Unit 2: Probability and distributions

4. Binomial distribution

Sta 101 - Fall 2015

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Slides posted at *http://bit.ly/sta101_f15*

1. Housekeeping

2. Main ideas

1. Binomial distribution is used for calculating the probability of exact number of successes for a given number of trials

2. Expected value and standard deviation of the binomial can be calculated using its parameters n and p

3. Shape of the binomial distribution approaches normal when the S-F rule is met

3. Summary

- Project 1 questions?
- PS2 due Friday night, PA2 due Sunday night
- RA3 on Monday

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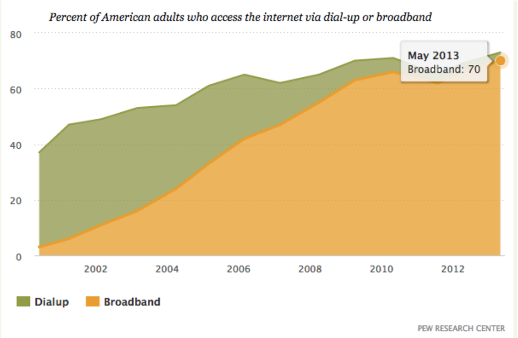
High-speed broadband connection at home in the US



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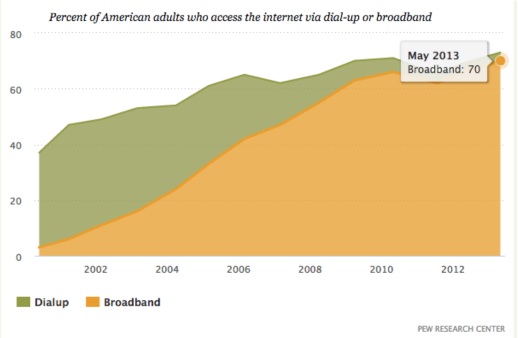

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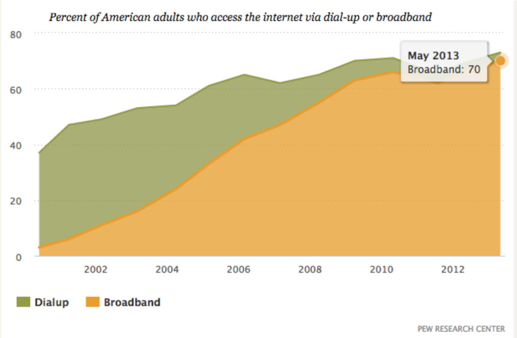

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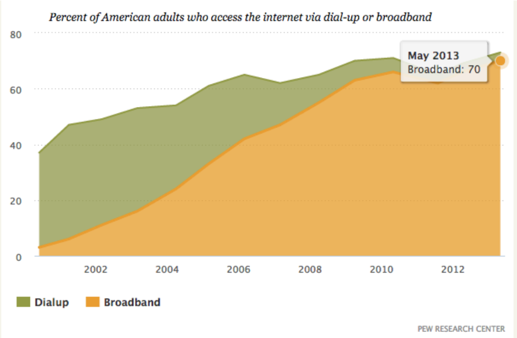

- Each person in the poll be thought of as a *trial*
- A person is labeled a *success* if s/he has high-speed broadband connection at home, *failure* if not
- Since 70% have high-speed broadband connection at home, *probability of success* is $p = 0.70$

Suppose we randomly select three individuals from the US, what is the probability that exactly 1 has high-speed broadband connection at home?

Let's call these people Anthony (A), Barry (B), Cam (C). Each one of the three scenarios below will satisfy the condition of "exactly 1 of them says Yes":

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The probability of exactly one 1 of 3 people saying Yes is the sum of all of these probabilities.

$$0.063 + 0.063 + 0.063 = 3 \times 0.063 = 0.189$$

The question from the prior slide asked for the probability of given number of successes, k, in a given number of trials, n, ($k = 1$ success in $n = 3$ trials), and we calculated this probability as

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The *Binomial distribution* describes the probability of having exactly k successes in n independent trials with probability of success p.

$$P(k\ successes\ in\ n\ trials) = \binom{n}{k}\ p^k\ (1-p)^{(n-k)}$$

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Note: You can also use R for the calculation of number of scenarios:

```
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Note: And to compute probabilities

```
> dbinom(1, size = 3, prob = 0.7)
```

```
[1] 0.189
```

Clicker question

Which of the following is false?

(a) There are n ways of getting 1 success in n trials, $\binom{n}{1} = n$.
(b) There is only 1 way of getting n successes in n trials, $\binom{n}{n} = 1$.
(c) There is only 1 way of getting n failures in n trials, $\binom{n}{0} = 1$.
(d) There are $n-1$ ways of getting $n-1$ successes in n trials, $\binom{n}{n-1} = n-1$.

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(d) *There are $n-1$ ways of getting $n-1$ successes in n trials, $\binom{n}{n-1} = n-1$.*

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Which of the following is not a condition that needs to be met for the binomial distribution to be applicable?

(a) the trials must be independent

(b) the number of trials, n, must be fixed

(c) each trial outcome must be classified as a *success* or a *failure*

(d) the number of desired successes, k, must be greater than the number of trials

(e) the probability of success, p, must be the same for each trial

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According to the results of the Pew poll suggesting that 70% of Americans have high-speed broadband connection at home, is the probability of exactly 2 out of 15 randomly sampled Americans having such connection at home pretty high or pretty low?

(a) pretty high

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(a) pretty high
(b) *pretty low*

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According to the results of the Pew poll 70% of Americans have high-speed broadband connection at home, what is the probability that exactly 2 out of 15 randomly sampled Americans have such connection at home?

(a) $0.70^2 \times 0.30^{13}$

(b) $\binom{2}{15} \times 0.70^2 \times 0.30^{13}$

(c) $\binom{15}{2} \times 0.70^2 \times 0.30^{13}$

(d) $\binom{15}{2} \times 0.70^{13} \times 0.30^2$

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(c) $\binom{15}{2} \times 0.70^2 \times 0.30^{13}$
$= \frac{15!}{13! \times 2!} \times 0.70^2 \times 0.30^{13} = 105 \times 0.70^2 \times 0.30^{13} = 8.2e-06$

(d) $\binom{15}{2} \times 0.70^{13} \times 0.30^2$

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- But this doesn't mean in every random sample of 100 Americans exactly 70 will have high-speed broadband connection at home. In some samples there will be fewer of those, and in others more. How much would we expect this value to vary?
  - $\sigma = \sqrt{np(1-p)} = \sqrt{100 \times 0.70 \times 0.30} \approx 4.58$

Note: Mean and standard deviation of a binomial might not always be whole numbers, and that is alright, these values represent what we would expect to see on average.

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You can use the normal distribution to approximate binomial probabilities when the sample size is large enough.

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You can use the normal distribution to approximate binomial probabilities when the sample size is large enough.

S-F rule: The sample size is considered large enough if the expected number of successes and failures are both at least 10

$$np \geq 10 \qquad \text{and} \qquad n(1-p) \geq 10$$

Clicker question

Below are four pairs of Binomial distribution parameters. Which distribution's shape can be approximated by the normal distribution?

(a) $n = 25, p = 0.45$

(b) $n = 100, p = 0.95$

(c) $n = 150, p = 0.05$

(d) $n = 500, p = 0.015$

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1. Using R:

```
> sum(dbinom(750:1000, size = 1000, prob = 0.7))
```

```
[1] 0.00026
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2. Using the normal approximation to the binomial: Since we have at least expected successes $(1000 \times 0.7 = 700)$ and 10 expected failures $(1000 \times 0.3 = 300)$,

$$Binom(n = 1000, p = 0.7) \sim$$
$$N(\mu = 1000 \times 0.7, \sigma = \sqrt{1000 \times 0.7 \times 0.3})$$

Application exercise: 2.4 Binomial distribution

See course website for details.

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