⇨ DISTRIBUTION FUNCTION / CUMULATIVE DISTRIBUTION FUNCTION:—

Definition:- Let X be a random variable defined on $(\Omega, \mathcal{Q}, P)$. Define a point function $F(\cdot)$ on $\mathbb{R}^1$ by

$F(x) = P\{\omega : X(\omega) \le x\}$, for all $x \in \mathbb{R}^1$, is called the distribution function of R.V. X.

Properties:- (Alternative Definition*)

A real valued function $F(x)$ defined on $\mathbb{R}$ [or $(-\infty, \infty)$] which satisfies the following properties:

⟨i⟩ $x_1 < x_2 \Rightarrow F(x_1) \le F(x_2) \ \forall \ x_1, x_2 \in \mathbb{R}^1$.
i.e. $F(x)$ is monotonically non-decreasing.

⟨ii⟩ $F(-\infty) = \lim_{x \to -\infty} F(x) = 0$

⟨iii⟩ $F(+\infty) = \lim_{x \to +\infty} F(x) = 1$.

⟨iv⟩ $F(x+0) = \lim_{h \to 0+} F(x+h) = F(x) \ \forall \ x \in \mathbb{R}^1$.
i.e. $F(x)$ is right continuous, is called a distribution function of X.

Proof of the properties of distribution function:-

⟨i⟩ $x_1 < x_2$

$\Rightarrow \{X \le x_1\} \subseteq \{X \le x_2\}$

so, by the monotonicity theorem of probability,

$$P(X \le x_1) \le P(X \le x_2)$$

i.e. $F(x_1) \le F(x_2)$

⟨ii⟩ Let us take a sequence of events $B_n = \{X \le -n\}$, $n = 1, 2, \dots$

$\therefore B_n$ is a contracting sequence of events, i.e., monotonically decreasing. Hence, by continuity theorem,

$$\lim_{n \to \infty} P(B_n) = P(\lim_{n \to \infty} B_n)$$

$$\lim_{n \to \infty} P(X \le -n) = P(\lim_{n \to \infty} \{X \le -n\})$$

or, $\lim_{n \to \infty} P(X \le -n) = P(\phi)$

or, $\lim_{n \to \infty} F(-n) = 0 \Rightarrow F(-\infty) = 0$.

⟨iii⟩ Let us take a sequence $A_n = \{X \le n\}$

$\therefore A_n$ is an expanding sequence of events, i.e., monotonically increasing. Hence, by continuity theorem,

$$\lim_{n \to \infty} P(A_n) = P(\lim_{n \to \infty} A_n)$$

$$\lim_{n \to \infty} P(X \le n) = P(\lim_{n \to \infty} (X \le n))$$

or, $\lim_{n \to \infty} P(X \le n) = P(\Omega)$

or, $\lim_{n \to \infty} F(n) = 1$.

or, $F(\infty) = 1$.

⟨iv⟩ Let us take a sequence of events $C_n = \{X \le x + \frac{1}{n}\}, n = 1, 2, \ldots$

$\therefore$ C_n is a contracting sequence of events, i.e., monotonically decreasing. Hence, by continuity theorum,

$$P(\lim_{n\to\infty} C_n) = \lim_{n\to\infty} P(C_n)$$

$$\therefore P\left(\lim_{n\to\infty}\{X \le x + \tfrac{1}{n}\}\right) = \lim_{n\to\infty} P\left(X \le x + \tfrac{1}{n}\right)$$

$$\text{i.e. } P(X \le x) = \lim_{n\to\infty} P\left(X \le x + \tfrac{1}{n}\right)$$

$$\text{i.e. } F(x) = \lim_{n\to\infty} F\left(x + \tfrac{1}{n}\right)$$

Take, $\frac{1}{n} = h$, as $n \to \infty$, $h \to 0$.

$$\lim_{n\to\infty} F(x+h) = F(x) \quad \text{or,} \quad F(x+0) = F(x).$$

Remark:- (1) <u>$F(x)$ is not necessary continuous to the left.</u>

Justification:- Define, $D_n = \{\omega : X(\omega) \le x - \frac{1}{n}\}, n \in \mathbb{N}$

Note that, $\lim_{n\to\infty} D_n = \lim_{n\to\infty}\{\omega : X(\omega) \le x - \frac{1}{n}\}$

$$= \{\omega : X(\omega) < x\}$$

By continuity theorem of probability,

$$\lim_{n\to\infty} P[D_n] = P[\lim_{n\to\infty} D_n]$$

$$\Rightarrow \lim_{n\to\infty} P[\{\omega : X(\omega) \le x - \tfrac{1}{n}\}] = P[\{\omega : X(\omega) < x\}]$$

$$\Rightarrow \lim_{n\to\infty} F\left(x - \tfrac{1}{n}\right) = P[\{\omega : X(\omega) \le x\}] - P[\{\omega : X(\omega) = x\}]$$

$$\Rightarrow \lim_{h\to 0+} F(x-h) = F(x) - P[X = x]$$

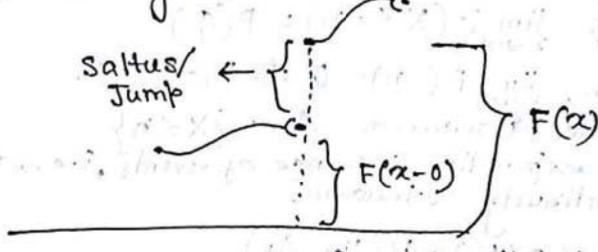
$$\Rightarrow F(x) - F(x-0) = P[X = x] \ge 0$$

Hence, $F(x-0)$ is not necessary equal to $F(x)$, i.e., $F(x)$ is not necessarily continuous to the left.

(2) <u>Jump or Saltus of a distribution function:-</u>

If $P[X = a] = 0$, then $F(a-0) = F(a)$ and $F(x)$ is continuous at $x = a$.

If $P[X = a] > 0$, then the quantity $F(a) - F(a-0) = P(X=a)$ is called the jump or saltus of the d.f. $F(x)$ at $x = a$.



$$\left[\begin{aligned} &F(x) - F(x-0) \\ &= P[X \le x] - P[X \le x-0] \\ &= P[X \le x] - P[X < x] \\ &= P[X = x]. \end{aligned}\right]$$

If $P[X = a] > 0$, then $F(x)$ has discontinuity at $X = a$ with saltus $P[X = a]$. So that the jump of a distribution function F at $X = x$ equals to the probability mass situated or concentrated at $X = x$.

(3) A necessary and sufficient condition for the r.v. X or its d.f. F to be continuous at $X = x$ is $P[X = x] = 0$.

Proof:- Let $P[X=x] = 0$

Then $F(x) - F(x-0) = 0$

i.e. $F(x) = F(x-0)$ ....(1)

Further since, F is d.f., $\therefore F(x) = F(x+0) \ \forall \ x \in \mathbb{R}^1$. ....(2).

From (1) and (2), we have

$$F(x) = F(x-0) = F(x+0)$$

i.e. F is continuous at $X = x$.

(Necessary):- F is continuous at $X = x$.

$$\therefore F(x-0) = F(x) = F(x+0)$$

$$\Rightarrow F(x) - F(x-0) = 0$$

i.e. $P[X = x] = 0$.

$\therefore$ The condition is necessary.

Ex.1. Let X be the R.V. denoting "the number of heads in tossing a fair coin thrice". Find the cdf.

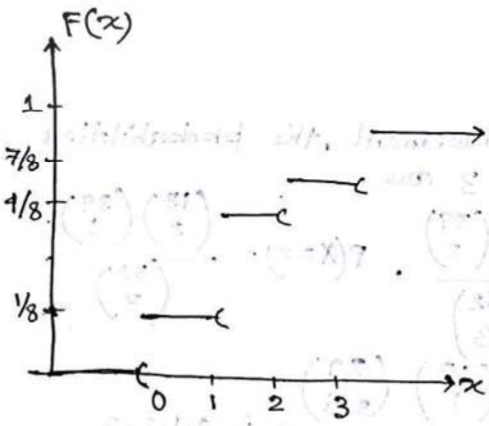
Solution:-

$$\Omega = \{\underset{\omega_1}{HHH}, \underset{\omega_2}{HTT}, \underset{\omega_3}{HHT}, \underset{\omega_4}{HTH}, \underset{\omega_5}{THH}, \underset{\omega_6}{TTH}, \underset{\omega_7}{THT}, \underset{\omega_8}{TTT}\}$$

Note that $X(\omega_i) = \begin{cases} 3 & \text{if } i=1 \\ 2 & \text{if } i = 3,4,5 \\ 1 & \text{if } i = 2,6,7 \\ 0 & \text{if } i = 8. \end{cases}$

Since the coin is fair, hence $P[\omega_i] = \frac{1}{8}$, $\omega_i \in \Omega$.

The CDF of X is $F(x) = P[\{\omega : X(\omega) \le x\}]$



$$= \begin{cases} P[\Phi] & \text{if } x < 0 \\ P[\{TTT\}] & \text{if } 0 \le x < 1 \\ P[\{HTT, THT, TTH, TTT\}] & \text{if } 1 \le x < 2 \\ P[\Omega - \{HHH\}] & \text{if } 2 \le x < 3 \\ P[\Omega] & \text{if } x \ge 3 \end{cases}$$

$$= \begin{cases} 0 & \text{if } x < 0 \\ 1/8 & \text{if } 0 \le x < 1 \\ 4/8 & \text{if } 1 \le x < 2 \\ 7/8 & \text{if } 2 \le x < 3 \\ 1 & \text{if } x \ge 3 \end{cases}$$

Note:- The set of values of X together with their corresponding probabilities is called the d.f. of X.

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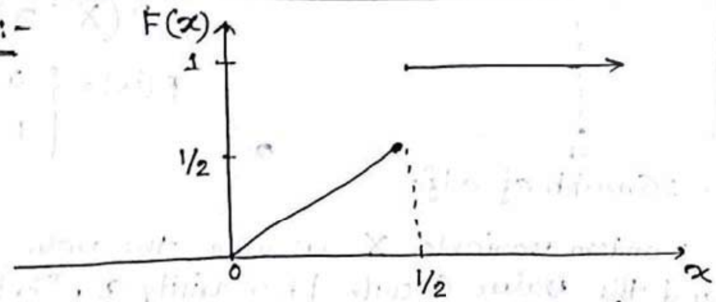
Problem:- 1. Check whether the following function are distribution function or not:

$$F(x) = \begin{cases} 0 & \text{if } x<0 \\ x & \text{if } 0 \le x < 1/2 \\ 1 & \text{if } x \geq 1/2 \end{cases}$$

Solution:-



(i) From the graph it is clear that the function is non-decreasing.

(ii) $F(-\infty) = \lim_{x\to -\infty} F(x) = \lim_{x\to-\infty} (0) = 0.$

(iii) $F(\infty) = \lim_{x\to \infty} F(x) = \lim_{x\to\infty} (1) = 1.$

(iv) $\lim_{h\to 0} F(0+h) = \lim_{h\to 0} h = 0 = F(0)$

$\lim_{h\to 0} F(\frac{1}{2}+h) = \lim_{h\to 0} (1) = 1 = F(1/2)$

So, $F(x)$ is right - continuous.

So, $F(x)$ is a cdf here..

Problem 2. Is the following function cdf or not?

$$F(x) = \begin{cases} 0 & \text{if } x \le 0 \\ \frac{x^2}{2} & \text{if } 0 < x \le 1 \\ \frac{1}{2} + \frac{(x-1)^3}{3} & \text{if } 1 < x \le 2 \\ \frac{6}{7} + \frac{1}{7}(x-2)^4 & \text{if } 2 < x \le 3 \\ 1 & \text{if } x > 3. \end{cases}$$

Solution:- (i) It is non-decreasing.

(ii) $F(-\infty) = 0$ if $x \le 0$ (iii) $F(\infty) = 1$ if $x > 3$.

(iv) $F(x+0) = \lim_{h\to 0+} F(0+h) = \lim_{h \to 0+} \frac{h^2}{2} = 0 = F(0).$

$\Rightarrow F(x)$ is continuous to the right at $x=0$.

$F(1+0) = \lim_{h\to 0+} F(1+h) = \lim_{h\to 0+} \left\{ \frac{1}{2} + \frac{h^3}{3} \right\} = \frac{1}{2} = F(1).$

$\Rightarrow F(x)$ is continuous to the right at $x=1$.

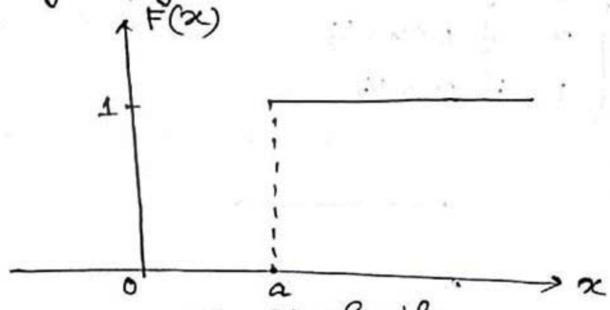
$F(2+0) = \lim_{h\to 0+} F(2+h) = \lim_{h\to 0+} \left\{ \frac{6}{7} + \frac{h^4}{7} \right\} = \frac{6}{7} \neq \frac{5}{6} = F(2).$

$\Rightarrow F(x)$ is not right continuous at $x=2$.

$\therefore F(x)$ can't be a CDF.

Problem:- 3. The random variable X assumes the value 'a' with probability unity, Sketch its d.f.

Solution:-



-: Graph of cdf :-

$P(X=a)=1$

$P(X<a)=0$

$P(X\leq a)=1$

$\therefore P(X\leq x)=1 \ \forall \ x\geq a$

$\therefore F(x)=\begin{cases}0 & \text{if } x<a\\ 1 & \text{if } x\geq a\end{cases}$

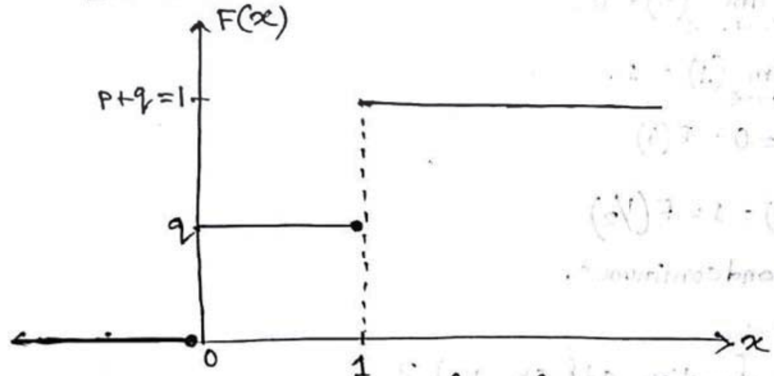
Problem:- 4 The random variable X assumes the value 1 with probability p and the value 0 with probability q. Sketch the d.f.

Solution:-

$P[X=0]=q$

$P[X=1]=p.$

So, $P(X\leq x)=F(x)$



-: Graph of cdf :-

$F(x)=\begin{cases}0, & \text{if } x<0\\ q, & \text{if } 0\leq x<1\\ 1, & \text{if } x\geq 1\end{cases}$

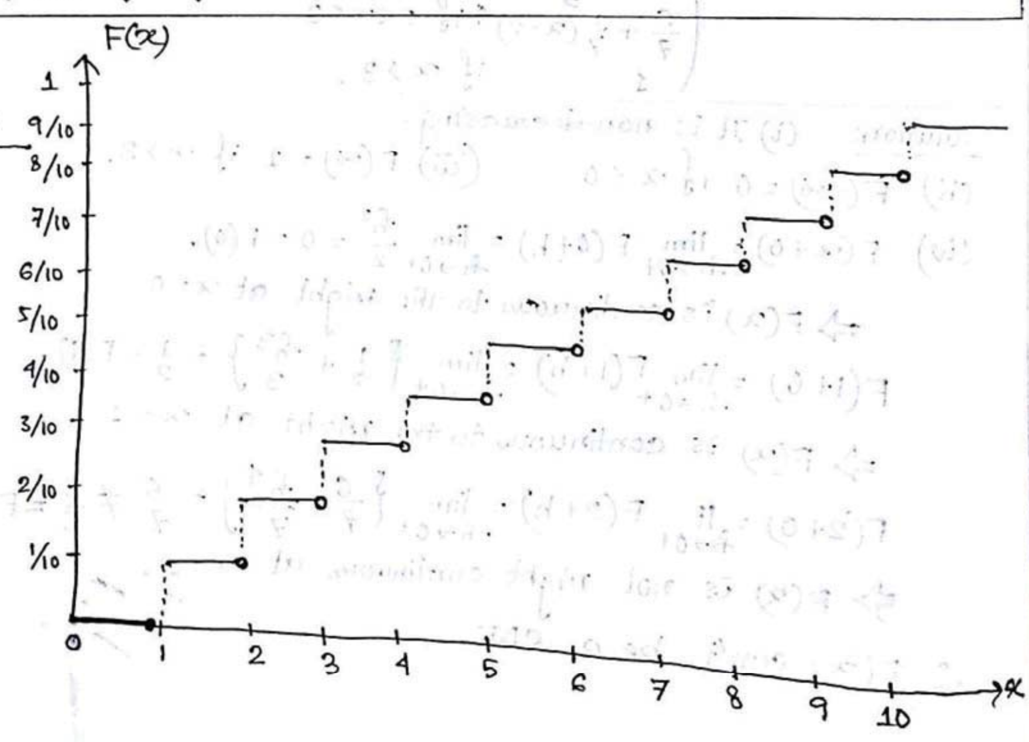
Since $F(0)=P(X\leq 0)$
$=q.$

$F(1)=P(X\leq 1)$
$=P(X=0)+P(X=1)$
$=p+q=1.$

Problem:- 5. A whole number is chosen at random between 1 and 10. Sketch the df of the related random variable.

Solution:-

x	$P(X=x)$	$F(x)$
1	1/10	1/10
2	1/10	2/10
3	1/10	3/10
4	1/10	4/10
⋮	⋮	⋮
10	1/10	1



Ex. 5. Suppose $G(x) = \begin{cases} 0 & \text{if } x < -1 \\ 1/2 & \text{if } -1 \le x \le 0 \\ a + be^{-x^2/2} & \text{if } x > 0 \end{cases}$

Determine the values of a and b so that $G(x)$ is a distribution function.

Solution:- To be a d.f., $G(x)$ needs to satisfy four properties.

(i) $G(-\infty) = 0$

(ii) $G(\infty) = 1 \Rightarrow \lim_{x\to\infty} \left(a + be^{-x^2/2}\right) = 1$

$\Rightarrow a + b.0 = 1 \Rightarrow a = 1.$

(iii) $G(x)$ is non-decreasing.

(iv) $G(x)$ is right continuous. So, we have

$$G(0) = \frac{1}{2} = \lim_{h\to 0+} G(0+h)$$

$$= \lim_{h\to 0+} \left(a + be^{-h^2/2}\right)$$

$$= a + b$$

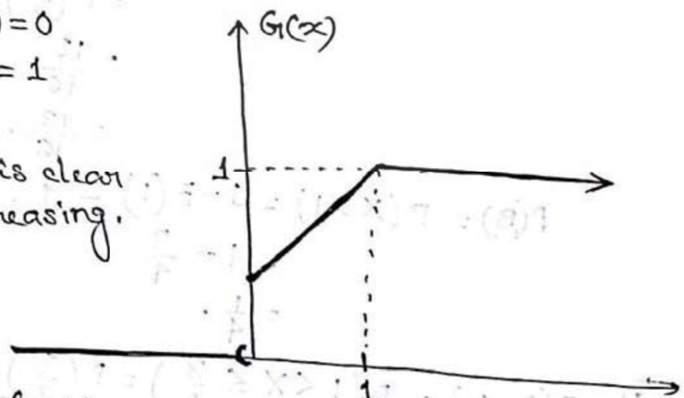
So, $b = -1/2$.

Ex. 6. Verify whether the following function $G(x)$ is a c.d.f. or not:

$$G(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1/2 & \text{if } x = 0 \\ 1/2 + x/2 & \text{if } 0 < x < 1 \\ 1 & \text{if } x \ge 1 \end{cases}$$

Solution:- (i) $G(-\infty) = 0$

(ii) $G(\infty) = 1$



(iii) From Graph it is clear that $G(x)$ is increasing.

(iv) $G(x+0) = \lim_{h\to 0+} F(0+h)$

$= \lim_{h\to 0} \left(\frac{1}{2} + \frac{h}{2}\right)$

$= \frac{1}{2} = G(0)$

$G(x+1) = \lim_{h\to 0} G(1+h)$

$= \lim_{h\to 0} (1)$

$= 1 = G(1).$

$\therefore G(x)$ is right continuous.

So, $G(x)$ is a c.d.f.

Ex. 7. Let F_1 and F_2 be two d.f.s. If a and b are non-negative integers whose sum is unity then show that $aF_1 + bF_2$ is also d.f.s.

Solution:-

(i) $F = aF_1 + bF_2$.

Let $x_1 < x_2$

Then since F_1 and F_2 are d.f.s, so, we have

$$F_1(x_1) \leq F_1(x_2) \quad \& \quad F_2(x_1) \leq F_2(x_2).$$

Since a and b are non-negative integers, so

$$aF_1(x_1) + bF_2(x_1) \leq aF_1(x_2) + bF_2(x_2)$$

$$\Rightarrow (aF_1 + bF_2)(x_1) \leq (aF_1 + bF_2)(x_2)$$

$$\Rightarrow F(x_1) \leq F(x_2).$$ So, F is non-decreasing.

(ii) $F_1(-\infty) = 0$, $F_2(-\infty) = 0$

$$F(-\infty) = (aF_1 + bF_2)(-\infty)$$
$$= aF_1(-\infty) + bF_2(-\infty)$$
$$= 0.$$

(iii) $F_1(\infty) = 1$, $F_2(\infty) = 1$.

$$F(\infty) = (aF_1 + bF_2)(\infty)$$
$$= aF_1(\infty) + bF_2(\infty)$$
$$= a + b = 1.$$

(iv) Now, $F_1(x+0) = \lim_{h \to 0} F_1(x+h) = F_1(x) \ \forall x$.

And, $F_2(x+0) = \lim_{h \to 0} F_2(x+h) = F_2(x) \ \forall x$.

Now, $F(x+0) = \lim_{h \to 0} (aF_1 + bF_2)(x+h)$

$$= \lim_{h \to 0} aF_1(x+h) + \lim_{h \to 0} bF_2(x+h)$$
$$= aF_1(x) + bF_2(x)$$
$$= F(x) \ \forall x.$$

Hence, $F(x)$ is right continuous.

so, F is a d.f.

Ex. 8. $F(x)$ is a d.f. Then show that $G(x)$ is also a d.f., where

$$G(x) = \left[1-(1-F(x))^n \right], \; n \in \mathbb{N}.$$

Solution:- (i) Let $x < y$

$$F(x) \leq F(y)$$

$$\Rightarrow 1 - F(x) \geq 1 - F(y)$$

$$\Rightarrow (1-F(x))^n \geq (1-F(y))^n$$

$$\Rightarrow 1-(1-F(x))^n \leq 1-(1-F(y))^n$$

$$\Rightarrow G(x) \leq G(y).$$

(ii) $$G(-\infty) = \lim_{x\to -\infty} G(x)$$

$$= \lim_{x\to -\infty} \left\{1-(1-F(x))^n\right\}$$

$$= 1 - \lim_{x\to -\infty} (1-F(x))^n$$

$$= 1 - \left\{\lim_{x\to -\infty} (1-F(x))\right\}^n$$

$$= 1 - \left(1 - \lim_{x\to -\infty} F(x)\right)^n$$

$$= 1 - (1-F(-\infty))^n$$

$$= 1 - 1 = 0.$$

(iii) $$G(\infty) = \lim_{x\to \infty} G(x)$$

$$= \lim_{x\to \infty} \left\{1-(1-F(x))^n\right\}$$

$$= 1 - \left(1 - \lim_{x\to\infty} F(x)\right)^n$$

$$= 1 - (1-F(\infty))^n$$

$$= 1.$$

(iv) $$\lim_{h\to 0} G(x+h)$$

$$= \lim_{h\to 0} \left\{1-(1-F(x+h))^n\right\}$$

$$= 1 - \left(1 - \lim_{h\to 0} F(x+h)\right)^n$$

$$= 1 - (1-F(x))^n$$

$$= G(x).$$

$\therefore$ $G(x)$ is also a d.f.

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Ex. 9. Show that every d.f. F has the following properties:

(a) $\lim_{x\to\infty} x\int_x^{\infty} \frac{1}{z}\, dF(z) = 0$ (b) $\lim_{x\to+0} x\int_x^{\infty} \frac{1}{z}\, dF(z) = 0$

(c) $\lim_{x\to-\infty} x\int_{-\infty}^{x} \frac{1}{z}\, dF(z) = 0$ (d) $\lim_{x\to-0} x\int_{-\infty}^{x} \frac{1}{z}\, dF(z) = 0$

Solution:-

(a) $x>0$

$$0 \le x\int_x^{\infty} \frac{1}{z}\, dF(z) \le x\int_x^{\infty} \frac{1}{x}\, dF(z)$$

$\left[\begin{matrix} x<z<\infty \\ 0<\frac{1}{z}<\frac{1}{x}\end{matrix}\right]$

$$= x\cdot\frac{1}{x}\int_x^{\infty} dF(z)$$

$$= 1-F(x).$$

Take limit both sides,

$$0 \le \lim_{x\to\infty} x\int_x^{\infty}\frac{1}{z}\, dF(z) \le \lim_{x\to\infty}(1-F(x))$$

$$=0.$$

$$\therefore \lim_{x\to\infty} x\int_x^{\infty}\frac{1}{z}\, dF(z) = 0.$$

(b) Let x be a positive proper fraction.

$$0 \le x\int_x^{\infty}\frac{1}{z}\, dF(z) = x\int_x^{\sqrt{x}}\frac{1}{z}\, dF(z) + x\int_{\sqrt{x}}^{\infty}\frac{1}{z}\, dF(z)$$

For, $x\to 0+$, first part $\to 0$.

$\sqrt{x}<z<\infty \Rightarrow 0<\frac{1}{z}<\frac{1}{\sqrt{x}}$.

$$0 \le x\int_x^{\infty}\frac{1}{z}\, dF(z) \le x\int_{\sqrt{x}}^{\infty}\frac{1}{z}\, dF(z)$$

$$\le x\int_{\sqrt{x}}^{\infty}\frac{1}{\sqrt{x}}\, dF(z)$$

$$\le \sqrt{x} \quad \left[\because 1-F(\sqrt{x}) < 1\right]$$

$$\therefore \lim_{x\to 0+} x\int_x^{\infty}\frac{1}{z}\, dF(z) = 0.$$

(c) $x<0$ and $0 \le z \le x$.

$\therefore$ z and x are both negative so

$$\frac{x}{z} \le 1.$$

$$0 \le x\int_{-\infty}^{x}\frac{1}{z}\, dF(z) \le \int_{-\infty}^{x} dF(z) = F(x)$$

$$\therefore 0 \le \lim_{x\to-\infty} x\int_{-\infty}^{x}\frac{1}{z}\, dF(z) \le \lim_{x\to-\infty} F(x)$$

$$=0.$$

$$\therefore \lim_{x\to-\infty} x\int_{-\infty}^{x}\frac{1}{z}\, dF(z) = 0.$$

(d) Taking x to be a negative proper fraction.

$$0 \ge x\int_{-\infty}^{x}\frac{1}{z}\, dF(z)$$

$$= x\int_{-\infty}^{\sqrt{x}}\frac{1}{z}\, dF(z) + x\int_{\sqrt{x}}^{x}\frac{1}{z}\, dF(z)$$