DISCRETE & CONTINUOUS RANDOM VARIABLES

[A]. Discrete Random Variable :-

Definition:— A random variable X takes only a countable (finite or infinite) number of isolated values $x_1, x_2, \ldots, x_n, \ldots$ with $P[X=x_i]>0$ $\forall i$, is called a discrete random variable.

The points $x_1, x_2, \ldots$ that have positive probabilities of occurance are called the jump or mass points of the r.v. X.

Probability Mass Function:— (PMF) Let X be a discrete R.V. with mass points $x_1, x_2, \ldots\ldots$ Then $\Omega = \bigcup_{i=1}^{\infty} \{\omega : X(\omega) = x_i\}$ and

$$1 = P(\Omega) = \sum_{i=1}^{\infty} P[\{\omega : X(\omega) = x_i\}] \quad [\text{By countable additivity of } P[\cdot]]$$

$$= \sum_{i=1}^{\infty} P[X = x_i].$$

Definition:- Let X be a discrete RV with mass points $\{x_1, x_2, \ldots\}$. Then the function

$$f(x) = \begin{cases} P[X=x_i] & \text{if } x = x_i,\ i=1,2,\ldots \\ 0 & \text{if } x \neq x_i, \end{cases}$$

is called the PMF of the RV X.

Theorem:- A function $f(x)$ is said to be a PMF of some discrete RV X if
(i) $f(x) \geq 0 \ \forall x \in \mathbb{R}$ (ii) $\sum_x f(x) = 1$.

Alternative definition:- The probability mass function $f(x)$ of a RV X whose set of possible values are $\{x_1, x_2, \ldots\}$ is a function from $\mathbb{R}$ to $\mathbb{R}$ that satisfies the following properties :

(i) $f(x) = 0$ if $x \neq x_i$
(ii) $f(x) = P[X = x_i]$ if $x = x_i$, $i = 1, 2, \ldots$
(iii) $\sum_x f(x) = 1$.

Given the pmf of a discrete distribution, we can get the distribution function by successive addition, i.e.,

$$F(x) = f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_n), \text{ where}$$

$$x_1 < x_2 < x_3 < \cdots < x_n < x < x_{n+1} < \cdots$$

On the other hand, given the df, we can get the pmf by successive subtraction, i.e.,

$$f(x_i) = F(x_i) - F(x_{i-1}) = P[X \leq x_i] - P[X < x_i]$$

is the probability at the point x_i.

Ex.1. For what values of θ and c is the function f given by

$$f(x)=\begin{cases} c.\dfrac{\theta^x}{x} & , x=1,2,3,\dots \\ 0 & , \text{otherwise} \end{cases}$$

a PMF ?

Solution:- (i) As $f(x)>0 \ \forall \ x=1,2,3,\dots$

Hence, $c>0$, $\theta>0$.

(ii) $$1=\sum_x f(x)=\sum_{x=1}^{\infty} c.\frac{\theta^x}{x}$$

$$= c\sum_{x=1}^{\infty}\frac{\theta^x}{x}$$

$$= c\left\{-\log_e(1-\theta)\right\} \quad , \text{if } 0<\theta<1.$$

$$\therefore \ c=-\frac{1}{\log_e(1-\theta)} \text{ and } 0<\theta<1.$$

Ex.2. Let $f(x)=\begin{cases} pq^x & , x=0,1,2,3,\dots ; \ p+q=1, \ 0<p<1. \\ 0 & , \text{OW} \end{cases}$

Does $f(x)$ define a PMF of some RV X? What is the DF of x? Find $P[n\le X\le m]$, $n,m\in\mathbb{N}$.

Solution:- (i) $0<p<1$

$\Rightarrow 1-p>0$

$\Rightarrow (1-p)^x>0 \quad [\because x=0,1,2,3,\dots]$

$\Rightarrow p(1-p)^x>0 \quad [\because 0<p<1]$

$\therefore f(x)\geqslant 0.$

(ii) $$\sum_{x=0}^{\infty} f(x)=\sum_{x=0}^{\infty} pq^x=p\sum_{x=0}^{\infty} q^x = \frac{p}{1-q}=\frac{p}{p}=1.$$

$\therefore f(x)$ defines a PMF of some RV X.

(iii) $$F(x)=P[X\le x]=p+pq+pq^2+\dots+pq^x$$

$$=p\left[1+q+q^2+\dots+q^x\right]$$

$$=p.\frac{1-q^{x+1}}{1-q} \quad [x=0,1,2,\dots]$$

$$=1-q^{x+1};$$

(iv) $$P[n\le X\le m]=P[X\le m]-P[X<n]$$

$$=P[X\le m]-P[X\le n-1]$$

$$=F(m)-F(n-1)=\left\{1-q^{m+1}\right\}-\left\{1-q^n\right\}$$

$$=(q)^n-q^{m+1}.$$

Ex.3. Find the PMF of the RV X whose DF is

$$F(x) = \begin{cases} 0 & , x < 0 \\ \frac{i(i+1)}{n(n+1)} & , i \le x \le i+1 , i = 0, 1, \dots, (n-1), \\ 1 & , x \geq n. \end{cases}$$

Solution:- Note that $i = 1, 2, 3, \dots, n$.

$$P[X=i] = P[X \le i] - P[X < i]$$

$$= F(i) - F(i-0)$$

$$= \frac{i(i+1)}{n(n+1)} - \frac{i(i-1)}{n(n+1)}$$

$$= \frac{2i}{n(n+1)}.$$

$$F(i-0) = P[X<i] = P[X \le i-1] = F(i-1) = \frac{(i-1)i}{n(n+1)}.$$

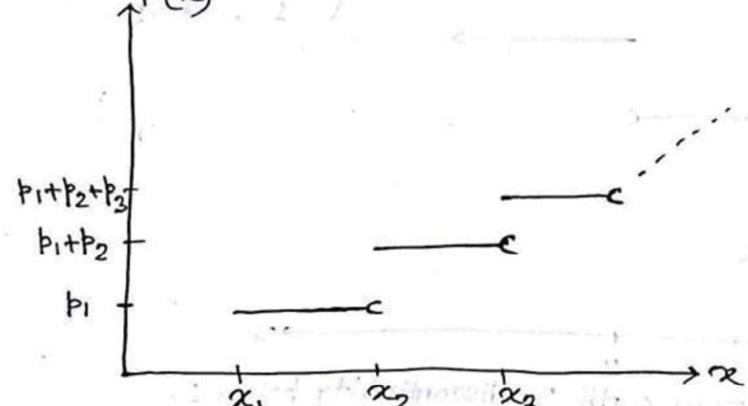
The PMF of X is $f(x) = \begin{cases} \frac{2x}{n(n+1)} & , x = 1(1)n, \\ 0 & , \text{ow}. \end{cases}$

Distribution Function of Discrete Random Variables:-

Let X be a discrete R.V. with mass points $x_1 < x_2 < \dots\dots$ Then the D.F. is

$$F(x) = P[X \le x] = \begin{cases} 0 & , x < x_1 \\ P[X=x_1] & , x_1 \le x < x_2 \\ \sum_{i=1}^{2} P[X=x_i] & , x_2 \le x < x_3 \\ \vdots & \end{cases}$$

$$= \begin{cases} 0 & , x < x_1 \\ \sum_{i=1}^{K} P[X=x_i] & , x_K \le x < x_{K+1} , K = 1, 2, 3, \dots \end{cases}$$



Hence, the DF F(x) of a discrete RV, the discontinuity points are the mass points of the RV. The number of discontinuity points is the same as the no. of mass points.

Ex.6. Can a function of the form $f(x) = \begin{cases} c\left(\frac{2}{3}\right)^x & , x = 1, 2, 3, \ldots \\ 0 & , \text{otherwise} \end{cases}$

be a probability mass function (PMF)?

Solution:-

Note that, here $f(x) \geq 0$ if $c \geq 0$.

And, to be a p.d.f, the below condition also needs to be satisfie

$$\sum_{i=1}^{\infty} c\left(\frac{2}{3}\right)^i = 1$$

$$\Rightarrow c \cdot \frac{2/3}{1 - 2/3} = 1$$

$$\Rightarrow c = 1/2 .$$

Thus, only for $c = 1/2$, $f(x)$ can be a PMF.

Ex.7. Let X be the number of births in a hospital until the first girl is born. Assume that the probability is $1/2$ that a baby born is a girl. Determine the PMF and DF of X.

Solution:- X is an r.v. that can assume any positive integer i, $f(i) = P(X = i)$, and X_i : i occurs if the first $i-1$ births are all boys and the i^{th} birth is a girl.

Thus $f(i) = \left(\frac{1}{2}\right)^{i-1}\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^i$ for $i = 1, 2, 3, \ldots\ldots$

and $f(x) = 0$ if $x \neq 1, 2, 3, \ldots\ldots$

$$F(t) = \begin{cases} 0 & \text{if } t < 1 \\ 1/2 & \text{if } 1 \leq t < 2 \\ 1/2 + 1/4 & \text{if } 2 \leq t < 3 \\ \frac{1}{2} + \frac{1}{4} + \frac{1}{8} & \text{if } 3 \leq t < 4 \\ \vdots & \\ \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}, & \text{if } (n-1) \leq t < n \end{cases}$$

$$\text{So, } F(t) = \begin{cases} 0 & \text{if } t < 1 \\ \sum_{i=1}^{n-1}\left(\frac{1}{2}\right)^i & \text{if } \overline{n-1} \leq t < n, \; n = 2, 3, 4, \ldots\ldots \end{cases}$$

$$= \begin{cases} 0 & \text{if } t < 1 \\ 1 - \left(\frac{1}{2}\right)^{n-1} & \text{if } \overline{n-1} \leq t < n \;\; \forall\, n = 2, 3, 4, \ldots\ldots \end{cases}$$