5) State and proof Bayes theorem.

Solⁿ

Statement: (Bayes' Theorem)

For a sequence of mutually exclusive and exhaustive events $A_1, A_2, \ldots \in \mathcal{A}$ with $P(A_i) > 0$ $\forall i = 1, 2, \ldots$

$$P(A_j|B) = \frac{P(A_j)\,P(B|A_j)}{\sum_{i=1}^{\infty} P(A_i)\,P(B|A_i)}$$, where B is any other event.

Proof: Since $A_1, A_2, \ldots$ are mutually exclusive and exhaustive events, $P(A_i) > 0$,

$$\therefore \sum_{i=1}^{\infty} P(A_i) = P\left(\sum_{i=1}^{\infty} A_i\right) = P(\Omega) = 1.$$

$$B = B \cap \Omega = B \cap \left(\sum_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} (B \cap A_i)$$

$$\therefore P(B) = P\left(\sum_{i=1}^{\infty} (B \cap A_i)\right)$$ [$(B \cap A_i)$ is a sequence of mutually disjoint events $\in \mathcal{A}$, applying Axiom III].

$$= \sum_{i=1}^{\infty} P(B \cap A_i)$$

$$= \sum_{i=1}^{\infty} P(A_i)\,P(B|A_i) \quad \text{—— (i)}$$

Now, $$P(A_j|B) = \frac{P(A_j \cap B)}{P(B)}$$

$$= \frac{P(A_j)\,P(B|A_j)}{\sum_{i=1}^{\infty} P(A_j)\,P(B|A_j)}$$, by (i) and since $P(B) > 0$.

Hence the theorem is proved.

Application of Baye's Theorem:-

(1) In answering a question on a multiple-choice test, an examinee either knows the answer (with probability p) or he guesses (with probability $1-p$). Let the probability of answering the question correctly be 1 for an examinee who knows the answer and $1/m$ for one who guesses (m being the number of multiple choice alternatives). What's the prob. that he really knows the answer? Supposing an examinee answers a question correctly.

Solution:- Let A_1 be the event that an examinee knows the answer
A_2 " " " " " guesses " "
Now B be the event that the answer is correct.

$P(A_1)=p$, $P(A_2)=1-p$, since A_1, A_2 are m.e. & exhaustive events.

$P(B|A_1)=1$ and $P(B|A_2)=\frac{1}{m}$.

By Bayes theorem, $$P(A_1|B)=\frac{P(A_1)P(B|A_1)}{P(A_1)P(B|A_1)+P(A_2)P(B|A_2)}$$

$$=\frac{p.1}{p.1+(1-p)\frac{1}{m}}$$

$$=\frac{mp}{1+(m-1)p}.$$

(2) There are two drawers in each of three boxes that are identical in appearence. The first one contains a gold coin in each drawer, the second contains a silver coin in each drawer, but the third contains a gold coin in one drawer and a silver coin in the other. A box is chosen, one of its drawers is opened and a gold coin is found. What's the probability that the other drawer too will have a gold coin?

Solution:- Let A_1 be the event that the first box is chosen.
A_2 " " " " second " " " .
A_3 " " " " " third " " " ,

and B be the event that the second one is a gold coin.

$P(B|A_1)=1$, $P(B|A_2)=0$, $P(B|A_3)=\frac{1}{2}$.

$P(A_1)=P(A_2)=P(A_3)=\frac{1}{3}$.

By Bayes' theorem,
$$P(A_1|B)=\frac{P(A_1)P(B|A_1)}{\sum_{i=1}^{3}P(A_i)P(B|A_i)}$$

$$=\frac{1/3\times 1}{1/3\times 1+1/3\times 0+1/3\times\frac{1}{2}}$$

$$=\frac{2}{3}.$$

(3) In a doll factory, machines M_1, M_2 and M_3 manufacture respectively 45, 25 and 30 percent of the total output. Of their output, 6, 8, 3 percent respectively are defective. What is the probability that it was manufactured by M_1?

<u>Sol.</u> B_i : Chosen ball is manufactured by machine M_i
A : Chosen doll is defective

By Baye's theorem,

$$P(B_i|A) = \frac{P(B_i)\,P(A|B_i)}{\sum_{i=1}^{3} P(B_i)\,P(A|B_i)}, \quad i=1,2,3$$

$$P(M_1|A) = \frac{45\times 6}{45\times 6 + 25\times 8 + 30\times 3} = \frac{27}{56}.$$

(4) An urn containing 5 balls has been filled up by taking 5 balls from another urn containing 5W & 5B balls. A ball is taken at random from urn 1 and it happens to be black. What's the prob. of drawing a white ball from the remaining?

<u>Sol.</u> Let B_i denotes that among 5 balls kept in urn 1, exactly i^{th} are white.
A : the first ball taken from Urn 1 is black.
C : the second ball drawn from Urn 1 is white.

$$P(C|A) = \frac{\sum_{i=0}^{4} P(B_i)\,P(A|B_i)\,P(C|A\cap B_i)}{\sum_{i=0}^{5} P(B_i)\,P(A|B_i)}, \text{ by Extended Baye's theorem.}$$

$$P(B_i) = \binom{5}{i}\binom{5}{5-i}\Big/\binom{10}{5}, \quad i=0(1)5$$

$$P(A|B_i) = \frac{5-i}{5}, \quad P(C|A\cap B_i) = \frac{i}{4}, \quad i=0(1)4.$$

i	$P(B_i)$	$P(A\|B_i)$	$P(C\|A\cap B_i)$
0	$1/\binom{10}{5}$	1	0
1	$25/\binom{10}{5}$	4/5	1/4
2	$100/\binom{10}{5}$	3/5	1/2
3	$100/\binom{10}{5}$	2/5	3/4
4	$25/\binom{10}{5}$	1/5	1
5	$1/\binom{10}{5}$	0	

$$\therefore P(C|A) = \frac{5/18}{1/2} = \frac{5}{9}.$$