INDEPENDENCE OF EVENTS

1) i) Define mutually exclusive, exhaustive and mutually independent events. Let the two events be mutually exclusive are they mutually independent.

ii) Show by an example that pairwise independence does not necessarily imply mutual independence.

iii) Distinguish between pairwise and mutual independence of a finite set of events.

iv) Show that if A_1, A_2, A_3 are mutually independent then A_1^c, A_2^c, A_3^c are also mutually where A_i^c is the complement of A.

Solⁿ → i) Mutually exclusive events : Several events $A_1, A_2, \ldots, A_n$ in relation to a random experiment are said to be mutually exclusive (or, disjoint) if any two of them can't occur simultaneously. Everytime the experiment is performed is $A_i \cap A_j = \phi \;\forall\; i \neq j,\; i<j = 1(1)n$.

Exhaustive Events : several events in relation to a random experiment are said to be exhaustive events if at least one of them necessarily occurs. Thus the events $A_1, A_2, \ldots, A_n$ or $A_1, A_2, \ldots$ are exhaustive if

$$\bigcup_{i=1}^{\infty} A_i = \Omega .$$

Pairwise Independence of a set of events :

A set of events $\{A_1, A_2, \ldots, A_n\}$ is said to be pairwise independent if

$$P(A_i \cap A_j) = P(A_i)P(A_j), \quad i \neq j,\; i<j.$$

Here we have $\binom{n}{2}$ restrictions.

Mutually independence of a set of events :

A set of events $\{A_1, A_2, \ldots, A_n\}$ is said to be mutually independent if

$$P(A_i \cap A_j) = P(A_i)P(A_j), \quad i \neq j$$

$$P(A_i \cap A_j \cap A_k) = P(A_i)P(A_j)P(A_k), \quad i<j<k.$$

$$\vdots$$

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)P(A_2)\cdots P(A_n).$$

$$\text{i.e. } P\left[\bigcap_{i=1}^{n} A_i\right] = \prod_{i=1}^{n} P[A_i]$$

Scanned by CamScanner

The idea of mutual independent emerges from the following fact.

$$P(A_1 \cap A_2 \cap \dots \cap A_n) = P(A_1)\,P(A_2|A_1)\,P(A_3|A_1 \cap A_2) \dots P(A_n | \bigcap_{i=1}^{n-1} A_i).$$

Under statistical independence if all the conditional probabilities become equal to the respective unconditional probabilities, then we get —

$$P(A_1 \cap A_2 \cap \dots \cap A_n) = P(A_1)\,P(A_2) \dots P(A_n)$$

Here we have $(2^n - 1 - n)$ restrictions.

If two events are mutually exclusive then they will not be mutually independent.

A fair coin is tossed twice, $\Omega = \{HH, TH, HT, TT\}$

A: Two head appears $= \{HH\}$

B: One head & one tail appear $= \{HT, TH\}$

("exactly one head appears")

This two events are mutually exclusive.

$$A \cap B = \{\phi\}$$

$$\therefore P(A \cap B) = 0.$$

$$P(A) = \frac{1}{4},\ P(B) = \frac{2}{4} = \frac{1}{2}.$$

$$\therefore P(A \cap B) \neq P(A)\,P(B) = \frac{1}{8}.$$

They are not mutually independent

Note: → Mutually exclusive events in general are no independent and also independent events are not in general mutually exclusive.

ii) & iii)

Distinction between Pairwise Independence and Mutually Independence:

$A_1, A_2, \dots, A_n$ are pairwise independent if $P(A_i \cap A_j) = P(A_i)\,P(A_j)\ \forall\, i,j\ (i<j)$, but for mutually independence it is necessary that all of the $(2^n - n - 1)$ equation hold as mentioned earlier. It is evident that mutually independence implies pairwise independence but the converse may not be true. An example to show that pairwise independence does not imply mutually independence.

Suppose a fair coin is tossed twice.

Let A: the first toss gives a head.

B: the second toss gives a head.

C: both give the same outcome.

$\Omega = \{HH, HT, TH, TT\}$

$A = \{HH, HT\}$ $\quad B \cap C = \{HH\}$ $\quad P(A) = P(B) = P(C) = \frac{1}{2}$

$B = \{HH, TH\}$ $\quad A \cap C = \{HH\}$

$C = \{HH, TT\}$ $\quad A \cap B \cap C = \{HH\}$

$A \cap B = \{HH\}$

$\therefore P(A \cap B) = \frac{1}{4} = P(A)P(B)$; $P(B \cap C) = \frac{1}{4} = P(B)P(C)$;

$P(A \cap C) = \frac{1}{4} = P(A)P(C)$.

$\therefore$ A, B, C are pairwise independent.

$\therefore P(A \cap B \cap C) = \frac{1}{4} \neq P(A)P(B)P(C)$.

$\therefore$ A, B, C are not mutually independent.

iv) We know that mutually independence does necessarily imply pairwise independence.

So, A_1, A_2, A_3 are both mutually and pairwise independent.

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3); \quad P(A_1 \cap A_2) = P(A_1)P(A_2).$$

$$P(A_2 \cap A_3) = P(A_2)P(A_3); \quad P(A_1 \cap A_3) = P(A_1)P(A_3).$$

$$\begin{aligned}
P(A_1^c \cap A_2^c \cap A_3^c) &= 1 - P(A_1 \cup A_2 \cup A_3) \\
&= 1 - [P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) \\
&\quad - P(A \cap C) + P(A \cap B \cap C)] \\
&= \{1 - P(A)\} - P(B)(1 - P(A)) - P(C)\{1 - P(A)\} \\
&\quad + P(B)P(C)\{1 - P(A)\} \\
&= \{1 - P(A)\}\{1 - P(B) - P(C) + P(B)P(C)\} \\
&= \{1 - P(A)\}\{(1 - P(B)) - P(C)\{1 - P(B)\}\} \\
&= \{1 - P(A_1)\}\{1 - P(A_2)\}\{1 - P(A_3)\} \\
&= P(A_1^c)P(A_2^c)P(A_3^c)
\end{aligned}$$

$\therefore A_1^c, A_2^c, A_3^c$ are mutually independent if A_1, A_2, A_3 are mutually independent.

Scanned by CamScanner

2) In a sample space of 8 equally likely points, find the following:

i) Three events that are pairwise independent, but not mutually independent.

ii) Three events that are mutually independent.

Solⁿ → i) Consider a random experiment of a ~~certain~~ coin is thrown thrice.

Sample space is, $\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

and $P[\omega] = \frac{1}{8}$; $\omega \in \Omega$.

Define, A_1 : At least two heads $= \{HHH, HHT, HTH, THH\}$

$A_2 : \{HHH, HHT, HTT, THT\}$

$A_3 : \{HHH, HHT, TTH, TTT\}$.

$P(A_i) = \frac{1}{2} \quad \forall\, i = 1, 2, 3$.

Now, $P(A_1 \cap A_2) = P(\{HHH, HHT\}) = \frac{2}{8} = \frac{1}{4}$

$= P(A_1) P(A_2)$

$P(A_2 \cap A_3) = P(\{HHH, HHT\}) = \frac{1}{4}$

$= P(A_2) P(A_3)$.

$\therefore A_1, A_2, A_3$ are pairwise independent.

Now, $P[A_1 \cap A_2 \cap A_3] = P[\{HHH, HHT\}]$

$= \frac{1}{4} \neq \frac{1}{8} = P(A_1) P(A_2) P(A_3)$.

ii) $A : \{HHH, THT, HTH, HHT\}$

$B : \{HHH, THT, HTT, THH\}$

$C : \{HHH, HTT, HTH, TTT\}$

$P(A) = P(B) = P(C) = \frac{1}{2}$

$P(A \cap B) = \frac{1}{4} = P(A) P(B)$; $P(B \cap C) = \frac{1}{4} = P(B) P(C)$

$P(A \cap C) = \frac{1}{4} = P(A) P(C)$.

$P(A \cap B \cap C) = \frac{1}{8} = P(A) P(B) P(C)$.

$\therefore A, B, C$ are pairwise independent as well as mutual independent.

Scanned by CamScanner