RANDOM VARIABLES

Variables

- Random/stochastic/Probabilistic
  - Discrete
  - Continuous
- Non-random/De-generate random variables

In any probability problem, we may associate with each outcome (elementary event) of the experiment a finite real number. In many cases the outcome themselves are finite real numbers. This will be the case in tossing a die. In other cases, the numbers are artificially introduced. Thus for example, in tossing a coin thrice, the outcomes are not numbers but we may be interested in the number of heads obtained from the three tosses.

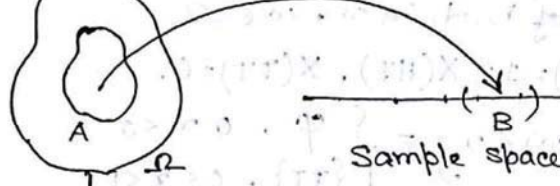
<u>Definitions of Random Variables:-</u>

(1) Let $(\Omega, \mathcal{A}, P)$ be a given probability space. Then a random variable is defined as a (Borel-measurable) function X w.r.t. $\mathcal{A}$, i.e., a random variable X is a function defined on the sample space Ω such that for every $x \in \mathbb{R}^1$, the inverse image $X^{-1}\{(-\infty, x]\} = \{\omega \mid -\infty < X(\omega) \leq x\}$ of the Borel set $(-\infty, x]$ under X is measurable w.r.t. $\mathcal{A}$ (i.e. belongs to $\mathcal{A}$).

(2) Let $(\Omega, \mathcal{A}, P)$ be a sample space of a random experiment. A real valued function $X(\omega)$ defined on Ω is called a Random Variable if $\{\omega : X(\omega) \leq x\} \in \mathcal{A} \ \forall x \in \mathbb{R}$.

(3) Let $(\Omega, \mathcal{A}, P)$ be a given probability space of a random experiment. A finite single-valued function X that maps Ω into $\mathbb{R}^1$ is called a random variable if the inverse image under X of all Borel sets in $\mathbb{R}^1$ are events, i.e. if

$$X^{-1}(B) = \{\omega : X(\omega) \in B\} \in \mathcal{A} \ \forall B \in \mathcal{B}.$$



Sample space $\rightarrow \Omega_X = \{X(\omega) \mid \omega \in \Omega\} \subseteq \mathbb{R}^1$

$\Omega \rightarrow \mathcal{A} \rightarrow P$ (Basic Prob. Measure)

$\Omega_X \rightarrow \mathcal{A}_X \rightarrow P_X$ (Probability measure introduced by X is called the prob. distn. of X)

$$P_X(B) = P\{X^{-1}(B)\}$$

Although the induced probability measure $P_X(\cdot)$ characterises the distribution of probability for X but this is a self-dependent concept and therefore not easy to understand. Let us, therefore, see in the following has a pointwise characterization of the distribution of probability for X can be developed.

Let us consider the Borel set $(-\infty, x]$ for $x \in \mathbb{R}^1$ instead of B and also let X is a random variable defined on a given probability space $(\Omega, \mathcal{A}, P)$ introduces the probability measure $P_X(\cdot)$. Now since $\{\omega \mid -\infty < X(\omega) \leq x\} = X^{-1}\{(-\infty, x]\} \ \forall x \in \mathbb{R}^1$

$$\therefore P_X\{(-\infty, x]\} = P[\omega \mid -\infty < X(\omega) \leq x] = F_X(x), \quad x \in \mathbb{R}^1.$$

Thus, for varying values of $x \in \mathbb{R}^1$, the (point) function $F_X(x)$ characterizes the same as the (set) for $P_X\{(-\infty, x]\}$ does and accordingly is called the (cumulative) distribution function (d.f.) of the probability distribution of X.

Remark:- (1) The notation of probability doesn't enter into the definition of a random variable.

(2) If X is a random variable, the sets $\{X = x\}$, $\{a < X < b\}$, $\{X < x\}$, $\{a \leq X < b\}$, $\{a < X \leq b\}$, $\{a \leq X \leq b\}$, etc are all events. Moreover, we could have used any of these events to define a r.v.

Example of R.V. :-

(1) Let E: tossing of a fair coin.
Then the sample space is: $\Omega = \{H, T\}$.
Let us define $X(H) = 1$, $X(T) = 0$. Then

$$X^{-1}(-\infty, x] = \{\omega : -\infty < X(\omega) \leq x\} = \begin{cases} \phi, & \text{if } x < 0 \\ \{T\}, & \text{if } 0 \leq x < 1 \\ \{H, T\}, & \text{if } 1 \leq x. \end{cases}$$

(2) Let E: tossing a coin twice.
Then the sample space is $\Omega = \{HH, TH, HT, TT\}$.
Define $X(\omega)$: the number of heads in ω, $\omega \in \Omega$.
Therefore, $X(HH) = 2$, $X(TH) = 1 = X(HT)$, $X(TT) = 0$.

$$X^{-1}(\infty, x] = \{\omega : -\infty < X(\omega) \leq x\} = \begin{cases} \phi, & x < 0 \\ \{TT\}, & 0 \leq x < 1 \\ \{TT\}, \{TH\}, \{HT\}, & 1 \leq x < 2 \\ \Omega, & x \geq 2. \end{cases}$$

Hence, $X(\omega)$ is a random variable defined on Ω.

(3) Let E: tossing a coin thrice.

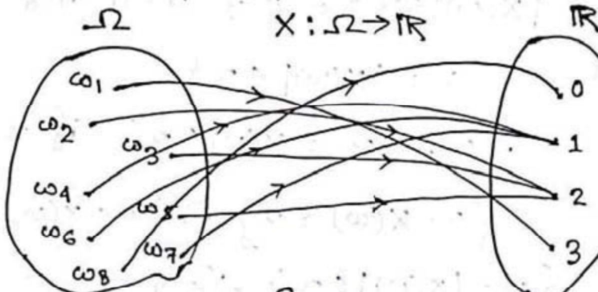
$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

Define $X(\omega)$: the number of heads in ω, $\omega \in \Omega$.

Then $X(HHH) = 3$, $X(HHT) = X(THH) = X(HTH) = 2$, $X(TTT) = 0$,

$X(HTT) = X(THT) = X(TTH) = 1$.

$\therefore$ X is a random variable with domain Ω and range $\{0,1,2,3\}$



Here, $$X^{-1}(-\infty, x] = \begin{cases} \phi & \text{if } x < 0 \\ \{TTT\} & \text{if } 0 \le x < 1 \\ \{HTT, THT, TTH\} & \text{if } 1 \le x < 2 \\ \{HHT, HTH, THH\} & \text{if } 2 \le x < 3 \\ \{HHH\} & \text{if } 3 \le x < 4 \\ \Omega & \text{if } 4 \le x. \end{cases}$$

Thus X is a random variable here.

Here Values of $X = \{3, 2, 2, 1, 2, 1, 1, 0\}$.

$$X(\omega_i) = \begin{cases} 0, & i = 8 \\ 1, & i = 4, 6, 7 \\ 2, & i = 2, 3, 5 \\ 3, & i = 1. \end{cases}$$

For any particular event $\{X \le 2.75\}$, the event space is

$\{HHT, HTT, HTH, THT, TTH, THH, TTT\}$.

If $\{0.5 \le X \le 1.72\}$, then event space $= \{HTT, THT, TTH\}$.

(4) Let E: a coin is tossed until a head appears.

X: Number of tosses required.

Here $\Omega = \{H, TH, TTH, \ldots\}$ and X assumes countably infinite number of values $1, 2, 3, \ldots$ with $X(\omega_1) = 1$, $X(\omega_2) = 2$, etc.

Here $$X^{-1}(-\infty, x] = \begin{cases} \phi, & \text{if } x < 1 \\ \{H\}, & \text{if } 1 \le x < 2 \\ \{TH\}, & \text{if } 2 \le x < 3 \\ \{TTH\}, & \text{if } 3 \le x < 4 \\ \ldots\ldots\ldots \end{cases}$$

Thus, X is a random variable.

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Problem: 1. Let X be a random variable, then
(a) Is $|X|$ also a random variable?
(b) Is X^2 also a random variable?

Solution:- Let X be an r.v. defined on $(\Omega, \mathcal{a})$.
Then $\{\omega : X(\omega) \le x\} \in \mathcal{a} \ \forall \ x \in \mathbb{R}^1$.

(a) Now, $|X(\omega)|$ is a real valued function defined on $(\Omega, \mathcal{a})$.
$\{\omega : |X(\omega)| \le x\} = \varphi$ if $x < 0$ and

Note that

$$\{\omega : |X(\omega)| \le x\}$$
$$= \{\omega : -x \le X(\omega) \le x\} \text{ if } x \geq 0$$
$$= \{\omega : X(\omega) \le x\} \cap \{x : X(\omega) < -x\}^c.$$

Hence, $\{\omega : |X(\omega)| \le x\} \in \mathcal{a} \ \forall \ x$.
So, $|X|$ is also an r.v defined on $(\Omega, \mathcal{a})$.

(b) Clearly, $X^2(\omega)$ is a real valued function on $(\Omega, \mathcal{a})$.
Note that,

$$\{\omega : X^2(\omega) \le x\}$$
$$= \begin{cases} \varphi & \text{if } x < 0 \\ \{\omega : -\sqrt{x} \le X(\omega) \le \sqrt{x}\} & \text{if } x \geq 0 \end{cases}$$
$$= \begin{cases} \varphi & \text{if } x < 0 \\ \{\omega : X(\omega) \le \sqrt{x}\} \cap \{\omega : X(\omega) < \sqrt{-x}\}^c, & \text{if } x \geq 0 \end{cases}$$
$$\in \mathcal{a}$$

Hence, $X^2(\omega)$ is a random variable defined on $(\Omega, \mathcal{a})$.

Problem: 2. If $X(\omega)$ is a random variable on $(\Omega, \mathcal{a})$, then show that $cX(\omega)$ is also a random variable on $(\Omega, \mathcal{a})$.

Proof:- Let x be any arbitrary but fixed real number.
Then $(-\infty, x] \in \mathcal{B}$.
For $c > 0$,

$$(cX)^{-1}(-\infty, x] = \{\omega : cX(\omega) \le x\} = \{\omega : X(\omega) \le \tfrac{x}{c}\}$$
$$= X^{-1}\left(-\infty, \frac{x}{c}\right] \in \mathcal{a}$$
$$(\because X \text{ is an r.v.})$$

So, $cX(\omega)$ is also a random variable.