THEORY OF PROBABILITY

Meanings of Probability :– It's a measure of chance of occurance of a phenomenon.

<1> The word 'Probability' may be used to mean 'the degree of belief' of a person making a statement or proposition. It is used in the sense when we say that a certain football team will be the champion in a league or we say that the 'Mahabharat' is very probably the work of several authors.

<2> On the other hand, the word has a different meaning, when we use it in the context of an experiment that can be repeated any no. of times under identical conditions. By the probability of any outcome of the experiment we shall now mean the long run relative frequency of any particular outcome of the experiment We use the probability in this sense when we say that the probability of getting a 'head' in tossing a coin is 3/4 or the probability that an article produced by a machine will defective is negligable. In statisties, we generally use the term in 2nd sense.

In probability and statistics, we concern ourselves to same special type of experiment.

(1) Random Experiment :—

A random experiment or statistical experiment is an experiment in which–

<i> all possible outcomes of the experiment are known in advance.

<ii> any performance of the experiment results in an outcome that is not known in advance.

<iii> The experiment can be repeated under identical or similar condition.

Ex : Consider an experiment of 'tossing a coin'. If the coin does not stand on the side there are two possible outcomes : Head (H), Tail (T). On any performance of the experiment, one does not know what the result will be. coin can be tossed as many times as desired under identical or similar condition. Hence, tossing of one is a random experiment.

(2) **Sample Space :—** The collection or set of all possible outcomes of a random experiment is called the sample space of the random experiment. It's noted by Ω (or S). The elements of the sample space (Ω) are called the 'Sample Point'.

Ex: (1) Consider a random experiment of 'tossing a coin' twice. Write down the sample space?

Sol. The sample space is — $\Omega = \{HH, HT, TH, TT\}$

The sample points are - HH, HT, TH, TT.

Ex: (2)

In each of the following experiment. What is the sample space?

i) a coin is tossed thrice.

ii) a die is rolled twice.

iii) a coin is tossed until a head appear.

Sol. i) $\Omega = \{HHH, HTH, THT, HHT, TTH, HTT, THH, TTT\}$

ii) $\Omega = \{(i,j) : i,j = 1(1)6\}$ [arithmatic progression[a(d)l]]

iii) $\Omega = \{H, TH, TTH, TTTH, ----\}$

Ex: (3) In each of the following experiments, what is the sample space?

i) In a survey of families with 3 children, the genders of the childrens are recorded in increasing of their age.

Sol. $\Omega = \{BBB, BBG, BGG, GBG, GGB, GBB, BGB, GGG\}$

ii) The experiment consists of selecting four items from a manufacturers output and observing whether or not each item is defective.

Sol. $\Omega = \{(a,b,c,d) : a, b, c, d$ is either defective or non-defective, consisting of 16 sample points$\}$

iii) Two cards are drawn from an ordinary deck of cards (a) with replacement ; (b) without replacements

Sol. (a) $\Omega = \{(x,y) : x,y = 1(1)52\}$ [cosisting 52^2 sample points]

(b) $\Omega = \{(x,y) : x,y = 1(1)52$ but $x \neq y\}$ [consisting 52×51 sample points]

Ex. (4) In each of the following experiments what is the sample space?

(i) Noting the lifetime of an electronic bulb.

(ii) A point is selected from a rod of unit length.

Sol. (i) $\Omega = \{x : 0 < x < \infty\}$ [continuous sample space]

(ii) $\Omega = \{x : 0 \leq x \leq 1\}$ [Here x is the distance of the selected point from the origin]

(3) **Trial :-** A trial refers to a special type of experiment in which there are two possible outcomes — 'success' and 'failure' with varying probability of success.

(4) **Outcome :-** Result of an experiment.

(5) **Sample :-** It is a part of the population and is supposed to represent the characteristic of the population.

(6) **Event :-** An event is a subset of sample space

(i) Elementary Event :- If an event contains only one sample point, it's known as an elementary event.

(ii) Composite Event :- If an event contain more than one sample points, it's known as a composite event.

Ex. (1). Consider the random experiment of 'tossing a fair coin twice'. Identify elementary & composite events.

Sol.

$\Omega = \{HH, HT, TH, TT\}$

The event (i) 'at least one head' is $A = \{HH, HT, TH\}$, is called a composite event.

(ii) 'no head' is $B = \{TT\}$, is called an elementary event.

Ex: 2: A club has 5 members A, B, C, D, E. It's required to select a chairman and a secretary. Assuming that 1 member can't occupy both positions. Write the sample space associated with this section. What's the event that member A is an officeholder.

Sol^n

sample space is, $\Omega = \{(x, y) : x, y = A, B, C, D, E \text{ but } x \neq y\}$

Here x stands for chairman and y stands for secretary.

Event is, $P = \{AB, BA, AC, AD, AE, CA, DA, EA\}$

$= \{(x, y) : \text{If } x = A \text{ then } y = B, C, D, E. \text{ If } y = A \text{ then } x = B, C, D, E\}$.

Mutually Exclusive Events :→ Several events $A_1, A_2, \ldots, A_n$ in relation to a random experiment are said to be mutually exclusive (or disjoint) if any two of them can't occur simultaneously, everytime the experiment is performed i.e. $A_i \cap A_j = \phi, \forall (i \neq j) / i<j = 1(1)n$.

Exhaustive Events :→ Several events $A_1, A_2, \ldots, A_n$ in relation to a random experiment are said to be exhaustive events if any of them must necessarily occur, everytime the experiment is performed that is $\bigcup_{i=1}^{n} A_i = \Omega$.

Equally Likely Cases (or events) :→ Several cases $A_1, A_2, A_3, \ldots$ are said to be equally likely if, after taking into consideration all relevant evidance, there is no reason to believe that one is more likely than the other.

Ex :→ For a random experiment of 'tossing a coin twice', the sample space is — $\Omega = \{HH, HT, TH, TT\}$.

Let A be the event of getting at least one head and B be the event of getting at most one head.

Then —
$$A = \{HT, TH, HH\}$$
$$B = \{HT, TH, TT\}$$
$A \cup B = \Omega$ and $A \cap B \neq \phi$.

Hence, the event A and B are exhaustive but not mutually exclusive.

Let C be the event of getting 'no head', then $C = \{TT\}$, $A \cup C = \Omega$, $A \cap C = \phi$,

Hence, the event A and C are exhaustive and mutually exclusive too.

The Classical Definition of Probability :→ If a random experiment can result in N (finite) mutually exclusive, exhaustive and equally likely cases and N(A) of them are favorable to the occurance of the event A, then the probability of occurance of A is —
$$P[A] = \frac{N(A)}{N}.$$

Remark: ① since $0 \le N(A) \le N$

$$0 \le P(A) \le 1$$

② By classical definition of probability of an event is a rational number between 0 and 1. But in general probability is a real no. between 0 and 1.

③ $P[A^c] = \frac{N - N(A)}{N} = 1 - \frac{N(A)}{N} = 1 - P(A)$.

Ex:

① A fair coin is tossed 3 times, what's the prob. of getting 'exactly 2 heads'.

② What's the prob. of getting 'at least on tail'?

Sol^n.

① $\Omega = \{HHH, HTH, THT, TTH, HTT, THH, TTT\}$ elementary cases are

Since the coin is fair, $N = 8$, elementary cases are equally likely. The events of getting two heads is $A = \{HHT, HTH, THH\}$. Hence the no. of favorable cases $N(A) = 3$.

By classical definition $P[A] = \frac{N(A)}{N} = \frac{3}{8}$.

② The event of getting 'at least one tail' is

$N(B) = \Omega - \{HHH\} = 8 - 1 = 7$.

$\therefore$ By classical definition, $P[B] = \frac{N(B)}{N} = \frac{7}{8}$.

Limitation of Classical Definition :→

1) It is assumed here that all the cases are equally likely. This def^n of probability is found useful when applied to the outcomes of the games of chance. If the outcomes of a random experiment are not equally likely then this def^n is not applicable.

2) This def^n breaks down if the no. of all possible cases is infinite.

3) In real life it is not easy to identify the outcomes as equally likely.

Statistical or Empirical (Approach) Definition of Probability :→ Suppose A is an event of a random experiment. Suppose it is possible to repeat the experiment a large number of times under essentially similar condition.

Denote by $n(A)$, the number of occurrance of A in 'n' repetition, $n(A)$ is called the frequency of A and $\frac{n(A)}{n}$, is the relative frequency. A kind of regularity is observed when a large number of repetition is considered. It is an observed fact that the relative frequencies stabilize to a certain value as 'n' become large. This tendency seems to be inherent in the nature of a random experiment and stability of relative frequencies for large values of n constitutes the basis of statistical theory or statistical definition of probability. This kind of regularity in a random experiment is known as statistical regularity. The limiting value of $\frac{n(A)}{n}$ as $n \to \infty$, is called the prob. of A, provided the limit exists.

Definition :→ If a random experiment is repeated under essentially similar conditions then the limiting value of the relative frequency of an event A, as the trials become indefinitely large, is called the probability of event A, provided the limit exists.

Consider the Question :→

1) If a coin is tossed, what is the probability that it will turn up head.

Ans :→ Examine the results of tosses given below:

No. of times the coin is tossed (n)	1 .... 10 .... 100 .... 1000 .... 2000 .... 3000 ----
No. of times the head turns up [N(A)]	0 ... 6 ... 61 .... 605 .... 1207 .... 1718 ..

Thus, we get the relative frequencies as :—

$\frac{6}{10}, \ldots, \frac{61}{100}, \ldots, \frac{605}{1000}, \ldots, \frac{1207}{2000}, \ldots, \frac{1718}{3000}, \ldots$

As the no. of tossing increases the relative frequency tents to stabilize at 0·6. Therefore the probability of getting a head in a tossing of a coin is 0·6.

Remark :→ If in a random experiment all possible cases are not equally likely, then we can't apply classical definition in this case, if the experiment can be repeated a large no. of times, then probability of an event A can be obtained by statistical definition, This is an improvement over the statistical defn.

Limitations :→

1) If an experiment is repeated a number of times, the experimental conditions may not remain identical or homogeneous.

2) The $\lim_{n \to \infty} \frac{n(A)}{n}$ may not be unique.

Subjective Probability :→ In everyday's life we hear or make statements such as "probably I shall miss the train", "probably Mr. Raj will be at home now." Such statements can be made more precise by "the chance of missing the train is 60%", "the chance that Mr. Raj will be at home now is 75%" etc. Here 60%, 75% etc. measures one's belief in the occurrence of the event. This Subjective method is another method of consigning probabilities of various events based on the personal beliefs.

When the experiment is not repeatable, this method may be adopted for assigning probabilities to events. Since, different persons may assign different probabilities, one can't arrive at objective conclusion using probabilities assigned by subjective methods.

PROBABILITY & STATISTICS:- The problem in Probability is —
"Given a stochastic model what we can say about the outcome".
The problem in statistics is —
"Given a sample what we can say about the population".

Set Theory	Probability Theory
1. Point / Element	Elementary Event
2. Set	Event
3. Universal Set	Sample Space
4. Null set	Impossible event
5. A is a subset of B	A implies B
6. A is a superset of B	A is implied by B

Ex. 1. Let A, B, C are 3 events. Then the expression of following events in set notations;

(i) Only A occurs : $A \cap B^c \cap C^c$

(ii) A occurs : A

(iii) Both A and B, but not C occur : $A \cap B \cap C^c$

(iv) All 3 events occur: $A \cap B \cap C$

(v) At least one occur: $A \cup B \cup C$

(vi) At least two occur: $(A \cap B) \cup (B \cap C) \cup (A \cap C)$

(vii) One and no more occur: $(A \cap B^c \cap C^c) \cup (B \cap A^c \cap C^c) \cup (C \cap A^c \cap B^c)$

(viii) Two and no more occur: $(A \cap B \cap C^c) \cup (B \cap C \cap A^c) \cup (A \cap C \cap B^c)$

(ix) None occurs: $A^c \cap B^c \cap C^c$

(x) If A occurs so does B: $A \subseteq B$.

Ex. 2. Eight students are arranged at random (a) in a row and (b) in a column
Find the probability that two given students will be next to each other.

Sol. (a) Req. prob. $= \dfrac{7!\,2!}{8!}$

(b) Req. prob. $= \dfrac{6!\,2!}{7!}$

Ex. 3. The nine digits 1, 2, 3, ..., 9 are arranged in random order to form a nine-digit number. Find the prob. that 1, 2 and 3 appears as neighbours in the order mentioned.

Sol. Req. prob $= \dfrac{7!}{9!} = \dfrac{1}{72}$.

Ex. 4. Find the prob. that seven people has birthdays on 7 different days of the week, assuming equal prob. for the seven days.

Sol. Req. prob. $= \dfrac{7!}{7^7}$.

No. of Distinguishable or distinct arrangement of r balls (objects) into n cells when —

(I) balls are distinguishable and exclusion principle followed.

(II) " " " but " " NOT " .

(III) " " indistinguishable and " " followed.

(IV) " " " but " " NOT " .

Exclusion Principle :- The principle of excluding a cell from taking more than one ball (object) while distributing r balls (objects) into n cells, i.e., to exclude or deban a ball (object) to be placed into a cell which is occupied.

CASE-I :- Let $u(r,n)$ denotes the no. of distinguishable distributions of r balls into n cells.

Hence, $u(r,n) = 0$ if $r > n$.

For $r \le n$, we have $u(r,n)$

$$= \begin{pmatrix}\text{no. of ways in which} \\ \text{1st ball can be placed} \\ \text{in any of the } n \\ \text{cells}\end{pmatrix} \times \begin{pmatrix}\text{no. of ways in} \\ \text{which 2nd ball} \\ \text{can be placed} \\ \text{in any of the} \\ (n-1) \text{ cells}\end{pmatrix} \times \cdots\cdots \times \begin{pmatrix}\text{no. of ways in} \\ \text{which } r\text{th ball} \\ \text{can be placed} \\ \text{in any of the} \\ (n-r+1) \text{ cells}\end{pmatrix}$$

$$= n(n-1)\cdots\cdots(n-r+1) = (n)_r .$$

CASE-II :- Maxwell-Boltzman Statistics

Here $u(r,n) = n.n.\ldots\ldots r \text{ times} = n^r$.

CASE-III :- Fermi-Dirac Statistics

Here $u(r,n) = 0$ for $r > n$

For $r \le n$, $u(r,n) = \dfrac{(n)_r}{r!} = \dbinom{n}{r}$.

CASE-IV :- Bose-Einstein Statistics

$u(r,n) =$ no. of distinguishable arrangements of r dots and $(n-1)$ bars

$$= \frac{(r+n-1)!}{r!\,(n-1)!}$$

$$= \binom{r+n-1}{r}.$$

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SOLVED EXAMPLES

Ex.1. 2 cards are drawn from a well-shuffled cards. What's the probability that both extracted cards are aces.

Sol. Here, total no. of cases, = no. of ways in which 2 cards can be drawn from 52 cards WOR.

$= 52 \times 51$

No. of favourable cases = No. of ways of getting two aces from 4 aces WOR

$= 4 \times 3$

So, Required probability $= \dfrac{\text{No. of favourable cases}}{\text{Total no. of cases}} = \dfrac{4 \times 3}{52 \times 51} = \dfrac{1}{221}$.

Ex.2. Two dice are thrown n times in succession. What's the prob. of obtaining double 6 at least one. Also determine the minimum no. of thrown required to accomplish the objective with a probability $> \frac{1}{2}$.

Sol. (i) No. of throws resulted in with required probability

$$= \frac{\text{a double six at least once}}{\text{total no. of all possible cases}}$$

$$= \frac{36^n - 35^n}{36^n} = 1 - \left(\frac{35}{36}\right)^n = p_n, \text{ say}$$

$$p_n > \frac{1}{2} \Rightarrow \left(\frac{35}{36}\right)^n < \frac{1}{n}$$

$$\Rightarrow n(\log 35 - \log 36) = -\log 2$$

$$\Rightarrow n < \frac{\log 2}{\log 36 - \log 35}.$$

$$\therefore n_{min} = \frac{\log 2}{\log 36 - \log 35} \cong 24.$$

Ex.3. A certain number h of distinguishable balls is distributed among N compartment. What is the prob. that a certain specified compartment will contain h balls?

Sol. Total no. of cases = No. of ways in which n distinguishable balls can be distributed among N compartment without following exclusion principle.

$= N^n$.

No. of favourable cases $= \begin{pmatrix}\text{No. of ways in which} \\ h \text{ balls can be chosen} \\ \text{from } n \text{ balls and} \\ \text{placed at the} \\ \text{specific compartment}\end{pmatrix} \times \begin{pmatrix}\text{No. of ways in} \\ \text{which the} \\ \text{remaining } (n-h) \\ \text{balls can be} \\ \text{distributed into} \\ (N-1) \text{ compartments}\end{pmatrix}$

$= \binom{n}{h} \times (N-1)^{n-h}$ $\therefore$ Req. prob. $= \dfrac{\binom{n}{h}(N-1)^{n-h}}{N^n}$.

Ex. 4. In an urn there are n groups of p objects in each. Objects in different groups are distinguished by some characteristic property. What's the prob. that among $(\alpha_1+\cdots+\alpha_n)$ objects taken, $[0 \le \alpha_i \le p \;\forall\, i=1(1)n]$, there are α_1 of one group, α_2 from another group, ... and so on.

Sol. The total no. of cases $= \binom{np}{\alpha_1+\alpha_2+\cdots+\alpha_n}$ ..

Favourable cases # are $= \begin{pmatrix}\text{no. of distinguishable} \\ \text{arrangements of} \\ \alpha_1,\ldots,\alpha_n\end{pmatrix}\begin{pmatrix}\text{no. of ways in which} \\ \alpha_1 \text{ comes from one} \\ \text{group, } \alpha_2 \text{ from } 2^{nd} \text{ and} \\ \text{so on}\end{pmatrix}$

$$= \frac{n!}{{}_1!\;{}_2!\cdots\cdots\;{}_n!}\binom{p}{\alpha_1}\binom{p}{\alpha_2}\cdots\binom{p}{\alpha_n}$$

Ex. 5. There are N tickets numbered $1, 2, \ldots, N$ of which n are taken at random in an increasing order of their numbers $x_1 < x_2 < \cdots < x_n$. What's the prob. that $x_m = M$.

Sol. The n tickets can be taken in $\binom{N}{n}$ ways. We assume that these are equally likely.

In order that $x_m = M$, it is necessary & sufficient that $(m-1)$ tickets have numbered from 1 to $M-1$. How $(n-m)$ tickets have numbers from $N-m$ to N and one ticket has the number M. Hence, the No. of favourable cases are $\binom{M-1}{m-1}\binom{N-M}{n-m}$.

$$\therefore \text{Req. prob. is } = \frac{\binom{M-1}{m-1}\binom{N-M}{n-m}}{\binom{N}{n}}.$$

Ex. 6. An urn contains 'a' white and 'b' black balls. Balls are drawn one by one until only those of the same colour are left. What's the prob. that they are white.

Sol. Let E be the given experiment and A be the desired event. Let E' be the desired experiment of drawing all the balls one by one and A' the event that the last ball drawn is white. Then A happens in E iff A' happens in E'. Hence, $P(A) = P(A')$. Since the balls are drawn at random in E', $P(A')$ is also the prob. that the first ball drawn is white and hence is $\frac{a}{a+b}$.

Ex.7. Three numbers are chosen from the first 30 natural numbers. What's the prob. that the chosen number will be in
(a) A.P. (b) G.P.

Solution:- (a) $N = \{1, 2, \ldots, 30\}$

Three numbers can be chosen from 30 natural numbers in $\binom{30}{3}$ ways which are assumed to be equally likely. In order that, the three numbers will be of the form $m, m+k, m+2k$ $(k \geq 1)$, the k must satisfy $1 \leq k \leq 14$ and for any such value of k, m must satisfy $1 \leq m \leq 30-2k$.
Hence, the number of such A.P.s is $\sum_{k=1}^{14} (30-2k)$

$= 30 \times 14 - 15 \times 14$
$= 14 \times 15.$

So, the required probability is $\dfrac{14 \times 15}{\binom{30}{3}} = 0.0517.$

(b) We count the triplets (arranged in increasing order) whose terms form a G.P. by listing them as follows:

Common ratio	Triplet
2	$\{(i, 2i, 4i),\ 1 \leq i \leq 7\}$
3	$\{(i, 3i, 9i),\ 1 \leq i \leq 3\}$
4	$(1, 4, 16)$
5	$(1, 5, 25)$
$3/2$	$(4, 6, 9)$, $(8, 12, 18)$, $(12, 18, 27)$
$5/2$	$(4, 10, 15)$
$4/3$	$(9, 12, 16)$
$5/3$	$(9, 15, 25)$
$5/4$	$(16, 20, 25)$

$\therefore$ Hence the required prob. is $\dfrac{19}{\binom{30}{3}} = 0.0047.$

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