VIP Refresher: Linear Algebra

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Matrix notations

❒ **Vector** – We note $x \in \mathbb{R}^n$ a vector with n entries, where $x_i \in \mathbb{R}$ is the i^{th} entry:

$$\boxed{x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n}$$

❒ **Matrix** – We note $A \in \mathbb{R}^{m\times n}$ a matrix with n rows and m, where $a_{i,j} \in \mathbb{R}$ is the entry located in the i^{th} row and j^{th} column:

$$\boxed{A = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{pmatrix} \in \mathbb{R}^{m\times n}}$$

Remark: the vector x defined above can be viewed as a $n \times 1$ matrix and is more particularly called a column-vector.

❒ **Matrix-vector multiplication** – The product of matrix $A \in \mathbb{R}^{m\times n}$ and vector $x \in \mathbb{R}^n$ is a vector of size $\mathbb{R}^n$, such that:

$$\boxed{Ax = \begin{pmatrix} \displaystyle\sum_{j=1}^{n} a_{1,j}x_j \\ \vdots \\ \displaystyle\sum_{j=1}^{n} a_{m,j}x_j \end{pmatrix} \in \mathbb{R}^m}$$

❒ **System of equations** – The system of equations

$$\begin{cases} y_1 &= a_{1,1}x_1 + a_{1,2}x_2 + ... + a_{1,n}x_n \\ y_2 &= a_{2,1}x_1 + a_{2,2}x_2 + ... + a_{2,n}x_n \\ \vdots & \\ y_m &= a_{m,1}x_1 + a_{m,2}x_2 + ... + a_{m,n}x_n \end{cases}$$

can be rewritten in matrix form $\boxed{y = Ax}$ with $y \in \mathbb{R}^m, A \in \mathbb{R}^{m\times n}$ and $x \in \mathbb{R}^n$.

Determinant

❒ **Definition** – The determinant of a square matrix $A \in \mathbb{R}^{n\times n}$, noted $|A|$ or $\det(A)$ is expressed recursively in terms of $A_{\backslash i,\backslash j}$, which is the matrix A without its i^{th} row and j^{th} column, as follows:

$$\boxed{\det(A) = |A| = \sum_{j=1}^{n} (-1)^{i+j} a_{i,j} |A_{\backslash i,\backslash j}|}$$

Remark: A is invertible if and only if $|A| \neq 0$. Also, $|AB| = |A||B|$ and $|A^T| = |A|$.

❒ **Characteristic equation** – The characteristic equation of a linear system of n equations represented by A is given by:

$$\boxed{\det(A - \lambda I) = 0}$$

For $n = 2$, this equation can be written as:

$$\boxed{\lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0}$$

❒ **Eigenvector, eigenvalue** – The roots λ of the characteristic equation are the eigenvalues of A. The solutions $\vec{v}$ of the equation $A\vec{v} = \lambda I$ are called the eigenvectors associated with the eigenvalue λ.

❒ **Computing the determinant in particular cases** – We have the following cases:

– For a 2×2 matrix – The determinant of a given matrix $A \in \mathbb{R}^{2\times 2} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ can be computed as follows:

$$\boxed{\det(A) = ad - bc}$$

– For a 3×3 matrix – The determinant of a given matrix $A \in \mathbb{R}^{3\times 3} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ can be computed as follows:

$$\boxed{\det(A) = a(ei - fh) - b(di - fg) + c(dh - eg)}$$

Partial fractions

❒ **Concept** – A fraction $\frac{P(x)}{Q(x)}$ with P and Q polynomial functions of x and $\deg(P) < \deg(Q)$ can be decomposed into partial fractions by distinguishing the types of roots that are in the factorized form of $Q(x)$, as detailed in the table below:

Factor of $Q(x)$	Type of root	Associated partial fraction
$(x-a)^n$	Real root of multiplicity $n \geqslant 1$	$\dfrac{A_1}{x-a} + ... + \dfrac{A_n}{(x-a)^n}$
$(ax^2+bx+c)^n$	Complex roots of multiplicity $n \geqslant 1$	$\dfrac{A_1x+B_1}{ax^2+bx+c} + ... + \dfrac{A_nx+B_n}{(ax^2+bx+c)^n}$