

Unit 6: Introduction to linear regression

2. Prediction, outliers, and inference for regression

Sta 101 - Fall 2015

Duke University, Department of Statistical Science

1. Housekeeping

2. Main ideas

1. Predict, but don't extrapolate
2. Predicted values also have uncertainty around them
3. R^2 assesses model fit -- higher the better
4. Inference for regression uses the t -distribution
5. Conditions for regression
6. Type of outlier determines how it should be handled

3. Summary

- ▶ PA 6 opens today, due Nov 22, Sun
- ▶ PS 6 due Nov 20, Fri
- ▶ RA 7 (last RA!) on Wednesday
- ▶ Project 2 questions?
 - If you want to see sample posters from previous years, stop by office hours
 - Most important advice: Sketch out a meeting / working plan with your team **TODAY**, keeping in mind Thanksgiving break

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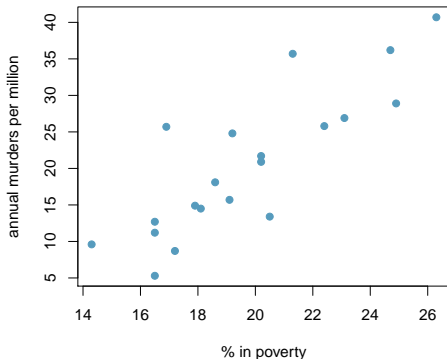
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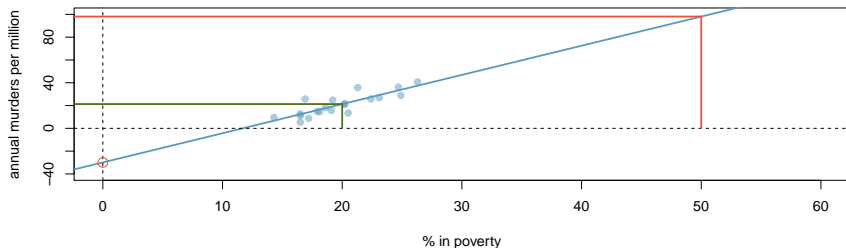
Suppose you want to predict annual murder count (per million) for a series of districts that were not included in the dataset. For which of the following districts would you be most comfortable with your prediction?

A district where % in poverty =

- (a) 5%
- (b) 15%
- (c) 20%
- (d) 26%
- (e) 40%



Sometimes the intercept might be an extrapolation: useful for adjusting the height of the line, but meaningless in the context of the data.



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In R:

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# load data
murder <- read.csv("https://stat.duke.edu/~mc301/data/murder.csv")
# fit model
m_mur_pov <- lm(annual_murders_per_mil ~ perc_pov, data = murder)
# create new data
newdata <- data.frame(perc_pov = 20)
# predict
predict(m_mur_pov, newdata)
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- ▶ With any prediction we can (and should) also report a measure of uncertainty of the prediction.

A *prediction interval* for y for a given x^* is

$$\hat{y} \pm t_{n-2}^* s \sqrt{1 + \frac{1}{n} + \frac{(x^* - \bar{x})^2}{(n-1)s_x^2}}$$

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- ▶ Prediction level: If we repeat the study of obtaining a regression data set many times, each time forming a XX% prediction interval at x^* , and wait to see what the future value of y is at x^* , then roughly XX% of the prediction intervals will contain the corresponding actual value of y .

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	fit	lwr	upr
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We are 95% confident that the annual murders per million for a county with 20% poverty rate is between 9.52 and 33.15.

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murder %>%  
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anova(m_mur_pov)
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Analysis of Variance Table

Response: annual_murders_per_mil

	Df	Sum Sq	Mean Sq	F value	Pr(>F)
perc_pov	1	1308.34	1308.34	43.064	3.638e-06 ***
Residuals	18	546.86	30.38		

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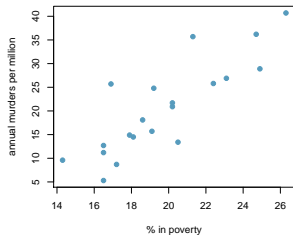
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$$R^2 = \frac{\text{explained variability}}{\text{total variability}} = \frac{SS_{reg}}{SS_{tot}} = \frac{1308.34}{1308.34 + 546.86} = \frac{1308.34}{1855.2} \approx 0.71$$

Clicker question

R^2 for the regression model for predicting annual murders per million based on percentage living in poverty is roughly 71%. Which of the following is the correct interpretation of this value?



- (a) 71% of the variability in percentage living in poverty is explained by the model.
- (b) 84% of the variability in the murder rates is explained by the model, i.e. percentage living in poverty.
- (c) 71% of the variability in the murder rates is explained by the model, i.e. percentage living in poverty.
- (d) 71% of the time percentage living in poverty predicts murder rates accurately.

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- ▶ Hypothesis testing for a slope: $H_0 : \beta_1 = 0$; $H_A : \beta_1 \neq 0$
 - $T_{n-2} = \frac{b_1 - 0}{SE_{b_1}}$
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 - p-value = P(observing a slope at least as different from 0 as the one observed if in fact there is no relationship between x and y)
- ▶ Confidence intervals for a slope:
 - $b_1 \pm T_{n-2}^* SE_{b_1}$
 - In R:

```
confint(m_mur_pov, level = 0.95)
```

	2.5 %	97.5 %
(Intercept)	-46.265631	-13.536694
perc_pov	1.740003	3.378776

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- ▶ Linearity → randomly scattered residuals around 0 in the residuals plot – important regardless of doing inference

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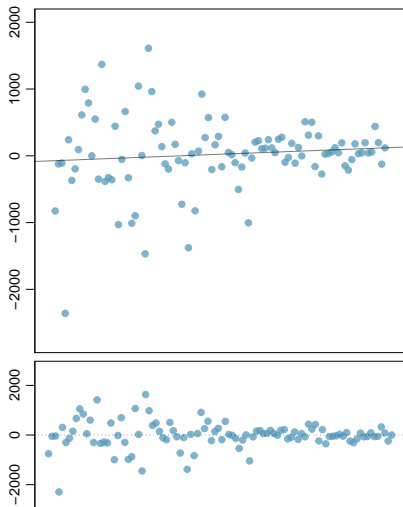
Important for inference

- ▶ Nearly normally distributed residuals → histogram or normal probability plot of residuals
- ▶ Constant variability of residuals (*homoscedasticity*) → no fan shape in the residuals plot
- ▶ Independence of residuals (and hence observations) → depends on data collection method, often violated for time-series data

Clicker question

What condition is this linear model obviously and definitely violating?

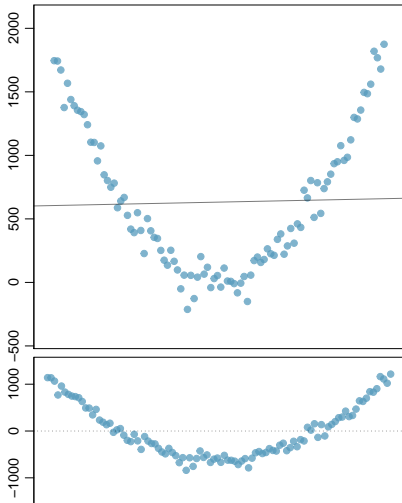
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- (c) Constant variability
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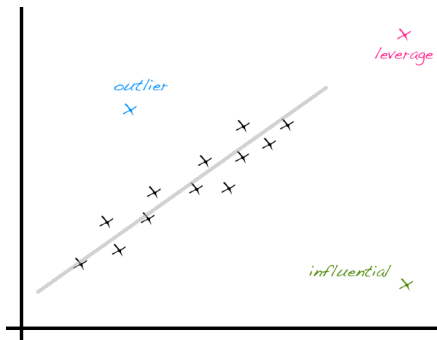
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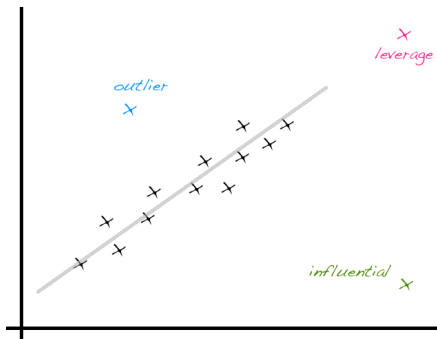
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- ▶ *Influential* point changes the slope (most likely also has high leverage) – run the regression with and without that point to determine



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- ▶ *Outlier* is an unusual point without these special characteristics (this one likely affects the intercept only)
- ▶ If clusters (groups of points) are apparent in the data, it might be worthwhile to model the groups separately.



Application exercise: 6.2 Linear regression

See course website for details

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