B. Continuous Random Variable :–

Definition:— A random variable X is said to be a continuous RV if it takes any value within its range of variation.

For a continuous RV X, $P[X=x]=0\ \forall x$,

By construction or axiomatic definition,

$$F(x)-F(x-0)=P[X=x]=0\ \forall x.$$

$\Rightarrow F(x)$ is continuous everywhere.

If $F(x)$ is continuous everywhere, then the associated R.V. X is known as Continuous Random Variable.

Absolutely continuous Random Variable:– An R.V. X with D.F. $F(x)$ is said to be an absolutely continuous RV, if $\exists$ a non-negative function $f(\cdot)$ such that

$$F(x)=\int_{-\infty}^{x} f(t)\,dt,\ \forall\ x\in\mathbb{R},$$

where $F(x)=P[X\leq x]$ is the distribution function of the RV X.

It may be noted that ——

(i) $F(-\infty)=\lim\limits_{x\to-\infty} F(x)=\lim\limits_{x\to-\infty}\int_{-\infty}^{x} f(x)\,dx=0.$

(ii) $F(\infty)=\lim\limits_{x\to\infty} F(x)=\lim\limits_{x\to\infty}\int_{-\infty}^{x} f(x)\,dx=1.$

(iii) $$P[a<X\leq b]=F(b)-F(a)$$

$$=\int_{-\infty}^{b} f(x)\,dx-\int_{-\infty}^{a} f(x)\,dx$$

$$=\int_{-\infty}^{a} f(x)\,dx+\int_{a}^{b} f(x)\,dx-\int_{-\infty}^{a} f(x)\,dx$$

$$=\int_{a}^{b} f(x)\,dx=P[a<X<b]=P[a\leq X<b]=P[a\leq X\leq b]$$

And the function $f(x)$ is called the probability density function (pdf).

Theorem:– A function $f(x)$ is said to be a PDF of some absolutely continuous R.V. X if it satisfies

(i) $f(x)\geq 0\ \forall x$

(ii) $\int_{-\infty}^{\infty} f(x)\,dx=1.$

Result:- If $F(x)$ is absolutely continuous and $f(x)$ is continuous at $X=x$, then $F'(x) = \frac{dF(x)}{dx} = f(x)$.

Proof:- Newton-Leibnitz Formula:-

$$I(\theta) = \int_{a(\theta)}^{b(\theta)} f(x,\theta)\,dx \text{, then } I'(\theta) \text{ is defined as}$$

$$I'(\theta) = \frac{dI(\theta)}{d\theta} = \int_{a(\theta)}^{b(\theta)} \frac{df}{d\theta}\cdot dx + \frac{db(\theta)}{d\theta} f(b(\theta),\theta) - \frac{da(\theta)}{d\theta} f(a(\theta),\theta)$$

Here, $F'(x) = \frac{dF(x)}{dx} = \frac{d}{dx}\int_{-\infty}^{x} f(x)dx = \int_{-\infty}^{x} \frac{df(x)}{dx}\,dx + 1.f(x) - 0.f(-$

$$= \int_{-\infty}^{x} 0.\,dx + f(x) = f(x).$$

Probability Density Function (PDF):- For an absolutely continuous RV X with D.F. $F(x)$, note that

$$\frac{d}{dx}[F(x)] = f(x),$$

$$\Rightarrow f(x) = \lim_{h\to 0+} \frac{F(x+h)-F(x)}{h} = \lim_{h\to 0+}\frac{P[x<X\le x+h]}{h},$$

since $P[X=x]=0$.

For small $h(>0)$, $f(x) \simeq \frac{P[x<X\le x+h]}{h}$, which is the ratio of the probability contained in $(x, x+h]$ for the distribution and the length of the interval, i.e., $f(x) \simeq \frac{P[x<X\le x+h]}{h}$ is the probability contained for the distribution per unit length in the interval $(x, x+h]$, where $h>0$ is small.

That is why, the quantity $f(x)$ is known as the probability density at the point x and the function $f(x)$ is called pdf of RV X.

Definition:- If X is an absolutely continuous RV X with D.F. $F(x)$, then $\exists$ a non-negative function $f(x) \ni$

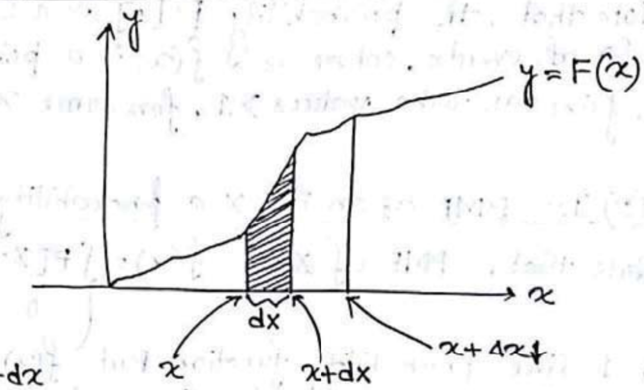
$$F(x) = \int_{-\infty}^{x} f(t)\,dt \quad \forall\, x \in \mathbb{R}$$ and then the function $f(x)$ is called the PDF of X. It satisfies the properties:

(i) $f(x) \geq 0 \;\forall\, x \in \mathbb{R}$

(ii) $\int_{-\infty}^{\infty} f(t)dt = 1$.

Question:- Why $f(x)$ is called a pdf? Justify your answer.

Solution:-



$$\int_x^{x+dx} f(x)\,dx = \int_x^{x+dx} F'(x)\,dx = \Big[F(x)\Big]_x^{x+dx}$$

$$= F(x+dx) - F(x)$$
$$= P[x < X \le x+dx]$$

dP = Elementary probability lies within the interval $(x, x+dx]$.

$= P[x < X \le x+dx]$

$= \lim_{\Delta x \to 0} P[x < X \le x+\Delta x]$

$= \lim_{\Delta x \to 0} \{F(x+\Delta x) - F(x)\}$

$= \lim_{\Delta x \to 0} \left\{ \frac{F(x+\Delta x) - F(x)}{\Delta x} \cdot \Delta x \right\}$

$= \left\{ \lim_{\Delta x \to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x} \right\} \left\{ \lim_{\Delta x \to 0} \Delta x \right\}$

$= F'(X)\,dx$

[Remark:-
* The geometrical interpretation of $\int_{-\infty}^{\infty} f(x)\,dx = 1$ is that the area bounded by the curve $y = f(x)$ and X-axis is unity.]

$\therefore \underbrace{dP}_{(M)} = f(x)\underbrace{dx}_{(V)}$, where $f(x)$ is the PDF of X.

In probability theory, 'probability' has been regarded as "mass" and "dx" is the one-dimensional analogue of "volume", so that "dP" plays the role of "mass(M)" and 'dx' plays the role of "volume(V)" in the relation M = PV of elementary physics.

Therefore, the function $f(x)$ is playing the role of "density" and that is why $f(x)$ is called the probability density function within an infinitisimal interval about $X = x$.

• Note:- For non-absolutely continuous RV, the PDF doesn't exist.

Scanned by CamScanner

Ex.1. Verify that the function $f(x)$ can be looked upon as the PDF of a continuous random variable.

$$f(x) = \begin{cases} x/2 & , 0 < x \leq 1 \\ 1/2 & , 1 < x \leq 2 \\ \frac{3-x}{2} & , 2 < x \leq 3 \\ 0 & , 3 < x \leq 4 \end{cases}$$

Obtain the Distribution function.

Solution:- Clearly, $f(x) \geq 0 \ \forall x \in \mathbb{R}$.

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} 0.dx + \int_{0}^{1} \frac{x}{2}dx + \int_{1}^{2} \frac{1}{2}dx + \int_{2}^{3} \frac{3-x}{2}dx + \int_{3}^{\infty} 0.dx$$

$= 1.$ Hence, $f(x)$ is a pdf.

The D.F. is $F(x) = \int_{-\infty}^{x} f(t)\,dt$

$$= \begin{cases} \int_{-\infty}^{x} 0.dt = 0 & , \text{if } x \leq 0 \\ \int_{-\infty}^{0} 0.dt + \int_{0}^{x} \frac{t}{2}dt = \frac{x^2}{4} & , \text{if } 0 < x \leq 1 \\ \int_{-\infty}^{1} f(t)dt + \int_{1}^{x} \frac{1}{2}dt = F(1) + \frac{x-1}{2} = \frac{2x-1}{4} & , \text{if } 1 < x \leq 2 \\ \int_{-\infty}^{2} f(t)dt + \int_{2}^{x} \frac{3-t}{2}dt = F(2) + \int_{2}^{x} \frac{3-t}{2}dt = \frac{6x-x^2-5}{4} & , \text{if } 2 < x \leq 3 \\ \int_{-\infty}^{3} f(t)dt + \int_{3}^{\infty} 0.dt = F(3) = 1 & , \text{if } x > 3 \end{cases}$$

Ex.2. Let $f(x) = \begin{cases} k & , 0 < x < 1/2 \\ 0 & , \text{ow} \end{cases}$ be a pdf of X. Find the constant k.

Solution:- $f(x) \geq 0 \Rightarrow k \geq 0$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{0}^{1/2} k.dx = \frac{k}{2} = 1 \Rightarrow k = 2.$$