Unit 2: Probability and distributions

2. Normal distribution

Sta 101 - Fall 2015

Duke University, Department of Statistical Science

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Slides posted at *http://bit.ly/sta101_f15*

Outline

1. Housekeeping

2. Main ideas
    1. Two types of probability distributions: discrete and continuous
    2. Normal distribution is unimodal, symmetric, and follows the 69-95-99.7 rule
    3. Z scores serve as a ruler for any distribution
    4. Z distribution is normal with $\mu = 0$ and $\sigma = 1$
    5. Normally distributed data plot as a straight line on the normal probability plot

3. Application exercise

4. Summary

5. Exercise [time permitting]

Announcements

- Formatting of problem set submissions
  - Bad: Scanned PDF handwriting, unreadable/too dark, haphazard orientation of pages, out of order
  - Better: Scanned PDF **neat** handwriting, good scan quality, one file containing all pages in order
  - Best: Typed using a word processor and PDFed
- Project 1: *https://stat.duke.edu/courses/Fall15/sta101.002/post/projects/project1.html*
  - Read over before next class and come with questions
  - Start searching for datasets
  - Set team weekly team meetings

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1. Two types of probability distributions: discrete and continuous

- A *discrete probability distribution* lists all possible events and the probabilities with which they occur
  - The events listed must be disjoint
  - Each probability must be between 0 and 1
  - The probabilities must total 1

1. Two types of probability distributions: discrete and continuous

- A *discrete probability distribution* lists all possible events and the probabilities with which they occur
  - The events listed must be disjoint
  - Each probability must be between 0 and 1
  - The probabilities must total 1
- A *continuous probability distribution* differs from a discrete probability distribution in several ways:
  - The probability that a continuous random variable will equal to any specific value is zero.
  - As such, they cannot be expressed in tabular form.
  - Instead, we use an equation or a formula to describe its distribution via a probability density function (pdf).
  - We can calculate the probability for ranges of values the random variable takes (area under the curve).

Discrete:
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Outcome ($)	X	P(X)
Win $10 (black aces)	10	$\frac{2}{52}$
Win $8 (red aces: 10 - 2)	8	$\frac{2}{52}$
Lose $2 (non-ace reds)	-2	$\frac{24}{52}$
No win / loss	0	$\frac{24}{52}$
		$\frac{52}{52} = 1$

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Continuous:
Distribution of female heights is unimodal and nearly symmetric with a mean of 65” and a standard deviation of 3.5” [source].

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- *68-95-99.7 Rule:*
  - about 68% of the distribution falls within 1 SD of the mean
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  - it is possible for observations to fall 4, 5, or more standard deviations away from the mean, but this is very rare if the data are nearly normal

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- Lots of variables are nearly normal, but few are actually normal.

Clicker question

Speeds of cars on a highway are normally distributed with mean 65 miles / hour. The minimum speed recorded is 48 miles / hour and the maximum speed recorded is 83 miles / hour. Which of the following is most likely to be the standard deviation of the distribution?

(a) -5
(b) 5
(c) 10
(d) 15
(e) 30

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(a) -5 $\rightarrow$ *SD cannot be negative*

(b) *5* $\rightarrow 65 \pm (3 \times 5) = (50, 80)$

(c) 10 $\rightarrow 65 \pm (3 \times 10) = (35, 95)$

(d) 15 $\rightarrow 65 \pm (3 \times 15) = (20, 110)$

(e) 30 $\rightarrow 65 \pm (3 \times 30) = (-25, 155)$

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How can we determine if it would be unusual for an adult woman in North Carolina to be 96” (8 ft) tall?

How can we determine if it would be unusual for an adult alien woman(?) to be 103 metreloots tall, assuming the distribution of heights of adult alien women is approximately normal?

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- Z score: number of standard deviations the observation falls above or below the mean
- Defined for distributions of any shape, but only when the distribution is normal can we use Z scores to calculate percentiles
- Observations with $|Z| > 2$ are usually considered *unusual*

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$$Z = \frac{X - \mu}{\sigma}, \text{ where } X \sim N(\mu, \sigma),$$

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- Z distribution is a special case of the normal distribution where $\mu = 0$ and $\sigma = 1$ (unit normal distribution).
- The Z distribution is also called the “standard normal” distribution.

Clicker question

Scores on a standardized test are normally distributed with a mean of 100 and a standard deviation of 20. If these scores are converted to standard normal Z scores, which of the following statements will be correct?

(a) The mean will equal 0, but the median cannot be determined.

(b) The mean of the standardized Z-scores will equal 100.

(c) The mean of the standardized Z-scores will equal 5.

(d) Both the mean and median score will equal 0.

(e) A score of 70 is considered unusually low on this test.

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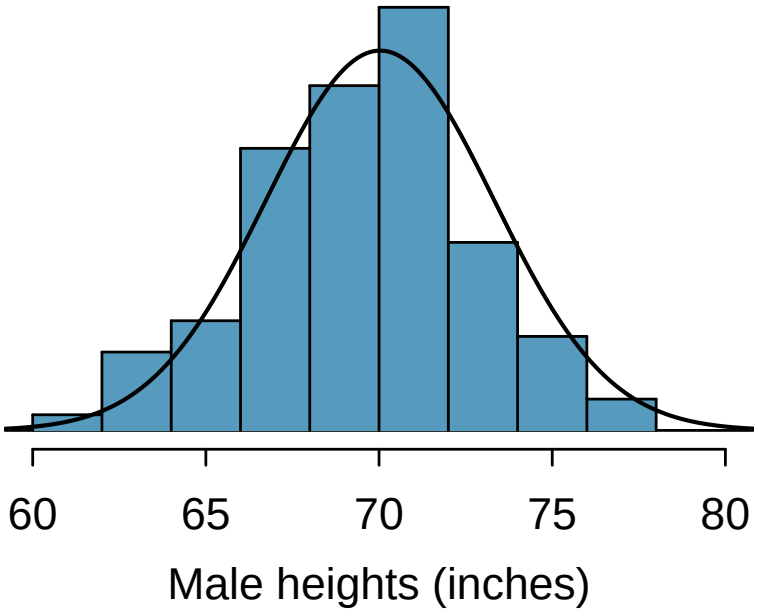
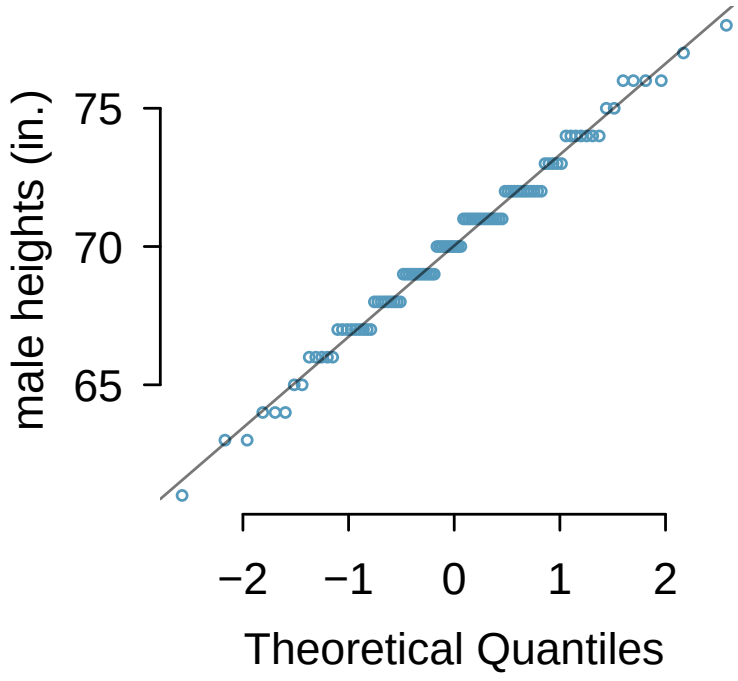
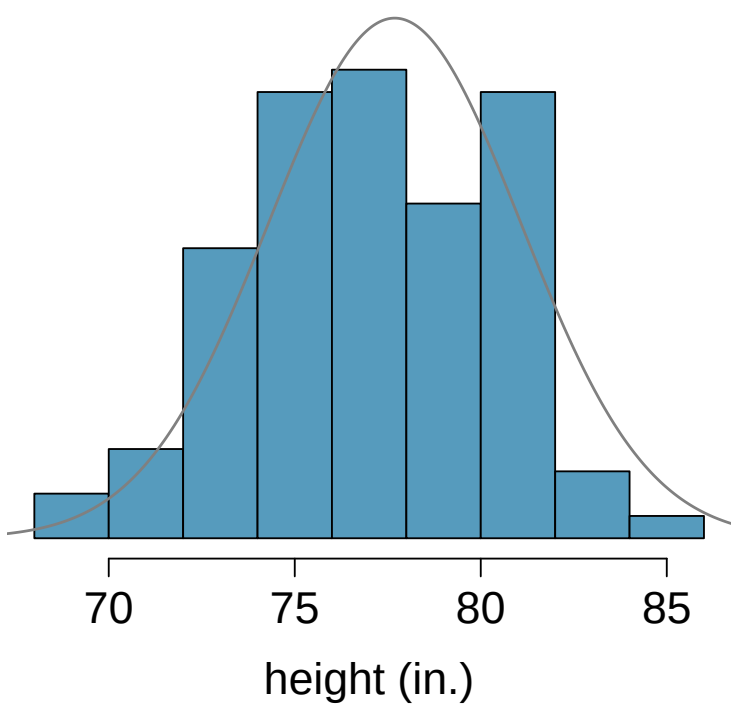
A histogram and *normal probability plot* of a sample of 100 male heights.

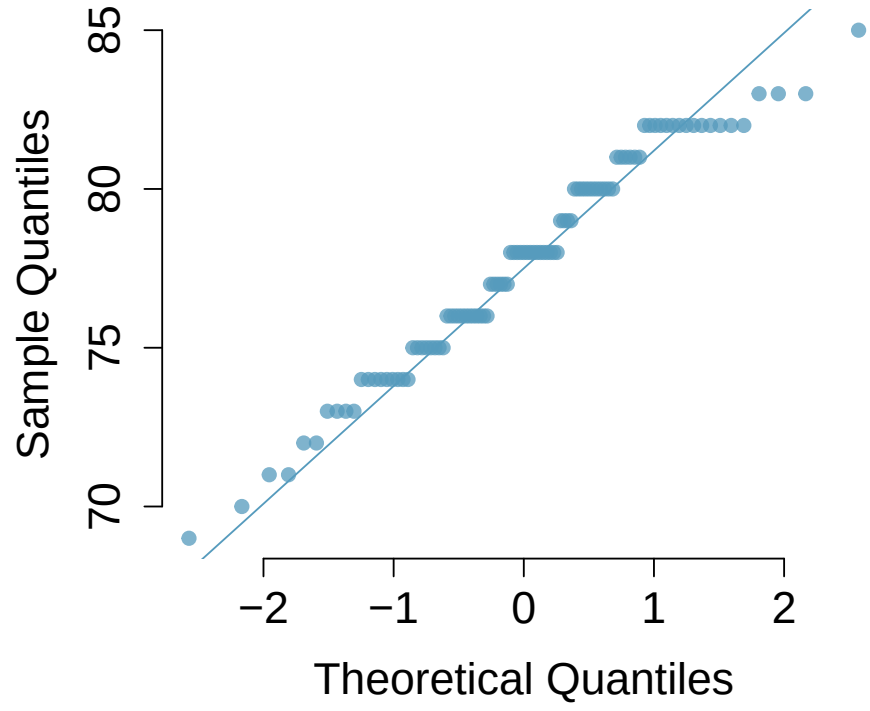




Why do the points on the normal probability have jumps?

Below is a histogram and normal probability plot for the heights of Duke men's basketball players (from 1990s and 2000s). Do these data appear to follow a normal distribution?





Source: GoDuke.com

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Application exercise: 2.3 Normal distribution

See the course website for instructions.

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Clicker question

Which of the following is false?

(a) Z scores are helpful for determining how unusual a data point is compared to the rest of the data in the distribution.

(b) Majority of Z scores in a right skewed distribution are negative.

(c) In a normal distribution, Q1 and Q3 are more than one SD away from the mean.

(d) Regardless of the shape of the distribution (symmetric vs. skewed) the Z score of the mean is always 0.

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(d) Regardless of the shape of the distribution (symmetric vs. skewed) the Z score of the mean is always 0.

At a pharmaceutical factory the amount of the active ingredient which is added to each pill is supposed to be 36 mg. The amount of the active ingredient added follows a nearly normal distribution with a standard deviation of 0.11 mg. Once every 30 minutes a pill is selected from the production line, and its composition is measured precisely. We know that the failure rate of the quality control is 3% at this factory. What are the bounds of the acceptable amount of the active ingredient?