Axiomatic Approach

1) Explain the concept of Kolmogorov's Axiomatic def^n of probability. Using this show that —
   i) $P(\phi) = 0$, when ϕ is null set.
   ii) $P(A) \leq 1$, for any event A.

Sol^n

Axiomatic Definition :— Let Ω be the sample space of a random experiment and $\mathcal{E}$ be a σ-field of events of Ω. A set function $P(\cdot)$ defined on $\mathcal{E}$ is called a probability measure if it satisfies the following conditions:

Axiom I (Axiom of non-negativity): $P(A) \geq 0 \ \forall A \in \mathcal{E}$.

Axiom II (Axiom of unit-norm): $P(\Omega) = 1$.

Axiom III (Axiom of countable additivity): If $A_i, i = 1(1)\infty$ be a disjoint sequence of events in $\mathcal{E}$, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

In axiomatic approach probability is regarded as a set function

i) Let $A_1, A_2, \ldots$ be events in $\mathcal{E} \ni A_i = \phi, \forall i$. Then $\bigcup_{i=1}^{\infty} A_i = \phi$ and since $A_i \cap A_j = \phi \cap \phi = \phi, \forall i \neq j$.

Then A_i's are also mutually exclusive (i.e. disjoint)

$\therefore$ By the axiom of countable additivity, we have

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

$$\text{or, } P(\phi) = P(\phi) + P(\phi) + P(\phi) + \cdots$$

But this can happen if either $P(\phi) = 0$ or, $P(\phi) = \infty$ or $-\infty$

But since P is a finite real valued function, so $P(\phi) = \infty$ or $-\infty$ is not possible.

So, $P(\phi) = 0$. (Proved)

ii) As $A \subset \Omega$ for each $A \in \mathcal{E}$.

$\Rightarrow P(A) \leq P(\Omega)$.

Now, from the axiom of unit norm, we know $P(\Omega) = 1$.

So, we get $\rightarrow P(A) \leq 1$ for any event A.

OR

$A \cup A^c = \Omega$, $A \cap A^c = \phi$ $\quad \because A \subset A^c$

by finite additivity of $P[\cdot]$,

$$P[A \cup A^c] = P[A] + P[A^c] - P[A \cap A^c]$$

$$\therefore P(\Omega) = P(A) + P(A^c)$$

$$0 \leq P(A^c) = 1 - P(A), \text{ by Axiom I.}$$

$$\therefore P[A] \leq 1.$$

2) (a) Let $A_1, \ldots, A_n$ be n events $\ni P(A_i) = 1$ $\forall i = 1(1)n$.

(b) Let $A_1, A_2, \ldots$ be the events $\ni P(A_i) = 0$, $\forall i = 1, 2, \ldots$ then show that $P\left(\bigcup_{i=1}^{\infty} A_i\right) = 0$.

(c) If the events A_i's are mutually exclusive and exhaustive events of Ω, $i = 1, 2, \ldots$. S.T. $\sum_i P(A_i) = 1$.

Sol^n →

(a) If A_i, $i = 1(1)n$ be events in $\mathcal{A}$, then Bonferroni inequality gives —

$$P\left(\bigcap_{i=1}^{n} A_i\right) \geq \sum_{i=1}^{n} P(A_i) - n + 1 \quad \text{——— (i)}$$

From the axiom of unit norm, $P(\Omega) = 1$.

As $A \subset \Omega$, $\forall A \in \mathcal{A}$.

$\therefore P(A) \leq P(\Omega) = 1 \quad \therefore P(A) \leq 1$ ——— (ii)

Here, $P(A_i) = 1 \;\forall i = 1(1)n$ ——— (iii)

So, From (i), (ii), (iii) we get $P\left(\bigcap_{i=1}^{n} A_i\right) = 1$.

(b) If $P(A_i) = 0$, we know from Boole's inequality

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i) \text{ and } P(A) \geq 0.$$

So, if $P(A_i) = 0 \;\forall i \geq 1$, we get, $P\left(\bigcup_{i=1}^{n} A_i\right) = 0$.

Hence the result is proved.

(c) Since, A_i's are exhaustive events, then $\bigcup_i A_i = \Omega$

$\therefore P\left(\bigcup A_i\right) = P(\Omega) = 1$.

Again A_i's are mutually exclusive, $P\left(\bigcup A_i\right) = P\left(\sum_i A_i\right)$

$\therefore P\left(\sum_i A_i\right) = 1$ i.e. $\sum_i P(A_i) = 1$. [By the principle of countable additivity of $P(\cdot)$]