1

22nd Jan

UNIT - IV

BRANCHING PROCESS

BRANCHING PROCESS - The history of the study of branching processes dates back to 1874 when a mathematical model was formulated by F. Galton and H. W. Watson for predicting whether a family name would become (समाप्त) extinct after a certain number of generation. The model didn't attract much attention for a long time. However the situation gradually changed and during last 70 yrs much attention has been devoted to it this is becoz of the develop-ment of interest in the application of proby theory in general and also becoz of the possibility of using the models in a variety of biological, physical and other fields of study where one is concern with objects (or with individuals) that can generate objects of similar kind such objects may be biological entities such as human beings, animals, genes, bacteria, etc which reproduce similar objects by biological methods or may be physical particles such as neutrons, which yield new neutrons under a nuclear chain reaction.

We consider the discrete time case. Suppose that we start with an initial set of objects (or individuals) which form the zeroth generation called Ancestors. The objects generated by the objects of zeroth generation are the direct decendents of the Ancestors and are said to form the first generation, the objects generated by those of the 1st generation from the 2nd generation & so on.

The no. of objects of the rth generation where ~~are~~ $r=0,1,2,\ldots n$ is a r.v. here. We assume that the objects reproduce independently of other objects. Let the r.vs $X_0, X_1, X_2, \ldots$ denote the size of (or the no. of individuals in) the zeroth, 1st, 2nd, ... so on. generations respectively. Let p_k be the probty that an object (irrespective of the generation to which it belongs) generates k similar objects where $p_k \geq 0$. $k=0,1,2,\ldots$

$$\sum_{\forall k} p_k = 1.$$

The sequence $\{X_n, n=0,1,2\ldots\}$ constitutes a Galton-Watson branching process with offspring (direct decendents) distribution p_k.

Our interest lies mainly in the

1) Probability distn of X_n

2) The probty that $X_n \to 0$ for some n i.e.

the probty of ultimate extinction.

For simplicity we assume that $X_0 = 1$ ie the process starts with a single ancestor.

The sequence $\{X_n\}$ forms a Markov's chain with transition probty matrix P_{ij} with probty that

$$P_{ij} = P_r[X_{n+1} = j / X_n = i] \quad \text{where } i, j = 0, 1, 2, \ldots$$

let $X_{n+1} = Z_1 + Z_2 + \ldots + Z_{X_n}$

where Z_r is the number of offspring of the rth member of the nth generation and Z_r's are independently identically distributed r.v.s with probty dist^n $\{P_k\}$.

let $P(s)$ be the p.g.f. of $\{Z_r\}$

then $P(s) = \sum_{k=0}^{\infty} P_r[Z_r = k] s^k$ Acc to def^n of pgf's

let pgf of X_{n+1} be $P_{n+1}(s)$, here

$n = 0, 1, 2, \ldots$

clearly, $X_1 = Z_1 + Z_2 + \ldots + Z_{X_0}$

since $X_0 = 1$

$X_1 = Z_1$

$P_1(s)$ is pgf of X_1 is

$$P_1(s) = \sum_{k=0}^{\infty} P_r[X_1 = k] s^k$$

$$= \sum_{k=0}^{\infty} P_r[Z_1 = k] s^k = P(s) \quad (\text{assumed})$$

Now $P_{n+1}(s) = \sum_{k=0}^{\infty} \underline{P_r[X_{n+1} = k]} s^k$

$$P_r[X_{n+1} = k] = \sum_{j=0}^{\infty} P_r[X_{n+1} = k, X_n = j]$$

$$= \sum_{j=0}^{\infty} P_r[X_n = j] P_r[X_{n+1} = k / X_n = j]$$

$$= \sum_{j=0}^{\infty} P_r[X_n = j] P_r[Z_1 + Z_2 + \dots + Z_j = k]$$

$$P_{n+1}(s) = \sum_{k=0}^{\infty} P_r[X_{n+1} = k] s^k$$

$$= \sum_{k=0}^{\infty} \left[\sum_{j=0}^{\infty} P_r[X_n = j] P_r[Z_1 + Z_2 + \dots + Z_j = k] \right] s^k$$

$$P_{n+1}(s) = \sum_{j=0}^{\infty} P_r[X_n = j] \sum_{k=0}^{\infty} P_r\left[\sum_{r=1}^{j} Z_r = k\right] s^k$$

~~therefor~~ Since, $Z_1, Z_2, \dots Z_j$ are j iid r.vs and $P(s)$ is pgf of each Z_r therefore pgf of $\sum_{r=1}^{j} Z_r = [P(s)]^j$

therefore,

$$P_{n+1}(s) = \sum_{j=0}^{\infty} P_r[X_n = j] [P(s)]^j$$

$$P_{n+1}(s) = P_n[P(s)] \quad \text{——— (1)}$$

now, from the fact that $P_{n+1}(s) = P_n(P(s))$. One may prove that

$$P_{n+1}(s) = P[P_n(s)]$$

for the prove of the above eqn we know that

$$P_1(s) = \sum_{k=0}^{\infty} P_r[X_1 = k] s^k$$

$$= \sum_{k=0}^{\infty} P_r[Z_1 = k] s^k \quad \because X_1 = Z_1, \; X_0 = 1$$

$$= P(s) \quad \text{——— (2)}$$

~~put~~ substituting $P(s) = P_1(s)$ in eq (1), we get

$P_{n+1}(s) = P_n(P_1(s))$

now substituting $n = n-1$ & $s = P(s)$ in above

$P_n(P(s)) = P_{n-1}(P_1(P(s)))$

After substituting $n=1$ in eq^n 1, we get

$P_2(s) = P_1(P(s))$

therefore,

$P_n(P(s)) = P_{n-1}(P_2(s)) \Rightarrow P_{n+1}(s) = P_{n-1}(P_2(s))$

again putting $n = n-1$ & $s = P(s)$, we get

$P_n(P(s)) = P_{n-2}(P_2(P(s)))$

After putting $n=2$. in eq^n 1

$P_3(s) = P_2(P(s))$.

$\therefore$ $P_n(P(s)) = P_{n-2}(P_3(s))$

$P_{n+1}(s) = P_{n-2}(P_3(s))$

~~Hence~~, proceeding in the similar manner

in general

$P_{n+1}(s) = P_{n-k+1}(P_k(s)) \quad k = 1, 2, \dots n$

& in particular

$k=n$ $\quad P_{n+1}(s) = P_1[P_n(s)]$

$P_1(s) = P(s)$ from eq^n 2

$\therefore$ $P_{n+1}(s) = P[P_n(s)]$

and hence

Proof 27 $\boxed{P_{n+1}(s) = P_n(P(s)) = P(P_n(s))}$ Imp

Thus. explicit form for the pgf of X_n depends upon the proby distn P_k.

EX- Consider for eg the organism which can either die or split into two so that

$P_0 = 1-p$

$P_2 = p$

$P_k = 0$ for all other values of k

pgf of z ie

$$P(s) = (1-p)s^0 + 0\cdot s^1 + p\cdot s^2 + 0\cdot + 0 + \dots$$

$$= (1-p) + ps^2$$

pgf of X_1 which is denoted by

$$P_1(s) = P(s) = (1-p) + ps^2.$$

now, pgf of X_2 which is $P_2(s)$

$$P_2(s) = P_1(P(s)) = (1-p) + p[(1-p) + ps^2]^2$$

pgf of X_3.

$$P_3(s) = (1-p) + p[(1-p) + p\{(1-p) + ps^2\}^2]^2$$

this simple eg shows that although the probties $P[X_n = j]$ can be obtained from $P_n(s)$ actual computation is quite difficult when n becomes large.

2020 - CAN 2019

Mean of X_n – ie Average size of the nth generation –

Av. size of 1st gen. is X_1

$$E(X_1) = m \text{ (say)}$$

since $X_0 = 1$.

$$X_1 = Z_1$$

$$E(Z_1) = E(X_1) = m$$

and let $V(Z_1) = V(X_1) = \sigma^2$

let $P_n(s)$ be the pgf of X_n so

$$E(X_n) = [P_n'(s)]_{s=1}$$

since $P_n(s) = P_{n-1}(P(s))$

therefore $P_n'(s) = [P_{n-1}'(P(s))]\,P'(s)$ $\quad$ ($n = n-1$ & $s = P(s)$)

putting $s=1$

$$P_n'(1) = [P_{n-1}'(P(1))]\, P'(1)$$

$$P(1) = \sum_k p_k 1^k = 1$$

$$P_n'(1) = [P_{n-1}'(1)]\, P'(1)$$

$$P'(1) = E(X_1) = E(Z_1)_{z=m} = m$$

$$P_n'(1) = m\, P_{n-1}'(1)$$

By hybration, we get

$$P_n'(1) = m.m.\, P_{n-2}'(1)$$

$$= m^2 P_{n-2}'(1)$$

$$= m^2 . m\, P_{n-3}'(1)$$

$$= m^3 P_{n-3}'(1)$$

$$\vdots$$

$$= m^{n-1} P_1'(1)$$

$$= m^{n-1} P'(1) \qquad P_1(s) = P(s)$$

$$= m^{n-1} m$$

$$= m^n.$$

we can derive the above result in an alternative way also,

here $X_{n+1} = Z_1 + Z_2 + \cdots + Z_{X_n} = \sum_{r=1}^{X_n} Z_r$

is sum of random number of discrete iid rvs Z_r we know that If

$S_N = Y_1 + Y_2 + \cdots + Y_N$

where N itself is a r.v. and Y_i's $i = 1, 2 \ldots N$ are iid r.vs. then

$E[S_N] = E[Y_i] E(N)$

using the above result, we can write

$$E(X_{n+1}) = E(Z_r)\, E(X_N)$$

$= m\,E(X_n)$

$E(X_{n+1}) - m\,E(X_n) = 0$

this is a homogeneous diff. eqn of order 1 of type

$$U_{n+1} - mU_n = 0$$

hence auxiliary eqn is

$$p - m = 0$$

$$\Rightarrow p = m$$

hence the soln is

$$E(X_n) = U_n = c\,m^n \quad \text{--- (1)}$$

const. c is determine by using a initial condn $E(X_1) = m$.

substituting $n=1$ in this eqn (1)

$$E(X_1) = U_1 = cm$$

$$m = cm$$

$$\Rightarrow c = 1.$$

hence, the soln is

$$E(X_n) = m^n.$$

where m denotes the average no. of objects generated per object.

Variance of X_n -

we know that

$$S_N = Y_1 + Y_2 + \dots + Y_N$$

where N, itself is a r.v. and Y_i's are iid r.vs. Then

$$V(S_N) = E(N)\,V(Y_i) + V(N)\left[E(Y_i)\right]^2$$

here $X_{n+1} = \sum_{r=1}^{X_n} Z_r$ Z_r's are iid r.vs. and X_n is a r.v. therefore

$$V(X_{n+1}) = E(X_n)\,V(Z_r) + V(X_n)\,[E(Z_r)]^2$$
$$= m^n \sigma^2 + V(X_n)\, m^2$$

$$V(X_{n+1}) - m^2 V(X_n) = m^n \sigma^2$$

this is a non homogeneous diff. eq^n of order 1 of type

$$U_{n+1} - m^2 U_n = m^n \sigma^2$$

so auxiliary eq^n is

$$p - m^2 = 0 \qquad p = m^2.$$

so, complementary fn is

$$U_n = A(m^2)^n$$

$V(S_N) = E(N)\,V(Y_i) + [E(Y_i)]^2 V(N)$

particular soln is

$$P.S. = \frac{1}{(E - m^2)}\, m^n \sigma^2$$

$$= \frac{m^n \sigma^2}{m - m^2}, \quad m \neq 1$$

so the soln is

$$U_n = CF + PS$$

$$V(X_n) = A(m^2)^n + \frac{m^n \sigma^2}{m - m^2}, \quad m \neq 1 \qquad (1)$$

the constant can be determine by

$$V(X_1) = \sigma^2$$

putting $n = 1$. in (1)

$$V(X_1) = A m^2 + \frac{m \sigma^2}{m - m^2}$$

$$A = \frac{1}{m^2}\left[\sigma^2 - \frac{m\sigma^2}{m(1-m)}\right]$$

$$= \frac{\sigma^2}{m(m-1)}$$

$V(X_n)$ will be

$$V(X_n) = \frac{\sigma^2}{m(m-1)} m^{2n} - \frac{m^n \sigma^2}{m(m-1)} \qquad m \neq 1$$

$$= \frac{\sigma^2 m^n}{m(m-1)} [m^n - 1)$$

$$= \frac{\sigma^2 m^{n-1}}{m-1} (m^n - 1), \quad \text{for } m \neq 1$$

for $m = 1$, we can get the result by taking the limit of above expression when $m \to 1$. Limit may be taken by differentiating numerator & denominator wrt m ie by L' Hospital's rule.

$$\lim_{m \to 1} \frac{(n-1) m^{n-2} \sigma^2 (m^n - 1) + m^{n-1} \sigma^2 n \cdot m^{n-1}}{1}$$

$$= 0 + \sigma^2 n$$

$$V(X_n) = n\sigma^2 \quad \text{if } m = 1.$$

Extinction of the Process.

Def^n - By extinction of process it is meant that the random sequence X_n consists of zeroes for all except a finite no. of values of n. In other words extinction occurs when probty that X_n takes value zero for some value of n. $P[X_n = 0] = 1$. for some n

Clearly, $X_n = 0$ for $n \geq m$ (say)

$X_n = 0$ for $n \geq m$

$P[X_{n+1}=0 \mid X_n=0] = 1.$

Theorem –

Statement If $E(Z_r) = m \le 1$. The probty of ultimate extinction is 1. and if

(ii) $E(Z_r) = m > 1$ the probty of ultimate extinction is the smallest +ve root less than unity of the eqn $s = P(s)$.

Proof let $q_n = P[X_n = 0]$

ie q_n is the probty that extinction occurs at and before the ~~ext~~ nth generation.

let $P_n(s)$ be the pgf of X_n so

$$P_n(s) = P[X_n=0] + P[X_n=1]\cdot s^1 + P[X_n=2]\cdot s^2 + \dots$$

$P_n(s) = ?$ at $s=0$

$$P_n(0) = P[X_n=0] + 0 + 0 + \dots$$

$$P_n(0) = q_n \quad \text{—— (1)}$$

therefore,

$$q_1 = P_1(0) \qquad \because P_1(s) = P(s)$$

$$q_1 = P(0) \quad \text{—— (2)} \quad \text{where } P(s) \text{ is the pgf } Z_r$$

therefore $q_1 = [P(s)]_{s=0}$

$$= [p_0 + p_1 s + p_2 s^2 + \dots]_{s=0}$$

$$q_1 = p_0 \quad \text{—— (3)}$$

Also, we know that

$$P_n(s) = P(P_{n-1}(s))$$

$$P_n(0) = P(P_{n-1}(0)) \qquad \text{Acc. to eq (1). we get}$$

$$q_n = P(q_{n-1}) \quad \text{—— (4)}$$

Case I If $p_0 = 0$

then $q_1 = 0$ (5) from eqn (3).

further $q_2 = P(q_1)$ from eq (4)

$$q_2 = P(0) \quad \text{from } (5)$$
$$= q_1 \quad \text{from } (2)$$
$$= 0 \quad \text{from } (5).$$

Similarly $q_3 = 0$

$$q_n = 0 \quad \forall n$$

ie probty of Extinction is zero for all n
hence, no extinction is possible when
$p_0 = 0$.

<u>Case II</u> - If $p_0 = 1$.
then $q_1 = 1$ from eq (3)
further $q_2 = P(q_1)$
$$= P(1)$$
$$= [P(s)]_{s=1}$$
$$= [p_0 + p_1 s + p_2 s^2 + \dots]_{s=1}$$
$$= \sum_{k=0}^{\infty} p_k = 1$$
$$q_2 = 1 \quad \text{———— (6)}.$$

in general $q_n = 1 \quad \forall n$

2) Extinction will occur right after the growth generation with probty 1.

Thus, we conclude that if $p_0 = 1$ population will never start, and if $p_0 = 0$

it will never become extinct thus we confined our sets to the next case when $0 < p_0 < 1$ ie.

case (ii) $0 < p_0 < 1$

$$P(s) = p_0 + p_1 s + p_2 s^2 + \dots$$

since p_k's are non-negative numbers.

therefore for $0 < p_0 < 1$, $P(s)$ is a strictly monotone increasing fn of s. Now

$$q_2 = P(q_1) \text{ from eq}^n \text{ (4)}$$

$$= P(p_0) \text{ from (3)}$$

$$q_2 > P(0) \text{ becoz } p_0 > 0$$

but $P(0) = q_1$ acc. to eq (2)

$$\therefore q_2 > q_1$$

similarily. it can be shown that

$$q_3 > q_2$$

therefore, we have.

$q_1 < q_2 < q_3 < \dots$ $\{q_n\}$ is monotonically

ie the sequence increasing and is bounded above by 1.

as q_n's are probties less than 1.

If the sequence $\{q_n\}$ must have a limit point. let the limiting value of q_n

$$\lim_{n \to \infty} q_n = q \text{ (say)} \quad 0 < q < 1$$

$$q_n = P(q_{n-1})$$

when $n \to \infty$

$$q = P(q) \quad \text{—— (7)}$$

this implies q is a root of eq^n

$s = P(s)$ in fact

q is the smallest +ve root less than unity of a eq^n $s = P(s)$

to prove this, let us suppose that

s_0 is an arbitrary +ve root of the eq^n $s = P(s)$,

We know that

$$q_1 = P(0) \text{ from eq}^n \text{ (5)}$$

but $P(0) < P(s_0)$ as $0 < s_0$

therefore $q_1 < P(s_0)$

$q_1 < s_0$ bcoz s_0 is the root of $P(s_0)$

now assuming

$$q_m < s_0 ,$$

we get $q_{m+1} = P(q_0)$

$< P(s_0)$ from eq (4)

$$q_{m+1} < s_0$$

so by ~~iduct~~ induction $q_n < s_0 \ \forall n$

therefore $\lim_{n \to \infty} q_n < s_0$

$q < s_0$ from eq (7)

Thus q is the smallest +ve root of a eq^n $s = P(s)$ // for further investigation, we consider the graph of $y = P(s)$ $0 \le s \le 1$

The following features of the curve may be noted

1) It starts with $(0, p_0)$ bcoz

$$P(s) = p_0 + p_1 s + \dots$$

when $s = 0$ $y = P(0)$

2) It ends with the point $(1, 1)$ bcoz

when ~~$s=0$~~ $s = 1$

$$y = P(1)$$

$$= \sum_k p_k = 1$$

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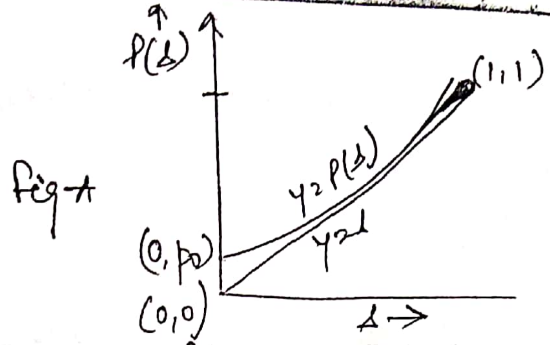

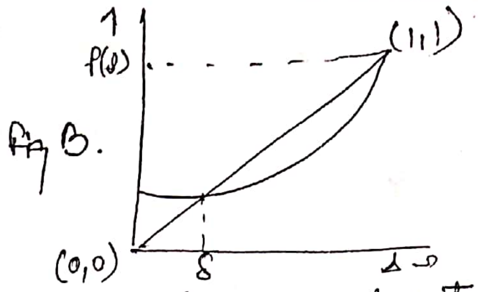

3) The curve lies entirely in the 1st quadrant

4) The curve is convex bcoz $P(s)$ is an increasing fn of s. So the curve $y = P(s)$ and the bisector line $y = s$, can intersect in at most 2 pts one of which is the end pt $(1,1)$

This implies the eqn $s = P(s)$ has two roots one of which is unity the two cases are considered with the help of figures A and B.

Now, case (A), the curve $y = P(s)$ lies entirely above the line $y = s$ in which case point $(1,1)$ is the only point of intersection this implies unity is the unique the root of the eqn $s = P(s)$, ie $q = 1$

Thus, the proby of ultimate extinction is 1, also $P(1) - P(s) = 1 - P(s)$

Since, in this case $y = P(s)$ lies entirely above

therefore, $P(s) \geq s$

$\therefore$ $1 - P(s) \leq 1 - s$

$P(1) - P(s) \leq 1 - s$

$\therefore$ $\dfrac{P(1) - P(s)}{1 - s} \leq 1$

$\therefore$ $\lim\limits_{s \to 1} \dfrac{P(1) - P(s)}{1 - s} \leq 1$

$$P'(1) \leq 1$$

but $P'(1) = E(Z_r) = m$ given

$\therefore$ $m \leq 1$

So for the case $m \leq 1$
the q is 1. the probty of ultimate extinction

Now case ⓑ.
the curve $y = P(s)$ intersects the line $y = s$
at another pt also say at $(\delta, P(\delta))$ such that
$\delta = P(\delta)$ and $\delta < 1$
therefore. two roots of the eqn
$s = P(s)$ one of which is
unity and the other is δ.
the curve $y = P(s)$ being convex lies below
the line $y = s$ in the range δ to 1 and
above the line $y = s$ in the range 0 to δ
$\Rightarrow$ $P(s) < s$ for $\delta < s < 1$
and $P(s) > s$ for $0 < s < \delta$

Now
$q_1 = P(0)$ acc. to eqn ②
since $0 < \delta$
and $P(s)$ is increasing fn of s
$\Rightarrow$ $q_1 < P(\delta)$
$P(\delta) = \delta$
$\Rightarrow$ $q_1 < \delta$

$q_n < \delta$

$q_{n+1} = P(q_n)$ acc. to eqn 4

$q_{n+1} < P(\delta$ $\quad q_n < \delta$

$q_{n+1} < \delta$

$q_n < \delta \quad \forall n$

$$\lim_{n\to\infty} q_n = q$$

$s = P(s)$ and δ is also a root of this eq^n which is the only +ve root other than unity of this eq^n

Hence, $q = \delta < 1$

by mean value theorem considered in the interval $[\delta, 1]$, there is a value z say $(\delta, 1)$ s.t.

$$P'(z) = \frac{P(1) - P(\delta)}{1-\delta} \text{ and}$$

$$= \frac{1-\delta}{1-\delta}$$

$$P'(z) = 1.$$

further, since, the derivative of $P(s)$ ie $P'(s)$ is monotone for therefore for $z < 1$, we have

$$P'(z) < P'(1)$$

$$1 < P'(1)$$

$\therefore$ $P'(1) > 1$

$\Rightarrow$ $E(Z_n) > 1$

$m > 1$

$\Rightarrow$ This shows that for $m > 1$ the probty of ultimate extinction is δ which is smallest +ve root less than unity of the eq^n

$$s = P(s)$$

~~$m \leq 1$~~

~~if $m > 1$, $s = P(s)$~~.

Ques (1) find the probty of extinction if $p_0 = \frac{1}{2}$, $p_1 = \frac{1}{4}$, and $p_2 = \frac{1}{4}$

$$E(z_r) = 0\cdot\frac{1}{2} + \frac{1}{4} + \frac{2}{4}$$

$$= \frac{3}{4} < 1$$

∴ Probty of ultimate extinction is 1.

(2) $p_0 = \frac{1}{4}$, $p_1 = \frac{1}{4}$ and $p_2 = \frac{1}{2}$

$$E(z_r) = 0\cdot\frac{1}{4} + \frac{1}{4} + \frac{2}{2}$$

$$= \frac{5}{4} > 1.$$

⇒ ~~pro~~

$$P(s) = \frac{1}{4}\cdot s^0 + \frac{1}{4}\cdot s + \frac{1}{2}\cdot s^2$$

$$= \frac{1}{4} + \frac{s}{4} + \frac{s^2}{2}$$

solve for $s = P(s)$.

$$s = \frac{1}{2} \text{ \& } 1.$$

$$= \frac{1}{4} + \frac{1}{8}.$$

first putting $s = \frac{1}{2}$

$P(s) =$

putting $s = 1$

$P(s) =$

⇒ Ans $\frac{1}{2}$

$$s = \frac{1}{4} + \frac{s}{4} + \frac{s^2}{2}$$

$$2s = \frac{1}{2} + \frac{s}{2} + s^2$$

$$4s = 1 + s + 2s^2$$

$$2s^2 - 3s + 1 = 0$$

$$2s^2 - 2s - s + 1 = 0$$

$$2s(s-1) - 1(s-1)$$

$$(s-1)(2s-1)$$

(3) If the dist^n of offspring is Binomial with par n, p find probty of extinction for mean $np = 1$

(1) $n = 2$, $p = 1/2$

(2) $n = 1$, $p = 1/2$

(3) $n = 2$, $p = 0.6$.

(1) probty of ulti. ext is 1.

(2) $\frac{1}{2} < 1$. probty of U.E. = 1.

(3) Mean = 1.2.

$$s = P(s)$$

$P(1) = q + p$

$n = 2$.

$(0.4 + 0.6)^2 1^2 = 1$ and 0.44

probty of u.e. is $\underline{\underline{0.44}}$.