Contents

1 Markov Chains

Many times we are concerned with independent trials which can be described as follow. A set of possible outcomes $E_1, E_2, \dots$ (finite or infinite in number) is given and with each there is associated a probability $p_k, k = 1, 2, 3. \dots$. The possibilities of sample sequences are defined by the multiplicative property

$$PE_{j0}, E_{j1}, \dots, E_{jn} = p_{j0}, p_{j1}, \dots, p_{jn}$$

In the theory of Markov Chains we consider the simplest generalization which consists in permitting the outcome of any trial to depend on the outcome of the directly preceding trial (and only on it). The outcome E_k is no longer associated with a fixed probability p_k but ot every pair (E_j, E_k) there corresponds a conditional probability p_{jk}, given that E_j has occurred at some trial, the probability of E_k at the next trial is p_{jk}. In addition to p_{jk} we must be given the probability a_k of the outcome of E_k at the initial trial, for p_{jk} to have the meaning of attributed to them. The probabilities of sample sequences corresponds to two, three or four trials must be defined by

$$\begin{aligned} P\{E_j, E_k\} &= a_j p_{jk} \\ P\{E_j, E_k, E_r\} &= a_j p_{jk} p_{kr} \\ P\{E_j, E_k, E_r, E_s\} &= a_j p_{jk} p_{kr} p_{rs} \end{aligned}$$

and generally

$$P\{E_{j_0}, E_{j_1}, \cdots, E_{j_n}\} = a_{j_0} p_{j_0 j_1} p_{j_1 j_2} \cdots p_{j_{n-2} j_{n-1}} p_{j_{n-1} j_n} \tag{1}$$

Here the initial trial is numbered as zero so that trial number one is the second trial.(This will be our notation.) If a_k is the probability of E_k at the initial trial, we must have

$$a_k \geq 0 \qquad \text{and} \qquad \sum a_k$$

More over E_j occurs it must be followed by some E_k and it is therefore necessary that for all j and k

$$p_{j1} + p_{j2} + p_{j3} + \cdots = 1, \qquad p_{jk} \geq 0 \tag{2}$$

We now show that for any number a_k and p_{jk} satisfying these conditions, the assignment 1 is a permissible definition of probabilities in the sample space corresponding to $(n+1)$ trials. The numbers defined in 1 are obviously non negative, we need only to prove that they add to unity. For fixed $j_0, j_1, \cdots, j_{n-1}$ and add the numbers 1 for all possible j_n using 2 with $j = j_{n-1}$ we see immediately that sum equals to $a_{j_0} p_{j_0 j_1} p_{j_1 j_2} \cdots p_{j_{n-2} j_{n-1}} p_{j_{n-1} j_n}$. thus for sum over all numbers 1 does not depend on n and since $\sum a_{j_0} = 1$, the sum equal unity for all n. The definition 1 depends formally on the number of trials but our assignment proves the mutual consistency of the definition 1 for all n. For example, to obtain the probability of event " the first two trial result in (E_j, E_k) " we have to fix $j_0 = j \text{and} j_1 = k$ and add the probabilities 1 for all possible $j_2, j_3, \cdots, j_n$. We have just shown that the sum add up to $a_j p_{jk}$ and thus is independent of n. This means that it is usually not necessary explicitly to refer to the number of trials, the event $(E_{j_0}, E_{j_1}, \cdots, E_{j_n})$ has the same probability in all the sample space of more then r trials.

Definition 1. *A sequence of trials with possible outcomes $E_1, E_2, \cdots$ is called a Markov Chain if the probabilities of sample sequences are defined by 1 in terms of a probability distribution distribution $\{a_k\}$ for E_k at the initial trial and fixed conditional probabilities p_{jk} of E_k given that E_j has occurred at the preceding trial.*

A slightly modified terminology is used for application of Markov Chains. The possible outcomes E_k are usually referred to as the possible ' states ' of the system. Instead of saying that the n^{th} step leads to E_k.

Finally p_{jk} is called the probability of a ' transition ' from E_j to E_k. We assume that the trials are performed at a uniform rate so that the number of the step serves as time parameter. The transition probabilities p_{jk} will be arranged in a matrix of transition probabilities.

$$\mathbf{P} = \begin{pmatrix} p_{11} & p_{12} & p_{13} & \cdots & \cdots \\ p_{21} & p_{22} & p_{23} & \cdots & \cdots \\ p_{31} & p_{32} & p_{33} & \cdots & \cdots \\ \multicolumn{5}{c}{\dotfill} \\ \multicolumn{5}{c}{\dotfill} \end{pmatrix}$$

where the first subscript stands for column. Clearly $\mathbf{P}$ is a square matrix with non-negative elements and unit row sums. Such a matrix is called a Stochastic Matrix. Any stochastic matrix can serve as a matrix of transition probabilities, together with our initial distribution $\{a_k\}$ it completely defines a Markov Chain with states $E_1, E_2, \cdots$. In many cases it is convenient to number the states starting with ' 0 ' rather than ' 1 '. A zero row and zero column is then to be added to p.

Example 1. *When there are only two possible states E_1 E_2, the matrix of transition probabilities is necessary of the form*

$$P = \begin{pmatrix} 1-p & p \\ \alpha & 1-\alpha \end{pmatrix}$$

Such a chain would be realised by the following conceptual experiment. A particle moves along the x axis in such a way that its speed remains constant but the direction of the motion can be reversed. The system is said to be in state E_1 if the particle moves in the right direction and in state E_2 if the motion is to left direction. The p is the probability of reversal when the particle moves to the right α the probability of a reversal when it moves to the left.

Example 2. ***Random Walk with absorbing barriers***
Let the possible state be $E_0, E_1, \dots E_s$ and consider the matrix of transition probabilities.

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ q & 0 & p & 0 & \cdots & 0 & 0 & 0 \\ 1 & q & 0 & p & \cdots & 0 & 0 & 0 \\ \multicolumn{8}{c}{\dots\dots\dots\dots\dots\dots\dots\dots} \\ 1 & 0 & 0 & 0 & \cdots & q & 0 & p \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}$$

Here from each of the ' interior ' states $E_0, E_1, \dots E_{s-1}$ transition are possible to right and left neighbors with

$$p_{i,i+1} = p \qquad \textit{and} \qquad p_{i,i-1} = q$$

However no transition is possible from either E_0 or E_s to any other state. The system may move from one state to another but once E_0 or E_s is reached, the system stays there fixed for ever.

Example 3. ***Random Walk with Reflecting barriers:***
An interesting variant of the proceeding example is represented by the chain

with possisble states $E_0, E_1, \ldots E_s$ and the transition probabilities

$$\mathbf{P} = \begin{pmatrix} q & p & 0 & 0 & \cdots & 0 & 0 & 0 \\ q & 0 & p & 0 & \cdots & 0 & 0 & 0 \\ 1 & q & 0 & p & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 & 0 & \cdots & q & 0 & p \\ 1 & 0 & 0 & 0 & \cdots & 0 & q & p \end{pmatrix}$$

This chain may be inter presented in gambling language by considering two players playing for unit strokes with the agreement that every time a player loses his lost dollar, his adversary returns it so that the game can continue forever.

2 Higher Transition Probabilities

We shall denote by $p_{jk}^{(n)}$ the probably of a transition from E_j to E_k in exactly n steps. In other words $p_{jk}^{(n)}$ is the conditional probability of entering E_k at the n_{th} step given the initial step E_j. This is the sum of the probabilities of all possible paths $E_j, E_{j_1}, \cdots, E_{j_{n-1}} E_k$ of length n starting at E_j and ending at E_k. In particular

$$p_{jk}^{(1)} = p_{jk}$$

and

$$p_{jk}^{(2)} = \sum_{\nu} p_{j\nu} p_{\nu k} \tag{3}$$

By induction we get the general recurrence formula

$$p_{jk}^{(n+1)} = \sum_{\nu} p_{j\nu} p_{\nu k}^{(n)} \tag{4}$$

A further indication on m leads to the basic identity.

$$p_{jk}^{(m+n)} = \sum_{\nu} p_{j\nu}^{(m)} p_{\nu k}^{(n)} \tag{5}$$

It reflect the simple fact that the first m steps lead from E_j to some intermediate state E_ν and that the probability of a subsequent passage from E_ν to E_k does not depend on the manner in which E_ν was reached. in th same way as the p_{jk} form the matrix P, we arrange the $p_{jk}^{(n)}$ in a matrix to be denoted by P^n. Then the relation 4 states that to obtain the element $p_{jk}^{(n+1)}$ of P^{n+1}, we have to multiply the j^{th} pf P by corresponding elements of the k^{th} column of P^n and add all the products. The operation is called row into column multiplication of the matrices P and P^n and is expressed symbolically by the equation

$$P^{n+1} = P.P^n$$

This suggests calling P^n the n^{th} power of P.Equation 5 expressed the familiar law

$$P^{m+n} = P^m.P^n$$

In order to have 5 true for all $n \geq 0$, we define $p_{jk}^{(0)}$ by $p_{jj}^{(0)} = 1$ and $p_{jk}^{(0)} = 0$ for $j \neq k$ as in natural.

Absolute Probabilities

Let a_j stands for the probability of the state E_j at the initial trial. The unconditional probability of entering E_k at the n^{th} step is then

$$a_k^{(n)} = \sum_j a_j p_{jk}^{(n)} \tag{6}$$

usually we let the process start from a fixed state E_i, i.e., we put $a_i = 1$. In this caes

$$a_k^{(n)} = p_{ij}^{(n)}$$

In many situation it is observed that influence of the initial state gradually wear off so that for large n, the distribution in 6 become nearly independent of the initial distribution $\{a_j\}$. This is the case if $p_{jk}^{(n)}$ converges to a limit independent of j that is if P^n converges to a matrix with identical rows. It should be noted that in the case of independent trials, all the rows of P are identical with a given probability distribution and this implies that

$$P^n = P \text{for all } n$$

.

Closures and Closed Sets

We say that E_k can be reached from E_j if there exists some $n \geq 0$ such that $p_{jk}^{(n)} > 0$ (i.e. if there is a positive probability of reaching E_k from E_j including the case $E_k > E_j$). For exam[ple, in an unrestricted random walk each state can be reached from every other state, but from an absorbing barrier no other state can be reached.

Definition 2. *A set C of states is closed if no state outside C can be reached from any state E_j in C. For an arbitrary set C of states, the smallest closed set containing C is called the closure of C.*

A single state E_k forming a closed set will be called absorbing. A Markov Chain is irreducible if there exists no closed set other than the set of all states. Clearly C is closed if and only if $p_{jk} = 0$ whenever j is in C and k is outside C, because in that case we see that $p_{jk}^{(n)} = 0$ for every n.

Theorem 1. *If in the matrix P^n all columns corresponding to states outside the closed set C are deleted, there remain stochastic matrices for which the fundamental relations 4 and 5 again hold.*

This means that we have a Markov Chain defined on C and this sub chain can be studied independently of all other states. The state E_k is absorbing iff $p_{kk} = 1$. In this case the matrix of the last theorem reduces to a single element. In general it is clear that the totality of all states E_k that can be reached from a given state E_j forms a closed set. An irreducible chain contains the proper closed subsets and so we have the simple but useful criterion.

Criterion : A chain is reducible if and only if every state can be reached from every other state.

Example 4. *In order to find all closed sets it suffices to know which p_{jk} vanish and which are positive. Accordingly we use a $*$ to denote positive* *elements and consider a typical matrix say*

$$\mathbf{P} = \begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \\ E_6 \\ E_7 \\ E_8 \\ E_9 \end{array} \begin{array}{c} \begin{array}{ccccccccc} E_1 & E_2 & E_3 & E_4 & E_5 & E_6 & E_7 & E_8 & E_9 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & * \\ 0 & * & * & 0 & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & * & * & 0 & 0 \\ 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & * \end{pmatrix} \end{array}$$

we number the states as $E_1, E_2, \cdots, E_9$. In the fifth row a $*$ appears only at the fifth place and therefore $p_{55} = 1$. Consequently E_5 is absorbing. The third and the eighth row contain only one positive element each, and it is clear that E_3 and E_8 form a closed set. From E_2 direct transitions are possible to itself and to E_3, E_5 and E_8. The pair (E_3, E_5) forms a closed set while E_5 is absorbing. Accordingly the closure of E_2 consists of the set (E_2, E_3, E_5, E_8). Similarly the closure of E_6 is $\{E_6, E_2, E_3, E_5, E_8\}$ and of the state E_7 is $\{E_2, E_3, E_5, E_6, E_7, E_8\}$.

Consider a chain with states $E_1, E_2, \cdots, E_s$ form a closed set $(r < s)$. The $r \times r$ sub matrix of P appearing in the left upper corner is the stochastic and we can exhibit P in the form of a partitioned matrix

$$\mathbf{P} = \begin{pmatrix} Q & 0 \\ U & V \end{pmatrix}$$

The matrix in the upper right corner has r rows and $s - r$ columns and only zero entries. Similarly U stands for a matrix with $s - r$ rows and r column while V is a sequence matrix. Similarly P^n can be written as

$$\mathbf{P} = \begin{pmatrix} Q^n & 0 \\ U^n & V^n \end{pmatrix}$$

2.1 Classification of States

The states of a Markov Chain can often be classified in a distinctive manner according to some fundamental properties of the system. By means of such classification it is possible to identify contain types of chains.

In this context, we have already discussed about closed sets and irreducible Markov Chains. We now proceed to obtain a classification of the states of a Markov Chain.

Many times the states $E_0, E_1, E_2, \cdots$ and thus to say that the system is in the state E_j, we say that the system is in the state j.

it reaches the state k for the first time at the n^{th} step and $p_{jk}^{(n)}$ is the probability that it reaches the state k at the n^{th} step (not necessarily for for the first time). A relation can be established between $f_{jk}^{(n)}$ and $p_{jk}^{(n)}$ as follows:

The probability that starting with j the state k is reached for the first time at the r^{th} step and again after that at the $(n-r)^{th}$ step is given by

$$f_{jk}^{(n)}.p_{jk}^{(n)} \text{for all} r \leq n$$

These cases are mutually exclusive. Hence

$$p_{jk}^{(n)} = \sum_{r=0}^{n} f_{jk}^{(r)} p_{kk}^{(n-r)}, n \geq 1 \tag{7}$$

with $p_{kk}^{0} = 1, f_{jk}^{0} = 0, f_{jk}^{1} = p_{jk}$ equation 7 can also be written as

$$p_{jk}^{(n)} = \sum_{r=0}^{n-1} f_{jk}^{(r)} p_{kk}^{(n-r)} + f_{jk}^{(n)}, n > 1 \tag{8}$$

Let f_{jk} denotes the probability that starting with the state j, the system will ever reach the state k. Clearly

$$f_{jk} = \sum_{n=1}^{\infty} f_{jk}^{(n)}$$

The mean first passage from the state j to the state k is given by

$$\mu_{jk} = \sum_{n=1}^{\infty} n.f_{jk}^{(n)}$$

We have to consider two cases $f_{jk} = 1$ and $f_{jk} < 1$. When $f_{jk} = 1$, it is certain that the system with the state j will reach the state k. In this case $f_{jk}^{(n)}, n = 1, 2, 3, \cdots$ is a proper probability distribution and this gives the first passage distribution for k (the system starting with j). In particular when

$k = j$, $f_{jj}^{(n)}, n = 1, 2, 3, \cdots$ will represent the distribution of the **recurrence time** of j and $j_{jj} = 1$ will imply that the return to state j is certain. In this case $\mu_{jj} = \sum_{n=1}^{\infty} n.f_{jj}^{(n)}$ is known as the **mean recurrence time** for the state j. Thus two questions concerning state j arise: first, whether the return to the state j is certain and secondly when this happens, whether the mean recurrence time μ_{jj} is finite.

Definition 3. *The state j is said to be **persistent** if $f_{jj} = 1$(i.e. the return to j is certain) and **transient** if $f_{jj} < 1$ (i.e. return to state j is uncertain).*

Definition 4. *A persistent state j is said to be **null** if $\mu_{jj} = \infty$.(i.e. the mean recurrence time is infinite and is said to be non-null of $\mu_{jj} < \infty$.)*

Definition 5. *A state j is said to be **periodic** with period $t(> 1)$ if return to state is possible only at $t, 2t, 3t, \cdots$ steps where t is the greatest integer with this property. In this case $p_{jj}^{(n)} = 0$ unless n is an integral multiple of t.*

Definition 6. *The state j is said to be **aperiodic**(or non-periodic) if number of such $t(> 1)$ exists.*

Definition 7. *A persistent non-null and aperiodic state (of a Markov Chain) is said to **ergodic**. A Markov Chain all of whose states are ergodic is said to be an ergodic chain.*

Theorem 2. *The state j is persistent or transient according as*

$$\sum_{n=0}^{\infty} p_{jj}^{(n)} = \infty \qquad or \qquad < \infty$$

.

Proof. We have the relation

$$p_{ij}^{(n)} = \sum_{m=i}^{n} f_{ij}^{(m)} p_{kk}^{(n-m)}$$

Hence

$$\begin{aligned}
\sum_{n=1}^{N} p_{ij}^{(n)} &= \sum_{n=1}^{N} \sum_{m=1}^{n} f_{ij}^{(m)} p_{jj}^{(n-m)} \\
&= f_{ij}^{(1)} p_{jj}^{(0)} + f_{ij}^{(1)} p_{jj}^{(1)} + f_{ij}^{(2)} p_{jj}^{(0)} \\
&+ f_{ij}^{(1)} p_{jj}^{(2)} + f_{ij}^{(2)} p_{jj}^{(1)} + f_{ij}^{(3)} p_{jj}^{(0)} \\
&+ \cdots + f_{ij}^{(1)} p_{jj}^{(N-1)} + f_{ij}^{(2)} p_{jj}^{(N-2)} + f_{ij}^{(N)} p_{jj}^{(0)}
\end{aligned}$$

Now let us consider

$$\begin{aligned}\sum_{m=1}^{N} f_{ij}^{(m)} \sum_{n=0}^{N-m} p_{jj}^{(n)} &= f_{ij}^{(1)}\{p_{jj}^{(0)} + p_{jj}^{(1)} + \cdots + p_{jj}^{(N-1)}\} \\ &+ f_{ij}^{(2)}\{p_{jj}^{(0)} + p_{jj}^{(1)} + \cdots + p_{jj}^{(N-2)}\} \\ &+ f_{ij}^{(N-1)}\{p_{jj}^{(0)} + p_{jj}^{(0)} + \cdots + p_{jj}^{(N-1)}\} \\ &+ f_{ij}^{(N)}\end{aligned}$$

Thus we have

$$\begin{aligned}\sum_{n=1}^{N} p_{ij}^{(n)} &= \sum_{n=1}^{N}\sum_{m=1}^{n} f_{ij}^{(m)} p_{jj}^{(n-m)} \\ &= \sum_{m=1}^{N} f_{ij}^{(m)} \sum_{n=0}^{N-m} p_{jj}^{(n-m)} \\ &\leq \sum_{m=1}^{N} f_{ij}^{(m)} \sum_{n=0}^{N} p_{jj}^{(n)}\end{aligned}$$

Also if $N > N'$, we have

$$\sum_{n=1}^{N} p_{ij}^{(n)} \geq \sum_{m=1}^{N'} f_{ij}^{(m)} \sum_{n=0}^{N-N'} p_{jj}^{(n)}$$

Hence

$$\sum_{m=1}^{N'} f_{ij}^{(m)} \sum_{n=0}^{N-N'} p_{jj}^{(n)} \leq \sum_{n=1}^{N} p_{ij}^{(n)} \leq \sum_{m=1}^{N} f_{ij}^{(m)} \sum_{n=0}^{N} p_{jj}^{(n)}$$

In this first let $N \to \infty$ and then $N' \to \infty$, we then obtain

$$
\begin{aligned}
f_{ij} \sum_{n=0}^{\infty} p_{jj}^{(n)} \leq \sum_{n=1}^{\infty} p_{ij}^{(n)} \leq f_{ij} \sum_{n=0}^{\infty} p_{jj}^{(n)} \\
f_{ij} \sum_{n=0}^{\infty} p_{jj}^{(n)} = \sum_{n=1}^{\infty} p_{ij}^{(n)} \\
f_{ij} &= \frac{\sum_{n=1}^{\infty} p_{ij}^{(n)}}{\sum_{n=0}^{\infty} p_{jj}^{(n)}} \text{in particular when } i = j \\
f_{ii} &= \frac{\sum_{n=1}^{\infty} p_{ii}^{(n)}}{1 + \sum_{n=1}^{\infty} p_{ii}^{(n)}} \\
1 - f_{ii} &= \frac{1}{1 + \sum_{n=1}^{\infty} p_{ii}^{(n)}}
\end{aligned}
$$

This implies that if $f_{ii} < 1$ i.e. i is transient. $\sum_{n=1}^{\infty} p_{ii}^{(n)}$ should be finite or $\sum_{n=0}^{\infty} p_{ii}^{(n)}$ should be finite. If $f_{ii} = 1$, this implies that $\sum_{n=0}^{\infty} p_{ii}^{(n)}$ should be infinite. □

Theorem 3. *With probability one, the system returns to a persistent state infinitely often, while it returns to a transient state only a finite number of times.*

Proof. Let $g_{ij}(m)$ be the probability that the system enters the state j at least m times from the initial state i, then we have

$$\begin{aligned} g_{ij} &= \lim_{m\to\infty} g_{ij}(m) \\ &= Pr\{\text{the system enters } j \text{ infinitely often from } i\} \end{aligned}$$

Now we have the relation

$$\begin{aligned} g_{ij}(1) &= f_{ij} \\ \text{also} \qquad g_{ij}(m+1) &= \sum_{n=1}^{\infty} f_{ij}^{(n)}.g_{jj}(m) \\ &= f_{ij}.g_{jj}(m) \\ \text{thus} \qquad g_{ij}(2) &= f_{ij}.g_{jj}(1) \\ &= f_{ij}.f_{jj} = (f_{jj})^2, \text{if } i=j \\ g_{ij}(3) &= f_{ij}.g_{jj}(2) = f_{ij}(f_{jj})^2 \end{aligned}$$

thus by induction we get the relation

$$g_{ij}(m) = f_{ij}.(f_{jj})^{m-1} \qquad m \geq 1$$

In particular when $i = j$

$$g_{ij}(m) = (f_{jj})^m$$

Hence, as $m \to \infty$, we obtain

$$g_{ij} = \begin{cases} 1 & \text{if } j \text{ is persistent} \\ 0 & \text{if } j \text{ is transient} \end{cases}$$

□

Corollary 3.1. *If j is transient*

$$\sum_{n=0}^{\infty} p_{jj}^{(n)} = (1 - f_{jj})^{-1}$$

and

$$\sum_{n=1}^{\infty} p_{ij}^{(n)} = f_{ij}(1 - f_{jj})^{-1} < \infty$$

Notation : We say that the state j can be reached from the state i, and write $i \to j$ if $p_{ij}^{(n)} > 0$ for some $n \geq 1$. If $i \to j$ and $j \to i$, we write $i \leftrightarrow j$.

Theorem 4. *If i is a persistent state and $i \leftrightarrow j$ then $f_{ji} = 1$.*

Proof. Let $g_{ij}(m)$ = probability that the system returns to state i at least m times and

$$\begin{aligned} g_{ii} &= \lim_{m\to\infty} g_{ii}(m) \\ &= \text{Prob. that the system returns to } i \text{ infinitely many times} \end{aligned}$$

Now for any state $k \in I$ (set of all states) and for any $n > 0$

$$g_{ii} = \sum_{k\in I} p_{ik}^{(n)}.g_{ki} \tag{1}$$

$$1 - g_{ii} = \sum_{k\in I} p_{ik}^{(n)}.(1 - g_{ki}) \qquad \text{for any } n > 0 \tag{2}$$

Since $\sum_{k\in I} p_{ik}^{(n)} = 1$ for any $n > 0$
Now since i is a persistent state, hence $1 - g_{ii} = 0$ Thus

$$\sum_{k\in I} p_{ik}^{(n)}(1 - g_{ki}) = 0 \qquad \text{for all } n > 0$$

$$p_{ik}^{(n)}(1 - g_{ki}) = 0 \qquad \text{for all } k \text{ and } n \tag{3}$$

(Since $p_{ik}^{(n)}(1 - g_{ki} \geq 0)$for all) We know that $i \to j$, Thus there exist an integer N such that $p_{ij}^{(N)} > 0$
In 3, substituting $k = j$ and $n = N$

$$p_{ij}^{(N)}(1 - g_{ji}) = 0$$

since $p_{ij}^{(N)} > 0 \qquad \Rightarrow (1 - g_{ji}) = 0$

$$g_{ji} = 1$$

Now we also know

$$g_{ji} = f_{ji}.g_{ii}$$

because $g_{ij}(m+1) = f_{ij}.g_{jj}^{(m)} i.e. g_{ij} = f_{ij}.g_{jj}$
Since $g_{ji} = 1$ and $g_{ii} = 1$ hence $f_{ji} = 1$ ∎

Theorem 5. *In an irreducible Markov Chain each state can be reached from any other state and they are all of the some type i.e they will be either all transient or persistent null or persistent non-null. In each case they will have the same period.*

Proof. Since the set of all states which can be reached from a particular state form a closed set, hence from the definition of an irreducible Markov Chain, each state can be reached from any other state. In order to prove that the states are all of the some type, we proceed as follows:
Let (i, j) be a pair of states. Since j is accessible from i and i is accessible from j, therefore there exist integers M and N such that

$$p_{ij}^{(M)} = \alpha > 0$$
$$p_{ji}^{(N)} = \beta > 0$$

$$\text{Now} \qquad p_{ii}^{(n+M+N)} \geq p_{ij}^{(M)}.p_{jj}^{(n)}.p_{ji}^{(N)} = \alpha\beta p_{jj}^{(n)} \tag{1}$$
$$p_{jj}^{(n+M+N)} \geq p_{ji}^{(N)}.p_{ii}^{(n)}.p_{ij}^{(M)} = \alpha\beta p_{ii}^{(n)} \tag{2}$$

from equation 1, we see that if the series $\sum p_{ii}^{(n)}$ converges [converges of $\sum p_{ii}^{(n)}$ is the same as converges of $\sum p_{ii}^{(n+N+M)}$] then the $\sum p_{ii}^{(n)}$ also converges.

Also from 2, if the divergence of the series $\sum p_{ii}^{(n)}$ implies the divergence of the series $\sum p_{jj}^{(n)}$ [i.e. of $\sum p_{ii}^{(n+N+M)}$].

Thus we see that the two series $\sum p_{ii}^{(n)}$ and $\sum p_{jj}^{(n)}$ converges or diverges together so the states (i, j) will be either both transient or both persistent together.

If the state i is persistent null then $p_{ii}^{(n)} \to 0 as n \to \infty$. ∎

3 Poisson Process

A stochastic process $\{X(t), t > 0\}$ is called Poisson process if $X(t)$ is a process with independent increments and the distribution of $X(t) - X(s), t > s$, is given by

$$P\{(X(t) - X(s)) = k\} = e^{-\lambda(t-s)} \frac{\{\lambda(t-s)\}^k}{k!} \qquad k = 0, 1, \cdots, \lambda$$

Let $X(t)$ denote the number of events occurring in the time interval $(0, t)$. Basic assumptions underlying the Poisson process are as follows:

1. The probability that an event will occur in the time interval $(t, t + \Delta t)$ is $\lambda\,\Delta t + 0\,(\Delta t)$ where λ is independent of t as well as the number of events occurred in the interval $(0, t)$.

2. The probability that more than one event will occur in the interval $(t, t + \Delta t)$ is $0\,\Delta t$.

Hence probability of number of no change in the interval $(t, t + \Delta t)$ is $1 - \lambda.\Delta t - 0(\Delta t)$. The probability of k events in timeinterval $(0, t)$ is denoted by $p_k(t)$

$$\text{Let} \qquad p_k(t) = P[X(t) = k], \qquad k = 0, 1, 2, \cdots$$

We are interested in finding out an expression for $p_k(t)$. For this purpose we extended the interval $(0, t)$ to point $t + \Delta t$. Now, we enumerate all the possible ways for computing the probability $p_k(t + \Delta t)$.

Now the occurrence of k events in the interval $t + \Delta t$ can happen in the following ways

1. Exactly k events occurred in $(0, t)$ and no event occurs during $(t, t+\Delta t)$. The probability of this event is

$$p_k(t)\,[1 - \lambda.\Delta t - 0(\Delta t)]$$

2. Exactly $(k - 1)$ events occurred in the interval $(0, t)$ and one event occurs in the interval $(t, t + \Delta t)$ and the probability of this event is

$$p_{k-1}(t)\,[\lambda.\Delta t + 0(\Delta t)]$$

3. Exactly $(k - i, i \geq 2$ events occurred in the interval $(0, t)$ and i events occurs in the interval $(t, t + \Delta t)$ and the probability of this event is $0(\Delta t)$.

Thus considering all the cases we get that

$$p_k(t+\Delta t) = p_k(t)[1-\lambda\,\Delta t] + p_{k-1}(t)\lambda\,\Delta t + 0(\Delta t) \tag{9}$$

$$\text{thus} \qquad \lim_{\Delta t\to 0} \frac{p_k(t+\delta t)-p_k(t)}{\Delta t} = -p_k(t)\lambda + p_{k-1}(t)\lambda \tag{10}$$

$$\frac{d}{dt}p_k(t) = -\lambda p_k(t) + \lambda p_{k-1}(t)\lambda \qquad \text{for } k \geq 1 \tag{11}$$

$$\frac{d}{dt}p_0(t) = -\lambda p_0(t) \qquad \text{for } k = 0 \tag{12}$$

$$\text{The initials conditions are } p_0(0) = 1 \text{ and } p_k(0) = 0, k \geq 1 \tag{13}$$

Thus from 12, we get

$$\begin{aligned} \frac{\frac{d}{dt}p_0(t)}{p_0(t)} &= -\lambda \\ \frac{d}{dt}[\log p_0(t)] &= -\lambda \\ \text{Thus} \qquad \log p_0(t) &= -\lambda t + C \\ p_0(t) &= e^{-\lambda t + C} \\ &= C' e^{-\lambda t} \end{aligned}$$

Putting the initial conditions $p_0(0) = 1$, we get $C' = 1$. Thus

$$p_0(t) = e^{-\lambda t}$$

Now from 11

$$\begin{aligned} \frac{d}{dt}p_1(t) &= -\lambda p_1(t) + \lambda p_0(t) \\ \frac{d}{dt}p_1(t) + \lambda p_1(t) &= \lambda e^{-\lambda t} \end{aligned}$$

Multiplying the above equation by $e^{\lambda t}$, we get

$$e^{\lambda t}\frac{d}{dt}p_1(t) + \lambda e^{\lambda t}p_1(t) = \lambda$$

$$\frac{d}{dt}[e^{\lambda t}p_1(t)] = \lambda$$
$$e^{\lambda t}p_1(t) = \lambda t + C$$

Now applying the condition $p_1(0) = 0$, we have $C = 0$, Thus

$$e^{\lambda t}p_1(t) = \lambda t$$
$$p_1(t) = \lambda t e^{-\lambda t}$$

Proceeding in the similar manner, we get

$$p_k(t) = e^{-\lambda t}\frac{(\lambda t)^k}{k!} \qquad k = 0, 1, 2, \cdots$$

Since the parameter λ is independent of t and number of events occurred prior to t, hence

$$\begin{aligned} P[X(t) - X(s) = k] &= P[X(t-s) = k] \\ &= e^{-\lambda(t-s)}\frac{[\lambda(t-s)]^k}{k!} \qquad k = 0, 1, 2, \cdots \end{aligned}$$

In many cases the reverse problem arises to determine P_k from P_s. Many situations arises when it is easer to find the p.g.f. P_s of a variable rather than the probably distribution $\{P_k\}$.Even without finding $\{P_k\}$, we can find the moments of the distribution. $\{P_k\}$ can obtained as

$$P_k = \frac{1}{k!}\left[\frac{d_k P_{(s)}}{ds^k}\right]_{s=0} \qquad k = 0, 1, 2, \ldots$$

Also P_x can be obtained as the coefficient of s^k in $P(s)$ as a power series in S.

Solution of $p_k(t)$ in Poisson process with the help of p.g.f.
Let $G_x(s,t)$ be the p.g.f. of $x(t)$ then

$$G_x(t) = \sum_{k=0}^{\infty} s^k.p_k(t)$$

Differentiating $G_x(s,t)$ w.r.to t ,we have

$$\frac{\partial}{\partial t}.G_x(s,t) = \sum_{k=0}^{\infty} s^k.\frac{d}{dt}.p_k(t)$$

Subtracting the value of $\frac{d}{dt}.p_k(t)$ from the relation

$$\frac{d}{dt}.p_k(t) = -\lambda p_k(t) + \lambda p_{k-1}(t)$$

We see

$$\begin{aligned}
\frac{\partial}{\partial t}.G_x(s,t) &= \sum_{k=0}^{\infty} -s^k.\lambda p_k(t) + \sum_{k=1}^{\infty} \lambda p_{k-1}(t).s^k \\
&= -\lambda.G_x(s,t) + \lambda s \sum_{k=1}^{\infty} s^{k-1}.p_{k-1}(t) \\
&= -\lambda.G_x(s,t) + \lambda s \sum_{r=0}^{\infty} s^r.p_r(t) \\
&= -\lambda.G_x(s,t) + \lambda s.G_x(s,t) \\
&= -\lambda(1-s)G_x(s,t)
\end{aligned}$$

The initial condition is $G_x(s,0) = 1$ as

$$G_x(s,t) = \sum_{k=0}^{\infty} s^k.p_k(t) = s^0 = 1$$

$$\begin{aligned}
\frac{1}{G_x(s,t)}.\frac{\partial}{\partial t}.G_x(s,t) &= -\lambda(1-s) \\
\frac{\partial}{\partial t}\log G_x(s,t) &= -\lambda(1-s) \\
\log G_x(s,t) &= -\lambda(1-s).t + C
\end{aligned}$$

By the initial condition

$$\log G_x(s,0) = c$$

i.e. $c = 0$

Then,

$$\begin{aligned}
\log G_x(s,t) &= -\lambda(1-s)t \\
G_x(s,t) &= e^{-\lambda t(1-s)}
\end{aligned}$$

This is the p.g,f. of a Poisson distribution with parameter λt. Consequently

$$p_k(t) = \frac{e^{-\lambda t}(\lambda t)^k}{k!} \qquad k = 0, 1, 2, \ldots$$

3.1 Time Dependent Poisson Process

If we assume that λ is a function of time in assumptions of Poisson process, then the differential equation for probability generating function reduces to the form

$$\frac{\partial}{\partial t} G_x(s,t) = -\lambda(t).(1-s).G_x(s,t)$$

i.e.

$$\frac{\partial}{\partial t} \log G_x(s,t) = -\lambda(t).(1-s)$$

i.e.

$$\log G_x(s,t) = -(1-s)\int_0^t \lambda(\tau)\, d\tau$$

From this we find

$$G_x(s,t) = e^{-(1-s).\int_0^t \lambda(\tau)\, d\tau}$$

This is the p.g.f. of a random variable having Poisson distribution with parameter

$$\int_0^t \lambda(\tau)\, d\tau$$

and

$$P\left[X(t)=k\right] = \frac{e^{-\int_0^t \lambda(\tau)\, d\tau}.[\int_0^t \lambda(\tau)\, d\tau]^k}{k!} \qquad k = 0,1,2,\ldots$$

3.2 Weighted Poisson Process

The Poisson Process describes the frequency distribution of occurrence of an event to an individual with risk parameter λ. If we are sampling a population of individuals, then variability of individuals with respect to this risk should be taken into account. For example, risk to accident varies through out the population according to the density function f(λ). Then the probability

$$p_k(t) = \frac{e^{-\lambda t}(\lambda t)^k}{k!} \qquad k = 0,1,2,\ldots$$

must be interpreted as the conditional distribution for given λ or of $\{X(t)|\lambda\}$ and the probability that an individual chosen at random from the population will experience k events in time interval of length 't'is

$$
\begin{aligned}
p_k(t) &= \int P\left[X(t)=k|\lambda\right].f(\lambda)\,d\lambda \\
&= \sum P\left[X(t)=k|\lambda\right].f(\lambda)
\end{aligned}
$$

Example 5. *Suppose λ has a type III distribution, i.e.*

$$
f(\lambda) = \frac{\beta^\alpha}{\Gamma\alpha} e^{-\beta\lambda}.\lambda^{\alpha-1} \qquad \lambda > 0,\ \alpha > 0, \beta > 0
$$

Then,

$$
\begin{aligned}
p_k(t) &= \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^k}{k!}.\beta^\alpha.\frac{e^{-\beta\lambda}\lambda^{\alpha-1}}{\Gamma\alpha}.\,d\lambda \\
&= \frac{\beta^\alpha}{\Gamma\alpha}.\frac{1}{k!}\int_0^\infty e^{-\lambda t}.(\lambda t)^k.e^{-\beta\lambda}.\lambda^{\alpha-1}.\,d\lambda \\
&= \frac{\beta^\alpha}{\Gamma\alpha}.\frac{t^k}{k!}\int_0^\infty e^{-(t+\beta)\lambda}.\lambda^{k+\alpha-1}.\,d\lambda
\end{aligned}
$$

Let $\lambda(t+\beta) = x$,then $\mathrm{d}\lambda = \frac{dx}{t+\beta}$

so,

$$
\begin{aligned}
p_k(t) &= \frac{\beta^\alpha}{\Gamma\alpha}.\frac{t^k}{k!}\int_0^\infty e^{-x}.\frac{x^{k+\alpha-1}}{(t+\beta)^k+\alpha}.\,dx \\
&= \frac{\beta^\alpha}{\Gamma\alpha}.\frac{t^k}{k!}\frac{\Gamma k+\alpha}{(t+\beta)^{\beta+k}} \qquad k = 0,1,2,\ldots \\
&= \frac{\Gamma k+\alpha}{\Gamma\alpha.k!}.\left[\frac{\beta}{(t+\beta)}\right]^\alpha.\left(\frac{t}{t+\beta}\right)^k \\
&= \binom{\alpha+k-1}{k} p^\alpha q^k \qquad k = 0,1,2,\ldots
\end{aligned}
$$

where $\frac{\beta}{t+\beta} = p$ and $q = 1-p$.

This is a negative Binomial distribution. We know that the mean of the negative binomial distribution is $r.\frac{q}{p}$ and the variance is $r.\frac{q}{p^2}$.Here $r = \alpha$.

Thus the mean of X(t) is $\alpha.(\frac{t}{t+\beta}).\frac{(t+\beta)}{\beta} = \frac{\alpha t}{\beta}$ and

Variance of X(t) is $\alpha.(\frac{t}{t+\beta}).\frac{(t+\beta)^2}{\beta^2} = \frac{\alpha.t.(\beta+t)}{\beta^2}$.

4 Birth Process

4.1 Pure Birth Process

In the study of some growth phenomena, birth may be introduced as an event where the probability of occurence of an event in (t,t+Δt) is dependent on the number of parent events already in existence.

For example

1. It may refer to literal birth.
2. It may refer to a new case in epidemic.
3. It may refer to the appearance of a new tumor cell etc.

Let $X(t)$ denote the number of births till point t given that initially there are k_0 births.Then we may be interested in computing

$$p_k(t) = p\left[X(t) = k|X(0) = k_0\right]$$

Where $X(0)$ represents the initial number of events (births) in existence.

Assumptions underlying the Pure Birth Process:

1. Given $X(t) = k$,the probability that a new event will occur in the time interval $(t, t + \Delta t)$ is $\lambda_k(t)\Delta t + 0(\Delta t)$. where $\lambda_k(t)$ is a function of k and t.
2. Probability that more than one event occur in the time interval $(t, t + \Delta t)$ is $0.(\Delta t)$

Hence probability of no change in $(t, t + \Delta t)$ is

$$[1 - \lambda_k(t).\Delta t - 0(\Delta t)]$$

In order to derive the differential equation for $p_k(t)$, we extend the time interval $(0, t)$ to a point $(t+\Delta t)$ and enumerate all the possible ways in which k events can happen in $(0, t + \Delta t)$ as follows:

	In time interval $(0,t)$	In time interval $(t,t+\Delta t)$
(i)	k events	no event
(ii)	$k-1$ events	one event
(iii)	$k-i$ events	i events
	$i \geq 2$	$i \geq 2$

and these probabilities are

(i) $p_k(t)$.[1-$\lambda_k(t)$.Δt - 0(Δ t)]
(ii) $p_{k-1}(t)$.[$\lambda_{k-1}(t)$.Δt + 0(Δt)]
(iii) 0(Δt) respectively.

Combining all these we get

$$\begin{aligned}
p_k(t+\Delta t) &= p_k(t)[1-\lambda_k(t).\Delta t - 0(\Delta t)] + p_{k-1}(t).\lambda_{k-1}(t).\Delta t + 0(\Delta t) \\
&= p_k(t) - p_k(t).\lambda_k(t).\Delta t + p_{k-1}(t).\lambda_{k-1}(t).\Delta t + 0(\Delta t) \\
p_k(t+\Delta t) - p_k(t) &= -p_k(t).\lambda_k(t).\Delta t + p_{k-1}(t).\lambda_{k-1}(t).\Delta t + 0(\Delta t) \\
\lim_{\Delta t \to 0} \frac{p_k(t+\Delta t) - p_k(t)}{\Delta t} &= -p_k(t).\lambda_k(t) + p_{k-1}(t).\lambda_{k-1}(t)
\end{aligned}$$

i.e.

$$\frac{d}{dt} p_k(t) = -p_k(t).\lambda_k(t) + p_{k-1}(t).\lambda_{k-1}(t)$$

Further if at time t, there are k_0 events then

$$\frac{d}{dt} p_{k_0}(t) = -p_{k_0}(t).\lambda_{k_0}(t)$$

The initial conditions are

$$\begin{aligned}
p_{k_0}(0) &= 1 \\
p_k(0) &= 0 \qquad \text{for} \qquad k > k_0
\end{aligned}$$

if initially there are k_0 events.

4.2 Homogeneous Pure Birth Process :

The pure birth process is said to be homogeneous if $\lambda_k(t)$ is independent of t. i.e.

$$\lambda_k(t) = \lambda_k$$

Then,

$$\frac{d}{dt}p_{k_0}(t) = -\lambda_{k_0}\, p_k(t) \tag{14}$$

and

$$\frac{d}{dt}p_k(t) = -\lambda_k.p_k(t) + \lambda_{k-1}.p_{k-1}(t) \qquad \text{for} \qquad k > k_0 \tag{15}$$

Initial conditions are: $p_{k_0}(0) = 1$, $p_k(0) = 0$, $k > k_0$

These equations can be solved successively assuming that all the $\lambda's$ are distinct. The solution for $p_k(t)$ is

$$p_k(t) = (-1)^{k-k_0}\lambda_{k_0}.\lambda_{k_0+1}.\lambda_{k_0+2}...\lambda_{k-1}\sum_{i=k_0}^{k}\left\{\frac{e^{-\lambda_i t}}{\prod\limits_{\substack{j=k_0\\ j\neq i}}^{k}(\lambda_i-\lambda_j)}\right\}, \quad k = k_0,\ k_{0+1},\ k_{0+2}\ldots \tag{16}$$

The above result will be proved by induction. For this we make use of the identity

$$\sum_{i=k_0}^{k}\frac{1}{\prod\limits_{\substack{j=k_0\\ j\neq i}}^{k}(\lambda_i-\lambda_j)} = 0 \qquad \text{if } \lambda_i\text{'s are distinct.}$$

We will not prove the identity. However , as an example this can be verified. Let k_0 =2 , k=4 . So we have,

$$\sum_{i=k_0}^{k}\frac{1}{\prod\limits_{\substack{j=k_0\\ j\neq i}}^{k}(\lambda_i-\lambda_j)} = \frac{1}{(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)} + \frac{1}{(\lambda_3-\lambda_2)(\lambda_3-\lambda_4)} + \frac{1}{(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)}$$

$$= \frac{(\lambda_3-\lambda_4)+\{-(\lambda_2-\lambda_4)\}+\{\lambda_2-\lambda_3\}}{(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)(\lambda_3-\lambda_4)}$$

$$= 0$$

Now, by solving equation 14, we have

$$\frac{d}{dt}\log p_{k_0}(t) = -\lambda_{k_0}$$
$$\log p_{k_0}(t) = -\lambda_{k_0} + C$$

Since $p_{k_0}(0) = 1$ so, $0 =$ C . Thus $p_{k_0}(t) = e^{-\lambda_{k_0}.t}$
Putting $k = k_0$ in 16 we get

$$p_{k_0}(t) = e^{-\lambda_{k_0}.t}$$

Which is the same as solution of 14. Thus we see that 16 is true for $k = k_0$
Let us suppose that 16, holds for $k = k - 1$, i.e.,

$$p_{k-1}(t) = (-1)^{k-k_0-1} \,.\, \lambda_{k_0} \,.\, \lambda_{k_0+1} \ldots \lambda_{k-2} \sum_{i=k_0}^{k-1} \left\{ \frac{e^{-\lambda_i t}}{\prod\limits_{\substack{j=k_0 \\ j\neq i}}^{k-1} (\lambda_i - \lambda_j)} \right\} \tag{17}$$

Now multiplying both sides of 15 by $e^{\lambda_k.t}$, we get

$$\begin{aligned} \frac{d}{dt}p_k(t).e^{\lambda_k.t} &= -\lambda_k.p_k(t).e^{\lambda_k.t} + \lambda_{k-1}.p_{k-1}(t).e^{\lambda_k.t} \\ \frac{d}{dt}[e^{\lambda_k.t}.p_k(t)] &= \lambda_{k-1}.p_{k-1}(t).e^{\lambda_k.t} \end{aligned}$$

Substituting the value of $p_{k-1}(t)$ from 17 we get,

$$\frac{d}{dt}[e^{\lambda_k.t}.p_k(t)] = (-1)^{k-k_0-1}.\lambda_{k_0}.\lambda_{k_0+1} \ldots \lambda_{k-2}.\lambda_{k-1} \sum_{i=k_0}^{k-1} \left\{ \frac{e^{-\lambda_i t}}{\prod\limits_{\substack{j=k_0 \\ j\neq i}}^{k-1} (\lambda_i - \lambda_j)} \right\} .e^{\lambda_k.t}$$

$$\boxed{\text{Note: the product } \lambda_{k_0}.\lambda_{k_0+1}.\ldots.\lambda_{k-1} \text{ goes upto } \lambda_{k-1}}$$

$$= (-1)^{k-k_0-1}.\lambda_{k_0}.\lambda_{k_0+1}\ldots\lambda_{k-1}\sum_{i=k_0}^{k-1}\frac{e^{-(\lambda_i-\lambda_k).t}}{\prod\limits_{\substack{j=k_0\\ j\neq i}}^{k-1}(\lambda_i-\lambda_j)}$$

$$\boxed{\text{Note: } e^{\lambda_k.t} \text{ is combined as } e^{-(\lambda_i-\lambda_k).t}}$$

$$= (-1)^{k-k_0-1}.\lambda_{k_0}.\lambda_{k_0+1}\ldots\lambda_{k-1}\sum_{i=k_0}^{k-1}\frac{\frac{d}{dt}\left[e^{-(\lambda_i-\lambda_k).t}\right]}{\left\{\prod\limits_{\substack{j=k_0\\ j\neq i}}^{k-1}(\lambda_i-\lambda_j)\right\}\{-(\lambda_i-\lambda_k)\}}$$

$$\boxed{\text{Note: } e^{-(\lambda_i-\lambda_k).t} \text{ is replaced by } \frac{\frac{d}{dt}\left[e^{-(\lambda_i-\lambda_k).t}\right]}{-(\lambda_i-\lambda_k)}}$$

$$= \quad (-1)^{k-k_0}.\lambda_{k_0}.\lambda_{k+1}\ldots\lambda_{k-1}\sum_{i=k_0}^{k-1}\frac{\frac{d}{dt}\left[e^{-(\lambda_i-\lambda_k).t}\right]}{\prod\limits_{\substack{j=k_0\\ j\neq i}}^{k}(\lambda_i-\lambda_j)}$$

$$\boxed{\text{Note: } \prod_{\substack{j=k_0\\ j\neq i}}^{k}(\lambda_i-\lambda_j) \text{ goes upto k}}$$

Integrating both sides with respect to t, we get,

$$e^{\lambda_k.t}.p_k(t) = (-1)^{k-k_0}.\lambda_{k_0}\ldots\lambda_{k-1}\left[\sum_{i=k_0}^{k-1}\frac{e^{-(\lambda_i-\lambda_k).t}}{\prod\limits_{\substack{j=k_0\\ j\neq i}}^{k}(\lambda_i-\lambda_j)} + C\right] \qquad (18)$$

Since $p_k(0) = 0$ (initial condition), so we get

$$0 = (-1)^{k-k_0}.\lambda_{k_0}\ldots\lambda_{k-1}\left[\sum_{i=k_0}^{k-1}\frac{1}{\prod\limits_{\substack{j=k_0\\ j\neq i}}^{k}(\lambda_i-\lambda_j)} + C\right]$$

From this and using the identity $\sum\limits_{i=k_0}^{k} \frac{1}{\prod\limits_{\substack{j=k_0 \\ j\neq i}}^{k} (\lambda_i - \lambda_j)} = 0$, we get

$C = \frac{1}{\prod\limits_{\substack{j=k_0 \\ j\neq i}}^{k-1} (\lambda_i - \lambda_j)}$

Substuting the value of 'C' in the above we get

$$e^{\lambda_k.t}.p_k(t) = (-1)^{k-k_0}.\lambda_{k_0}\ldots\lambda_{k-1}\left[\sum_{i=k_0}^{k-1} \frac{e^{-(\lambda_i-\lambda_k).t}}{\prod\limits_{\substack{j=k_0 \\ j\neq i}}^{k} (\lambda_i - \lambda_j)} + \frac{e^{-(\lambda_k t-\lambda_k.t)}}{\prod\limits_{j=k_0}^{k-1} (\lambda_k - \lambda_j)}\right]$$

$\boxed{\text{because } e^{-(\lambda_k t-\lambda_k.t)} = e^0 = 1}$

Thus , we get

$$p_k(t) = (-1)^{k-k_0}.\lambda_{k_0}\ldots\lambda_{k-1}\sum_{i=k_0}^{k}\left\{\frac{e^{-(\lambda_i.t)}}{\prod\limits_{\substack{j=k_0 \\ j\neq i}}^{k} (\lambda_i - \lambda_j)}\right\}$$

Thus if the result holds for $k-1$, then it also holds for k . Since it holds for k_0 and hence for k_0+1 , and so on .

Thus the required solution is

$$p_k(t) = (-1)^{k-k_0}.\lambda_{k_0}\ldots\lambda_{k-1}\sum_{i=k_0}^{k}\left\{\frac{e^{-(\lambda_i.t)}}{\prod\limits_{\substack{j=k_0 \\ j\neq i}}^{k} (\lambda_i - \lambda_j)}\right\} \qquad for\ k = k_0, k_{0+1}, k_{0+2}\ldots$$

4.3 Linear Birth Process (Yule Process):

We consider a population of members which can (by splitting or other wise) give birth to new members but can not die. Assume that in any short interval of length Δt, each member has the probability $\lambda\Delta t + 0(\Delta t)$ to create a new member. The constant λ determines the rate of increase of population. If there is no interaction among the members of population and at time t, the population size is k, then the probability of birth of a new individual in the population in the time interval $(t, t+\Delta t)$ is $k\lambda\Delta t + 0.(\Delta t)$ and the probability of more than one birth is $0.(\Delta t)$.

Example 6. *Suppose there are k individuals at time t , then the probability that each individual will give a birth (occurrence a new event) in time $(t, t+\Delta t)$ is $\lambda\Delta t + 0.(\Delta t)$. So probability of occurrence of j events in time interval $(t, t+\Delta t)$ is*

$$= {}^{k}c_j[\lambda.\Delta t + 0.(\Delta t)]^j[1 - \lambda.\Delta t - 0.(\Delta t)]^{k-j}$$

so for $j = 0$, the probability is

$$[1 - \lambda.\Delta t - 0.(\Delta t)]^k = 1 - k.\lambda.\Delta t + 0.(\Delta t)$$

for $j = 1$, the probability is

$$k[\lambda.\Delta t + 0.(\Delta t)][1 - \lambda.\Delta t - 0.(\Delta t)]^{k-1}$$

$$k[\lambda.\Delta t + 0.(\Delta t)][1 - (k-1).\lambda.\Delta t - 0.(\Delta t)]$$

$$= k.\lambda.\Delta t + 0.(\Delta t)$$

for $j \geq 2$, the probability is $0.(\Delta t)$.

Now the probability $p_k(t)$ satisfies the equation (in general)

$$\frac{d}{dt}p_k(t) = -\lambda_k.p_k(t) + \lambda_{k-1}p_{k-1}(t)$$
$$\frac{d}{dt}p_{k_0}(t) = -\lambda_{k0}.p_{k0}(t)$$

Here $\lambda_k = k\lambda$.

Thus the equations are

$$\frac{d}{dt}p_k(t) = -k\lambda.p_k(t) + (k-1)\lambda.p_{k-1}(t) \tag{19}$$

$$\frac{d}{dt}p_{k0}(t) = -k_0\lambda.p_{k0}(t) \tag{20}$$

The solution of the above equation can be obtained with the help of the solution of homogeneous pure birth process. In the pure birth process, we have

$$p_k(t) = (-1)^{k-k_0}.\lambda_{k_0}\ldots\lambda_{k-1}\sum_{i=k_0}^{k}\left\{\frac{e^{-(\lambda_i.t)}}{\prod\limits_{\substack{j=k_0\\ j\neq i}}^{k}(\lambda_i-\lambda_j)}\right\} \qquad for\ k = k_0, k_{0+1}, k_{0+2}\ldots$$

Now putting $\lambda_i = i\lambda$,we have

$$\begin{aligned}(-1)^{k-k_0}.\lambda_{k_0}\ldots\lambda_{k-1} &= (-1)^{k-k_0}.k_0\lambda.(k_0+1)\lambda\ldots(k-1)\lambda\\ &= (-1)^{k-k_0}\lambda^{k-k_0}.k_0.(k_0+1)\ldots(k-1)\\ &= (-1)^{k-k_0}\lambda^{k-k_0}\frac{1.2.3\ldots k_0-1.k_0\ldots(k-1)}{1.2.3\ldots k_0-1}\\ &= (-1)^{k-k_0}\lambda^{k-k_0}.\frac{(k-1)!}{(k_0-1)!}\frac{(k-k_0)!}{(k-k_0)!}\end{aligned}$$

$$(-1)^{k-k_0}.\lambda_{k_0}\ldots\lambda_{k-1} = (-1)^{k-k_0}\lambda^{k-k_0}\binom{k-1}{k_0-1}.(k-k_0)! \tag{21}$$

Now,

$$\begin{aligned}\prod_{\substack{j=k_0\\ j\neq i}}^{k}(\lambda_i-\lambda_j) &= \prod_{\substack{j=k_0\\ j\neq i}}^{k}(i\lambda-j\lambda)\\ &= \lambda^{k-k_0}\prod_{\substack{j=k_0\\ j\neq i}}^{k}(i-j)\\ &= \lambda^{k-k_0}\left[(i-k_0)(i-k_0-1)\ldots 3.2.1.(-1)(-2)\ldots\{-(k-i)\}\right]\end{aligned}$$

Example 7. *The above expression seems to be some what complicated. We give one example here to clarify the equation. Let $i = 3$ $k_0 = 1$ $k = 5$.*

$$\prod_{\substack{j=k_0\\ j\neq i}}^{k}(i\lambda - j\lambda) = (3\lambda-\lambda)(3\lambda-2\lambda)(3\lambda-4\lambda)(3\lambda-5\lambda)$$

$$= \lambda(2).\lambda(1).\lambda(-1).\lambda(-2)$$
$$= \lambda^4.2.1.(-1)(-2)$$

Hence

$$\prod_{\substack{j=k_0\\ j\neq i}}^{k}(\lambda_i - \lambda_j) = \lambda^{k-k_0}[(i-k_0)(i-k_0-1)\dots 3.2.1.(-1)^{k-i}\ (k-i)!\ ]$$

$$= \lambda^{k-k_0}(-1)^{k-i}(k-i)!\ (i-k_0)!$$
$$= \lambda^{k-k_0}(-1)^{k-i}\frac{(k-i)!\ (i-k_0)!}{(k-k_0)!}\ (k-k_0)!$$

$\boxed{\text{Note: Multiplying the numerator and denominator by } (k-k_0)!}$

$$= \lambda^{k-k_0}(-1)^{k-i}\left[\binom{k-k_0}{i-k_0}\right]^{-1}(k-k_0)! \tag{22}$$

Thus

$$p_k(t) = (-1)^{k-k_0}\lambda^{k-k_0}\binom{k-1}{k_0-1}(k-k_0)!\sum_{i=k_0}^{k}\frac{e^{-i\lambda t}.\binom{k-k_0}{i-k_0}}{\lambda^{k-k_0}(-1)^{k-i}(k-k_0)!}$$
$$= (-1)^{k-k_0}\binom{k-1}{k_0-1}\sum_{i=k_0}^{k}\frac{e^{-i\lambda t}}{(-1)^{k-i}}.\binom{k-k_0}{i-k_0}$$
$$= \binom{k-1}{k-k_0}e^{-k_0\lambda t}\sum_{i=k_0}^{k}\frac{e^{-i\lambda t}}{(-1)^{k-i}}.(e^{-\lambda t})^{i-k_0}\binom{k-k_0}{i-k_0}$$

$\boxed{\text{Note: } \binom{k-1}{k_0-1} \text{ is the same as } \binom{k-1}{k-k_0}}$

$$= \binom{k-1}{k-k_0} e^{-k_0\lambda t} \sum_{i=k_0}^{k} (-1)^{k-k_0-k+i} (e^{-\lambda t})^{i-k_0} \binom{k-k_0}{i-k_0}$$

$$p_k(t) = \binom{k-1}{k-k_0} e^{-k_0\lambda t} \sum_{i=k_0}^{k} (-e^{-\lambda t})^{i-k_0} \binom{k-k_0}{i-k_0} \tag{23}$$

Now let us evaluate

$$= \sum_{i=k_0}^{k} (-e^{-\lambda t})^{i-k_0} \binom{k-k_0}{i-k_0}$$

putting $l = i - k_0$ in the above we get

$$= \sum_{l=0}^{k-k_0} (-e^{-\lambda t})^{l} \binom{k-k_0}{l}$$

It is known that in general

$$\sum_{k=0}^{n} \binom{n}{k} (-x)^k = (1-x)^n$$

Hence LHS

$$= (1-e^{-\lambda t})^{k-k_0}$$

Thus

$$p_k(t) = \binom{k-1}{k-k_0} e^{-k_0\lambda t} (1-e^{-\lambda t})^{k-k_0} \qquad for\ k = k_0, k_{0+1}, k_{0+2} \ldots$$

From the above it can be shown that

$$Y(t) = X(t) - k_0$$

has a negative binomial distribution with parameters $r = k_0$ and $p = e^{-\lambda t}$

Because

$$P[X(t) = k] = P[Y(t) = k - k_0] = P[Y(t) = r] \quad i.e.\ r = k - k_0 \quad i.e.\ k = k_0 + r$$

$$
\begin{aligned}
P[Y(t)=r] &= \binom{k-1}{r}(e^{-\lambda t})^{k_0}(1-e^{-\lambda t})^r \\
&= \binom{k_0+r-1}{r}(e^{-\lambda t})^{k_0}(1-e^{-\lambda t})^r \qquad r=0,1,2,3,\ldots \quad \boxed{\text{because } k=k_0+r}
\end{aligned}
$$

Thus, Y(t) has a negative binomial distribution with parameters k_0 and $e^{-\lambda . t}$

$$
\begin{aligned}
E[Y(t)] &= k_0.\frac{1-e^{-\lambda t}}{e^{-\lambda t}} \\
&= k_0(e^{\lambda t}-1) \\
E[X(t)] &= k_0 + E[Y(t)] \\
&= k_0 + (e^{\lambda t}-1) \\
&= k_0 e^{\lambda t}
\end{aligned}
$$

Similarly

$$V[X(t)] = V[Y(t)] = k_0.\frac{1-e^{-\lambda t}}{e^{-2\lambda t}}$$

$$
\begin{aligned}
&= k_0[e^{2\lambda t}-e^{\lambda t}] \\
&= k_0[e^{\lambda t}-1].e^{\lambda t}
\end{aligned}
$$

Another Method for Solution of Linear Birth Process:(Method of P.G.F.)

For the linear birth process we have the equations

$$\frac{d}{dt}p_{k_0}(t) = -k_0\lambda p_{k_0}(t) \tag{24}$$

$$\frac{d}{dt}p_k(t) = -k\lambda p_k(t) + (k-1)\lambda p_{k-1}(t) \qquad k>k_0 \tag{25}$$

subject to initial conditions

$$p_{k_0}(0)=1 \qquad p_k(0)=0 \qquad k>k_0 \tag{26}$$

Let $G_x(s,t)$ be the p.g.f. of random variable $X(t)$

$$G_x(s,t) = \sum_{k=k_0}^{\infty} p_k(t).s^k \tag{27}$$

$$\frac{\partial}{\partial t}G_x(s,t) = \sum_{k=k_0}^{\infty} \frac{d}{dt}p_k(t).s^k$$

$$= \sum_{k=k_0}^{\infty} -k\lambda p_k(t).s^k + \sum_{k=k_0+1}^{\infty} (k-1)\lambda p_{k-1}(t)s^k \tag{28}$$

$$\begin{aligned}
\frac{\partial}{\partial t}G_x(s,t) &= -\lambda \sum_{k=k_0}^{\infty} p_k(t)k.s^k + \lambda \sum_{k=k_0+1}^{\infty} (k-1)s^k.p_{k-1}(t) \\
&= -\lambda s \sum_{k=k_0}^{\infty} p_k(t)k.s^{k-1} + \lambda s^2 \sum_{k=k_0+1}^{\infty} (k-1)s^{k-2}.p_{k-1}(t) \\
&= -\lambda s \frac{\partial}{\partial s}G_x(s,t) + \lambda s^2 \frac{\partial}{\partial s}G_x(s,t)
\end{aligned}$$

$$\frac{\partial}{\partial t}G_x(s,t) + \lambda s(1-s)\frac{\partial}{\partial s}G_x(s,t) = 0 \tag{29}$$

Suppose Z is a function of x and y and we have

$$p\frac{\partial z}{\partial x} + Q\frac{\partial z}{\partial y} = R$$

then we have

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

If

$$u(x,y,z) = c_1$$

$$v(x,y,z) = c_2$$

are independent solutions then the general solution is $u = \phi(v)$.

Then

$$\frac{dt}{1} = \frac{ds}{\lambda s(1-s)} = \frac{dG_x(s,t)}{0}$$

Now we obtain the solution of equation 29 using the above described technique.Considering the auxiliary equation

$$\frac{dt}{1} = \frac{dG_x(s,t)}{0}$$

we get

$$G_x(s,t) = C \qquad \text{(C is constant)} \tag{30}$$

Also considering the second auxiliary equation, we have

$$\frac{dt}{1} = \frac{ds}{\lambda s(1-s)}$$

i.e.

$$\lambda dt = \frac{ds}{\lambda s(1-s)} = ds\left[\frac{1}{s} + \frac{1}{1-s}\right]$$

After integration, we get

$$\lambda t = \log s - \log(1-s) + \log k \quad \text{where } k \text{ is constant}$$

$$\begin{aligned} &= \log\left[\frac{s}{1-s}.k\right] \\ or \lambda t &= \log\frac{s}{1-s} + C_1 \\ \log\frac{s}{1-s} &= \lambda t - C_1 \\ \frac{s}{1-s} &= e^{\lambda t - C_1} = C_2 e^{\lambda t} \end{aligned}$$

$$\frac{s}{1-s}e^{-\lambda t} = C_2 \tag{31}$$

Which is the solution obtained from above.

Then the general solution will be

$$G_x(s,t) = \phi\left[\frac{s}{1-s}.e^{-\lambda t}\right] \tag{32}$$

Where ϕ is an arbitrary function.To obtain the particular solution, we use the initial condition given in 26.

$$G_x(s,0) = \phi\left[\frac{s}{1-s}.e^0\right]$$

$$\text{but } G_x(s,0) = s_0^k$$

$$s^{k_0} = \phi\left(\frac{s}{1-s}\right)$$

Let us write

$$\frac{s}{1-s} = \theta$$

$$s = \theta - s.\theta$$

$$s + s\theta = \theta$$

$$s(1+\theta) = \theta$$

$$s = \frac{\theta}{1+\theta}$$

$$\text{Thus} \quad \phi(\frac{s}{1-s}) = \phi(\theta) = (\frac{\theta}{1+\theta})^{k_0}$$

$$\text{now} \quad G_x(s,t) = \phi\left[\frac{se^{-\lambda t}}{1-s}\right]$$

$$\text{Thus} \quad \phi\left[\frac{se^{-\lambda t}}{1-s}\right] = \left[\frac{\frac{se^{-\lambda t}}{1-s}}{1+\frac{se^{-\lambda t}}{1-s}}\right]^{k_0}$$

$$G_x(s,t) = \left[\frac{se^{-\lambda t}}{1-s+se^{-\lambda t}}\right]^{k_0}$$

$$= \left[\frac{se^{-\lambda t}}{1-s(1-e^{-\lambda t})}\right]^{k_0}$$

$$= s^{k_0}\left[\frac{e^{-\lambda t}}{1-s(1-e^{-\lambda t})}\right]^{k_0}$$

Let us consider a new variable

$$Y(t) = X(t) - k_0$$

Thus

$$X(t) = Y(t) + k_0$$

Now p.g.f. of $X(t) = s^{k_0}$. p.g.f. of $Y(t)$. Now $\left[\frac{e^{-\lambda t}}{1-s(1-e^{-\lambda t})}\right]^{k_0}$ is the p.g.f. of a negative binomial distribution with parameter $r = k_0$ and $p = e^{-\lambda t}$.

Now $P[X(t) = k] = P[Y(t) = k - k_0]$
We know that in case of negative binomial distribution with parameters α and p

$$p_x = \binom{x + \alpha - 1}{x} .p^{\alpha}.q^{x}$$

so

$$\begin{aligned} p_k(t) &= \binom{k - k_0 + k_0 - 1}{k - k_0} (e^{-\lambda t})^{k_0} (1 - e^{-\lambda t})^{k-k_0} \\ &= \binom{k-1}{k-k_0} e^{-k_0 \lambda t} (1 - e^{-\lambda t})^{k-k_0} \qquad for\ k = k_0, k_{0+1}, k_{0+2} \dots \end{aligned}$$

4.4 Time Dependent Linear Birth Process

In linear birth process, we assume that $\lambda_k = k\lambda$.

However if we assume $\lambda_k(t) = k\lambda(t)$ then we get time dependent linear birth process. In this case differential equation becomes

$$\frac{d}{dt}\ p_{k_0}(t) = -k_0\ \lambda(t)\ p_{k_0}(t)$$

$$\frac{d}{dt}\ p_k(t) = -k\ \lambda(t)\ p_k(t) + (k-1)\lambda(t) p_{k-1}(t)$$

In this case

$$G_x(s,t) = s^{k_0} \left[\frac{e^{-\int_0^t \lambda(\tau) d\tau}}{1 - s(1 - e^{-\int_0^t \lambda(\tau) d\tau})}\right]^{k_0}$$

Thus the distribution of $X(t)$ is of the same form as discussed above with $\lambda.t$ replaced as $\int_0^t \lambda(\tau) d\tau$.

5 Death Process

5.1 Pure Death Process

Let$X(t)$ denote the number of individuals present at time t given that initially there are k_0 individuals. The basic assumptions underlying the pure death process are as follows:

1. Give $X(t) = k$, probability that there will be a death during interval $(t, t+\Delta t)$ is $\mu_k(t).\Delta t + 0.(\Delta t)$.
   $\mu_k(t)$ is known as the force of mortality and is function of k as well as t.

2. Probability that there will be more than one death in the interval$(t, t+\Delta t)$ is $0.(\Delta t)$.

Hence probability of no change is $[1 - \mu_k(t).\Delta t - 0.(\Delta t)]$

In order to write the differential equation for $p_k(t)$, we extend the interval $(0, t)$ upto $(t + \Delta t)$ and consider all the probabilities which may lead to presence of k individuals in the interval$(0, t + \Delta t)$. Thus we have

$$\begin{aligned} p_k(t+\Delta t) &= p_{k+1}(t)[\mu_{k+1}(t)\Delta t + 0.(\Delta t)] \\ &\quad + p_k(t)[1 - \mu_k(t)\Delta t - 0.(\Delta t)] + 0.(\Delta t) \qquad k \leq k_0 \\ &= \mu_{k+1}(t)p_{k+1}(t).\Delta t + p_k(t) - \mu_k(t)p_k(t) + 0.(\Delta t) \quad k \leq k_0 \\ &= [1 - \mu_{k_0}(t) - 0.(\Delta t)]p_{k_0}(t) \end{aligned}$$

Thus

$$\begin{aligned} &\frac{d}{dt}p_{k_0}(t) = -\mu_{k_0}(t)p_{k_0}(t) \\ &\frac{d}{dt}p_k(t) = \mu_{k+1}(t)p_{k+1}(t) - \mu_k(t)p_k(t). \qquad k = k_0, k_{0-1}, k_{0-2}, \ldots \end{aligned}$$

Initial conditions are $p_{k_0}(0) = 1, \ p_k(0) = 0 \qquad k < k_0$.

5.2 Linear Death Process (Homogeneous)

Here we assume

$$\mu_k(t) = k\mu$$

Then the differential equation become

$$\frac{d}{dt}p_{k_0}(t) = -k_0.\mu p_{k_0}(t) \tag{33}$$

$$\frac{d}{dt}p_k(t) = (k+1)\mu p_{k+1}(t) - k\mu.p_k(t). \qquad k < k_0 \tag{34}$$

Let $G_x(s,t)$ be the p.g.f. of random variable $X(t)$.

$$G_x(s,t) = \sum_{k=0}^{k_0} s^k.p_k(t) \tag{35}$$

Now

$$\begin{aligned}
\frac{\partial}{\partial t}G_x(s,t) &= \sum_{k=0}^{k_0} s^k \frac{d}{dt}p_k(t) \\
&= \sum_{k=0}^{k_0-1}(k+1)\mu.p_{k+1}(t).s^k - \sum_{k=0}^{k_0} k.\mu.p_k(t).s^k \\
&= \sum_{k=0}^{k_0-1} p_{k+1}(t).(k+1)s^k - \mu s\sum_{k=0}^{k_0} p_k(t)k.s^{k-1} \\
&= \frac{\partial}{\partial s}G_x(s,t) - \mu s\frac{\partial}{\partial s}G_x(s,t) \\
&= -\mu(s-1).\frac{\partial}{\partial s}G_x(s,t)
\end{aligned}$$

Thus

$$\frac{\partial}{\partial s}G_x(s,t) + \mu(s-1).\frac{\partial}{\partial s}G_x(s,t) = 0 \tag{36}$$

The above equation is solved by the technique discussed earlier. Then the auxiliary equation are

$$\frac{dt}{1} = \frac{ds}{\mu(s-1)} = \frac{dG_x(s,t)}{0} \tag{37}$$

Solving $\frac{dt}{1} = \frac{dG_x(s,t)}{0}$, we get

$$G_x(s,t) = C \tag{38}$$

Also considering

$$\begin{aligned}
\frac{dt}{1} &= \frac{ds}{\mu(s-1)} \\
\mu.dt &= \frac{ds}{s-1} \\
\mu t &= \log(s-1) + C_2 \\
(s-1) &= C_2.e^{\mu t}
\end{aligned}$$

$$e^{-\mu t}(s-1) = C_2 \tag{39}$$

Thus the general solution is

$$G_x(s,t) = \phi(e^{-\mu t}.(s-1)) \tag{40}$$

For particular solution, set $t = 0$, in $G_x(s,t)$, we get

$$G_x(s,0) = s^{k_0} = \phi(s-1) \tag{41}$$

put $s - 1 = \theta \Rightarrow s = 1 + \theta$

$$\phi(\theta) = (1+\theta)^{k_0} \tag{42}$$

Let $\theta = e^{-\mu t}(s-1)$ from (8) and(10) we get

$$G_x(s,t) = [1 + e^{-\mu t}(1-s)]^{k_0}$$

$$= [1 - e^{\mu t} + se^{-\mu t}]^{k_0} \tag{43}$$

This is of the form $[q + ps]^n$.

Thus, it is the p.g.f. of a binomial distribution with parameters k_0 and $e^{-\mu t}$.

Thus

$$p_k(t) = P[X(t) = k | X(0) = k_0]$$

$$p_k(t) = \binom{k_0}{k} e^{-k\mu t}(1 - e^{-\mu t})^{k_0 - k} \qquad k = 0, 1, 2, \ldots k_0 \tag{44}$$

$$E[X(t)] = k_0.e^{-\mu t} \tag{45}$$

$$V[X(t)] = k_0.e^{-\mu t}(1 - e^{-\mu t}) \tag{46}$$

5.3 Time Dependent Linear Death Process

Here $\mu_k(t) = k.\mu(t)$.

we have the relationship

$$\frac{\partial}{\partial t}G_x(s,t) + (s-1)\mu(t)\frac{\partial}{\partial s}G_x(s,t) = 0$$

and

$$p_k(t) = \binom{k_0}{k} e^{-k\int_0^t \mu(\tau)d\tau}\left[1 - e^{-\int_0^t \mu(\tau)d\tau}\right]^{k_0-k} \qquad k = 0, 1, 2, \dots k_0$$

6 The Generalized Birth and Death Process

In the birth and death process, we make the following assumptions.

$$Given\ X(t) = k$$

1. The probability that a birth occur in the interval$(t, t+\Delta t)$ is $\lambda_k(t)\Delta t + 0.(\Delta t)$.
2. probability that a death occur in the interval$(t, t+\Delta t)$ is $\mu_k(t)\Delta t + 0.(\Delta t)$.
3. probability that more than one change will occur in $(t, t+\Delta t)$ is $0.(\Delta t)$.
4. Hence probability of no change is

$$[1 - \lambda_k(t)\Delta t - \mu_k(t)\Delta t - 0.(\Delta t)]$$

Consequently we can write

$$\begin{aligned} p_k(t, t+\Delta t) = {} & p_k(t)[1 - \lambda_k(t)\Delta t - \mu_k(t)\Delta t] \\ & + p_{k-1}(t)[\lambda_{k-1}(t).\Delta t] + p_{k+1}(t)[\mu_{k+1}(t).\Delta t] \\ & + 0.(\Delta t) \end{aligned}$$

and

$$\begin{aligned} p_0(t, t+\Delta t) = {} & p_0(t)[1 - \lambda_0(t)\Delta t - \mu_0(t)\Delta t] \\ & + p_1(t).\mu_1(t).\Delta t + 0.(\Delta t) \end{aligned}$$

consequently we get

$$\frac{d}{dt}p_k(t) = -[\lambda_k(t) + \mu_k(t)]p_k(t) + \lambda_{k-1}(t).\mu_{k-1}(t) + \mu_{k+1}(t)p_{k+1}(t) \tag{47}$$

and

$$\frac{d}{dt}p_0(t) = -[\lambda_0(t) + \mu_0(t)]p_0(t) + p_1(t).\mu_1(t) \tag{48}$$

Initial conditions are

$$p_{k_0}(0) = 1, p_k(0) = 0 \qquad k \neq k_0 \tag{49}$$

It is quite difficult to obtain general solution from these equations. However some special cases mat be considered.
The case of Linear Growth:
If $\lambda_k(t) = k\lambda$ and $\mu_k(t) = k\mu$ then the process is known as linear birth and death process.

Thus in this case the differential equations become

$$\frac{d}{dt}p_0(t) = \mu p_1(t) \tag{50}$$

$$\frac{d}{dt}p_k(t) = (k-1)\lambda p_{k-1}(t) + (k+1)\mu p_{k+1}(t) - k(\lambda + \mu).p_k(t) \tag{51}$$

Initial condition

$$p_{k_0}(0) = 1, p_k(0) = 0 \qquad k \neq k_0 \tag{52}$$

Let $G_x(s,t)$ be the p.g.f. of random variable $X(t)$.

$$G_x(s,t) = \sum_{k=0}^{\infty} p_k(t).s^k \tag{53}$$

$$\frac{\partial}{\partial t}G_x(s,t) = \sum_{k=0}^{\infty} \frac{d}{dt}p_k(t).s^k$$

$$= \sum_{k=1}^{\infty}(k-1)\lambda p_{k-1}(t).s^k$$

$$+ \sum_{k=0}^{\infty}(k+1)\mu p_{k+1}(t)$$

$$- \sum_{k=1}^{\infty} k.(\lambda+\mu)p_k(t).s^k$$

$$= s^2\lambda \sum_{k=1}^{\infty}(k-1)p_{k-1}(t).s^{k-2}$$

$$+ \mu \sum_{k=0}^{\infty}(k+1)p_{k+1}(t).s^k$$

$$- s(\mu+\lambda)\sum_{k=1}^{\infty} kp_k(t).s^{k-1}$$

$$\frac{\partial}{\partial t}G_x(s,t) = s^2\lambda\frac{\partial}{\partial s}G_x(s,t) + \mu\frac{\partial}{\partial s}G_x(s,t) - s(\mu+\lambda)\frac{\partial}{\partial s}G_x(s,t) \tag{54}$$

so we see that the p.g.f. of $G_x(s,t)$ satisfies the differential equation

$$\frac{\partial}{\partial t}G_x(s,t) + (\lambda s - \mu)(1-s)\frac{\partial}{\partial s}G_x(s,t) = 0 \tag{55}$$

The auxiliary equation are given by

$$\frac{dt}{1} = \frac{ds}{(\lambda s-\mu)(1-s)} = \frac{dG_x(s,t)}{0} \tag{56}$$

Now $\frac{dt}{1} = \frac{dG_x(s,t)}{0}$, gives

$$G_x(s,t) = C \tag{57}$$

Also from $\frac{dt}{1} = \frac{ds}{(\lambda s-\mu)(1-s)}$, using method of partial fractions, we get, from the case $\lambda \neq \mu$ as

$$\frac{dt}{1} = \left[\frac{ds}{(\lambda-\mu)(\lambda s-\mu)} + \frac{1}{(\lambda-\mu)(1-s)}\right]$$

$$(\lambda-\mu)dt = \left[\frac{\lambda}{(\lambda s-\mu)} + \frac{1}{(\lambda-\mu)(1-s)}\right]ds$$

After integration,

$$\begin{aligned}(\lambda-\mu)t &= \log(\lambda s-\mu)-\log(1-s)+C_2\\ &= \log\left[\frac{\lambda s-\mu}{1-s}\right]+C_2\\ \frac{\lambda s-\mu}{1-s} &= e^{(\lambda-\mu)t}.C_3\end{aligned}$$

$$e^{(\lambda-\mu)t}.\frac{\lambda s-\mu}{1-s}=C_4 \tag{58}$$

Thus the general solution is

$$G_x(s,t)=\phi\left[\frac{(1-s)}{(\lambda s-\mu)}.e^{(\lambda-\mu)t}\right] \tag{59}$$

Where ϕ is an arbitrary differential function.

Now using the initial condition that at $t=0$, we see from 59

$$G_x(s,0)=s^{k_0}=\phi\left(\frac{1-s}{\lambda s-\mu}\right) \tag{60}$$

holds at least for all s with $|s|<1$

put $\theta=\frac{1-s}{\lambda s-\mu}$

$$\lambda^{s\theta-\theta\mu}=1-s$$

$$s(1+\lambda\theta)=1+\theta\mu$$

$$s=\frac{(1+\mu\theta)}{(1+\lambda\theta)} \tag{61}$$

so

$$\phi(\theta)=\left(\frac{1+\mu\theta}{1+\lambda\theta}\right) \tag{62}$$

Let $\theta=\left[\frac{(1-s)}{(\lambda s-\mu)}.e^{(\lambda-\mu)t}\right]$, from59 and62 we get

$$g_x(s,t)=\left[\frac{1+\mu.\frac{1-s}{\lambda s-\mu}.e^{(\lambda-\mu)t}}{1+\lambda.\frac{1-s}{\lambda s-\mu}.e^{(\lambda-\mu)t}}\right]^{k_0} \tag{63}$$

$$= \left[\frac{\lambda s - \mu + \mu(1-s).e^{(\lambda-\mu)t}}{\lambda s - \mu + \lambda(1-s).e^{(\lambda-\mu)t}}\right]^{k_0} \tag{64}$$

Let us put

$$\alpha(t) = \mu.\frac{1-e^{(\lambda-\mu)t}}{(\mu-\lambda)e^{(\lambda-\mu)t}}$$

$$\beta(t) = \frac{\lambda}{\mu}.\alpha(t) = \frac{\lambda - \lambda.e^{(\lambda-\mu)t}}{\mu - \lambda.e^{(\lambda-\mu)t}}$$

Then

$$G_x(s,t) = \left[\frac{\alpha(t) + \{1-\alpha(t)-\beta(t)\}\, s}{1-\beta(t).s}\right]^{k_0} \tag{65}$$

now

$$\begin{aligned}
1-\beta(t).s &= 1 - \left(\frac{\lambda - \lambda.e^{(\lambda-\mu)t}}{\mu - \lambda.e^{(\lambda-\mu)t}}\right).s \\
&= \left\{\frac{\mu - \lambda.e^{(\lambda-\mu)t} - \lambda s + \lambda s.e^{(\lambda-\mu)t}}{\mu - \lambda.e^{(\lambda-\mu)t}}\right\} \\
&= \left\{\frac{-[\lambda s.\mu + \lambda.(1-s).e^{(\lambda-\mu)t}]}{\mu - \lambda.e^{(\lambda-\mu)t}}\right\}
\end{aligned}$$

Also

$$\begin{aligned}
\alpha(t) + \{1-\alpha(t)-\beta(t)\}\, s &= \alpha(t) + \left\{1 - \frac{\mu - \mu.e^{(\lambda-\mu)t}}{\mu - \lambda.e^{(\lambda-\mu)t}} - \frac{\lambda - \lambda.e^{(\lambda-\mu)t}}{\mu - \lambda.e^{(\lambda-\mu)t}}\right\}.s \\
&= \alpha(t) + \left\{\frac{\mu - \lambda.e^{(\lambda-\mu)t} - \mu + \mu.e^{(\lambda-\mu)t} - \lambda + \lambda.e^{(\lambda-\mu)t}}{\mu - \lambda.e^{(\lambda-\mu)t}}\right\}.s \\
&= \alpha(t) + \left\{\frac{-\lambda + \mu.e^{(\lambda-\mu)t}}{\mu - \lambda.e^{(\lambda-\mu)t}}\right\}.s \\
&= \left\{\frac{\mu - \mu.e^{(\lambda-\mu)t} - \lambda s + \mu s.e^{(\lambda-\mu)t}}{\mu - \lambda.e^{(\lambda-\mu)t}}\right\} \\
&= -\left\{\frac{(\lambda s - \mu) + \mu(1-s).e^{(\lambda-\mu)t}}{\mu - \lambda.e^{(\lambda-\mu)t}}\right\}
\end{aligned}$$

Thus 65 is $\left[\frac{\alpha(t)+\{1-\alpha(t)-\beta(t)\}s}{1-\beta(t).s}\right]^{k_0}$.

Now the numerator is

$$[\alpha(t) + \{1-\alpha(t)-\beta(t)\}\, s]^{k_0} = \sum_{j=0}^{k_0} \binom{k_0}{j} \{\alpha(t)\}^{k_0-j} \{1-\alpha(t)-\beta(t)\}^{j} .s^{j}$$

The denominator is

$$\begin{aligned}[1-\beta(t).s]^{-k_0} &= \sum_{i=0}^{\infty} \binom{-k_0}{i} (-1)^i [\beta(t)]^i .s^i \\ &= \sum_{i=0}^{\infty} \binom{-k_0+i-1}{i} [\beta(t)]^i .s^i\end{aligned}$$

Now in

$$(a_0 + a_1 s + a_2 s^2 + \ldots a_{k_0} s^{k_0})(b_0 + b_1 s + b_2 s^2 + \ldots b_{k_0} s^{k_0})$$

the coefficient of s^k is $\sum\limits_{j=0}^{Min(k_0,k)} a_j.b_{k-j}$.

Example 8.

$$(a_0 + a_1 s + a_2 s^2 + a_3 s^3 + \ldots)(b_0 + b_1 s + b_2 s^2 + b_3 s^3 + \ldots)$$

the coefficient of s^2 *is*

$$a_0.b_2 + a_1.b_1 + a_2.b_2$$

coefficient of s^5 *is*

$$a_0.b_5 + a_1.b_4 + a_2.b_3 + a_3.b_2$$

Therefore

$$\phi_k(t) = \sum_{j=0}^{Min(k_0,k)} \binom{k_0}{j} [\alpha(t)]^{k_0-j} [1-\alpha(t)-\beta(t)]^j \binom{k_0+k-j-1}{k-j} [\beta(t)]^{k-j}$$

$$p_k(t) = \sum_{j=0}^{Min(k_0,k)} \binom{k_0}{j} \binom{k_0+k-j-1}{k-j} [\alpha(t)]^{k_0-j} [\beta(t)]^{k-j} [1-\alpha(t)-\beta(t)]^j$$

and $p_0(t) = [\alpha(t)]^{k_0}$.
Now $E[X(t)] = [\frac{\partial}{\partial t}.G_x(s,t)]_{s=1}$

$$E[X(t)] = \left[k_0 \left\{ \frac{\alpha(t) + \{1 - \alpha(t) - \beta(t)\} s}{1 - \beta(t).s} \right\}^{k_0 - 1} \right.$$
$$\left. . \frac{\{1 - \beta(t).s\} \{1 - \alpha(t) - \beta(t)\} - \{\alpha(t) + \{1 - \alpha(t) - \beta(t)\} s\} \{-\beta(t)\}}{\{1 - \beta(t).s\}^2} \right]_{s=1}$$
$$= k_0 \left\{ \frac{1 - \beta(t)}{1 - \beta(t)} \right\}^{k_0 - 1} . \left\{ \frac{\{1 - \beta(t)\} \{1 - \alpha(t) - \beta(t)\} \{1 - \beta(t)\} . \beta(t)}{\{1 - \beta(t)\}^2} \right\}$$
$$= k_0 \left\{ \frac{1 - \alpha(t) - \beta(t) + \beta(t)}{1 - \beta(t)} \right\}$$

$$E[X(t)] = k_0 \left[\frac{1 - \alpha(t)}{1 - \beta(t)} \right]$$

$$E[X(t)] = k_0 \left\{ \frac{1 - \mu . \frac{1 - e^{(\lambda - \mu).t}}{\mu - \lambda . e^{(\lambda - \mu).t}}}{1 - \lambda . \frac{1 - e^{(\lambda - \mu).t}}{\mu - \lambda . e^{(\lambda - \mu).t}}} \right\}$$
$$= k_0 \left\{ \frac{\mu - \lambda . e^{(\lambda - \mu).t} - \mu + \mu . e^{(\lambda - \mu).t}}{\mu - \lambda . e^{(\lambda - \mu).t} - \lambda + \lambda . e^{(\lambda - \mu).t}} \right\}$$
$$= k_0 \left\{ \frac{e^{(\lambda - \mu).t} . (\mu - \lambda}{(\mu - \lambda)} \right\}$$
$$= k_0 \left\{ e^{(\lambda - \mu).t} \right\}$$

$$V[X(t)] = \left[\frac{\partial^2}{\partial s^2} G_x(s,t) \right]_{s=1} + \left[\frac{\partial}{\partial s} G_x(s,t) \right]_{s=1} - \left[\frac{\partial}{\partial s} G_x(s,t) | s = 1 \right]^2$$

It can be shown that

$$V[X(t)] = \frac{k_0 [1 - \alpha(t)] . [\alpha(t) + \beta(t)]}{[1 - \beta(t)]^2}$$

Thus

$$V[X(t)] = k_0 . e^{(\lambda - \mu)t} . \left\{ \frac{\alpha(t) + \beta(t)}{1 - \beta(t)} \right\}$$

Since $\left[\frac{1-\alpha(t)}{1-\beta(t)}\right] = e^{(\lambda-\mu)t}$

$$= k_0.e^{(\lambda-\mu)t}.\left\{\frac{\mu-\mu.e^{(\lambda-\mu)t}+\lambda-\lambda.e^{(\lambda-\mu)t}}{\mu-\lambda.e^{(\lambda-\mu)t}+\lambda.e^{(\lambda-\mu)t}-\lambda}\right\}$$
$$= k_0.e^{(\lambda-\mu)t}.\left\{\frac{(\mu+\lambda)[1-e^{(\lambda-\mu)t}]}{(\lambda-\mu)}\right\}$$
$$= k_0.e^{(\lambda-\mu)t}.\left\{\frac{(\mu+\lambda)[e^{(\lambda-\mu)t}-1]}{(\lambda-\mu)}\right\}$$

Thus

$$E[X(t)] = k_0.e^{\lambda-\mu}t$$
$$V[X(t)] = k_0.e^{(\lambda-\mu)t}.\left\{\frac{(\mu+\lambda)[e^{(\lambda-\mu)t}-1]}{(\lambda-\mu)}\right\}$$

Now if $\lambda = \mu$, then

$$E[X(t)] = k_0$$

Also for finding $V[X(t)]$ in case of $\lambda = \mu$,we proceed as

$$V[X(t)] = \left[\frac{k_0.\left\{1+(\lambda-\mu)t\frac{(\lambda-\mu)^2.t^2}{2!}+\dots\right\}(\mu+\lambda)\left\{(\lambda-\mu)t+\frac{(\lambda-\mu).t^2}{2!}+\dots\right\}}{(\lambda-\mu)}\right]$$
$$= k_0.\left[1+(\lambda-\mu)t\frac{(\lambda-\mu)^2.t^2}{2!}+\dots\right](\mu+\lambda)\left[(\lambda-\mu)t+\frac{(\lambda-\mu).t^2}{2!}+\dots\right]$$

if $\lambda \to \mu \Rightarrow \lambda = \mu$

$$= k_0.2\lambda.t$$
$$= 2.k_0.\lambda t$$

6.1 Limiting Behavior of Birth and Death Process

$$G_x(s,t) = \left[\frac{\lambda s-\mu+\mu(1-s).e^{(\lambda-\mu)t}}{\lambda s-\mu+\lambda(1-s).e^{(\lambda-\mu)t}}\right]^{k_0}$$

i.e. $\lim_{t\to\infty} p_0(t)$

$\left\{p_0(t) = \{\alpha(t)\}^{k_0}\right\}$

Thus

$$\lim_{t\to\infty} p_0(t) = \left[\frac{\mu}{\lambda}\right]^{k_0} \qquad \text{if} \quad \mu < \lambda$$
$$= 1 \qquad \text{if} \quad \mu \geq \lambda$$

See page 125, Medhi.
Further as $t \to \infty$

$$E[X(t)] = 0 \qquad \text{if} \quad \mu > \lambda$$
$$= k_0 \qquad \text{if} \quad \mu = \lambda$$
$$= \infty \qquad \text{if} \quad \mu < \lambda$$

Also

$$V[X(t)] = 0 \qquad \text{if} \quad \mu > \lambda$$
$$= \infty \qquad \text{if} \quad \mu < \lambda$$

6.2 Effect of Migration on Birth and Death Process:

For the migration there are two situations:

1. Emigration
2. Immigration

In emigration individuals go out from the population consequently the size of the population decreases due to emigration.

In immigration individuals come from out side to the population.Consequently the population increases due to immigration.

It can be thought that the probability that a person will emigrate from the population in the time interval $(t, t + \Delta t)$ may depend on the size of population. Thus the probability may be represented as

$$\mu_k^1(t).\Delta t + 0.(\Delta t)$$

Hence it can be adjusted force of mortality and we can write

$$\mu_k^{11}(t) = \mu_k(t) + \mu_k^1(t)$$

Thus in the birth and death process,the force of mortality $\mu_k(t)$ can be replaced by $\mu_k^{11}(t)$.

However in the case of immigration it may not dependent on the size of population and consequently the effect of immigration may not be adjusted with $\lambda_k(t)$.

6.3 The Effect of Immigration

In many biological populations, some form of migration is an essential characteristic. consequently we introduce this phenomenon in the birth and death process. So for as the emigration is concerned, it is clear that this can be allowed by a suitable adjustment of the death rate since for deaths and emigration we can take the chance of a single loss in the time interval $(t, t+\Delta t)$ as proportional to $X(t)$ where $X(t)$ denote the size of the population at time t.

With immigration on the other hand the situation is different. For the simplest reasonable assumption about immigration is that it occurs randomly without being affected by the size of population at time$t$.

It can be considered as a Poisson process independent of the population size.

We will consider a time homogeneous birth and death process with a random accession of immigrants (with Poisson Process) with immigration rate ν.

We denote

$$\begin{aligned} X(t) &= k \qquad \text{size of population at time } t \\ X(0) &= k_0 \qquad \text{Initial population is } k_0 \\ p_t(t) &= P[X(t) = k | X(0) = k_0] \end{aligned}$$

The basic assumption underlying in a time homogeneous linear birth and death process with the effect of immigration:

$$\text{Given}\, X(t) = k$$

1. The probability that there will be an increase in the population either due to birth or due to immigration in the interval$(t, t+\Delta t)$ is

$$k.\lambda.\Delta t + \nu.\Delta t + 0.(\Delta t)$$

2. The probability that the size of population will decrease by one unit in the interval$(t, t+\Delta t)$ is

$$k.\mu.\Delta t + 0.(\Delta t)$$

3. Probability that more than one change will occur in the time interval$(t, t+\Delta t)$ is $0.(\Delta t)$.

4. Probability of no change in the interval $(t, t+\Delta t)$ is

$$1 - (k.\lambda + \nu).(\Delta t) - k.\mu.\Delta t - 0.(\Delta t)$$

$$\begin{aligned} p_(t+\Delta t) = p_k(t)[1 - (k.\lambda + \nu).(\Delta t) - k.\mu.\Delta t] \\ + p_{k+1}(t)[(k+1)\mu.\Delta t] \\ + p_{k-1}(t)[(k-1)\lambda\Delta t + \nu.\Delta t] + 0.(\Delta t) \end{aligned}$$

$$p_(t+\Delta t) = p_0(t)[1 - \nu.(\Delta t)] + p_1(t)[\mu.(\Delta t)] + 0.(\Delta t)$$

By transferring $p_k(t)$ in LHS, dividing by Δt and taking the limit as $\Delta t \to 0$ we have

$$\begin{aligned} \frac{d}{dt}p_k(t) = -k(\lambda + \mu)p_k(t) + (k+1)\mu p_{k+1}(t) \\ + (k-1)\lambda p_{k-1}(t) + \nu p_{k-1}(t) - \nu p_k(t) \end{aligned}$$

$$\frac{d}{dt}p_0(t) = \mu p_1(t) - \nu p_0(t)$$

with the initial condition
$p_{k_0}(0) = 1 \qquad p_k(0) = 0 \qquad k \neq k_0$
Let $G_x(s,t)$be the p.g.f. of random variable $X(t)$

$$\begin{aligned} G_x(s,t) &= \sum_{k=0}^{\infty} p_k(t).s^k \\ \frac{\partial}{\partial t}G_x(s,t) &= \sum_{k=0}^{\infty} \frac{d}{dt}p_k(t).s^k \\ G_x(s,t) &= \sum_{k=0}^{\infty} \big[-k(\lambda+\mu)p_k(t) + (k+1)\mu p_{k+1}(t) \\ &\quad +(k-1)\lambda p_{k-1}(t) + \nu p_{k-1}(t) - \nu p_k(t) \big].s^k \end{aligned}$$

$$= \sum_{k=0}^{\infty} -k(\lambda+\mu)p_k(t).s^k + \sum_{k=0}^{\infty}(k+1)\mu p_{k+1}(t).s^k$$
$$+ \sum_{k=1}^{\infty}(k-1)\lambda p_{k-1}(t).s^k + \sum_{k=1}^{\infty} \nu p_{k-1}(t).s^k - \sum_{k=0}^{\infty} p_k(t).s^k$$
$$= -\sum_{k=0}^{\infty}(\lambda+\mu).s.k.p_k(t).s^{k-1} + \sum_{k=0}^{\infty} \mu p_{k+1}(t).(k+1).s^k$$
$$+ \sum_{k=1}^{\infty} \lambda.s^2_{k-1}p_{k-1}(t).s^{k-2} + \sum_{k=1}^{\infty} \nu s p_{k-1}(t).s^{k-1} - \sum_{k=0}^{\infty} .\nu.p_k(t).s^k$$
$$= -(\lambda+\mu).s\frac{\partial}{\partial t}G_x(s,t) + \mu\frac{\partial}{\partial t}G_x(s,t) + \lambda.s^2\frac{\partial}{\partial t}G_x(s,t)$$
$$+ \nu.sG_x(s,t) - \nu G_x(s,t)$$

Thus we see that $G_x(s,t)$ satisfies the differential equation.

$$\frac{\partial}{\partial t}G_x(s,t) + (\lambda s - \mu)(1-s)\frac{\partial}{\partial t}G_x(s,t) = \nu.(s-1)G_x(s,t)$$

so the auxiliary equation for solving the equation will be

$$\frac{dt}{1} = \frac{ds}{(\lambda s - \mu)(1-s)} = \frac{dG_x(s,t)}{\nu.(s-1)G_x(s,t)}$$

now considering

$$\frac{dt}{1} = \frac{ds}{(\lambda s - \mu)(1-s)}$$

we get

$$\frac{dt}{1} = ds\left[\frac{\lambda}{(\lambda-\mu)(\lambda s-\mu)} + \frac{1}{(\lambda-\mu)(1-s)}\right]$$

so we have

$$
\begin{aligned}
(\lambda-\mu).t+C &= \log(\lambda s-\mu)-\log(1-s)\\
&= \log\left(\frac{\lambda s-\mu}{1-s}\right)\\
\left(\frac{\lambda s-\mu}{1-s}\right) &= e^{(\lambda-\mu)t+C}\\
\left(\frac{1-s}{\lambda s-\mu}\right)e^{(\lambda-\mu)t} &= C_1
\end{aligned}
$$

Now considering

$$
\begin{aligned}
\frac{ds}{(\lambda s-\mu)(1-s)} &= \frac{dG_x(s,t)}{\nu.(s-1)G_x(s,t)}\\
\frac{ds}{(\lambda s-\mu)} &= -\frac{dG_x(s,t)}{\nu.G_x(s,t)}\\
or\ \frac{\log(\lambda s-\mu)}{\lambda} &= -\frac{1}{\nu}.\log G_x(s,t)+C_2\\
or\ \frac{\nu}{\lambda}\log(\lambda s-\mu)+\log G_x(s,t) &= C_3\\
or\ \log\left\{(\lambda s-\mu)^{\frac{\nu}{\lambda}}.G_x(s,t)\right\} &= C_3\\
(\lambda s-\mu)^{\frac{\nu}{\lambda}}.G_x(s,t) &= C_4
\end{aligned}
$$

Consequently the most general solution is given as

$$
(\lambda s-\mu)^{\frac{\nu}{\lambda}}.G_x(s,t)=\psi\left[\left(\frac{1-s}{\lambda s-\mu}\right)e^{(\lambda-\mu)t}\right]
$$

where ψ is an arbitrary function which is obtained from the initial condition i.e.

$$
G_x(s,0)=s^{k_0}
$$

i.e.

$$
(\lambda s-\mu)^{\frac{\nu}{\lambda}}.s^{k_0}=\psi\left[\frac{1-s}{\lambda s-\mu}\right]
$$

Let us put

$$
\begin{aligned}
\frac{1-s}{\lambda s-\mu} &= \theta \\
1-s &= (\lambda s-\mu).\theta = \lambda.s.\theta - \mu.\theta \\
1+\mu\theta &= s+\lambda.s.\theta \\
&= (1+\lambda\theta).s \\
or\ s &= \frac{1+\mu\theta}{1+\lambda\theta} \\
so\ \psi(\theta) &= \left\{\lambda.\left(\frac{1+\mu\theta}{1+\lambda\theta}-\mu\right)\right\}^{\frac{\nu}{\lambda}}.\left(\frac{1+\mu\theta}{1+\lambda\theta}\right)^{k_0} \\
&= \left\{\frac{\lambda+\lambda\mu\theta-\mu-\mu\lambda\theta}{1+\lambda\theta}\right\}^{\frac{\nu}{\lambda}}.\left(\frac{1+\mu\theta}{1+\lambda\theta}\right)^{k_0} \\
&= \left(\frac{\lambda-\mu}{1+\lambda\theta}\right)^{\frac{\nu}{\lambda}}.\left(\frac{1+\mu\theta}{1+\lambda\theta}\right)^{k_0}
\end{aligned}
$$

Thus the general solution is

$$
\begin{aligned}
(\lambda s-\mu)^{\frac{\nu}{\lambda}}.G_x(s,t) &= \left\{\frac{\lambda-\mu}{1+\lambda.\frac{1-s}{\lambda s-\mu}.e^{(\lambda-\mu)t}}\right\}^{\frac{\nu}{\lambda}}.\left\{\frac{1+\mu.\frac{1-s}{\lambda s-\mu}.e^{(\lambda-\mu)t}}{1+\lambda.\frac{1-s}{\lambda s-\mu}.e^{(\lambda-\mu)t}}\right\}^{k_0} \\
(\lambda s-\mu)^{\frac{\nu}{\lambda}}.G_x(s,t) &= \left\{\frac{(\lambda-\mu)^{\frac{\nu}{\lambda}}\left[(\lambda s-\mu)+\mu(1-s).e^{(\lambda-\mu)t}\right]^{k_0}.(\lambda s-\mu)^{\frac{\nu}{\lambda}}}{\left[(\lambda s-\mu)+\lambda(1-s).e^{(\lambda-\mu)t}\right]^{\frac{\nu}{\lambda}+k_0}}\right\} \\
G_x(s,t) &= \left\{\frac{(\lambda-\mu)^{\frac{\nu}{\lambda}}\left[(\lambda s-\mu)+(1-s)\mu.e^{(\lambda-\mu)t}\right]^{k_0}}{\left[(\lambda s-\mu)+\lambda(1-s).e^{(\lambda-\mu)t}\right]^{k_0+\frac{\nu}{\lambda}}}\right\}
\end{aligned}
$$

Special Case When $k_0 = 0$

$$
\begin{aligned}
G_x(s,t) &= \left\{ \frac{(\lambda-\mu)^{\frac{\nu}{\lambda}}}{[(\lambda s-\mu)+(1-s)\lambda.e^{(\lambda-\mu)t}]^{\frac{\nu}{\lambda}}} \right\} \\
&= (\lambda-\mu)^{\frac{\nu}{\lambda}} \left\{ (\lambda s-\mu)+(1-s)\lambda.e^{(\lambda-\mu)t} \right\}^{-\frac{\nu}{\lambda}} \\
&= (\lambda-\mu)^{\frac{\nu}{\lambda}} \left\{ \lambda.e^{(\lambda-\mu)t}-\mu-\lambda(e^{(\lambda-\mu)t}-1).s \right\}^{-\frac{\nu}{\lambda}} \\
&= (\lambda-\mu)^{\frac{\nu}{\lambda}} \left[\left(\lambda.e^{(\lambda-\mu)t}\right) \left\{ 1-\frac{\lambda(e^{(\lambda-\mu)t}-1).s}{\lambda.e^{(\lambda-\mu)t}-\mu} \right\}^{-\frac{\nu}{\lambda}} \right] \\
&= \left[\frac{\lambda-\mu}{\lambda.e^{(\lambda-\mu)t}-\mu} \right]^{\frac{\nu}{\lambda}} \left[1-\frac{\lambda(e^{(\lambda-\mu)t}-1).s}{\lambda.e^{(\lambda-\mu)t}-\mu} \right]^{-\frac{\nu}{\lambda}}
\end{aligned}
$$

Which is the p.g.f. of a negative binomial distribution with $r = \frac{\nu}{\lambda}$ and $\left[p = \frac{\lambda-\mu}{\lambda.e^{(\lambda-\mu)t}-\mu}\right]$.

Since the p.g.f. of a negative binomial distribution is $\left[\frac{p}{1-qs}\right]^r$ or $[p^r.(1-qs)^{-r}]$.
obviously if $p = \frac{\lambda-\mu}{\lambda.e^{(\lambda-\mu)t}-\mu}$
Then

$$
\begin{aligned}
q &= \left\{ 1-\frac{\lambda-\mu}{\lambda.e^{(\lambda-\mu)t}-\mu} \right\} \\
&= \left\{ \frac{\lambda.e^{(\lambda-\mu)t}-\mu-\lambda+\mu}{\lambda.e^{(\lambda-\mu)t}-\mu} \right\} \\
&= \left\{ \frac{\lambda\left\{e^{(\lambda-\mu)t}-1\right\}}{\lambda.e^{(\lambda-\mu)t}-\mu} \right\}
\end{aligned}
$$

$$
\begin{aligned}
Mean &= r.\frac{q}{p} \\
&= \left[\frac{\nu}{\lambda}.\frac{\lambda\left\{e^{(\lambda-\mu)t}-1\right\}}{\lambda.e^{(\lambda-\mu)t}-\mu}.\frac{\lambda.e^{(\lambda-\mu)t}-\mu}{\lambda-\mu}\right] \\
&= \frac{\nu}{(\lambda-\mu)}.\left\{e^{(\lambda-\mu)t}-1\right\} \\
and\ variance &= r.\frac{q}{p^2} \\
&= \frac{\nu}{(\lambda-\mu)}.\left\{e^{(\lambda-\mu)t}-1\right\}.\frac{\lambda.e^{(\lambda-\mu)t}-\mu}{\lambda-\mu} \\
&= \frac{\nu}{(\lambda-\mu)^2}.\left\{e^{(\lambda-\mu)t}-1\right\}\left\{\lambda.e^{(\lambda-\mu)t}-\mu\right\} \\
Mean &= \frac{\nu}{(\lambda-\mu)}.\left\{e^{(\lambda-\mu)t}-1\right\} \\
&= \frac{\nu}{(\lambda-\mu)}\left\{1+(\lambda-\mu)t+\frac{(\lambda-\mu)^2t^2}{2!}+\frac{(\lambda-\mu)^3t^3}{3!}+\ldots-1\right\} \\
&= \nu\left\{t+\frac{(\lambda-\mu)t^2}{2!}+\frac{(\lambda-\mu)^2t^3}{3!}+\ldots\right\}
\end{aligned}
$$

$$\text{if} \quad \mu=\lambda \quad \text{then}$$

$$\text{Mean} = \nu t$$

6.4 Limiting Size of Population When $t \to \infty$

$$
\begin{aligned}
E[X(t)] &= \frac{\nu}{(\lambda-\mu)}.\left\{e^{(\lambda-\mu)t}-1\right\} \\
\lim_{t\to\infty} E[X(t)] &= \frac{\nu}{(\lambda-\mu)}[0-1] \qquad \text{if} \quad \lambda<\mu \\
&= \frac{\nu}{(\lambda-\mu)} \qquad \text{if} \quad \lambda<\mu \\
\lim_{t\to\infty} E[X(t)] &= \infty \qquad \text{if} \quad \lambda>\mu
\end{aligned}
$$

Also for $\lambda=\mu \quad E[X(t)] = \nu t \to \infty$ as $\quad t\to\infty$

Further

$$
\begin{aligned}
V[X(t)] &= \frac{\nu}{(\lambda-\mu)^2}.\left\{e^{(\lambda-\mu)t}-1\right\}\left\{\lambda.e^{(\lambda-\mu)t}-\mu\right\} \\
\lim_{t\to\infty} V[X(t)] &= \frac{\nu(-1)(-\mu)}{(\lambda-\mu)^2} \qquad \text{if} \quad \mu > \lambda \\
&= \frac{\nu.\mu}{(\lambda-\mu)^2} \\
&= \infty \qquad \text{if} \quad \lambda > \mu
\end{aligned}
$$

Also when $\lambda = \mu$

$$
\begin{aligned}
V[X(t)] &= \nu\left\{t+\frac{(\lambda-\mu)t^2}{2!}+\frac{(\lambda-\mu)^2t^3}{3!}+\ldots\right\}\lambda\left\{t+\frac{(\lambda-\mu)t^2}{2!}+\frac{(\lambda-\mu)^2t^3}{3!}+\ldots\right\} \\
&= \nu.[t][\lambda t]
\end{aligned}
$$

7 Random Walk Model

Suppose a particle is moving on the axis of X. Let at time $t = 0$, the position of the particle is at Z_0 and letZ_1 be the jump of the particle at first trial then at time $t = 1$,the particle will be at $Z_0 + Z_1$. i.e.

$$X_1 = Z_0 + Z_1$$

Similarly $Z_2, Z_3, \ldots$are also defined.

Here obviously$Z_0, Z_1, Z_2, \ldots$ are all random variables and

$$
\begin{aligned}
X_0 &= Z_0 \\
X_1 &= Z_0 + Z_1 \\
X_2 &= Z_0 + Z_1 + Z_2 \\
&\vdots \\
X_n &= Z_0 + Z_1 + Z_2 + \ldots + Z_n \\
&= X_{n-1} + Z_n
\end{aligned}
$$

Definition 8. *Let$Z_i = 0, 1, 2, \ldots$ be a sequence of independent discrete random variables and let*

$$X_n = Z_0 + Z_1 + Z_2 + \ldots + Z_n$$

then $X_n$is a Markov Chain and the chain is called random walk model.

The transition probability for random walk model are given as

$$\begin{aligned} p_{ij}^{(m)} &= p[X_{m+1} = j | X_m = i] \\ &= p[Z_{m+1} = j - i] \end{aligned}$$

If random variables $Z_i s$ are identical then this Markov Chain becomes homogeneous Markov Chain and in that case

$$p_{ij} = p_{j-i} \qquad where\ p_{j-i} = p[Z = j - i]$$

7.1 Classical Random Walk or Simple Random Walk

If$Z_i s$ are independent and identically distributed random variables such that

$$p[Z_i = 1] = p \qquad p(Z_i = 1) = q \qquad p = q = 1$$

$$\begin{aligned} \Rightarrow p_{i,i+1} &= p \\ p_{i,i-1} &= q \\ p_{ij} &= o \qquad forj \neq i+1 \end{aligned}$$

Then $\{X_n\} = Z_0 + Z_1 + Z_2 + \ldots + Z_n$ with initially $X_0 = Z_0$is called simple random walk.

A simple random walk is said to be symmetric if the transition probabilities are equal for left as well as right moment.i.e.$p = q = 1/2$

we say that simple random walk has a drift towards right if $p > 1/2$ and a drift towards left if $p < 1/2$.

7.2 Unrestricted Simple Random Walk

Consider the moments of the particle are X axis such that

$$\begin{aligned} p_{i,i+1} &= p \\ p_{i,i-1} &= q \qquad i \ldots -3, -2, -1, 0, 1 \\ p_{ij} &= o \qquad forj \neq i-1, i+1 \end{aligned}$$

which gives the transition probability matrix of the form

$$p = \begin{pmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \cdots & \cdots & q & 0 & p & \cdots & \cdots & \cdots \\ \cdots & \cdots & 0 & q & 0 & p & \cdots & \cdots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{pmatrix}$$

Let $X_n$denotes the position of the particle at time $t = n$ then

$$X_n = Z_0 + Z_1 + Z_2 + \ldots + Z_n$$

Thus we get that if there is no restriction on the movement of the particle in the manner described above then the movement of the particle is known as unrestricted simple random walk.

We now study the behavior of this phenomenon.

Let initially the position of the particle on the X axis is at $x = i$ such that

$$p_{i,i+1} = p \qquad p_{i,i-1} = q$$

Let Z_n denote the jump of the particle at n^{th} trial then

$p_{ij}^{(n)}$ = the probability that the position of the particle will be at j in n trials starting from i.

Then

$$p_{ij}^{(n)} = p[Z_1 + Z_2 + \ldots + Z_n = j - i]$$

In order that the sum of n jumps is $j - i$, these should be $\frac{n+j-i}{2}$ position jumps and $\frac{n-j+i}{2}$ negative jumps and its probability is

$$p_{ij}^{(n)} = \begin{cases} \binom{n}{\frac{n+j-i}{2}}.p^{\frac{n+j-i}{2}}.q^{\frac{n-j+i}{2}} & \text{if } \frac{n+j-i}{2} \text{ is integer} \\ 0 & \text{Otherwise} \end{cases}$$

Also since the position of the particle at a particular moment depends only upon the position of the particle as previous moment, so an unrestricted random walk is obviously a Markov Chain.

Also for $0 < p < 1$, the chain is irreducible because any state can be reached from any other state.

In particular

$$p_{00}^{(n)} = \begin{cases} \binom{n}{\frac{n}{2}}.p^{\frac{n}{2}}.q^{\frac{n}{2}} & \text{if n is integer} \\ 0 & \text{Otherwise} \end{cases}$$

Thus we see that state $'0'$ has period $'2'$. Since the chain is irreducible, so all the states has period 2.

Thus

$$p_{00}^{(2n)} = \binom{2n}{n}.p^n.q^n$$
$$= \frac{(2n)!}{n!\,n!}.p^n.q^n$$

using sterling formula for factorials as

$$n! = \sqrt{2\pi}\, e^{-n}.n^{n+\frac{1}{2}}$$

We get

$$\begin{aligned}
p_{00}^{(2n)} &= \frac{(2\pi)^{\frac{1}{2}}.e^{-2n}.(2n)^{2n+\frac{1}{2}}.p^n.q^n}{(2\pi)^{\frac{1}{2}}.e^{-n}.n^{n+\frac{1}{2}}(2\pi)^{\frac{1}{2}}.e^{-n}.n^{n+\frac{1}{2}}} \\
&= \frac{2^{2n+\frac{1}{2}}.n^{2n+\frac{1}{2}}.p^n.q^n}{n^{2n+1}.2^{\frac{1}{2}}.\pi^{\frac{1}{2}}} \\
&= \frac{2^{2n}.p^n.q^n}{\pi^{\frac{1}{2}}n^{\frac{1}{2}}} \\
&= \frac{(2^2)^n.(p\,q)^n}{(\pi.n)^{\frac{1}{2}}} \\
&= \frac{(4\,p\,q)^n}{(n\pi)^{\frac{1}{2}}}
\end{aligned}$$

If $p \neq q$, then $4\,p\,q < 1$ and so on. And consequently the series $\sum_{n=0}^{\infty} \frac{(4\,p\,q)^n}{(n\pi)^{\frac{1}{2}}}$ is convergent as $p_{00}^{(0)} = 1$ and $\sum_{n=0}^{\infty}(4\,p\,q)^n$ is convergent. This shows that state(0) is transient.

Also for

$$\begin{aligned}
p &= q \\
p_{00}^{(2n)} &= \frac{1}{(n\pi)^{\frac{1}{2}}} \\
\text{Hence} \qquad \sum_{n=0}^{\infty} p_{00}^{(2n)} &= \sum_{n=0}^{\infty} \frac{1}{(n\pi)^{\frac{1}{2}}} \\
&= \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{1}{\sqrt{n}} \qquad \Rightarrow \sqrt{n} \to \infty \quad \text{as} \quad n \to \infty
\end{aligned}$$

is a divergent series showing that the state $'0'$ will be persistent if $p = q = \frac{1}{2}$.

Again $\lim_{n\to\infty} p_{00}^{(2n)} = 0$ which shows that $'0'$ is persistent null for $p = q = \frac{1}{2}$.

Thus we conclude that

1. All the states are transient if $p \neq q$.
2. All the states are persistent null if$p = q = \frac{1}{2}$.

7.3 Simple Random Walk With One Absorbing Barrier

Consider a random walk on the X axis where there is an absorbing barrier at $x = a$.

Let

$$X_n = \begin{cases} X_{n-1} + Z_n & \text{if} X_{n-1} = a+1, a+2 \\ a & \text{if} X_{n-1} = a \end{cases}$$

From each of the intermediate state $i = a+1, a+2, + \ldots$ transition are possible to the neighboring states. i.e.

$$\begin{aligned} p_{i,i+1} &= p \qquad \text{if} \quad i = a+1, a+2 \ldots \\ p_{i,i-1} &= q \\ p_{a\,a} &= o \end{aligned}$$

Hence from each of the intermediate states, it can be reached to the point $'a'$ but from $'a'$, no other state can be reached. So one state is persistent and all the remaining states are transient. (Because return from a particular state to that state is not certain.)

7.4 Random walk with two absorbing berries

Consider the movement of the particles on X-axis between the points 'a' and '$b'(a > b)$ From one of the intermediate points between a and b ,the particle can move one unit to the right with probability p and one unit to the left with probability q.But when even the particle reaches the points '$a' or$'b',the motion stops and particle remains there. If X_n denote the position of the particle at time $t = n$, the sequence of random variables can be characterized as

$$X_n = \begin{cases} X_{n-1} + Z_n, & \text{if } a < X_{n-1} < b; \\ a, & \text{if } X_{n-1} = a; \\ b, & \text{if } X_{n-1} = b. \end{cases}$$

Where Z_i's are i.i.d. random variable with

$$P(Z_i = 1) = p \qquad \text{and} \qquad P(Z_i = -1) = q$$

obviously the stochastic process is a Markov Chain with transition probabilities.

$$\begin{aligned} p_{ii+1} &= p \qquad \text{where } a+1 \leq i \leq b-1 \\ p_{ii-1} &= q \\ p_{aa} &= 1 \\ p_{bb} &= 1 \end{aligned}$$

In the Markov chain there are two absorbing barriers 'a' and 'b' so that these states are persistent and remaining all states are transient.

7.5 Simple Random walk with One Reflecting barriers:

Let us consider the movement of a particle on the X-axis. The particle moves one unit right with probability p and one unit left with probability q.Let X_n denote the position of the particle at t=n and Z_n is the jump of the particle at time $t = n$ then

$$X_n = \begin{cases} X_{n-1} + Z_n, & \text{if } 0 < X_{n-1} < \infty; \\ X_{n-1} + Z', & \text{if } X_{n-1} = 0. \end{cases}$$

where Z_i 's are i.i.d. random variables and each takes the value 1 with probability p and -1 with probability q. Z' is such that it takes value 1 with probability p and 0 with probability q obviously. X_n depend upon X_{n-1} and hence it is a Markov chain and the transition probabilities for the chain are given as

$$\left.\begin{aligned} p_{ii+1} &= p \\ p_{ii-1} &= q \end{aligned}\right\} i = 1, 2, 3, \ldots$$

$$\begin{aligned} p_{00} &= q \\ p_{01} &= p \end{aligned}$$

Thus we see that whenever the particle is at $x = 0$, it can go for the left also but it is immediately reflected back to $x = -1$.Hence -1 is a reflecting barrier.

The state space is the

$$I : \{0, 1, 2, 3, \ldots\}$$

and the transition probability matrix is

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \ldots \\ \ldots \\ \ldots \\ \ldots \\ \ldots \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & 3 & \ldots & \ldots & \ldots \end{array} \\ \begin{pmatrix} q & p & 0 & \ldots & \ldots & \ldots & \ldots \\ q & 0 & p & 0 & \ldots & \ldots & \ldots \\ 0 & q & 0 & p & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix} \end{array}$$

Let us consider the system of equations

$$\begin{aligned} y_i &= \sum_{j=1}^{\infty} p_{ij} y_j \qquad i \geq 1 \\ y_i &= p y_{i+1} + q y_{i-1} \qquad \text{i} > 1 \\ y_1 &= p y_2 \end{aligned}$$

Since

$$\begin{aligned} (p+q)\, y_i &= p y_{i+1} + q y_{i-1} \\ p\,(y_{i+1} - y_i) &= q\,(y_i - y_{i-1}) \\ (y_{i+1} - y_i) &= \tfrac{q}{p}\,(y_i - y_{i-1}) \\ \Rightarrow \qquad (y_2 - y_1) &= \tfrac{q}{p} y_1 \\ (y_3 - y_2) &= \tfrac{q}{p}\,(y_2 - y_1) \\ &= \tfrac{q}{p}\tfrac{q}{p} y_1 \\ (y_4 - y_3) &= \tfrac{q}{p}\,(y_3 - y_2) \\ &= \tfrac{q}{p}\tfrac{q^2}{p^2} y_1 \\ (y_i - y_{i-1}) &= \left(\tfrac{q}{p}\right)^{i-1} y_1 \end{aligned}$$

Adding all these equations we get

$$
\begin{aligned}
(y_i - y_1) &= \left[\frac{q}{p} + \frac{q^2}{p^2} + \frac{q^3}{p^3} + \ldots + \frac{q^{i-1}}{p^{i-1}}\right] y_1 \\
y_i &= \left[1 + \frac{q}{p} + \left(\frac{q}{p}\right)^2 + \left(\frac{q}{p}\right)^3 + \ldots + \left(\frac{q}{p}\right)^{i-1}\right] y_1 \\
y_i &= \frac{1-\left(\frac{q}{p}\right)^i}{1-\frac{q}{p}} \qquad \ldots\ldots\ldots (A)
\end{aligned}
$$

From (A), we see that y's are bounded and have non zero solution if $p > q$.

Theorem 6. *Theorem The necessary and sufficient conditions that the states 0,1,2,... of an irreducible Markov chain will be also transient in that the system of equation*

$$y_i = \sum_{j=1}^{\infty} p_{ij} y_j \qquad i \geq 1$$

admits a non zero bounded solution.

Hence all the states are transient if $p > q$ and the states are persistent if $p \leq q$.

It is seem that if p=q,then the states are persistent null and for $p < q$,the states are persistent non-null.

Thus we have

1. All the states are transient if $p > q$.
2. All the states are persistent if $p = q$.
3. All the states are persistent if $p < q$.

7.6 Simple Random Walk With Two reflecting barriers:

Consider the movement of the particle on the axis of X between 0 and a whenever the particle is at any point between 0 and a, it moves one unit right with probability p and one unit left with probability q.$(p + q = 1)$.

If the particle is at'0', it moves one unit right with probability p and remains there with probability q.

Also if particle is at'a',it moves one unit left with probability q and remains there with probability p.

Thus if random variable X_n is used for denote the position of point at true $t = n$ then obviously $\{X_n\}$
from a Markov-chain with following transition probability

$$\left.\begin{array}{rcl} p_{ii+1} & = & p \\ p_{ii-1} & = & q \end{array}\right\} i = 1, 2, 3, \ldots, a-1$$

$$\begin{array}{rcl} p_{00} & = & q \\ p_{01} & = & p \\ p_{aa} & = & p \\ p_{aa-1} & = & q \end{array}$$

Thus the motion of the particle also form on irreducible Markov Chain with state space

$$I : \{0, 1, 2, 3, \ldots, a\}$$

Since it is a finite irreducible Markov chain, hence all its states are persistent non-null.

8 Gambler's Ruin Problem

Historically the classical random walk has been the gambler's ruin problem.

For this problem ,let us suppose that a gambler with interval capital K,plays against an opponent whose interval capital is $(a-k)$. The game proceeds by stages and at each stag the first gambler can win one unit with probability p and lose one unit with probability q against his adversary (*opponent*). The actual capital possessed by the first gambler is thus represented by the random walk on the non-negative integers with with absorptions at $x=0$ and $x=a$ being interpreted as the ruin of the gamblers.

8.1 Probability of Ruin:

The probability q_K of the gambler's ruin, when he starts with initial capital k can be obtained by quite elementary consideration as follows. After the first trial,the capital of the first gambler is either$(k+1)$ (if he wins) with probability p and $k-1$ if he looses (with probability$q$).

Hence probability of his ruin after first trial will be q_{k+1} and q_{k-1} respectively So that

$$\begin{aligned} q_k &= p\cdot q_{k+1}+q\cdot q_{k-1} \qquad \text{1<k<a-1} \\ and \\ q_1 &= p\cdot q_2+q \\ q_{a-1} &= q\cdot q_{a-2} \end{aligned}$$

All these equations can be written in a general form as

$$q_k = p\cdot q_{k+1}+q\cdot q_{k-1} \qquad 1\leq k\leq a-1 \qquad \ldots(1)$$

provided we adopt the convention that

$$q_0 = 1, \qquad q_a=0 \qquad \ldots(2)$$

The equ^n (1) is a difference equation of order two and is homogeneous

The auxiliary equation (1) is given as

$$\begin{aligned} pm^2-m+q &= 0 \qquad \ldots(3) \\ \text{which can be written as} \\ pm^2-(p+q)\,m+q &= 0 \\ pm\,(m-1)-q\,(m-1) &= 0 \\ (pm-q)\,(m-1) &= 0 \\ \Rightarrow m &= 1 \qquad and \qquad m=\tfrac{q}{p} \end{aligned}$$

Thus if $p \neq \frac{1}{2} \neq q$,the equation(3) has two different roots and we can write the general solution of(1) as

$$\begin{aligned}
q_k &= c_1\,(1)^k + c_2 \cdot \left(\frac{q}{p}\right)^k \qquad \ldots(4) \\
\text{or} & \\
q_k &= c_1 + c_2 \cdot \left(\frac{q}{p}\right)^k
\end{aligned}$$

where c_1 and c_2 are constants.

Applying the initial conditions
$q_0 = 1$ and $q_a = 0$,we have

$$\begin{aligned}
1 &= c_1 + c_2 \qquad \Rightarrow c_1 = 1 - c_2 \\
0 &= c_1 + c_2 \left(\frac{q}{p}\right)^a \\
c_1 + (1 - c_1)\left(\frac{q}{p}\right)^a &= 0 \\
c_1\left[1 - \left(\frac{q}{p}\right)^a\right] &= -\left(\frac{q}{p}\right)^a \\
c_1 &= \frac{\left(\frac{q}{p}\right)^a}{\left(\frac{q}{p}\right)^a - 1} \qquad \ldots(5) \\
\textit{and} & \\
c_2 &= \frac{-1}{\left(\frac{q}{p}\right)^a - 1}
\end{aligned}$$

Thus the required solution for q_k is

$$\begin{aligned}
q_k &= \frac{\left(\frac{q}{p}\right)^a}{\left(\frac{q}{p}\right)^a - 1} - \frac{1 \cdot \left(\frac{q}{p}\right)^k}{\left(\frac{q}{p}\right)^a - 1} \qquad \ldots(6) \\
\textit{suppose} & \\
p &= \frac{1}{3} \\
q &= \frac{2}{3} \\
k &= 3, \qquad a = 4 \\
q_3 &= \frac{(2)^4 - (2)^3}{2^4 - 1} \\
&= \frac{16-8}{15} \\
&= \frac{8}{15}
\end{aligned}$$

Similarly we can calculate the probability p_k of the success of first gambler i.e. ruin of his opponent by interchanging p and q and replacing k by $a - k$

so

$$\begin{aligned}
p_k &= \frac{\left(\frac{p}{q}\right)^a - \left(\frac{q}{p}\right)^{a-k}}{\left(\frac{p}{q}\right)^a - 1} \\
&= \frac{\left(\frac{p}{q}\right)^a \left[1 - \left(\frac{p}{q}\right)^{-k}\right]}{\left(\frac{p}{q}\right)^a \left[1 - \left(\frac{p}{q}\right)^{-k}\right]} \\
p_k &= \frac{1 - \left(\frac{q}{p}\right)^k}{1 - \left(\frac{q}{p}\right)^a} \qquad \ldots (7) \\
Now & \\
p_k + q_k &= \frac{\left(\frac{q}{p}\right)^a - \left(\frac{q}{p}\right)^k}{\left(\frac{q}{p}\right)^a - 1} + \frac{1 - \left(\frac{q}{p}\right)^k}{1 - \left(\frac{q}{p}\right)^a} \\
&= \frac{\left(\frac{q}{p}\right)^a - \left(\frac{q}{p}\right)^k - 1 + \left(\frac{q}{p}\right)^k}{\left(\frac{q}{p}\right)^a - 1} \\
p_k + q_k &= 1
\end{aligned}$$

So the chance of an unending contest between the two gamblers is zero i.e. the game has to terminate.

Let us consider the case when $p = q = \frac{1}{2}$

Then the auxiliary equation has two equal roots and the general solution may be written as

$$q_k = c_1 + c_2 k$$

using the boundary condition

$q_0 = 1$ and $q_a = 0$, we have

$$\begin{aligned}
1 &= c_1 \\
0 &= c_1 + c_2 a \\
0 &= 1 + c_2 a \\
\text{hence} \quad c_2 &= -\frac{1}{a} \\
\text{so} \quad q_k &= 1 - \frac{k}{a}
\end{aligned}$$

By changing q to p and k to $a - k$,we get

$$\begin{aligned}
p_k &= 1 - \frac{a-k}{a} \\
&= \frac{k}{a}
\end{aligned}$$

so that $q_k + p_k + 1$

Thus in this case also, the chance of an ending contest is zero.

8.2 Expected Duration of the game:

Suppose d_k denotes the expected duration of game starting the random walk from the point $x = k$. If the first trial leads to a win for the first gambler,the

conditional duration of the game from that point is d_{k+1}.So the expected duration of the whole game of the first trial is a win is $(1 + d_{k+1})$.
Similarly the expected duration of the whole game if the first trial is a failure for the first gambler is $(1 + d_{k-1})$.
Therefore we have

$$\begin{aligned} d_k &= p\,(1 + d_{k+1}) + q\,(1 + d_{k-1}) \qquad \ldots (1) \\ & \qquad\qquad\qquad\qquad where \qquad 1 \leq k \leq a - 1 \end{aligned}$$

The extreme value $k = 1$ and $k = a-1$ being included on the assumption that

$$d_0 = 0 \qquad and \qquad d_a = 0 \qquad \ldots (2)$$

The equation (1)is obviously non-homogeneous difference equation of order '2' which can be written as

$$p \cdot 1 + d_{k+1} - d_k + q \cdot d_{k-1} = -1 \qquad \ldots (3)$$

The auxiliary equation for (1) is written as

$$pm^2 - m + q = 0$$

which again gives the roots as

$$m = 1 \qquad and \qquad m = \tfrac{q}{p}$$

Thus if p and q are not equal, than the complimentary function of(1) is given as

$$C.F. = A_1 + A_2 \left(\frac{q}{p}\right)^k$$

and the particular integral is $(P.I.)$

$$\begin{aligned}
\left[\frac{1}{(pE^2-E+q)}\right]^{-1} &= \left[\frac{1}{(pE-q)(E-1)}\right]^{-1} \\
&= \frac{1}{(pE-q)}\left[\frac{1}{\Delta}(-1)\right] \\
&= \frac{1}{(pE-q)\Delta}(-1) \\
&= \frac{1}{\Delta[p(1+\Delta)-q]}(-1) \\
&= \frac{1}{\Delta[p-q+p\Delta]}(-1) \\
&= \frac{1}{\Delta(p-q)\left(1+\frac{p}{p-q}\Delta\right)}(-1) \\
&= \frac{1}{(p-q)\Delta}\left[1+\frac{p}{p-q}\Delta\right]^{-1}(-1) \\
&= \frac{1}{(p-q)\Delta}\left[1-\frac{p}{p-q}\Delta+\frac{p^2}{(p-q)^2}\Delta^2\ldots\ldots\right](-1) \\
&= -\frac{k}{p-q}+\frac{p}{p-q} \\
&= \frac{k}{p-q}+C
\end{aligned}$$

Thus the general solution is given as $(for p \neq q)$

$$d_k = A_1 + A_2\left(\frac{q}{p}\right)^k + \frac{k}{q-p}$$

Using boundary conditions as $d_0 = 0$ and $d_a = 0$,we have

$$\begin{aligned}
0 &= A_1 + A_2 + 0 \\
\text{i.e.} \quad A_2 &= -A_1 \\
\text{and} \quad 0 &= A_1 + A_2 \cdot \left(\frac{q}{p}\right)^a + \frac{a}{q-p} \\
0 &= -A_2 + A_2 \cdot \left(\frac{q}{p}\right)^a + \frac{a}{q-p} \\
A_2\left[\left(\frac{q}{p}\right)^a - 1\right] &= -\frac{a}{q-p} \\
A_2 &= \frac{\frac{a}{p-q}}{\left(\frac{q}{p}\right)^a-1} \\
\text{so} \quad A_1 &= \frac{\frac{a}{p-q}}{1-\left(\frac{q}{p}\right)^a}
\end{aligned}$$

So the general solution is

$$\begin{aligned}
d_k &= \frac{\frac{a}{q-p}}{\left(\frac{q}{p}\right)^a-1} + \frac{\frac{a}{p-q}}{1-\left(\frac{q}{p}\right)^a}\left(\frac{q}{p}\right)^k + \frac{k}{q-p} \\
&= \frac{k}{q-p} - \frac{a}{q-p}\left[\frac{1-\left(\frac{q}{p}\right)^k}{1-\left(\frac{q}{p}\right)^a}\right]
\end{aligned}$$

If $p = \frac{1}{2}$,then the

$$\begin{aligned}
C.F. &= A_1 + A_2 k \\
\text{and} \qquad P.I. &= \tfrac{1}{(pE-q)(E-1)}\,(-1) \\
&= \tfrac{1}{p(E-1)^2}\,(-1) \\
&= \tfrac{2}{\Delta^2}\,(-1) \\
&= \tfrac{-2 \cdot k(k-1)}{2!} \\
\text{and the game solution is} & \\
d_k &= A_1 + A_2 k - k\,(k-1) \\
\text{using the boundary condition} & \\
0 &= A_1 \\
0 &= A_1 + A_2 a - a\,(a-1) \\
A_2 \cdot a &= a\,(a-1) \\
A_2 &= (a-1) \\
\text{so the game solution is} & \\
d_k &= (a-1) \cdot k - k \cdot (k-1) \\
&= ak - k - k^2 + k \\
d_k &= k\,(a-k)
\end{aligned}$$

9 Chapman-Kolmogorov Equation:

9.1 Continuous Parameter Markov Chain :

Let $\{X(t), t \in T, t > 0\}$ be a continuous parameter Markov chain with At most a countable number of states $i = 0, 1, 2, \ldots$
Let $I = \{i, i = 0, 1, 2, \ldots\}$
we define

$$P_{ij}(\tau, t) = P[X(t) = j | X(\tau) = i] \qquad t > \tau \qquad \ldots (1)$$

This is known as transition probability function for the process which gives the probability that the system is in state j at time t given that at timeτ it was in state$i$.This Markov chain will be called homogeneous if the transition probability depends on the difference of t and τ.If the transition probabilities defines in (1) simply depends upon the difference $(t-\tau)$ and not on particular values of t and τ,then the Markov chain is said to be homogeneous with respect to time and we can write(1) as

$$\begin{aligned} P_{ij}(t-\tau) &= P[X(t) = j | X(\tau) = i] \\ &= P[X(t+t_1) = j | X(\tau + t_1) = i] \qquad forall \qquad t_1 \geq 0 \end{aligned}$$

Let us consider three points of time i.e.
$\tau < \xi < t$ and let

$$P[X(t) = j, X(\xi) = k | X(\tau) = i] = P[X(\xi) = k | X(\tau) = i] \cdot P[X(t) = j | X(\xi) =$$

(since we are dealing with Markov chain)

$$= P_{ik}(\tau, \xi) \cdot P_{kj}(\xi, t)$$

$$P_{ij}(\tau, t) = \sum_{k=0}^{\infty} P_{ik}(\tau, \xi) \cdot P_{kj}(\xi, t)$$

This is known as Chapman -Kolmogorov equation.

For homogeneous case

$$P_{ij}(t-\tau, t) = \sum_{k=0}^{\infty} P_{ik}(\xi - \tau) \cdot P_{kj}(t - \xi)$$

9.2 Kolmogorov System of differential equations:

In order to derive the Kolomogorov system of differential equations from the Chapman-Kolmogorov equation, we make the following assumption known as regularity assumptions.

9.2.1 Regularity Assumptions:

1. For any integer i,there exists a function $\nu_{ii}(\tau) < 0$ such that.

$$\lim_{\Delta\to 0} \frac{1-P_{ii}(\tau,\tau+\Delta)}{\Delta} = -\nu_{ii}(\tau)$$

2. For any pair of integers $i, j, i \neq j$,there exists a function $\nu_{ij}(\tau) > 0$ such that

$$\lim_{\Delta\to 0} \frac{1-P_{ii}(\tau,\tau+\Delta)}{\Delta} = -\nu_{ii}(\tau)$$

3. For any fixed i, the passage in 2 is uniform with respect to j.The function $\nu_{ii}(\tau)$ is known as intensity function.If we agree to write that

$$P_{ij}(\tau,\tau) = \delta_{ij}$$

where δ_{ij} is the Kronecker delta i.e.

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j; \end{cases}$$

Now from 1 and 2,we have

$$\begin{aligned} v_{ii}(\tau) &= \frac{\partial}{\partial t} P_{ii}(\tau,t)\,|t = \tau \\ \textit{and} \qquad \nu_{ij}(\tau) &= \frac{\partial}{\partial t} P_{ij}(\tau,t)\,|t = \tau \end{aligned}$$

The probabilities meaning of the intensity function emerges if we write 1 and 2 as follows:

$$\begin{aligned} P_{ii}(\tau,\tau+\Delta) &= 1 - \nu_{ii}(\tau)\cdot +0(\Delta) \\ P_{ij}(\tau,\tau+\Delta) &= \nu_{ij}(\tau)\cdot +0(\Delta) \end{aligned}$$

where $0(\Delta)$ is such that

$$\lim_{\Delta\to 0} \frac{0(\Delta)}{\Delta} = 0$$

9.3 Kolmogorov's Forward System of Differential Equation:

Let us consider three points $P < t < t + \Delta$
Now from Chapman -Kolmogorov equations, we have

$$\begin{aligned}
P_{ik}(\tau, t+\Delta) &= \sum_j P_{ij}(\tau,t)\cdot P_{jk}(t,t+\Delta) \\
&= P_{ik}(\tau,t)\cdot P_{kk}(t,t+\Delta) + \sum_{j\neq k} P_{ij}(\tau,t)\cdot P_{jk}(t,t+\Delta)
\end{aligned}$$

Substituting the value of $P_{kk}(t,t+\Delta)$from equation 6 in equation 8,we have

$$\begin{aligned}
P_{ik}(\tau, t+\Delta) &= P_{ik}(\tau,t)\left[1 + v_{kk}(\tau)\cdot\Delta + 0(\Delta)\right] + \sum_{j\neq k} P_{ij}(\tau,t)\cdot P_{jk}(t,t+\Delta) \\
\tfrac{P_{ik}(\tau,t+\Delta)-P_{ik}(\tau,t)}{-\Delta} &= -P_{ik}(\tau,t+\Delta) - \tfrac{0(\tau)}{\Delta} + \tfrac{\sum\limits_{j\neq k} P_{ij}(\tau,t)\cdot P_{jk}(t,t+\Delta)}{-\Delta}
\end{aligned}$$

From regularity assumption 2, we see that the R.H.S. has limit as $\Delta \to 0$ and hence L.H.S. should also have limit as $\Delta \to 0$,we get

$$\tfrac{\partial}{\partial\tau}P_{ik}(\tau,t) = -\nu_{ii}(\tau)P_{ik}(\tau,t) - \sum_{j\neq k}\nu_{ij}(\tau)P_{ik}(\tau,t)$$

and hence in general ,we have

$$\tfrac{\partial}{\partial\tau}P_{ik}(\tau,t) = -\sum_j \nu_{ij}(\tau)P_{ik}(\tau,t)$$

Here t , is fixed and hence the initial equation will be

$$P_{ij}(\tau,\tau) = \delta_{ij}$$

the above equation is known as the Kolmogorov's Backward system of different equation. For a homogeneous Markov chain the intensity function is independent of time i.e. $v_{jk} = \nu_{jk}(t)$ and the system of differential equation can be written as

$$\begin{aligned}
\tfrac{\partial}{\partial t}P_{ik}(t) &= \sum_j P_{ij}(t)\nu_{jk} \\
\tfrac{\partial}{\partial t}P_{ik}(t) &= -\sum_j \nu_{ij}P_{jk}(t)
\end{aligned}$$

Where t is the length of the interval between τ and t and not a particular point of time. with common initial condition

$$P_{ik}(0) = \delta_{ik}$$

We can write

$$P(\tau,t) = \begin{pmatrix} P_{00}(\tau,t) & P_{01}(\tau,t) & \dots & P_{0j}(\tau,t) & \dots \\ P_{10}(\tau,t) & P_{11}(\tau,t) & \dots & P_{1j}(\tau,t) & \dots \\ \dots & \dots & \dots & \dots & \dots \\ P_{i0}(\tau,t) & P_{i1}(\tau,t) & \dots & P_{ij}(\tau,t) & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

and

$$\nu(\tau,t) = \begin{pmatrix} \nu_{00}(\tau) & \nu_{01}(\tau) & \dots & \nu_{0j}(\tau) & \dots \\ \nu_{10}(\tau) & \nu_{11}(\tau) & \dots & \nu_{1j}(\tau) & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \nu_{i0}(\tau) & \nu_{i1}(\tau) & \dots & \nu_{ij}(\tau) & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

then the forward system can be written as

$$\begin{aligned} \tfrac{\partial}{\partial t}P(\tau,t) &= P(t)\cdot\nu(\tau,t) \\ \tfrac{\partial}{\partial t}P(\tau,t) &= -\nu(\tau)\cdot P(\tau,t) \end{aligned}$$

with the initial condition $P(\tau,\tau) = I$ and $P(t,t) = I$.

9.3.1 The case Poisson Process :

We have in general

$$\begin{aligned} \tfrac{\partial}{\partial t}P_{ik}(\tau,t) &= \sum_j P_{ij}(t)\cdot\nu_{jk}(t) \\ \tfrac{\partial}{\partial t}P_{ik}(\tau,t) &= -\sum_j \nu_{jk}(t)\cdot P_{ij}(\tau,t) \end{aligned}$$

In the case

$$\underline{\nu} = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & \dots \\ 0 & -\lambda & \lambda & 0 & \dots \\ 0 & 0 & -\lambda & \lambda & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

i.e $\quad \nu_{ii}(\tau) = -\lambda\ , \quad \nu_{ii+1}(\tau) = -\lambda$

$\nu_{ij}(\tau) = 0 \quad$ for all other j.

9.4 Kolmogorov Forward System of Differential Equation:

$$\begin{aligned}
\frac{\partial}{\partial t} P_{ik}(\tau,t) &= \sum_j P_{ij}(\tau,t)\cdot \nu_{jk}(t) \\
&= P_{i1}(\tau,t)\cdot \nu_{1k}(t) + P_{i2}(\tau,t)\cdot \nu_{2k}(t) \\
&\quad + \ldots\ldots + \; P_{ik-1}(\tau,t)\cdot \nu_{k-1k}(t) \\
&\quad + P_{kj}(\tau,t)\cdot \nu_{kk}(t) + P_{ik+1}(\tau,t)\cdot \nu_{k+1k}(t) + \ldots\ldots \\
&= P_{ik-1}(\tau,t)\cdot \nu_{k-1k}(t) + P_{ik}(\tau,t)\cdot \nu_{kk}(t) \\
\frac{\partial}{\partial t} P_{ik}(\tau,t) &= \lambda\cdot P_{ik-1}(\tau,t) - \lambda\cdot P_{ik}(\tau,t)
\end{aligned}$$

9.5 Kolmogorov Backward System of Differential Equation:

$$\begin{aligned}
\frac{\partial}{\partial t} P_{ik}(\tau,t) &= -\sum_j \nu_{ij}(\tau)\cdot P_{jk}(\tau,t) \\
&= -\nu_{i1}(\tau)\cdot P_{1k}(\tau,t) - \ldots\ldots \\
&\quad -\nu_{ii-1}(\tau)\cdot P_{i-1k}(\tau,t) - \nu_{ii}(\tau)\cdot P_{ik}(\tau,t) \\
&\quad -\nu_{ii+1}(\tau)\cdot P_{i+1k}(\tau,t) - \ldots\ldots \\
&= -\left[-\lambda\cdot P_{ik}(\tau,t) + \lambda\cdot P_{i+1k}(\tau,t)\right] \\
&= \lambda\cdot P_{ik}(\tau,t) - \lambda\cdot P_{i+1k}(\tau,t)
\end{aligned}$$

10 Branching Process

The history of branching processes dates back to 1874 when mathematical model was formulated by Galton and Watson for the problem of 'extinction of families'. The model did not attract much attention for a long time, the situation gradually changed and during last 40 years much attention has been devoted to it, This is because of the development of interest in the application of probability theory ;in general and also because of the probability of using the model in a variety of biological, physical and other problems when one is concerned with objects that can generate objects of similar kind, such as human beings, animals, genes, bacteria and so on which reproduce similar objects by biological methods or may be physical particles such as neutrons which yield new neutrons under a nuclear chain reaction or in the process of nuclear fission.

We consider the discrete time case;Suppose that we start with an initial set of objects (or individuals) which form the $0-th$ generation- these objects are called ancestors. The offspring's reproduced or the objects generated by objects of the 0^{th} generation are the direct 'descendant's 'of the ancestors and are said to form the 1^{st} generation ; the object generated by these of the 1^{st} generation form the 2^{nd} generation and so on. The number of objects of the r^{th} generation $(r = 0, 1, 2, \ldots)$ is a random variable. We assume that the objects reproduce independent of other objects i.e. there is no interference.

Definition 9. *Let the random variables* $X_0, X_1, X_2, \ldots$*denote the size of* (*with number of objects in*) *the* 0^{th}, 1^{st}, 2^{nd} *,... generations respectively.Let the probability that an object* (*irrespective of the generation to which it belongs*) *generates k similar objects be denoted by* p_k*where* $p_k \neq 0, \qquad k = 0, 1, 2, \ldots$ *and* $\sum\limits_k P_k = 1$ *The sequence* $\{x_m, n = 0, 1, 2, \ldots\}$ *constitutes a Galton-Watson branching process with offspring distribution* $\{p_k\}$.

Our interest lies mainly in the probability distribution of X_n and the probability that $X_n \to 0$ some n i.e the probability of ultimate extinction of the family.

We assume that $X_0 = 1$'i.e' the process starts with a single ancestor .

The sequence $\{X_n\}$ forms a Markov Chain with transition probabilities

$p_{ij} = P_r \{X_{n+1} = j | X_n = i\} \qquad i, j = 0, 1, 2, \ldots$

The generating function prove very useful in the study of branching processes.

10.1 Properties Of Generating Functions Of Branching Process:

We have

$$X_{n+1} = \sum_{r=1}^{X_n} \xi_r$$

where ξ_r are i.i.d. random variables with distribution $\{p_k\}$. Let

$$\begin{aligned} P(s) &= \sum_k P_r\{\xi_r = k\} \cdot s^k \\ &= \sum_k P_k \cdot s^k \end{aligned}$$

be the p.g.f. of $\{\xi_r\}$ and let

$$P(s) = \sum_k P_r\{X_n = k\} \cdot s^k, \qquad n = 0, 1, 2, \ldots$$

be the p.g.f. of$\{X_n\}$

We assume that $X_0 = 1$

Clearly $P_0(s) = s$ and $P_1(s) = P(s)$

The random variables X_1 and ξ_r both give offspring distribution.

Theorem

$$\begin{aligned} P_n(s) &= P_{n-1}(P(s)) \\ P_n(s) &= P(P_{n-1}(s)) \end{aligned} \tag{66}$$

<u>Proof</u> we have

$$\begin{aligned} P_r\{X_n = k\} &= \sum_{j=0}^{\infty} P_r\{X_n = k | X_{n-1} = j\} \cdot P\{X_{n-1} = j\} \\ &= \sum_{j=0}^{\infty} P_r\left\{\sum_{r=1}^{j} \xi_r = k\right\} \cdot P\{X_{n-1} = j\} \end{aligned}$$

so that for $n = 1, 2, \ldots$

$$\begin{aligned} P_n(s) &= \sum_{k=0}^{\infty} P[X_n = k] \cdot s^k \\ &= \sum_{k=0}^{\infty} s^k \left[\sum_{j=0}^{\infty} P_r\left\{\sum_{r=1}^{j} \xi_r = k\right\} \cdot P\{X_{n-1} = j\}\right] \\ &= \sum_{j=0}^{\infty} P\{X_{n-1} = j\} \left[\sum_{k=0}^{\infty} P\{(\xi_1 + \xi_2 + \ldots + \xi_j) = k\} \cdot s^k\right] \end{aligned}$$

The expression in the sequence bracket is the p.g.f. of the sum $\xi_1+\xi_2+\ldots+\xi_j$, j i.i.d. random variables each with p.g.f. $P(s)$.Hence it is equal to$[P(s)]^j$

Thus

$$\begin{aligned} P_n(s) &= \sum_{k=0}^{\infty} P_r\{X_{n-1}=j\}\cdot[P(s)]^j \\ &= P_{n-1}(P(s)) \end{aligned}$$

Thus we get 1 .

Now putting $n=2,3,4,\ldots$,we get

$$\begin{aligned} P_2(s) &= P_1(P(s)) \\ \text{But} \qquad P_1(s) &= P(s) \qquad \text{hence} \\ P(s) &= P_1(P(s)) \\ \text{Now} & \\ P_3(s) &= P_2(P(s)) \\ \text{and similarly} & \\ P_4(s) &= P_3(P(s)) \\ \text{and so on} & \\ \text{We know that} & \\ P_n(s) &= P_{n-1}(P(s)) \\ \text{Since} \qquad P_n(s) &= P_{n-1}(P(s)) \\ \text{hence} \qquad P_n(P(s)) &= P_{n-1}(P(P(s))) \\ \text{since} \qquad P_2(s) &= P(P(s)) \\ \text{Now} & \\ P_{n-1}(P(s)) &= P_{n-2}(P(P(s))) \\ &= P_{n-2}(P_2(s)) \\ \text{So} \qquad P_n(s) &= P_{n-2}(P_2(s)) \\ \text{So in the above if we put } n=3, \text{ we get} & \\ P_3(s) &= P_1(P_2(s)) \\ &= P(P_2(s)) \\ \text{Now} \qquad P_{n-2}(P_2(s)) &= P_{n-3}(P(P_2(s))) \\ &= P_{n-3}(P_3(s)) \\ \text{so} \qquad P_n(s) &= P_{n-3}(P_3(s)) \\ \text{and in general , we get} & \\ P_n(s) &= P_{n-k}(P_k(s)) \qquad k=0,1,2,\ldots,n \\ \text{and in particular for} \qquad k &= n-1 \\ P_n(s) &= P_1(P_{n-1}(s)) \\ &= P(P_{n-1}(s)) \end{aligned}$$

10.1.1 Moments of X_n :

The above theorem would be used to fixed moments of X_n.

We have $P'(1) = E(\xi_r) = E(X_1) = m\ (say)$

Theorem 7. *If* $E(X_1) = m = \sum_{k=0}^{\infty} k \cdot p_k$ *and* $\sigma^2 = V(X_1)$

Then $E(X_1) = m^n$

$$\begin{aligned} \textit{and} \qquad V(X_n) &= \frac{m^{n-1}(m^n - 1)}{m - 1} \cdot \sigma^2 \qquad \textit{if} \quad m \neq 1 \\ &= n \cdot \sigma^2 \qquad \textit{if} \quad m = 1 \end{aligned}$$

Proof

Differentiating the equations

$$P_n(s) = P_{n-1}(P(s))$$

we get

$$P_n'(s) = P_{n-1}'(P(s)) \cdot P'(s)$$

$$\text{therefore} \qquad P_n'(1) = P_{n-1}'(P(1)) \cdot P'(1)$$

$$\text{But} \qquad P(1) = 1 \qquad \text{and} \qquad P'(1) = m$$

$$\text{so} \qquad P_n'(1) = m \cdot P_{n-1}'(1)$$

Iterating

$$\begin{aligned} P_n'(1) &= m \cdot P_{n-2}'(1) \cdot m \\ &= m^2 \cdot P_{n-2}'(1) \\ &\vdots \\ &= m^{n-1} \cdot P_1'(1) = m^{n-1} \cdot P'(1) \\ &= m^n \end{aligned}$$

$$\text{Then} \qquad E(X_n) = P_n'(1) = m^n$$

We can derive the result in an alternative way also

for $m \neq 1$

$$\text{Since} \qquad X_{n+1} = \sum_{r=1}^{X_n} \xi_r$$

We have a theorem $(From Previous Results)$

Let $X_i, i = 1, 2, \ldots$ be i.i.d. and let

$$S_N = X_1 + X_2 + \ldots + X_N$$

where N itself is a random variable independent of X_i's then

$$E\left(S_N\right) = E\left(X_i\right) + E\left(N\right)$$

Similarly for this situation

$$V\left(S_N\right) = E\left(N\right) \cdot V\left(X_i\right) + V\left(N\right) + \left\{E\left(X_i\right)\right\}^2$$

Thus

$$\begin{aligned} E\left(X_{n+1}\right) &= E\left(X_i\right) \cdot E\left(X_n\right) \\ &= m \cdot E\left(X_n\right) \end{aligned}$$

This is homogeneous difference equation

It is of the type

$$u_{n+1} - m \cdot u_n = 0$$

The root is m and the solution becomes

$$u_n = c \cdot m^n$$

$$\text{i.e} \qquad E\left(X_n\right) = c \cdot m^n$$

The constant c is determined by the initial condition.

$$(X_1) = m$$

$$(X_1) = c \cdot m = m$$

this gives $c = 1$

and hence the solution becomes

$$(X_n) = m^n$$

$$\begin{aligned} \text{Also} \qquad V\left(X_{n+1}\right) &= (X_n) \cdot (\xi_r) + V\left(X_n\right) + \left\{E\left(\xi_r\right)\right\}^2 \\ &= m^n \cdot \sigma^2 + (X_n) \cdot (m)^2 \\ (X_{n+1}) &= m^2 \cdot (X_n) + m^n \cdot \sigma^2 \end{aligned}$$

This is non -homogeneous difference equation of the type

$$u_{n+1} - m^2 \cdot u_n = m^n \cdot \sigma^2$$

The solution will become

$$u_n = G.F. + P.I.$$

There the root is m^2.

so G.F. is given as

$$u_n = A \cdot \left(m^2\right)^n$$

The particular solution is

$$P.I. = \frac{1}{(E - m^2)} \cdot m^n \cdot \sigma^2$$
$$= \frac{m^n \cdot \sigma^2}{(m - m^2)}$$

as the solution is obtained by putting m in place of E.

Form of Difference equation

$$A = \frac{\sigma^2}{m\,(m-1)}$$

so the general solution.

$$V\,(X_n) = A \cdot \left(m^2\right)^n + \frac{m^n \cdot \sigma^2}{m - m^2}$$

The constant A is determined by using the initial condition i.e

$$V\,(X_1) = \sigma^2$$

$$\sigma^2 = A \cdot \left(m^2\right)^n + \frac{m^n \cdot \sigma^2}{m - m^2}$$

$$\text{Hence} \qquad A = \frac{\sigma^2 - \frac{\sigma^2 \cdot m}{m - m^2}}{m^2}$$
$$= \frac{\frac{m \cdot \sigma^2 - m^2 \cdot \sigma^2 - m \cdot \sigma^2}{m - m^2}}{m^2}$$
$$A = \frac{\frac{-m^2 \cdot \sigma^2}{m - m^2}}{m^2} = \frac{-\sigma^2}{m - m^2}$$
$$= \frac{\sigma^2}{m^2 - m} = \frac{\sigma^2}{m\,(m-1)}$$

$$\text{so} \qquad V\,(X_n) = \frac{\sigma^2}{m\,(m-1)} \cdot m^{2n} + m^n \cdot \frac{\sigma^2}{m - m^2}$$
$$= \frac{\sigma^2}{m\,(m-1)} \cdot m^{2n} - m^n \cdot \frac{\sigma^2}{m\,(m -}$$
$$= \frac{m^{n-1}}{m-1} \cdot (m^n - 1) \cdot \sigma^2 \qquad n = 1,$$

This is true for all n and $m \neq 1$.

We can get the result for $m = 1$,by taking the limit as $m \to 1$.Limit may be taken by differentially

$$limit_{m\to 1}\frac{(n-1)\cdot m^{n-2}\cdot (m^n - 1) + m^{n-1}\cdot n\,(m^{n-1})}{1}\cdot \sigma^2 = n\sigma^2$$

10.2 Probability of Extinction:

Definition 10. *By extinction of process , it is meant that the random sequence $\{X_n\}$ consists of zeros for all except a finite number of values of n.In other words, extinction occurs when $P\{X_n = 0\} = 1$ for some value of n clearly if $X_n = 0$ for $n = m$,then $X_n = 0$ for $n > m$.*

Also $P\{X_{n+1} = 0|X_n = 0\} = 1$

Theorem 8. *If $m \leq 1$,the probability of ultimate extinction is 1.If $m > 1$, the probability of ultimate extinction is the positive root less than unity of the equation*

$$s = P(s)$$

Proof. Let $q_n = P\{X_n = 0\}$ i.e. q_n is the probability that extinction occurs at or before the n^{th} generation.

Clearly $q_n = P_n(0)$

Since $P_n(s)$ is the p.g.f. of X_n

$$\begin{aligned} P_n(s) &= \sum_{k=0}^{\infty} P\{X_n = k\}\, s^k \\ &= P[X_n = 0]\cdot s^0 + P[X_n = 1]\cdot s^1 + P[X_n = 2]\cdot s^2 + \ldots \\ P_n(0) &= P[X_n = 0] = q_n \\ \text{Also} \quad q_1 &= P_1(0) = P(0) = p_0 \\ \text{Since} \quad q_1 &= P[X_1 = 0] \quad \text{and} \\ P_1(s) &= p_0\cdot s^0 + p_1\cdot s^1 + p_2\cdot s^2 + \ldots \end{aligned}$$

so $\quad P_1(0) = p_0$

Also $\quad P_1(0) \quad$ is the same as $P(0) \quad$ as $\quad X_1 \quad$ and $\quad \xi_r \quad$ are same .

We also know that

$$\begin{aligned} P_n(s) &= P(P_{n-1}(s)) \\ \text{so} \qquad P_n(0) &= P(P_{n-1}(0)) \\ q_n &= P(q_{n-1}) \end{aligned}$$

Finite if $p_0 = 0$,then $q_1 = 0$ because $q_1 = p_0$

Also q_2 is also equal to zero because

$q_2 = P(q_1) = P(0) = p_0 = 0$ since $q_1 = 0$

Thus for $p_0 = 0$,$q_1 = q_2 = q_3 = \ldots = 0$ i.e. if the probability of no offspring is zero,extinction can never occur.

However If $p_0 = 1$, then $q_1 = 1$,$q_2 = 1$,...

i.e. if the probability of no offspring is one then the extinction is certain to occur after the $zero^{th}$ generation .

so we confine ourselves to the case $0 < p_0 < 1$. As $p(s)$ is a strictly increasing function of s ,

$q_2 = P(q_1) = P(p_0) > P(0) = q_1$

so $q_2 > q_1$

Similarly it can be shown that,

$q_3 > q_2$ and so on i.e

$q_1 < q_2 < q_3 \ldots$

Thus the monotone increasing sequence $\{q_n\}$ is bounded above by 1.Hence q_n must have a limit, $lim_{n\to\infty} q_n = q \ (say)$

$0 \leq q \leq 1$; $\qquad q$ is the probability of ultimate extinction.

Now we have seen that

$q_n = P(q_{n-1})$

Thus,we see that

$q = P(q)$

i.e $\qquad q$ is a root of the equation

$s = P(s) \qquad \ldots (A^*)$

we now further investigate about the root .

First we show that q is the smallest positive root of (A^*).

Let s_0 be an arbitrary positive root of (A^*), than

$q_1 = P(0) < P(s_0) = s_0$

$q_2 = P(q_1) < P(s_0) = s_0$

so $q_2 < s_0$

and assuming $q_m < s_0$, we get

$q_{m+1} = P(q_m) < P(s_0) = s_0$ and
by induction $q_n < s_0$ for all n.
Thus $q = \lim_{n\to\infty} q_n \leq s_0$.
which implies that q is the smallest positive root of the equation $s = P(s)$
For this we consider the graph of
$s = P(s)$ in $0 \leq s \leq 1$.
It starts with the

1. Point $(0, p_0)$ because

   $P(s) = p_0 + p_1 s + p_2 s^2 + \ldots$

   so $P(0) = p_0$

2. It ends with the point $(1, 1)$ because

   $P(1) = 1$

   The curve lies entirely in the first quadrant. the curve is convex as $P(s)$ is an increasing function of s.

   So the curve $y = P(s)$ can intersect the line $y = s$ in at most two points,one of which is the end point $(1, 1)$ i.e. the equation (A^*) has at most two roots,one of which is unity.

   The two cases are now considered with the help of the figures

   <u>Case 1:</u>

   The curve $y = P(s)$ lies entirely above the line $y = s$ in which case $(1, 1)$ is the only point of intersection i.e. unity is the unique root of the equation $s = P(s)$

   so that $q = \lim_{n\to\infty} q_n = 1$

   Then

   $P(1) - P(s) = 1 - P(s) \leq 1 - s$

   because for this case $P(s) \geq s$

   i.e. $1 - P(s) \leq 1 - s$ (from to given)

   so that

$$\lim_{s\to 1} \frac{P(1) - P(s)}{1 - s} \leq 1$$

but the L.H.S. is $P'(1)$ i.e.

$$P'(1) \leq 1$$

But $P'(1) = m$,which implies that $m \leq 1$

This show that if $m \leq 1$,then $q = 1$ i.e. ultimate extinction is certain.

Case2:

The curve $y = P(s)$ intersects the line $y = s$ at another point $(\delta, P(\delta))$ such that $\delta = P(\delta)$,$\delta < 1$.

i.e. there is another root of equation (A^*) which is $\delta < 1$.

The curve $y = P(s)$,being convex, lies below the line $y = s$ in $(\delta, 1)$ and above $y = s$ in $(0, \delta)$ i.e.

$$P(s) < s \qquad \text{in} \qquad \delta < s < 1$$

and

$$P(s) > s \qquad \text{in} \qquad 0 < s < \delta$$

Then

$$q_1 = P(0) < P(\delta) = \delta$$

(because $0 < \delta$ and δ is a root of $s = p(s)$)

Now assuming $q_m < \delta$,we get

$q_{m+1} = P(q_m) < P(\delta) = \delta$ and by induction $q_n < \delta$ for all n.

Hence $\lim\limits_{n\to\infty} q_n = q = \delta$ because q and δ both are the only root ,less then unity of equation (A^*).

Now by the mean value theorem considered in the interval $[\delta, 1]$,there is a value ξ in $\delta < \xi < 1$ such that

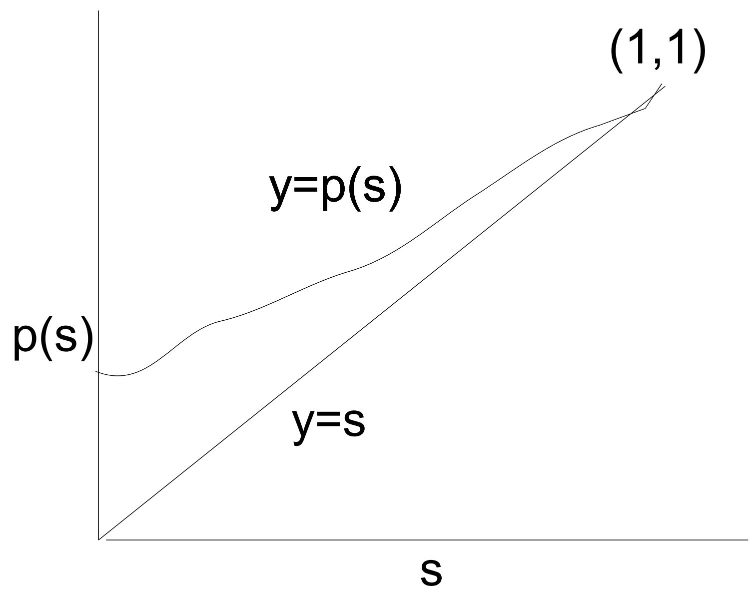
$$P'(\xi) = \frac{P(1) - P(\delta)}{1 - \delta}$$

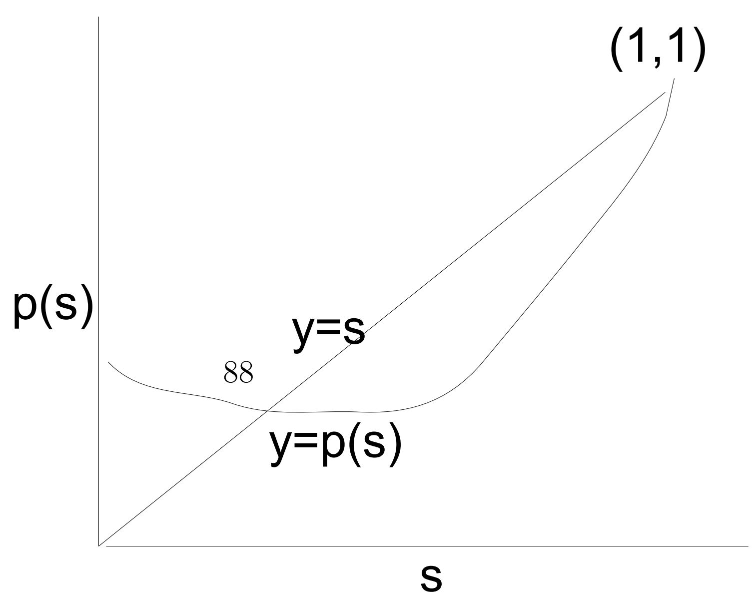
But $P(\delta)$,hence R.H.S.=1 i.e. $P'(\xi) = 1$

Further since the derivative of $P'(s)$ is monotone, $P'(1) > 1$ because $\xi < 1$ and $P'(\xi) = 1$

However $P'(1) = m$ i.e $m > 1$

This show that for $m > 1$,the probability of ultimate extinction is $\delta < 1$ as it is the smallest positive root of the equation $s = P(s)$.


(1,1)
y=p(s)
p(s)
y=s
s


(1,1)
p(s)
y=s
y=p(s)
s

$\square$

Example 9. *Stationary distribution*

$$\begin{array}{cc} & \begin{array}{cc} S & F \end{array} \\ \begin{array}{c} S \\ F \end{array} & \begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \end{pmatrix} \end{array}$$

Then the probability distribution of first passage time from S to S.

$$\begin{aligned}
P[x=1] &= p_1 \\
P[x=2] &= q_1 p_2 \\
P[x=3] &= q_1 q_2 p_2 \\
P[x=4] &= q_1 q_2 q_2 p_2 \\
P[x=5] &= q_1 q_2 q_2 q_2 p_2 \\
\vdots \quad & \quad \vdots
\end{aligned}$$

obviously sum of probabilities is

$$\begin{aligned}
&= p_1 + q_1 p_2 \left\{1 + q_2 + q_2^2 + q_2^3 + \dots\right\} \\
&= p_1 + \frac{q_1 p_2}{1-q_2} = p_1 + q_1 = 1
\end{aligned}$$

The mean first passage time μ_{ss}

$$\begin{aligned}
\mu_{ss} &= p_1 + 2q_1p_2 + 3q_1p_2q_2 + 4q_1p_2q_2^2 + 5q_1p_2q_2^3 + \dots \\
&= p_1 + k\,(say) \\
k &= 2q_1p_2 + 3q_1p_2q_2 + 4q_1p_2q_2^2 + 5q_1p_2q_2^3 + \dots \\
kq_2 &= \qquad 2q_1p_2q_2 + 3q_1p_2q_2^2 + 4q_1p_2q_2^3 + \dots \\
k\cdot(1-q_2) &= 2q_1p_2 + q_1p_2q_2 + q_1p_2q_2^2 + q_1p_2q_2^3 + \dots \\
k &= q_1 + \frac{q_1}{1-q_2} = q_1 + \frac{q_1}{p_2} \\
\textit{Hence} \qquad \mu_{ss} &= p_1 + q_1 + \frac{q_1}{p_2} = 1 + \frac{q_1}{p_2} = \frac{p_2+q_1}{p_2}
\end{aligned}$$

Similarly probability distibution of first passage time from F to F

$$
\begin{aligned}
P\left[x=1\right] &= q_2 \\
P\left[x=2\right] &= p_2q_1 \\
P\left[x=3\right] &= p_2p_1q_1 \\
P\left[x=4\right] &= p_2p_1p_1p_1q_1 \\
P\left[x=5\right] &= p_2p_1 \\
\vdots \quad & \quad \vdots
\end{aligned}
$$

obviously sum of probabilities is

$$
\begin{aligned}
&= p_1 + q_1p_2\left\{1+q_2+q_2^2+q_2^3+\dots\right\} \\
&= p_1 + \frac{q_1p_2}{1-q_2} = p_1 + q_1 = 1
\end{aligned}
$$

The mean first passage time μ_{FF}

$$
\begin{aligned}
\mu_{FF} &= q_2 + 2p_2q_1 + 3p_2q_1p_1 + 4p_2q_1p_1^2 + 5p_2q_1p_1^3 + \dots \\
&= q_2 + K\,(say) \\
K &= 2p_2q_1 + 3p_2q_1p_1 + 4p_2q_1p_1^2 + 5p_2q_1p_1^3 + \dots \\
kp_1 &= \qquad 2p_2q_1p_1 + 3p_2q_1p_1^2 + 4p_2q_1p_1^3 + 5p_2q_1p_1^4 + \dots \\
k\cdot(1-p_1) &= 2q_1p_2 + p_2q_1p_1 + q_1p_2p_1^2 + p_2q_1p_1^3 + \dots \\
k &= p_2 + \frac{p_2}{1-p_1} \\
\text{Hence} \qquad \mu_{FF} &= q_2 + p_2 + \frac{p_2}{q_1} = 1 + \frac{p_2}{q_1} = \frac{q_1+p_2}{q_1} \\
\text{hence} \qquad u_S &= \frac{1}{\mu_{SS}} = \frac{p_2}{p_2+q_1} \\
u_F &= \frac{1}{\mu_{FF}} = \frac{q_1}{q_1+p_2}
\end{aligned}
$$

So stationary distribution is given as $\frac{p_2}{p_2+q_1}, \frac{q_1}{q_1+p_2}$ *obviously* $u_S + u_F = 1$.

<u>*Let us check*</u>

$$v_j = \sum_i u_i p_{ij}$$

so here put $S = 1$ *and* $F = 2$

$u_1 = \frac{p_2}{p_2+q_1} \qquad u_2 = \frac{q_1}{q_1+p_2}$

$$\begin{aligned} v_1 &= u_1 \cdot p_{11} + u_2 \cdot p_{21} \\ &= \frac{p_2}{p_2+q_1} \cdot p_{11} + \frac{q_1}{q_1+p_2} \cdot p_{21} \\ &= \frac{1}{p_2+q_1}\left[p_2 p_1 + q_1 p_2\right] \\ &= \frac{1}{p_2+q_1}\left[p_2\left(p_1+q_1\right)\right] \\ &= \frac{p_2}{p_2+q_1} = u_1 \qquad (\textit{verified}) \end{aligned}$$

10.3 Generating Functions

In dealing with integral valued random variables, it is often of great convenience to apply the powerful method of generating function. Many stochastic process that we come across involve integral valued random variable and quite often we can use generating functions for their studies. The principle advantage of its use is that a single function may be used to represent a whole set of individual items.

Definition 11. *Let* $a_0, a_1, a_2, \cdots$ *be a sequence of real numbers. Using a variable* s*, we may define a function*

$$\begin{aligned} A(s) &= a_0 s^0 + a_1 s^1 + a_2 s^2 + \cdots \\ &= \sum_{k=0}^{\infty} a_k s^k \end{aligned} \tag{67}$$

If this power series converges in some interval $-s_0 < s < s_0$*, then* $A(s)$ *is called the generating function of the sequence* $a_0, a_1, a_2, \cdots$.

The variable s itself has no particular significance. Here we assume s to be real but generating function with complex variable is also used sometimes. Differentiating 67, k times, putting $s = 0$ and dividing by $k!$ we get $a_k i.e.$

$$a_k = \frac{1}{k!}\left[\frac{d^k A(s)}{ds^k}\right]_{s=0} \tag{68}$$

10.4 Probability Generating Function

Suppose X is a random variable which assumes non-negative integral values $0, 1, 2, \ldots$ and that

$$P[X = k] = p_k, \qquad k = 0, 1, 2, \ldots \qquad \sum p_k = 1 \tag{69}$$

If we take a_k to be the probability p_k, $k = 0, 1, 2, \ldots$ then the corresponding generating function $P(s)$ of the sequence of probabilities $\{p_k\}$ is known as probability generating function (p. d. f) of the random variable X.

It sometimes also called the s - transformation or geometric transformation of X. Thus we have

$$\begin{aligned} P(s) &= \sum_{k=0}^{\infty} p_k s^k \\ &= E(s^k) \end{aligned} \tag{70}$$

Where $E(s^k)$ is the expectation of the function s^k (a random variable) of the random variable X. The series $P(s)$ converges for at least $-1 \leq s \leq 1$. The first two derivatives of $P(s)$ are given by

$$\begin{aligned} P'(s) &= \sum_{k=1}^{\infty} k p_k s^{k-1} \\ \text{and} \qquad P''(s) &= \sum_{k=1}^{\infty} k(k-1) p_k s^{k-2} \end{aligned} \tag{71}$$

Now the expectation of X i.e. $E(X)$ is given as

$$E(X) = \sum_{k=1}^{\infty} k p_k = P'(1) \tag{72}$$

also

$$E[X(X-1)] = \sum_{k=1}^{\infty} k(k-1) p_k = P''(1) \tag{73}$$

and

$$
\begin{aligned}
E(X^2) &= E[X(X-1)] + E(X) \\
&= P''(1) + P'(1) \\
V(X) &= E(X^2) - [E(X)]^2 \\
&= P''(1) + P'(1) - [P'(1)]^2
\end{aligned}
$$

This mean and variance can be obtained with a knowledge of p. g. f. . In fact moments and cumulants etc. can be expressed in terms of generating functions.

The k^{th} factorial moments of X is given as

$$
E[X(X-1)\ldots(X-k+1)] = \left[\frac{d^k P(s)}{ds^k}\right]_{s=1} \qquad \text{for } k = 1, 2, \ldots
$$

Also $P(e^t)$ is the moment generating function as

$$
\begin{aligned}
M_X(t) &= E[e^{tX}] \\
&= \sum_{k=0}^{\infty} p_k . e^{tk} \\
&= \sum_{k=0}^{\infty} p_k . s^k, \qquad \text{where } s = e^t
\end{aligned}
$$

Example 10. ***Poisson Distribution***

$$
\begin{aligned}
p_k &= \frac{e^{-\lambda}\lambda^k}{k!}, \qquad k = 0, 1, \ldots \\
P(s) &= \sum_{k=0}^{\infty} p_k . s^k \\
&= \sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} . s^k \\
&= \sum_{k=0}^{\infty} \frac{e^{-\lambda}(\lambda s)^k}{k!} \\
&= e^{-\lambda} . e^{\lambda s} = e^{\lambda(s-1)}
\end{aligned}
$$

Now

$$
\begin{aligned}
P'(s) &= e^{-\lambda}.\lambda e^{\lambda s} \\
P'(1) &= \lambda \\
P''(s) &= e^{-\lambda}.\lambda^2 e^{\lambda s} \\
P''(1) &= \lambda^2 \\
\textit{thus} \qquad E(X) &= \lambda \\
V(X) &= \lambda^2 + \lambda - \lambda^2 = \lambda
\end{aligned}
$$

Example 11. ***Geometric Distribution***

$$
\begin{aligned}
p_k &= p.q^k, \qquad k = 0, 1, \ldots \ \textit{and } p + q = 1 \\
P(s) &= \sum_{k=0}^{\infty} p_k.s^k \\
&= \sum_{k=0}^{\infty} p.q^k.s^k \\
&= p \sum_{k=0}^{\infty} .q^k.s^k \\
&= \frac{p}{1 - qs}
\end{aligned}
$$

we have

$$
\begin{aligned}
P'(s) &= \frac{pq}{(1 - qs)^2} \\
P'(1) &= \frac{p}{q} \\
P''(s) &= \frac{2pq^2}{(1 - qs)^3} \\
P''(1) &= \frac{2q^2}{p^2} \\
\textit{thus} \qquad E(X) &= \frac{p}{q} \\
V(X) &= P''(1) + P'(1) - [P'(1)]^2 \\
&= \frac{2q^2}{p^2} + \frac{p}{q} - \left[\frac{p}{q}\right]^2 = \frac{p}{q^2}
\end{aligned}
$$

Example 12. ***Binomial Distribution***

$$\begin{aligned}
p_k &= \tbinom{n}{k} p^k q^{n-k} \qquad k = 0, 1, 2, \ldots \\
P(s) &= \sum_{k=0}^{\infty} p_k s^k \\
&= \sum_{k=0}^{n} p_k s^k \\
&= \sum_{k=0}^{n} \tbinom{n}{k} p^k q^{n-k} s^k \\
&= (q + sp)^n \\
P^1(1) &= n(q + sp)^{n-1}.p \\
P^{11}(1) &= n(n-1).(q + sp)^{n-2}.p^2 \\
E(x) &= P^1(1) = np \\
V(x) &= P^{11}(1) + P^1(1) - [P^1(1)]^2 \\
&= n(n-1).p^2 + np - n^2 p^2 \\
&= npq
\end{aligned}$$

Example 13. *Let X be a random variable with p.g.f. $P(s)$. To find the p.g.f. of the random variable $Y = mx + n$*

Let$P_x(s)$ and $P_y(s)$ be the p.g.f.of X and Y respectively. We have,

$$\begin{aligned}
P_y(s) &= E[s^y] \\
&= E[s^{mx+n}] \\
&= E[s^{mx}.s^n] \\
&= s^n E[(s^m)^x] \\
&= s^n.P_x(s^m)
\end{aligned}$$

Similarly if X and Y are independent random variables then the p.g.f. of X and Y.

$$\begin{aligned}
P_z(s) &= E[s^z] \\
&= E[s^{x+n}] \\
&= E[s^x s^y] \\
&= E[s^x].E[s^y] \\
&= P_x(s).P_y(s)
\end{aligned}$$

In the above examples we have discussed the problem of finding P_s for a given set of p_k's.