IMPORTANT THEOREMS

1) Define conditional probability. Show that it satisfy all the axioms of probability.

Solⁿ.

- **Conditional Probability :-**

Classical Defⁿ: Conditional probability of the occurance of the event B given that A has already been occured, denoted by $P(B|A)$, is defined as,

$$P(B|A) = \frac{N(A \cap B)}{N(A)}, \text{ provided } N(A) > 0.$$

where, $N(A)$ is the no. of cases favorable to the event A, $N(A \cap B)$ is the no. of cases favorable to the simultaneous occurance of A and B.

If N be the total no. of equally likely elementary cases then

$$P(B|A) = \frac{N(A \cap B)/N}{N(A)/N} = \frac{P(A \cap B)}{P(A)}, \quad P(A) > 0.$$

$$\Rightarrow P(A \cap B) = P(B|A)\,P(A).$$

Axiomatic Defⁿ: Consider the probability space $(\Omega, \mathscr{A}, P)$ where Ω is the sample space, $\mathscr{A}$ is the σ-field of the subspace of Ω and P is the probability function defined on $\mathscr{A}$.

Let $A \in \mathscr{A} \ni P(A) > 0$, then conditional probability of occurance of any event B belonging to $\mathscr{A}$ given that A has already been occured is defined as.

$$P(B|A) = \frac{P(A \cap B)}{P(A)}.$$

- **Conditional Probability satisfies all the axioms of Probability:**

i) We have $P(A \cap B) \geq 0 \;\forall B$ and $(B|A)$; and $P(A) > 0$.

So, $\frac{P(A \cap B)}{P(A)} \geq 0 \;\forall B$.

i.e.. $P(B|A) \geq 0$ for any $B \in \mathscr{A}$.

$\Rightarrow$ Axiom I of probability.

ii) Since, $(\Omega \cap A) = A$.

$\therefore P(\Omega | A) = \frac{P(\Omega \cap A)}{P(A)} = \frac{P(A)}{P(A)} = 1 \quad (\because P(A) > 0)$.

$\Rightarrow$ Axiom II of probability.

iii) Let us consider a sequence of disjoint events $\{C_n\}$, $C_n \in \mathcal{E} \; \forall n$.

Now, $P\left[\bigcup_{n=1}^{\infty} C_n | A\right] = \frac{P\left[\left(\bigcup_{n=1}^{\infty} C_n\right) \cap A\right]}{P(A)}$, $P(A) > 0$, where $\{C_n \cap A\}$ is also a sequence of disjoint event.

$= \frac{P\left[\bigcup_{n=1}^{\infty} (C_n \cap A)\right]}{P(A)}$ as $\{C_n\}$ is so, and $(C_n \cap A) \in \mathcal{E} \; \forall n$.

$= \frac{\sum_{n=1}^{\infty} P(C_n \cap A)}{P(A)}$ [by the axiom of unconditional prob

$= \sum_{n=1}^{\infty} \frac{P(C_n \cap A)}{P(A)} = \sum_{n=1}^{\infty} P(C_n | A)$

$\Rightarrow$ Axiom III of probability

Hence the proof.

2) What do you mean by stochastic independence of events?

Soln $\rightarrow$ The event A is said to be stochastically independent of the event B if occurance of A does not depend upon the occurance or non-occurance of B, i.e. $P(A|B) = P(A)$, $P(B) > 0$.

$$\Rightarrow \frac{P(A \cap B)}{P(B)} = P(A).$$

$$\Rightarrow P(A \cap B) = P(A)\,P(B) \quad \text{— (i)}$$

Similarly B is said to be stochastically independent of the event A if:

$$P(B|A) = P(B), \; P(A) > 0.$$

$$\Rightarrow P(A \cap B) = P(A)\,P(B) \quad \text{— (ii)}$$

Note that the expression (i) is symmetric in A & B, Hence instead of saying A is independent of B or B is independent of A, one must say A & B are independent of each other.

Remark: If two events are mutually exclusive then they will not be stochastically independent of each other.

3) State and proof Compound Probability Theorem.

Sol^n

Statement :→ (Compound Probability) The probability of simultaneous occurance of A and B is given by the product of the unconditional probability of the event A by the conditional probability of B, supposing that A actually occurred. In other words.

$$P(A\cap B) = P(A)\,P(B|A).$$

Proof :→

Let there be n no. of all possible outcomes, of these

n_A = no. of outcomes favorable to A.

n_B = no. of outcomes favorable to B.

n_{AB} = no. of outcomes favorable to A and B.

Then, $P(A) = \frac{n_A}{n}$, $P(A\cap B) = \frac{n_{AB}}{n}$ and $P(B|A) = \frac{n_{AB}}{n_A}$.

$$P(A\cap B) = \frac{n_{AB}}{n}$$

$$= \frac{n_A}{n} \times \frac{n_{AB}}{n_A}$$ [It is supposed that A has actually been occured, i.e., $P(A) > 0$ and hence $n_A > 0$]

$$= P(A)\,P(B|A).$$

Hence the theorem is proved.

In general case, if $A_1, \dots, A_n$ be any events in $\mathscr{E}$, then by induction

$$P\left(\bigcap_{i=1}^{n} A_i\right) = P(A_1)\,P(A_2|A_1)\,P(A_3|A_1\cap A_2) \cdots P(A_n|A_1\cap \dots \cap A_{n-1}),$$

provided $P(A_1\cap \dots \cap A_{n-1}) > 0$.

— This is called 'Law of Multiplication'.

4) State & proof the theorem of total probability.

$\underline{\text{Sol}^n} \rightarrow$

Total Probability

Theorem: Let $(\Omega, \mathcal{A}, P)$ be the probability space, suppose $\{H_n\}$ is a sequence of mutually exclusive and exhaustive events such that $P(H_n) > 0 \ \forall n$, $H_n \in \mathcal{A} \ \forall n$.

Then the probability of any event $B \in \mathcal{A}$ is given by

$$P(B) = \sum_{n=1}^{\infty} P(H_n) P(B \mid H_n)$$

Proof: Since $\{H_n\}$ is a sequence of mutually exclusive and exhaustive events,

$$\therefore \bigcup_{n=1}^{\infty} H_n = \Omega.$$

Now, $B = B \cap \Omega$.

$$\therefore P(B) = P\left[B \cap \left(\bigcup_{n=1}^{\infty} H_n\right)\right] = P\left[\bigcup_n (B \cap H_n)\right]$$

Note that $H_i \cap H_j = \phi \ \forall i \neq j$

$$\Rightarrow (B \cap H_i) \cap (B \cap H_j) = \phi \ \forall i \neq j.$$

clearly, $\{B \cap H_n\}$ is also a sequence of mutually disjoint events $\in \mathcal{A}$.

Hence by Axiom-III, we have $\rightarrow P\left(\bigcup_n (B \cap H_n)\right) = \sum_n P(B \cap H_n)$

Thus, $P(B) = \sum_n P(B \cap H_n)$.

So, $P(B) = \sum_n P(H_n) P(B \mid H_n)$ [From the axiom of compound probability]

Hence the proof.

Implication:- The implication of this result is that the unconditional probability of the event B can be obtained as the weighted average of the conditional probabilities.

Application of Total Probability Theorem:-

1. A box has 12 red and 6 black balls. A ball is selected from the box. If it is red, it is returned to box. If the ball is black, it and 2 additional balls are added to the box. Find the probability that a second ball drawn from the box is
   (i) red (ii) black.

Sol. Let R_i and B_i respectively be the event that the i^{th} ball drawn is red and that the i^{th} ball drawn is black for $i=1,2$.

$$P(R_1) = \frac{12}{18}, \quad P(B_1) = \frac{6}{18}$$

$$P(R_2|R_1) = \frac{12}{18}, \quad P(R_2|B_1) = \frac{12}{20}$$

$$P(B_2|R_1) = \frac{6}{18}, \quad P(B_2|B_1) = \frac{8}{20}$$

(i) $P(R_2) = P(R_1)\,P(R_2|R_1) + P(B_1)\,P(R_2|B_1)$

$$= \frac{12}{18} \times \frac{12}{18} + \frac{6}{18} \times \frac{12}{20}$$

$$= \frac{29}{45}.$$

(ii) $P(B_2) = P(R_1)\,P(B_2|R_1) + P(B_1)\,P(B_2|B_1)$

$$= \frac{12}{18} \times \frac{6}{18} + \frac{6}{18} \times \frac{8}{20}$$

$$= \frac{16}{45}.$$