process with $\lambda = 2$

Distribution of $N(t)$

$$P(N(t) = n) = \frac{e^{-\lambda t}\,(\lambda t)^n}{n!}, n = 0, 1, \ldots$$

For arbitrary n,

$$P(X_n > x) = P(N(x) = 0 \mid N(t_{n-1}) = n - 1) = e^{-\lambda x}, x \geq 0$$

Hence, the distribution of X is $F(x) = 1 - e^{-\lambda x}, x \geq 0$.

Sample path of Poisson process with $\lambda = 2$ is shown in Figure 7.1.

Suppose that in a system, an unit fails, according with a Poisson process with rate $\lambda = 3$ per day. Suppose that there are 6 spare units in an inventory and the next supply is not due in 4 days. The probability that the system will be out of order in the next 4 days is

$$\sum_{n=7}^{\infty} P(N(4) = n) = 1 - \sum_{n=0}^{6} P(N(4) = n) = 0.954\ .$$

In this module, we discuss various stochastic processes as shown in Figure 7.2.

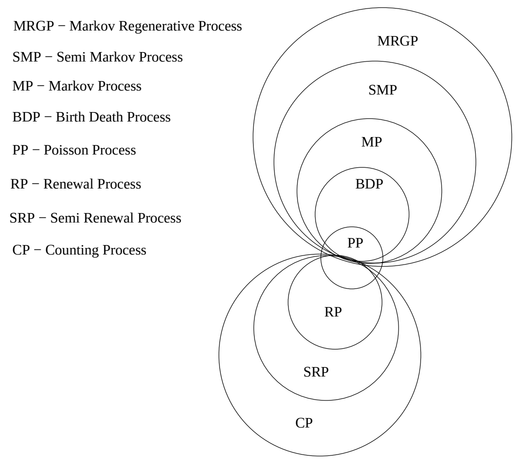

Figure 7.2: Various Stochastic Processes

Renewal Function

Let

$$M(t) = E[N(t)] = \sum_{n=1}^{\infty} P(N(t) \geq n) = \sum_{n=1}^{\infty} F^{(n)}(t)$$

Then $M(t)$ is called a renewal function. The function $m(t) = M'(t)$ is called the renewal density function of the renewal process.

$$\begin{aligned} m(t) &= \lim_{h \to 0^+} \frac{P(\text{one or more renewals in } (t, t+h))}{h} \\ &= \lim_{h \to 0^+} \frac{1}{h} \sum_{n=1}^{\infty} P(n \text{ renewals in } (t, t+h)) \\ &= \sum_{n=1}^{\infty} \lim_{h \to 0^+} \frac{1}{h} \left[F^{(n)}(t+h) - F^{(n)}(t) \right] = M'(t). \end{aligned}$$

For example, the renewal process whose inter-arrival time follows i.i.d. with uniform distribution between 0 and 1, the renewal function for $0 \leq t \leq 1$ is given by

$$M(t) = \sum_{n=1}^{\infty} F^{(n)}(t) = \sum_{n=1}^{\infty} \frac{t^n}{n!} = e^t - 1.$$

$$M(t) = \sum_{n=1}^{\infty} F^{(n)}(t)$$

In equivalent form using the Laplace transform,

$$M^*(s) = \sum_{n=1}^{\infty} F^{(n)*}(s) = \sum_{n=1}^{\infty} \frac{f^{(n)*}(s)}{s}$$

where $f^{(n)}(t) = F^{(n)'}(t)$.

$$M^*(s) = \frac{1}{s}\sum_{n=1}^{\infty} [f^*(s)]^n = \frac{f^*(s)}{s\,[1-f^*(s)]}.$$

Hence,

$$m^*(s) = \frac{f^*(s)}{1-f^*(s)} \text{ and } f^*(s) = \frac{m^*(s)}{1+m^*(s)}.$$

Renewal Equation

It can be shown that the renewal function $M(t), 0 \leq t < \infty$, uniquely determine the inter-arrival time distribution F. For example, $M(t) = \lambda t$ corresponds to the exponential distribution with parameter λ. Renewal equations are useful for deriving the quantity of interest associated with a renewal process at a function of time. A renewal equation is expressed by a recursive form through an integral equation. We know that

$$M(t) = E[N(t)].$$

This can be evaluated by conditioning on X_1, the time of first renewal, i.e.,

$$M(t) = \int_0^{\infty} E[N(t) \mid X_1 = x]dF(x).$$

This integral can be evaluated by dividing into two cases: one is the case where the first renewal occurs after time t and the other is the case where the first renewal occurs before time t. In the former case, since there are no renewals observed by t,

$$E[N(t) \mid X_1 = x > t] = 0.$$

In the later case, the first renewal occurs before time t and the expected number of renewals between x and t will be $M(t-x)$ from the definition of the renewal function. Hence,

$$E[N(t) \mid X_1 = x < t] = 1 + M(t-x).$$

Hence,

$$M(t) = \int_0^t (1 + M(t-x))dF(x) = F(t) + \int_0^t M(t-x)dF(x).$$

This integral equation is called a renewal equation if $M(.)$ is considered as unknown.The above equation can be written as:

$$M = F + M * f.$$

Now, taking Laplace transform on both sides,

$$M^* = F^* + (M * f)^* = F^* + M^* f^*.$$

Substituting $F^* = \frac{f^*(s)}{s}$ and simplifying, we get

$$f^*(s) = \frac{sM^*(s)}{1 + sM^*(s)}.$$

The renewal equation can be generalized for $Z(t)$, the unknown function associated with a renewal process with distribution function F,

$$Z(t) = Q(t) + \int_0^t Z(t-x)dF(x)$$

where $Q(t)$ is a known function.

Renewal Times

Let $X_1, X_2, \ldots$ be the time between its successive occurrences. Then

$$S_0 = 0; S_{n+1} = S_n + X_{n+1}, n = 1, 2, \ldots$$

define the times of occurrence assuming that the time origin is taken to be an instant of such an occurrence. The sequence $\{S_n, n = 0, 1, \ldots\}$ is called a renewal process provided that $X_1, X_2, \ldots$ be i.i.d. non-negative random variables. Then the S_n, are called renewal times.

Note that, the renewal process $\{S_n, n = 0, 1, \ldots\}$ is said to be recurrent if $X_n < \infty$ almost surely for every n; otherwise is called transient.

Age, Excess and Spread at Time t
Suppose $\{N(t), t \geq 0\}$ is a renewal process.

1. **Age:** The age at t of the renewal process is defined by

$$A(t) = t - S_{N(t)}.$$

2. **Excess:** The excess at t of the renewal process is defined by

$$Y(t) = S_{N(t)+1} - t.$$

3. **Spread:** The spread at t of the renewal process is defined by

$$X_{N(t)+1} = A(t) + Y(t).$$

For example, let $Y(t)$ be the excess at time t. Let $g(t) = E[Y(t)]$. Using the above renewal equation, we can find $g(t)$. Conditioning on X_1 gives

$$g(t) = \int_0^\infty E[Y(t) \mid X_1 = x] dF(x).$$

Now,

$$E[Y(t) \mid X_1 = x > t] = x - t$$

while for $X_1 = x < t$

$$E[Y(t) \mid X_1 = x < t] = g(t - x).$$

Hence,

$$g(t) = \int_t^\infty (x - t) dF(x) + \int_0^t g(t - x) dF(x)$$

is the renewal equation for $g(t)$.