MECHANICAL VIBRATIONS

MECHANICAL VIBRATIONS

(M.K.S. SYSTEM)

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ROORKEE

First Edition 1970
Second Edition 1972

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Printed in India
By N. C. Jain at the Roorkee Press Roorkee (U.P.)

FOREWORD

During recent years the subject of Mechanical Vibrations has gained considerable importance in the teaching curriculum of Mechanical Engineering students. Formerly Mechanical Vibrations used to form a small part of Theory of Machines, but at present it is taught as separate full subject to students of Mechanical Engineering. Unfortunately, no textbook on this subjec in the M.K.S. system fulfilling the requirements of Indian students was available. The publication of the present rather comprehensive volume on Mechanical Vibrations fills a long-felt need and is, therefore, very welcome.

This book which has been written in a remarkably lucid style covers the syllabii of mechanical Vibrations prescribed by various institutions for the degree course in Mechanical Engineering. The outstanding virtue of this book is that the student is given a clear understanding of the fundamental concepts in vibrations. Emphasis has been given throughout the book to explain the physical picture before deriving the mathematical equations for various vibrating systems. A large number of solved examples have been given to illustrate the text. The problems at the end of various chapters are of practical nature and are carefully designed to stimulate thinking rather than merely provide practice in numerical computation.

The author, Dr. G. K. Grover, possesses a long experience of teaching the subject to under-graduate and post-graduate students. Besides, he has been an active research worker in the field of Mechanical Vibrations. His teaching and research experience has greatly enhanced the value of the book.

I am confident that this book will be widely welcomed and used as a textbook on Mechanical Vibrations in various engineering institutions. The book will also prove to be of great use to practising engineers.

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Roorkee. U.P.

Jan. 14, 1970

PREFACE TO THE SECOND EDITION

The author is extremely grateful to all those teachers, students and other reviewers who have given various suggestions, from time to time, as regards inclusion of certain topics which were not included in the first edition. On their suggestions two more chapters on "Transient Vibrations" and "Non-linear Vibrations" have been included in this edition. Considering the enormity of these fields, the two chapters included are but an introduction which cover only the basic concepts which can be used for further application to vibration problems in engineering. Besides, some additional portion on dashpots and other types of dampings has been added in chapter 3. It is hoped this text now will also satisfy to a good amount the requirement of post graduate students in this particular subject.

A lot more problems for practice have been added at the end of each chapter to give more practice and confidence to the students preparing for examinations in this subject. The few mistakes and printing errors of the first edition have also been corrected.

Further suggestions and comments from the readers will be gratefully received.

July 1, 1972 G. K. Grover

PREFACE TO THE FIRST EDITION

The primary purpose of this book is to make the students have a clear insight into the phenomenon of vibrating systems under different conditions. Detailed explantions have been given where the students have usually been found to stumble due to the lack of clear concepts. Much stress has been laid on the fundamentals to make sure that they grasp the ideas throughly and have no subsequent difficulty in the later advanced chapters. The physical interpretation for various mathematical derivations, which is so very essential for this subject, has no where been lost sight of. A large number of solved examples have been included for better understanding of the principles. Practice problems included with answers, will make the students develope further confidence in the subject.

The contents of this book conform to the syllabii contents of the most of the Indian Universities and Colleges on this subjects for the Degree Course in Mechanical Engineering. Since no other book on this subject in M.K.S. systems is commonly available in the Indian market, it is hoped that this gap will be satisfactorily filled up by this book. Besides the degree students of Mechanical Engineering, this book will be equally useful for the practising engineers as the systems are discussed from the practical view-point.

It is not possible to acknowledge all contributors to this book. Ideas drawn from various books, technical papers, persons in this field and the personal experience of the author have in some manner or the other been woven into the fabric of this text. The constant encouragement received at various sources has gone a long way in the preparation of this text. The author is deeply indebted to Prof. C. P. Gupta for his generous and kind "Foreword". The pains taken by the publishers to bring this book out in its present get-up is also gratefully acknowledged.

Suggestions from the readers regarding any modifications, alterations, additions, ommissions and their pointing out of any printing errors or those which might have crept in because of large amount of numerical work or due to oversight, will be gratefully received.

Jan 13, 1970. G.K. Grover

CONTENTS

CHAPTER 1

FUNDAMENTALS OF VIBRATIONS

1.1 Introduction.

Vibration problems occur wherever there are rotating or moving parts in a machinery. Apart from the machinery itself, the sorrounding structure also faces the vibration hazard because of this vibrating machinery. The common examples are locomotives, diesel engines mounted on unsound foundations, whirling of shafts etc. The main causes of vibration are as follow.

1 Unbalanced forces in the machine. These forces are produced from within the machine itself.

2 Dry friction between the two mating surfaces. This produces what are known as self excited vibrations.

3 External excitations. These excitations may be periodic, random or of the nature of an impact produced external to the vibrating system.

4 Earthquakes. These are responsible for the failure of many buildings, dams, etc.

5. Winds. These may cause the vibrations of transmission and telephone lines under certain conditions.

The effects of vibrations are excessive stresses, undesirable noise, looseness of parts and partial or complete failure of parts. Inspite of these harmful effects, the vibration phenomenon does have some uses also, *e.g.* in musical instruments, vibrating screens, shakers, stress relieving, etc.

Elimination or reduction of the undesirable vibrations can be obtained by one or more of the following methods.

1 Removing the cause of vibrations.

2 Putting in screens if noise is the only objection.

3 Resting the machinery on proper type of isolators.

4 Shock absorbers.

5 Dynamic vibration absorbers.

Although above methods are available to reduce vibrations at a stage where no changes in design are possible, anticipation of trouble in the original planning and design can make possible the avoidance of vibration problems at little cost.

1.2 Definitions.

Explained below are some of the terms which will be used over and again in this text.

Periodic Motion—A motion which repeats itself after equal intervals of time.

Time Period—Time taken to complete one cycle.

Frequency—Number of cycles per unit time.

Simple Harmonic Motion—A periodic motion of a particle whose acceleration is always directed towards the mean position and is proportional to its distance from the mean position. Alternatively, it may be defined as the motion of the projection of a particle moving round a circle with uniform angular velocity, on a diameter.

Amplitude—The maximum displacement of a vibrating body from the mean position.

Free Vibrations—The vibration of a system because of its own elastic properties. No external exciting force acts in this case.

Forced Vibrations—The vibrations which the system executes under an external periodic force. The frequency of vibration in this case is the same as that of excitation.

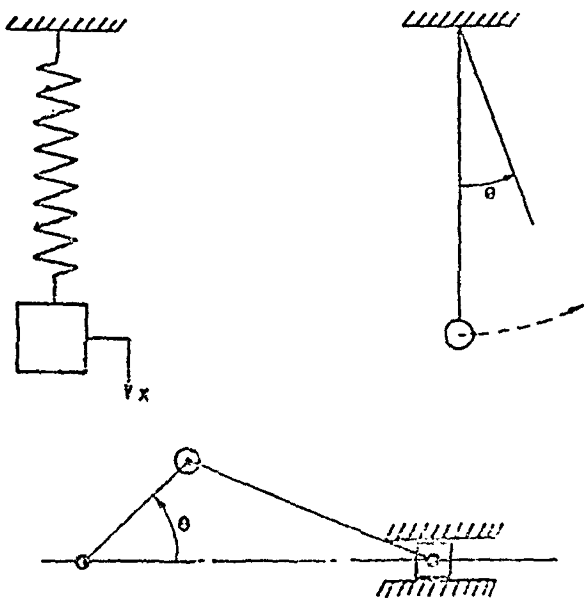
Natural Frequency—Frequency of free vibration of the system. It is a constant for a given system.

Resonance—The vibration of the system when the frequency of the external force is equal to the natural frequency

of the system. The amplitude of vibration at resonance becomes excessive.

Damping—Resistance to the motion of the vibrating body.

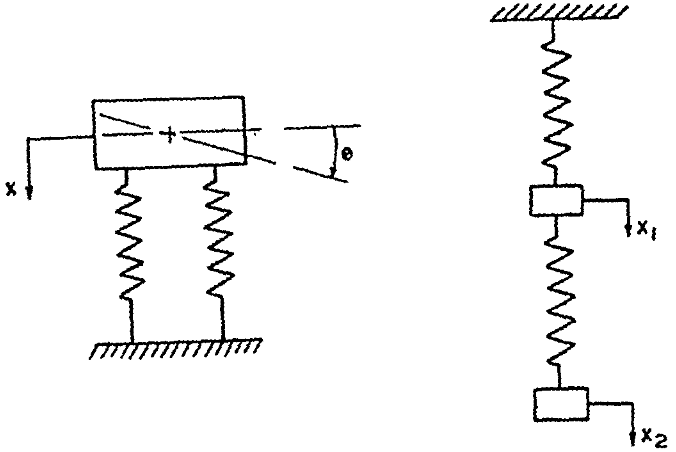
Degree of Freedom—A system is said to be *n-degrees of freedom system* if it needs n independent coordinates to specify completely the configuration of the system at any instant. A mass supported by a spring and constrained to move in one direction without rotation is a single degree of freedom system. The same is true for a simple pendulum oscillating in one plane. A crank-slider mechanism is also a single degree freedom system since only the crank angle is sufficient to define the system completely. These vibratory systems are illustrated in Fig. 1.2.1 (a).



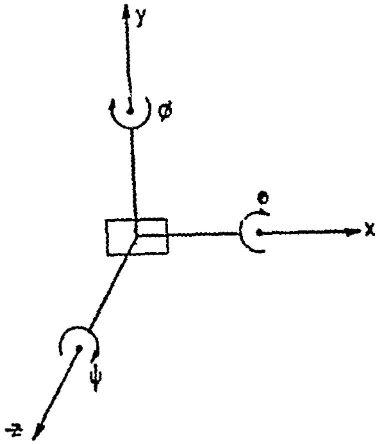
(a) Single degree of freedom systems.

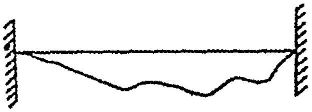
Fig. 1.2.1. Degrees of freedom.

On the other hand a spring-supported rigid mass which can move in the direction of the springs and can also have angular motion in one plane has two degrees of freedom. A two-mass two-spring system constrained to move in one direction without rotation has also two degrees of freedom [see Fig. 1.2.1 (b)]. A



(b) Two degrees of freedom systems.





(c) Six degrees of freedom system.

(d) Infinite degrees of freedom system.

Fig. 1.2.1. Degrees of freedom.

ody in space has six degrees of freedom, three translational nd three rotational. A flexible beam between two supports as an infinite number of degrees of freedom. These systems re shown in Fig. 1.2.1 (c) and (d).

Phase Difference—It is the angle between two rotating vectors

representing simple harmonic motions of the same frequency. This is further clarified in Sec. 1.3.

1.3 Vector method of representing harmonic motions.

Let a particle having simple harmonic motion be represented by the equation

$$x = X \sin \omega t \tag{1.3.1}$$

In this equation X is the amplitude of vibration and ω is the circular frequency in radians per second. Consider a vector X rotating in the anticlockwise direction, occuping at time t, a position as shown in Fig. 1.3.1. At time $t=0$, the vector would be along the abscissa. At any time t, the angle turned by the vector in anticlockwise direction is ωt from the abscissa (shown in the figure). The projection of this vector on the ordinate gives the displacement of the particle from the mean position at the particular instant. This vector is known as the *Displacement Vector.* The total angle turned by this vector per cycle is 2π radians. Since the angular velocity is ω radians/second,

the time period per cycle $= 2\pi/\omega$ sec,

and the frequency $f = \omega/2\pi$ cycles/sec.

Differentiating equation (1.3.1),

$$\begin{aligned}\dot{x} &= \omega X \cos \omega t \\ &= \omega X \sin (\omega t + \pi/2) \qquad (1.3.2)\end{aligned}$$

Equation (1.3.2) is similar to equation (1.3.1) except that the vector position at time t is $(\omega t + \pi/2)$ from the abscissa instead of ωt and the length of this vector is ωX instead of X. This is called the *Velocity Vector* and its projection on the ordinate gives the velocity $\dot{x}$ of the particle at time t. This is also shown in Fig. 1.3.1.

Differentiating once again equation (1.3.2),

$$\begin{aligned}\ddot{x} &= \omega^2 X \cos (\omega t + \pi/2) \\ &= \omega^2 X \sin (\omega t + \pi) \qquad (1.3.3)\end{aligned}$$

The amplitude of this *Acceleration Vector* is $\omega^2 X$ and its position from the abscissa at time t is $(\omega t + \pi)$. Its projection on the ordinate gives the acceleration $\ddot{x}$ of the particle at time t (also shown in Fig. 1.3.1.

It may be seen from the figure that the velocity vector leads the displacement vector by $\pi/2$ and the acceleration vector leads the velocity vector by another $\pi/2$. It may also be noted that all the three vectors are rotating in the anticlockwise direction with the same angular velocity, and their relative positions remain unchanged. If $\omega > 1$, the velocity vector is longer than the displacement vector and the acceleration vector is longer than velocity vector. The reverse is true for $\omega < 1$.

These vectors are commonly termed as *Rotating Vectors* and the angle between the two rotating vectors is termed as *Phase Difference*. Angle α, shown in Fig. 1.3.2, is the phase difference between the vectors X_1 and X_2

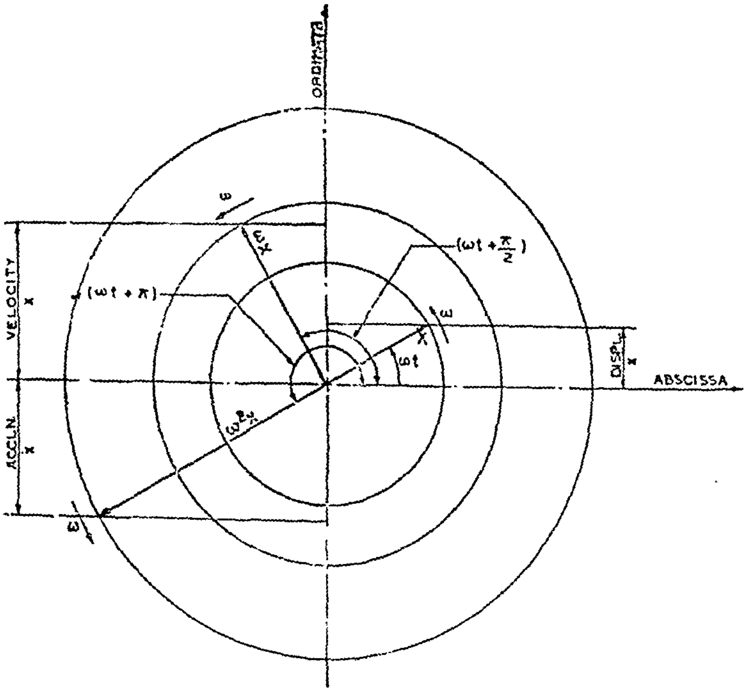
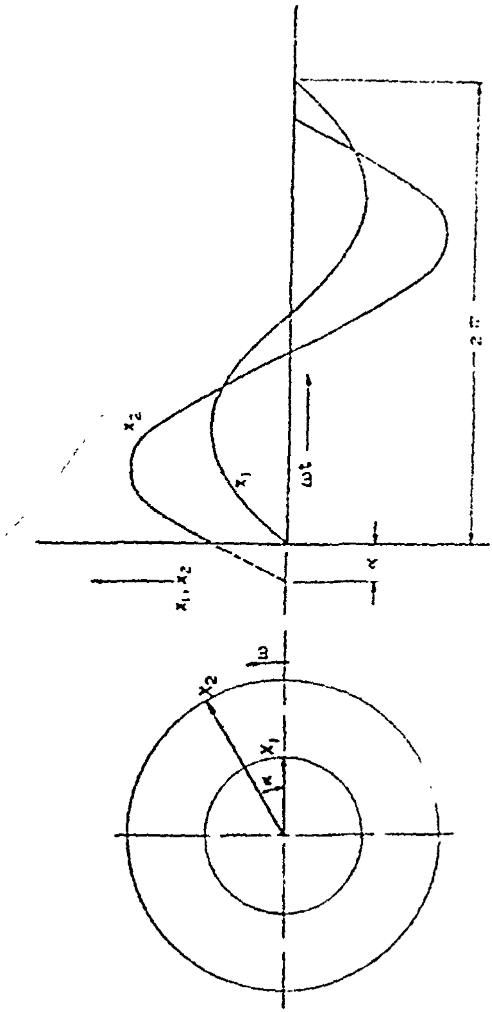


Fig. 1.3.1. Displacement, velocity and acceleration vectors.



$x_1 = X_1 \sin \omega t$

$x_2 = X_2 \sin (\omega t + \alpha)$

Fig. 1.3.2. Rotating vectors.

Illustrative Example 1.3.1

The motion of a particle is represented by the equation $x = 10 \sin \omega t$. Show the relative positions and magnitudes of the displacement, velocity and acceleration vectors at time $t = 0$, for the case when

(i) $\omega = 2.0$ rad/sec.

(ii) $\omega = 0.5$ rad/sec.

Solution

We have

$$x = 10 \sin \omega t$$

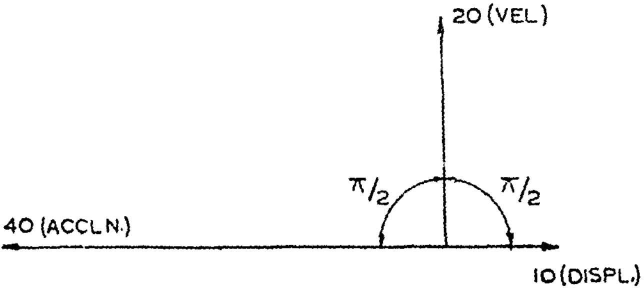
$$\dot{x} = 10\omega \sin\left(\omega t + \frac{\pi}{2}\right), \text{ from equation (1.3.2)}$$

$$\ddot{x} = 10\omega^2 \sin(\omega t + \pi), \text{ from equation (1.3.3)}$$

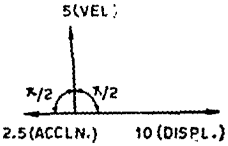
The magnitudes of displacement, velocity and acceleration vectors are 10, 10ω and $10\omega^2$ respectively. The phase difference is such that the velocity vector leads the displacement vector by $\pi/2$ and the acceleration vector leads the velocity vector by another $\pi/2$.

Case (*i*) $\omega = 2.0$ rad/sec.

The rotating vector diagram is shown in Fig. 1.3.3 (i).



(i) $\omega = 2{\cdot}0$ rad/sec.



(ii) $\omega = 0.5$ rad/sec.

Fig. 1.3.3. Displacement, velocity and acceleration vectors.

The time period is the inverse of the frequency and is given by

$$\tau = \frac{2\pi}{\omega} = \frac{2\pi}{2} = \pi \text{ sec.} \qquad \textbf{Ans.}$$

Case (*ii*) $\omega = 0.5$ rad/sec.

The rotating vector diagram is shown in Fig. 1.3.3 (ii).

$$\tau = \frac{2\pi}{0{\cdot}5} = 4\pi \text{ sec.}$$ **Ans.**

1.4 Addition of two simple harmonic motions of the same frequency.

A particle may be subjected to two simple harmonic motions of the same frequency simultaneously, in the same direction. In general, the amplitudes and the phase angle of these motions will be different. The resultant is still a simple harmonic motion.

Let the two motions be represented by

$$\left.\begin{aligned} x_1 &= X_1 \sin \omega t \\ x_2 &= X_2 \sin (\omega t + \alpha) \end{aligned}\right\} \tag{1.4.1}$$

Then the resultant motion is given by

$$x = x_1 + x_2$$

or
$$\begin{aligned} x &= X_1 \sin \omega t + X_2 \sin (\omega t + \alpha) \\ &= X \sin (\omega t + \beta), \text{ (say).} \end{aligned} \tag{1.4.2}$$

Then $X \sin (\omega t + \beta) \equiv X_1 \sin \omega t + X_2 \sin (\omega t + \alpha)$

Expanding the two sides and equating $\cos \omega t$ and $\sin \omega t$ terms on one side to the corresponding terms on the other side, we have

$$X \cos \beta = X_1 + X_2 \cos \alpha$$
$$X \sin \beta = X_2 \sin \alpha$$

This gives the amplitude of resultant motion and the phase difference as

$$\left.\begin{aligned} X &= \sqrt{(X_1 + X_2 \cos \alpha)^2 + (X_2 \sin \alpha)^2} \\ \text{and} \quad \beta &= \tan^{-1}\left(\frac{X_2 \sin \alpha}{X_1 + X_2 \cos \alpha}\right) \end{aligned}\right\} \tag{1.4.3}$$

for the resultant motion given by

$$x = X \sin (\omega t + \beta)$$

The graphical method for the addition of two simple harmonic motions is illustrated in Fig. 1.4.1. The addition of X_1 and X_2 is done by graphical method, and X and β measured.

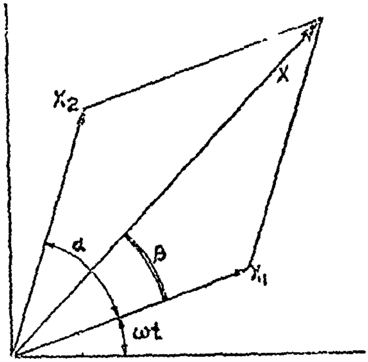


Fig. 1.4.1. Addition of vectors.

Illustrative Example 1.4.1

Add the following motions analytically.

$x_1 = 2 \cos (\omega t + 0{\cdot}5)$
$x_2 = 5 \sin (\omega t + 1.0)$

Check the solution graphically.

Solution

(i) *Analytical.* The resultant motion is given by

$$x = x_1 + x_2$$
$$= 2 \cos (\omega t + 0.5) + 5 \sin (\omega t + 1.0)$$

After expanding and grouping the cos ωt and sin ωt terms separately, we have

$$x = [5 \cos 1.0 - 2 \sin 0.5] \sin \omega t + [2 \cos 0.5 + 5 \sin 1.0] \cos \omega t$$

Putting down the values of the terms within the brackets, remembering that the angles are in radians, we have

$$x = 1.74 \sin \omega t + 5.90 \cos \omega t$$

If $x = X \sin (\omega t + \beta)$
$$= (X \cos \beta) \sin \omega t + (X \sin \beta) \cos \omega t$$

we have

$X \cos \beta = 1.74$, and $X \sin \beta = 5.90$, giving
$X = 6.17$

and $\beta = 73.6° = 1.28$ rad.

$\therefore$ $x = 6.17 \sin(\omega t + 1.28)$ **Ans.**

(ii) *Graphical.* For adding the two motions graphically, let us put the two given equations as

$$x_1 = 2 \sin(\omega t + 0.5 + \pi/2) = 2 \sin(\omega t + 2.07)$$

and $x_2 = 5 \sin(\omega t + 1.0)$

or $x_1 = 2 \sin(\omega t + 118.8°)$

and $x_2 = 5 \sin(\omega t + 57.3°)$

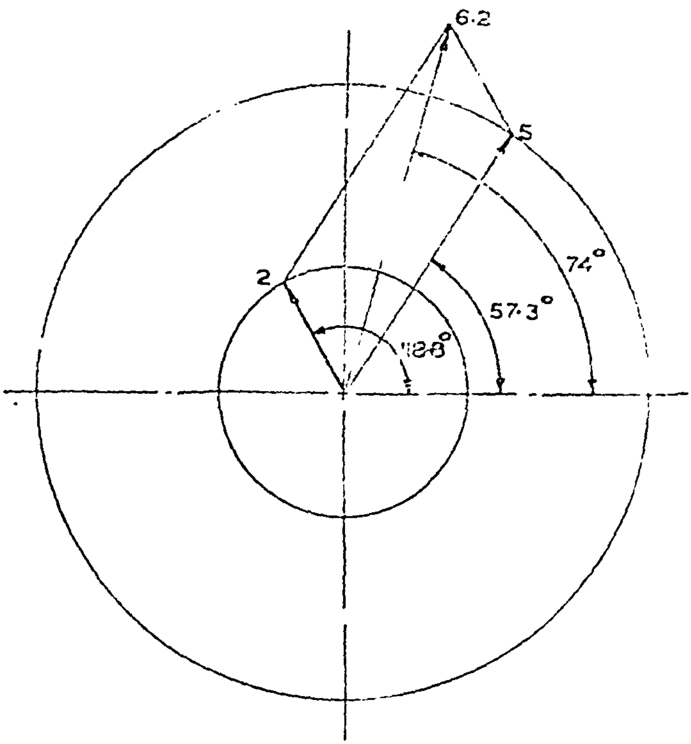


Fig. 1.4.2. Addition of vectors.

Since both the equations are in the same form, the vector diagram can now be drawn as shown in Fig. 1.4.2. The sum of the two vectors as obtained from measurement is 6.2 at an angle of 74°

$\therefore$ $x = 6.2 \sin(\omega t + 74°)$, which agrees closely with the analytical result obtained earlier. **Ans.**

Illustrative Example 1.4.2

Consider the following harmonic motions ;

(i) $x_1 = \frac{1}{2} \cos \frac{\pi}{2} t, \qquad x = \sin \pi t$

(ii) $x_1 = 2 \cos \pi t, \qquad x_2 = 2\cos 2t$

Is the sum $(x_1 + x_2)$, in each case, a periodic motion ? If so, what is its period ?

Solution

Case (*i*) Period of first wave $= \frac{2\pi}{\pi/2} = 4$ sec

Period of second wave $= \frac{2\pi}{\pi} = 2$ sec

The period of first wave is 4 seconds and that of the second wave is 2 seconds. If there is a certain time during which integral number of cycles are performed for each wave, then the combined motion is periodic since the motion repeats itself after that time. Thus the lowest common multiple of the two individual time periods is the period of the periodic motion, which, in this case is 4 sec. **Ans.**

Case (*ii*) Period of first wave $= \frac{2\pi}{\pi} = 2$ sec

Period of second wave $= \frac{2\pi}{2} = \pi$ sec

The time periods of 2 sec and π sec do not have a common multiple, hence the motion is not periodic. **Ans.**

1.5 Beats phenomenon.

Let us consider a particle subjected to two different harmonic motions given by

$$\left.\begin{aligned} x_1 &= a \sin \omega_1 t \\ x_2 &= b \sin \omega_2 t \end{aligned}\right\} \tag{1.5.1}$$

The resultant motion is given by

$$x = x_1 + x_2 = a \sin \omega_1 t + b \sin \omega_2 t \tag{1.5.2}$$

If ω_1 and ω_2 are different, then the resultant motion is not sinusoidal. An interesting special case occurs when the two frequencies are only slightly different from each other. Under these conditions the phase difference between the rotating

vectors keeps on shifting slowly and continuously. At a time when they are in phase with each other, the amplitude of resultant vibration is equal to the sum of the amplitudes of individual motions, that is $(a + b)$; and at a time when they are out of phase, the amplitude is equal to the difference of the individual amplitudes, that is $(a - b)$. Thus the resultant amplitude continuously keeps on changing from maximum of $(a + b)$ to minimum of $(a - b)$ with a frequency equal to the difference between the individual component frequencies (Fig. 1.5.1). This phenomenon is known as *Beats*. The frequency of the beats is $(\omega_2 - \omega_1)$ and it is necessary that this frequency be small in order to experience the phenomenon. Also the amplitudes a and b should be approximately equal to get clear and distinct beats.

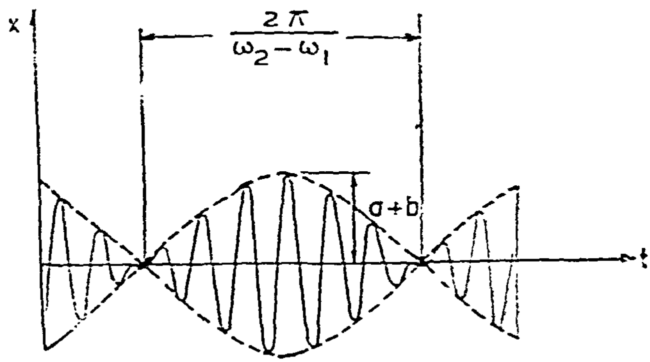


Fig. 1.5.1. Beats phenomenon.

The existance of beats can also be shown mathematically.

$$\text{Let} \quad \omega_2 - \omega_1 = \triangle\omega \tag{1.5.3}$$

Then from equation (1.5.2),

$$x = a \sin \omega_1 t + b \sin (\omega_1 + \triangle\omega)t$$

$$= a \sin \omega_1 t + b\,[\sin \omega_1 t \cos \triangle\omega t + \cos \omega_1 t \sin \triangle\omega t]$$

$$\text{or} \quad x = (a + b \cos \triangle\omega t) \sin \omega_1 t + (b \sin \triangle\omega t) \cos \omega_1 t \tag{1.5.4}$$

Equation (1.5.4) can be considered as a sum of two harmonic motions of frequency ω_1, 90° out of phase, and having time dependent amplitudes as bracketed terms. Therefore the amplitude of resultant motion is given by

$$x = \sqrt{(a + b\cos\Delta\omega t)^2 + b\sin\Delta\omega t)^2}$$

or $$x = \sqrt{a^2 + b^2 + 2\,ab\cos\Delta\omega t}$$

This expression is seen to vary between $(a + b)$ and $(a-b)$ with a frequency $\Delta\omega$.

Illustrative Example 1.5.1

A body performs, simultaneously, the motions

$x_1 = 1.90 \sin 9.5\, t$

$x_2 = 2.00 \sin 10.0\, t$

the units being centimeters, radians and seconds. Find the maximum and minimum amplitudes of the combined motion and the time period of the periodic motion.

Solution

The maximum and minimum amplitudes of motion are

$$\left.\begin{aligned} r_{max} &= 2.00 + 1.90 = 3.90 \text{ cm} \\ r_{min} &= 2.00 - 1.90 = 0.10 \text{ cm} \end{aligned}\right\}$$ **Ans.**

The beat frequency is given by

$$f = \frac{10.0-9.5}{2\pi} = \frac{0{\cdot}5}{2\pi} = 0.0795 \text{ c.p.s., and the period}$$

$$\tau = \frac{1}{f} = 4\pi = 12.57 \text{ seconds.}$$ **Ans.**

Illustrative Example 1.5.2

A body has two simultaneous motions given by the equations

$x_1 = a \sin \omega_1 t$

$x_2 = a \sin \omega_2 t$

where ω_1 is only slightly higher than ω_2. Show that for the resultant beat phenomenon curve, the slope at t $= \pi/(\omega_1-\omega_2)$ is given by

$$\frac{dx}{dt} = -\,a\,(\omega_1-\omega_2)\sin\left(\frac{\omega_1+\omega_2}{\omega_1-\omega_2}\right)\cdot\frac{\pi}{2}$$

and that the slope becomes zero only when

$$\frac{\omega_1+\omega_2}{\omega_1-\omega_2} = 2n, \text{ where } n = 1, 2, 3 \quad \ldots \quad \ldots \quad \ldots$$

Solution

The resultant motion is given by

$$x = x_1 + x_2 = a\,(\sin \omega_1 t + \sin \omega_2 t)$$

$$= 2a \sin\left(\frac{\omega_1+\omega_2}{2}\right) t .\ \cos\left(\frac{\omega_1-\omega_2}{2}\right) t.$$

$$\frac{dx}{dt} = 2a \sin\left(\frac{\omega_1+\omega_2}{2}\right) t . \left[-\left(\frac{\omega_1-\omega_2}{2}\right) \sin\left(\frac{\omega_1-\omega_2}{2}\right) t\right]$$

$$+\ 2a \cos\left(\frac{\omega_1-\omega_2}{2}\right) t . \left[\left(\frac{\omega_1+\omega_2}{2}\right) \cos\left(\frac{\omega_1+\omega_2}{2}\right) t\right]$$

Substituting for $t = \dfrac{\pi}{\omega_1-\omega_2}$, we have

$$\left[\frac{dx}{dt}\right]_{t=\pi/(\omega_1-\omega_2)} = -a\,(\omega_1-\omega_2) \sin\left(\frac{\omega_1+\omega_2}{\omega_1-\omega_2}\right)\frac{\pi}{2}$$

The slope becomes zero when

$$\frac{\omega_1+\omega_2}{\omega_1-\omega_2} . \frac{\pi}{2} = n\pi$$

or $$\frac{\omega_1+\omega_2}{\omega_1-\omega_2} = 2n$$ **Ans.**

1.6 Complex method of representing harmonic vibrations.

Let there be a vector V in the x-y plane represented by a complex number

$$V = a + jb \tag{1.6.1}$$

where $j = \sqrt{-1}$. See Fig. 1.6.1.

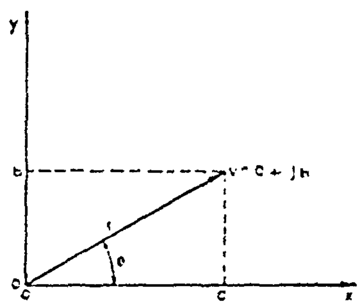


Fig. 1.6.1. Representation of vectors by complex numbers.

Let $r = \sqrt{a^2+b^2}$ be the modulus of the complex number. This is also equal to the magnitude of the given vector.

Let $\theta = \tan^{-1}\frac{b}{a}$ be its argument. This is also the angle that the vector makes with x-axis

Then $V = a + jb = r[\cos\theta + j\sin\theta]$ (1.6.2)

or $V = re^{j\theta}$ (1.6.3)

by Euler's formula.

If for a particular vibrating particle, r is the amplitude and ω its circular frequency, then the displacement of the particle can be written down as

$$x = r[\cos\omega t + j\sin\omega t] = r\,e^{j\omega t} \quad (1.6.4)$$

Differentiating the above equation,

$$\dot{x} = \omega r[-\sin\omega t + j\cos\omega t]$$

or $\dot{x} = j\omega r[\cos\omega t + j\sin\omega t] = j\omega r\,e^{j\omega t}$ (1.6.5)

Differentiating again.

$$\ddot{x} = j^2\omega^2 r[\cos\omega t + j\sin\omega t] = j^2\omega^2 re^{j\omega t} \quad (1.6.6)$$

$$= -\omega^2 re^{j\omega t}$$

Camparing equations (1.6.4), (1.6.5) and (1.6.6) with equations (1.3.1), (1.3.2) and (1.3.3), it can be seen that multiplication of complex number by j is equivalent to rotation of the corresponding vector by 90° around the origin.

Illustrative Example 1.6.1

Represent the following complex numbers in exponential form.

(i) $3 + j7$

(ii) $-5 + j4$

Solution

(i) $V = 3 + j7 = re^{j\theta}$

where $r = \sqrt{3^2+7^2} = 7.62$

and $\theta = \tan^{-1} \dfrac{7}{3} = 66.8° = 1.17$ rad

$\therefore \quad V = 7.62 e^{j\,1.17}$ **Ans.**

(ii) $V = -5 + j4 = r\,e^{j\theta}$

where $r = \sqrt{(-5)^2 + (4)^2} = 6.40$

and $\theta = \tan^{-1}\left(\dfrac{4}{-5}\right) = 141.3° = 2.47$ rad.

$\therefore \quad V = 6.40\,e^{j2.47}$ **Ans.**

Illustrative Example 1.6.2

Represent the following complex numbers in rectangular form.

(i) $5\,e^{j0.1}$

(ii) $17\,e^{-j3.74}$

Solution

(i) $V = 5\,e^{j0.1}$, 0.1 being the angle in radians,

$= 5\,[\cos 0.1 + j \sin 0.1]$

$= 5\,[0{\cdot}925 + j\,0.099]$

or $V = 4{\cdot}97 + j0.49$ **Ans.**

(ii) $V = 17\,e^{-j3.74}$, 3.74 being the angle in radians.

$= 17\,[\cos 3.74 - j \sin 3.74]$

$= 17\,[(-.829) - j\,(-0\,559)]$

or $V = -14.08 + j\,9.50$ **Ans**

1.7 Work done by a harmonic force on a harmonic motion.

Let a harmonic force $P = P_0 \sin \omega t$ be acting upon a body. Under this harmonic force the body will vibrate with the same frequency and with some phase lag, in general. Let the motion of the body be given by $x = x_0 \sin (\omega t - \phi)$. The work done by the force P during an interval when the body moves through a displacement dx is given by

$$dW = P.\,dx = P.\,\frac{dx}{dt}\,.\,dt$$

Over a period of one cycle ωt varies from 0 to 2π, and therefore t varies from 0 to $2\pi/\omega$. Therefore the work done per cycle is given by

$$W = \int_0^{2\pi/\omega} P \cdot \frac{dx}{dt} \cdot dt$$

$$= \int_0^{2\pi/\omega} [P_0 \sin \omega t]\, [\omega x_0 \cos (\omega t - \phi)]\, dt$$

$$= P_0 \omega x_0 \int_0^{2\pi/\omega} \sin \omega t\, [\cos \omega t\, \cos\phi + \sin \omega t \sin \phi] dt$$

$$= P_0 \omega x_0 \left[\cos \phi \int_0^{2\pi/\omega} \sin \omega t \cos \omega t\, dt + \sin \phi \int_0^{2\pi/\omega} \sin^2 \omega t\, dt\right]$$

$$= P_0 \omega x_0 \left[0 + \frac{\pi}{\omega} \sin \phi \right]$$

or $$W = \pi P_0 x_0 \sin \phi \tag{1.7.1}$$

This means, if $\phi = 0$, i. e. the force is in phase with the displacement, no work is done per cycle. On the other hand if $\phi = 90°$, i. e. the force is ahead of displacement by 90° or in phase with the velocity, the work done per cycle is $\pi P_0 x_0$. This also concludes that for a system with zero damping, no work will be done by the external force except at resonance and for a damped system there will always be energy required to keep the system vibrating at any frequency.

Illustrative Example 1.7.1

A force $P_0 \sin \omega t$ acts on a displacement $x_0 \sin (\omega t - \pi/6)$, where

$P_0 = 2.5$ kg

$x_0 = 5.0$ cm

and $\omega = 20\pi$ rad/sec

What is the work done during

(i) the first second ?

(ii) the first 1/40 second ?

Solution

$$\text{Work done} = \int_0^{t_1} P.\frac{dx}{dt}\,dt$$

$$= P_0\,x_0\omega\int_0^{t_1} \sin\omega t\cos\left(\omega t - \frac{\pi}{6}\right)dt$$

$$= P_0\,x_0\omega\int_0^{t_1} \frac{1}{2}\left[\sin\left(2\omega t - \frac{\pi}{6}\right) + \sin\frac{\pi}{6}\right]dt$$

$$= P_0\,x_0\,\omega\left[-\frac{\cos\left(2\omega t - \frac{\pi}{6}\right)}{2\,\omega} + \frac{1}{2}\,t\right]_0^{t_1}$$

Substituting for P_0, x_0 and ω, we have

$$\text{Work done} = \frac{2.5\times5\times62.8}{2}\left[-\frac{\cos\left(40\pi t - \frac{\pi}{6}\right)}{40\pi} + \frac{\cos\pi/6}{40\,\pi} + \frac{1}{2}\,t_1\right]$$

(i) when $t_1 = 1$

$$\text{Work done} = \frac{2{\cdot}5\times5\times62{\cdot}8}{2}\left[-\frac{\cos\pi/6}{40\pi} + \frac{\cos\pi/6}{40\pi} + \frac{1}{2}\right]$$

$= 196{\cdot}5$ kg-cm. **Ans.**

(ii) when $t_1 = 1/40$

$$\text{Work done} = \frac{2{\cdot}5\times5\times62{\cdot}8}{2}\left[-\frac{\cos\left(\pi - \frac{\pi}{6}\right)}{40\pi} + \frac{\cos\pi/6}{40\pi} + \frac{1}{2}\times\frac{1}{40}\right]$$

$= 11.1$ kg-cm. **Ans.**

1.8 Fourier series and harmonic analysis.

A periodic motion is one which repeats itself in all details after a certain time interval called the period of motion τ. From the mathematical theory it can be shown that any periodic curve $f(t)$ of frequency ω $(=2\pi/\tau)$ can be represented by a series of harmonic functions the frequencies of which are the integral multiples of the frequency ω. Or,

$$f(t) = a_0 + a_1 \cos \omega t + a_2 \cos 2\omega t + a_3 \cos 3\omega t + \ldots\ldots \quad \ldots$$
$$+ b_1 \sin \omega t + b_2 \sin 2\omega t + b_3 \sin 3\omega t \ldots \quad \ldots(1.8.1)$$

The various amplitudes a_1, $a_2 \ldots b_1$, $b_2 \ldots\ldots$ of individual sine waves and the constant a_0 can be determined analytically when $f(t)$ is given. The series given by equation (1.8.1) is called the *Fourier Series*. The harmonic of frequency ω is called the fundamental or the first harmonic of $f(t)$ and the harmonic of frequency $n\omega$ is called the n^{th} harmonic.

In determining a_n and b_n, use is made of the following formulae.

$$\left.\begin{aligned}
\int_t^{t+2\pi/\omega} \cos(n\omega t)\cos(m\omega t)\,dt &= \begin{cases} 0, & \text{for } m \neq n \\ \dfrac{\pi}{\omega}, & \text{for } m = n \end{cases} \\
\int_t^{t+2\pi/\omega} \sin(n\omega t)\sin(m\omega t)\,dt &= \begin{cases} 0, & \text{for m} \neq n \\ \dfrac{\pi}{\omega}, & \text{for } m = n \end{cases} \\
\int_t^{t+2\pi/\omega} \sin(n\omega t)\cos(m\omega t)\,dt &= 0, \text{ for all } m, n \\
\int_t^{t+2\pi/\omega} \sin(n\omega t)\,dt &= 0, \text{ for all } n \\
\int_t^{t+2\pi/\omega} \cos(n\omega t)\,dt &= 0, \text{ for all } n
\end{aligned}\right\} \quad (1.8.2)$$

where m and n are non-zero integers.

The coefficients a_0, a_1, $a_2 \ldots\ldots b_1$, $b_2 \ldots$ are obtained as follows.

(i) To find a_0, integrate both sides of equation (1.8.1) over any interval of length $\tau = 2\pi/\omega$. Then, according to equations (1.8.2), all the integrals on the right hand side of the equation are zero except the one containing a_0, that is

$$\int_t^{t+2\pi/\omega} f(t)\,dt = a_0 \frac{2\pi}{\omega}$$

or
$$a_0 = \frac{\omega}{2\pi}\int_t^{t+2\pi/\omega} f(t)\, dt \qquad (1.8.3)$$

(ii) To find a_n, multiply both sides of equation (1.8.1) by cos $(n\omega t)$ and integrate over any interval of time $\tau = 2\pi/\omega$. Then, according to equations (1.8.2), all the integrals on the right are zero except the one containing a_n, that is

$$\int_t^{t+2\pi/\omega} f(t) \cos (n\omega t)\, dt = a_n \int_t^{t+2\pi/\omega} \cos^2 (n\omega t)\, dt$$
$$= a_n \frac{\pi}{\omega}$$

or
$$a_n = \frac{\omega}{\pi}\int_t^{t+2\pi/\omega} f(t) \cos (n\omega t)\, dt \qquad (1.8.4)$$

(iii) In the same way if we multiply equation (1.8.1) by sin $(n\omega t)$ and integrate, we have

$$b_n = \frac{\omega}{\pi}\int_t^{t+2\pi/\omega} f(t) \sin (n\omega t)\, dt \qquad (1.8.5)$$

This branch of mathematics which deals with splitting up a periodic function into a series of harmonic functions is called *Harmonic Analysis.*

Illustrative Example 1.8.1

A periodic motion observed on the oscilloscope is illustrated in Fig. 1.8.1. Represent this motion by harmonic series.

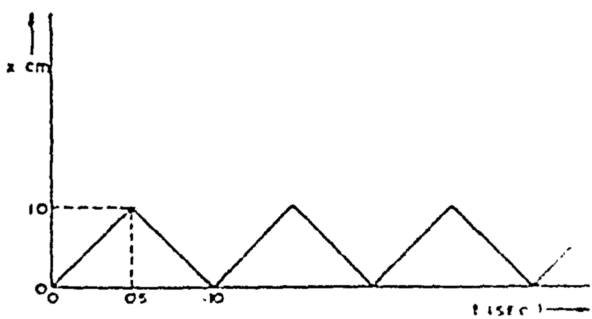


Fig. 1.8.1. Periodic motion.

Solution

Let the equation representing the above periodic motion be represented by equation (1.8.1), that is,

$$x(t) = a_0 + a_1 \cos \omega t + a_2 \cos 2\omega t + \dots$$
$$+ b_1 \sin \omega t + b_2 \sin 2\omega t + \dots$$

The equation of the curve for one cycle can be written as

$$x(t) = 200t, \qquad 0 \leqslant t \leqslant \cdot 05$$
$$= -200t + 20, \qquad \cdot 05 \leqslant t \leqslant \cdot 1$$

$$\omega = \frac{2\pi}{\tau} = \frac{2\pi}{0 \cdot 1} = 20\pi$$

Equation (1.8.3), now gives

$$a_0 = 10\left[\int_0^{.05} 200t \; dt + \int_{.05}^{0.1} \left(-200\, t + 20\right) dt\right]$$

or $a_0 = 5$

Equation (1.8.4) gives

$$a_n = 20\left[\int_0^{.05} 200t \cos 20\pi nt \; dt + \right.$$
$$\left. \int_{.05}^{0.1} \left(-200t + 20\right) \cos 20\pi nt \; dt\right]$$

or $a_n = \dfrac{20}{\pi^2 n^2}\left[\cos \pi n - 1\right]$

that is

$$a_n = -\frac{40}{\pi^2 n^2}, \text{ if } n \text{ is odd}$$
$$= 0, \text{ if } n \text{ is even.}$$

Equation (1.8.5) gives

$$b_n = 20\left[\int_0^{\cdot 05} 200t \sin 20\pi nt \, dt + \int_{\cdot 05}^{0.1} \left(-200t + 20\right) \sin 20\pi nt \, dt\right]$$

or $\quad b_n = 0$

Hence, we can represent the harmonic series as

$$x(t) = 5 - \frac{40}{\pi^2} \cos 20\pi t - \frac{40}{\pi^2 3^2} \cos 60\pi t \ldots$$

or $\quad x(t) = 5 - \dfrac{40}{\pi^2}\left[\cos 20\pi t + \dfrac{1}{3^2} \cos 60\pi t + \ldots\right]$ **Ans.**

PROBLEMS FOR PRACTICE

1.1 A harmonic motion has an amplitude of 5.0 cm and a frequency of 25 cps. Find the time period, maximum velocity and maximum acceleration.

1.2 An instrument has a natural frequency of 10 cps. It can stand a maximum acceleration of 1000 cm/sec^2. Find the maximum displacement amplitude.

1.3 The motion of a particle is respresented by the equation $x = 4 \sin \omega t$. Sketch roughly, the variation of the maximum values of

(i) displacement,

(ii) velocity,

(iii) acceleration, and

(iv) jerk

with the change in exciting frequency.

1.4 A particle is simultaneously subjected to two motions

$$x_1 = 5 \sin 2\pi t$$
$$x_2 = 7 \sin (2\pi t + \pi/3)$$

Sketch a displacement versus time plot for these two motions and show the phase difference on this sketch. What is the final motion like ?

1.5 The displacement of the slider in the slider-crank mechanism is given by

$$x = 24 \cos 8\pi t + \frac{3}{2} \cos 16\pi t.$$

Plot a displacement versus time diagram. What is the acceleration of the piston at $t = \frac{1}{8}$ sec ?

1.6 Add the following vectors analytically:

$$x_1 = 4 \cos (\omega t + 10°)$$
$$x_2 = 6 \sin (\omega t + 60°).$$

Check the solution graphically :

1.7 Add the following vectors analytically :

$$x_1 = 8 \sin (\omega t + 30°)$$
$$x_2 = 10 \sin (\omega t - 60°)$$

Check the solution graphically.

1.8 Find the amplitude of the sum of the two harmonic motions

$$x_1 = 3 \cos (2t + 1)$$
$$x_2 = 4 \cos (2t + 1.5).$$

1.9 Show that the resultant motion of three harmonic motions given below is zero.

$$x_1 = a \sin \omega t$$
$$x_2 = a \sin \left(\omega t + \frac{2\pi}{3}\right)$$
$$x_3 = a \sin \left(\omega t + \frac{4\pi}{3}\right)$$

1.10 A body is subjected to two harmonic motions as given below :

$$x_1 = 15 \sin (\omega t + \pi/6)$$
$$x_2 = 8 \cos (\omega t + \pi/3)$$

What extra harmonic motion should be given to the body to bring it to a static equilibrium ?

1.11 Split up the harmonic motion

$x = 10 \sin \left(\omega t + \frac{\pi}{6}\right)$, into two harmonic motions, one having a phase angle of zero and the other of 45°.

1.12 Split up the harmonic motion

$$x = 8 \sin (\omega t + \pi/4)$$

into two harmonic motions, one of which has an amplitude of 10 and phase difference zero.

1 13 Split up the harmonic motion

$$x = 8 \cos (\omega t + \pi/4)$$

into two harmonic motions, one of them having a phase angle of zero and the other having a phase angle of 60°

1.14 Consider x_1 and x_2 to be two harmonic motions of periods T_1 and T_2, respectively. Show that the resultant motion, $x_1 + x_2$, can be periodic only if two integers m and n can be found such that

$$m T_1 = n T_2 = T \text{ (say).}$$

Under these conditions, show that T is she period of the resultant periodic motion.

1.15 Consider the harmonic motions

(i) $x_1 = \cos \frac{\pi}{2} t, \quad x_2 = \frac{1}{2} \sin \pi t$

(ii) $x_1 = \frac{1}{2} \cos 2t, \quad x_2 = \frac{1}{3} \cos \left(2t + \frac{\pi}{4}\right)$

(iii) $x_1 = \frac{1}{2} \cos 2t, \quad x_2 = \frac{1}{3} \cos \left(2 + \frac{\pi}{4}\right)t$

(iv) $x_1 = 3 \sin 7t, \quad x_2 = 7 \cos 3t$

Is the sum $(x_1 + x_2)$ in each of the above cases a periodic motion ? If so, what is its period ?

1.16 A body describes simultaneously two motions,

$$x_1 = 3 \sin 40t$$
$$x_2 = 4 \sin 41t$$

What is the maximum and minimum amplitude of combined motion and what is the beat frequency ?

1.17 A particle is subjected to two simulaneous harmonic motions in x and y direction as given below.

$$x = a \sin \omega t$$
$$y = b \sin (\omega t - \phi)$$

1.5 The displacement of the slider in the slider-crank mechanism is given by

$$x = 2\tfrac{1}{2} \cos 8\pi t + \frac{3}{2} \cos 16\pi t.$$

Plot a displacement versus time diagram. What is the acceleration of the piston at $t = \frac{1}{4}$ sec ?

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$$x_1 = 4 \cos (\omega t + 10^\circ)$$
$$x_2 = 6 \sin (\omega t + 60^\circ).$$

Check the solution graphically :

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$$x_1 = 8 \sin (\omega t + 30^\circ)$$
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(iii) $x_1 = \frac{1}{2} \cos 2t, \quad x_2 = \frac{1}{3} \cos \left(2 + \frac{\pi}{4}\right) t$

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Under these conditions, show that T is she period of the resultant periodic motion.

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(i) $x_1 = \cos\frac{\pi}{2}\,t, \quad x_2 = \frac{1}{2}\sin \pi t$

(ii) $x_1 = \frac{1}{2}\cos 2t, \quad x_2 = \frac{1}{3}\cos\left(2t + \frac{\pi}{4}\right)$

(iii) $x_1 = \frac{1}{2}\cos 2t, \quad x_2 = \frac{1}{3}\cos\left(2 + \frac{\pi}{4}\right)t$

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1.17 A particle is subjected to two simulaneous harmonic motions in x and y direction as given below.

$$x = a \sin \omega t$$
$$y = b \sin (\omega t - \phi)$$

Describe the path of the particle when $a=b$ and ϕ is (i) zero, (ii) 45°, (iii) 90°, (iv) 135° and (v) 180°.

What will happen when $a \neq b$.

.18 Show that the motion of the piston of a reciprocating engine is periodic with terms containing the fundamental and even harmonics.

..19 Represent the following complex numbers in exponential form.

(i) $3 + j\,4$

(ii) $3 - j\,4$

(iii) $-3 + j\,4$

(iv) $-3 - j\,4$

..20 Represent the following complex numbers in rectangular form.

(i) $9\,e^{j0\cdot 3}$

(ii) $5\,e^{j2\cdot 1}$

(iii) $14\,e^{-j2\cdot 8}$

(iv) $10\,e^{-j1\cdot 1}$

..21 A force $P_0 \sin \omega t$ acts on a displacement $x_0 \sin (\omega t - \pi/3)$.

If $P_0 = 10$ kg

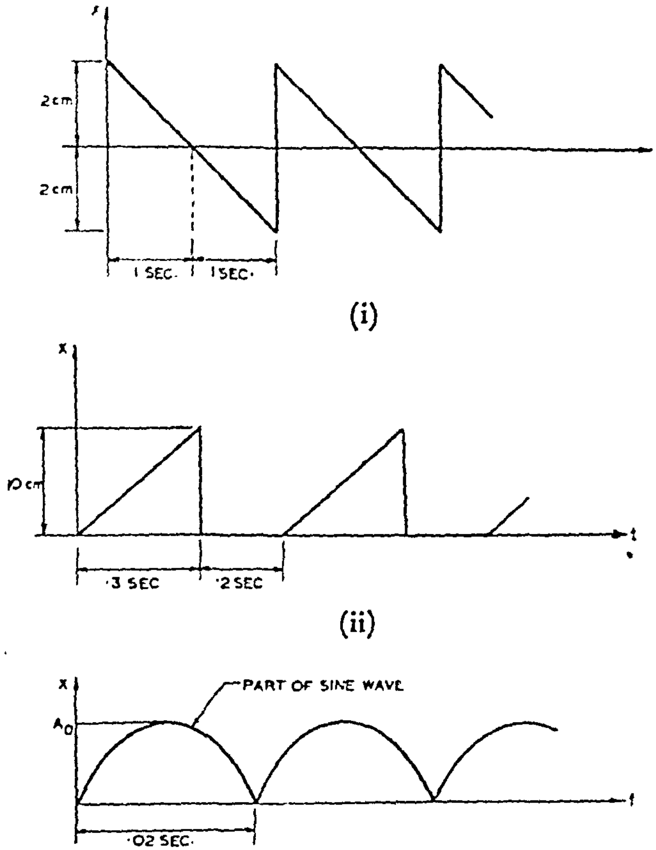
$x_0 = 2$ cm

$\omega = 2\pi$ rad/sec, find the work done during

(i) the first cycle,

(ii) the first second,

(iii) the first quarter second.



(iii)

Fig. P. 1.22.

1.22 Represent the periodic motions given in Fig. P. 1.22 ... harmonic series.

1.23 Show that the fourier series expansion for ... $x(t)$ defined in the finite interval $-\pi \leqslant t \leqslant$ ...

$$x(t) = 0, \qquad -\pi \leqslant t \leqslant 0$$
$$= \sin t, \qquad 0 \geqslant t \geqslant \pi$$

is given by

$$x(t) = \frac{1}{\pi} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\cos 2nt}{4n^2 - 1} \ldots$$

CHAPTER 2

UNDAMPED FREE VIBRATIONS OF SINGLE DEGREE OF FREEDOM SYSTEMS

2.1 Introduction.

For the proper design of machines or machine parts which are subjected to vibratory forces, it is essential to estimate their natural frequencies to avoid resonance conditions. In most cases there is always certain amount of damping associated with the system. But in some cases the damping is so small that it can be neglected. Under these conditions natural frequency based on undamped free vibration is a critical factor.

First in this chapter, the principles underlying the determination of natural frequency are discussed for a classical spring-mass system. Later on, mathematical models are represented for certain physical systems for the purpose of determining their natural frequencies.

2.2 Derivation of differential equation.

Let us consider a spring-mass system of Fig. 2.2.1 (a), constrained to move in a rectilinear manner along the axis of the spring. Let k and m be the stiffness of the spring and the mass of the block respectively. At any instant, let it occupy any displacad position as shown in Fig. 2.2.1 (b). We can locate its postion with reference to the support or any other reference mark. The reference mark considered here is the equilibrium position of the mass, from where the position of mass at any instant is x (say). Let us consider x to be positive in the downward direction and negative in the upward direction.

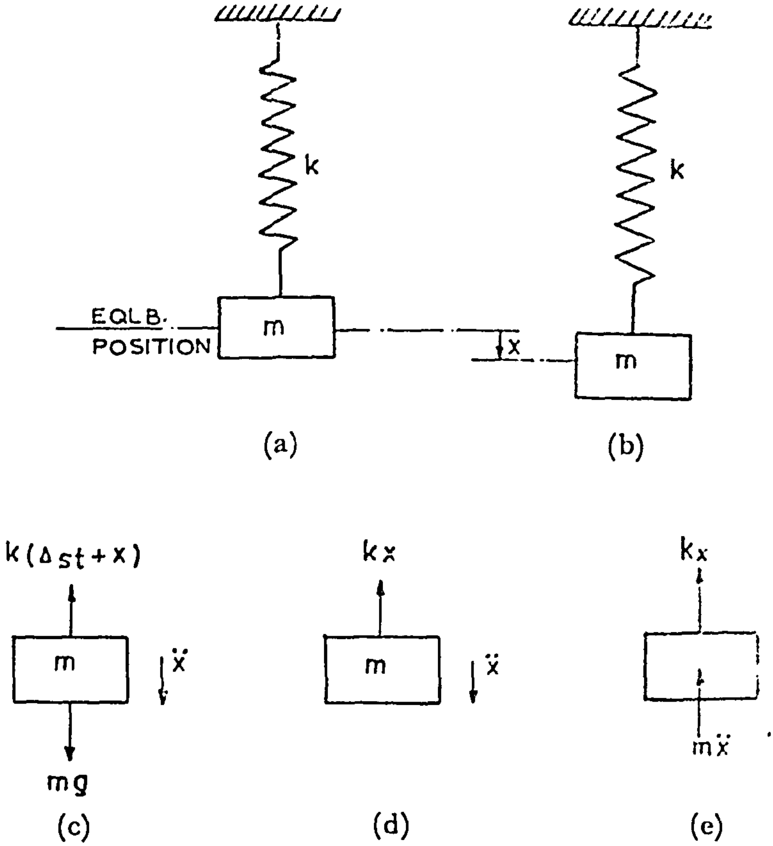


Fig. 2.2.1. Derivation of differential equation for a spring-mass system.

In the equilibrium position the forces acting on the mass are

(i) mg, vertically downwards, and

(ii) $k.\Delta_{st}$, vertically upwards,
where Δ_{st} is the static deflection of the spring under the load mg.

For equilibrium, we have

$$k.\Delta_{st} = mg \tag{2.2.1}$$

In the displaced position, Fig. 2.2.1 (b), the external forces acting on the mass are as shown in Fig. 2.2.1 (c). The spring force now is $k\,(\Delta_{st}+x)$ because $(\Delta_{st}+x)$ is the total deflection of the spring in this position. Under the forces

shown the body has an acceleration $\ddot{x}$ downwards. It should be clearly understood that we are not talking of numerical value of acceleration which may be positive or negative. We are simply saying that we have chosen a direction sign for acceleration as positive downwards and negative upwards from equilibrium position, and these directions are the same as those chosen for x. If at any instant the value of $\ddot{x}$ comes out to be negative, all it means is that it has a negative or upward acceleration. Therefore, from Newton's second law of motion,

$$m\ddot{x} = mg - k(\Delta_{st} + x)$$
$$= -k.x,$$

because $mg - k\Delta_{st} = 0$ from equation (2.2.1)

Therefore, $m\ddot{x} + kx = 0$ (2.2.2)

This is the differential equation of motion for a single degree of freedom spring-mass system having free vibrations. It may be pointed out here that it was not necessary to consider the force mg and $k\Delta_{st}$ in the displaced system since these forces neutralize each other at all instants and therefore, we need consider only those forces which have come into picture from beyond the equilibrium position. In this case the only external force acting on the body in the displaced position is kx in the upward direction because x is the displacement from the equilibrium position [see Fig. 2.2.1 (d)].

Therefore, $m\ddot{x} = -kx$, which is the same as equation (2.2.2).

Equation (2.2.2) has been derived by the application of Newton's second law of motion. The same equation can be derived by the application of *D'Alemberts Principle* which states that a body which is not in static equilibrium by virtue of some acceleration which it possesses, can be brought to static equilibrium by introducing on it the inertia force which can be considered to be an extra external force. This inertia force is equal to mass times the acceleration of the body and acts through the centre of gravity of the body in the direction opposite to that of the acceleration.

If now we consider the displaced system of Fig. 2.2.1 (b), the spring force on the body is kx in the upward direction.

Since the body has an acceleration $\ddot{x}$ in the downward direction, it can be brought to static equilibrium by introducing inertia force $m\ddot{x}$ acting in the upward direction, i. e. in the direction opposite to that of acceleration. So the body is in static equilibrium under the action of two forces as shown in Fig. 2.2.1 (e).

Therefore, $m\ddot{x}+kx=0$, which equation is the same as obtained earlier in this section. Certain problems are very much simplified by the application of D'Alemberts principle.

Consider again the case of a system having angular motion. Let us take the example of a light, stiff rod of length l, pivoted at one end and having a concentrated mass m at the other end, as shown in Fig. 2.2.2, so as to form a simple pendulum.

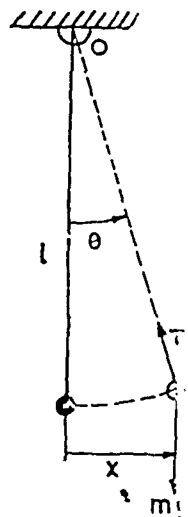


Fig. 2.2.2. Simple pendulum.

Consider at any instant, that the rod is displaced through an angle θ. The external forces acting on the bob are shown in the figure. For small amplitudes of vibration, the displacement of the bob may be considered to be linear,

i. e. $x \simeq l\,\theta$ (2.2.3)

Now taking moments about the pivot point O, and applying Newton's second law, we have

(Mass M. I. of the system about O) × (Angular acceleration)

= Algebraic sum of the external moments about O in the direction of angular acceleration.

If θ is positive anticlockwise, $\ddot{\theta}$ is also positive anticlockwise, then we have

$$J_0 \ddot{\theta} = -mgx$$
$$= -mgl\theta, \text{ from equation (2.2.3)}$$

If the bob is considered to be a concentrated mass and the rod to be of negligible mass, then

$$J_0 = ml^2$$

Therefore, $ml^2 \ddot{\theta} = -mgl\theta$

or $l\ddot{\theta} + g\theta = 0$ (2.2.4)

Equation (2.2.4) is similar to equation (2.2.2) in all respects and thus represents a similar motion.

Illustrative Example 2.2.1

Write down the differential equation of motion for the system shown in Fig. 2.2.3.

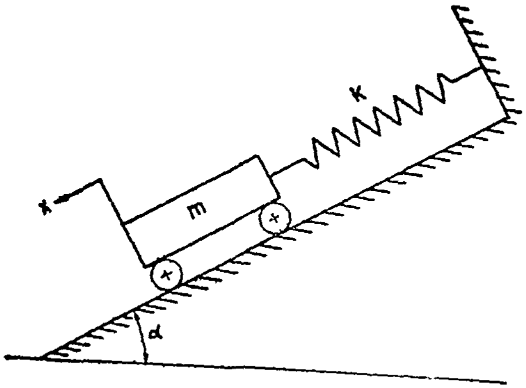


Fig. 2.2.3. Inclined spring-mass system.

Solution

In the equilibrium position the static deflection of the spring is Δ_{st}, such that

$$k.\ \Delta_{st} = mg \sin \alpha$$

For any subsequent deflection x of the spring beyond the equilibrium position, the additional spring force acting on the mass is kx in the upward direction along the incline. The gravity component down the incline always balances the initial force of the spring due to static deflection.

Thus, applying Newton's second law of motion, we have

$$m\ddot{x} = -kx$$

or $$m\ddot{x} + kx = 0$$

which equation is the same as equation (2.2.2) whether $\alpha = 0$ or $90°$ or any other angle. **Ans.**

Illustrative Example 2.2.2

Derive the differential equation of motion for a spring controlled simple pendulum shown in Fig. 2.2.4. The spring is in its unstretched position when the pendulum rod is vertical.

Solution

Consider the system displaced through an angle θ. The forces acting on the system are shown in the figure. Applying Newton's second law for the angular acceleration and moments about point O, we have

$$J_0\ddot{\theta} = -mgl\theta - ka\theta \cdot a$$

or $$ml^2\ddot{\theta} + (mgl + ka^2)\,\theta = 0$$

which is the required differential equation of motion.

It may be noted that an approximation has been taken in the above derivation, with is justified for small amplitudes of vibration. **Ans.**

2.3 Solution of differential equation.

The differential equation derived for the motion of a classical spring-mass system in Sec. 2.2. was

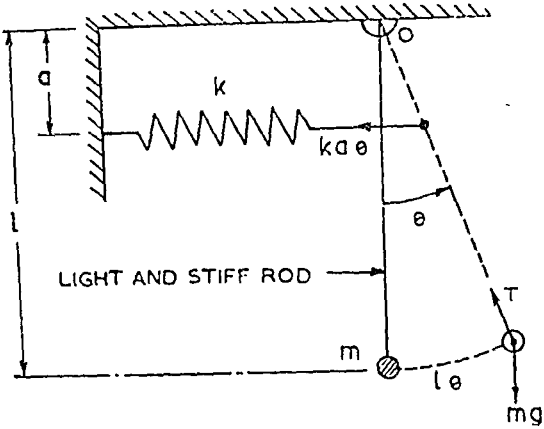


Fig. 2.2.4. Spring controlled simple pendulum.

$$m\ddot{x} + kx \quad 0$$

Writing this equation as

$$\ddot{x} + \frac{k}{m} \quad = 0$$

and letting

$$\frac{k}{m} = \omega_n^2 \tag{2.3.1}$$

we have

$$\ddot{x} + \omega_n^2 x = 0 \tag{2.3.2}$$

Writing ω_n^2 for k/m has a special significance which will be clear in the following paragraphs.

Equation (2.3.2) is the standard differential equation for a single degree of freedom system having undamped free vibrations. The standard solution for this differential equation can be written in either of the following three ways.

$$\left.\begin{aligned} x &= A \sin \omega_n t + B \cos \omega_n t \\ x &= A_1 \sin (\omega_n t + \phi_1) \\ x &= A_2 \cos (\omega_n \quad + \phi_2) \end{aligned}\right\} \tag{2.3.3}$$

It may be noticed that a second order differential equation ıas in its solution two arbitrary constants which have to be ıetermined from the initial conditions. Each one of the olutions given in equations (2.3.3) is a harmonic motion of

circular frequency ω_n. This is the angular velocity with which the vector rotates about the origin, as shown in Fig. 1.3.1.

Regarding the determination of two constants in any of the equations (2.3.3), we have to make use of two initial conditions. Let us say

$$\left.\begin{array}{ll} & x = X_0 \text{ at } t = 0 \\ \text{and} & \dot{x} = 0 \quad \text{at } t = 0 \end{array}\right\} \tag{2.3.4}$$

It is most convenient to pick up the first of the equations (2.3.3) and determine the constants A and B, although the other two equations can also be used, and the final solution, of course, has to be the same.

Writing the first of equations (2.3.3) again,

$$x = A \sin \omega_n t + B \cos \omega_n t \tag{2.3.5}$$

Differentiating it,

$$\dot{x} = A\omega_n \cos \omega_n t - B\omega_n \sin \omega_n t$$

Substituting in the above two equations the initial conditions as given in equations (2.3.4), we have

$$\begin{array}{ll} & X_0 = 0 + B \\ \text{and} & 0 = A\omega_n - 0 \\ \text{giving} & A = 0 \\ \text{and} & B = X_0 \end{array}$$

Substituting these constants back in equation (2.3.5), we have

$$x = X_0 \cos \omega_n t \tag{2.3.6}$$

as the final solution for the specified initial conditions. The time-displacement curve for equation (2.3.6) is shown in Fig. 2.3.1. This figure shows the vibratory feature of an undamped, single degree of freedom system.

The time period of this oscillatory motion is that time during which the corresponding rotating vector having a circular frequency ω_n completes one cycle of 2π radians.

Therefore, $$\tau = \frac{2\pi}{\omega_n} \tag{2.3.7}$$

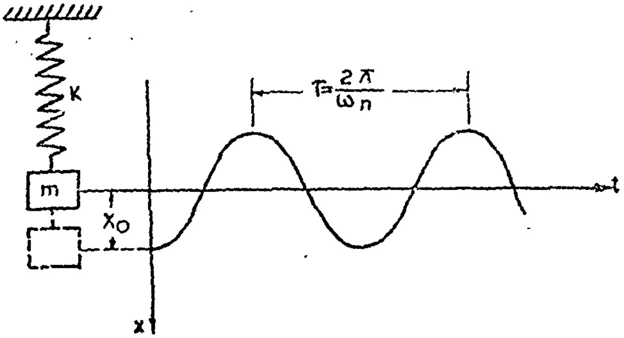


Fig. 2.3.1. Undamped free vibration.

Also the natural frequency in cycles per second denoted by f_n is the inverse of the time period, i.e.,

$$f_n = \frac{\omega_n}{2\pi} \tag{2.3.8}$$

Another way of writing ω_n, is

$$\omega_n = \sqrt{\frac{k}{m}}, \text{ from equation (2.3.1)}$$

$$= \sqrt{\frac{kg}{mg}}$$

But $\qquad \dfrac{mg}{k} = \Delta_{st}$, from equation (2.2.1)

Therefore, $\qquad \omega_n = \sqrt{\dfrac{g}{\Delta_{st}}} \qquad (2.3.9)$

Or, in terms of cycles per second we have,

$$f_n = \frac{1}{2\pi}\sqrt{\frac{g}{\Delta_{st}}} \tag{2.3.10}$$

If the static deflection is in centimeters, we have, after substituting 980 cm/sec² for g,

$$f_n = \frac{\sqrt{980}}{2\pi}\sqrt{\frac{1}{\Delta_{st}}}$$

$$f_n = \frac{4{\cdot}982}{\sqrt{\Delta_{st}}} \tag{2.3.10a}$$

The above equation gives the relationship between the natural frequency and the static deflection of the system and is plotted in Fig. 2.3.2 on the log-log scale. This plot is very useful as the natural frequency of a system can be read out straightaway if its static deflection is known

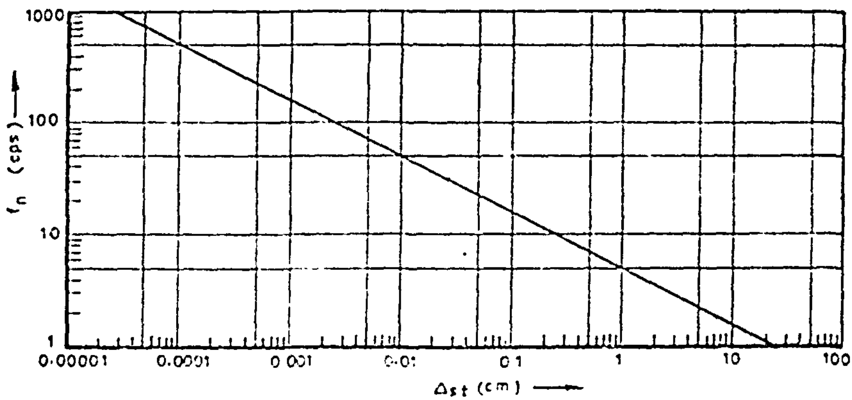


Fig. 2.3.2. Natural frequency versus static deflection.

Illustrative Example 2.3.1

A light cantilever of length l has a weight W fixed at its free end. Find the frequency of lateral vibrations in the vertical plane.

Solution

Fig. 2.3.3 shows the schematic of the system. For the deflection at the free end of the cantilever, we have

$$\Delta_{st} = \frac{Wl^3}{3EI} \tag{2.3.11}$$

where E is the modulus of elasticity of the material of the cantilever and I is the M.I. of the section of the beam about its neutral axis.

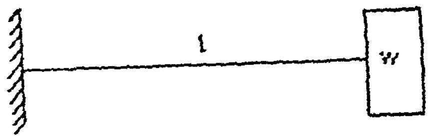


Fig. 2.3.3. A light cantilever system.

Now $k = W/\Delta_{st}$

$$= \frac{3EI}{l^3}, \text{ from equation (2.3.11)}$$

Therefore, $$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{3EI/l^3}{W/g}} = \sqrt{\frac{g}{\Delta_{st}}} \qquad (2.3.12)$$

or $\omega_n = \sqrt{\dfrac{g}{\Delta_{st}}}$ rad/sec. **Ans.**

Illustrative Example 2.3.2

Find the time period of vibration of a *compound* pendulum shown in Fig. 2.3.4.

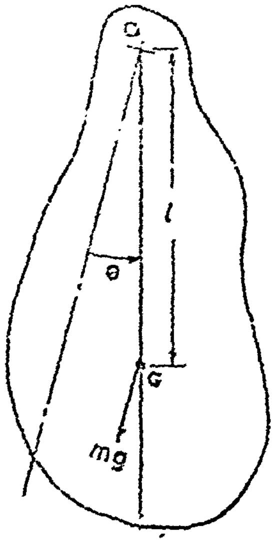


Fig. 2.3.4 Compound pendulum.

Solution

Taking small amplitude of vibration and applying Newton's second law, we have

$$J_0 \ddot{\theta} = - mgl\,\theta$$

Where J_0 is the mass moment of inertia of the body about O and is equal to $(J_G + ml^2)$, J_G being the mass moment of inertia of the body about its centre of gravity G.

Therefore, $(J_G + ml^2)\ \ddot{\theta} + mgl\,\theta = 0$

which gives $\omega_n = \sqrt{\dfrac{mgl}{J_G + ml^2}}$

$$\text{and } \tau = 2\pi \sqrt{\frac{J_G + ml^2}{mgl}}$$

If $J_G = mr^2$, r being the radius of gyration of the body about its c. g., then

$$\tau = 2\pi \sqrt{\frac{r^2 + l^2}{gl}} \qquad \textbf{Ans.}$$

Illustrative Example 2.3.3

Determine the differential equation of motion for the system shown in Figure 2.3.5 (a), where the mass moment of inertia of the weight W and the bell crank lever about O is J_0. What is the time period of vibration of this system in the vertical plane ?

Is there any limitation on the value of b ?

Solution

Assume that in equilibrium position, weight W is vertically above A. Consider the displaced position of the system at any instant as shown in Fig. 2.3.5 (b). If $\triangle_{st}$ is the static extension of the spring in equilibrium position, its total extension in the displaced position is $(\triangle_{st} + a\theta)$.

From the Newton's second law, we have

$$J_0 \ddot{\theta} = W(l + b\,\theta) - k\,(\triangle_{st} + a\theta).a \tag{2.3.13}$$

But in the equilibrium position

$$Wl = k.\triangle_{st}.a$$

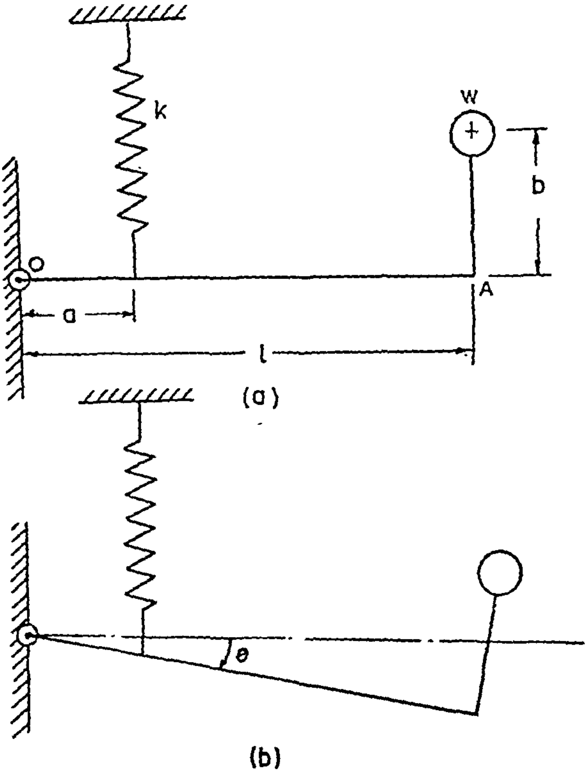


Fig. 2.3.5. Time period determination of a vibratory system.

Substituting the above equation in equation (2.3.13), we have

$$J_0\,\ddot{\theta} = (Wb - ka^2)\,\theta$$

or
$$J_0\,\ddot{\theta} + (ka^2 - Wb)\theta = 0$$

which gives

$$\omega_n = \sqrt{\frac{ka^2 - Wb}{J_0}}$$

or
$$\tau = 2\pi\sqrt{\frac{J_0}{ka^2 - Wb}}$$

The time period becomes an imaginary quantity if $ka^2 < Wb$. This makes the system unstable. Thus for the system to vibrate, the limitation is

$$ka^2 > Wb$$

$$\text{or} \qquad b < \frac{ka^2}{W} \qquad \textbf{Ans.}$$

Illustrative Example 2.3.4.

A flywheel weighing 35 kg was allowed to swing as a pendulum about a knife edge at inner side of the rim, as shown in Fig. 2.3.6. If the measured period of oscillation was 1.22 seconds, determine the moment of inertia of the flywheel about its geometric axis.

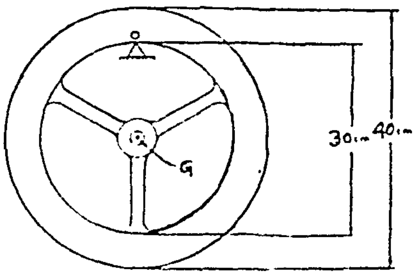


Fig. 2.3.6 Determination of moment of inertia.

Solution

If J_0 is the mass M.I. of the flywheel about the point of suspension O, then for small displacement θ, we have

$$J_0 \ddot{\theta} = -W.\frac{30}{2}.\,\theta$$

$$\text{or} \quad J_0 \ddot{\theta} + 35 \times 15\,\theta = 0$$

$$\text{or} \qquad \omega_n = \sqrt{\frac{35 \times 15}{J_0}}$$

$$\text{or} \qquad \tau = 2\pi \sqrt{\frac{J_0}{35 \times 15}} = 1.22 \text{ (given)}$$

This gives $J_0 = 19.8$ kg-cm-sec².

If J_G is the mass M. I. of the flywheel about its geometric axis, then

$$J_G = J_0 - \frac{W}{g} r^2$$

$$\text{or} \quad J_G = 19.8 - \frac{35}{980} \times \left(\frac{30}{2}\right)^2 = 11.8 \text{ kg-cm-sec}^2 \ \textbf{Ans.}$$

Illustrative Example 2.3.5

Fig. 2.3.7 (a) shows a wire AB, at the ends of which two equal rods are fixed centrally. These rods are then suspended by means of strings at points C and D such that in equilibrium position the rods are parallel and in the same horizontal plane,

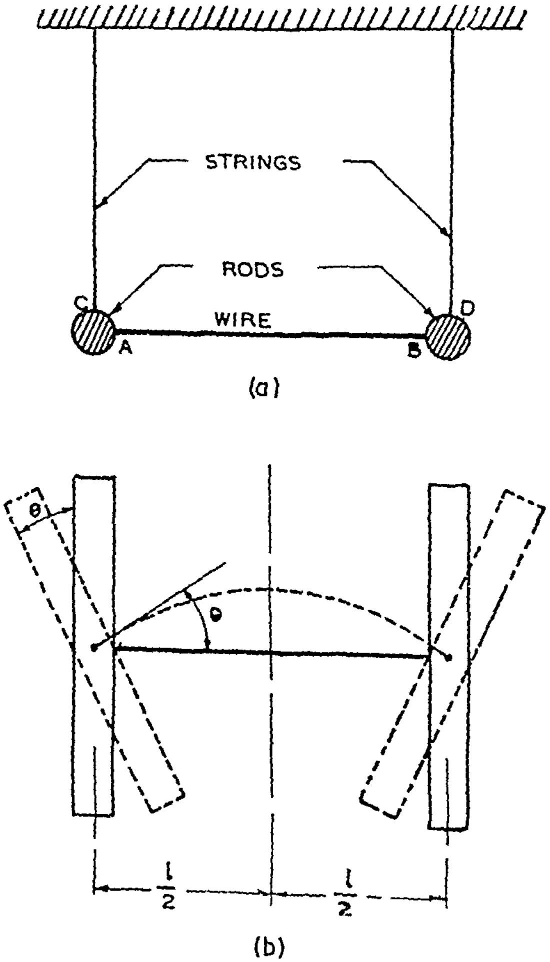


Fig. 2.3.7. Determination of modulus of elasticity from free vibration test.

and the strings are vertical. Now the rods are displaced slightly as shown in Fig. 2.3.7 (b), care being taken that they still remain in the same horizontal plane. On releasing, the rods oscillate about the string axes, and the wire bends to and fro in the horizontal plane. Find the modulus of elasticity of the material of the wire in terms of the time period of oscillation and the other constants of the system.

Solution

We know from the Strength of Materials, that

$$\frac{d^2 y}{dx^2} = \frac{I}{R} = \frac{M}{EI} \tag{2.3.14}$$

where, M, E and I are the bending moment, modulus of elasticity and the moment of inertia of the cross-section of the wire about the neutral axis.

Integrating equation (2.3.14),

$$\frac{dy}{dx} = \frac{M}{EI} . x + C_1 \tag{2.3.15}$$

Integrating again,

$$y = \frac{M}{EI} \frac{x^2}{2} + C_1 x + C_2$$

Applying the boundary conditions

$$y = 0, \text{ at } \left[\begin{array}{l} x = 0 \\ x = l \end{array} \right.$$

we have $\quad C_2 = 0$

$$C_1 = -\frac{Ml}{2EI}$$

Substituting the above constants in equation (2.3 15)

$$\frac{dy}{dx} = \frac{M}{EI} x - \frac{Ml}{2EI}$$

and $\quad \left(\frac{dy}{dx} \right)_{x=0} = \theta = -\frac{Ml}{2EI}$

which gives $M = -\frac{2EI}{l} \theta \qquad (2.3.16)$

The negative sign in the above equation signifies that the direction of the moment at the two ends of the wire is such as to make it convex upwards. Taking the left hand end the moment on the wire is, therefore, anticlockwise. This gives a clockwise reaction moment on the left rod. If θ is taken positive anticlockwise for the rod, then the moment on the rod is clockwise *i. e.* negative$\left(= -\frac{2\,EI}{l}\theta\right)$.

Now, differential equation of motion for the left rod is, therefore, given by

$$J\,\ddot{\theta} = -\frac{2EI}{l}\,\theta \tag{2.3.17}$$

where J is the mass moment of inertia of each rod about the string axis about which it oscillates. Equation (2.3.17) is, in fact, true for either rod.

Therefore, $$J\,\ddot{\theta} + \frac{2\,EI}{l}\,\theta = 0$$

Hence, $$\omega_n = \sqrt{\frac{2EI}{J\,l}}$$

or $$\tau = 2\pi\sqrt{\frac{Jl}{2EI}}$$

giving $$E = \frac{2\pi^2 Jl}{\tau^2 I}$$ **Ans.**

2.4 Torsional vibrations.

Imagine a system consisting of a rotor of mass moment of inertia J connected to a shaft of torsional stiffness k_t, as shown in Fig. 2.4.1. When the rotor is displaced slightly in the angular manner about the axis of the shaft, and released, it executes torsional oscillations. Its natural frequency may be obtained as derived below.

When, at any instant, the rotor occupies a position θ with reference to the equilibrium position the torque acting on the rotor through the twisted shaft is $-k_t.\theta$. The negative sign is included because the torque on the rotor acts in a direction opposite to its twist. From Newton's second law of motion, therefore

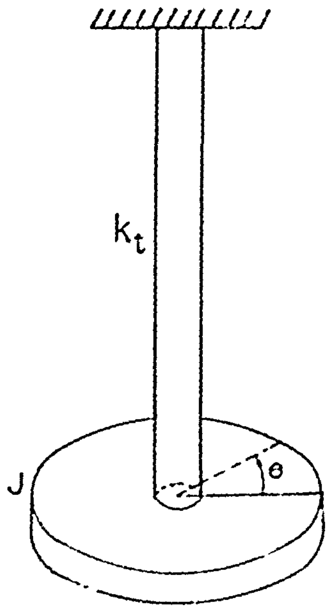


Fig. 2.4.1. Torsional system.

$$J\ddot{\theta} = -k_t.\theta$$

or $$J\ddot{\theta} + k_t.\theta = 0$$

or $$\ddot{\theta} + (k_t/J)\,\theta = 0 \tag{2.4.1}$$

Putting $$\omega_n^2 = k_t/J \tag{2.4.2}$$

the equation (2.4.1) becomes

$$\ddot{\theta} + \omega_n^2\theta = 0 \tag{2.4.3}$$

This equation is of the same form as equation (2.3.2), and therefore the solution is of harmonic type as given in equations (2.3.3). The natural frequency of vibration of this system is obtained from equation (2.4 2), i. e.

$$\omega_n = \sqrt{\frac{k_t}{J}} \tag{2.4.4}$$

Illustrative Example 2.4.1

Calculate the natural frequency of vibration of a torsional pendulum of Fig. 2.4.1 with the following dimensions.

Length of the rod,	l =	1m
Diameter of the rod,	d =	5 mm
Diameter of the rotor,	D =	20 cm
Weight of the rotor,	W =	2 kg

The modulus of rigidity for the material of the rod may be assumed to be 0.85×10^6 kg/cm².

Solution

$$J = \frac{W}{g} \cdot \frac{[D/2]^2}{2} = \frac{2}{980} \times \frac{10^2}{2} = 0.102 \text{ kg-cm-sec}^2.$$

From Strength of Materials, we know that

$$\frac{T}{I_p} = \frac{G\,\theta}{l}$$

or $$k_t = \frac{T}{\theta} = \frac{G.I_p}{l}$$

or $$k_t = \frac{(0.85 \times 10^6) \times [\,(\pi/32) \times (0.5)^4\,]}{100}$$

$$= 52.1 \text{ kg-cm/rad.}$$

Applying equation (2.4.4),

$$\omega_n = \sqrt{\frac{k_t}{J}} = \sqrt{\frac{52.1}{0.102}} = 22.6 \text{ rad/sec}$$

or $$f_n = \frac{22.6}{2\pi} = 3.6 \text{ cycles/sec.}$$ **Ans.**

2.5 Equivalent stiffness of spring combinations.

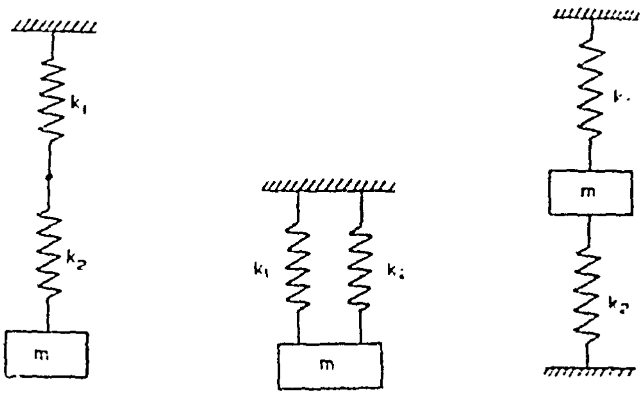
In certain systems more than one spring may be used. To convert these systems into equivalent mathematical models it is necessary to find out the equivalent stiffness of the spring combinations. The equivalent stiffness depends upon whether the springs are in series or parallel.

2.5A Springs in series. Fig. 2.5.1(a) shows a system having two springs k_1 and k_2 in series. If these springs are replaced by an equivalent spring of stiffness k, then the total static deflection of the body in two cases under the same load must be the same. The total deflection in the actual case is the sum of deflections

in the individual springs, each spring has a force mg acting on it.

Therefore,
$$\Delta_{st} = \frac{mg}{k} = \frac{mg}{k_1} + \frac{mg}{k_2}$$

or
$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2} \tag{2.5.1}$$



(a) Springs in series. (b) (c) Springs in parallel.

Fig. 2.5.1. Spring combinations.

That is, in the case of series springs, the reciprocal of the equivalent spring stiffness is equal to the sum of the reciprocals of individual spring stiffnesses.

2.5 B Springs in parallel. The springs are said to be in parallel when the absolute deflection in each of the individual springs is equal to the deflection of the system. Fig. 2.5.1 (b) and (c) are the two cases with springs in parallel. Considering these to be single degree of freedom systems, each of the springs is carrying a part of the total load supported, such that the total load supported is equal to the sum of the loads carried by individual springs. Let the spring k_1 be carrying a load m_1g and the spring k_2 a load m_2g.

Therefore, $m_1g + m_2g = mg.$

If k is the stiffness of the equivalent spring, then the total deflection of the body in both cases must be the same,

i. e. $$\Delta_{st} = \frac{mg}{k} = \frac{m_1 g}{k_1} = \frac{m_2 g}{k_2}$$

$$= \frac{m_1 g + m_2 g}{k_1 + k_2} = \frac{mg}{k_1 + k_2}$$

Therefore, $$k = k_1 + k_2 \tag{2.5.2}$$

That is, in the case of parallel springs, the equivalent stiffness is equal to the sum of the individual spring stiffnesses.

Illustrative Example 2.5.1

For the system shown in Fig. 2.5.2,

$k_1 = 2.0$ kg/cm,

$k_2 = 1.5$ kg/cm,

$k_3 = 3.0$ kg/cm, and

$k_4 = k_5 = 0.5$ kg/cm.

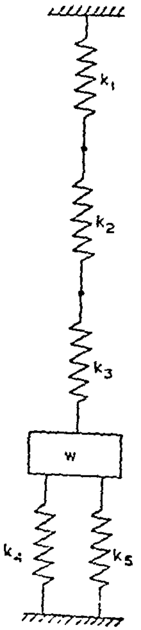


Fig. 2.5.2 Combined series and parallel springs.

Find W such that the system has a natural frequency of 10 cps.

Solution

If k_{e1} is the effective spring stiffness of the top three springs in series, then

$$\frac{1}{k_{e1}} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}$$

$$= \frac{1}{2.0} + \frac{1}{1.5} + \frac{1}{3.0} = 1.5$$

or $k_{e1} = 0.667$ kg/cm.

If k_{e2} is the effective spring stiffness of the lower two springs in prallel, then

$$k_{e2} = k_4 + k_5 = 0.5 + 0.5 = 1.0$$

or $k_{e2} = 1.0$ kg/cm

Now k_{e1} and k_{e2} are two springs in parallel, therefore effective stiffness

$$k_e = k_{e1} + k_{e2} = 0.667 + 1.0 = 1.667 \text{ kg/cm}$$

$$f_n = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k_e g}{W}} = 10 \text{ cps (given)}.$$

Therefore $W = \dfrac{k_e g}{4\pi^2 \times 10^2} = \dfrac{1.667 \times 980}{4\pi^2 \times 100}$ **Ans.**

or $W = 0.414$ kg.

Illustrative Example 2.5.2

Find the natural frequency of torsional oscillations for the system shown in Fig 2.5.3. Take $G = 0.85 \times 10^6$ kg/cm^2. Neglect the inertia effect of the shaft.

Solution

We know, from Strength of Materials. that $\dfrac{T}{I_p} = \dfrac{G\theta}{l}$, giving the torsional stiffness

$$k_t = \frac{T}{\theta} = \frac{GI_p}{l} = \frac{G}{l} \cdot \frac{\pi}{32} d^4$$

If k_{t1}, k_{t2} and k_{t3} are the torsional stiffnesses of the three different sections of the shaft, then

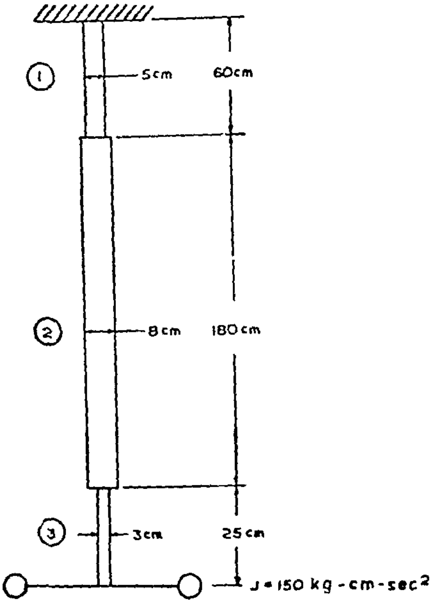


Fig. 2.5.3.
Multi-shaft torsional system.

$$k_{t1} = \frac{G}{l_1} \cdot \frac{\pi}{32} \cdot d_1^4$$

$$k_{t2} = \frac{G}{l_2} \cdot \frac{\pi}{32} \cdot d_2^4$$

$$k_{t3} = \frac{G}{l_3} \cdot \frac{\pi}{32} \cdot d_3^4$$

Since the three sections of the shaft are in series, the effective stiffness k_t is given by

$$\frac{1}{k_t} = \frac{1}{k_{t1}} + \frac{1}{k_{t2}} + \frac{1}{k_{t3}}$$

$$= \frac{32}{\pi G}\left[\frac{l_1}{d_1^4} + \frac{l_2}{d_2^4} + \frac{l_3}{d_3^4}\right]$$

$$= \frac{32}{\pi \times 0.85 \times 10^6}\left[\frac{60}{5^4} + \frac{180}{8^4} + \frac{25}{3^4}\right]$$

giving $\quad k_t = 1.88 \times 10^5$

energy) due to its elevation from a reference level and the potential energy (strain energy) of the spring. Let us denote the above three quantities by T, G and S.

The kinetic energy of the body with rectilinear motion is given by

$$T = \tfrac{1}{2}\, m\dot{x}^2 \tag{2.6.1}$$

If the reference level is chosen as the equilibrium position, then the potential energy due to the elevation of the weight from the reference level is given by

$$G = -mgx \tag{2.6.2}$$

The negative sign is included because there is a loss of energy as the level is lowered by a distance x.

The strain energy of the spring is equal to the work done in deforming the spring through a distance x. In the equilibrium position the spring force is $k\Delta_{st}$. In the displaced position the spring force is $k(\Delta_{st}+x)$. The average spring [illegible] during the deformation is $k(\Delta_{st}+x/2)$. The work [illegible] ing the spring is the strain energy of the spring a [illegible] average spring force times the deformation.

Since no energy is being dissipated and no external force is acting on the body, the total energy of the system at any instant is constant. If E denotes the total energy of the system, then

$$E = T + U = \text{constant}$$

or $\frac{1}{2} m\dot{x}^2 + \frac{1}{2} kx^2 = \text{constant}$ (2.6.5)

Differentiating the above equation with respect to time t, we have

$$m\dot{x}\ddot{x} + kx\dot{x} = 0,$$

and after cancelling out the common factor $\dot{x}$, the following equation results,

$$m\ddot{x} + kx = 0 \tag{2.6.6}$$

This equation is the same as equation (2.2.2).

For finding out the natural frequency of a vibratory system by energy method, it is sometimes convenient to equate the maximum kinetic energy of the system (which occurs when the system passes through the mean position where potential energy is zero) to the maximum potential energy of the system (which occurs in the extreme position where kinetic energy is zero), i.e.

$$\frac{1}{2}m\,(V_{max})^2 = \frac{1}{2}\,k\,X^2 \tag{2.6.7}$$

where X is the amplitude of vibration. Since $V_{max} = \omega_n X$ (Sec 1.3, ω_n being used because the free vibrations are excecuted at the natural frequency), equation (2.6.7) reduces to

$$\frac{1}{2}\,m\,(\omega_n X)^2 = \frac{1}{2}\,k\,X^2$$

or $$\omega_n = \sqrt{\frac{k}{m}} \tag{2 6.8}$$

the same as equation (2.3.1).

Illustrative Example 2.6.1

A cylinder of weight W and radius r rolls without slipping on a cylindrical surface of radius R. Find the natural frequency for small oscillations about the lowest point.

Solution

Refer to Fig. 2.6.2. When the rolling cylinder is in the lowest position, the point P' coincides with P. Consequently,

energy) due to its elevation from a reference level and the potential energy (strain energy) of the spring. Let us denote the above three quantities by T, G and S.

The kinetic energy of the body with rectilinear motion is given by

$$T = \tfrac{1}{2} m\dot{x}^2 \tag{2.6.1}$$

If the reference level is chosen as the equilibrium position, then the potential energy due to the elevation of the weight from the reference level is given by

$$G = -mgx \tag{2.6.2}$$

The negative sign is included because there is a loss of energy as the level is lowered by a distance x.

The strain energy of the spring is equal to the work done in deforming the spring through a distance x. In the equilibrium position the spring force is $k\Delta_{st}$. In the displaced position the spring force is $k(\Delta_{st}+x)$. The average spring force during the deformation is $k(\Delta_{st}+x/2)$. The work done in deforming the spring is the strain energy of the spring and is equal to the average spring force times the deformation.

Therefore $$\begin{aligned} S &= k(\Delta_{st}+x/2)\,x \\ &= k\,\Delta_{st}.\,x + \tfrac{1}{2}k\,x^2 \end{aligned}$$

But $$k.\Delta_{st} = mg$$

Therefore, $$S = mgx + \tfrac{1}{2}kx^2 \tag{2.6.3}$$

If the total potential energy due to gravitational effect and the spring straining is denoted by U, then

$$\begin{aligned} U &= G + S \\ &= -mgx + mgx + \tfrac{1}{2}kx^2 \end{aligned}$$

or $$U = \tfrac{1}{2}kx^2 \tag{2.6.4}$$

Equation (2.6.4) shows that the total potential energy may be obtained by imagining that the spring is deformed through a distance x from its unstretched length, i. e. when the initial force in the spring is zero and the final force kx. The average force during this deformation is $\tfrac{1}{2}kx$ and the work done being equal to this average force times the deformation, is equal to $\tfrac{1}{2}kx^2$. This takes into account the work done by deforming the spring against the initial force and the gravitational work, both of which neutralize each other.

Since no energy is being dissipated and no external force is acting on the body, the total energy of the system at any instant is constant. If E denotes the total energy of the system, then

$$E = T + U = \text{constant}$$

or $\frac{1}{2} m\dot{x}^2 + \frac{1}{2} kx^2 = \text{constant}$ (2.6.5)

Differentiating the above equation with respect to time t, we have

$$m\dot{x}\,\ddot{x} + kx\,\dot{x} = 0,$$

and after cancelling out the common factor $\dot{x}$, the following equation results,

$$m\ddot{x} + kx = 0 \tag{2.6.6}$$

This equation is the same as equation (2.2.2).

For finding out the natural frequency of a vibratory system by energy method, it is sometimes convenient to equate the maximum kinetic energy of the system (which occurs when the system passes through the mean position where potential energy is zero) to the maximum potential energy of the system (which occurs in the extreme position where kinetic energy is zero), i.e.

$$\tfrac{1}{2}m\,(V_{max})^2 = \tfrac{1}{2}\,k\,X^2 \tag{2.6.7}$$

where X is the amplitude of vibration. Since $V_{max} = \omega_n X$ (Sec 1.3, ω_n being used because the free vibrations are excecuted at the natural frequency), equation (2.6.7) reduces to

$$\tfrac{1}{2}\,m\,(\omega_n X)^2 = \tfrac{1}{2}\,k\,X^2$$

or $$\omega_n = \sqrt{\frac{k}{m}} \tag{2.6.8}$$

the same as equation (2.3.1).

Illustrative Example 2.6.1

A cylinder of weight [illegible] a cylindrical surface of [illegible] for small oscillations [illegible]

Solution

Refer to Fig. 2.6.2. [illegible] lowest position, the [illegible]

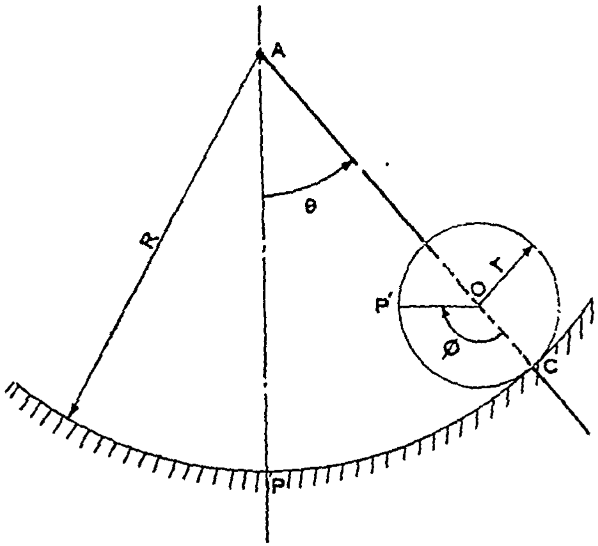


Fig. 2.6.2. A cylinder oscillating on another cylindrical surface.

Arc CP' = Arc CP

or $\qquad r\phi = R\theta \qquad$ (2.6.9)

Translational velocity of the centre of the cylinder

$$= (R-r)\,\dot{\theta}$$

Rotational velocity of the cylinder $= (\dot{\phi}-\dot{\theta})$

$$\text{K.E.} = (\text{K.E.})_{Tr} + (\text{K.E.})_{Rot}$$

$$= \frac{1}{2}\,\frac{W}{g}\left[(R-r)\,\dot{\theta}\right]^2 + \frac{1}{2}J_0\,(\dot{\phi}-\dot{\theta})^2$$

where $\quad J_0$ = mass M.I. of the cylinder about its axis.

$$\text{P. E.} = W\,(R-r)\,(1-\cos\theta)$$

Now, Total Energy = K. E. + P. E. = constant

or $$\frac{1}{2}\,\frac{W}{g}\left[(R-r)\,\dot{\theta}\right]^2 + \frac{1}{2}J_0\,(\dot{\phi}-\dot{\theta})^2 + W(R-r)\,(1-\cos\theta) = \text{const.} \qquad (2.6.10)$$

Replacing $\dot{\phi}$ by $\frac{R}{r}\dot{\theta}$ [as obtained from equation (2 6 9)], and

J_G by $\frac{W}{g} \cdot \frac{r^2}{2}$ in equation (2.6.10), we have

$$\frac{1}{2}\,\frac{W}{g}(R-r)^2\,\dot{\theta}^2+\frac{1}{2}\;\frac{W}{g}\,\frac{(R-r)^2\dot{\theta}^2}{2}+W\,(R-r)\,(1-\cos\theta)=\text{const.}$$

or $$\frac{3}{4}\;\frac{(R-r)\dot{\theta}^2}{g}+(1-\cos\theta)=\text{const.}$$

Differentiating the above equation,

$$\frac{3}{4}\;\frac{(R-r)}{g}2\dot{\theta}\,\ddot{\theta}+(\sin\theta)\,\dot{\theta}=0$$

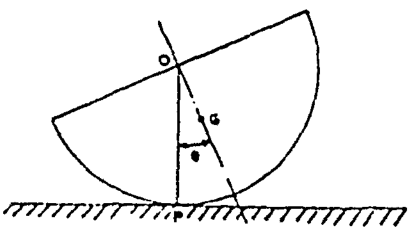
Cancelling out $\dot{\theta}$ throughout and replacing $\sin\theta$ by θ for small values of θ, we have

$$\frac{3}{2}\;\frac{(R-r)}{g}\,\ddot{\theta}+\theta=0$$

giving $$\omega_n=\sqrt{\frac{2g}{3\,(R-r)}}$$ **Ans.**

Illustrative Example 2.6.2

Find the natural frequency of vibration of the half solid cylinder shown in Fig. 2.6.3, when slightly displaced from the equilibrium position and released.



Weight $= W$
Radius $= r$

Fig. 2.6.3. A half cylinder oscillating on a flat surface.

Solution

$$OG=\bar{x}=\frac{4r}{3\pi}$$

$$J_G=J_0-\frac{W}{g}\,\bar{x}^2=\frac{W}{g}\;\frac{r^2}{2}-\frac{W}{g}\left(\frac{4r}{3\pi}\right)^2$$

$$=\frac{W}{g}\,r^2\left[\frac{1}{2}-\frac{16}{9\pi^2}\right]$$

In the displaced position at any instant, the energies of the system are as follow.

$$\text{P. E.} = W\bar{x}\ (1-\cos\theta)$$

$$\text{K. E.} = \frac{1}{2} J_p.\ \Omega^2,\ \text{where}\ \Omega = \frac{d\theta}{dt}$$

Therefore, $W.\dfrac{4r}{3\pi}\ (1-\cos\theta)+\frac{1}{2} J_p\dot{\phi}^2 = \text{const.}$

Differentiating with respect to t,

$$W.\frac{4r}{3\pi}\ \sin\theta.\dot{\theta} + J_p\,\dot{\theta}\ \ddot{\theta} = 0$$

or $\quad J_p\ddot{\theta} + W.\dfrac{4r}{3\pi}\,\theta \quad = 0$

which gives $\qquad \omega_n = \sqrt{\dfrac{W.\,4r}{3\pi.\,J_p}} \qquad$ (2.6.11)

To find J_p, assume $GP = OP - OG$, which is true for small amplitudes of vibration.

Now $\qquad J_p = J_G + \dfrac{W}{g}GP^2 = J_G + \dfrac{W}{g}(OP-OG)^2$

$$= \frac{Wr^2}{g}\left[\frac{1}{2}-\frac{16}{9\pi^2}\right]+\frac{W}{g}\left[r-\frac{4r}{3\pi}\right]^2$$

which gives $\qquad J_p = \dfrac{W}{g}r^2\left[\dfrac{3}{2}-\dfrac{8}{3\pi}\right]$

Substituting in equation (2.6.11), gives

$$\omega_n = \sqrt{\frac{8g}{r(9\pi-16)}}$$ **Ans.**

Illustrative Example 2.6.3

A U-tube, open to atmosphere at both ends contains a column length l of a certain liquid. Find the natural period of oscillation of the liquid column.

Solution

Let, at any instant, the liquid column be displaced from the equilibrium position through a distance x as shown in Fig. 2.6.4. Then, if ρ and A are the weight density of the liquid and the cross-section area of the tube respectively, then the mass of the liquid column is $\rho Al/g$ and each particle in the liquid column has a velocity $\dot{x}$ at that instant.

Therefore, K. E. $= \frac{1}{2}\left(\frac{\rho A l}{g}\right)\dot{x}^2$

For writing down the expression for P.E. of the system at this instant, it may be considered that a length x of the liquid column has been physically taken away from the top of the left side of the U-tube, and put down on top of the right side of the U-tube, i.e. a weight of $\rho A x$ is raised through a distance x.

Therefore, P.E. $= (\rho A x).x$

The total energy of system is

$$\frac{1}{2}\left(\frac{\rho A l}{g}\right)\dot{x}^2 + \rho A x^2 = \text{constant}$$

Differentiating with respect to t, and cancelling out the common terms, we have

$$\frac{1}{2}\frac{l}{g}\ddot{x} + x = 0$$

giving $\qquad \omega_n = \sqrt{\frac{2g}{l}}$

and $\qquad \tau = 2\pi\sqrt{\frac{l}{2g}}$ **Ans.**

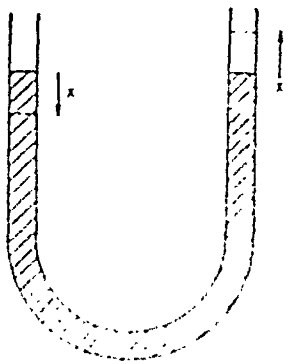


Fig. 2.6.4. Vibrating liquid column in a U-tube.

Illustrative Example 2.6.4

Show that for finding the natural frequency of a spring-mass system, the mass of the spring can be taken into account by adding one-third its mass to the main mass.

Solution

Let l be the length of the spring under equilibrium condition Consider an element dy of the spring at a distance y from the support (see Fig. 2.6.5). It ρ is the mass per unit length of the spring in equilibrium condition, then the mass of the spring $m_s = \rho\, l$ and the mass of the element dy is equal to $\rho\, dy$

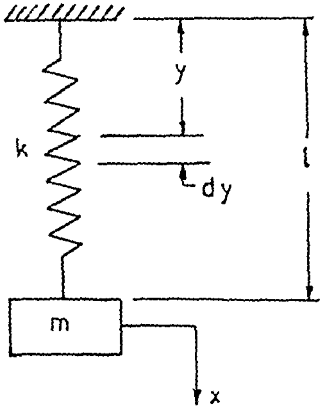


Fig. 2.6.5. Effect of mass of the spring on natural frequency.

At any instant, let the mass be displaced from the equilibrium position through a distance x. Then the potential energy of the system is

$$\text{P.E.} = \tfrac{1}{2} kx^2 \qquad (2.6.12)$$

The kinetic energy of vibration of the system at this instant consists of K.E. of the main mass plus the K.E. of the spring. The K.E. of the main mass is equal to $\frac{1}{2} m\dot{x}^2$. The kinetic energy of the element dy of the spring is $\frac{1}{2}(\rho\, dy)\left(\frac{y}{l}\dot{x}\right)^2$.

Therefore the total kinetic energy of the system is given by

$$\text{K.E.} = \tfrac{1}{2} m\dot{x}^2 + \int_0^l \tfrac{1}{2}(\rho\, dy)\left(\frac{y}{l}\dot{x}\right)^2$$

$$= \frac{1}{2} m\dot{x}^2 + \frac{1}{2}\rho\frac{\dot{x}^2}{l^2}\frac{l^3}{3}$$

$$= \frac{1}{2} m \dot{x}^2 + \frac{1}{2} \frac{m_s}{3} \dot{x}^2$$

where $m_s = \rho l$ is the mass of the spring.

$$\text{Or} \quad \text{K. E.} = \frac{1}{2}\left[m + \frac{m_s}{3}\right]\dot{x}^2 \tag{2.6.13}$$

Adding equations (2.6.12) and (2.6.13), we have

$$\frac{1}{2} kx^2 + \frac{1}{2}\left[m + \frac{m_s}{3}\right]\dot{x}^2 = \text{constant}$$

Differentiating and cancelling out the common factor $\dot{x}$, we have

$$kx + \left[m + \frac{m_s}{3}\right]\ddot{x} = 0$$

$$\text{giving} \quad \omega_n = \sqrt{\frac{k}{m + m_s/3}} \tag{2.6.14}$$

which shows that for finding the natural frequency of the system, the mass of the spring can be taken into account by adding one-third its mass to the main mass. **Ans.**

PROBLEMS FOR PRACTICE

2.1 A weight of 10 kg when suspended from a spring, causes a static deflection of 1 cm. Find the natural frequency of the system.

2.2 A car weighing 1000 kg deflects its springs 4 cm under its load. Determine the natural frequency of the car in vertical direction.

2.3 A spring-mass system has spring stiffness of k kg/cm and the weight of the mass W kg. It has natural frequency of vibration as 12 c.p.s. An extra 2 kg weight is coupled to W and the natural frequency reduces by 2 c.p.s. Find k and W.

2.4 A steel wire ($E = 2.0 \times 10^6$ kg/cm^2) is of 2 mm dia and is 30 meters long. It is fixed at the upper end and carries a weight W kg at its lower end. Find W so that the frequency of longitudinal vibrations is 4 cycles/sec.

2.5 A load of 10 kg is supported by a steel wire 1 mm in diameter and 3 meters long. The system is made to move upwards with a uniform velocity of 10 cm/sec when the upper end is suddenly stopped. Determine the frequency and the amplitude of the resulting vibrations of the load and the maximum stress in the wire.

2.6 A light cantilever of rectangular section (5 cm deep by 2.5 cm wide) has a weight fixed at its free end. Find the ratio of the frequency of free lateral vibrations in vertical plane to that in the horizontal plane.

2.7 The connecting rod shown in Fig. P.2.7 is supported at the wirst pin end. It is displaced and allowed to oscillate. The weight of the rod is 5 kg and the *CG* is 20 cm from the pivot point O. If the frequency of oscillation is 40 cycles per minute, calculate the moment of inertia of the system about its *CG*.

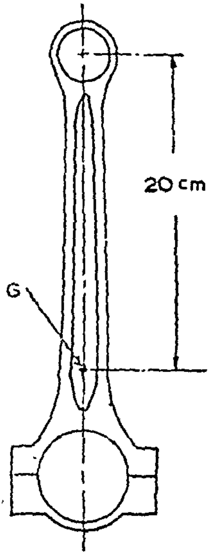


Fig. P.2.7.

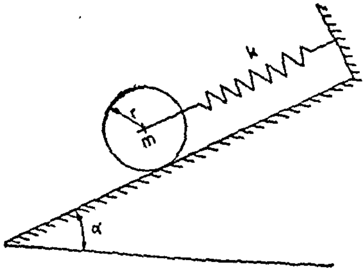


Fig. P.2.8.

2.8 A homogenous solid cylinder of mass m is linked by a spring of constant k and is resting on an inclined plane as

shown in Fig. P.2.8. If it rolls without slipping, find ou its frequency of oscillation.

2.9 An inclined shaft carries an eccentric as shown in Fig P.2.9. The shaft is inclined at an $\angle \alpha$ with the horizonta and the moment of inertia of the system about the shaf axis is J. Determine the frequency of oscillation due to a small unbalance w kg at a radius b in the disc.

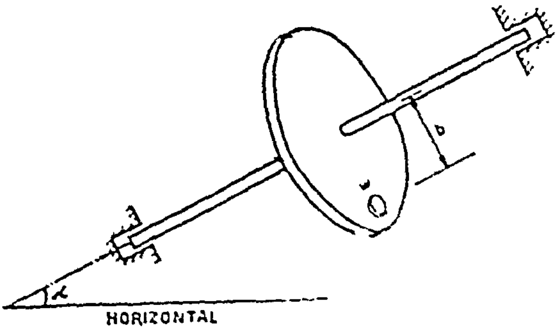


Fig. P.2.9.

2.10 Find the frequency of vibration for the system discussed in Illustrative Example 2.2 2. What is the frequency of vibration in the following special cases ?

(i) $k = 0$

(ii) $a = 0$

(iii) $l = a = \infty$

What physical models do these cases correspond to ?

2.11 Find the natural frequency of oscillation for the system shown in Fig. P.2.11. Assume the bell crank lever to be light and stiff, and the mass m to be concentrated.

2.12 (a) Find the time period of small vibrations of an inverted pendulum and spring system shown in Fig.

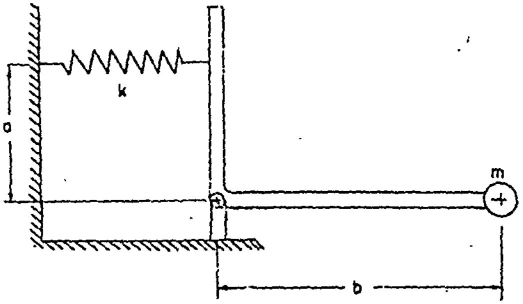
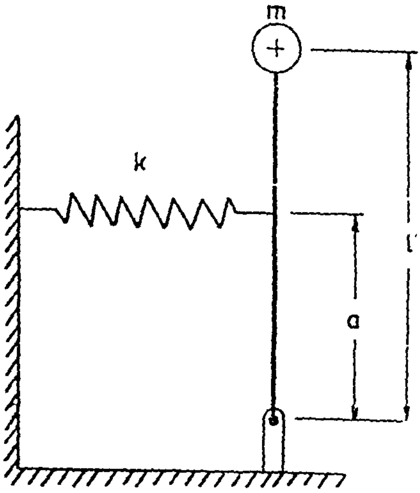


Fig. P.2.11.

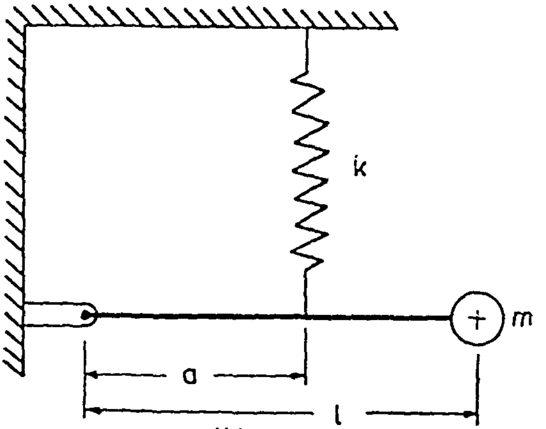
P.2.12 (a), given that the pendulum is vertical in the equilibrium position. Is there any limitation on the value of k ? Discuss.



(a)

Fig. P.2.12.

(b) Compare the time period of vibration of the above system with that of one shown in Fig. P. 2.12 (b).



(b)

Fig. P.2.12.

2.13 A uniform bar AB, 60 cm long and weighing 100 kg is supported on a hinge at one end A and on a spring support at the other end B, so that it can vibrate in a vertical plane. The stiffness of the spring is 20 kg/cm. When in static equilibrium, the bar is horizontal. The bar may be assumed to be flexurally rigid. The end B of the bar is depressed one centimeter and released. Calculate

(i) the frequency of resulting vibrations,

(ii) the maximum bending moment at the mid-point of the bar.

2.14 Determine the expression for the natural frequency of the system shown in Fig. P.2.14. Assume that the wires

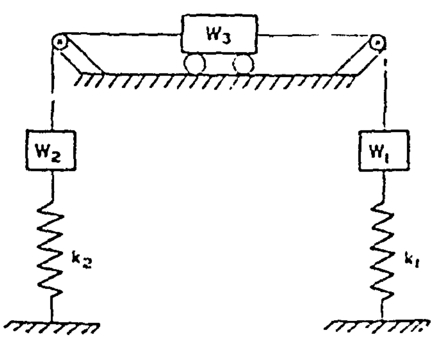


Fig. P.2.14.

connecting the weights do not stretch and are always in tension.

2.15 Determine the torsional natural frequency of the system shown in Fig. P.2.15. Neglect the mass moment of inertia of the shaft. J = 0·1 kg-cm-sec^2

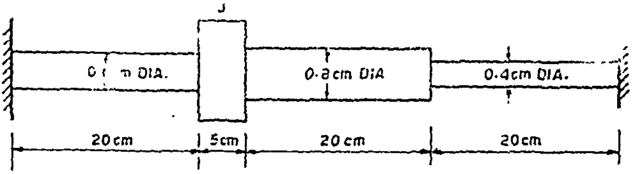


Fig. P.2.15.

2.16 A torsion pendulum has to have a natural frequency of 5 cps. What length of a steel wire of diameter 2 mm should be used for this pendulum. The inertia of the mass fixed at the free end is 0.1 kg-cm-sec^2. Take G = $0{\cdot}85 \times 10^6$ kg/cm^2.

2.17 The mass moment of inertia J_G of complicated bodies about their centres of gravity may be found by suspending a light disc by three strings of length l each, attached at points distant a from the centre of gravity as shown in Fig. P.2.17. This device is known as *Trifilar Suspension* and the body is placed in this suspension such that its C.G. is vertically above the C.G. of the suspension. Show that for small angle of oscillation θ, the value of the mass moment of inertia is given by

$$J_G = \frac{Wa^2\tau^2}{4\pi^2 l}$$

Where W=weight of the body

and τ =time period of oscillation.

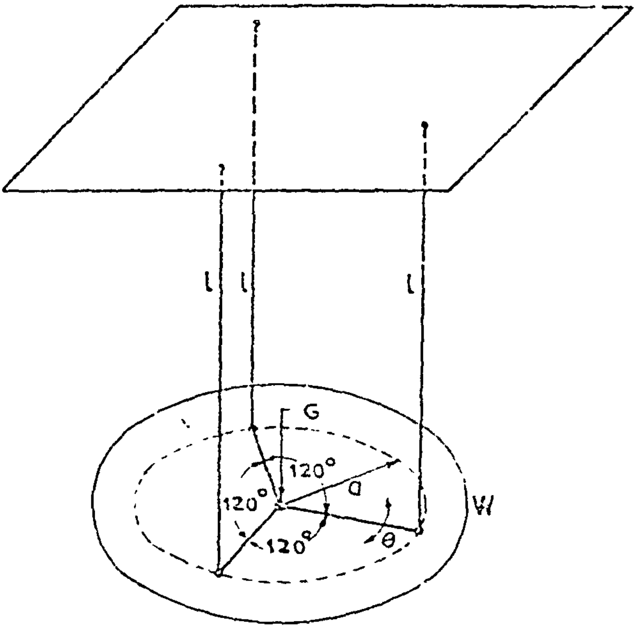


Fig. P.2.17.

2.18 A bifiliar suspension consists of a thin cylindrical rod of weight W suspended symmetrically by two equal strings as shown in Fig. P.2.18. Find the period of motion for small angular oscillations of the rod about the vertical axis y-y.

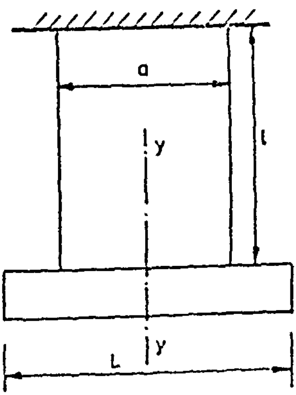


Fig. P.2.18.

2.19 A mass m is suspended from a spring system as shown in Fig. P.2.19. Determine the natural frequency of the system.

$$k_1 = 5 \text{ kg/cm}$$

$$k_2 = k_3 = 8 \text{ kg/cm}$$

Weight of the mass = 25kg.

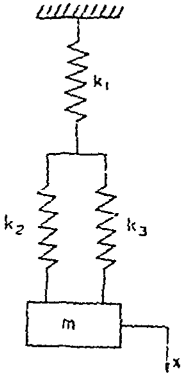


Fig. P.2.19.

2.20 Determine the equivalent length of shafting 2 cm diameter which will have the same torsional sliffness as the stepped shaft shown in Fig. P. 2.20.

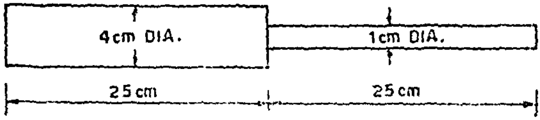


Fig. P.2.20.

2.21 Find an expression for the natural frequency of the weight W for the system shown in Fig.P.2.21. Neglect the weight of the cantilever beam. Study the special cases when

(i) $k = \infty$

(ii) $I = \infty$

What physical models do these cases correspond to ?

2.22 Determine the natural frequency of the spring-mass-pulley system shown in Fig. P. 2.22.

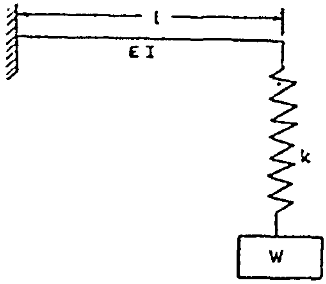


Fig. P.2.21.

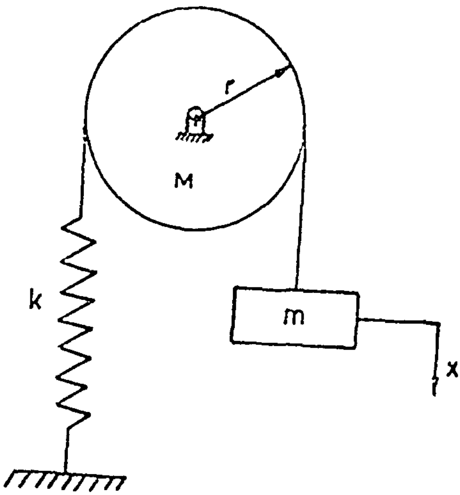


Fig. P.2.22.

2.23 A spherical body of radius r rolls without slipping on a concave spherical surface of radius R. Find the frequency of small vibration of the sphere about the equilibrium position.

2.24 A cylinder of diameter D and weight W floats vertically in a liquid of weight density ρ as shown in Fig. P.2.24. It is depressed slightly and released. Find the period of its oscillation.

2.25 A sphere of diameter D floats half submerged in water. If the sphere is depressed slightly and released, determine

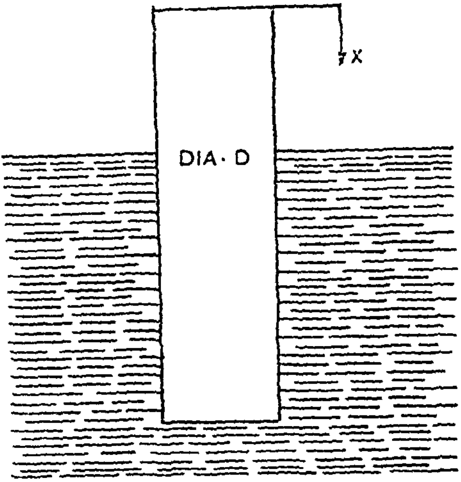


Fig. P.2.24.

the period of vibration. What is this period if $D = 1$ meter.

2.26 A pipe of length l and cross-sectional area A_0 connects two tanks of cross-sectional areas A_1 and A_2 as shown in Fig. P.2.26. Show that the period of motion for small oscillations of the fluid between the two tanks is given by

$$\tau = 2\pi \sqrt{\frac{h\,A_0\,(A_1 + A_2) + l\,A_1\,A_2}{g\,A_0\,(A_1 + A_2)}}$$

where h is the height of the fluid in the tanks above the level of the connecting pipe.

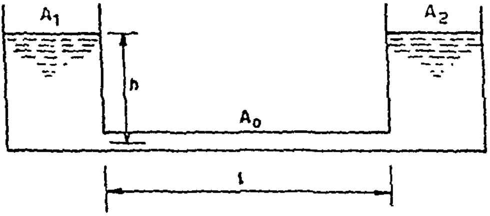


Fig. P.2.26.

2.27 Show that for finding the natural frequency of torsional oscillations of a shaft and disc system, the inertia of the

shaft can be taken into account by adding one-third the inertia of the shaft to that of the disc.

2.28 A uniform rod of weight W and length l rests on the curved surface of a fixed cylinder as shown in Fig. P.2.28. It is depressed slightly on one end and released. Find the frequency of resulting vibrations.

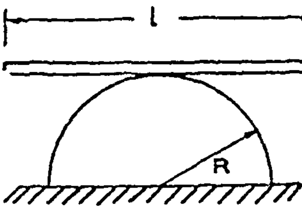


Fig. P.2.28.

2.29 A vibration pick-up consists of a spring-mass system suspended in an enclosure filled with a fluid as shown in Fig. P.2.29. Show that its natural frequency is given by

$$\omega_n = \sqrt{\frac{K}{M+(A_1/A_2)^2\, m}}$$

where m is the mass of the fluid in the annular space between the suspended mass and the enclosure.

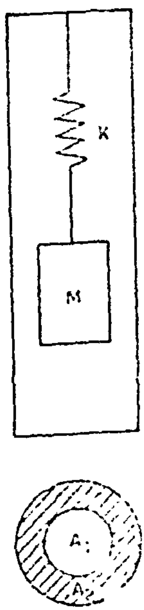


Fig. P.2.29.

CHAPTER 3

DAMPED FREE VIBRATIONS OF SINGLE DEGREE OF FREEDOM SYSTEMS

3.1 Introduction.

In general, all physical systems are associated with one or the other type of damping. In certain cases the amount of damping may be small and in other cases large. When damped free vibrations take place, the amplitude of vibration gradually becomes small and finally is completely lost. The rate at which the amplitude decays, depends upon the type and amount of damping in the system. The aspects we are primarily interested in damped free vibration are

(i) the frequency of damped oscillations, and

(ii) the rate of decay.

3.2 Different types of dampings.

The damping in a physical system may be one of the several types. In the following paragraphs only some of the important types of dampings are discussed.

Viscous damping. This is the most important type of damping and occurs for small velocities in lubricated sliding surfaces, dashpots with small clearances, etc. Eddy current damping is also of viscous nature. The amount of damping resistance will depend upon the relative velocity and upon the parameters of the damping system. For a particular system the damping resistance is always proportional to the relative velocity.

One of the reasons for so much importance of this type of damping is that it affords an easy analysis of the system by virtue of the fact that the differential equation for the system

becomes linear with this type of damping. That is the reason why the systems are often represented to include an equivalent viscous damper even though the damping may not be truly viscous.

Dry friction or Coulomb damping. This type of damping occurs when two machine parts rub against each other, dry or unlubricated. The damping resistance in this case is practically constant and is independent of the rubbing velocity.

Solid or structural damping. This type of damping is due to the internal friction of the molecules. The stress-strain diagram for a vibrating body is not a straight line but forms a hysterisis loop the area of which represents the energy dissipated due to molecular friction per cycle per unit volume. The size of the loop depends upon the material of the vibrating body, frequency and the amount of dynamic stress.

Slip or interfacial damping : Energy of vibration is dissipated by microscopic slip on the interfaces of machine parts in contact under fluctuating loads. Microscopic slip also occurs on the interfaces of the machine elements forming various types of joints. The amount of damping depends amongst other things upon the surface roughness of the mating parts, the contact pressure, and the amplitude of vibration. This type of damping is essentially of a non linear type.

3.3 Free vibrations with viscous damping.

Consider a classical spring-mass-dashpot system as shown in Fig. 3.3.1 (a). In this figure, k is the stiffness of the spring, m the mass of the body and c the damping coefficient. The damping resistance at any instant is equal to $c\dot{x}$ where $\dot{x}$ is the relative velocity between the piston and the cylinder of the dashpot. The spring and the dashpot are in parallel in this case.

Imagine, at any instant, the system to be displaced through a distance x from the equilibrium position as shown in Fig. 3.3.1 (b). The body has, at this instant, a velocity $\dot{x}$ in the downward direction and an accleration $\ddot{x}$ in the downward direction also, i.e. the direction of positive x. The external forces acting on the body at this instant are

(i) the spring force kx acting in the upward direction, and

(ii) the damping force $c\dot{x}$ acting in the upward direction. These forces are shown in Fig. 3.3.1 (c).

Therefore, from Newton's second law of motion, we have

$$m\ddot{x} = -c\dot{x} - kx$$

or $$m\ddot{x} + c\dot{x} + kx = 0 \tag{3.3.1}$$

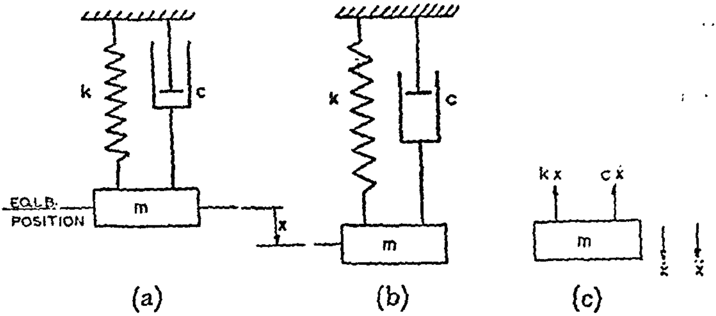


Fig. 3.3.1. Free vibrations with viscous damping.

This is the fundamental differential equation of motion for a single degree of freedom system having damped, free vibrations.

It may be noticed here that the gravitational force was not included as one of the external forces; neither the spring force due to static deflection was considered. These two forces always neutralize eachother and so, were eliminated from the very beginning. From now onwards, the forces considered in any system will be the fo-ces on the body that will have come into the picture beyond the equilibrium position.

Equation (3.3.1) is a linear differential equation of the second order and its solution can be written as

$$x = e^{st} \tag{3.3.2}$$

where, e = base of natural logarithms

$= 2.718$

t = time

and $\quad s$ = a constant to be determined.

Differentiating equation (3.3.2) twice with respect to time, we have

$$\frac{dx}{dt} = \dot{x} = se^{st}$$

$$\frac{d^2x}{dt^2} = \ddot{x} = s^2e^{st}$$

Substituting these expressions in equation (3.3.1), we get

$$ms^2e^{st} + cse^{st} + ke^{st} = 0$$

or $\quad (ms^2 + cs + k)\, e^{st} = 0$

or $\quad ms^2 + cs + k = 0 \qquad (3.3.3)$

The above equation is called the *Characteristic Equation* of the system. For a second order system (as the one under study) this equation is quadratic in s, and in this case the two values of s are given by

$$s_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \qquad (3.3.4)$$

Hence $x = e^{s_1t}$ and $x = e^{s_2t}$ may both be the solutions of equation (3.3.1), and therefore, the most general solution may be written as

$$x = C_1e^{s_1t} + C_2e^{s_2t} \qquad (3.3.5)$$

where C_1 and C_2 are the two arbitrary constants to be determined from the initial conditions. It may be pointed out here that writing the solution in the form of equation (3.3.5) is possible only where differential equations are linear, and so the principle of superposition holds good. Such superposition does not hold good in the case of non-linear differential equations.

In order to proceed further it is desirable to define a new term called the *critical damping coefficient*, denoted by c_c. It is that value of the damping coefficient c that makes the expression within the radical sign of equation (3.3.4) vanish and thereby igves two equal roots of s.

Therefore, $\left(\frac{c_c}{2m}\right)^2 = \frac{k}{m}$

or $\frac{c_c}{2m} = \sqrt{\frac{k}{m}} = \omega_n$ (3.3.6)

Another dimensionless factor ζ, called the *damping factor* or the *damping ratio* may now be defined as the ratio of the damping coefficient to the critical damping coefficient,

i. e., $\zeta = \frac{c}{c_c}$ (3.3.7)

It may be made clear at this stage that the critical damping coefficient is a constant depending upon the mass and stiffness in the system, and is independent of the actual amount of damping.

In equation (3.3.4), $\frac{c}{2m}$ can be written as

$$\frac{c}{2m} = \frac{c}{c_c} \cdot \frac{c_c}{2m}$$

$$= \zeta \cdot \omega_n \text{ [from equations (3.3.7) \& (3.3.6)]}$$

Therefore,

$$s_{1,2} = \left[-\zeta \pm \sqrt{\zeta^2 - 1}\right]\omega_n \qquad (3.3.8)$$

If $c > c_c$ or $\zeta > 1$, there is said to be over-damping in the system.

If $c = c_c$ or $\zeta = 1$, there is said to be critical damping in the system,

If $c < c_c$ or $\zeta < 1$, there is said to be under-damping in the system.

Depending upon these different amounts of damping in the system, the values of s as seen from equation (3.3.8) will be real and unequal, real and equal, and complex conjugate respectively. Discussed below are the solutions for different amount of damping in the system.

3.3A Over-damped system ($\zeta > 1$). In this system the damping is comparatively large and the two values of s as given by equation (3.3.8) are

$$s_1 = [-\zeta + \sqrt{\zeta^2 - 1}]\ \omega_n$$
$$s_2 = [-\zeta - \sqrt{\zeta^2 - 1}]\ \omega_n$$

and both are real and negative numbers. The solution of the differential equation as given by equation (3.3.5) becomes

$$x = C_1 e^{[-\zeta + \sqrt{\zeta^2 - 1}]\omega_n t} + C_2 e^{[-\zeta - \sqrt{\zeta^2 - 1}]\omega_n t} \qquad (3.3.9)$$

This is the final solution for an over-damped system and Fig. 3.3.2. shows the general behaviour of the system governed by the above equation. The system is non-vibratory.

The procedure to obtain the values of the arbitrary constants C_1 and C_2 will be explained by taking some initial conditions. The procedure is quite general for any initial conditions.

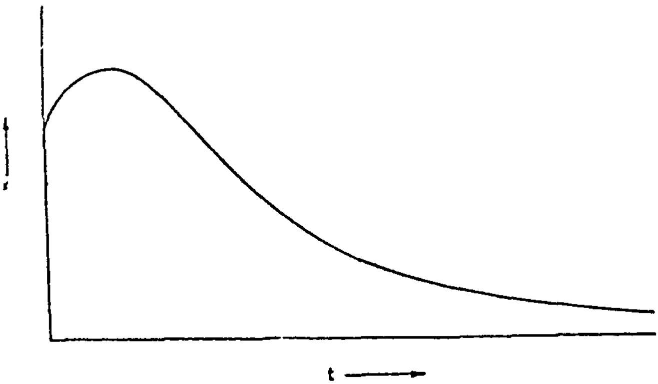


Fig. 3.3.2. Displacement-time plot of an over-damped system with general initial conditions.

Let us imagine the body to be displaced through a distance X_0 from the equilibrium position and released without any initial velocity. Then

$$\left.\begin{aligned} x &= X_0 \quad \text{at } t=0 \\ \dot{x} &= 0 \quad \text{at } t=0 \end{aligned}\right\} \qquad (3.3.10)$$

Differentiating equation (3.3.9) with respect to t, we have

$$\frac{dx}{dt} = \dot{x} = C_1 \left[-\zeta + \sqrt{\zeta^2 - 1}\right]\omega_n e^{[-\zeta + \sqrt{\zeta^2 - 1}]\omega_n t}$$

$$+ C_2 \left[-\zeta - \sqrt{\zeta^2 - 1}\right]\omega_n e^{[-\zeta - \sqrt{\zeta^2 - 1}]\omega_n t}$$

Substituting in the equations giving x and $\dot{x}$, the initial conditions of equation (3.3.10), we have

$$X_0 = C_1 + C_2$$

$$0 = C_1\,[-\zeta+\sqrt{\zeta^2-1}]\,\omega_n + C_2\,[-\zeta-\sqrt{\zeta^2-1}]\,\omega_n$$

giving $\quad C_1 = \dfrac{[\zeta+\sqrt{\zeta^2-1}]\,X_0}{2\sqrt{\zeta^2-1}}$

and $\quad C_2 = \dfrac{[-\zeta+\sqrt{\zeta^3-1}]X_0}{2\sqrt{\zeta^2-1}}$

Substituting the values of C_1 and C_2 in equation (3.3.9), we have

$$x = \frac{X_0}{2\sqrt{\zeta^2-1}}\left[\begin{array}{l}\left[\zeta+\sqrt{\zeta^2-1}\right]e^{[-\zeta+\sqrt{\zeta^2-1}]\omega_n t}\\ +\left[-\zeta+\sqrt{\zeta^2-1}\right]e^{[-\zeta-\sqrt{\zeta^2-1}]\omega_n t}\end{array}\right] \quad (3.3.11)$$

Since the power of e is negative in both the terms in th^ above equation, they both decrease exponentially with t, and so x decreases as t increases, becoming zer as $t \to \infty$. The non-dimensional displacement-time plots of the system as governed by equation (3.3.11), i.e. for zero starting velocity, for various values of ζ ($\zeta > 1$) are shown in Fig. 3.3.3. In all

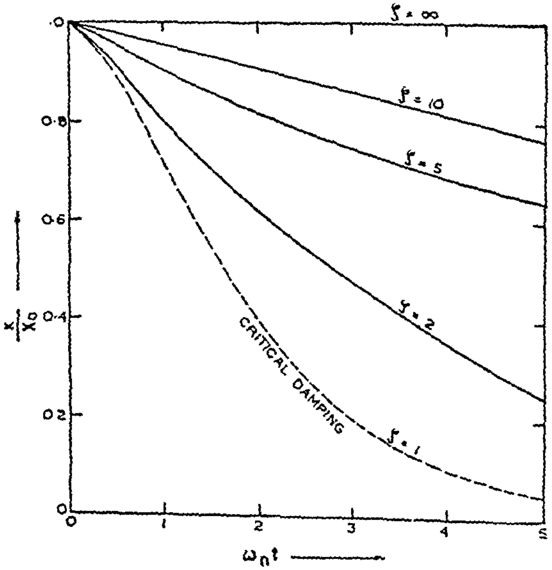


Fig. 3.3.3. Displacement-time plots of over-damped and critically damped systems with zero starting velocity.

these curves the system comes to the equilibrium position in nearly an exponential manner. Higher the damping, more sluggish is the response of the system. This may even be seen physically that higher the damping, more resistance to motion is there; and therefore slower is the movement. Theoretically, however, the system will take infinite time to come back to the equilibrium position once it is disturbed from it. This type of motion is called *aperiodic motion.*

3.3 B Critically damped system ($\zeta=1$). For the system having critical damping, the two values of s as given by equation (3.3.8) are equal to each other.

i.e. $s_1=s_2=-\zeta\omega_n=-\omega_n$.

The solution of equation (3.3.3), for $s_1=s_2$, is given by

$$x=C_1 e^{s_1t}+C_2t\, e^{s_1t}$$

Or $$x=C_1e^{-\omega_nt}+C_2te^{-\omega_nt}$$

Or $$x=[C_1+C_2t]e^{-\omega_nt} \tag{3.3.12}$$

The above equation is the solution for a system having critical damping. For finding the values of the two arbitrary constants C_1 and C_2, we take the same set of initial conditions as given by equation (3.3.10). Differentiating equation (3.3.12) with respect to t,

$$\dot{x}=C_2e^{-\omega_nt}-(C_1+C_2t)\,\omega_ne^{-\omega_nt}$$

Substituting the initial conditions of equation (3.3.10) in the equations for x and $\dot{x}$ above, we have

$$X_0=C_1$$
$$0=C_2-\omega_nC_1$$

which give

$$C_1=X_0$$

and $$C_2=\omega_nX_0$$

Substituting the values of C_1 and C_2 in equation (3.3.12),

$$x=X_0\,(1+\omega_nt)\,e^{-\omega_nt} \tag{3.3.13}$$

The value of x in the above equation can be shown to decrease as t increases, and ultimately tends to zero as t tends

to infinity. The non-dimensional *displacement-time* plot of the system as given by equation (3.3.13) is shown in Fig. 3.3.3 as a dotted curve. This is also an aperiodic motion, and in this case the displacement-time curve lies below any of the curves for over-damped system.

3.3C Under-damped system ($\zeta<1$). This is by far the most usual case that exists in physical systems and is therefore the most important. The system in this case is said to have small damping.

The two values of s as given by equation (3.3.8) can be written as

$$s_1 = [-\zeta+j\sqrt{1-\zeta^2}]\omega_n$$
$$s_2 = [-\zeta-j\sqrt{1-\zeta^2}]\omega_n$$

Here the roots are complex conjugate and the solution of the differential equation as given by equation (3.3.5) becomes

$$x = C_1e^{[-\zeta+j\sqrt{1-\zeta^2}]\omega_nt}+C_2e^{[-\zeta-j\sqrt{1-\zeta^2}]\omega_nt} \tag{3.3.14}$$

The equation in this form is not very useful and so is changed as below.

$$x = e^{-\zeta\omega_nt}\left[C_1e^{j\sqrt{1-\zeta^2}\,\omega_nt}+C_2e^{-j\sqrt{1-\zeta^2}\,\omega_nt}\right]$$

Applying the relationships

$$e^{ja} = \cos a + j\sin a$$

$$e^{-ja} = \cos a - j\sin a$$

to the equation for x, we have

$$x = e^{-\zeta\omega_nt}\begin{bmatrix} C_1\,[\cos\sqrt{1-\zeta^2}\,\omega_nt+j\sin\sqrt{1-\zeta^2}\,\omega_nt] \\ +C_2\,[\cos\sqrt{1-\zeta^2}\,\omega_nt-j\sin\sqrt{1-\zeta^2}\,\omega_nt]\end{bmatrix}$$

$$\text{or } x = e^{-\zeta\omega_nt}\,[(C_1+C_2)\cos\sqrt{1-\zeta^2}\,\omega_nt + j\,(C_1-C_2)\sin\sqrt{1-\zeta^2}\,\omega_nt]$$

The constants (C_1+C_2) and $j\,(C_1-C_2)$ in the above equation are real quantities which make C_1 and C_2 complex conjugate quantities.

The equation derived above can be written in either of the following three forms.

$$\left.\begin{aligned} x &= e^{-\zeta\omega_n t}[A\cos\sqrt{1-\zeta^2}\,\omega_n t \\ &\qquad + B\sin\sqrt{1-\zeta^2}\,\omega_n t] \\ x &= A_1 e^{-\zeta\omega_n t}\cos[\sqrt{1-\zeta^2}\,\omega_n t+\phi_1] \\ x &= A_2 e^{-\zeta\omega_n t}\sin[\sqrt{1-\zeta^2}\,\omega_n t+\phi_2] \end{aligned}\right\} \quad (3.3.15)$$

The motion governed by the above equations is of the oscillatory type with the damped natural frequency of vibration given by

$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n \quad (3.3.16)$$

and an amplitude which decreases in an exponential manner with the increase in time (see Fig. 3.3.4).

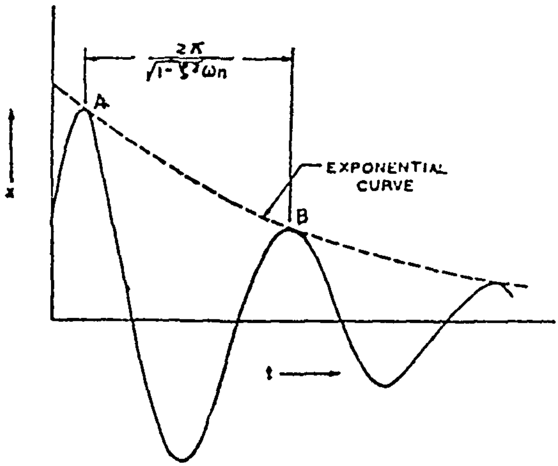


Fig. 3.3.4. Displacement-time plot of an under-damped system with general initial conditions.

Take the same set of initial conditions as given in equation (3.3.10). Differentiating one of the equation (3.3.15), say the third one, with respect to t, we have

$$\begin{aligned} \dot{x} = {} & A_2\sqrt{1-\zeta^2}\,\omega_n e^{-\zeta\omega_n t}\cos[\sqrt{1-\zeta^2}\,\omega_n t+\phi_2] \\ & - A_2\zeta\omega_n e^{-\zeta\omega_n t}\sin[\sqrt{1-\zeta^2}\,\omega_n t+\phi_2] \end{aligned}$$

Substituting the initial conditions of equation (3.3.10) in the third equation (3.3.15) and the equation above for $\dot{x}$, we have

$$X_0 = A_2\sin\phi_2$$

$$0 = A_2 \sqrt{1-\zeta^2}\, \omega_n \cos \phi_2 - A_2 \zeta\, \omega_n \sin \phi_2$$

which give

$$A_2 = \frac{X_0}{\sqrt{1-\zeta^2}}$$

and $$\phi_2 = \tan^{-1} \left[\frac{\sqrt{1-\zeta^2}}{\zeta}\right].$$

Substituting these values in the starting equation, i.e. the third equation (3.3.15), we finally have

$$x = \frac{X_0}{\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_n t} \sin\left[\sqrt{1-\zeta^2}\, \omega_n t + \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta}\right] \quad (3.3.17)$$

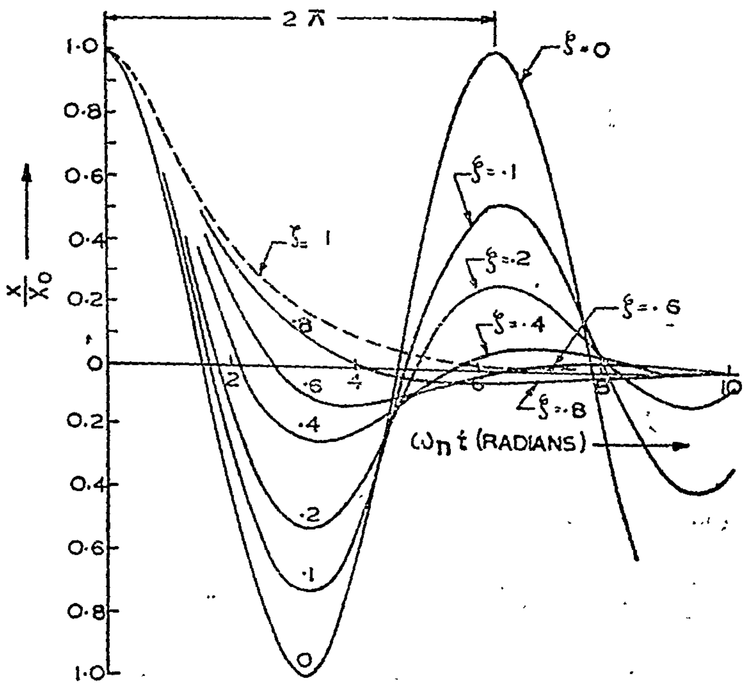


Fig. 3.3.5 Displacement-time plots of underdamped system with zero velocity.

This equation is of the oscillatory type and has an amplitude $\frac{X_0}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}$ which is seen to decay exponentially with time Theoretically, the system will never come to rest although the amplitude of vibration may become infinitely small.

The non-dimensional displacement-time plots of the system as governed by equation (3.3.17) for various values of ζ ($\zeta<1$) are shown in Fig. 3.3.5. All these curves show the decay of the vibratory motion. Higher the damping value, faster is the decay. Form this figure it is also seen that the frequency of damped free oscillations decreases with the increase in ζ. This is according to relation (3.3.16) which is plotted in Fig.

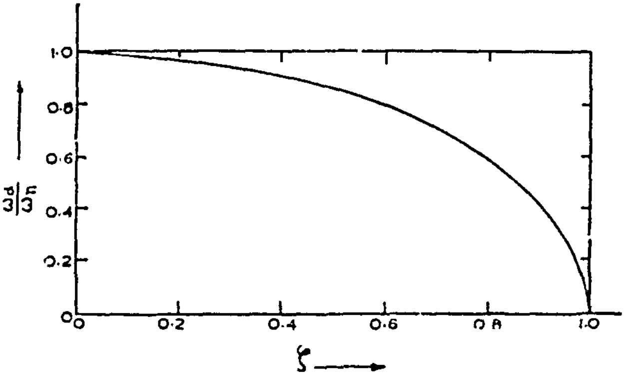


Fig. 3.3.6. Variation of damped natural frequency with damping.

3.3.6. The decrease in the damped natural frequency with the increase in ζ is small initially but is very steep as ζ increases further. This shows, therefore, that the damped natural frequency of a lightly damped system may be taken as approximately equal to the undamped natural frequency.

From the three cases discussed in the preceding paragraphs it is seen that the overdamped and critically damped systems have aperiodic motion while the underdamped system has oscillatory motion. Critical damping is the least amount of damping that a system can have for non-oscillatory or aperiodic motion.

Illustrative Example 3.3.1

The mass of a spring-mass-dashpot system is given an i velocity (from the equilibrium position) of $A\omega_n$ where ω_n undamped natural frequency of the system. Find the equ of motion for the system, for the cases, when

(i) $\zeta = 2.0$

(ii) $\zeta = 1.0$

(iii) $\zeta = 0.2$

Plot displacement-time graphs for the three cases.

Solution

Case (i) $\zeta=2.0$ (over damped).

Putting $\zeta=2.0$ in equation (3.3.9), we have

$$x=C_1 e^{-0.27\omega_n t}+C_2 e^{-3.73\omega_n t} \tag{3.3.18}$$

Differentiating,

$$\dot{x}=-0.27\,\omega_n C_1 e^{-0.27\omega_n t} -3.73\omega_n C_2 e^{-3.73\,\omega_n t}$$

Putting down the initial conditions

$$x = 0 \quad \text{at} \quad t = 0$$

$$\dot{x} = A\omega_n \quad \text{at} \quad t = 0$$

in the above two equations, we have

$$0 = C_1+C_2$$

$$A\omega_n = -0.27\omega_n C_1-3.73\omega_n C_2$$

which give

$$C_1 = 0.288A$$

and

$$C_2 = -0.288A$$

Therefore the equation (3.3.18) becomes

$$x = 0.288A\left[e^{-0.27\,\omega_n t} - e^{-3.73\,\omega_n t}\right] \tag{3.3.19}$$

Case (ii) $\zeta = 1.0$ (critically damped)

Equation (3.3.12) is re-written below.

$$x =\left[C_1 + C_2 t\right] e^{-\omega_n t} \tag{3.3.20}$$

Differentiating,

$$\dot{x} = -\left[C_1 + C_2 t\right]\omega_n e^{-\omega_n t}+C_2 e^{-\omega_n t}$$

Putting down the initial conditions as before, we have

$$0 = C_1$$
$$A\omega_n = -C_1\,\omega_n + C_2$$

which give

$$C_1 = 0$$

and $\quad C_2 = A\omega_n$

Hence equation (3.3.20) becomes

$$x = A\,\omega_n t\,e^{-\omega_n t} \tag{3.3.21}$$

Case (iii) $\quad \zeta = 0.2$ (under damped).

Substituting $\zeta = 0.2$ in the third of equations (3.3.15), we have

$$x = A_2\,e^{-0.2\,\omega_n t}\sin\,(0{\cdot}98\,\omega_n t + \phi_2) \tag{3.3.22}$$

Differentiating,

$$\dot{x} = -0.2\,\omega_n A_2\,e^{-0.2\,\omega_n t}\sin\,(0.98\,\omega_n t + \phi_2)$$
$$+0{\cdot}98\,\omega_n\,A_2\,e^{-0.2\,\omega_n t}\cos\,(0.98\,\omega_n t + \phi_2)$$

Substituting the initial conditions in the above two equations, we have

$$0 = A_2\sin\phi_2$$
$$A\omega_n = -0.2\,\omega_n\,A_2\sin\phi_2 + 0.98\,\omega_n\,A_2\cos\phi_2$$

or $\quad A_2\sin\phi_2 = 0$

$$A_2\cos\phi_2 = 1.02\,A$$

which give

$$A_2 = 1.02\,A$$
$$\phi_2 = 0$$

Hence equation (3.3.22) becomes

$$x = 1.02Ae^{-0.2\,\omega_n t}\sin\,(0.98\,\omega_n t) \tag{3.3.23}$$

Equations (3.3.19), (3.3.21) and (3.3 23) are plotted as three different curves in Fig. 3.3.7. **Ans.**

Illustrative Example 3.3.2

Between a solid weight of 10 kg and the floor are kept two slabs of isolators, natural rubber and felt, in series, as shown in Fig. 3.3.8 (a). The natural rubber slab has a stiffness of 3 kg/cm and an equivalent viscous damping coefficient of 0.1

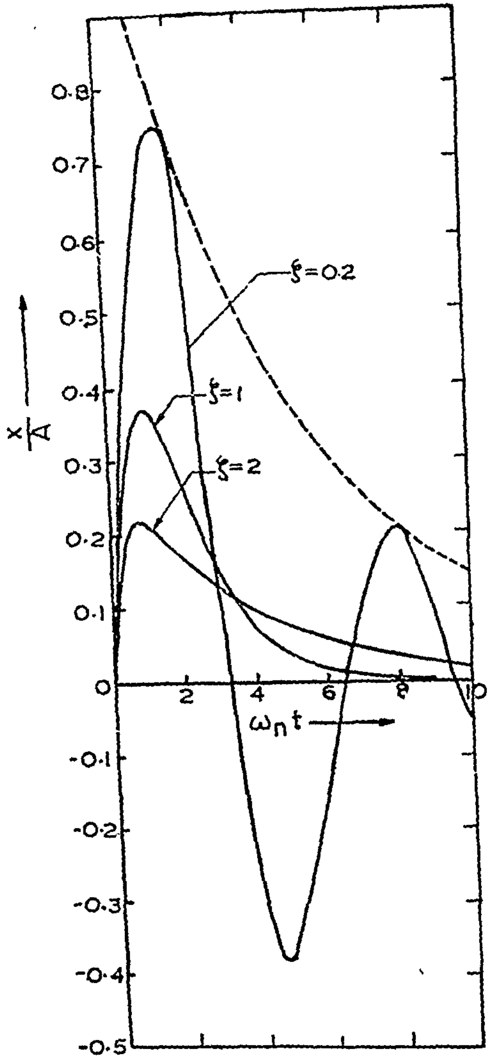


Fig. 3.3.7 Time-displacement plots for over-damped, critically damped and under-damped systems with zero initial displacement.

kg-sec/cm. The felt slab has a stiffness of 12 kg/cm and an equivalent viscous damping coefficient of 0.33 kg-sec/cm. Determine the undamped and the damped natural frequencies

of the system in vertical direction. Neglect the weights of the isolators.

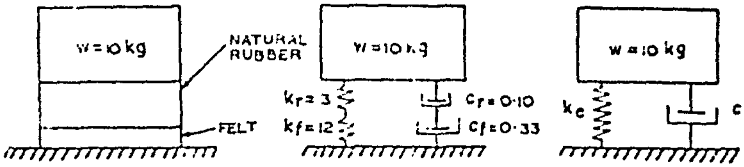


Fig. 3.3.8. A system with two isolators in series.

Solution

The isolators being in series, the system can be schematically represented by Fig. 3.3.8 (b). This can be further reduced to that shown in Fig. 3.3.8. (c), where k_e and c_e are the equivalent stiffness and the equivalent damping coefficient for the system, given by

$$\frac{1}{k_e} = \frac{1}{k_r} + \frac{1}{k_f} = \frac{1}{3} + \frac{1}{12} = \frac{5}{12}$$

or $\quad k_e = \dfrac{12}{5} = 2.4 \text{ kg/cm}$

$$\frac{1}{c_e} = \frac{1}{c_r} + \frac{1}{c_f} = \frac{1}{0.1} + \frac{1}{0.33} = 13$$

or $\quad c_e = \dfrac{1}{13} = 0.077 \text{ kg-sec/cm}$

Now, $\quad \omega_n = \sqrt{\dfrac{k_e}{m}} = \sqrt{\dfrac{2.4 \times 980}{10}} = 15.33 \text{ rad/sec}$

or $\quad f_n = \dfrac{15.33}{2\pi} = 2.44 \text{ c.p.s.}$ **Ans.**

$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n$$

$$\zeta = \frac{c_e}{2\sqrt{k_e m}} = \frac{0.077}{\sqrt{\dfrac{2.4 \times 10}{980}}} = 0{\cdot}246$$

Therefore, $\omega_d = \sqrt{1-{\cdot}246^2} \times 15{\cdot}33 = 14.85 \text{ rad/sec.}$

or $\quad f_d = \dfrac{14{\cdot}85}{2\pi} = 2{\cdot}36 \text{ c.p.s.}$ **Ans.**

Illustrative Example 3.3.3

A gun barrel weighing 600 kg has a recoil spring of stiffness

30,000 kg/ meter. If the barrel recoils 1.3 meters on firing, determine,

(a) the initial recoil velocity of the barrel,

(b) the critical damping coefficient of the dashpot which is engaged at the end of the recoil stroke,

(c) the time required for the barrel to return to a position 5 cm from the initial position.

Solution

(a) Energy stored at the end of the recoil

$$= \tfrac{1}{2} kx^2 = \tfrac{1}{2} \times 30{,}000 + 1.3^2$$
$$= 25{,}300 \text{ kg-meters.}$$

This should be equal to the initial kinetic energy of the barrel since no energy is lost in the recoil of the barrel.

Therefore, $\frac{1}{2} \dfrac{W}{g} \quad V_0^2 = 25{,}300$

or $\qquad \frac{1}{2} \cdot \dfrac{600}{9.8} V_0^2 = 25{,}300$

or $\qquad V_0^3 = 827$

giving $\qquad V_0 = 28{\cdot}75$ meters/sec. **Ans.**

(b) The critical damping coefficient is given by

$$c_c = 2\sqrt{km}$$
$$= 2\sqrt{\frac{30{,}000 \times 600}{9.8}}$$

or $\qquad c_c = 2710$ kg-sec/meter **Ans.**

(c) $\omega_n = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{30{,}000 \times 9.8}{600}} = 22{\cdot}1$ rad/sec

$$\tau = \frac{2\pi}{\omega_n} = \frac{2\pi}{22{\cdot}1} = 0.284 \text{ sec.}$$

Time for the recoil or outward stroke

$$= \frac{1}{4} \times \text{time period}$$
$$= 0{\cdot}284 = 0{\cdot}071 \text{ sec.}$$

During the return stroke the system has critical damping, whose equation (3.3.12) is re-written below.

$$x = (C_1 + C_2\, t)\, e^{-\omega_n t}$$

Differentiating,

$$\dot{x} = C_2\, e^{-\omega_n t} - (C_1 + C_2 t)\, \omega_n e^{-\omega_n t}$$

Counting time from the end of the recoil stroke or the beginning of the return travel, we have the initial conditions as

$$\left.\begin{aligned} x &= 1.3 \\ \dot{x} &= 0 \end{aligned}\right\} \text{ at } \quad t = 0$$

Substituting these initial conditions in the equations for x and $\dot{x}$ above, we have

$$1.3 = C_1$$
$$0 = C_2 - C_1\, \omega_n$$

giving

$$C_1 = 1.3$$
$$C_2 = 28.8$$

Therefore the equation of motion becomes

$$x = (1.3 + 28.8\, t)\, e^{-22.1\, t} \tag{3.3.24}$$

Now it is required to find t when $x = 0.05$. This can be done by trial and error as follows.

t	$28.8\, t$	$1.3 + 28.8\, t$	$e^{-22 \cdot 1\, t}$	x
0.10	2.88	4.18	0.11	0.460
0.20	5.76	7.06	0.012	0.085
0.21	6.05	7.35	0.0096	0.071
0.22	6.35	7.65	0.0077	0.049

Or, the approximate time = 0.22 seconds when $x = 0{\cdot}5$ meters (actually x will be 0·049 instead of 0·05).

Therefore, the total time required by the barrel to move out and to return to a position given by $x = 0{\cdot}05$ meters is given by

Total time = 0.071 + 0·22

Or $\quad T \simeq 0{\cdot}29$ seconds. **Ans.**

The displacement-time plot of system under the above mentioned conditions is shown in Fig. 3.3.9.

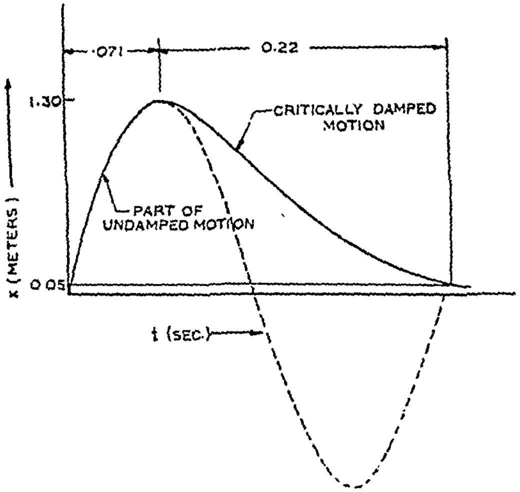


Fig. 3.3.9. Displacement-time plot of a system with zero damping in the outward stroke and critical damping in the return stroke.

Illustrative Example 3.3.4.

The system shown in Fig. 3.3.10 is displaced from its static equilibrium position to the right a distance of 1 cm. An implusive force acts towards the left on the mass at the instant of its release to give it an initial velocity V_0 in that direction.

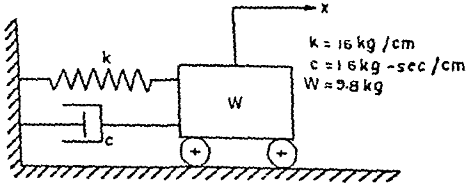


Fig. 3.3.10. Motion of a system with initial displacement and an initial velocity.

(a) Derive an expression for the displacement of the mass from the equilibrium position in terms of time t and initial velocity V_0.

(b) What value of V_0 would be required to make the mass pass the position of static equilibrium $\frac{1}{100}$ sec after it is applied ?

(c) What would be the maximum displacement of the mass to the left of the static equilibrium position for (b) ?

Solution

The equation of motion of the system is given by

$$\frac{W}{g}\ddot{x} + c\dot{x} + kx = 0$$

For this system

$$\omega_n = \sqrt{\frac{kg}{W}} = \sqrt{\frac{16\times 980}{9\cdot 8}} = 40 \text{ rad/sec}$$

$$\zeta = \frac{c}{2m\,\omega_n} = \frac{1\cdot 6\times 980}{2\times 9\cdot 8\times 40} = 2\cdot 0$$

Therefore, the system is an over-damped one, and its equation (3.3.9) is

$$x = C_1\, e^{[-\zeta+\sqrt{\zeta^2-1}]\,\omega_n t} + C_2\, e^{[-\zeta-\sqrt{\zeta^2-1}]\,\omega_n t}$$

Substituting the values of ζ and ω_n we have

$$x = C_1\, e^{-10\cdot 8\, t} + C_2\, e^{-149\cdot 2\, t} \tag{3.3.25}$$

The initial conditions are

$$\left.\begin{aligned} x &= 1 \\ \dot{x} &= -V_0 \end{aligned}\right\} \text{ at } \quad t=0$$

Differentiating equation (3.3.25) substituting the initial conditions in both the equations,

$$1 = C_1 + C_2$$
$$-V_0 = -10\cdot 8\, C_1 - 149\cdot 2\, C_2$$

giving

$$C_1 = 1\cdot 08 - 0\cdot 00723\, V_0$$
$$C_2 = -0\cdot 08 + 0\cdot 00723\, V_0$$

(a) Substituting the values of the constants C_1 and C_2 in equation (3.3.25), we have,

$$\left.\begin{aligned} x &= (1.08-0.00723V_0)e^{-10.8t} \\ &\quad + (0.00723\,V_0-0.08)e^{-149.2\,t} \end{aligned}\right\} \qquad (3.3.26)$$

which is the final solution. **Ans.**

(b) At $t = \frac{1}{100}$ sec, $x = 0$

Applying this condition to equation (3.3.26)

$$0 = (1.08-0.00723\ V_0)e^{-0.108}$$
$$+ (0.00723\ V_0-0.08)\ e^{-1 \cdot 49}$$

Solving, gives

$$V_0 = 196 \text{ cm/sec.}$$ **Ans.**

(c) Substituting the above value of V_0 in equation (3.3.26),

$$x = -\ 0.34\ e^{-10.8t} + 1.34\ e^{-149.2\ t} \qquad (3.3.27)$$

In order to find the maximum displacement, we differentiate the above equation and equate it to zero for zero velocity (corresponding to the maximum displacement). i.e.,

$$\dot{x} = 3.66\ e^{-10.8\ t} - 200\ e^{-149.2\ t} = 0$$

or $$\frac{200}{3.66} = e^{-10.8\,t + 149.2\,t} = e^{138.4t}$$

which gives $t = 0.029$ sec.

This value of t is substituted in equation (3.3.27) to get the maximum displacement, or

$$x_{max} = -\ 0.34\ e^{-10.8 \times 0.029} + 1.34\ e^{-149.2 \times 0.029}$$

or $$x_{max} = -\ 0.23 \text{ cm.}$$

The negative sign shows that this displacement is on the left side of the equilibrium position. **Ans.**

3.4 Logarithmic decrement.

Let us go back to Fig. 3.3.4. for the free vibrations of an underdamped system. Consider two points, A and B, corresponding to the times t_A and t_B, where

$$t_B - t_A = \frac{2\pi}{\sqrt{1-\zeta^2}\,\omega_n}$$

If we look at the equation (3.3.17) for an underdamped system the amplitude of the damped oscillation is given by the expression $\frac{X_0}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}$ which is the envelope of the maximas of the displacement-time curve. It is shown as a dotted curve in Fig. 3.3.4. Now the height of the displacement-time curve at $t = t_A$ is equal to the height of the envelope at the same time.

Therefore, $x_A = \frac{X_0}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t_A}$

and $x_B = \frac{X_0}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t_B}$

Dividing one by the other,

$$\frac{x_A}{x_B} = e^{-\zeta\omega_n (t_A - t_B)}$$

$$= e^{\zeta\omega_n (t_B - t_A)}$$

But, $t_B - t_A = \frac{2\pi}{\sqrt{1-\zeta^2}\,\omega_n}$

Therefore, $\frac{x_A}{x_B} = e^{2\pi\zeta/\sqrt{(1-\zeta^2)}}$

or $\log_e \frac{x_A}{x_B} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$

This is called the *Logarithmic Decrement* and is denoted by δ.

$$\text{Therefore, } \delta = \log_e \frac{x_A}{x_B} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \tag{3.4.1}$$

This shows that the ratio of any two successive amplitudes for an underdamped system, vibrating freely, is constant and is a function of the damping only. For small values of ζ the above equation reduces to

$$\delta \simeq 2\pi\zeta \tag{3.4.2}$$

The logarithmic decrement is also given by the equation

$$\delta = \frac{1}{n} \log_e \frac{x_0}{x_n} \tag{3.4.3}$$

(a) Substituting the values of the constants C_1 and C_2 in equation (3.3.25), we have,

$$\left.\begin{aligned} x &= (1.08-0.00723V_0)e^{-10.8t} \\ &\quad + (0.00723\ V_0-0.08)e^{-149.2\ t} \end{aligned}\right\} \qquad (3.3.26)$$

which is the final solution. **Ans.**

(b) At $t=\frac{1}{100}$ sec, $x=0$

Applying this condition to equation (3.3.26)

$$0 = (1.08-0.00723\ V_0)e^{-0.108} + (0.00723\ V_0-0.08)\ e^{-1 \cdot 49}$$

Solving, gives

$$V_0 = 196 \text{ cm/sec.}$$ **Ans.**

(c) Substituting the above value of V_0 in equation (3.3.26),

$$x = -\ 0.34\ e^{-10.8t} + 1.34\ e^{-149.2\ t} \qquad (3.3.27)$$

In order to find the maximum displacement, we differentiate the above equation and equate it to zero for zero velocity (corresponding to the maximum displacement). i.e.,

$$\dot{x} = 3.66\ e^{-10.8\ t} - 200\ e^{-149.2\ t} = 0$$

or $$\frac{200}{3.66} = e^{--10.8\ t\ +\ 149.2\ t} = \text{e}^{138.4t}$$

which gives $t = 0.029$ sec.

This value of t is substituted in equation (3.3.27) to get the maximum displacement, or

$$x_{max} = -\ 0.34\ e^{-10.8\times0.029} + 1.34\ e^{-149.2\times0.029}$$

or $$x_{max} = -\ 0.23 \text{ cm.}$$

The negative sign shows that this displacement is on the left side of the equilibrium position. **Ans.**

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Therefore, $x_A = \dfrac{X_0}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t_A}$

and $\quad x_B = \dfrac{X_0}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t_B}$

Dividing one by the other,

$$\frac{x_A}{x_B} = e^{-\zeta\omega_n (t_A - t_B)}$$

$$= e^{\zeta\omega_n (t_B - t_A)}$$

But, $\quad t_B - t_A = \dfrac{2\pi}{\sqrt{1-\zeta^2}\,\omega_n}$

Therefore, $\dfrac{x_A}{x_B} = e^{2\pi\zeta/\sqrt{(1-\zeta^2)}}$

or $\qquad \log_e \dfrac{x_A}{x_B} = \dfrac{2\pi\zeta}{\sqrt{1-\zeta^2}}$

This is called the *Logarithmic Decrement* and is denoted by δ.

$$\text{Therefore,}\quad \delta = \log_e \frac{x_A}{x_B} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \tag{3.4.1}$$

This shows that the ratio of any two successive amplitudes for an underdamped system, vibrating freely, is constant and is a function of the damping only. For small values of ζ the above equation reduces to

$$\delta \simeq 2\pi\zeta \tag{3.4.2}$$

The logarithmic decrement is also given by the equation

$$\delta = \frac{1}{n} \log_e \frac{x_0}{x_n} \tag{3.4.3}$$

where x_0 represents the amplitude at a particular maxima and x_n represents the amplitude after a further n cycles. This can be proved easily as below.

$$\delta = \log_e \frac{x_0}{x_1} = \log_e \frac{x_1}{x_2} \ldots = \log_e \frac{x_{n-1}}{x_n}$$

or $$n\,\delta = \log_e \left[\frac{x_0}{x_1} \cdot \frac{x_1}{x_2} \quad \ldots \quad \frac{x_{n-1}}{x_n}\right]$$

$$= \log_e \frac{x_0}{x_n}$$

or $$\delta = \frac{1}{n} \log_e \frac{x_0}{x_n}$$

The logarithmic decrement method is often used to find the amount of damping in a physical system.

Illustrative Example 3.4.1

The disc of a torsional pendulum has a moment of inertia of 600 kg-cm^2 and is immersed in a viscous fluid. The brass shaft attached to it is of 10 cm diameter and 40 cm long. When the pendulum is vibrating, the observed amplitudes on the same side of the rest position for successive cycles are 9°, 6° and 4°. Determine.

(a) logarithmic decrement,

(b) damping torque at unit velocity, and

(c) the periodic time of vibration.

Assume for the brass shaft $G = 4.5 \times 10^5$ kg/cm^2.

What would the frequency be if the disc is removed from the viscous fluid ?

Solution

(a) Logarithmic decrement

$$\delta = \log_e \frac{9}{6} \left(\text{or } \log_e \frac{6}{4}\right)$$

$$= \log_e 1.5 = 0\;405$$

or $$\delta = 0.405$$ **Ans.**

(b) Since, $\delta = \dfrac{2\pi\zeta}{\sqrt{1-\zeta^2}}$

therefore, $(1-\zeta^2)\,\delta^2 = 4\,\pi^2\,\zeta^2$

or, $\delta^2 = (4\,\pi^2+\delta^2)\,\zeta^2$

or
$$\zeta = \frac{\delta}{\sqrt{4\pi^2+\delta^2}}$$
$$= \frac{0{\cdot}405}{\sqrt{4\pi^2+{\cdot}405^2}} = 0{\cdot}0645$$

Now damping torque at unit velocity is the torsional damping coefficient c of the system, given by

$$\frac{c}{c_c} = \zeta$$

or $c = \zeta\, c_c$

But $c_c = 2\sqrt{k_t J}$ [Similar to equation (3·3.6)]

Therefore $c = \zeta\, 2\,\sqrt{k_t J}$ (3.4.4)

The moment of inertia of the system has been given as 600 kg-cm². This is the weight moment of inertia as seen from the units. We need mass moment of inertia J to be used in vibration equations.

Therefore $J = \dfrac{600}{980}$ kg-cm-sec²

$= 0{\cdot}611$ kg-cm-sec².

Torsional stiffness of the shaft,

$$k_t = \frac{G.\,I_p}{l} = \frac{G}{l}\cdot\frac{\pi}{32}d^4$$
$$= \frac{4{\cdot}5\times10^5}{40}\times\frac{\pi}{32}\times10^4 = 1{\cdot}1\times10^7 \text{ kg-cm/rad.}$$

Substituting the values of ζ, J and k_t as found above, in equation (3.3.4), we have,

$$c = 0{\cdot}0645\times2\times\sqrt{1{\cdot}1\times10^7\times0{\cdot}611}$$
$$= 334 \text{ kg-cm/rad.}$$

This is the damping torque at unit velocity. **Ans.**

(c) Periodic time of vibration,

$$\tau = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}}$$

Now, $\omega_n = \sqrt{\frac{k_t}{J}} = \sqrt{\frac{1{\cdot}1 \times 10^7}{0{\cdot}611}} = 4240$ rad/sec

Therefore $\tau = \dfrac{2\pi}{4240 \times \sqrt{1 - 0{\cdot}0645^2}} = 0{\cdot}00149$ sec. **Ans.**

The frequency when the disc is removed from the viscous fluid, is the natural frequency of the system, and is given by

$$f_n = \frac{\omega_n}{2\pi}$$

But $\omega_n = 4240$ rad/sec as found above.

Hence $f_n = \dfrac{4240}{2\pi} = 675$ c.p.s. **Ans.**

3·5 Viscous dampers.

There are two important types of viscous dampers that are invariably used for providing viscous damping in vibrating systems. These are described below.

3.5.1 Fluid dashpot. This consists of a piston moving to and fro in a cylinder full of viscous fluid as shown in Fig. 3.5.1. There are three components of damping that are experienced when the piston moves in the cylinder. These are:—

(i) Damping due to the drag of the fluid.

(ii) Damping produced by the pressue flow of the fluid through the clearance space as a result of piston displacement.

(iii) Damping resistance due to the pressure difference on the two sides of the piston. This pressure difference is caused by the restriction to the fluid flow due to the piston motion.

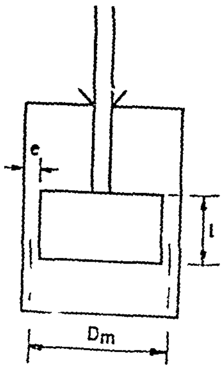


Fig. 3.5.1. Fluid dashpot.

It can be shown that if the clearance between the piston and the cylinder is small, the first two components of the damping are negligible and the total damping is wholly due to the third component, and is given by

$$c = \frac{12\ \mu}{\pi} \cdot \frac{A_p^2 l}{D_m\ e^3} \tag{3.5.1}$$

where c = viscous damping coefficient (kg-sec/cm),

μ = coefficient of viscosity of the fluid (kg-sec/cm²),

A_p = area of one flat side of the piston (cm²),

l = length of the piston (cm),

D_m = mean diameter of the piston and the cylinder (cm),

e = clearance between the piston and the cylinder (cm).

The assumptions used in deriving the above relations are:— damping medium being a perfect fluid, sharp edge orifice effects neglected, piston rod dia small as compared to piston dia, laminar flow in the clearance space, piston and cylinder being concentric.

Because of the above assumptions, the equation (3.5.1) is only approximate.

In practice some adjusting device is used in order to adjust the damping to the required value.

3.5.2 Eddy current damping. Consider a non-ferrous conducting rectangular plate being moved in a direction perpendicular to the lines of magnetic flux as shown in Fig. 3.5.2 As the plate moves, a current is induced in the plate

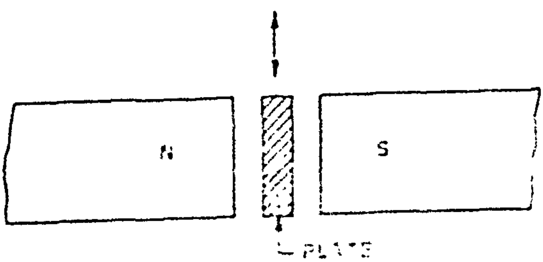


Fig. 3.5.2. Principle of eddy current damping.

and is proportional to the velocity of the plate. This current is in the form of eddy current that set up a magnetic field in a direction opposing the original magnetic field that causes

them. Thus, there is a resistance to the motion of the plate in a magn tic field. The resisting force produced by the flux field from the eddy currents is proportional to the velocity. Hence, mechanical damping of the viscous type is achieved.

This type of damping is used in many vibrometers and also in some other vibration control systems.

3.6 Dry friction or Coulomb damping.

Consider two dry sliding surfaces with a normal reaction N between them. Then the force of friction acting on each of the two mating surface is given by

$$F = \mu N \tag{3.6.1}$$

where μ is defined as the coefficient of friction between the two mating surfaces. The dependence of μ on the relative rubbing velocity is shown in Fig. 3.6.1. For ideally smooth surfaces, dry friction coefficient μ is independent of velocity. For rough surfaces, dry friction coefficient decreases somewhat initially with the increase in velocity (shown exaggarated in the figure), and then is practically constant, allhough it does decrease very slightly with further increasing velocities also. For the lubricated surfaces, μ is approximately proportional to velocity giving an approximately viscous damping.

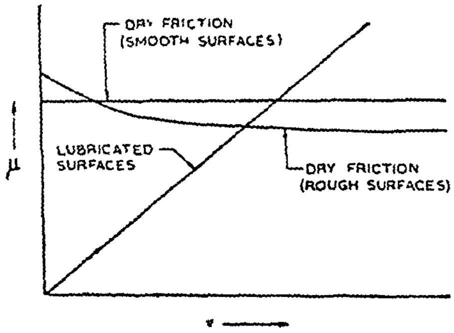


Fig. 3.6.1. Dependence of the coefficient of friction on relative velocity.

For the case under study we shall take μ as constant throughout the velocity range. The slight decrease in friction coefficient with increase in velocity is responsible for many cases of the phenomenon known as *Self-Excited* vibrations.

The aspects that we are interested in, in connection with free vibrations with Coulomb damping, are

(i) the frequency of damped oscillations, and

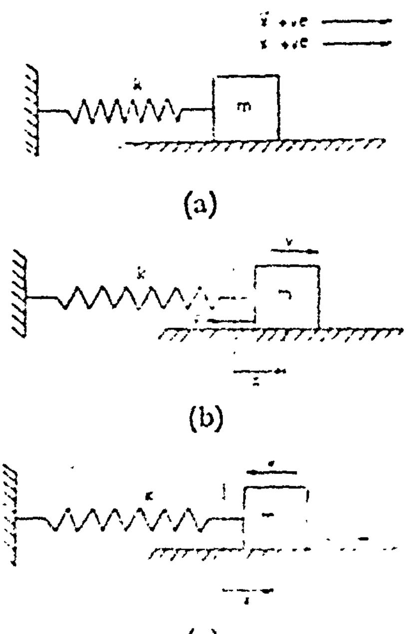
(ii) the rate of decay of these oscillations.

These are analysed in the following paragraphs.

3.6A Frequency of damped oscillations. Imagine a spring-mass system as shown in Fig. 3.6.2 (a), with the mass capable of sliding on a dry surface, μ being the coefficient of dry friction between the two surfaces. In the equilibrium position shown, the spring is unstretched and no friction force acts on the mass. Take the positive direction of x towards the right and therefore, the positive direction of $\ddot{x}$ also towards the right.

Mass displaced towards right moving towards right.

Let the mass at any instant be displaced towards the right of



(a) (b) (c)

Fig. 3.6.2. Free vibration with dry friction or Coulomb damping (a) Equilibrium position. (b) Mass displaced towards right and moving towards right. (c) Mass displaced towards right and moving towards left.

the equilibrium position and be moving towards the right as shown in Fig. 3.6.2 (b). Then friction force F on the mass is acting towards the left (i.e. in the negative direction). The spring force on the mass is also acting towards the left (negative direction). Therefore the equation of motion of the mass for this part of the motion is as follows.

$$m\ddot{x} = -kx - F$$

or $$\ddot{x} + \frac{k}{m}\left(x + \frac{F}{k}\right) = 0 \qquad (3.6.1)$$

Let $$x + \frac{F}{k} = y$$

Therefore, $$\ddot{x} = \ddot{y}$$

Substituting for x and $\ddot{x}$ in equation (3.6.1), we have

$$\ddot{y} + \frac{k}{m}y = 0 \qquad (3.6.2)$$

This is simple harmonic motion about $y = 0$ $\left(\text{or } x + \frac{F}{k} = 0\right)$, and is true only for that quarter of the cycle when the mass *is displaced towards right and is moving towards right.* The natural frequency of vibration for this part of the simple harmonic motion is obtained from equation (3.6.2) as,

$$\omega_n = \sqrt{\frac{k}{m}}$$

Mass displaced towards right and moving towards left. In this case, the friction force on the body acts towards the right (i.e. positive direction), because the body is now moving towards the left. This is shown in Fig. 3.6.2 (c). Every thing else remains the same as in the previous case. The equation of motion is

$$m\ddot{x} = -kx + F$$

or $$\ddot{x} + \frac{k}{m}\left(x - \frac{F}{k}\right) = 0 \qquad (3.6.3)$$

Now let $$x - \frac{F}{k} = y$$

Therefore, $$\ddot{x} = \ddot{y}$$

Substituting for x and $\ddot{x}$ in equation (3.6.3), we have

$$\ddot{y} + \frac{k}{m} y = 0$$

i.e. the same as equation (3.6.2)

This is again simple harmonic motion about $y = 0$ $\left(\text{or}\quad x - \frac{F}{k} = 0\right)$, and is true only for that quarter of the cycle when the mass is displaced towards the right and moving towards the left. The natural frequency for this part of the simple harmonic motion is again

$$\omega_n = \sqrt{\frac{k}{m}}$$

Similarly it can be shown that the frequency of vibration in the other two quarters, which also form parts of simple harmonic motion, is also

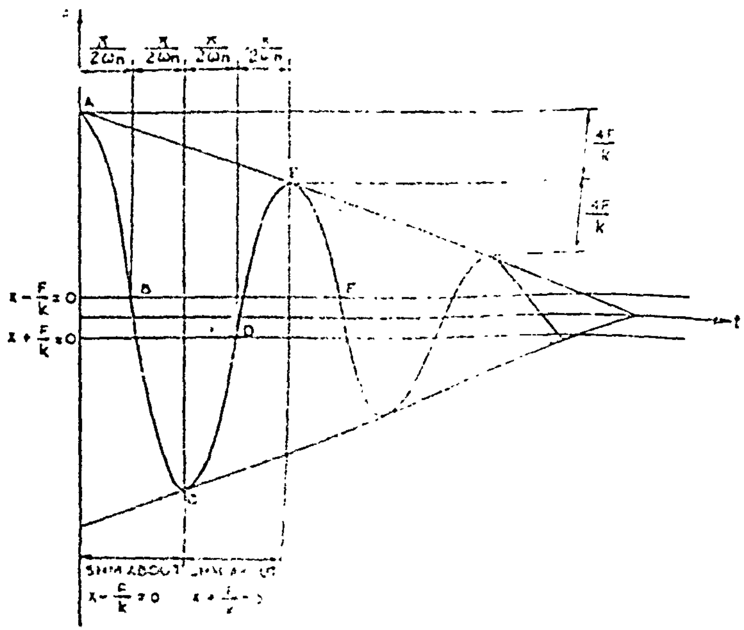


Fig. 3.6.3. *Displacement-time plot of a system hav... free vibrations with Coulomb damp...*

$$\omega_n = \sqrt{\frac{k}{m}}$$

While the system is displaced towards the left and moving towards the left, it has simple harmonic motion about $x - \frac{F}{k} = 0$, and while it is displaced towards the left and moving towards the right, it has simple harmonic motion about $x + \frac{F}{k} = 0$. The displacement-time plot for such a system is shown in Fig. 3.6.3.

It can be clearly seen from this figure that each half cycle corresponding to the body moving in one direction, is a part of a simple harmonic motion about a point other than the equilibrium position. The frequency of vibration for each one of these half cycles is $\sqrt{\frac{k}{m}}$. Hence, the frequency of vibration for a system having Coulomb damping is the same as that of an undamped system,

$$\text{i.e.} \quad \omega_n = \sqrt{\frac{k}{m}} \tag{3.6.4}$$

or the time period of one complete cycle is

$$\tau = \frac{2\pi}{\omega_n} \tag{3.6.5}$$

Each of the portions AB, BC, CD, DE, etc. of the plot in Fig. 3.6.3 is traversed in time $\tau/4$ or $\pi/2\omega_n$

3.6 B Rate of decay of oscillations. Fig. 3.6.3 also shows the rate of decay of free vibrations with Coulomb damping. It is seen that when the body moves from point A to point C, the motion is a part of simple harmonic motion about B. If we consider the motion about the equilibrium position $x = 0$, it is clear from the figure that in moving from A to C, the body has lost an amplitude of $2F/k$. Similarly there is another loss of amplitude equal to $2F/k$ about the equilibrium position in the next half cycle CDE. Thus the amplitude loss per half cycle is $\frac{2F}{k}$ and therefore, the amplitude loss per cycle is equal to $\frac{4F}{k}$

This conclusion can also be drawn from energy viewpoint.

In Fig. 3.6.3, let x_A be the amplitude of the body from the mean position, to start with. After half a cycle, when it has reached the other extreme, let the amplitude be x_C. Since in these two extreme positions the velocities of the mass is zero, the total energies in two positions are the potential energies and are respectively equal to $\frac{1}{2} kx_A^2$ and $\frac{1}{2} kx_C^2$. The difference between the two total energies must be equal to the energy dissipated or work done against friction.

The friction force F is constant throughout the motion A to C. The distance moved by the body is equal to (x_A+x_C). Therefore, work done against friction is equal to $F\,(x_A+x_C)$. Equating this to the loss of total energy of system, we have

$$\tfrac{1}{2}kx_A^2-\tfrac{1}{2}kx_C^2 = F\,(x_A+x_C)$$

or, $$x_A-x_C = 2F/k$$

Similarly, it can be shown that

$$x_C-x_E = 2F/k$$

Adding the above two equations,

$$x_A-x_E = 4F/k \tag{3.6.6}$$

that is, the amplitude loss per cycle is equal to $4F/k$

In the case of viscous damping, the *ratio* of any two successive amplitudes was constant and the envelope of the maximas of the displacement-time plot was an exponential curve. In the case of Coulomb damping, the *difference* between any two successive amplitudes is constant and the envelope of the maximas of the displacement-time plot is a straight line (Fig. 3.6.3).

Another point of difference between the two types of dampings is that in the case of viscous damping, the body once disturbed from the equilibrium position finally comes to rest in the equilibrium position (although it takes theoretically infinite time to do so), whereas in the case of Coulomb damping the body may finally come to rest in the equilibrium position or in a displaced position depending upon the initial amplitude and the amount of friction present.

Illustrative Example 3.6.1

A horizontal spring mass system with Coul

a weight of 5.0 kg attached to a spring of stiffness 1.0 kg/cm. If the coefficient of friction is 0·025, calculate

(a) the frequency of free oscillations,

(b) the number of cycles corresponding to 50% reduction in amplitude if the initial amplitude is 5·0 cm, and

(c) the time taken to achieve this 50% reduction.

Solution

For the horizontal system the normal reaction is equal to W, the weight of the mass.

Therefore, $F = \mu W = 0{\cdot}025 \times 5 = 0{\cdot}125$ kg.

(a) $$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1\times 980}{5}} = 14{\cdot}0 \text{ rad/sec}$$

$$f_n = \frac{\omega_n}{2\pi} = \frac{14{\cdot}0}{2\,\pi} = 2.23 \text{ cycles/sec.}$$ **Ans.**

(b) Initial amplitude $= 5{\cdot}0$ cm

Amplitude after 50% reduction $= 2{\cdot}5$ cm

Total reduction in amplitude $= 5.0 - 2.5 = 2{\cdot}5$ cm

Reduction in amplitude/cycle $= \dfrac{4F}{k} = \dfrac{4\times 0{\cdot}125}{1}$

$= 0{\cdot}5$ cm

Number of cycles for 50% reduction $= \dfrac{2{\cdot}5}{0{\cdot}5} = 5$ cycles. **Ans.**

(c) Time taken to achieve the 50% reduction

$=$ time taken to perform 5 cycles

$$= 5 \times \frac{2\pi}{\omega_n} = \frac{5 \times 2\pi}{14} = 2{\cdot}242 \text{ seconds.}$$ **Ans.**

Illustrative Example 3.6.2

A verticle spring of stiffness 10 kg/cm supports a mass of 40 kg. There is a friction force of 5 kg which always resists the vertical displacement whether upwards or downwards. The mass is released from a position in which the total extension of the spring is 12·6 cm. Determine the final extension of the spring in the position in which system comes to rest.

Solution

Extension of the spring under equilibrium condition when no no friction is present $= \dfrac{W}{k} = \dfrac{40}{10} = 4$ cm.

Total extension of the spring when initially relased = 12.6 cm.

Therefore, initial extension of the spring from the equilibrium position $= 12.6 - 4 = 8.6$ cm.

Loss of amplitude per cycle $= \dfrac{4F}{k} = \dfrac{4 \times 5}{10} = 2{\cdot}0$ cm.

This shows the number of complete cycles that the system undergoes is only 4.

Amplitude at the end of four cycles

$$= 8.6 - 4 \times 2.0 = 0.6 \text{ cm.}$$

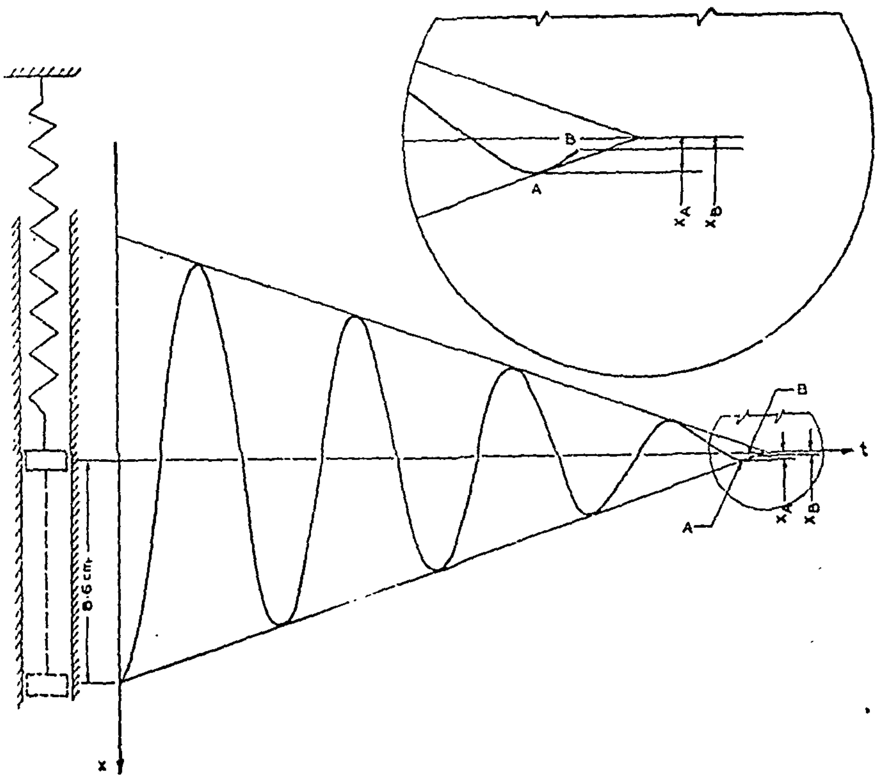


Fig. 3.6.4. Free vibrations with Coulomb damping showing the final rest position.

Refering to Fig. 3.6.4, it is necessary now to check whether the mass will come to rest in this extended position at the end of 4 cycles (Point A) or will move up to start the fifth cycle.

Considering the forces from the equilibrium position, the spring force acting on the mass in the upward direction

$$= k \times \text{extension}$$

$$= 10 \times 0{\cdot}6 = 6 \text{ kg.}$$

The resisting friction force $= 5$ kg.

Thus the spring force is more than the friction force and therefore the mass starts moving up. Let the mass start up from point A and finally come to rest in any position, say point B. At points A and B, the system has only the potential energy of the spring. The difference in these potential energies must be equal to the work done in moving the distance $(x_A - x_B)$ against the friction force.

Therefore, $\frac{1}{2}\, k x_A^2 - \frac{1}{2}\, k x_B^2 = F\,(x_A - x_B)$

Substituting the values of $x_A = 0.6$, $F = 5$ and $k = 10$, and solving for x_B gives

$$x_B = 0{\cdot}6 \text{ or } 0{\cdot}4 \text{ cm.}$$

The two values of x_B correspond to the two stationary points for which the above energy equation is applicable. However, it has been shown that the body does start from $x_B = 0.6$. Therefore, it finally comes to rest in the extended position of 0·4 cm beyond the equilibrium position.

Hence, the final extension of the spring in the final rest position $= 4 + 0{\cdot}4 = 4{\cdot}4$ cm. The displacement-time plot of the motion is shown in Fig. 3.6.4. **Ans.**

3.7 Solid or structural damping

This is one of the most important types of dampings since it occurs in all vibrating systems subject to elastic restoring forces. The amount of damping, however, is small.

Elastic materials' stress-strain plots are different for loading and for unloading. When such a material is subjected to cyclic reversal of loading, a hysterisis loop appears on the stress-strain plot and the area of this loop is the energy

dissipated per unit volume per cycle. All this means, is that more work is done on the system while straining it than what is recovered during its relaxation. This type of damping is also called *Hysteresis Damping.* It is proportional to the stress amplitude of the elastic material and therefore proportional to the amplitude of the vibrating body. This proportionality holds good upto a certain stress amplitude after which it is found to increase much more rapidly with the stress amplitude. Cast iron has much more hysterisis damping than mild steel.

If two similar systems are set vibrating such that their initial stress amplitudes are the same, then each will diminish equally in amplitude on a per cycle basis. The system of higher natural frequency, however, will be damped out earlier with respect to time since it executes more number of cycles per unit time.

Illustrative Example 3.7.1

A system 'A' having structural damping executes 12 cps in free vibrations. Another similar system 'B' executes 15 cps in free vibrations. Both systems are set vibrating with the same initial stress level. If system 'A' takes 4.5 seconds to damp out completely, how long will the system 'B' take to damp out completely ?

Solution

Both systems execute the same number of cycles after being set into vibration with the same initial stress level, since the systems are similar.

Therefore, $12 \times 4{\cdot}5 = 15 \times t$

where t is the time taken for the second system to damp out completely.

Hence, $t = \dfrac{12 \times 4{\cdot}5}{15} = 3{\cdot}6$ seconds **Ans.**

3.8 Slip or interfacial damping.

Consider a cantilever B on which another bar A is placed. The two are pressed together by a number of C-clamps or any other means to provide a pressure P between the two

contact surfaces, as shown in Fig. 3.8.1 (a). Now when the system vibrates in the vertical plane as shown, notice the continuous slip between the two surfaces, shown exaggerated

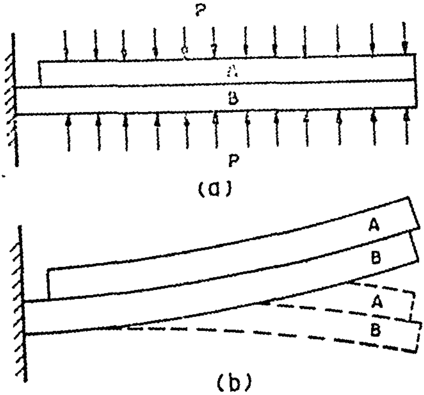


Fig. 3.8.1. Mechanism of slip damping.

in Fig. 3.8.1 (b). The energy dissipated per cycle depends upon the coefficient of friction, the pressure between the plates and the amplitude. This variation is shown in Fig. 3.8.2.

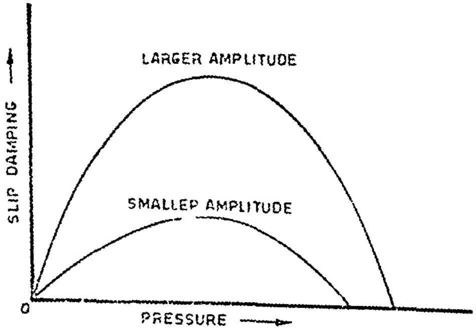


Fig. 3.8.2. Variation of slip damping with contact pressure.

At zero pressure there is large slip but no energy is dissipated in friction because of zero pressure, and therefore no energy loss. At very high pressure, there is essentially no slip and hence no energy loss There is an optimum value of pressure for which the energy dissipated is maximum. This is different

for different amplitudes of vibration. Larger the energy dissipation, larger is the effective damping in the system.

In the bolted or rivetted joints as shown in Fig. 3.8.3, the damping is essentially due to microscopic slip at the interfaces

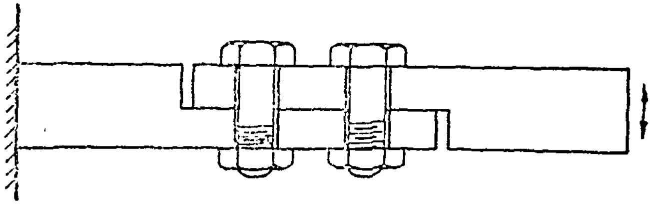


Fig. 3.8.3. Damping in joints.

of the parts in contact including the bolt head and the nut surfaces in contact with the members. The trend in the variation of damping with the amplitude of vibration, is shown in Fig. 3.8.4.

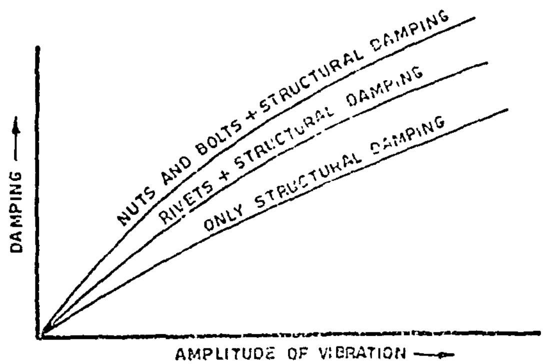


Fig. 3.8.4. Variation of joints damping with vibration amplitude.

The nut and bolt joint provides greater amount of damping than the rivetted joint.

PROBLEMS FOR PRACTICE

3.1 A spring-mass-dashpot system consists of a spring of stiffness 0.35 kg/cm. The weight of the mass is 3.43 kg. The s is displaced 2 cm beyond the equilibrium position

and released. Find the equation of motion for the system, if the damping coefficient of the dashpot is equal to

(i) 0.140 kg-sec/cm,
(ii) 0.070 kg-sec/cm,
(iii) 0.014 kg-sec/cm.

3.2 In prob. 3.1, if the system is given an initial velocity of 5 cm/sec from the equilibrium position, find the equations of motion of the system for three different values of the damping coefficient.

3.3 A 25 kg mass is resting on a spring of 5 kg/cm and dashpot of 0.15 kg-sec/cm in parallel. If a velocity of 10 cm/sec is applied to the mass at the rest position, what will be its displacement from the equilibrium position at the end of first second ?

3.4 Show that the mass of a system having overdamping, will never pass through the equilibrium position if it is given

(i) an initial displacement only,
(ii) an initial velocity only.

3.5 Two deshpots of coefficients c_1 and c_2 are connected in (i) series (ii) parallel. Find their equivalent damping coefficients from first principles.

3.6 For an underdamped system, plot a graph of phase angle versus damping factor, in the case of damped free vibration.

3.7 For the system shown in Fig. P. 3.7,

$W = 1.5$ kg; $k=5$ kg/cm; $a=6$ cm; and $L=14$ cm. Taking the rod on which the weight is fixed, as light and stiff, determine the value of c for the system to be critically damped.

3.8 Write the differential equation of motion for the system shown in Fig. P. 3.8 and find the natural frequency of damped vibrations and the critical damping coefficient of th^ dashpot.

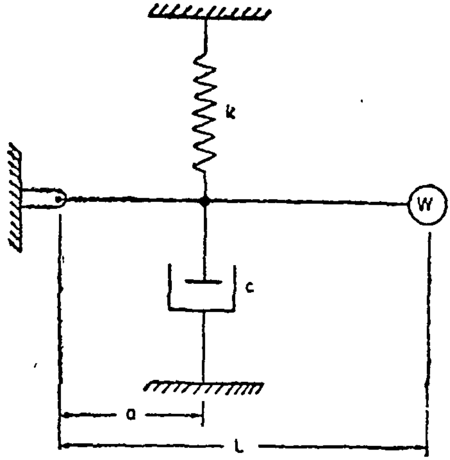


Fig. P.3.7.

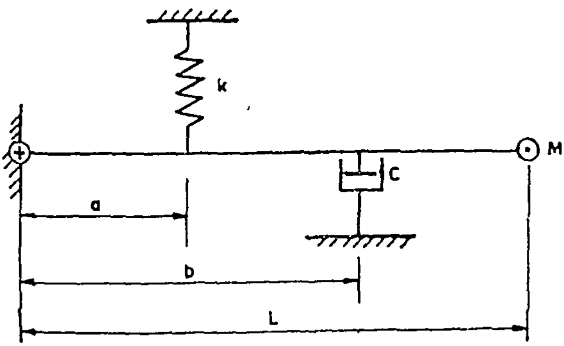


Fig. P.3.8.

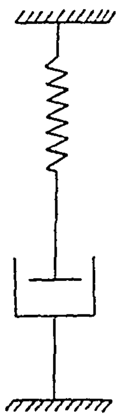


Fig. P. 3.9.

3.9 A system as shown in Fig. P. 3.9 consists of a weightless spring of scale k and a weightless piston in a dashpot of damping coefficient c. The piston is displaced a distance X_0 and released. For this ideal case, derive an expression for the motion of the piston. Discuss the result.

3.10 A weight of 1 kg is to be supported on a spring having a stiffness of 10 kg/cm. The damping coefficient is 0.005 kg-sec/cm. Determine the natural frequency of the system. Find also the logarithmic decrement and the amplitude after three cycles if the initial displacement is 0.30 cm.

3.11 The damped vibration record of a spring-mass-dashpot system shows the following data.

Amplitude on second cycle $=1.20$ cm.

Ampliude on third cycle $=1.05$ cm.

Spring constant, k $=8$ kg/cm.

Weight on the spring, W $=2$ kg.

Determine the damping constant, assuming it to be viscous.

3.12 A weight of 2 kg is supported on an isolator having a spring scale of 3 kg/cm and viscous damping. If the amplitude of free vibration of the weight falls to one half its original value in 1.5 seconds, determine the damping coefficient of the isolator.

3.13 A body of weight 5 kg is supported on a spring of stiffness 2 kg/cm and has dashpot connected to it which produces a resistance of 0.002 kg at a velocity of 1 cm/sec. In what ratio will the amplitude of vibration be reduced after 5 cycles.

3.14 For a system having viscous damping, plot a curve for the number of cycles elapsed for the amplitude to decay to 50% of the initial value, against the damping factor.

3.15 For the system shown in Fig. P.3.15 the characteristic of the dashpot is such that when a constant force of 5 kg is applied to the piston its velocity is found to be constant at 12 cm/sec.

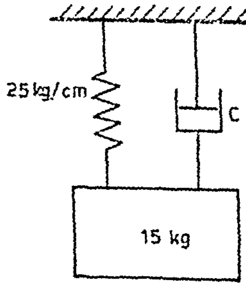


Fig. P.3.15.

(i) Determine the value of c.

(ii) Would you expect the complete system to be periodic or aperiodic ?

3.16 A spring-mass-dashpot system has a static deflection of 1 cm. Would the natural frequency obtained from this static deflection be the undamped natural frequency or damped natural frequency ? Give reasons.

3.17 For the case of a dashpot, plot the dimensionless damping against the dimensionless clearance.

3.18 A weight of 1 kg is attached to a spring having a stiffness of 4 kg/cm. The weight slides on a horizontal surface, the coefficient of friction between the weight and the surface being 0.1. Determine the frequency of vibrations of the system and the amplitude after one cycle if the initial amplitude is 0.25 cm. Determine the final rest position.

8.19 A vertical spring-mass system has a mass of weight 0.5 kg and an initial deflection of 0.2 cm. Find the spring stiffness and the natural frequency of the system. The system is subjected to Coulomb damping. When displaced by 2 cm from the equilbrium position and released, it undergoes complete 10 cycles and comes to rest in the extreme position on the side on which it was displaced. Calculate the Coulomb damping and the final rest position.

3.20 A body of weight $W = 1$ kg, lies on a dry horizontal plane and is connected by a spring to a rigid support. The body is displaced from the unstressed position by the amount equal to 25.5 cm with the tension in the spring at this displacement equal to 5 W, and then released with zero velocity. How long will the body vibrate and at what distance from the unstressed position will it stop if the coefficient of friction is 0.25 ?

3.21 A body of weight 1500 kg is suspended on a leaf spring. The system was set into vibration and the frequency of vibration was measured as 0.982 cps. The successive

amplitudes were measured to be 4.8 cm, 4.1 cm, 3.4 cm, 2.7 cm. Determine the spring stiffness and the coefficient of Coulombs damping.

3.22 The system of Problem 3.21 is given a sudden velocity of 25 cm/sec from its natural equilibrium position. Find

(a) the maximum displacement,

(b) the number of cycles counting from the instant of maximum displacement, and

(c) the final rest position.

3.23 Two steel cantilevers with end weights have the following physical data.

$$b_1 = b, \qquad b_2 = 2b$$
$$d_1 = d, \qquad d_2 = 2d$$
$$l_1 = l, \qquad l_2 = 2l$$
$$W_1 = W, \qquad W_2 = 8W$$

Find the ratio of their natural frequencies.

Show that if the weight W_1 is given an initial lateral displacement of y and the weight W_2 is given an initial displacement of $2y$, the maximum stress levels of the two systems are the same. Under these conditions, if both the systems are released from their respective initial displacements simultaneously, which system will damp out earlier ?

CHAPTER 4

FORCED VIBRATIONS OF SINGLE DEGREE OF FREEDOM SYSTEMS

4.1 Introduction.

Uptill now we have considered the free vibrations of systems where a system once disturbed from its equilibrium position executes vibrations because of its elastic properties. This system will come to rest, sooner or later, depending upon the amount of damping present in the system. In the case of forced vibrations there is an impressed force on the system which keeps it vibrating. The vibrations of air compressors, internal combustion engines, machine tools and various other machinery are all examples of forced vibrations. It is very important for the designer to be able to make a mathematical model for a system subjected to forcing functions, and then to analyse it.

There are, in general, three types of forcing functions commonly encountered in engineering practice.

(i) Periodic forcing function.
(ii) Impulsive type of forcing function.
(iii) Random forcing function.

Periodic forces can be further divided into two categories, harmonic and nonharmonic. This chapter is mainly devoted to harmonic forcing functions which is the most common type. Analysis for nonharmonic forcing functions is possible by expansion of these into harmonic forcing functions and is treated in brief in this chapter.

Impulsive types of forces are also quite common and they give rise to transient vibrations. This will be discussed later in this text.

The most typical of the random forcing functions is the earthquake excitation. This is not within the scope of this text and so will not be discussed.

4.2 Forced vibrations with constant harmonic excitation.

Consider a spring-mass system having viscous damping, excited by a sinusoidal forcing function $F_0 \sin \omega t$, as shown in Fig. 4.2.1 (a). At any instant, when the mass is displaced from

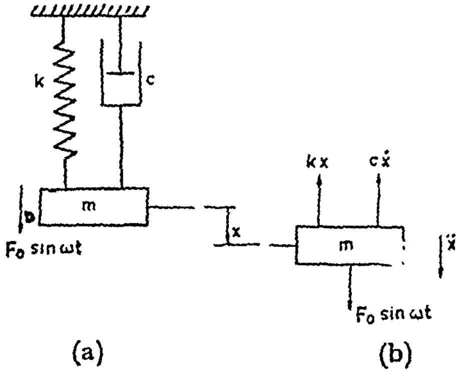


Fig. 4.2.1. Forced vibrations with harmonic excitation.

the mean position through a distance x in the downwa[rd] direction (positive direction of x), the external forces acting [on] the system are shown in Fig. 4.2.1 (b). These forces are

(i) kx, in the upward direction,

(i i) $c\dot{x}$, in the upward direction, and

(iii) $F_0 \sin \omega t$, in the downward direction.

From Newton's second law of motion,

$$m\ddot{x} = -c\dot{x} - kx + F_0 \sin \omega t$$

or,
$$m\ddot{x} + c\dot{x} + kx = F_0 \sin \omega t \qquad ($$

This is a linear, non-homogenous, second order differential equation. The solution of this equation consists of two parts, complementary function and particular integral. The complementary function is obtained from the differential equation

$$m\ddot{x} + c\dot{x} + kx = 0$$

i.e., by considering no forcing function. The solution of this equation has already been obtained in the previous chapter and is given by,

$$x_c = A_2\, e^{-\zeta\,\omega_n t} \sin\left[\sqrt{1-\zeta^2}\,\omega_n t + \phi_2\right] \tag{4.2.2}$$

This is the same as the last of equations (3.3.15), except that x is replaced by x_c as this is not the complete solution but only the *complementary* solution. We can also write the complementary solution in any of the other two alternative forms of equations (3.3.15). And secondly, this complementary solution is based on the system being an under-damped one. In case the system is over-damped or critically damped, the complementary solution will change to equations (3.3.9) or (3.3.12) respectively. Any of these equations contains two arbitrary constants which have to be determined from the initial conditions, but not at this stage. The initial conditions have to be applied to the *complete solution.*

The particular solution of equation (4.2.1) is dealtwith in all text books of fundamental differential equations. That approach can be used to find the particular solution of this equation. In this text, the *Vector Method* of finding the particular solution will be used to give more insight into the behaviour of the system.

The particular solution is a steady state harmonic oscillation having a frequency equal to that of the excitation, and the displacement vector lags the force vector by some angle. Let us therefore, assume that the particular solution is

$$x_p = X \sin(\omega t - \phi) \tag{4.2.3}$$

where X is the amplitude of vibration of the system, ϕ is the angle by which the displacement vector lags the force vector and x_p corresponds to the particular solution. If our assumption is not correct at this stage we will land into some absurd results.

Differentiating equation (4.2.3) twice, we have

$$\dot{x}_p = \omega X \cos(\omega t - \phi) = \omega X \sin(\omega t - \phi + \pi/2)$$
$$\ddot{x}_p = -\omega^2 X \sin(\omega t - \phi) = \omega^2 X \sin(\omega t - \phi + \pi)$$

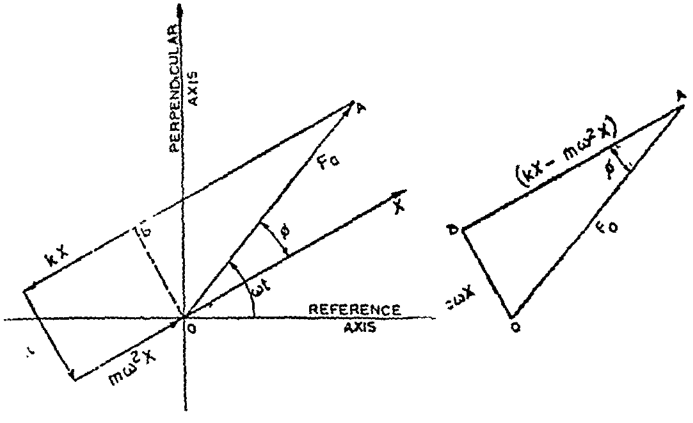
Substituting the values of x_p, $\dot{x}_p$ and $\ddot{x}_p$ as given by equation (4.2.3) and the two equations above, for x, $\dot{x}$ and $\ddot{x}$ in equation (4.2.1), we have

$$m\omega^2 X \sin(\omega t - \phi + \pi) + c\omega X \sin(\omega t - \phi + \pi/2)$$
$$+ kX \sin(\omega t - \phi) = F_0 \sin \omega t$$

Or, we can write the above equation as

$$F_0 \sin \omega t - kX \sin(\omega t - \phi) - c\omega X \sin(\omega t - \phi + \pi/2)$$
$$- m\omega^2 X \sin(\omega t - \phi + \pi) = 0 \qquad (4.2.4)$$

The four terms in equation (4.2.4), including their signs, represent the four forces, in magnitude and direction, acting on the body. These forces are the impressed force, spring force, damping force and intertia force respectively, and their sum is equal to zero. This is nothing but the *D'Alemberts Principle.* The vector representation of equation (4.2.4) is shown in Fig. 4.2.2 (a).



(a) (b)

Fig. 4.2.2. Vector representation of forces on the system having forced vibrations.

The impressed force on the body is $F_0 \sin \omega t$ [from eqaution (4.2.4)], i.e., the force vector F_0 is acting at an angle ωt from the reference axis and is so shown in the figure. The displacement vector lags the force vector by an angle ϕ [from equation (4.2.3)], and therefore this vector acts at an angle $(\omega t-\phi)$ from the reference axis. The spring force acting on the body is $-kX \sin(\omega t-\phi)$ [from equation (4.2.4)], which means that a vector $-kX$ acting at an angle $(\omega t-\phi)$ or vector kX acting in opposite direction to $(\omega t-\phi)$. Thus the spring force on the body acts in a direction opposite to that of displacement. In a similar way the damping and the inertia force vectors are represented in the vector diagram. The actual forces acting on the body are the components of the above four force vectors along the perpendicular axis. Since the body is in equilibrium under the four forces, these vectors must form a closed polygon.

The following points are observed from the vector diagram.

1. The displacement lags the impressed force by an angle ϕ.

2. The spring force is always opposite in direction to the displacement.

3. The damping force lags the displacement by 90°. Since the velocity leads the displacement by 90° (Sec. 1.3), it follows that the damping force is always opposite in direction to the velocity.

4. The inertia force is in phase with the displacement. Since the acceleration is out of phase with the displacement (Sec. 1.3), therefore the inertia force is always opposite in direction to the acceleration.

It may be noted that the relative positions of the vectors and their magnitudes do not change with time.

In order to find out the values of X and ϕ in equation (4.2.3) we consider the right angled triangle OAB after droping OB perpendiculer to AB. This triangle is shown separately also in Fig. 4.2.2 (b), from which, we have

$$X=\frac{F_0}{\sqrt{[(k-m\omega^2)^2+(c\omega)^2]}} \tag{4.2.5}$$

and $$\phi = \tan^{-1}\left[\frac{c\omega}{k-m\omega^2}\right] \tag{4.2.6}$$

In order to obtain the above equations in non-dimensional form, we divide the numerators and denominators by k,

Therefore, $$X = \frac{F_0/k}{\sqrt{\left(1-\frac{m\omega^2}{k}\right)^2+\left(\frac{c\omega}{k}\right)^2}} \tag{4.2.5a}$$

and $$\phi = \tan^{-1}\left[\frac{c\omega/k}{1-\frac{m\omega^2}{k}}\right] \tag{4.2.6a}$$

Now, $$\frac{m\omega^2}{k} = \frac{\omega^2}{\omega_n^2} \text{ [from equation (2.3.1)]} \tag{4.2.7}$$

$$\frac{c\omega}{k} = \frac{c}{c_c} \cdot \frac{c_c}{2m} \cdot \frac{2m}{k} \cdot \omega$$

$$= \zeta\, \omega_n \frac{2}{\omega_n^2}\, \omega \text{ [from equations (3.3.7), (3.3.6) and (2.3.1)]}$$

or $$\frac{c\omega}{k} = 2\zeta \frac{\omega}{\omega_n} \tag{4.2.8}$$

In the above equations (4.2.7) and (4.2.8), ω_n is the undamped natural frequency of the system and ζ is the damping factor.

Let $$\frac{F_0}{k} = X_{st} \tag{4.2.9}$$

where X_{st} may be defined as zero frequency deflection of the spring-mass system under a steady force F_0. It may not be confused with Δ_{st}, the static deflection of the spring-mass system under the supporting load. The value of Δ_{st} is equal to $\frac{W}{k}$ or $\frac{mg}{k}$ [equation (2.2.1)]. Substituting the expressions (4.2.7), (4.2.8) and (4.2.9) in equations (4.2.5a) and (4.2.6a), we have

$$X = \frac{X_{st}}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\zeta\frac{\omega}{\omega_n}\right]^2}} \tag{4.2.10}$$

and $$\phi = \tan^{-1}\left[\frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right] \tag{4.2.11}$$

Hence the particular solution of equation (4.2.3) may be

written as

$$x_p = \frac{X_{st} \sin(\omega t - \phi)}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\frac{\omega}{\omega_n}\right]^2}} \tag{4.2.12}$$

ϕ being given by equation (4.2.11).

Therefore the complete solution is $x = x_c + x_p$, the values of x_c and x_p being given by equations (4.2.2) and (4.2.12) respectively,

or,
$$x = A_2 e^{-\zeta\omega_n t} \sin[\sqrt{1-\zeta^2}\,\omega_n t + \phi_2] + \frac{X_{st} \sin(\omega t - \phi)}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\frac{\omega}{\omega_n}\right]^2}} \tag{4.2.13}$$

This is the complete solution to an under-damped system subjected to sinusoidal excitation. It is found to contain two arbitrary constants A_2 and ϕ_2, which have to be determined from this complete solution from the two initial conditions.

The first part of the complete solution, i.e. complementary function, is seen to decay with time, and vanishes ultimately. This part, in engineering practice, is commonly called as *Transient Vibrations*. The second part, i.e. the particular solution, is seen to be a sinusoidal vibration with a constant amplitude and is called as *Steady State Vibrations*. The reason is that after the transients die out, the complete solution consists of steady vibrations only. The complete solution is a superposition of transient and steady state vibrations and is shown in Fig. 4.2.3.

It may also be noticed that the trasnient vibrations take place at the damped natural frequency of the system, whereas the steady state vibrations occur at the frequency of excitation.

In case of forced vibrations without damping, equation (4.2.13) changes to

$$x = A_2 \sin(\omega_n t + \phi_2) + \frac{X_{st} \sin(\omega t - \phi)}{\left|1 - \left(\frac{\omega}{\omega_n}\right)^2\right|}$$

which ϕ as given by equation (4.2.11) is either 0° or 180° depending upon whether $\omega < \omega_n$ or $\omega > \omega_n$. This equation can

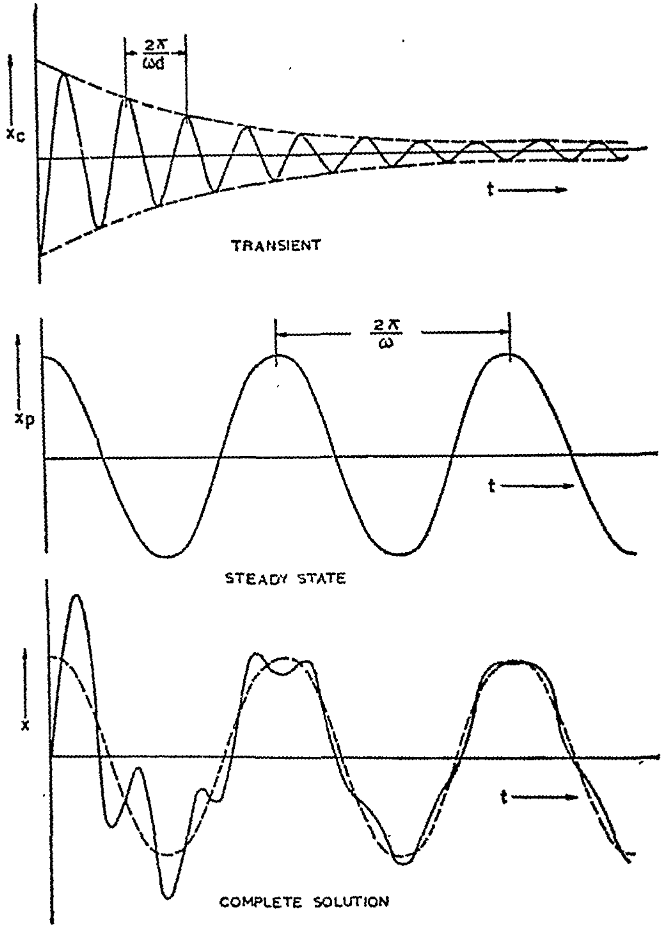


Fig. 4.2.3. Superposition of transient and steady state vibrations.

therefore be further simplified and written as

$$x = A_2 \sin\ (\omega_n t + \phi_2) + \frac{X_{st} \sin \omega t}{[1-(\omega/\omega_n)^2]} \tag{4.2.14}$$

In the equation (4.2.14), the first term, i.e. the transient term, does not die out theoretically. But under all practical conditions the transient vibrations do die out sooner or later.

4.2 A Steady state vibrations. For all practical systems subject to harmonic excitation, the transient vibrations die out

within a matter of short time, leaving only the steady state vibrations. Thus it is important to know the steady state behaviour of the system when subjected to different excitation frequencies. By the behaviour of the system we mean its steady state amplitude and the phase lag. Equations (4.2.10) and (4.2.11) are of much significance in this regard. In equation (4.2.10), the ratio of the steady state amplitude to the zero frequency deflection, i.e. $\frac{X}{X_{st}}$ is defined as the *Magnification Factor* and is denoted by M. F. It is the factor by which the zero frequency deflection is to be multiplied to get the amplitude. Equation (4.2.10) can be re-written in the following form.

$$\text{M.F.} = \frac{X}{X_{st}} = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\zeta\frac{\omega}{\omega_n}\right]^2}} \tag{4.2.15}$$

The equation (4.2.11) for the phase lag is re-written again for convenience.

$$\phi = \tan^{-1}\left[\frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right] \tag{4.2.16}$$

The dimensionless plots of magnification versus frequency ratio and phase lag versus frequency ratio for different values of damping factor are shown in Figs. 4.2.4 and 4.2.5. These curves reveal a lot of interesting and useful information regarding the behaviour of the system to sinusoidal excitation.

Curves of Fig. 4.2.4 are also known as *Frequency-Response curves*, since they give the response of the system to various frequencies. It is seen from these curves that the response of a particular system at any particular frequency is lower for higher value of damping. In other words, the curves for higher values of damping lie below those for lower values of damping. At zero frequency the magnification is unity and is independent of the damping, i.e. $X = X_{st}$, which itself is the definition of zero frequency deflection. At very high frequency the magnification tends to zero or the amplitude of vibration becomes very small. At resonance ($\omega = \omega_n$), the [illegible]

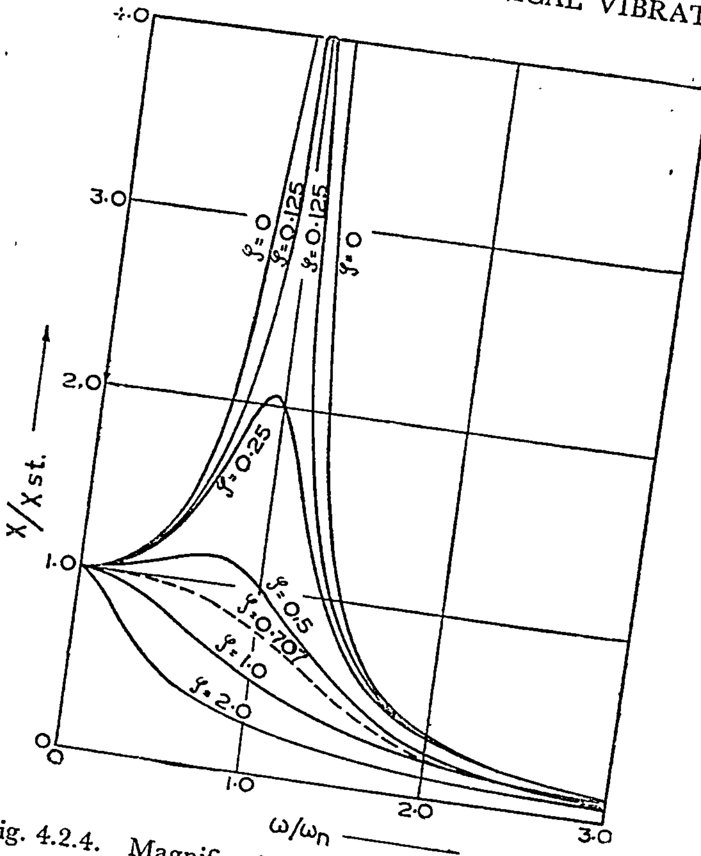


Fig. 4.2.4. Magnification v/s frequency ratio for different amounts of damping.

of vibration becomes excessive for small damping and decreases with the increase in damping. For zero damping at resonance the amplitude is infinite theoretically [(equation (4.2.15)]. Practically, however the system may go into destruction much before that or the amplitude may be limited because of some other factors.

The phase angle also varies from zero at low frequencies to 180° at very high frequencies. It is 90° at resonance and is independent of damping. Over a small frequency range containing the resonance point, the variation of phase angle is more abrupt for lower values of damping than for higher

values. More abrupt the change in phase angle about resonance, more sharp is the peak in the frequency response curve. For zero damping, the phase lag suddenly changes from zero to 180° at resonance. The corresponding zero-damping frequency-response curve is also infinitely sharp at resonance.

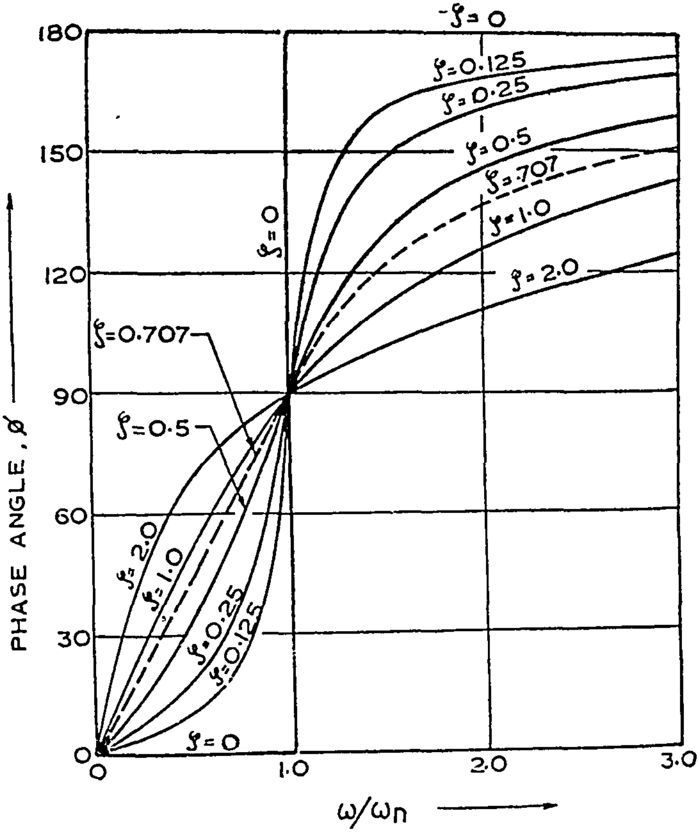


Fig. 4.2.5. Phase lag v/s frequency ratio for different amounts of damping.

Let us now study the phenomenon of Figs. 4.2.4 and 4.2.5 by means of the vector diagram and gain some more insight into what is happening in the system.

With reference to Fig. 4.2.2, at very low frequencies (ω very small), the inertia term $m\omega^2 X$ becomes negligibly small and damping term $c\omega X$ is also small. This gives rise to small

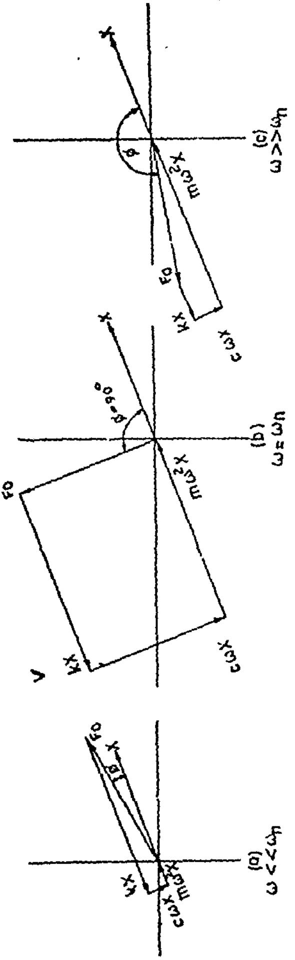


Fig. 4.2.6. Vector diagrams for forced vibrations for various operating condtions.

value of ϕ as shown in Fig. 4.2.6 (a). The impressed force F_0 is almost equal and opposite to the spring force kX under these conditions. Thus, for very low frequencies, the phase angle tends to zero and the impressed force wholly balances the spring force.

With increase in frequency, the damping force vector $c\omega X$ grows larger. Angle ϕ has also to increase so that component of F_0 perpendicular to x-direction may balance the increasing damping force. The inertia force vector grows much more

rapidly with increase in frequency because of the factor ω^2 in its expression. If we continue to increase the frequency, time comes when the spring force and the inertia force vectors are equal and opposite. This is shown in Fig. 4.2.6 (b) Under these conditions

$$kX = m\omega^2 X, \text{ or } \omega = \sqrt{\frac{k}{m}} = \omega_n$$

This is the resonance condition of the system and the vector diagram becomes a rectangle. The impressed force completely balances the damping force and $\phi = 90°$.

Therefore, $c\omega X = F_0$

or the amplitude at resonance is

$$X_r = \frac{F_0}{c\omega} = \frac{F_r/k}{c\omega/k}$$

$$= \frac{X_{st}}{2\zeta\left(\frac{\omega}{\omega_n}\right)}, \text{ from equations (4.2.9) and (4.2.8)}$$

$$= \frac{X_{st}}{2\zeta}, \text{ since at resonance } \omega = \omega_n.$$

Therefore, $$\boxed{\frac{X_r}{X_{st}} = \frac{1}{2\zeta}} \qquad (4.2.17)$$

Equation (4.2.17) can also be obtained from equation (4.2.15) by putting $(\omega/\omega_n) = 1$

At very high frequencies, inertia vector becomes very large and the damping force and the spring force vectors are negligibly small. Angle ϕ tends to 180° and the impressed force is wholly utilized to balance the inertia force, as shown in Fig. 4.2.6 (c).

Thus, the following conclusions may be mentioned.

1. At very low frequencies, the phase angle is zero and the impressed force balances the spring force.

2. At resonant frequency, the phase angle is 90° and the impressed force balances the damping force. Th amplitude at resonance is inversely proportional to the damping factor and is as given by equation (4.2.17). Also the spring force and the inertia force are equal and opposite at resonance.

3. At very high frequencies the phase angle is 180° and the impressed force balances the inertia force.

It may also be concluded that the phase angle lies between zero and 90° for frequencies below resonance and lies between 90° and 180° for frequencies above resonance.

Going back to Fig. 4.2.4, it is seen that the maximum amplitude occurs not at the resonant frequency but a little toward its left. This shift increases with the increase in damping. For zero damping, the maximum amplitude (infinite value), of course, is obtained at the resonant frequency.

The frequency at which the maximum amplitude occurs can be obtained from equation (4.2.15) by differentiating this equation with respect to (ω/ω_n) and equating this differential to zero.

$$\text{Or,}\quad \frac{d\,(\text{MF})}{d\left(\frac{\omega}{\omega_n}\right)}=-\frac{2\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]\left[-2\left(\frac{\omega}{\omega_n}\right)\right]+2\left[2\zeta\frac{\omega}{\omega_n}\right][2\zeta]}{2\left\{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\zeta\frac{\omega}{\omega_n}\right]^2\right\}^{3/2}}$$

$$=0$$

$$\text{which gives,}\quad \left(\frac{\omega_p}{\omega_n}\right)=\sqrt{1-2\zeta^2} \tag{4.2.18}$$

where ω_p means the frequency corresponding to the peak amplitude. No maxima or peak will occur when the expression within the radical sign is negative, i.e. for $\zeta > (\sqrt{\frac{1}{2}})$ or for $\zeta > 0.707$. It may also be seen in Fig. 4.2.4 that for value of $\zeta > 0{\cdot}707$, the response curve is always below the unity magnification line.

The peak frequency is different from the damped natural frequency of the system which is given by equation (3.3.16), as

$$\left(\frac{\omega_d}{\omega_n}\right)=\sqrt{1-\zeta^2} \tag{4.2.19}$$

Illustrative Example 4.2.1

A periodic torque having a maximum value of 6 kg-cm at a frequency corresponding to 4 radians per second is impressed upon a flywheel suspended from a wire. The wheel has a moment of inertia of 1200 kg-cm² and the wire has a stiffness

of 12 kg-cm/rad. A viscous dashpot applies damping couple of 4 kg-cm at an angular velocity of 1 rad/sec. Calculate

(a) the maximum angular displacement from rest position,

(b) the maximum couple applied to dashpot and

(c) the angle by which the angular displacement lags the torque.

Solution

The external excitation is

$$T = T_0 \sin \omega t$$

where,

$$T_0 = 6 \text{ kg-cm},$$

$$\omega = 4 \text{ rad/sec}.$$

The moment of inertia of the system is given to be 1200 kg-cm². This is the weight moment of inertia. The mass moment of inertia, therefore, is

$$J = \frac{1200}{980} = 1.223 \text{ kg-cm-sec}^2$$

$$k_t = 12 \text{ kg-cm/rad}$$

$$c_t = 4 \text{ kg-cm-sec/rad}$$

It is required to calculate θ and $c_t\omega\theta$, where θ is the torsional amplitude of vibration.

(a) With reference to Fig. 4.2.2 and equation (4.2.5), the equation for the torsional vibration amplitude can be obtained by replacing the translational terms in the above mentioned equation by torsional terms, i.e.

$$\theta = \frac{T_0}{\sqrt{(k_t - J\omega^2)^2 + (c_t\omega)^2}}$$

or
$$\theta = \frac{6}{\sqrt{(12 - 1.223 \times 16)^2 + (4 \times 4)^2}} = 0.338 \text{ radian.} \quad \textbf{Ans.}$$

(b) Maximum damping couple $= c_t\omega\theta = 4 \times 4 \times \cdot 338$

$= 5.4$ kg-cm. **Ans.**

(c) Applying equation (4.2 6) after changing the translational terms to torsional terms,

$$\phi = \tan^{-1}\left[\frac{c_t\omega}{k_t - J\omega^2}\right]$$

or $\quad \phi = \tan^{-1}\left[\frac{4\times 4}{12-1.223\times 16}\right] = \underbrace{\tan^{-1}(-2.114)}$ Low

or $\quad \phi = \underline{115.3°}$ **Ans.**

Illustrative Example 4.2.2

The damped natural frequency of a system as obtained from a free vibration test, is 9.8 cps. During the forced vibration test, with constant exciting force, on the same system, the maximum amplitude of vibration is found to be at 9.6 cps. Find the damping factor for the system and its natural frequency.

Solution

$$\omega_p = 9.6 \times 2\pi \text{ rad/sec}$$
$$\omega_d = 9.8 \times 2\pi \text{ rad/sec.}$$

Substituting the above quan. es in equations (4.2.18) and (4.2.19), we have

$$\frac{9.6\times 2\pi}{\omega_n} = \sqrt{1-2\zeta^2}$$

$$\frac{9.8\times 2\pi}{\omega_n} = \sqrt{1-\zeta^2}$$

Dividing one by the other,

$$\frac{9.6}{9.8} = \sqrt{\frac{1-2\zeta^2}{1-\zeta^2}}$$

Squaring,

$$0.96 = \frac{1-2\zeta^2}{1-\zeta^2}, \text{ which gives } \zeta = 0.196 \quad \textbf{Ans.}$$

Substituting this value of . n any of the above two equations, we get

$$\omega_n = 10\times 2\pi \text{ rad/sec}$$

or $\quad f_n = \dfrac{\omega_n}{2\pi} = 10$ cps. **Ans.**

4.3 Forced vibrations with rotating and reciproc ting unbalance.

An electric motor, a turbine and in fact, all other rot.ting machinery have some amount of unbalance left in them even

after correcting their unbalance on precision balancing machines. The final unbalance is measured in terms of an equivalent mass m_0 rotating with its centre of gravity at a distance e from the axis of rotation. This centrifugal force generated because of the rotation of the body is proportional to square of the frequency of rotation. This centrifugal force is the maximum value of the sinusoidal excitation in any direction. It varies with the speed of rotation and is different from the harmonic excitation discussed in Sec. 4.2, where the maximum force was independent of the frequency.

Fig. 4.3.1 represents an elastically supported machine rotating at ω rad/sec. Let the unbalance mass m_0 have an eccentricity e. Let m be the total mass of the machine including the unbalance mass m_0, and k and c the spring stiffness and the damping coefficient respectively in the direction of vibration. Let the mass m_0 make an angle ωt with the reference axis, at any instant. Then the centrifugal force $m_0\omega^2 e$ acts outward from the centre of rotation as shown in the figure. The equation of motion in the vertical direction can now be written as follows.

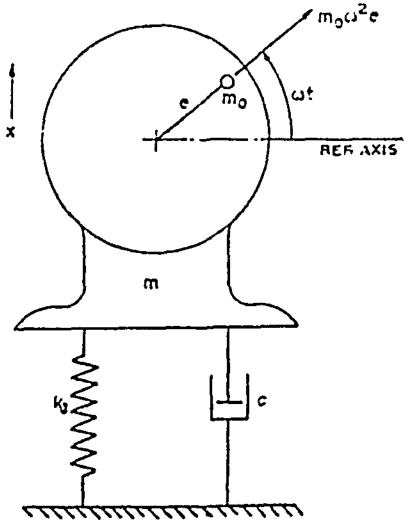


Fig. 4.3.1. Rotating unbalance.

$$(m-m_0)\frac{d^2x}{dt^2} + m_0\frac{d^2}{dt^2}(x+e\sin\omega t) = -kx - c\frac{dx}{dt}$$

or $$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = m_0e\omega^2\sin\omega t \tag{4.3.1}$$

If we compare this equation with equation (4.2.1), the only difference is that F_0 in the previous equation is replaced by $m_0e\omega^2$. Every thing else remains the same. Therefore, the transient part of the solution is the same as equation (4.2.2), considering, of course, that the system is under-damped. The steady state amplitude corresponding to equation (4.2.5a) becomes

$$X = \frac{m_0e\omega^2/k}{\sqrt{\left(1-\frac{m\omega^2}{k}\right)^2+\left(\frac{c\omega}{k}\right)}}$$

The above equation reduces to the following dimensionless equation

$$\frac{X}{\left(\frac{m_0e}{m}\right)} = \frac{\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\zeta\frac{\omega}{\omega_n}\right]^2}} \tag{4.3.2}$$

The equation gives the dimensionless steady state amplitude as a function of frequency ratio and damping factor. The equation for phase angle remains the same as equation (4.2.11), i.e.

$$\phi = \tan^{-1}\left[\frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right] \tag{4.3.3}$$

The plot of equation (4.3.2) is shown in Fig. 4.3.2. The plot for the phase lag remains the same as shown in Fig. 4.2.5.

At low speeds, the centrifugal exciting force $m_0e\omega^2$ is small, and therefore, all the response curves of Fig. 4.3.2. start from zero. At resonance when $\frac{\omega}{\omega_n} = 1$, we have

$$\frac{X}{\left(\frac{m_0e}{m}\right)} = \frac{1}{2\zeta} \tag{4.3.4}$$

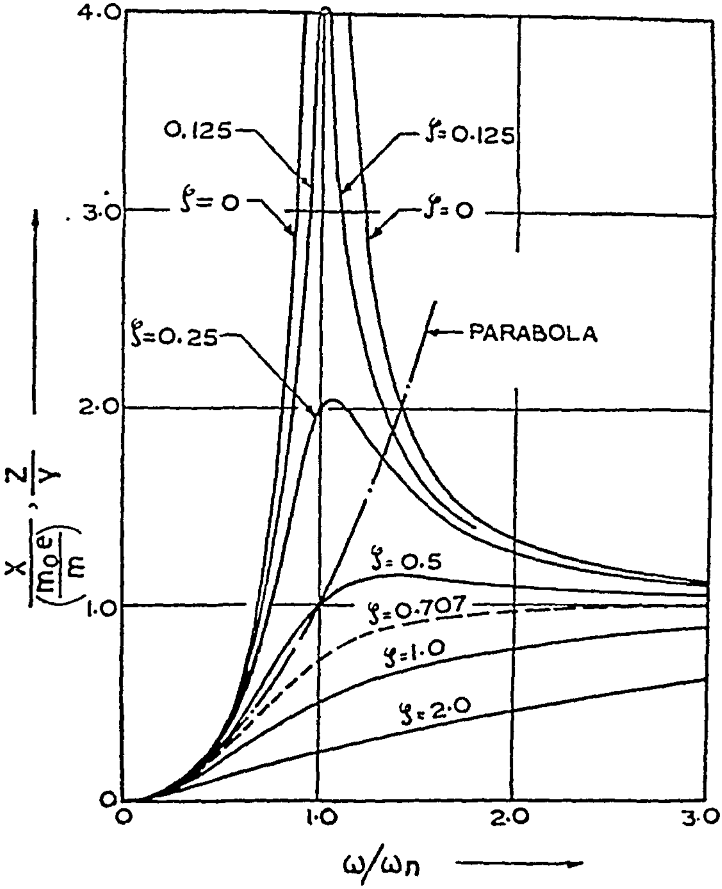


Fig. 4.3.2. Dimensionless amplitude v/s frequency ratio for different amounts of damping.

and the amplitude is limited by the damping present in the system. Under these conditions, the motion of mass $(m-m_o)$ lags that of mas m_o by 90° (see phase angle plot of Fig. 4.2.5) When (ω/ω_n) is very large, the ratio $X/\left(\frac{m_0 e}{m}\right)$ tends to unity and the main mass $(m-m_o)$ has an amplitude $X=\frac{m_o e}{m}$. This motion is 180° out of phase with the exciting force, i.e. when the unbalanced mass moves up, the main mass moves down and vice-versa. In such a case it can easily be shown that

the amplitude of vibration is such that the centre of gravity of the total system remains stationary.

This analysis of vibration with rotating unbalance can easily be extended to the case of reciprocating unbalanee.

Consider a reciprocating engine as shown in Fig. 4.3.3. Here, the equivalent mass of reciprocating parts is m_0 and

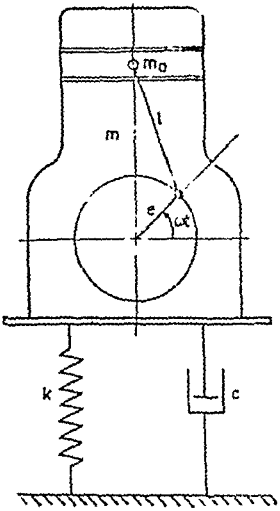


Fig. 4.3.3. Reciprocating unbalance.

the total mass of the engine including the reciprocating parts is m. The crank length and the connecting rod length are e and l respectively. The inertia force due to the reciprocating mass is approximately equal to $me\omega^2\,[\sin \omega t + (e/l) \sin 2\omega t]$. If e is small as compared to l, the second harmonic may be neglected and the exciting force becomes equal to $me\omega^2 \sin \omega t$, which is the same as that for the rotating unbalance discussed earlier in this section. Hence, if e is small, the previous analysis of rotating unbalance is applicable to this case of reciprocating unbalance.

Illustrative Example 4.3.1

A system of beams supports a motor weighing 1200 kg. The motor has an unbalanced weight of 1 kg located at 6.0

cm radius. It is known that the resonance occurs at 2210 r.p.m. What amplitude of vibration can be expected at the motor's operating speed of 1440 r.p.m. if damping factor is assumed to be less than 0·1.

Solution

Equation (4.3.2) will be used to find the amplitude of vibration.

In this case,

$$\frac{\omega}{\omega_n}=\frac{1440}{2210}=0{\cdot}652$$

$$\frac{m_0}{m}=\frac{1}{1200}$$

$$e=6.0$$

If $\zeta=0.1$,

$$\frac{X}{\frac{1}{1200}\times 6}=\frac{(\cdot 652)^2}{\sqrt{[1-(.652)^2]^2+[2\times 0{\cdot}1\times \cdot 652]^2}}$$

Solving the above equation gives $X=0.00362$ cm.

If $\zeta=0$,

$$\frac{X}{\frac{1}{1200}\times 6}=\frac{(\cdot 652)^2}{[1-(\cdot 652)^2]}$$

or $X=0.00370$ cm.

So, if the damping is less than 0.1 (that means it is between 0·1 and zero), the amplitude of vibration will lie between 0·00362 and 0·00370 cm. **Ans.**

Illustrative Example 4.3.2

A single cylinder vertical petrol engine of total weight 320 kg is mounted upon a steel chassis frame and causes a vertical static deflection of 0·2 cm. The reciprocating parts of the engine weigh 24 kg and move through a vertical stroke a 15 cm with S.H.M. A dashpot is provided, the damping resistance of which is directly proportional to the velocity and amounts to 50 kg at 30 cm/sec. Determine

(a) the speed of the driving shaft at which resonance will occur, and

(b) the amplitude of steady state forced vibrations when the driving shaft ot the engine rotates at 480 r.p.m.

Solution

(a) $W = 320$ kg

$\Delta_{st} = 0.2$ cm

Equation (2.3.9) gives

$$\omega_n = \sqrt{\frac{g}{\Delta_{st}}}$$

$$= \sqrt{\frac{980}{0.2}} = 70 \text{ rad/sec}$$

Therefore, resonant speed

$$= \frac{70}{2\pi} \times 60 = 670 \text{ c.p.m.}$$ **Ans.**

(b) $$\omega = \frac{480 \times 2\pi}{60} = 50{\cdot}4 \text{ rad/sec.}$$

$$\frac{\omega}{\omega_n} = \frac{50.4}{70} = 0{\cdot}72$$

$$\boxed{\zeta = \frac{c}{2m\,\omega_n}}$$

$$= \frac{(50/30)}{2 \times \left(\dfrac{320}{980}\right) \times 70} = 0{\cdot}0364$$

$$\frac{m_0}{m} = \frac{24}{320} = 0{\cdot}075$$

$$e = \frac{15}{2} = 7.5 \text{ cm}$$

Substituting these values in equation (4.3.2), we have

$$\frac{X}{{\cdot}075 \times 7.5} = \frac{(0{\cdot}72)^2}{\sqrt{[1-(0{\cdot}72)^2]^2 + [2 \times 0{\cdot}0364 \times 0{\cdot}72]^2}}$$

which, on solving gives

$$X = 0{\cdot}6 \text{ cm.}$$ **Ans.**

4.4 Forced vibrations due to excitation of the support.

In many cases, the excitation of the system is through the support or the base instead of being applied to the mass. In

this case, the support will be considered to be excited by a regular sinusoidal motion

$$y = Y \sin \omega t \tag{4.4.1}$$

as shown in Fig. 4.4.1.

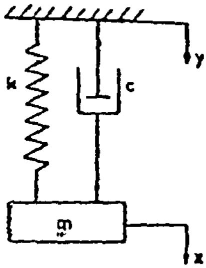


Fig. 4.4.1. Forced vibrations due to the excitation of the support.

4.4 A Absolute amplitude. If x is the absolute motion of the mass m, then the equation of motion can be written as follows.

$$m\ddot{x} = -c(\dot{x}-\dot{y}) - k(x-y)$$

or,
$$m\ddot{x} + c\dot{x} + kx = ky + c\dot{y} \tag{4.4.2}$$

Substituting for y from equation (4.4.1), we have,

$$m\ddot{x} + c\dot{x} + kx = Y[k \sin \omega t + c\omega \cos \omega t]$$

or
$$m\ddot{x} + c\dot{x} + kx = Y\sqrt{k^2+(c\omega)^2}\,\sin(\omega t+\alpha) \tag{4.4.3}$$

where
$$\alpha = \tan^{-1}\frac{c\omega}{k} = \tan^{-1}\left(2\zeta\frac{\omega}{\omega_n}\right) \tag{4.4.4}$$

Equation (4.4.3) is of the same form as equation (4.2.1), therefore the steady state solution is given by the equation similar to equation (4.2.3) or

$$x = X \sin(\omega t + \alpha - \phi) \tag{4.4.5}$$

where X, the steady state amplitude is given by the equation similar to equation (4.2.5), or

$$X = \frac{Y\sqrt{k^2+(c\omega)^2}}{\sqrt{(k-m\omega^2)^2+(c\omega)^2}}$$

Or, in dimensionless form,

(b) the amplitude of steady state forced vibrations when the driving shaft ot the engine rotates at 480 r.p.m.

Solution

(a) $W = 320$ kg

$\Delta_{st} = 0.2$ cm

Equation (2.3.9) gives

$$\omega_n = \sqrt{\frac{g}{\Delta_{st}}}$$

$$= \sqrt{\frac{980}{0.2}} = 70 \text{ rad/sec}$$

Therefore, resonant speed

$$= \frac{70}{2\pi} \times 60 = 670 \text{ c.p.m.}$$ **Ans.**

(b) $$\omega = \frac{480 \times 2\pi}{60} = 50{\cdot}4 \text{ rad/sec.}$$

$$\frac{\omega}{\omega_n} = \frac{50.4}{70} = 0{\cdot}72$$

$$\boxed{\zeta = \frac{c}{2m\,\omega_n}}$$

$$= \frac{(50/30)}{2 \times \left(\dfrac{320}{980}\right) \times 70} = 0{\cdot}0364$$

$$\frac{m_0}{m} = \frac{24}{320} = 0{\cdot}075$$

$$e = \frac{15}{2} = 7.5 \text{ cm}$$

Substituting these values in equation (4.3.2), we have

$$\frac{X}{{\cdot}075 \times 7.5} = \frac{(0{\cdot}72)^2}{\sqrt{[1-(0{\cdot}72)^2]^2 + [2 \times 0{\cdot}0364 \times 0{\cdot}72]^2}}$$

which, on solving gives

$$X = 0{\cdot}6 \text{ cm.}$$ **Ans.**

4.4 Forced vibrations due to excitation of the support.

In many cases, the excitation of the system is through the support or the base instead of being applied to the mass. In

this case, the support will be considered to be excited by a regular sinusoidal motion

$$y = Y \sin \omega t \tag{4.4.1}$$

as shown in Fig. 4.4.1.

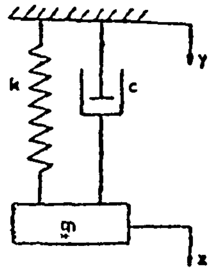


Fig. 4.4.1. Forced vibrations due to the excitation of the support.

4.4 A Absolute amplitude. If x is the absolute motion of the mass m, then the equation of motion can be written as follows.

$$m\ddot{x} = -c(\dot{x}-\dot{y}) - k(x-y)$$

or,
$$m\ddot{x} + c\dot{x} + kx = ky + c\dot{y} \tag{4.4.2}$$

Substituting for y from equation (4.4.1), we have,

$$m\ddot{x} + c\dot{x} + kx = Y[k \sin \omega t + c\omega \cos \omega t]$$

or
$$m\ddot{x} + c\dot{x} + kx = Y\sqrt{k^2+(c\omega)^2}\sin(\omega t+a) \tag{4.4.3}$$

where
$$a = \tan^{-1}\frac{c\omega}{k} = \tan^{-1}\left(2\zeta\frac{\omega}{\omega_n}\right) \tag{4.4.4}$$

Equation (4.4.3) is of the same form as equation (4.2.1), therefore the steady state solution is given by the equation similar to equation (4.2.3) or

$$x = X \sin(\omega t + a - \phi) \tag{4.4.5}$$

where X, the steady state amplitude is given by the equation similar to equation (4.2.5), or

$$X = \frac{Y\sqrt{k^2+(c\omega)^2}}{\sqrt{(k-m\omega^2)^2+(c\omega)^2}}$$

Or, in dimensionless form,

$$\frac{X}{Y} = \frac{\sqrt{1+\left(2\zeta\frac{\omega}{\omega_n}\right)^2}}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\zeta\frac{\omega}{\omega_n}\right]^2}} \tag{4.4.6}$$

And ϕ is given by equation (4.2.11), or

$$\phi = \tan^{-1}\left[\frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right] \tag{4.4.7}$$

Comparing equations (4.4.1) and (4.4.5), it is seen that the motion of the mass lags that of the support through an angle $(\phi-\alpha)$, or the angle of lag

$$\phi-\alpha = \tan^{-1}\left[\frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right] - \tan^{-1}\left[2\zeta\frac{\omega}{\omega_n}\right] \tag{4.4.8}$$

as obtained from equations (4.4.7) and (4.4.4).

Equations (4.4.5), (4.4.6) and (4.4.8) completely define the absolute motion of the body because of the support or base excitation.

From equation (4.4.6) we see that when $\omega << \omega_n$, $X/Y \simeq 1$, or the complete system moves as rigid body at slow frequencies. When $\omega >> \omega_n$, $X/Y \simeq 0$, or the body is stationary at high frequencies. The ratio X/Y as given by equation (4.4.6) is sometimes called *Displacement Transmissibility*. The curves for transmissibility and phase lag are drawn and discussed in Sec. 4.10.

4.4 B Relative amplitude. If z represents the relative motion of the mass with respect to the support, we have

$$z = x - y$$

or

$$x = y + z$$

Substituting for x from the above equation in equation (4.4.2), we have

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{y} \tag{4.4.9}$$

Substituting for y from equation (4.4.1),

$$m\ddot{z} + c\dot{z} + kz = m\omega^2 Y \sin\omega t \tag{4.4.10}$$

Equation (4.4.10) is of the same form as equation (4.3.1), and therefore, the steady state relative amplitude Z and the phase lag ϕ between the excitation and the relative displacement are given by

$$\frac{Z}{Y}=\frac{\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\zeta\frac{\omega}{\omega_n}\right]^2}} \tag{4.4.11}$$

$$\text{and } \phi=\tan^{-1}\left[\frac{2\zeta\,\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right] \tag{4.4.12}$$

The plots for these equations are shown in Figs. 4.3.2 and 4.2.5, respectively.

Illustrative Example 4.4.1

The time of free vibration of a mass hung from the end of a helical spring is 0.8 seconds. When the mass is stationary, the upper end is made to move upwards with displacement y centimeters given by

$$y=1.8\sin 2\pi t$$

where t is the time in seconds measured from the beginning of the motion.

Neglecting the mass of spring and any damping effects, determine the vertical distance through which the mass is moved in the first 0.3 seconds.

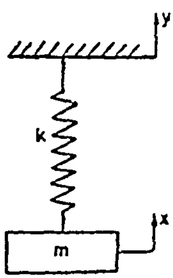


Fig. 4.4.2. An undamped spring-mass system with vibrating support.

Solution

Referring to Fig. 4.4.2, the differential equation of motion for the system is $m\ddot{x}=k\,(y-x)$

$$\text{or} \qquad m\ddot{x} + kx = ky$$

$$\text{or} \qquad m\ddot{x} + kx = kY \sin \omega t \tag{4.4.13}$$

where $Y = 1.8$ cm

and $\omega = 2\pi$ rad/sec.

The complete solution of equation (4.4.13) consists of complementary solution plus particular solution, i.e.

$$x = x_c + x_p$$

Since there is no damping in the system the complementary solution consists of either of the equations (3.3.15). Choosing the first of these and putting $\zeta = 0$,

$$x_c = A \cos \omega_n t + B \sin \omega_n t \tag{4.4.14}$$

The steady state solution is given by the equation (4.4.5), where the values of X and $(\phi - \alpha)$ are given by equations (4.4.6) and (4.4.8). Rewriting these equations for $\zeta = 0$, we have

$$x_p = X \sin (\omega t + \alpha - \phi)$$

$$\text{where} \quad X = \frac{Y}{\left| 1 - \left(\frac{\omega}{\omega_n} \right)^3 \right|}$$

$$(\phi - \alpha) = 0, \qquad \text{if } \frac{\omega}{\omega_n} < 1$$

$$= 180°, \quad \text{if } \frac{\omega}{\omega_n} > 1$$

$$\text{Now } \omega_n = \frac{2\pi}{\tau} = \frac{2\pi}{0.8}$$

$$\omega = 2\pi$$

$$\text{Therefore,} \quad \frac{\omega}{\omega_n} = 0.8$$

$$\text{and} \quad \phi - \alpha = 0$$

Hence

$$x_p = \frac{Y}{\left[1 - \left(\frac{\omega}{\omega_n} \right)^2 \right]} \sin \omega t \tag{4.4.15}$$

is the particular or the steady state solution.

The complete solution, then, is

$$x = A \cos \omega_n t + B \sin \omega_n t + \frac{Y}{\left[1 - \left(\frac{\omega}{\omega_n} \right)^2 \right]} \sin \omega t. \tag{4.4.16}$$

Putting in equation (4.4.16) and its derivative, the initial conditions

$$\left.\begin{array}{l} x = 0 \\ \dot{x} = 0 \end{array}\right] \text{at } t = 0,$$

we have,

$$0 = A$$

$$0 = B\,\omega_n + \frac{\Upsilon\omega}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]}$$

giving, $A = 0$

and $$B = -\frac{\Upsilon\left(\frac{\omega}{\omega_n}\right)}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]}$$

Substituting the above values of constants in equation (4.4 16), we have

$$x = -\frac{\Upsilon\left(\frac{\omega}{\omega_n}\right)}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]}\sin\omega_n t + \frac{\Upsilon}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]}\sin\omega t$$

or $$x = \frac{\Upsilon}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]}\left[\sin\omega t - \left(\frac{\omega}{\omega_n}\right)\sin\omega_n t\right] \qquad (4{\cdot}4.17)$$

when $t = 0.3$

$$\sin\omega t = \sin(2\pi\times 0.3) = \sin 108° = 0.95$$

$$\sin\omega_n t = \sin\left(\frac{108}{0{\cdot}8}\right) = \sin 135° = 0.707$$

Substituting in equation (4.4.17),

$$x = \frac{1.8}{[1-(0.8)^2]}[0.95 - 0.8\times 0.707]$$

$$= 5\,[0.95 - 0.566] = 1.92 \text{ cm}$$

or $x = 1.92$ cm. **Ans.**

Illustrative Example 4.4.2

The support of a spring-mass system is vibrating with an amplitude of 5 mm and a frequency of 1150 cycles/min. If the mass weighs 0.9 kg and the spring has a stiffness of 2 kg/cm,

determine the amplitude of vibration of the mass. What amplitude will result if a damping factor of 0.2 is included in the system.

Solution

We are interested in the steady state amplitude only.

$$W = 9.0 \text{ kg}$$

$$k = 2 \text{ kg/cm}$$

Therefore, $\omega_n = \sqrt{\frac{kg}{W}} = \sqrt{\frac{2\times 980}{0.9}} = 46.7 \text{ rad/sec}$

$$\omega = 1150\times\frac{2\pi}{60} = 120.3 \text{ rad/sec}$$

$$\frac{\omega}{\omega_n} = \frac{120.3}{46.7} = 2.58$$

Applying equation (4.4.6) after putting $\zeta = 0$,

$$\frac{X}{0.5} = \left|\frac{1}{1-(2.58)^2}\right| = \frac{1}{5.65}$$

or $\quad X = 0.0886$ cm. **Ans.**

when $\zeta = 0.2$, applying again equation (4.4.6),

$$\frac{X}{0.5} = \frac{\sqrt{1+(1.03)^2}}{\sqrt{(5.65)^2+(1.03)^2}} = \frac{1.435}{5.75}$$

or $\quad X = 0.125$ cm. **Ans.**

Illustrative Example 4.4.3

The springs of an automobile trailer are compressed 10 cm under its own weight. Find the critical speed when the trailer is travelling over a road with a profile approximated by a sine wave of amplitude 8 cm. and wave length of 14 meters. What will be the amplitude of vibration at 60 km/hour.

Solution

Equation (2.3.9) gives

$$\omega_n = \sqrt{\frac{g}{\Delta_{st}}} = \sqrt{\frac{980}{10}} = 9.9 \text{ rad/sec.}$$

or $\quad f_n = \dfrac{9.9}{2\pi}$ cycles/sec.

Let the critical speed of the trailer be v meters/sec.

Then the corresponding forcing frequency (see Fig. 4.4.3) is $v/14$ cycles/sec. This must be equal to the natural frequency f_n.

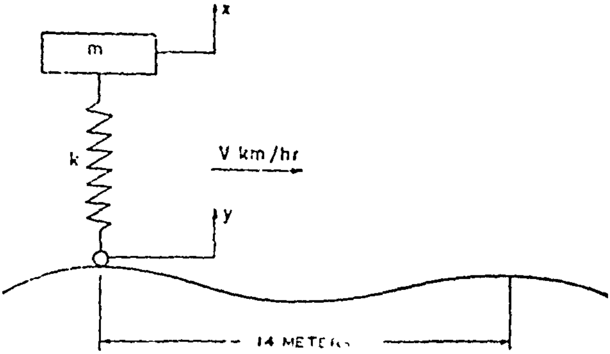


Fig. 4.4.3. Automobile trailer travelling on a wavy road.

Therefore $\dfrac{v}{14} = \dfrac{9.9}{2\pi}$

or $\quad v = 22{\cdot}1$ meters/sec.

This gives $V = \dfrac{22{\cdot}1 \times 60 \times 60}{1000}$ km/hr

or $\quad V = 79{\cdot}5$ km/hour. **Ans.**

The value of the exciting frequency ω corresponding to a speed of 60 km/hr is given by

$$\omega = \frac{60 \times 1000}{60 \times 60} \times \frac{2\pi}{14} \text{ rad/sec.}$$

or $\quad \omega = 7{\cdot}47$ rad/sec.

$$\frac{\omega}{\omega_n} = \frac{7{\cdot}47}{9{\cdot}9} = 0{\cdot}755$$

Applying equation (4·4·6) after putting $\zeta=0$, we have

$$\frac{X}{8} = \left|\frac{1}{1-(0{\cdot}755)^2}\right|$$

which gives $X = 18.6$ cm. **Ans.**

4.5 Energy dissipated by damping.

When a system is undergoing steady state forced vibrations with viscous damping, energy is continuously being absorbed by the dashpot. This energy absorbed per cycle can be determined from Sec. 1·7 where the expression for work done per cycle by a harmonic force on a harmonic motion was found out to be

$$\text{W.D.} = \pi P_0 x_0 \sin \phi$$

where, x_0 = amplitude of vibratory motion,

P_0 = amplitude of vibratory force,

and ϕ = the angle by which the motion lags the force.

For the case under study,

$$x_0 = X$$

$$\left.\begin{aligned} P_e &= c\omega X \\ \phi &= -90^\circ \end{aligned}\right\} \text{See the vector diagram of Fig. 4.2.2.}$$

Therefore,

$$\begin{aligned} \text{W.D./cycle} &= \pi\,(c\omega X) \times (-1) \\ &= -\pi\, c\omega X^2 \end{aligned}$$

or the energy dissipated per cycle is given by

$$\text{E.D./cycle} = \pi\, c\omega X^2 \qquad (4.5.1)$$

From this, the energy dissipated per second or the horse power can be calculated.

Illustrative Example 4.5.1

Determine the horse power required to vibrate a spring-mass-dashpot system with an amplitude of 1·5 cm and at a frequency of 100 cps. The system has a damping factor of 0·05 and a damped natural frequency of 22 cps as found out from the free vibration record. The mass of the system weighs 0·5 kg.

Solution

Applying equation (4·5·1),

energy dissipated per cycle $= \pi\, c\omega X^2$

Now, $\omega = 100 \times 2\pi$ rad/sec

$X = 1{\cdot}5$ cm

To find c, we apply equation (3.3.6) and (3.3.7),

or $\qquad c = 2m\,\omega_n\;\zeta$

but $\qquad \omega_n = \omega_d$ when ζ is small (see Fig. 3.3.6).

Therefore $\qquad c = 2 \times \dfrac{0.5}{980} \times (22\times 2\pi) \times 0.05$

$$= 0.00707 \text{ kg-sec/cm.}$$

Hence E.D./cycle $= \pi \times \cdot 00707 \times (200\pi) \times 1.5^2$ kg-cm.

E.D./sec $= \pi \times .00707 \times 200\pi \times 2.25 \times 100$kg-cm.

$= \pi \times .00707 \times 200\pi \times 2.25$ kg-meters.

H.P. $= \dfrac{\pi \times .00707 \times 200\pi \times 2.25}{75} = 0.42$ **Ans.**

4.6 Forced vibrations with Coulomb damping.

In the case of forced vibrations with Coulomb damping we will consider an equivalent viscous damping c_e such that the energy absorbed per cycle is the same in both the cases.

If X is the amplitude of steady state vibration and F the constant friction force, the energy absorbed per cycle is $4Fx$. For the same amplitude of vibration, the energy absorbed per cycle for the case of equivalent viscous damping is $\pi c_e \omega X^2$ [from equation (4.5.1)].

Equating the two expressions,

$$\pi c_e \omega X^2 = 4FX$$

or
$$c_e = \frac{4F}{\pi \omega X} \qquad (4.6.1)$$

The steady state amplitude for a system having viscous damping is given by equation (4.2.5a),

i.e.
$$X = \frac{F_0/k}{\sqrt{\left[1 - \frac{m\omega^2}{k}\right]^2 + \left[\frac{c\omega}{k}\right]^2}}$$

Substituting for c the value of equivalent viscous damping as obtained in equation (4.6.1), we have

$$X = \frac{F_0/k}{\sqrt{\left[1 - \frac{m\omega^2}{k}\right]^2 + \left[\frac{4F}{\pi X k}\right]^2}}$$

After putting $(k/m) = \omega_n^2$ in the above equation and solving for X, gives,

$$\frac{X}{F_0/k} = \frac{\sqrt{1-\left(\frac{4F}{\pi F_0}\right)^2}}{1-\left(\frac{\omega}{\omega_n}\right)^2} \tag{4.6.2}$$

The amplitude will have a real value if

$$\frac{4F}{\pi F_0} < 1$$

or

$$\frac{F}{F_0} < \frac{\pi}{4} \tag{4.6.3}$$

In most cases the friction force F is small and the equation (4.6.2) holds good. As the friction force increases, this equation becomes approximate only, till when $(F/F_0 > \pi/4)$, the equation (4.6.2) caeses to hold good.

It may be noticed that at the resonance $(\omega = \omega_n)$, the amplitude becomes infinite although friction damping is present in the system. This may look to be a little strange, but can be explained by consideration of energy input and energy dissipated. The energy input per cycle is proportional to the amplitude of the system and the energy dissipated per cycle by Coulomb damping is also proportional to the amplitude $(= 4\,FX)$. Thus if the friction damping force is small, the energy dissipated is always less than the input energy and therefore, the amplitude increases without limit. In the case of viscous damping, the energy dissipated is proportional to the square of the amplitude $(= \pi c \omega X^2)$, and even if the damping is small, increasing amplitude makes the dissipation of energy increase rapidly with amplitude and the stage comes when the input and absorbed energies balance. That decides the amplitude in a system with viscous damping.

Illustrative Example 4.6.1

A horizontal spring-mass system subjected to dry friction damping has the following physical data.

Weight of the mass	= 3.7 kg
Spring constant of the spring	= 7.7 kg/cm.

Coefficient of friction between the mass and horizontal plane on which it slides $= 0.22$

The mass is subjected to a sinusoidal forcing functions of amplitude 2 kg and frequency 5 c.p.s. Find the amplitude of vibration of the mass.

Also calculate the equivalent viscous damping.

Solution

$$F = \mu R = 0.22 \times 3{\cdot}7 = 0{\cdot}813 \text{ kg}$$

$$F_0 = 2 \text{ kg.}$$

$$k = 7{\cdot}7 \text{ kg/cm.}$$

$$\omega = 5 \times 2\pi \text{ rad/sec.}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{7{\cdot}7 \times 980}{3{\cdot}7}} = 45{\cdot}1 \text{ rad/sec.}$$

$$\frac{\omega}{\omega_n} = \frac{5 \times 2\pi}{45{\cdot}1} = 0{\cdot}697$$

Substituting the above values in equation (4.6.2), we have

$$\frac{X}{(2/7{\cdot}7)} = \frac{\sqrt{1 - \left(\frac{4}{\pi} \times \frac{0.813}{2}\right)}}{1-(0.697)^2}$$

which gives, $X = 0.41$ cm. **Ans.**

Equivalent viscous damping

$$c_e = \frac{4F}{\pi\omega X}$$

$$= \frac{4 \times 0{\cdot}813}{\pi \times 10\pi \times 0.41} = 0{\cdot}0805$$

or $c_0 = 0{\cdot}0805$ kg-sec/cm. **Ans.**

4.7 Forced vibrations with Coulomb and viscous damping.

Consider a system having a compound damping consisting of Coulomb damping force F, and viscous damping coefficient c_1 in parallel. Then the equivalent damping coefficient is given by

$$c_e = c_1 + \frac{4F}{\pi\omega X}$$

the second part of the equivalent damping coefficient being the same as that given in equation (4.6.1).

Substituting in equation (4.2.5a) for c, the value of c_e as given in equation (4.7.1), we have

$$X = \frac{F_0/k}{\sqrt{\left[1 - \frac{m\omega^2}{k}\right]^2 + \left[\left(c_1 + \frac{4F}{\pi\omega X}\right)\frac{\omega}{k}\right]^2}}$$

The above equation can be written down in the following manner.

$$\left\{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right] + \left(\frac{c_1\,\omega}{k}\right)^2\right\}X^2 + \left[\frac{2F\,c_1\,\omega}{\pi k^2}\right]X$$
$$\times\left[\left(\frac{4F}{\pi k}\right)^2 - \left(\frac{F_0}{k}\right)^2\right] = 0 \qquad (4.7.2)$$

Solving the above equation for X, gives the amplitude of vibration of the system having both Coulomb and viscous damping.

If, in this equation, $c_1 = 0$ (i.e. no viscous damping), the amplitude obtained corresponds to that of equation (4.6.2). If on the other hand, $F = 0$ (i.e. no Coulomb damping), the amplitude obtained is the same as that given by equation (4.2.5a).

4.8 Determination of equivalent viscous damping from frequency-response curve.

The damping in a system, as was pointed out in Chap. 3, can be obtained from free vibration decay curve. Where the free vibration test is not practical, the damping may be obtained from the frequency-response curve of forced vibration test.

Suppose the frequency-response curve as obtained for a system excited with a constant force, is that shown in Fig. 4.8.1. The magnification at resonance is given by $1/2\zeta$ as obtained in equation (4.2.17). It is a little difficult, however, to get the exact resonance point since the peak point occurs slightly away from the resonance. If the amplitude of vibration or the magnification can be found out at resonance, then the damping factor is given by

$$\zeta = \frac{1}{2(\text{M.F.})_{\text{res}}}$$

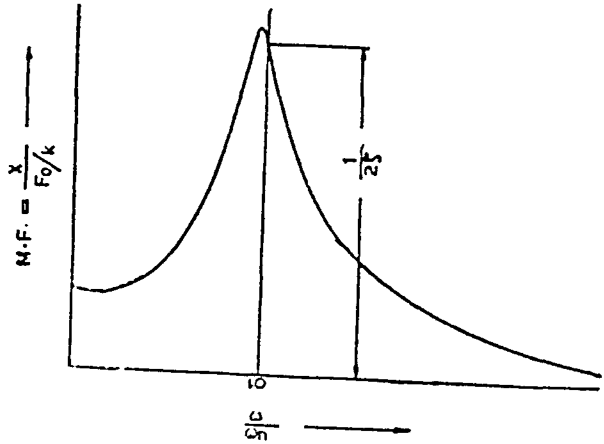


Fig. 4.8.1. Determination of equivalent viscous damping from the resonant amplitude.

The fact that the phase difference between the exciting force and the displacement is 90° at resonance, is made use of in locating the resonant point. Two signals, one corresponding to the exciting force and the other corresponding to the displacement, are fed to an X-Y plotter or to the two beams of

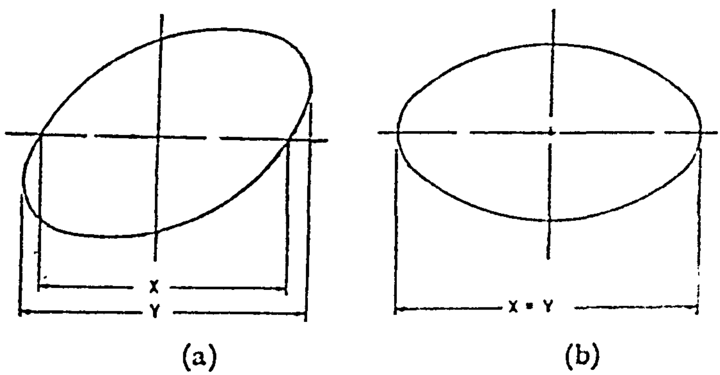


Fig. 4.8.2. Determination of phase difference between two harmonic signals of the same frequency.

a double beam oscilloscope. Since the two signals are of the same frequency. the Lissajous figure obtained on the oscilloscope screen is an ellipse, as shown in Fig. 4.8.2 (a) The phase

difference ϕ between the force and the displacement, then, is given by

$$\phi = \sin^{-1} \frac{X}{Y} \tag{4.8.1}$$

where, X = intercept of the ellipse on one of the central axes, and

Y = total length of the ellipse along the same axis.

The phase difference will be 90° when $X = Y$, i.e. when the ellipse becomes symmetrical, as shown in Fig. 4.8.2 (b). The excitation frequency can be adjusted till the ellipse obtained is completely symmetrical. The symmetricity of the ellipse can easily be checked visually. At this excitation, the amplitude can be obtained, which is the amplitude at resonance. The damping factor can be obtained from the corresponding magnification.

There is another method available for finding the damping in the system provided the damping is small and is of viscous nature. This is illustrated in the following example.

Illustrative Example 4.8.1

Describe how would you determine the damping in a system if you have just a frequency-response curve of the system with constant excitation. It being assumed that the damping is of viscous nature.

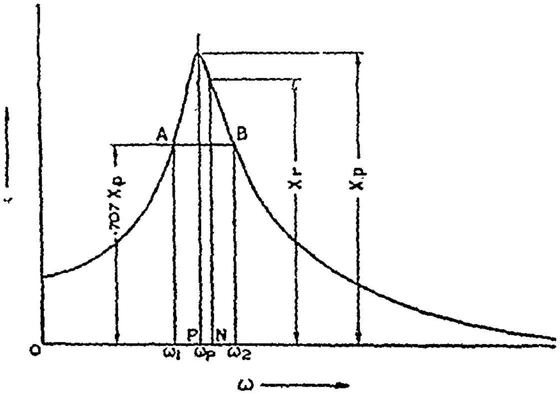


Fig. 4.8.3. Determination of equivalent viscous damping from the frequency-response curve.

Solution

Let Fig. 4.8.3 be the frequency response curve of the system obtained from forced vibration test. As a first approximation let the resonance frequency be the peak frequency. Then eqation (4.2.17) becomes

$$\frac{X_p}{X_{st}} = \frac{1}{2\zeta} \tag{4.8.2}$$

Now draw a horizontal line at $X = 0.707 \ X_p$, cutting the response curve at two points, the corresponding values of abscissae being ω_1 and ω_2.

Rewriting equation (4.2.15) for points A and B of Fig. 4.8.3, we have

$$\frac{0.707\, X_p}{X_{st}} = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_p}\right)^2\right]^2+\left[2\,\zeta\frac{\omega}{\omega_p}\right]^2}} \tag{4.8.3}$$

In the above equation ω_p has been written in place of ω_n because it is taken that $\omega_n = \omega_p$ as a first approximation. Also the values of ω in this equation correspond to ω_1 and ω_2.

From equations (4.8.2) and (4.8.3), we have

$$\frac{0.707}{2\zeta} = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_p}\right)^2\right]^2+\left[2\,\zeta\frac{\omega}{\omega_p}\right]^2}}$$

which yields the equation

$$\left(\frac{\omega}{\omega_p}\right)^4 - 2\ (1-2\,\zeta^2)\left(\frac{\omega}{\omega_p}\right)^2 + (1-8\zeta^2) = 0$$

Solving, we get

$$\left(\frac{\omega}{\omega_p}\right)^2 = (1-2\zeta^2) \pm 2\zeta\sqrt{1+\zeta^2}$$

Assuming $\zeta << 1$, we have

$$\left(\frac{\omega}{\omega_p}\right)^2 = 1 \pm 2\,\zeta$$

that is, $\left(\frac{\omega_1}{\omega_p}\right)^2 = 1-2\,\zeta$

and, $\left(\frac{\omega_2}{\omega_p}\right)^2 = 1+2\,\zeta$

From the above two equations, we get

$$\frac{\omega_2^2 - \omega_1^2}{\omega_p^2} = 4\zeta$$

Putting $\frac{\omega_2 + \omega_1}{2} = \omega_p$

we finally have

$$\frac{\omega_2 - \omega_1}{\omega_p} = 2\zeta \tag{4.8.4}$$

or $$\frac{AB}{OP} = 2\zeta$$

or $$\zeta = \frac{1}{2}\frac{AB}{OP}$$

Taking these measurements from the frequency response curve after making necessary construction, we get the first approximate value of ζ.

To get an accurate value of ζ, use ζ as found above to determine the natural frequency of the system from equation (4.2.18), i.e.

$$\omega_n = \frac{\omega_p}{\sqrt{1-2\zeta^2}}$$

Mark this point as N on the abscissa and draw an ordinate. The height of this ordinate upto the curve gives the resonance amplitude X_r.

Now repeat the process by first drawing a horizontal line at a height $X = 0{\cdot}707\ X_r$ and then finding the frequencies ω_1' and ω_2' corresponding to the points of intersection. Then in a similar manner it can be shown that

$$2\zeta = \frac{\omega_2' - \omega_1'}{\omega_n} \tag{4.8.4a}$$

Hence an accurate value of ζ can be found by construction and measurements.

4.9 Forced vibrations of an undamped system having non-harmonic excitation.

If the external force $F(t)$ acting on an undamped single degree of freedom system is not sinusoidal but is periodic

alright, then $F(t)$ may be expanded in a Fourier series as shown in Sec. 1.8, or

$$F(t) = P_0 + \sum_{n=1}^{\infty} P_n \cos n\omega t + \sum_{n=1}^{\infty} Q_n \sin n\omega t \tag{4.9.1}$$

where the constants P_0, P_n and Q_n are to be determined from $F(t)$.

The differential equation of the system now becomes,

$$m\ddot{x} + kx = P_0 + \sum_{n=1}^{\infty} P_n \cos n\omega t + \sum_{n=1}^{\infty} Q_n \sin n\omega t \tag{4.9.2}$$

If we are interested in the steady state vibrations only, the steady state response to the complete forcing function $F(t)$ is the vector sum of the responses to the individual forcing functions as on the right hand side of equation (4.9.2).

The steady state response to the constant force P_0 is equal to P_0/k. This can be obtained from equation (4.2.5) by putting $c=0$, $\omega=0$ and replacing F_0 by P_0. There might be a slight confusion here because by putting $\omega=0$, the forcing function $F_0 \sin \omega t$ used in Sec. 4.2 reduces to zero. But for that matter we can consider the forcing function there as $F_0 \cos \omega t$, and the steady state amplitude does not depend upon whether the forcing is of sine or cosine type.

The steady state response to the other cosine and sin functions of equation (4.9.2) can be individually obtained in a manner similar to that discussed in Sec. 4.2. The complete response will be the sum of all the individual responses. The following example will further clarify the procedure.

Illustrative Example 4.9.1

Obtain the steady state response of an undamped single degree of freedom system to an alternating square wave excitation.

Solution

The fourier series representation of a square wave function $F(t)$ of amplitude F_0 is given by

$$F(t) = \frac{4F_0}{\pi}\left[\sin \omega t + \frac{1}{3} \sin 3\omega t + \frac{1}{5} \sin 5\omega t \ldots\right] \tag{4.9.3}$$

The equation of motion of the system becomes

$$m\ddot{x} + kx = \frac{4F_0}{\pi}\left[\sin \omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{5}\sin 5\omega t + \cdots\right] \tag{4.9.4}$$

The steady state solution for each one of the component forces in the above equation can be written as second part of equation (4.2.14) where X_{st} and ω for each component solution will be replaced by P_n/k and $n\omega$ respectively.

Therefore, the steady state solution is

$$x = \frac{4F_0}{\pi k}\frac{\sin \omega t}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]} + \frac{4F_0}{3\pi k}\frac{\sin 3\omega t}{\left[1-\left(\frac{3\omega}{\omega_n}\right)^2\right]} + \frac{4F_0}{5\pi k}\frac{\sin 5\omega t}{\left[1-\left(\frac{5\omega}{\omega_n}\right)^2\right]} + \cdots\cdots$$

or

$$x = \frac{4F_0}{\pi k}\left[\frac{\sin \omega t}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]} + \frac{1}{3}\frac{\sin 3\omega t}{\left[1-\left(\frac{3\omega}{\omega_n}\right)^2\right]} + \frac{1}{5}\frac{\sin 5\omega t}{\left[1-\left(\frac{5\omega}{\omega_n}\right)^2\right]} + \cdots\right] \tag{4.9.5}$$

If in the above equation any odd integral multiple of ω is equal to ω_n, then the system will go into resonance at the corresponding fractional value of $\frac{\omega}{\omega_n}$. In the present case, the frequency-response curve will consist of a series of peaks at $\frac{\omega}{\omega_n} = 1, \frac{1}{3}, \frac{1}{5}$ and so on. If $\frac{\omega}{\omega_n}$ is near $\frac{1}{3}$, the second term of equation (4·9·5) governs most of the motion and the other terms may be neglected in comparison to it. Similarly if $\frac{\omega}{\omega_n}$ is near $\frac{1}{5}$, the third term predominates, and so on.

4.10 Vibration isolation and transmissibility.

In the beginning of Chap. 1 it was stated that many kinds of vibrations are undesirable and therefore should be eliminated or, at least, reduced. For example, the inertia forces developed

in a reciprocating engine or unbalanced forces produced in any other rotating machinery should be isolated from the foundation so that the adjoining structure is not set into heavy vibrations. Another example may be the isolation of delicate instruments from their supports which may be subjected to certain vibrations.

In either case the effectiveness of isolation may be measured in terms of the force or motion transmitted to that in existance. The first type is known as *force isolation* and the second type as *mo tion isolation.* The lesser the force or motion transmitted the greater is said to be the isolation.

In this section, we discuss the ways of reducing transmitted vibrations by vibration isolation, which is obtained by placing properly-chosen isolator materials between the vibrating body and the supporting structure. The isolating materials may be pads of rubber, felt or cork, or metallic springs. All these isolating materials are elastic and possess damping properties. The force or motion transmitted can be evaluated as in the following paragraphs.

4.10 A. Force transmissibility. The term transmissibility in the case of force-excited system is defined as the ratio of the force transmitted to the foundation to that impressed upon the system. Imagine a mass m supported on the foundation by means of an isolator having an equivalent stiffness and damping coefficient k and c respectively, and excited by a force $F_0 \sin \omega t$, as shown in Fig. 4.10.1 (a). Under steady state conditions, force acting on the mass can be represented by means of a vector diagram as shown in Fig. 4.2.2 (a), and reproduced in Fig. 4.10.1 (b). Out of these four force acting on the mass, the spring force kX and the dashpot forces $c\omega X$ are the two common forces acting on the mass and also on the foundation. Therefore the force transmitted to the foundation is the vector sum of these two forces acting in directions opposite to that on the mass, as has been shown in Fig. 4.10.1 (c). These forces are 90° out of phase with each other and their vector sum F_{tr} is the force transmitted to the foundation.

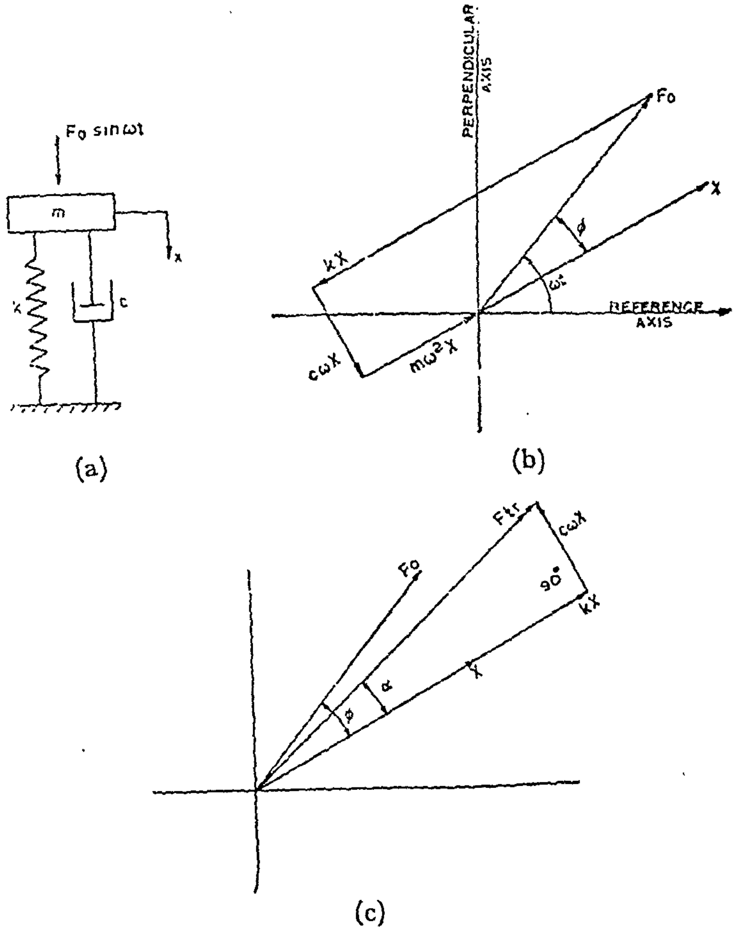


Fig. 4.10.1. Transmissibility determination from vector diagram.

Therefore $\quad F_{tr} = \sqrt{(kX)^2+(c\omega X)^2}$

or $\quad F_{tr} = X\sqrt{k^2+(c\omega)^2}$

Substituting the value of X from equation (4.2.5), we have.

$$F_{tr} = \frac{F_0\sqrt{k^2+(c\omega)^2}}{\sqrt{(k-m\omega^2)^2+(c\omega)^2}} \tag{4.10.1}$$

which, in dimensionless form, can be written as

$$\mathcal{T}_r = \frac{F_{tr}}{F_0} = \frac{\sqrt{1+\left(2\,\zeta\,\dfrac{\omega}{\omega_n}\right)^2}}{\sqrt{\left[1-\left(\dfrac{\omega}{\omega_n}\right)^2\right]^2+\left[2\,\zeta\dfrac{\omega}{\omega_n}\right]^2}} \tag{4.10.1}$$

$\mathcal{T}_r$ being the transmissibility.

The angle through which the transmitted force lags the impressed force can be seen from Fig. 4.10.1 (c) to be $(\phi-a)$,

$$\text{where}\quad a = \tan^{-1}\left[\frac{c\omega X}{kX}\right]$$

$$= \tan^{-1}\left[\frac{c\omega}{k}\right]$$

$$= \tan^{-1}\left[2\,\zeta\frac{\omega}{\omega_n}\right]$$

Angle ϕ *is given by equation* (4.2.11). Therefore the angle of lag is equal to

$$\phi-a = \tan^{-1}\left[\frac{2\zeta\dfrac{\omega}{\omega_n}}{1-\left(\dfrac{\omega}{\omega_n}\right)^2}\right]-\tan^{-1}\left[2\zeta\,\frac{\omega}{\omega_n}\right] \tag{4.10.3}$$

Equations (4.10.2) and (4.10.3) give the transmissibility and the phase lag of transmitted force from the impressed force, and are plotted in Figs. 4.10.2 and 4.10.3 respectively, with different damping factors.

The transmissibility curve of Fig. 4.10.2 gives us a lot of useful information. The first thing that we see is that all the curves start from unity value of transmissibility, pass through the unity transmissibility at $(\omega/\omega_n) = \sqrt{2}$ and after that they tend to zero as $(\omega/\omega_n)\to\infty$. These curves can be divided into two ranges, one from $(\omega/\omega_n)=0$ to $(\omega/\omega_n)=\sqrt{2}$, and the second from $(\omega/\omega_n) = \sqrt{2}$ to $(\omega/\omega_n)\to\infty$.

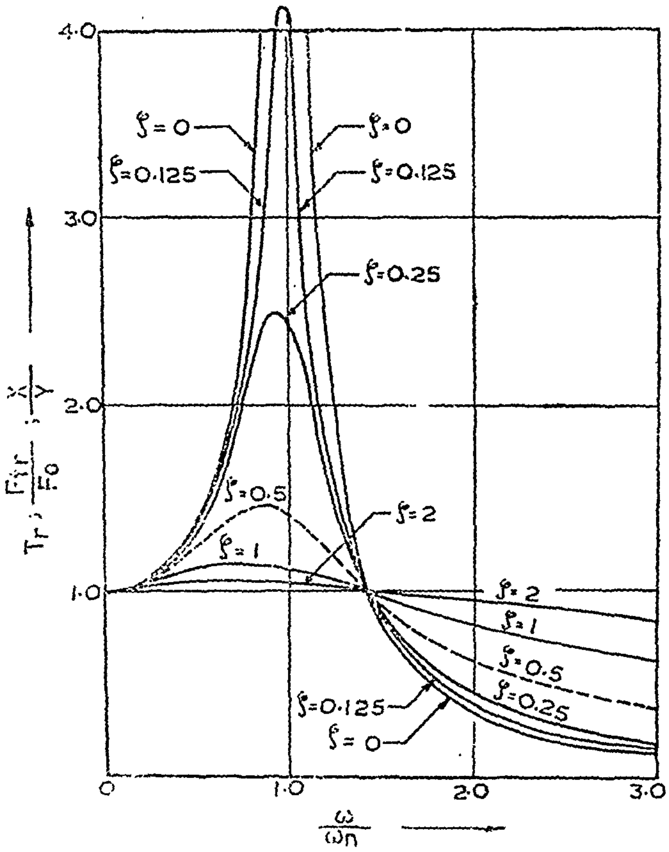


Fig. 4.10.2. Transmissibility v/s frequency ratio for different amounts of damping.

In the first range greater amount of damping gives lower transmissibility, which always remains greater than unity. In the second range greater amount of damping gives greater transmissibility, which is always less than unity. This means that damping is favourable in the first range and unfavourable in the second range. In order to have low value of transmissibility, the operating range is generally kept far away in the second range. Under these conditions zero damping will be ideally suitable as this would give extremely low value of transmissibility. But since the system has to pass through the resonance in reaching the operating point and zero damping will

give very high transmissibility (though for a moment only). some amount of damping is generally incorporated in the system at the cost of a little higher transmissibility at the operating point. Some systems operate in the first range at very low values of ω/ω_n. For these systems higher the damping, lesser the transmissibility.

From the phase angle plot of Fig. 4.10.3 it is interesting to note that, all curves start from zero and tend to 90° when $\omega/\omega_n \rightarrow \infty$. When $\zeta < 0.5$, the phase angle continues to increase to a maxima which depends upon the amount of damping and then gradually becomes 90° when $(\omega/\omega_n) \rightarrow \infty$ When $\zeta \geqslant 0.5$, the phase angle gradually increases from 0 to 90° without having a maxima point.

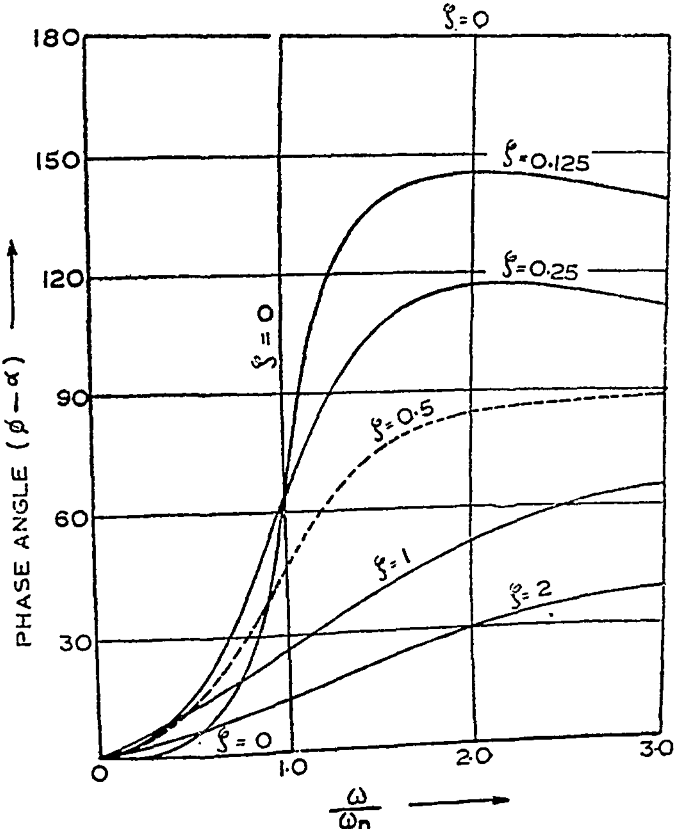


Fig. 4.10.3. Phase angle v/s frequency ratio for different amounts of damping.

4.10 B Motion transmissibility. In Sec. 4.4 on the 'Forced vibrations due to excitation of the support,' equations (4.4.6) and (4.4.8) give respectively, the *ratio of absolute* amplitude of the mass to the base excition amplitude, and the phase angle. The former is the motion transmissibility and the *latter phase lag of the absolute motion of the body from the* exciting motion. These two equations are exactly the same as equations (4.10.2) and (4.10.3) with the corresponding plots in Figs. 4.10.2 and 4.10.3.

4.10 C Materials used in vibration isolation. Materials commonly used for vibration isolation, as has already been mentioned, are rubber, felt cork and metallic spring. The effectiveness of each depends on the operating conditions.

Rubber is loaded in compression or in shear, the latter mode gives higher flexibity. With loading greater than about 6 kg per sq cm, it undergoes much faster deterioration. Its damping and stiffness properties vary widely with applied load, temperature, shape factor, excitation frequency and the amplitude of vibration. The maximum temperature upto which rubber can be used satisfactorily is about 65°C. It must not be used in presence of oil which attacks rubber. It is found very suitable for high frequency vibrations.

Felt is used in compression only and is capable of taking *extremely high loads. It has very high damping and so is* suitable in the range of low frequency ratio. It is mainly used in conjuction with metallic springs to reduce noise transmission.

Cork is very useful for accoustic isolation and is also used in small pads placed underneath a large concrete block. For satisfactory working it must be loaded from 100 to 250 kg/sq cm. It is not affected by oil products or moderate temperature changes. However, its properties change with the frequency of excitation.

Metallic springs are not affected by the operating conditions or the environments. They are quite consistent in their behaviour and can be accurately designed for any desired conditions. They have high sound transmissibility which can be reduced by loading felt in conjunction with it. It has neg-

ligible damping and so is suitable for working in the range of high frequency ratio.

Illustrative Example 4.10.1

A machine weighting 1000 kg is mounted on four identical springs of total spring constant k and having negligible damping. The machine is subjected to a harmonic external force of amplitude $F_0 = 50$ kg and frequency 180 r.p.m. Determine

(a) the amplitude of motion of the machine and maximum force transmitted to the foundation because of the unbalanced force when $k = 2000$ kg/cm.

(b) the same as in (a) for the case when $k = 100$ kg/cm.

Solution

(a) $k = 2000$ kg/cm.

$$m = \frac{1000}{980} \text{ kg-sec}^2/\text{cm}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2000 \times 980}{1000}} = 44.3 \text{ rad/sec}$$

$$\frac{\omega}{\omega_n} = \frac{180 \times 2\pi}{60 \times 44{\cdot}3} = 0.425$$

$$F_0 = 50 \text{ kg.}$$

$$\zeta = 0$$

$$X_{st} = \frac{F_0}{k} = \frac{50}{2000} = 0.025 \text{ cm.}$$

Amplitude

Substituting in equation (4.2.15), we have

$$\frac{X}{0.025} = \frac{1}{|1-0.425^2|} = \frac{1}{0.819}$$

or Amplitude $X = \dfrac{0.025}{0.819} = 0.0305$ cm. **Ans.**

Transmitted Force

Substituting in equation (4.10.2), we have

$$\frac{F_{tr}}{50} = \frac{1}{|1-0.425^2|} = \frac{1}{0.819}$$

or Transmitted force $F_{tr} = \dfrac{50}{0.819} = 61.2$ kg. **Ans.**

(b) $k = 100$ kg/cm.

$$m = \frac{1000}{980} \text{ kg-sec}^2\text{/cm.}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{100 \times 980}{1000}} = 9.9 \text{ rad/sec.}$$

$$\frac{\omega}{\omega_n} = \frac{180 \times 2\pi}{60 \times 9.9} = 1.90$$

$$F_0 = 50 \text{ kg.}$$

$$\zeta = 0$$

$$X_{st} = \frac{F_0}{k} = \frac{50}{100} = 0.5$$

Amplitude

Substituting in equation (4.2.15), we have

$$\frac{X}{0.5} = \frac{1}{|(1-1.90^2)|} = \frac{1}{2.61}$$

or Amplitude $X = \dfrac{0.5}{2.61} = 0.191$ cm. **Ans.**

Transmitted Force

Substituting in equation (4.10.2), we have

$$\frac{F_{tr}}{50} = \frac{1}{|(1-1.90^2)|} = \frac{1}{2.61}$$

or Transmitted force $F_{tr} = \dfrac{50}{2.61} = 19.2$ kg. **Ans.**

Illustrative Example 4.10.2

A machine weighing 75 kg is mounted on springs of stiffness $k = 1200$ kg/cm with an assumed damping factor of $\zeta = 0.20$. A piston within the machine weighing 2 kg has a reciprocating motion with a stroke of 8 cm and a speed of 3000 c.p.m. Assuming the motion of the piston to be harmonic, determine the amplitude of vibration of the machine and the vibratory force transmitted to the foundation.

Solution

$W = 75$ kg.

$k = 1200$ kg/cm.

$$\zeta = 0{\cdot}20$$

$$\omega_n = \sqrt{\frac{kg}{W}} = \sqrt{\frac{1200 \times 980}{75}} = 125 \text{ rad/sec.}$$

$$\omega = \frac{3000 \times 2\pi}{60} = 100\pi \text{ rad/sec.}$$

$$\frac{\omega}{\omega_n} = \frac{100\pi}{125} = 2{\cdot}51$$

$$W_0 = 2 \text{ kg.}$$

$$e = \frac{8}{2} = 4 \text{ cm.}$$

$$\frac{m_0}{m}\, e = \left(\frac{2/980}{75/980}\right) \times 4 = 0{\cdot}1067 \text{ cm.}$$

$$F_0 = m_0\, \omega^2 e = \frac{2}{980} \times (100\,\pi)^2 \times 4 = 806 \text{ kg.}$$

For finding the amplitude X of vibration of the machine, substitute above quantities in equation (4.3.2).

$$\frac{X}{0{\cdot}1067} = \frac{(2{\cdot}51)^2}{\sqrt{[1-(2.51)^2]^2+[2\times 0{\cdot}2\times 2.51]^2}}$$

which gives

$X = 0.125$ cm. **Ans.**

For finding the vibratory force F_{tr} transmitted to the foundation, substitute the above quantities in equation (4.10.2).

$$\frac{F_{tr}}{806} = \sqrt{\frac{1 + (2\times 0{\cdot}2\times 2.51)^2}{[1-(2{\cdot}51)^2]^2+[2\times 0{\cdot}2\times 2{\cdot}51]^2}}$$

which gives

$F_{tr} = 212$ kg. **Ans.**

Illustrative Example 4.10.3

A radio set weighing 20 kg must be isolated from a machine vibrating with an amplitude of 0.005 cm at 500 cpm. The set is mounted on four isolators, each having a spring scale of 32 kg/cm and damping factor of 0·4 kg-sec/cm.

(a) What is the amplitude of vibration of the radio?

(b) What is the dynamic load on each isolator due to vibration ?

Solution

Let m be the mass of the radio and, k and c represent the overall equivalent striffness and the damping coefficient of the four isolators.

$$m = \frac{20}{980} = 0{\cdot}0204 \text{ kg-sec}^2/\text{cm}$$

$$k = 4 \times 32 = 128 \text{ kg/cm.}$$

$$c = 4 \times 0{\cdot}4 = 1{\cdot}6 \text{ kg-sec/cm.}$$

$$y = Y \sin \omega t$$

$$Y = 0{\cdot}005 \text{ cm.}$$

$$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 500}{60} = \text{rad/sec}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{128}{0{\cdot}0204}} = 79{\cdot}2 \text{ rad/sec.}$$

$$\frac{\omega}{\omega_n} = \frac{52.5}{79{\cdot}2} = 0{\cdot}662$$

$$\zeta = \frac{c}{2\sqrt{km}} = \frac{1{\cdot}6}{2\sqrt{128 \times 0{\cdot}0204}} = 0{\cdot}496$$

(a) To find the absolute amplitude X of vibration of the radio, substitute the above quantities in equation (4.4.6)

$$\frac{X}{0.005} = \sqrt{\frac{1 + (2 \times 0.496 \times 0.662)^2}{[1 - (0.662)^2]^2 + [2 \times 0.496 \times 0.662]^2}}$$

which gives

$$X = 0.0069 \text{ cm.}$$ **Ans.**

(b) The dynamic load on isolators due to vibration can be obtained by first finding the relative amplitude Z of vibration and then

$$F_{\text{Dyn}} = Z\sqrt{(c\omega)^2 + k^2}$$

Applying equation (4.4.11), we have

$$\frac{Z}{0.005} = \frac{(0.662)^2}{\sqrt{[1-(0.662)^2]^2 + [2 \times 0.496 \times 0.662]^2}}$$

which gives

$$Z = 0.0025 \text{ cm}$$

Therefore, $F_{Dyn} = 0.0025 \sqrt{(1.6 \times 52.4)^2 + 128^2}$

or $F_{Dyn} = 0.385$ kg.

Hence, the dynamic load on each isolator is

$$\frac{0.385}{4} = 0.096 \text{ kg.}$$ **Ans.**

The maximum force transmitted through the isolators can also be found by equating it to the maximum inertia force on the radio, i.e.

Max. dynamic load on the isolators

= Max. force transmitted through the isolators

= Max. inertia force on the radio = $m\omega^2 X$

$$= \frac{20}{980} \times (52.4)^2 \times 0.0069$$

$$= 0.39 \text{ kg}$$

Hence the dynamic load on each isolator is

$$\frac{0.39}{4} = 0.097 \text{ kg.}$$ **Ans.**

4.11 Vibration measuring instruments.

The primary purpose of a vibration measuring instrument is to give an output signal which represents as closely as possible the vibration phenomenon. This phenomenon may be displacement, velocity or acceleration of the vibrating system and accordingly the instrument which reproduces signals proportional to these are called vibrometers, velometers or accelerometers.

Fig. 4.11.1 shows the schemetic of a *seismic instrument* which is used to measure any of the vibration phenomenon. It consists of a frame in which the seismic mass m is supported by means of a spring k and dashpot c. The frame is mounted on a vibrating body and vibrates along with it. The system reduces to a spring mass dashpot system having base or support excitation as discussed in Sec. 4.4. Consider the vibrating body (base) to have a sinusoidal motion

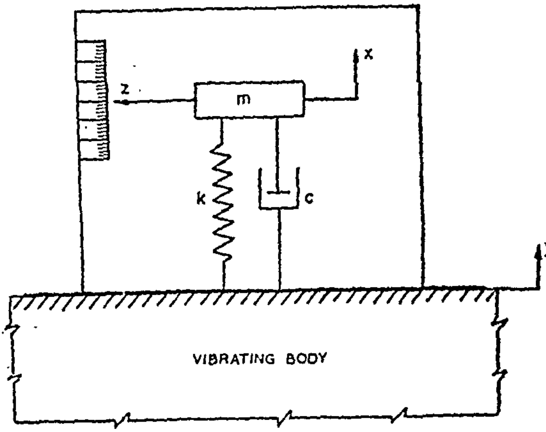


Fig. 4.11.1. Seismic instrument.

$$y = Y \sin \omega t.$$

Then the steady state relative amplitude Z of the seismic mass with respect to the frame is obtained from equation (4.4.11) and the phase difference between the exciting motion and the relative motion is given by equation (4.4.12).

Imagine that a scale is fixed on the frame and a pointer on the seismic mass. Then the amplitude of motion of the mass over the scale represents the relative motion z having an amplitude Z. This motion is also harmonic.

The plots of equations (4.4.11) and (4.4.12) for the relative response and the phase shift respectively, are already shown in Figs. 4.3.2 and 4.2.5.

4.11 A Displacement measuring instruments or vibrometers. If the input motion y in Fig. 4.11.1 is harmonic, then the relative motion z that can be recorded by means of a secondary strain sensing transducer, is also harmonic. The ratio of recorded motion to exciting motion is given by equation (4.4.11) and plot by Fig. 4.3.2. If in equation (4.4.11), $\frac{\omega}{\omega_n} >> 1$, then Z/Y tends to unity irrespective of the value of damping. This is also seen in Fig. 4.3.2. The ratio $\frac{Z}{Y} = 1$ means that the

relative amplitude recorded Z, is equal to the excitation amplitude Y or the amplitude of the vibrating system. Thus, provided ω/ω_n is large, the amplitude recorded is approximately equal to the amplitude of the vibrating body. Further more, it can be seen from Fig. 4.3.2 that if damping factor is about 0.7 or a little lower, it is possible to have a better approximation of the relation ($Z/Y \simeq 1$) over a larger range of frequency ratio. However, in most vibrometers, damping is kept as small as possible (for reasons of reduced distortion, discussed later in this section), but ω/ω_n is large enough to ensure that the recorded motion is a good approximation of the input motion. The ratio ω/ω_n can be made large by having the instrument of low natural frequency, the average value of which may be about 4 c.p.s.

We have seen that if the vibrating body has a *harmonic motion* of a frequency such that $(\omega/\omega_n) > 3.0$ (say), then the amplitude recorded is a good approximation of the amplitude of the vibrating body. However, the output signal is not in phase with the input motion and so, there is some time delay depending upon the amount of damping in the system. But that is immaterial as long as the output signal is a true representation of the input signal.

Now consider that the vibrating body has a non-harmonic periodic motion of fundamental frequency ω such that $(\omega/\omega_n) > 3{\cdot}0$ (as before). Then the fundamental will be transmitted with the same accuracy as before and any higher harmonic has a higher frequency than the fundamental and will be recorded still more precisely. But if we look back to the phase angle plot of Fig 4.2.5. we see that the phase shift for different values of ω/ω_n (corresponding to different harmonics) is different, in general, depending upon the amount of damping. This will mean that although each harmonic separately, is recorded accurately, it has different phase angle relationship with the fundamental in the recorded motion than what it had in the input motion. The result is that the output motion will be a distorted motion and not a true reproduction of the input motion. This can be seen in Fig. 4.11.2, where the fundamental in the recorded wave has suffered a lag of ϕ_1 from that in the input wave, and the third harmonic has suffered a lag of

ϕ_2. The final output wave is distorted. This difficulty can be overcome by making the damping in the system to be *zero*. Under this condition and when $(\omega/\omega_n) > 1$ the phase difference is 180° whatever the frequency ratio. So each harmonic, seperately, apart from being recorded accurately is also transmitted

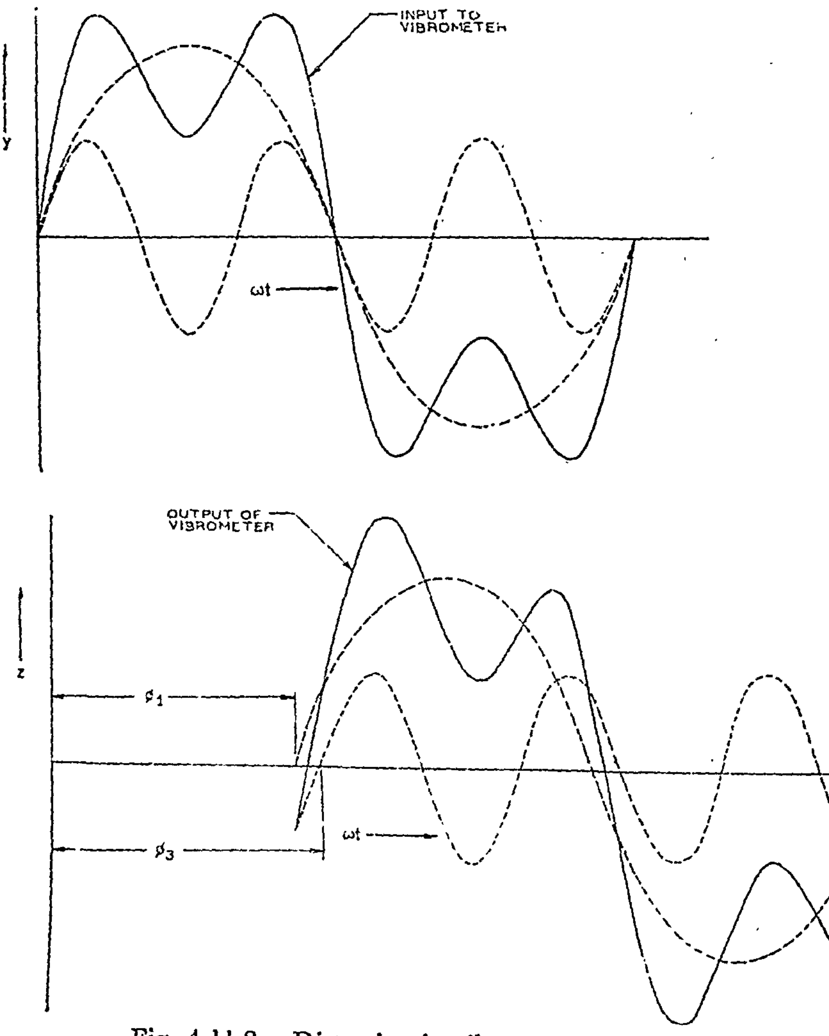


Fig. 4.11.2. Distortion in vibrometers having damping.

with the same phase angle relationship. The resulting output signal is a true reproduction of the input motion.

4.11 B Velocity measuring instruments or velometers. In Sec 4.11 A it was said that the relative motion Z could be measured by means of a secondary strain sensing transducer. If, however, we can have a velocity sensing secondary transducer of the type of a magnet rigidly fixed to the seismic mass moving in a coil fixed to the frame, then the output voltage at the two ends of the coil will be proportional to the relative velocity since the voltage output is proportional to the rate at which the lines of force are cut. This relative velocity is equal to the input velocity of the support or the vibrating system at large values of ω/ω_n. Hence the instrument behaves as a *Velometer* and everything else said in regard to the vibrometer holds good for velometer.

4.11 C Acceleration measuring instruments or accelerometers. The instrument shown in Fig. 4.11.1 can also behave as an accelerometer under certain conditions.

In equation (4.4.11) if $(\omega/\omega_n) << 1$, then

$$\frac{Z}{Y} \simeq \left(\frac{\omega}{\omega_n}\right)^2 \tag{4.11.1}$$

or

$$Z \simeq \frac{(\omega^2 Y)}{\omega_n^2} \tag{4.11.1a}$$

The expression $\omega^2 Y$ in the above equation is equal to the acceleration amplitude of the body vibrating with frequency ω and having a displacement amplitude Y. Hence, the amplitude recorded Z, under these conditions is proportional to the acceleration of the vibrating body since ω_n is a constant of the instrument. Therefore the extreme left hand part of the frequency-response curve of the instrument actually represents the accelerations at various frequencies. The ratio ω/ω_n can be made small by having ω_n as large as possible.

The curve represented by equation (4.11.1) is a parabola and this curve is also shown in Fig. 4.3.2. It is seen that this curve agrees fairly well with other curves for small values of ω/ω_n and almost coincides upto the range of $(\omega/\omega_n) = 0$ to 0·5, with the frequency-response curve for damping factor

of 0·7 and a little lower. And this is the damping ratio usually kept in accelerometers.

The natural frequency of the accelerometer should be at least twice as high as the highest frequency of the acceleration to be recorded. There is a possibility of some difficulty in the case of non-harmonic periodic vibrations where the harmonics of higher frequency may not be recorded accurately unless ω_n is much higher than the highest frequencies of the harmonics. For this reason the natural frequency of most of the good accelerometers is above 10,000 cps. much above the range of the frequency of mechanical vibrations.

Regarding the distortion, if the input acceleration is not sinusoidal but is of complex nature as of the form

$$f_i(t) = A_1 \cos \omega t + A_2 \cos 2\omega t + A_3 \cos 3\omega t + \ldots \ldots \quad (4.11.2)$$

each harmonic now has a phase shift depending upon the damping present in the system and the frequency of the fundamental. Fortunately for us, it is seen from Fig. 4.2.5 that for a damping ratio of about 0·7 or a little lower, the phase shift is almost linear with frequency of operation in the working range of the accelerometer; i.e. for successive components of equation (4.11.2), the phase shift is approximately ϕ, 2ϕ, 3ϕ ... etc. Hence the output acceleration curve will be of the form

$$f_0(t) = A_1 \cos (\omega t - \phi) + A_2 \cos (2\omega t - 2\phi) + A_3 \cos (3\omega t - 3\phi) + \ldots$$

or $$f_0(t) = A_1 \cos \beta + A_2 \cos 2\beta + A_3 \cos 3\beta + \ldots \quad (4.11.3)$$

where $\beta = \omega t - \phi$

From equations (4.11.2) and (4.11.3) we see that the input and output acceleration curves are of the same form where each term retains the same relative harmonic relationship with the other term. Therefore there will be no phase distortion and the output signal will be similar to the input acceleration in everyway.

4.11 D Frequency measuring instruments. One of the methods of measuring the frequency of vibration of a system is by means of *Frahm's Reed Tachometer*. It consists of a

system having a number of reeds fixed over it in the form of cantilevers carrying small masses at their free ends, as shown in Fig. 4.11.3. The natural frequencies of the set of these reeds is adjusted to give a definite series of known frequencies. When this instruments is attached to the body whose frequency of vibration is to be measured, then the reed whose natural frequency is nearest to the excitation frequency vibrates near resonant condition and has a large amplitude of vibration The frequency of the vibrating body is then given by the known frequency of reed vibrating with maximum amplitude.

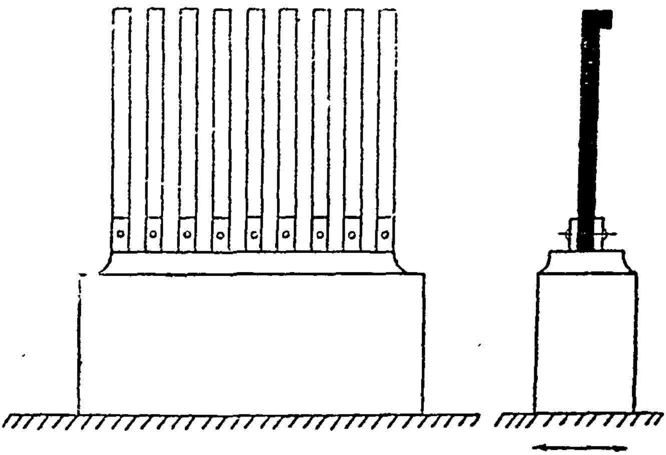


Fig. 4.11.3. Frahm's reed tachometer.

The accuracy of this instrument depends upon the difference between the natural frequencies of the successive reeds. The smaller the difference, more accurate is the instrument and vice-versa. Of course, with a more accurate instrument of this type, the range of frequencies that can be measured, will be smaller.

Illustrative Example 4.11.1

A vibrometer has a period of free vibration of 2 seconds. It is attached to a machine with a vertical hormonic frequency of 1 cps. If the vibrometer mass has an amplitude of 0.25 cm relative to the vibrometer frame, what is the amplitude of vibration of the machine ?

Solution

$$\omega_n = \frac{2\pi}{2} = \pi \text{ rad/sec.}$$

$$\omega = 1 \times 2\pi = 2\pi \text{ rad/sec.}$$

$$\frac{\omega}{\omega_n} = 2.0$$

$$\zeta \simeq 0, \text{ for vibrometers.}$$

$$Z = 0.25 \text{ cm}$$

Substituting the above quantities in equation (4.4.11), we have

$$\frac{0.25}{Y} = \frac{(2.0)^2}{\sqrt{[1-(2\cdot0)^2]^2}} = \frac{4}{3}$$

or $$Y = \frac{3}{4} \times 0\cdot25 = 0\cdot1875 \text{ cm},$$

which is the amplitude of vibration of the support or the machine as in this case. ***Ans.***

Illustrative Example 4.11.2.

A commercial type vibration pick-up has a natural frequency of 5.75 cps, and a damping factor of 0.65. What is the lowest frequency beyond which the amplitude can be measured within

(i) one percent error.

(ii) two percent error.

Solution

$$f_n = 5\cdot75 \text{ cps.}$$

$$\zeta = 0\cdot65$$

(i) We want to find that value of ω/ω_n for which $(Z/Y) = 1\cdot01$ (see Fig. 4.3.2). This is the lowest value of ω/ω_n beyond which the amplitude can be measured within one percent error.

Using equation (4.4.11), and letting $(\omega/\omega_n) = r$ we have

$$1\cdot01 = \frac{r^2}{\sqrt{(1-r)^2+(2\times0\cdot65\times r)^2}}$$

Simplification leads to

$$0{\cdot}02\, r^4 - 0{\cdot}31\, r^2 + 1 = 0$$

giving $\qquad r = 3{\cdot}30$ and $2{\cdot}02$.

These are the two values of r at which $(Z/Y) = 1{\cdot}01$. In between these two values (Z/Y) will be greater than $1{\cdot}01$. Therefore, the lowest frequency beyond which the amplitude can be measured within 1% error is given by

$$\frac{f}{f_n} = \frac{\omega}{\omega_n} = r = 3{\cdot}30$$

or $\qquad f = 3{\cdot}30 \times 5{\cdot}75 = 19$ cps. **Ans.**

(ii) In this case $(Z/Y) = 1{\cdot}02$. When we use equation (4.4.11) for finding the value or r $(=\omega/\omega_n)$ for the given value of damping we get imaginary value of r^2. This means that the curve in Fig. 4.3.2, for $\zeta = 0{\cdot}65$, does not go as high as $Z/Y = 1{\cdot}02$. Therefore, to get the frequency for 2% error we will have to take $Z/Y = 0{\cdot}98$,

or $$0{\cdot}98 = \frac{r^2}{\sqrt{(1-r^2)^2 + (2 \times 0{\cdot}65 \times r)^2}}$$

Simplification leads to

$$0{\cdot}04\, r^4 + 0{\cdot}31\, r^2 - 1 = 0$$

giving the only positive value for

$$r = 1{\cdot}55$$

Hence $$\frac{f}{f_n} = \frac{\omega}{\omega_n} = r = 1{\cdot}55$$

or $\qquad f = 1{\cdot}55 \times 5{\cdot}75 = 8{\cdot}9$ cps. **Ans.**

Illustrative Example 4.11.3

A device used to measure torsional accelerations consists of a ring having a mass moment of inertia of 0·5 kg-cm-sec² connected to a shaft by a spiral spring having a scale of 10 kg-cm/rad, and a viscous damper having a constant of 1.12 kg-cm-sec/rad. When the shaft vibrates with a frequency of 15 cpm, the relative amplitude between the ring and the shaft is found to be 2°. What is the maximum acceleratiom of the shaft ?

Solution

$$J = 0.5 \text{ kg-cm-sec}^2.$$

$$k_t = 10 \text{ kg-cm/rad.}$$

$$c_t = 1.12 \text{ kg-cm-sec/rad.}$$

$$\omega = 15 \times 2\pi/60 = \pi/2 \text{ rad/sec.}$$

$$\omega_n = \sqrt{\frac{k_t}{J}} = \sqrt{\frac{10}{0.5}} = 4.47 \text{ rad/sec.}$$

$$\zeta = \frac{c_t}{2\sqrt{k_t J}} = \frac{1.12}{2\sqrt{10 \times 0.5}} = 0.25$$

$$\theta_z = 2^\circ = \frac{2}{57.3} \text{ rad} = 0.0349 \text{ radians.}$$

$$\frac{\omega}{\omega_n} = \frac{\pi/2}{4.47} = 0.352$$

Rewriting equation (4.4.11) for the torsional system, we have

$$\frac{\theta_z}{\theta_y} = \frac{\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\frac{\omega}{\omega_n}\right]^2}}$$

Substituting the above quantities in this equation,

$$\frac{0.0349}{\theta_y} = \frac{(0.352)^2}{\sqrt{[1-(0.352)^2]^2 + [2 \times 0.25 \times 0.352]^2}}$$

which gives

$$\theta_y = 0.253$$

Therefore maximum acceleration of the shaft $= \omega^2 \theta_y$

$$= \left(\frac{\pi}{2}\right)^2 \times 0{\cdot}253 = 0\,62 \text{ rad/sec}^2.$$

Illustrative Example 4.11.4

Determine the weight W of the mass to be placed at th[e end] of one of the reeds of a Frahm tachometer in order th[at the] reed be in resonance at a frequency of 1800 cpm. Th[e] reed is 5 cm long, 0.6 cm wide and 0.075 cm thick. [The] modulus of the material of the reed is 2.0×10^6 kg/cm²

Solution

$$\Delta_{st} = \frac{Wl^3}{3EI}$$

$$\therefore \quad k = \frac{W}{\Delta_{st}} = \frac{3EI}{l^3}$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{3EI}{l^3}\frac{g}{W}} \text{ cps,}$$

$$= \frac{1800}{60} \text{ (given)}$$

$$= 30 \text{ cps.}$$

$$\therefore \quad \frac{3EIg}{l^3 W} = 30^2\ (2\pi)^2$$

or
$$W = \frac{3EIg}{900 \times 4\pi^2 \times l^3}$$

Now,
$$I = (1/12) \times 0.6 \times 0.075^3 = 21.2 \times 10^{-6} \text{ [illegible]}$$

Therefore,
$$W = \frac{3 \times (2 \times 10^6) \times (21.2 \times 10^{-6}) \times 981}{900 \times 4 \times \pi^2 \times 5^3}$$

or
$$W = 0.028 \text{ kg.}$$ Ans.

PROBLEMS FOR PRACTICE

4.1 A weight of 100kg is suspended on a spring having a scale of 20 kg/cm, and is acted upon by a harmonic force of 4 kg at the undamped natural frequency. The damping may be considered to be viscous with a coefficient of 0.1 kg-sec/cm. Determine

(a) the undamped natural frequency,

(b) the amplitude of the weight, and

(c) the phase difference between the force and the displacement.

4.2 During the installation of a 4-pole, 50-cycle induction motor weighing 250 kg, it is determined by means of a level that the deflection of the foundation under the motor is 0.012 cm. Would you consider this foundation safe? Explain your answer.

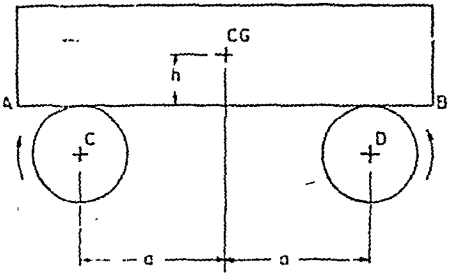


Fig. P. 4.3

4.3 The apparatus shown in Fig. P. 4 3 is intended to determine the coefficient of sliding friction between dry surfaces. A metal block is supported with its lower plane face AB horizontal and in contact with two cylindrical rollers which revolve, as shown in the figure, in opposite directions about fixed axes C and D. When initially disturbed so that the c.g. of the block which is at a height h above AB, is displaced from the central position between the rollers, it performs simple harmonic motion.

If the coefficient of friction μ between the block and the rollers is the same for both rollers, show that its magnitude is

$$\mu = \frac{4\,\pi^2 a}{(g\tau^2 + 4\pi^2 h)}$$

where τ is period of oscillations.

What will happen if the two rollers rotate in directions opposite to those shown in the figure.

4.4 Plot a curve showing dimensionless peak frequency against damping factor for the forced vibrations of a spring-mass system with a variable dashpot.

4 5 Show that the maximum velocity of the mass of a vibrating spring-mass-dashpot system occurs at $(\omega/\omega_n) = 1$ irrespective of the amount of damping.

4.6 An eccentric weight exciter is used to determine the vibratory characteristics of a structure weighing 200 kg. At a speed of 1000 rpm, a stroboscope showed the eccen-

tric weight to be at the bottom position at the instant the structure was moving downward through its static equilibrium position and the corresponding amplitude was 2.00 cm. If the unbalance of the eccentric is 5 kg-cm, determine,

(a) the undamped natural frequency of the structure,

(b) the damping factor of the structure, and

(c) the angular position of the eccentric at 1300 rpm at the instant when the structure is moving downward through its equilibrium position.

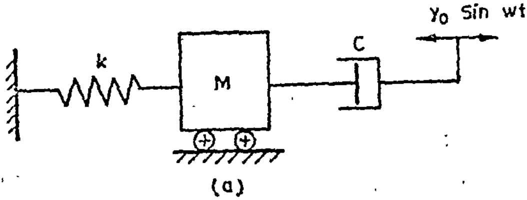
4.7 The point of suspension of a simple pendulum of length l and mass m is given a horizontal excitation $y = Y \sin \omega t$. Derive an expression for the angular amplitude of vibration of the mass and plot a non-dimensional frequency-response curve.

4.8 A motor weighing 60 kg is mounted on a simple beam that has a stiffness of 40 kg/cm at that point. The rotor of the motor weighs 10 kg and has an eccentricity of 0.01 cm. What will be the amplitude of vibration of the motor when it runs at 1460 rpm ? Neglect damping, the weight of the beam and the deflection of the motor shaft.

4.9 X A vertical single stage air compressor weighing 500 kg is mounted on springs having stiffness of 200 kg/cm and dashpots with a damping factor of 0.2. The rotating parts are completely balanced and the equivalent reciprocating parts weigh 20 kg. The stroke is 20 cm. Determine the dynamic amplitude of vertical motion and the phase difference between the motion and the excitation force if the compressor is operated at 200 rpm.

4.10 A spring mass system has a natural frequency of 4 cycles/sec. When the mass is at rest the support is made to move up with displacement $y = 3 \sin 3\pi t$ (where t is in seconds and y in centimeters) measured from the beginning of the motion. Determine the distance through which the mass moves in the first 0 1 second.

4.11 Analyse the problems of Fig. P. 4.11 for steady state response of the mass.



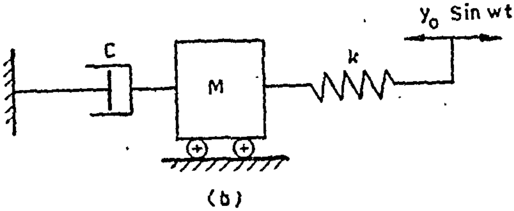


Fig. P. 4.11.

4.12 Determine the critical speed of a 1000 kg automobile travelling on a concrete road with expansion joints spaced 12 meters apart, if the static deflection of the spring system is 5 cm.

4.13 The two rear wheels of an automobile are subjected to a load of 500 kg through springs of 20 kg/cm. What is the amplitude of vibration of the rear of the automobile at a speed of 80 km/hr on a road having waves 2 cm total depth whose crests are 1.5 meter apart. At what speed will there be resonance.

4.14 A vehicle weighs 490 kg and the total spring constant of its suspension system is 60 kg/cm. The profile of the road may be approximated to a sine wave of amplitude 4.00 cm and wave length of 4.0 meters. Determine

(a) the critical speed of the vehicle,

(b) the amplitude of the steady state motion of the mass when the vehicle is driven at critical speed and the damping factor is 0.5, and

(c) the amplitude of steady state motion of the mass

when the vehicle is driven at 57 km/hr and the damping factor same as in (b).

4.15 For the cases shown in Fig. P. 4.15 (a) and (b), determine the steady state response of the point between the dashpot and the spring to a sinusoidal input $y = Y \sin \omega t$. Also derive an expression for the phase lag.

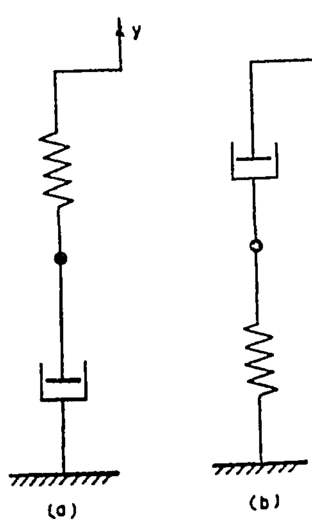


Fig. P. 4.15.

4.16 For the data given in problem 4.9, find the horse power dissipated in vibration of the compressor unit.

4.17 Draw a graph between the magnification factor against the phase angle for various values of damping ratios.

4.18 Determine the H.P. required to run a motor carrying an eccentric at 3600 rpm, when the motor is mounted on a concrete block and the whole assembly is supported by felt pads placed on the foundation. The damping coefficient of the system is 30 kg-sec/cm and the amplitude of vibration is 0.12 cm.

4.19 A spring-mass system is guided to move only in the vertical direction. The guideways cause a dry frictional force of 2 kg on the mass. Mounted on the mass is a

counter rotating eccentric with a total unbalance of 1.7 kg-cm. The spring-mass system has a natural frequency of 8.5 cps and the mass weighs 3.8 kg.

Find the amplitude of vibration of the mass

(i) at 400 rpm,

(ii) at 1000 rpm.

For the frequency response curve shown in Fig. P.4.20 find out the damping factor after necessary construction, and scaling the quantities directly from the curve.

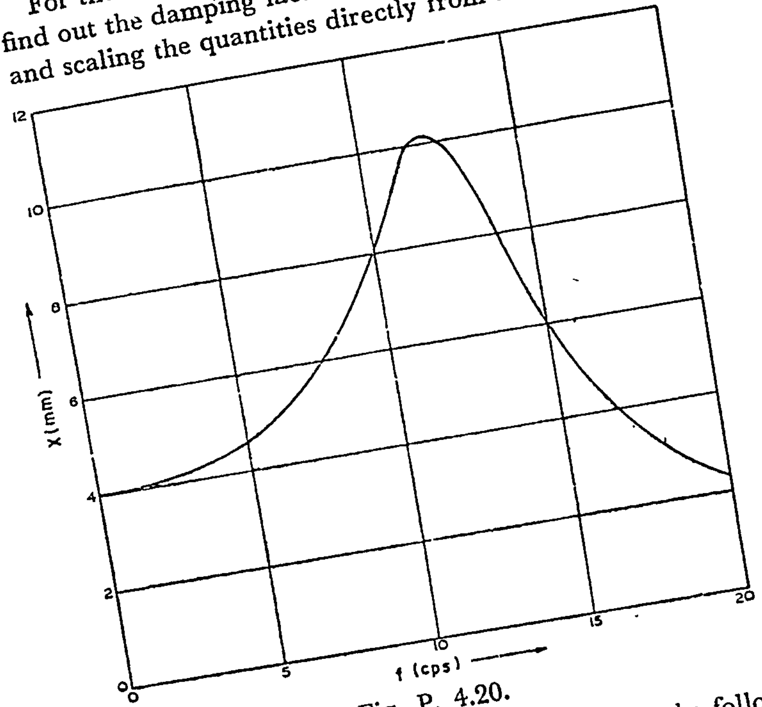


Fig. P. 4.20.

4.21 A vertical spring-mass-dashpot system has the follow physical data

$$W = 2.4 \text{ kg},$$

$$k = 1.44 \text{ kg/cm}$$

$$\zeta = 0.2$$

While the system is in equilibrium position a co force $F_0 = 3.6$ kg acts on the mass. Find the complete sc

to the equation of motion under these conditions. What is the steady state displacement of the mass from the equilibrium position ?

4.22 The equation of motion of a reciprocating engine impressed upon by its inertia forces is given by

$$m\ddot{x} + kx = P_1 \sin \omega t + P_2 \sin 2\omega t.$$

Find the general solution to this equation.

4.23 The saw-tooth tpye excitation as shown in Fig. P. 4.23 acts on the support of a spring-mass-dashpot system. Determine the resultant steady state motion of the mass when $W = 2000$ kg, $K = 8$ kg/cm and $c = 4$ kg-sec/cm.

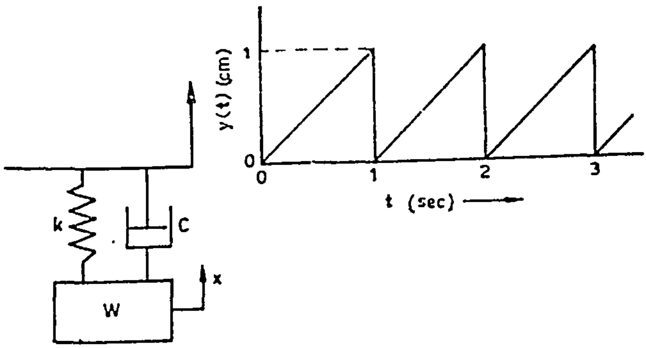


Fig. P. 4.23.

4.24 A machine weighing 100 kg and supported on springs of total stiffness 800 kg/cm has an unbalanced rotating element which results in a disturbing force of 40 kg at a speed of 3000 rpm. Assuming a damping factor of $\zeta = 0.20$ determine

(a) the amplitude of motion due to the unbalance,

(b) the transmissibility, and

(c) the transmitted force.

4.25 A machine weighing one tonne is acted upon by an external force of 250 kg at a frequency of 1500 rpm. To reduce the effects of vibration. isolators of rubber having a static defletcion of 0.2 cm under the given load and an estimated damping factor $\zeta = 0.2$ are used. Determine :

(i) the force transmitted to the foundation.

(ii) the amplitude of vibration of machine.

(iii) the phase lag.

4.26 A huge machinery is mounted on a bed plate which is supported on four elastic members, each having a stiffness of 4000 kg/cm. The total weight to be supported is 1 ton. It is estimated that the total damping force exerted on the system is 20% of the critical and is of viscous nature.

When the speed of rotation of the machine is 2000 rpm, the amplitude of vertical motion of the bed plate is 0.006 cm. Calculate the total maximum force transmitted through each mounting to the foundation.

4.27 A compressor unit weighing 200 kg is mounted rigidly on a concrete bed weighing 500 kg. The disturbing force whose frequency is the same as the compressor speed and which is sinusoidal, has a maximum value of 30 kg. If the compressor speed is 1000 rpm, determine the stiffness of rubber pads to be used beneath the concrete bed such that the force transmitted is 0.5% of the disturbing force. Neglect damping.

4.28 A centrifugal compressor weighing 100 kg is supported on isolators having a damping factor of 0.20. It runs at a constant speed of 1500 rpm and has a rotating unbalance of 10 kg-cm. What should be the stiffness of the isolators if the force transmitted to the foundations is to be less than 10% of the unbalanced force.

4.29 An instrument weighing 50 kg is located in an airplane cabin which vibrates at 2000 cpm with an amplitude of 0.01 cm. Determine the stiffness of the four steel springs required as supports for the instruments to reduce its amplitude to 0.0005 cm. Also calculate the maximum total load for which each spring must be designed.

4.30 An aircraft instrument weighing 10 kg is to be isolated from the engine vibrations. The engine runs at speeds ranging from 1800 rpm to 2500 rpm. Natural rubber

isolators with negligible damping are used. Determine the rubber stiffness for 90% isolation.

4.31 Discuss very briefly the advanges and the disadvantages of the various isolator materials.

In the case of isolators consisting of springs only what is the transmissibiliy when the spring constant is

(i) $k = \infty$;

(i i) $k = 0$;

(iii) k = any intermediate value ?

In what range of speed is the spring isolator desirable and in what range it is undesirable ?

4.32 What are the principles on which a vibrometer and an accelrometer are based ? What should be the range of natural frequencies for a vibrometer and for an accelerometer for a frequency f of vibration and of acceleration respectively ?

Explain how the result is affected in case the motion contains higher harmonics.

4.33 An undamped vibration pick-up having a natural frequency of 1 cps is used to measure a harmonic vibration of 4 cps. If the amplitude recorded is 0.052 cm, what is the correct amplitude ?

4.34 A vibrometer with a natural frequency of 2 cps and with negligible damping is attached to a vibrating system which performs a harmonic motion. Assuming the difference between the maximum and minimum recorded values as 0.06 cm, determine the amplitude of motion of the vibrating system when its frequency is

(i) 20 cps,

(ii) 4 cps.

4·35 A commercial vibration pick up has a damped natural frequency of 4.5 cps and a damping ratio of 0.5. What is the range of impressed frequency at which the amplitude can be read directly from the pick up with an error not exceeding 2% of the actual amplitude.

4.36 An instrument for measuring accelerations records 30 oscillation/sec. The natural frequency of the instrument is 800 cycles/sec. What is the acceleration of the machine part to which the instrument is attached if the amplitude recorded is 0.002 cm ? What is the amplitude of vibration of the machine part ?

4.37 An accelerometer having a natural frequency of 1000 cpm and a damping factor of 0.7 is attached to a vibrating system. Determine the maximum acceleration of the system when the recorded amplitude is

$$\omega^2 Z = 50 \text{ cm/sec}^2$$

when the system performs a hormonic motion at

(i) 400 cpm,

(ii) 800 cpm.

4.38 The static deflection of the weight of a vibrometer is 2 cm. The instrument when attached to a machine vibrating with a frequency of 125 cpm record to relative amplitude of 0.003 cm. Find out for the machine

(a) the amplitude of vibration,

(b) the maximum velocity of vibration, and

(c) the maximum acceleration of vibration.

4.39 A seismic instrument with a natural frequency of 6 cps is used to measure the vibration of a machine running at 120 rpm. The instrument gives the reading for the relative displacement of the seismic mass as 0.005 cm. Determine the amplitudes of displacement, velocity and acceleration of the vibratitng machine. Neglect damping.

4.40 Determine the torsional stiffness of a spring for a torsiograph with a ring having moment of inertia of 0.02 kg-cm-sec², so that the difference in the relative motion and that of the vibrating shaft will not be greater than 3% when the shaft vibrates with a frequency of 1000 cpm or above. Neglect damping. If the shaft amplitude is 0.01 radian, determine the corresponding dynamic tonque on the spring.

4.41 It is desired to study the vibration of the foundation of a two cycle diesel engine between speeds of 300 and 1200 rpm by means of a vibrometer. It is known that the vibration consists of two harmonics, because of primary and secondary inertia forces in the engine. Find the maximum natural frequency that the vibrometer may have in order to keep the amplitude distortion below 5%.

4.42 A reed tachometer consists of 16 reeds and covers a range of 5 cps to 20 cps. Each reed, made of steel, is 1 mm thick, 5 mm wide and 6 cm long. Find the end weights for the reeds corresponding to the two extreme frequencies.

CHAPTER 5

TWO DEGREES OF FREEDOM SYSTEMS

5.1 Introduction.

The discussions so far have been limited to systems having single degree of freedom and therefore having one natural frequency and requiring one independent coordinate to define the system completely at any instant. Although the ideal system with which the theory dealt, occurs rarely in practice, but a number of actual cases are sufficiently close to the ideal ones to give us a reasonably correct estimate of certain conclusions of importance.

The systems having two degrees of freedom are important in as far as they introduce us to the coupling phenomenon where the motion of any of the two independent coordinates, in general, depends also on the motion of the other coordinate through the coupling springs or dashpots. A system having two degrees of freedom has two natural frequencies and the free vibration of any point in the system, in general, is a combination of two harmonics of these two natural frequencies respectively. Under certain conditions, any point in the system may execute harmonic vibrations at any of the two natural frequencies, and these are known as the *Principal Modes of Vibration.* The two degrees of freedom system analysed in this chapter also explains certain other phenomenon of interest.

5.2 Principal modes of vibration.

Let us consider an ideal two degrees of freedom system as shown in Fig. 5.2.1 (a) where the masses are constrained to move in the direction of spring axes. Let x_1 and x_2 be the

displacements of the masses at any instant measured from the equilibrium positions of these masses respectively, positive in the direction shown. In order to write the differential equations for these two masses, let us assume $x_2 > x_1$. Then the spring forces acting on these masses are as shown in Fig. 5.2.1 (b), giving us the following differential equations.

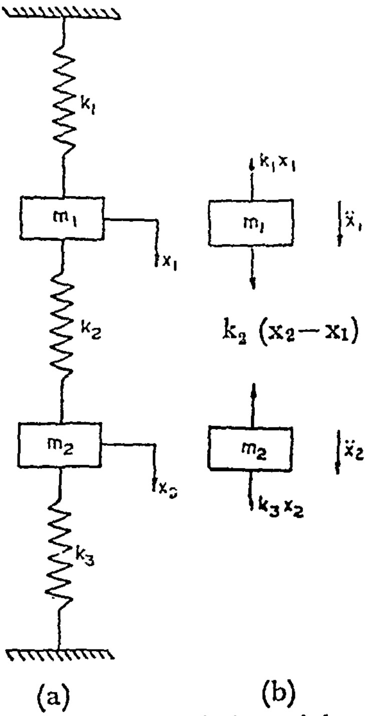


Fig. 5.2.1 Derivation of differential equation for a two-degree system.

$$m_1\ddot{x}_1 = -k_1 x_1 + k_2\,(x_2 - x_1)$$
$$m_2\ddot{x}_2 = -k_2\,(x_2 - x_1) - k_3\, x_2$$

or, these equations may be re-written as,

$$\left.\begin{aligned} m_1\ddot{x}_1 + (k_1 + k_2)\,x_1 &= k_2\, x_2 \\ m_2\ddot{x}_2 + (k_2 + k_3)\,x_2 &= k_2\, x_1 \end{aligned}\right\} \qquad (5.2.1)$$

The reason why we considered $x_2 > x_1$ was to have some physical concept and ease in writing down the differential equations. We could as well have considered $x_2 < x_1$ and the final equations (5.2.1) would, of course, be the same.

Let us now make a guess work regarding the solutions for x_1 and x_2 in considering that they can have harmonic vibrations under steady state conditions. We will then consider the case when mass m_1 executes harmonic vibration at frequency ω_1 and mass m_2 executes harmonic vibration at frequency ω_2, that is

$$\left.\begin{aligned} x_1 &= X_1 \sin \omega_1 t \\ \text{and} \qquad x_2 &= X_2 \sin \omega_2 t \end{aligned}\right] \tag{5.2.2}$$

where X_1 and X_2 are the amplitudes of vibration of the two masses under steady conditions. This is just a guess work and we donot know yet if such a solution is possible. If our guess is wrong we will land into some absurd results.

Substituting equations (5.2.2) in one of the equations (5.2.1), say the first one, we have

$$-m_1 \omega_1^2 X_1 \sin \omega_1 t + (k_1 + k_2) X_1 \sin \omega_1 t = k_2 X_2 \sin \omega_2 t$$

or $$\frac{X_1}{X_2} = \frac{k_2 \sin \omega_2 t}{[(k_1 + k_2) - m_1 \omega_1^2] \sin \omega_1 t}$$

Since X_1 and X_2 are the amplitudes of two harmonic motions, therefore their ratio must be constant and independent of time.

Therefore, $$\frac{\sin \omega_2 t}{\sin \omega_1 t} = C \text{ (a constant).}$$

This should be true at all times.

Let us see what happens if $C > 1$.

At time $t = \dfrac{\pi}{2\omega_1}$, $\sin \omega_1 t = \sin \dfrac{\omega_1 \pi}{2\omega_1} = 1$

$$\therefore \quad \frac{\sin \omega_2 t}{\sin \omega_1 t} = \frac{\sin \omega_2 t}{1} = C > 1$$

or $\sin \omega_2 t > 1$

which is not possible.

Thefore $C \ngtr 1$.

Similarly it can be shown that $C \nless 1$.

The only possibility, therefore is that $C = 1$.

Hence $$\frac{\sin \omega_2 t}{\sin \omega_1 t} = 1$$

which is only possible if $\omega_1 = \omega_2$.

So, the two harmonic motions, if at all they exist, have to be of the same frequency. Let us then modify the assumed solutions of equations (5.2.2) as:—

$$\left.\begin{aligned} x_1 &= X_1 \sin \omega t \\ x_2 &= X_2 \sin \omega t \end{aligned}\right] \tag{5.2.3}$$

Substituting equations (5.2.3) in equations (5.2.1) and cancelling out the common terms $\sin \omega t$, we have

$$\left.\begin{aligned} [-m_1 \omega^2 + (k_1 + k_2)]\, X_1 &= k_2\, X_2 \\ [-m_2 \omega^2 + (k_2 + k_3)]\, X_2 &= k_2\, X_1 \end{aligned}\right] \tag{5.2.4}$$

Equations (5.2.4) give the following two equations, respectively.

$$\frac{X_1}{X_2} = \frac{k_2}{[(k_1+k_2)-m_1\omega^2]} \tag{5.2.5}$$

$$\frac{X_1}{X_2} = \frac{[(k_2 + k_3) - m_2\omega^2]}{k_2} \tag{5.2.6}$$

From the above two equations, we have

$$\frac{k_2}{[(k_1 + k_2) - m_1\omega^2]} = \frac{[(k_2 + k_3) - m_2\omega^2]}{k_2} \tag{5.2.7}$$

which gives

$$[(k_1 + k_2) - m_1\omega^2]\,[(k_2 + k_3) - m_2\,\omega^2] = k_2^2$$

or

$$m_1m_2\omega^4 - [m_1\,(k_2+k_3)+m_2(k_1+k_2)]\omega^2 + [k_1\,k_2+k_1\,k_3+k_2\,k_3] = 0 \tag{5.2.8}$$

The above equation is a quadratic in ω^2 and gives two values of ω^2, and therefore two positive values of ω corresponding to the two natural frequencies ω_{n1} and ω_{n2} of the system. Equation (5.2.8) is called the *Frequency Equation* since the roots of this equation give the natural frequencies of the system.

Let us take a special case to gain more insight into the behaviour of the system.

Let
$$\left.\begin{aligned} k_1 &= k_3 = k \\ m_1 &= m_2 = m \end{aligned}\right] \tag{5.2.9}$$

Then the frequency equation (5.2.8) reduces to

$$m^2\omega^4 - 2m\,(k+k_2)\omega^2 + (k^2+2kk_2) = 0$$

which gives

$$\omega_{n1}, \omega_{n2} = \sqrt{\frac{(k+k_2)\pm k_2}{m}}$$

$$\left.\begin{aligned} \text{or} \quad \omega_{n1} &= \sqrt{\frac{k}{m}} \\ \text{and} \quad \omega_{n2} &= \sqrt{\frac{k+2k_2}{m}} \end{aligned}\right] \tag{5.2.10}$$

ω_{n1} and ω_{n2} being the first and the second natural frequencies of the system.

For this special case the equations (5.2.5) and (5.2.6), are reduced as below :—

$$\frac{X_1}{X_2} = \frac{k_2}{(k+k_2)-m\omega^2} \tag{5.2.11}$$

$$\frac{X_1}{X_2} = \frac{(k_2+k)-m\omega^2}{k_2} \tag{5.2.12}$$

In either of the equations (5.2.11) or (5.2.12), substituting for ω the value of ω_{n1} from the first of equations (5.2.10), we have

$$\left(\frac{X_1}{X_2}\right)_1 = +1$$

This means that when the system is vibrating with the first natural frequency ω_{n1}, the mode shape is such that the ratio of the amplitude of the two masses is equal to + 1. The suffix '1' after (X_1/X_2) means the ratio in the *first* mode shape corresponding to the first natural frequency.

Again, in either of the equations (5.2.11) or (5.2.12), substituting for ω the value of ω_{n2} from the second of equations (5.2.10), we have

$$\left(\frac{X_1}{X_2}\right)_2 = -1$$

which means that when the system is vibrating with the second natural frequency ω_{n2}, the mode shape is such that the ratio of the amplitudes of the two masses is equal to − 1. The suffix '2' after (X_1/X_2) means the ratio in the *second* mode shape corresponding to the second natural frequency.

The ratio of the amplitude of the two masses being + 1 means that the amplitudes are equal and the two motions are in phase, that is, the two masses move up or down together.

The ratio of the amplitudes of the two masses being -1 means that the amplitudes are equal but the two motions are out of phase, that is, one mass moving down and the other moving up, and vice-versa. In the first case every point in the system executes harmonic motion of frequency ω_{n1} and in the second case that of frequency ω_{n2}. This motion where every point in the system executes harmonic motion with one of the natural frequencies of the system, is called the *Principal Mode of Vibration.* A system having two degrees of freedom can vibrate in two principal modes of vibrations corresponding to its two natural frequencies, the mode shape being given by either of the equations (5.2.11) or (5.2.12) after substituting for ω the corresponding value of the natural frequency.

We have seen that in the first principal mode of vibration, the mode shape is given by $(X_1/X_2)_1 = +1$. Here the two masses move in phase with the same amplitude. Thus, the coupling spring k_2 of Fig. 5.2.2. (a) is neither stretched nor compressed during vibrations, It moves bodily with both the masses, and is therefore completely ineffective. Even if this coupling spring is removed [Fig. 5.2.2 (b)], the two masses will vibrate as two single degree of freedom systems with natural frequency $\sqrt{k/m}$ which was obtained to be the first natural frequency of the system in the first of equations (5.2.10).

Again, the second mode shape is given by $(X_1/X_2)_2 = -1$. Here two masses move out of phase with the same amplitude. Thus the spring k_2 is symmetrically compressed and stretched alternately. The middle point of this spring does not have any motion and is effectively grounded, giving us an equivalent system as in Fig. 5.2.2 (c). Each half of this spring has double the stiffness $2k_2$. And the original system is now converted to two single degree of freedom systems each having the natural frequency $\sqrt{(k+2k_2)/m}$, which was obtained to be the second natural frequency of the system in the second of equations (5.2.10).

And lastly, we may conclude that, if the two masses of Fig. 5.2.2 (a) are given equal initial displacements in the same direction and released, they will vibrate in the first principal mode of vibration with the first natural frequency. Also, if

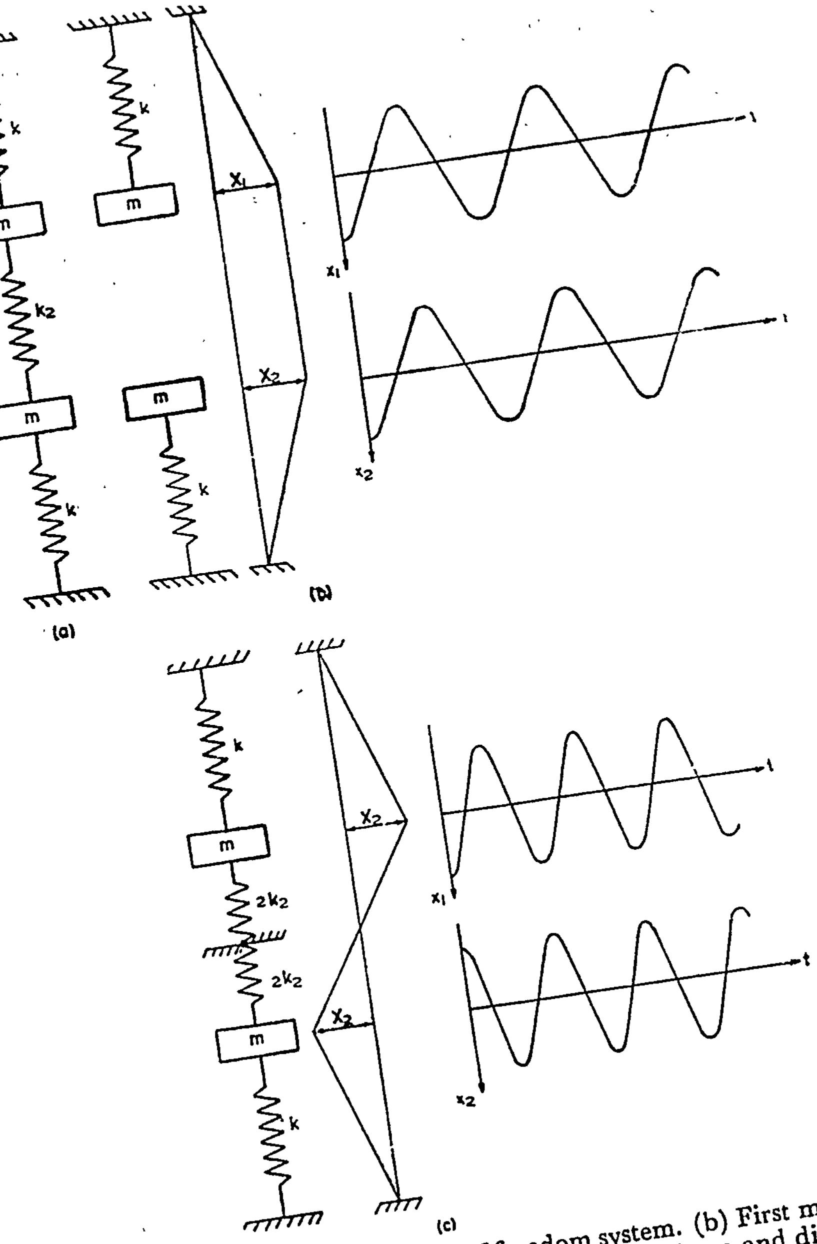


Fig. 5.2.2. (a) Two degrees of freedom system. (b) First mo vibration with equivalent system, mode shape and dis ment-time plot. (c) Second mode of vibration equivalent system, mode shape and displacement-tim

they are given equal initial displacements in the opposit direction and released, they will vibrate in the second principa mode of vibration with the second natural frequency. If however, the two masses are given unequal initial displacement in any direction, their motion will be superposition of tw harmonic motions corresponding to the two natural frequencie as given below.

$$\left.\begin{aligned} x_1 &= X_1' \cos \omega_{n1}t + X_1'' \cos \omega_{n2}t \\ x_2 &= X_2' \cos \omega_{n1}t + X_2'' \cos \omega_{n2}t \end{aligned}\right] \tag{5.2.13}$$

where X_1' and X_1'' are the amplitudes of mass m_1 at lower and higher natural frequencies respectively ; and X_2' and X_2'' are the amplitudes of mass m_2 at lower and higher natural frequencies respectively. They have the following relationships.

$$\left.\begin{aligned} \frac{X_1'}{X_2'} &= \left(\frac{X_1}{X_2}\right)_1 \\ \frac{X_1''}{X_2''} &= \left(\frac{X_1}{X_2}\right)_2 \\ X_1' + X_1'' &= \text{initial displacement of } m_1 \\ X_2' + X_2'' &= \text{initial displacement of } m_2 \end{aligned}\right] \tag{5.2.14}$$

Any motion can be broken up into the sum of two principal motions having two different natural frequencies ω_{n1} and ω_{n2}. In the general system of Fig. 5.2.1 (a), the expressions for the two principal mode shapes are not so simple and the coupling element can not be eliminated or split up as in the special case discussed above. However, equations (5.2.13) are applicable to the general case.

Further, in the principal mode of vibration, no mention has been made to the absolute values of the amplitudes, which depend upon the initial conditions. If the amplitude for one of the masses is taken as unity, the principal mode is said to be *Normal Mode of Vibration.*

Illustrative Example 5.2.1

For the system shown in Fig. 5.2.1 (a), find the two natural frequencies when

$$m_1 = m_2 = m = 0.01 \text{ kg-sec}^2/\text{cm}$$
$$k_2 = K_3 = k = 9\,0 \text{ kg/cm}$$
$$k_2 = 3.5 \text{ kg/cm}.$$

Find out the resultant motions of m_1 and m_2 for the following different cases.

The displacements mentioned below are from the equilibrium positions of the respective masses.

(a) Both masses are displaced $\frac{1}{2}$ cm in the downward direction and released simultaneously.

(b) Both masses are displaced $\frac{1}{2}$ cm; m_1 in the downward direction and m_2 in the upward direction, and released simultaneously.

(c) Mass m_1 is displaced $\frac{1}{2}$ cm downward and mass m_2 is displaced $\frac{3}{4}$ cm downward. Both masses are released simultaneously.

(d) Mass m_1 is displaced $\frac{1}{2}$ cm upward while mass m_2 is held fixed. Both masses are then released simultaneously.

Solution

The above system corresponds to the special case of Fig. 5.2.2 (a) for which the two natural frequencies are obtained in equations (5.2.10); or

$$\omega_{n1} = \sqrt{\frac{k}{m}} = \sqrt{\frac{9}{0.01}} = 30 \text{ rad/sec.}$$

$$\omega_{n2} = \sqrt{\frac{k+2k_2}{m}} = 1, \sqrt{\frac{9+2\times 3.5}{0.01}} = 40 \text{ rad/sec.}$$

Mode shapes for this system have been shown to be

$$\left(\frac{X_1}{X_2}\right)_1 = 1, \left(\frac{X_1}{X_2}\right)_2 = -1$$

Since the system is given only initial displacements, the equations (5.2.13) apply for these conditions. If, however, the system is given only initial velocities, the *cosine* functions will have to be replaced by *sine* functions.

(a) Applying the relationship of equations (5.2.14), we have.

$$\frac{X_1'}{X_2'} = 1$$

$$\frac{X_1''}{X_2''} = -1$$

$$X_1' + X_1'' = \frac{1}{2}$$

$$X_2' + X_2'' = \frac{1}{2}$$

These give $\quad X_1' = X_2' = \frac{1}{2}$

and $\quad X_1'' = X_2'' = 0$

Substituting these in equations (5.2.13), we obtain

$$x_1 = \tfrac{1}{2} \cos 30t$$
$$x_2 = \tfrac{1}{2} \cos 30t$$

This is obvious also since initial displacement are such as to give the first mode shape to the system. **Ans.**

(b) In this case the displacements are such as to give the second mode shape to the system. The equations of motion, therefore, are

$$x_1 = \tfrac{1}{2} \cos 40t$$
$$x_2 = -\tfrac{1}{2} \cos 40t$$

These can also be obtained by applying equations (5 2.14). The negative sign included in the expression for x_2 above means that its initial displacement is in the negative direction (upwards). **Ans.**

(c) Applying equations (5.2.14)

$$\frac{X_1'}{X_2'} = 1$$

$$\frac{X_1''}{X_2''} = -1$$

$$X_1' + X_1'' = \frac{1}{2}$$

$$X_2' + X_2'' = \frac{3}{4}$$

These give $\quad X_1' = \frac{5}{8},\ X_1'' = -\frac{1}{8},$

$\quad X_2' = \frac{5}{8},\ X_2'' = \frac{1}{8}$

Substituting these values in equation (5.2.13), we have

$$x_1 = \tfrac{5}{8} \cos 30t - \tfrac{1}{8} \cos 40t$$
$$x_2 = \tfrac{5}{8} \cos 30t + \tfrac{1}{8} \cos 40t$$ **Ans.**

(d) The solution for this case, now, can be written down by inspection, as

$$x_1 = -\tfrac{1}{4}\cos 30t - \tfrac{1}{4}\cos 40t$$
$$x_2 = -\tfrac{1}{4}\cos 30t + \tfrac{1}{4}\cos 40t$$

This may be checked by the usual method. **Ans.**

5 3 Other cases of simple two degrees of freedom systems.

In the following paragraphs we will study three different systems of two degrees of freedom and find out the two natural frequencies and the corresponding mode shapes.

5.3 A Two masses fixed on a tightly stretched string. Consider two masses m_1 and m_2 fixed on a tight string stretched between two supports as shown in Fig. 5.3.1 (a), and having a tension T. Let the amplitude of vibration of the two masses be small and tension T large so that it remains appreciably constant during the vibrations of the two masses. At any instant let y_1 and y_2 be the displacements of the two masses respectively as shown in Fig. 5.3.1 (a). The free body diagrams of the two masses are shown in Fig. 5.3.1 (b). The components of the tension T along the original direction of the string are $T\cos\phi_1$, $T\cos\phi_3$ and $T\cos\phi_3$, and each one of these is approximately equal to T for small amplitudes since the angles ϕ_1, ϕ_2 and ϕ_3 are small. Therefore, there is no resultant force on the masses in the original direction of the string.

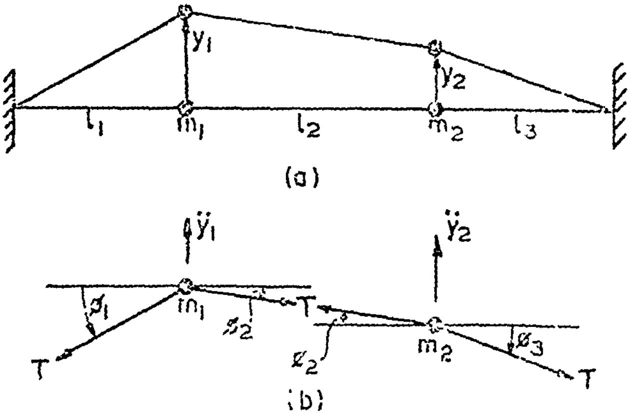


Fig. 5.3.1. Analysis of a system consisting of two masses fixed on a tightly stretched string.

Writing down the equations for lateral motion of the masses, we have

$$m_1\ddot{y}_1 = -\,T\sin\phi_1 - T\sin\phi_2$$
$$m_2\ddot{y}_2 = +\,T\sin\phi_2 - T\sin\phi_3$$

Substituting in the above two equations the values of

$$\left.\begin{aligned}\sin\phi_1 &= \frac{y_1}{l_1}\\ \sin\phi_2 &= \frac{y_1 - y_2}{l_2}\\ \sin\phi_3 &= \frac{y_1}{l_3}\end{aligned}\right]\quad \text{Ture for small } \phi_1, \phi_2, \phi_3$$

and rearranging, we get

$$\left.\begin{aligned}m_1\ddot{y}_1 + \left(\frac{T}{l_1} + \frac{T}{l_2}\right)y_1 &= \frac{T}{l_2}\,y_2\\ m_2\ddot{y}_2 + \left(\frac{T}{l_2} + \frac{T}{l_3}\right)y_2 &= \frac{T}{l_2}\,y_1\end{aligned}\right] \tag{5.3.1}$$

Assume, for the principal mode of vibration, the solutions to be

$$\left.\begin{aligned}y_1 &= Y_1\sin\omega t\\ y_2 &= Y_2\sin\omega t\end{aligned}\right] \tag{5 3.2}$$

Substituting the above solutions in equations (5.3.1) and cancelling out the common term $\sin\omega t$, we have

$$\left.\begin{aligned}\left[-m_1\omega^2 + \left(\frac{T}{l_1} + \frac{T}{l_2}\right)\right]Y_1 &= \frac{T}{l_2}Y_2\\ \left[-m_2\omega^2 + \left(\frac{T}{l_2} + \frac{T}{l_3}\right)\right]Y_2 &= \frac{T}{l_2}Y_1\end{aligned}\right] \tag{5.3.3}$$

Equations (5.3.3) give the following two equations respectively.

$$\frac{Y_1}{Y_2} = \frac{T/l_2}{\left(\dfrac{T}{l_1} + \dfrac{T}{l_2}\right) - m_1\omega^2} \tag{5.3.4}$$

$$\frac{Y_1}{Y_2} = \frac{\left(\dfrac{T}{l_2} + \dfrac{T}{l_3}\right) - m_2\omega^2}{T/l_2} \tag{5.3.5}$$

The frequency equation is obtained by equating the two expressions in the above two equations and cross multiplying.

To make the analysis simple, let us take a special case when

$$m_1 = m_2 = m$$
$$l_1 = l_2 = l_3 = l$$

as shown in Fig. 5.3.2 (a). The equations (5.3.4) and (5.3.5) become

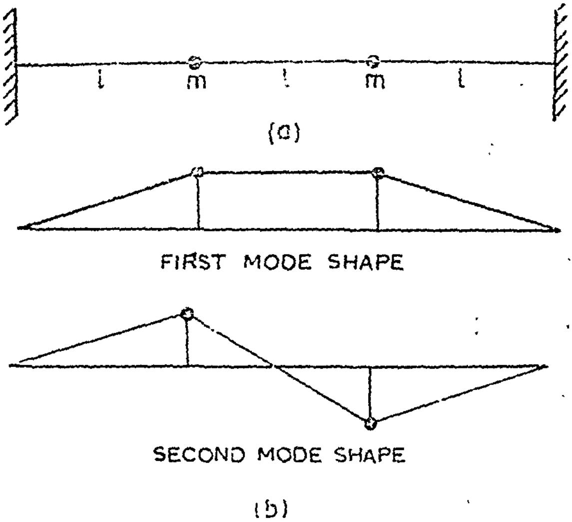


Fig. 5.3.2. Mode shapes.

$$\frac{Y_1}{Y_2} = \frac{T/l}{\frac{2T}{l} - m\omega^2} \tag{5.3.6}$$

$$\frac{Y_1}{Y_2} = \frac{\frac{2T}{l} - m\omega^2}{T/l} \tag{5.3.7}$$

from which we have

$$\frac{T/l}{\frac{2T}{l} - m\omega^2} = \frac{\frac{2T}{l} - m\omega^2}{T/l}$$

or
$$\frac{2T}{l} - m\omega^2 = \pm \frac{T}{l} \tag{5.3.8}$$

which is the frequency equation.

Solving for ω, we have the two values of the natural frequencies given by

$$\left.\begin{aligned} \omega_{n1} &= \sqrt{\frac{T}{ml}} \\ \omega_{n2} &= \sqrt{\frac{3T}{ml}} \end{aligned}\right] \tag{5.3.9}$$

The corresponding principal mode shapes are obtained by substituting in either of equations (5.3.6) or (5.3.7) for ω the values of ω_{n1} or ω_{n2} from equations (5.3.9). These mode shapes come out to be

$$\left(\frac{Y_1}{Y_2}\right)_1 = +1$$

$$\left(\frac{Y_1}{Y_2}\right)_2 = -1$$

and are shown in Fig. 5.3.2 (b).

5.3B Double pendulum. It consists of two point masses m_1 and m_2 suspended by strings of length l_1 and l_2 as shown in Fig. 5.3.3 (a). Let the system vibrate in a vertical plane with small amplitudes, under which conditions the masses may be considered to have only horizontal motion. Let θ_1 and θ_2 be the angles the strings at any instant make with the vertical and let x_1 and x_2 be the horizontal displacements of the two masses, such that the following relationships hold good for small amplitudes.

$$\left.\begin{aligned} \sin\theta_1 &= \theta_1 = \frac{x_1}{l_1} \\ \sin\theta_2 &= \theta_2 = \frac{x_2 - x_1}{l_2} \end{aligned}\right] \tag{5.3.10}$$

Fig. 5.3 3 (b) shows the free body diagrams for the two masses. Considering no motion in the vertical direction, the vertical components of the forces on each of the two masses must balance. Therefore,

$$T_2 \cos\theta_2 = m_2 g$$

and $$T_1 \cos\theta_1 = m_1 g + T_2 \cos\theta_2$$

For small values of θ_1 and θ_2, the above relations reduce to

$$\left.\begin{aligned} T_2 &= m_2 g \\ \text{and} \quad T_1 &= m_1 g + T_2 = (m_1 - m_2) g \end{aligned}\right] \tag{5.3.11}$$

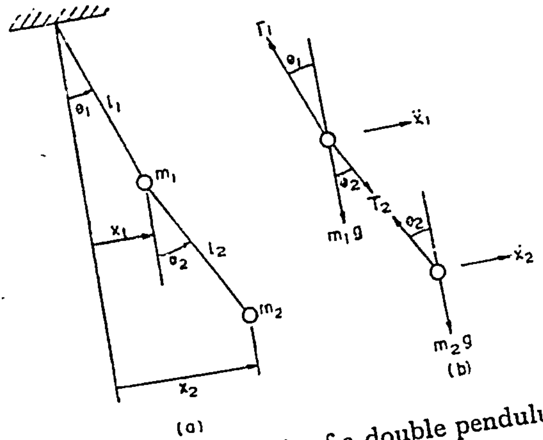


Fig. 5.3.3. Analysis of a double pendulum.

Writing down the differential equations of tho two masses for motion in horizontal direction, we have

$$m_1 \ddot{x}_1 = T_2 \sin \theta_2 - T_1 \sin \theta_1$$

$$m_2 \ddot{x}_2 = -T_2 \sin \theta_2$$

With the help of equations (5.3.10) and (5.3.11), the above two equations become

$$\left.\begin{aligned} m_1 \ddot{x}_1 + \left[\frac{m_1+m_2}{l_1} + \frac{m_2}{l_2}\right] g x_1 &= \frac{m_2}{l_2}\, g x_2 \\ m_2 \ddot{x}_2 + \frac{m_2}{l_2}\, g x_2 &= \frac{m_2}{l_2}\, g x_1 \end{aligned}\right] \quad (5.3.12$$

Assume, for the principal modes of vibration, the solutio (5.2.

to be

$$\left.\begin{aligned} x_1 &= X_1 \sin \omega t \\ x_2 &= X_2 \sin \omega t \end{aligned}\right]$$

Substituting the above solutions in equations (5.3.12) cancelling out the common term $\sin\omega t$, we get

$$\left.\begin{aligned} \left\{-m_1 \omega^2 + \left[\frac{m_1+m_2}{l_1} + \frac{m_2}{l_2}\right] g\right\} X_1 &= \frac{m_2}{l_2} g\, X_2 \\ \left[-m_2 \omega^2 + \frac{m_2}{l_2}\, g\right] X_2 &= \frac{m_2}{l_2} g\, X_1 \end{aligned}\right] \quad (5$$

Equations (5.3.14) give the following two equ respectively.

$$\frac{X_1}{X_2} = \frac{\dfrac{m_2}{l_2}\, g}{\left[\dfrac{m_1+m_2}{l_1} + \dfrac{m_2}{l_2}\right] g - m_1\, \omega^2} \tag{5.3.15}$$

$$\frac{X_1}{X_2} = \frac{\dfrac{m_2}{l_2}\, g - m_2\, \omega^2}{\dfrac{m_2}{l_2}\, g} \tag{5.3.16}$$

The frequency equation is obtained by equating the two expressions in the above two equations and cross multiplying.

Let us again consider a special case when

$$m_1 = m_2 = m$$
$$l_1 = l_2 = l$$

as shown in Fig. 5.3.4 (a).

The equations (5.3.15) and (5.3.16) become

$$\frac{X_1}{X_2} = \frac{\dfrac{g}{l}}{\dfrac{3g}{l} - \omega^2} \tag{5.3.17}$$

$$\frac{X_1}{X_2} = \frac{\dfrac{g}{l} - \omega^2}{\dfrac{g}{l}} \tag{5.3.18}$$

from which we have

$$\frac{\dfrac{g}{l}}{\dfrac{3g}{l} - \omega^2} = \frac{\dfrac{g}{l} - \omega^2}{\dfrac{g}{l}}$$

or
$$\frac{\omega^4}{g^2} - \frac{4\,\omega^2}{lg} + \frac{2}{l^2} = 0 \tag{5.3.19}$$

which is the frequency equation.

Solving for ω, we have the two values of the natural frequencies as

$$\left.\begin{aligned} \omega_{n1} &= \sqrt{\frac{g}{l}\,(2-\sqrt{2})} \\ \omega_{n2} &= \sqrt{\frac{g}{l}\,(2+\sqrt{2})} \end{aligned}\right] \tag{5.3.20}$$

The corresponding principal modes are obtained as

$$\left(\frac{X_1}{X_2}\right)_1 = -1 + \sqrt{2} = +0.414$$

$$\left(\frac{X_1}{X_2}\right)_2 = -1 - \sqrt{2} = -2.414$$

These mode shapes are shown in Fig. 5.3.4 (b).

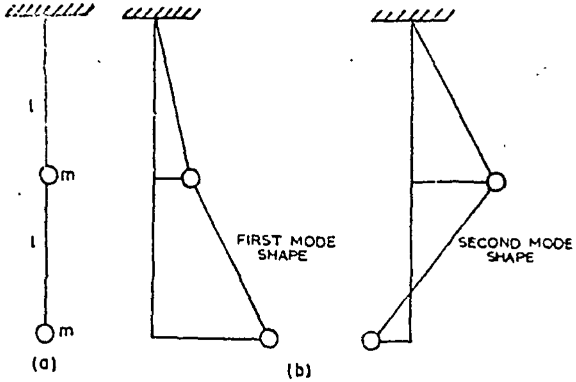


Fig. 5.3.4. Mode shapes.

5.3C Torsional system. Consider a two rotor torsional system as shown in Fig. 5 3.5 (a) in which J_1 and J_2 are the two respective mass moments of inertias coupled by a shaft of torsional stiffness k_t. Let, at any instant, the displacements of the two respective inertias be θ_1 and θ_2 from a certain reference position, positive in the clockwise direction when viewed from the left, as shown in Fig. 5.3.5 (b). The twist in the shaft is $(\theta_2-\theta_1)$, and for the relative displacements of the rotors shown, the shaft exerts a torque of $k_t(\theta_2-\theta_1)$ on J_1 in the direction of rotation, and the same torque on J_2 opposite to the direction of rotation. Therefore, the differential equations of motion are

$$\left.\begin{aligned} J_1\ddot{\theta}_1 &= k_t\,(\theta_2-\theta_1) \\ \text{and} \qquad J_2\ddot{\theta}_2 &= -k_t\,(\theta_2-\theta_1) \end{aligned}\right] \qquad (5.3.21)$$

These equations may be written as

$$\left.\begin{aligned} J_1\ddot{\theta}_1 + k_t\,\theta_1 &= k_t\,\theta_2 \\ \text{and} \qquad J_2\ddot{\theta}_2 + k_t\,\theta_2 &= k_t\,\theta_1 \end{aligned}\right] \qquad (5.3.22)$$

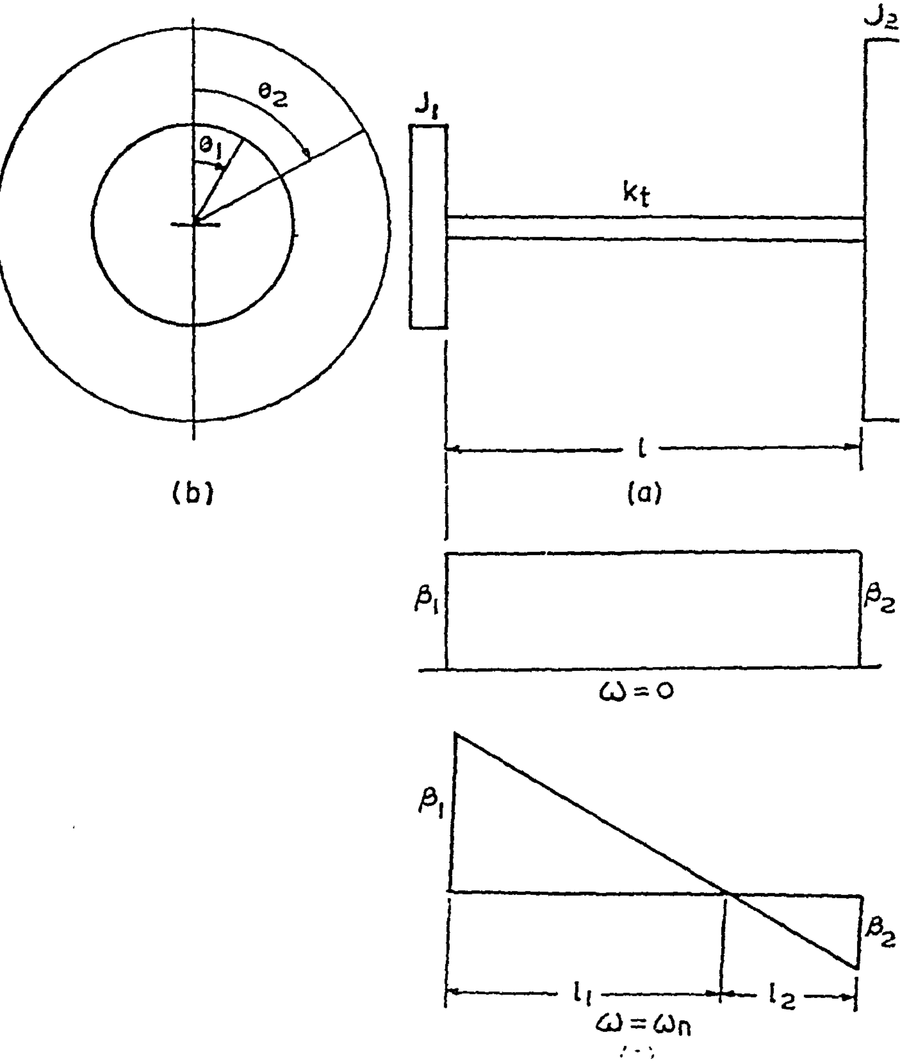


Fig. 5.3.5. Two rotor torsional system.

Let, for the principal modes of vibration, the solutions be

$$\left.\begin{aligned} \theta_1 &= \beta_1 \sin \omega t \\ \theta_2 &= \beta_2 \sin \omega t \end{aligned}\right] \tag{5.3.23}$$

Substituting these solutions in equations (5.3.22) and cancelling out the common term $\sin \omega t$, we get

$$\left.\begin{aligned}(-J_1\,\omega^2 + k_t)\,\beta_1 &= k_t\,\beta_2 \\ (-J_2\,\omega^2 + k_t)\,\beta_2 &= k_t\,\beta_1\end{aligned}\right] \tag{5.3.24}$$

which further give the following two equations

$$\frac{\beta_1}{\beta_2} = \frac{k_t}{-J_1\omega^2+k_t} \tag{5.3.25}$$

$$\frac{\beta_1}{\beta_2} = \frac{-J_2\omega^2+k_t}{k_t} \tag{5.3.26}$$

Equating the two expressions on the right hand sides of the above two equations and cross multiplying, we have the frequency equation as

$$\omega^2\,[J_1\,J_2\,\omega^2 - k_t\,(J_1+J_2)] = 0 \tag{5.3.27}$$

which gives

$$\left.\begin{aligned}\omega_{n1} &= 0 \\ \omega_{n2} &= \sqrt{\frac{k_t\,(J_1+J_2)}{J_1\,J_2}} = \omega_n \text{ (say)}\end{aligned}\right] \tag{5.3.28}$$

One of the two natural frequencies of this system is, therefore, zero. The fact is further clarified when we obtain the mode shapes, which come out to be

$$\left.\begin{aligned}\left(\frac{\beta_1}{\beta_2}\right) &= +1, \text{ when } \omega = \omega_{n1} = 0 \\ \left(\frac{\beta_1}{\beta_2}\right) &= -\frac{J_2}{J_1}, \text{ when } \omega = \omega_{n2} = \omega_n\end{aligned}\right] \tag{5.3.29}$$

These mode shapes are shown in Fig. 5.3.5 (c). Here we see that the mode shape corresponding to $\omega = 0$ is such that the shaft is not twisted at all. Therefore no elastic force are set up in the shaft and so the vibrations can not take place. Thus, the system is not really a two degree freedom system although, loosely speaking, it is said to be so since it has two inertia points. This system has only one natural frequency ω_n other than zero.

This system could have been tackled on the basis of relative displacement of the two rotors which is also equal to the twist of the shaft.

Multiplying the first of equations (5.3.21) by J_2 and the second by J_1, and subtracting from the other, we have

$$J_1 J_2 (\ddot{\theta}_1 - \ddot{\theta}_2) = k_t (J_2 + J_1)(\theta_2 - \theta_1)$$

Replacing $(\theta_2 - \theta_1)$, the relative displacement, by θ, we have

$$J_1 J_2 \ddot{\theta} + k_t (J_1 + J_2)\, \theta = 0 \tag{5.3.30}$$

This is the differential equation of relative motion which corresponds to a single degree of freedom system, giving,

$$\omega_n = \sqrt{\frac{k_t (J_1 + J_2)}{J_1 J_2}}$$

which is the same as obtained in the second equation (5.3.28).

For the only mode in which the vibrations are possible, the mode shape is given by the second equation (5.3.29), which means that the displacements of the two rotors are in opposite directions and are inversely proportional to their inertias. Thus if l is the total length of the connecting shaft, then the node point divides the shaft in two lengths l_1 and l_2 such that

$$\frac{l_1}{l_2} = \frac{J_2}{J_1}$$

or $$\frac{l_1}{l_1 + l_2} = \frac{J_2}{J_1 + J_2}$$

$$\left.\begin{aligned} \text{or} \quad l_1 &= \left(\frac{J_2}{J_1 + J_2}\right) l \\ \text{and} \quad l_2 &= \left(\frac{J_1}{J_1 + J_2}\right) l \end{aligned}\right] \tag{5.3.31}$$

The torsional stiffness of the shaft is obtained from the well known equation in the Strengh of Materials text-books. This equation is

$$\frac{T}{I_p} = \frac{G\theta}{l}$$

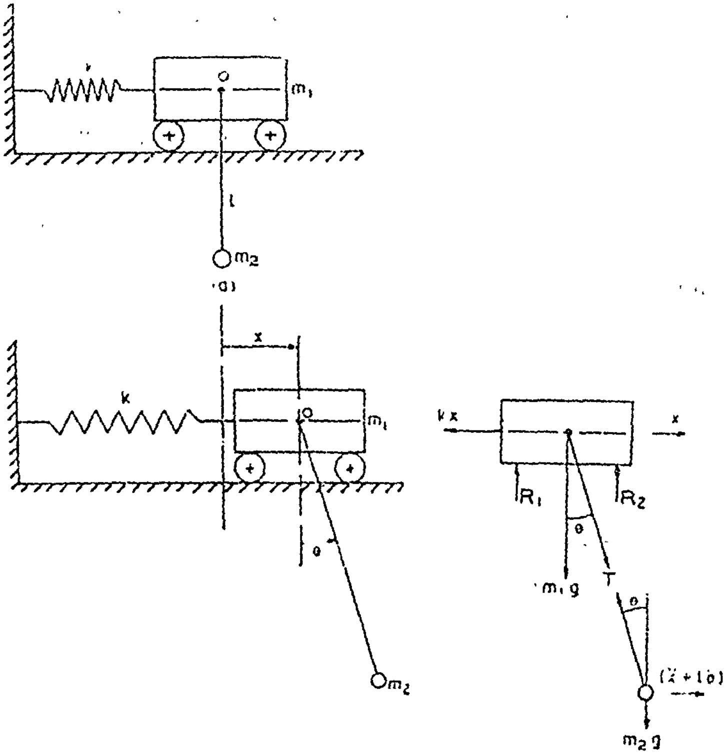
Therefore, the stiffness, which is the torque per unit twist, is given by

$$k_t = \frac{T}{\theta} = \frac{G I_p}{l} \tag{5.3.32}$$

showing that the stiffness is inversely proportional to the length of the shaft. Also the stiffness is directly proportional to I_p (or d^4).

Illustrative Example 5.3.1

Derive an expression for the natural frequencies and amplitude ratios for the system shown in Fig. 5.3.6 (a), for small displacement in the plane of the paper. The pendulum rod is stiff and weightless, and is pivoted at point O.



(b) (c)

Fig. 5.3.6. Combined pendulum and spring-mass system.

Solution

Consider the displaced position of the system at any instant as shown in Fig. 5.3.6 (b). The free body diagrams for both the masses are shown in Fig. 5.3.6 (c). For small angular displacements the motion of the mass m_2 is horizontal only.

Therefore,

$$m_2 g = T \cos\theta \simeq T$$

$$m_2\,(\ddot{x} + l\,\ddot{\theta}) = -\,T\sin\theta \simeq -\,T\,\theta$$

$$m_1\,\ddot{x} = -\,k\,x + T\sin\theta \simeq -\,k\,x + T\,\theta$$

The vertical forces on the mass m_1 balance and it is not necessary to write the corresponding equation.

Eliminating T from the above three equations,

we have

$$m_2\,(\ddot{x} + l\,\ddot{\theta}) + m_2 g\,\theta = 0$$

and $$m_1\ddot{x} + kx - m_2 g\,\theta = 0$$

or $$l\,\ddot{\theta} + g\,\theta = -\,\ddot{x}$$

and $$m_1\ddot{x} + kx = m_2 g\,\theta$$

Let the solution for the principal modes be

$x = X\sin\omega t$

$\theta = \beta\sin\omega t$

Substituting these solutions in the above two differential equations and cancelling out $\sin\omega t$, we have

$$-l\omega^2\beta + g\beta = \omega^2 X$$

and $$-\,m_1\,\omega^3 X + kX = m_2 g\beta$$

or $$\frac{X}{\beta} = \frac{g-l\omega^2}{\omega^3} = \frac{m_2 g}{k-m_1\omega^2}$$

The frequency equation obtained is

$$m_1 l\omega^4-(m_1 g + m_2 g + kl)\,\omega^2 + kg = 0$$

which gives

$$\omega^2 = \frac{(m_1 + m_2)g + kl \pm \sqrt{[(m_1 + m_2)g + kl]^2-4m_1 l\,kg}}{2\,m_1 l}$$

Ans.

These are the two natural frequencies of the system. It is of interest to study the following special cases and to compare the results obtained with the corresponding physical systems. These special cases are the natural frequencies obtained from the above equation when

(i) $k = \infty$;

(i i) $m_2 = 0$; and

(iii) $l = 0$.

This is left for the students to study.

Illustrative Example 5.3.2

Two uniform rods AB and CD are pivoted at their upper ends as shown in Fig. 5.3.7 (a). Their lower ends are at the same level and are connected by a spring. Each rod weighs 5 kg/m and is vertical in equilibrium position with the spring unstrained. The spring has a stiffness of 3 kg/cm.

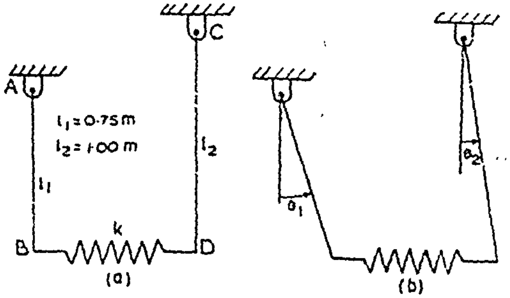


Fig. 5.3.7. Spring-connected double pendulum.

The spring is now compressed slightly and released. Find the frequency of the resulting vibrations if the effect of gravity is neglected. If AB moves through 1° on either side of the vertical, find the corresponding angular amplitude of CD and the maximum force in the spring.

Solution

Consider the displaced position of the system at any instant, as shown in Fig. 5.3.7 (b).

$$J_A = \frac{M_1 l_1^2}{3}$$

$$J_C = \frac{M_2 l_2^2}{3}$$

where M_1 and M_2 are the masses of the two rods respectively.

Neglecting gravity, the equations of motion can be written as below.

$$J_A \ddot{\theta}_1 = -(l_1 \theta_1 - l_2 \theta_2) k l_1$$

$$J_C \ddot{\theta}_2 = +(l_1 \theta_1 - l_2 \theta_2) k l_2$$

or $$\frac{M_1 l_1}{3}\ddot{\theta}_1 + kl_1\theta_1 = kl_2\theta_2$$

$$\frac{M_2 l_2}{3}\ddot{\theta}_2 + kl_2\theta_2 = kl_1\theta_1$$

Letting $\theta_1 = \beta_1 \sin \omega t$
and $\theta_2 = \beta_2 \sin \omega t$
for principal modes of vibration, and substituting these in the differential equations, leads to the following equations after simplification.

$$\frac{\beta_1}{\beta_2} = \frac{kl_2}{kl_1 - \frac{M_1 l_1}{3}\omega^2} = \frac{kl_2 - \frac{M_2 l_2}{3}\omega^2}{kl_1}$$

These give

$$\omega_n^2 = \frac{3(M_1 + M_2)k}{M_1 M_2}$$ besides a zero value.

Now $M_1 = \frac{5 \times 0.75}{980}$ kg-sec²/cm

and $M_2 = \frac{5 \times 1.00}{980}$ kg-sec²/cm

Substitution in the above equation gives

$\omega_n = 64.1$ rad/sec

or $f_n = 10.2$ cycles/sec.

The reason that one of the natural frequencies is zero, is because the gravity has been neglected. If gravity is considered two non-zero frequencies will result. **Ans.**

Now $$\frac{\beta_1}{\beta_2} = \frac{kl_2}{kl_1 - \frac{M_1 l_1}{3}\omega^2} = \frac{3 \times 100}{3 \times 75 - \frac{5 \times 0.75}{980} \times \frac{75}{3} \times 64.1^2}$$

or $$\frac{\beta_1}{\beta_2} = -\frac{1}{0.55}$$ **Ans.**

Angular movement of CD = 0.55°, and it is out of phase with the movement of AB.

Maximum force in the spring

$$= k(l_1\beta_1 - l_2\beta_2)$$

$$= 3\left[\frac{75 \times 1}{57.3} + \frac{100 \times 0.55}{57.3}\right] = 6.83 \text{ kg.}$$ **Ans.**

Illustrative Example 5.3.3

Determine the natural frequency of torsional vibrations of a shaft with two circular discs of uniform thickness at the ends. The weights of the discs are $W_1 = 500$ kg and $W_2 = 1000$ kg, and their outer diameters are $D_1 = 125$ cm and $D_2 = 190$ cm. The length of the shaft is $l=300$ cm and its diameter $d=10$ cm. Modulus of rigidity for the material of the shaft is $G=0.85\times 10^6$ kg/cm².

Also find in what proportion will the natural frequency of this shaft change if along half the length of the shaft the diameter is increased from 10 cm to 20 cm.

Solution

The system is shown in Fig. 5.3.8.

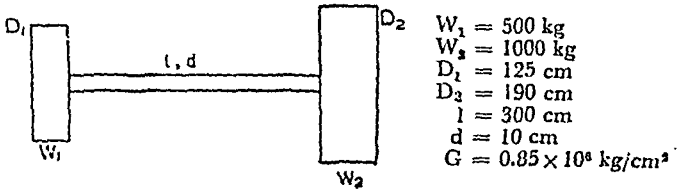


Fig. 5.3.8. Two-rotor torsional system.

$$J_1 = \frac{W_1}{g}\frac{r_1^2}{2} = \frac{500}{980} \times \frac{(125/2)^2}{2} = 1000 \text{ kg-cm-sec}^2$$

$$J_2 = \frac{W_2}{g}\frac{r_2^2}{2} = \frac{1000}{980} \times \frac{(190/2)^2}{2} = 4620 \text{ kg-cm-sec}^2$$

$$k_t = \frac{G\,I_p}{l} = \frac{(0.85\times 10^6)}{300} \times \left(\frac{\pi}{32}\times 10^4\right) = 2.78\times 10^6 \text{ kg-cm/rad}$$

Applying equation (5.3.28), i.e.,

$$\omega_n = \sqrt{\frac{k_t\,(J_1+J_2)}{J_1\,J_2}}\text{, we have}$$

$$\omega_n = \sqrt{\frac{2.78\times 10^6\times(1000+4620)}{1000\times 4620}}$$

or $\omega_n = 58.1$ rad/sec.

or $$f_n = \frac{58.1}{2\pi} = 9.25 \text{ c.p.s.}$$ **Ans.**

Now along half the lengh of the shaft the dia is doubled. Consider two halves of the shaft in series; one of the original diameter and the other of double the diameter. The stiffness of half the shaft with the original diameter will be $2k_t$. The stiffness of the other half of the shaft with twice the diameter will be $2k_t \times 2^4$ since the stiffness is propertional to the fourth power of the diameter. These two half shafts are in series. Therefore the equivalent stiffness is given by

$$\frac{1}{k_{te}} = \frac{1}{2\,k_t} + \frac{1}{2\,k_t \times 2^4}$$

Substituting for $k_t = 2.78 \times 10^6$ and solving, we have
$k_{te} = 5.23 \times 10^6$ kg-cm/rad.

Now, $$\frac{(\omega_n)_{\text{modified}}}{(\omega_n)_{\text{original}}} = \sqrt{\frac{k_{te}}{k_t}} = \sqrt{\frac{5.23}{2.78}} = 1.37$$

Hence there is 37% increase in the natural frequency of the system. **Ans.**

5.4 Combined rectilinear and angular modes.

In the previous section, cases have been discussed where two coordinates have been either both rectilinear to both angular. In this section we will discuss a system having combined rectilinear and angular modes.

Consider a body having mass M and mass moment of inertia J ($= Mr^2$, r being radius of gyration about the CG of the body) supported as shown in Fig. 5.4.1 (a), and capable of oscillating in the directions x and θ. Let, at any instant, the body be displaced through a rectilinear distance x and angular distance θ as shown in Fig. 5.4.1 (b). At this instant, taking θ to be small, the springs k_1 and k_2 are compressed through $(x - l_1\theta)$ and $(x + l_2\theta)$ respectively, beyond their equilibrium position.

The free body diagram of the system is shown in Fig. 5.4.1 (c). The differential equations of the system are written for motions in the x and θ directions by taking the forces and the moments in the respective directions acting on the system. These equations are

$$M\ddot{x} = -k_1(x - l_1\theta) - k_2(x + l_2)$$

$$J\ddot{\theta} = k_1(x - l_1\theta)\,l_1 - k_2(x + l_2\theta)\,l_2$$

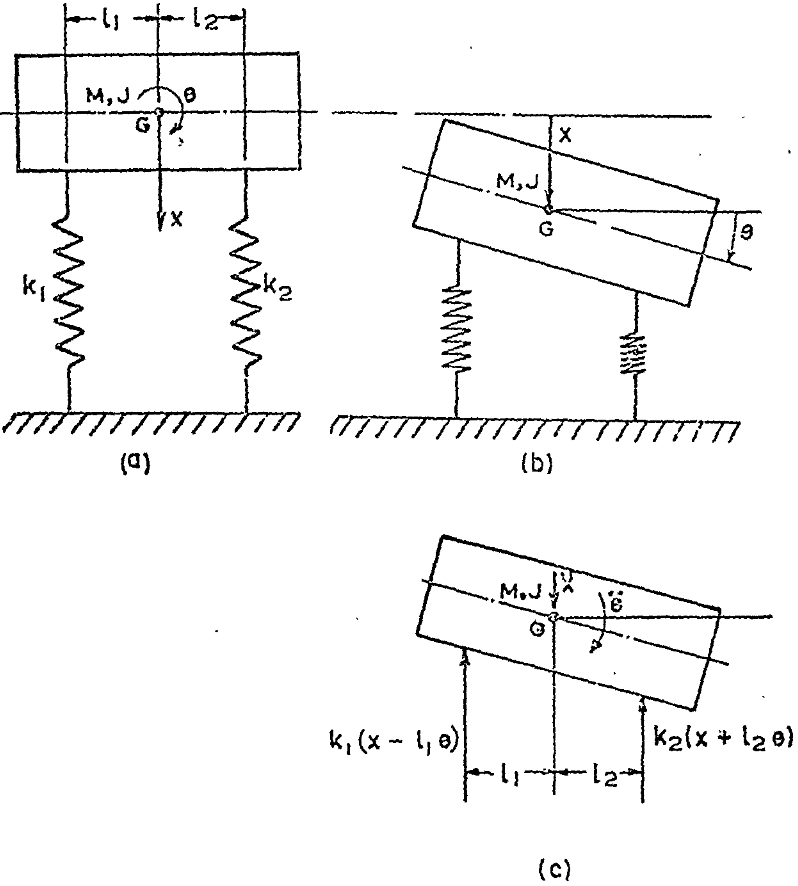


Fig. 5.4.1. Analysis of a combined rectilinear and angular mode system.

or
$$\left.\begin{aligned} M\ddot{x} + (k_1 + k_2)\,x &= (k_1 l_1 - k_2 l_2)\,\theta \\ J\,\ddot{\theta} + (k_1 l_1^2 + k_2 l_2^2)\,\theta &= (k_1 l_1 - k_2 l_2)x \end{aligned}\right] \tag{5.4.1}$$

Letting
$$\left.\begin{aligned} \frac{k_1 + k_2}{M} &= a \\ \frac{k_1 l_1 - k_2 l_2}{M} &= b \\ \text{and} \quad \frac{k_1 l_1^2 + k_2 l_2^2}{J} &= c \end{aligned}\right] \tag{5.4.2}$$

and remembering that $J = Mr^2$, the equations (5.4.1) reduce as below

$$\left.\begin{aligned} \ddot{x} + a\,x &= b\,\theta \\ \ddot{\theta} + c\,\theta &= (b/r^2)\,x \end{aligned}\right] \qquad (5.4.3)$$

As in the previous cases we end up with two differential equations which are coupled with respect to the two coordinates. Here b is called the coupling coefficient since, if $b = 0$, the two equations are independent of each other (uncoupled) and therefore the two motions, rectilinear and angular, can exist independently of each other with their respective natural frequencies $\sqrt{a}$ and $\sqrt{c}$. Thus for the case of uncoupled system when $b = 0$, i.e., $k_1\,l_1 = k_2\,l_2$, the natural frequencies in the rectilinear and angular modes respectively, are

$$\left.\begin{aligned} \omega_{n1} &= \sqrt{a} = \sqrt{\frac{k_1 + k_2}{M}} \\ \text{and} \qquad \omega_{n2} &= \sqrt{c} = \sqrt{\frac{k_1\,l_1^2 + k_2\,l_2^2}{J}} \end{aligned}\right] \qquad (5.4.4)$$

We could have written down these two natural frequencies in the very beginning by considering the system to be having single degree of freedom, successively, in rectilinear mode and angular mode.

Going bock to the coupled equations (5.4.3), let us assume the following solutions for the principal mode of vibration.

$$\left.\begin{aligned} x &= X \sin \omega t \\ \theta &= \beta \sin \omega t \end{aligned}\right] \qquad (5.4.5)$$

Substituting the above solutions in equation (5.4 3) and cancelling out the common term $\sin \omega t$, we have

$$\left.\begin{aligned} [-\omega^2 + a]\ X &= b\beta \\ [-\omega^2 + c]\ \beta &= (b/r^2)\ X \end{aligned}\right] \qquad (5.4.6)$$

From equations (5.4.6) we get the following two equations respectively.

$$\frac{X}{\beta} = \frac{b}{a-\omega^2} \qquad (5.4.7)$$

$$\frac{X}{\beta} = \frac{c-\omega^2}{b/r^2} \qquad (5.4.8)$$

Therefore,

$$\frac{b}{a-\omega^2}=\frac{c-\omega^2}{b/r^2}$$

which gives the frequency equation as

$$\omega^4-(a+c)\,\omega^2+[ac-(b^2/r^2)]=0 \tag{5.4.9}$$

The roots of the above equation give the following two natural frequencies of the system.

$$\left.\begin{aligned}\omega_{n1}^2&=\tfrac{1}{2}(a+c)-\sqrt{\tfrac{1}{4}(c-a)^2+\frac{b^2}{r^2}}\\ \omega_{n2}^2&=\tfrac{1}{2}(a+c)+\sqrt{\tfrac{1}{4}(c-a)^2+\frac{b^2}{r^2}}\end{aligned}\right] \tag{5.4.10}$$

The above two natural frequencies reduce to those of equations (5.4.4) for the uncoupled case when $b=0$.

The expressions for the mode shapes can be obtained in usual manner, but these expressions will not be much meaningful because of their complexity.

Illustrative Example 5.4.1

For the system shown in Fig. 5.4.2, find ratio l/d such that the two natural frequencies in the plane of the paper for the figure on the right are the same.

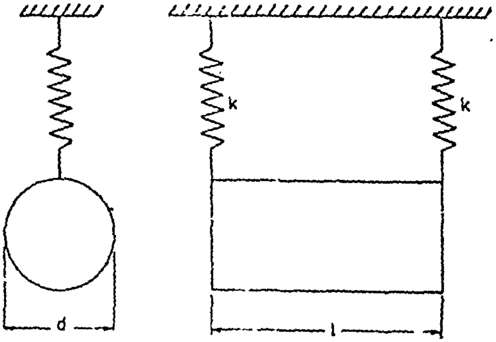


Fig. 5.4.2. A cylinder suspended by two springs.

Solution

Let M be the mass of the cylinder and J the mass moment of inertia about an axis through the CG perpendicular to the

plane of the figure on the right. Then

$$J = M\left[\frac{l^2}{12} + \frac{d^2}{16}\right]$$

The value of b as given in the second equation (5.4.2) is zero because of symmetry. Therefore the two natural frequencies are given by equations (5·4.4), and these should be equal.

Therefore, $$\sqrt{\frac{2\,k}{M}} = \sqrt{\frac{2\,k\,(l/2)^2}{M[(l^2/12)+(d^2/16)]}}$$

which gives $$\frac{l}{d} = \frac{\sqrt{3}}{2\sqrt{2}}$$ **Ans.**

5.5 Systems with damping.

Consider a two degrees of freedom system with damping as shown in Fig. 5.5.1 (a). At an instant when the masses m_1 and m_2 are displaced through distance x_1 and x_2 from the

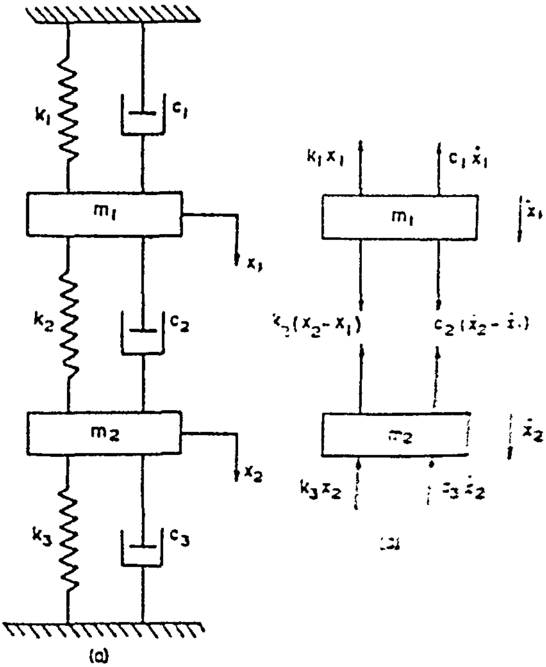


Fig. 5.5.1. A two degree of freedom system with multiple damping elements.

equilibrium position, the forces acting on the two masses are as shown in Fig. 5.5.1 (b).

Writing the differential equations of motion for the two masses,

$$m_1 \ddot{x}_1 = -k_1 x_1 - c_1 \dot{x}_1 + k_2 (x_2 - x_1) + c_2 (\dot{x}_2 - \dot{x}_1)$$
$$m_2 \ddot{x}_2 = -k_2 (x_2 - x_1) - c_2 (\dot{x}_2 - \dot{x}_1) - k_3 x_2 - c_3 \dot{x}_2$$

or, on rearranging,

$$\left.\begin{aligned} &[m_1 \ddot{x}_1 + (c_1+c_2)\dot{x}_1 + (k_1+k_2)x_1] - [c_2 \dot{x}_2 + k_2 x_2] = 0 \\ &[c_2 \dot{x}_1 + k_2 x_1] - [m_2 \ddot{x}_2 + (c_2+c_3)\dot{x}_2 + (k_2+k_3)x_2] = 0 \end{aligned}\right] \quad (5.5.1)$$

These are two coupled linear differential equations of second order and their solutions can be put down as

$$\left.\begin{aligned} x_1 &= A_1 e^{st} \\ x_2 &= A_2 e^{st} \end{aligned}\right] \quad (5.5.2)$$

where A_1, A_2 and s are constants.

Substituting equations (5.5.2) in equations (5.5.1) and cancelling out the common term e^{st}, we have

$$\left.\begin{aligned} &[m_1 s^2 + (c_1+c_2)s + (k_1+k_2)]\, A_1 - [c_2 s + k_2]\, A_2 = 0 \\ &[c_2 s + k_2]\, A_1 - [m_2 s^2 + (c_2+c_3)\, s + (k_2+k_3)]\, A_2 = 0 \end{aligned}\right] \quad (5.5.3)$$

The above equations will have values of A_1 and A_2 different from zero only if the determinant formed from their coefficients is zero, i. e. if

$$\begin{vmatrix} [m_1 s^2 + (c_1+c_2)s + (k_1+k_2)] & -[c_2 s + k_2] \\ [c_2 s + k_2] & -[m_2 s^2 + (c_2+c_3)s + (k_2+k_3)] \end{vmatrix} = 0$$

or, $$-[m_1 s^2 + (c_1+c_2)s + (k_1+k_2)]\,[m_2 s^2 + (c_2+c_3)s + (k_2+k_3)] + [c_2 s + k_2]^2 = 0$$

which after expanding and rearranging, becomes

$$s^4 + \left[\frac{c_1+c_2}{m_1} + \frac{c_2+c_3}{m_2}\right]s^3 + \left[\frac{k_1+k_2}{m_1} + \frac{k_2+k_3}{m_2} + \frac{c_1 c_2 + c_2 c_3 + c_3 c_1}{m_1 m_2}\right]$$
$$+ \left[\frac{k_1(c_2+c_3) + k_2(c_3+c_1) + k_3(c_1+c_2)}{m_1 m_2}\right] s + \left[\frac{k_1 k_2 + k_2 k_3 + k_3 k_1}{m_1 m_2}\right] = 0 \quad (5.5.4)$$

This equation is called the *characteristic equation* of the system and the values of s have to be obtained from this equation. There will be four values of s for which equations

(5.5.2) will be solutions of equations (5.5.1), and so the general solutions are

$$\left.\begin{aligned} x_1 &= A_{11}\, e^{s_1 t} + A_{12}\, e^{s_2 t} + A_{13}\, e^{s_3 t} + A_{14}\, e^{s_4 t} \\ x_2 &= A_{21}\, e^{s_1 t} + A_{22}\, e^{s_2 t} + A_{23}\, e^{s_3 t} + A_{24}\, e^{s_4 t} \end{aligned}\right] \qquad (5.5.5)$$

where the coefficients A_{1i} in first of these equations are four arbitrary constants to be determined from the initial conditions and the coefficients A_{2i} in the second equation are related to A_{1i} from either of the equations (5.5.3) by substituting for s the corresponding value of s_i obtained from equation (5.5.4).

Hence from the first of equations (5.5.3),

$$A_{2i} = \frac{[m_1 s_i^2 + (c_1 + c_2) s_i + (k_1 + k_2)]}{(c_2 s_i + k_2)} A_{1i} \qquad (5.5.6)$$

For a physical system with damping we can anticipate that the motion will die out after a certain time. Looking back at the solutions as obtained in equations (5.5.5), we can conclude that s_1, s_2, s_3 and s_4 must either be real negative numbers or complex numbers with negative real parts. When s_i is a real negative number, that part of the solution containing s_i will decay exponentially and so the corresponding motion will be aperiodic. When s_i is a complex number with negative real part, that part of the solution containing s_i will be an oscillatory motion which will also decay exponentially. If all the values of s are real negative numbers, the final motion will be a dead beat motion or aperiodic motion, being a superposition of four aperiodic motions. If two valus of s are real negative numbers and the other two complex conjugate numbers with real negative parts, then the final motion will be a superposition of two aperiodic motions and a damped oscillatory motion. For two complex conjugate roots of s the solution can be put in the form of damped free vibration as discussed in Sec. 3.3C.

Illustrative Example 5.5.1

For the system whose schematic diagram is shown in Fig. 5.5.2, find the mode shapes and the general equations of motion for the two masses for the case when

vibrations with undamped natural frequency $\sqrt{k/m}$ and damping coefficient $2c$ (or damping factor 2ζ). This is reflected in the values of $s_{3,4}$ obtained.

5.6 Undamped forced vibrations with harmonic excitation.

When a harmonic forcing function acts on a system, the solutions consist of the transient part and the steady state part. In the steady state part the vibrations of any point in the system take place at the frequency of excitation. It will be shown in the analysis that resonance in the system will occur when the exciting frequency equals any of the natural frequencies of the system.

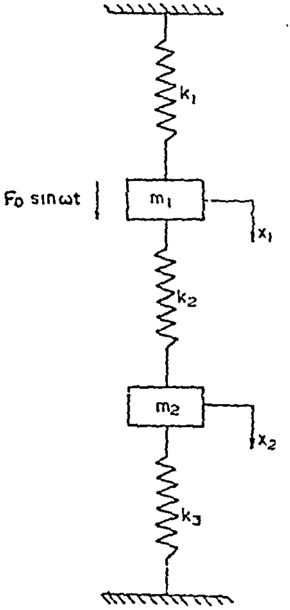


Fig. 5.6.1. Forced vibrations of a two degree of freedom system.

Consider again the system shown in Fig. 5.6.1 with the exciting force $F_0 \sin \omega t$ acting on mass m_1. The differential equations of motion for the two masses will be same as equations (5.2.1) except that the term $F_0 \sin \omega t$ will now be

introduced on the right hand side of the first of equations (5.2.1). Doing this and rearranging, gives

$$\left.\begin{aligned} [m_1\ddot{x}_1 + (k_1+k_2)\,x_1] - [k_2x_2] &= F_0 \sin \omega t \\ [k_2x_1] - [m_2\ddot{x}_2 + (k_2+k_3)x_2] &= 0 \end{aligned}\right] \quad (5.6.2)$$

For the case of no damping in the system each mass will be either in phase or out of phase with the exciting force. Let us then assume, for the steady state, the solutions as

$$\left.\begin{aligned} x_1 &= X_1 \sin \omega t \\ x_2 &= X_2 \sin \omega t \end{aligned}\right] \quad (5.6.2)$$

Substituting these in equations (5.6.1) and cancelling out the common term sin ωt, we have

$$\left.\begin{aligned} [-m_1\omega^2+(k_1+k_2)]\,X_1-k_2X_2 &= F_0 \\ k_2X_1-[-m_2\omega^2 + (k_2+k_3)]\,X_2 &= 0 \end{aligned}\right] \quad (5.6.3)$$

Solving for X_1 and X_2 from tne above two equations, we get

$$X_1 = \frac{[k_2+k_3-m_2\omega^2]\,F_0}{\begin{bmatrix} m_1m_2\omega^4-\{m_1\,(k_2+k_3)+m_2\,(k_1+k_2)\}\omega^2 \\ +\{k_1k_2+k_2k_3+k_3k_1\} \end{bmatrix}} \quad (5.6.4)$$

$$X_2 = \frac{k_2\,F_0}{\begin{bmatrix} m_1m_2\omega^4 - \{m_1\,(k_2+k_3) + m_2\,(k_1+k_2)\}\,\omega^2 \\ +\{k_1k_2+k_2k_3+k_3k_1\} \end{bmatrix}} \quad (5.6.5)$$

The above two equations give the steady state amplitude of vibration of the two masses respectively, as a function of ω. The denominator of both the expressions is the same and it is no coincidence. Further, this denometor is equal to the expression for the frequency equation (5.2.8) which is for the same system being considered here. And this expression for the frequency equation is zero when ω is equal to any of the two natural frequencies. That means when the exciting frequency is equal to any of the two natural frquencies, the denominators of equations (5.6.4) and (5.6.5) vanish simultaneously, or the amplitudes X_1 and X_2 are infinite. which is the resonance condition. Thus, we have two resonance frequencies, each corresponding to the one natural frequency of the system. At resonance, all the points in the system have infinite amplitude of vibration.

A damped dynamic vibration absorber can take care of the entire frequency range of excitation but at the cost of reduced effectiveness. This will not be treated in this text as it is not within its scope.

Various other types of absorbers have been used under various other conditions. These are subsequenly introduced in the following sub-sections.

5.7A Undamped dynamic vibration absorber. The undamped dynamic vibration absorber is also known as *Frahm Vibration Absorber* after the name of its inventor. The principle of working of this absorber was briefly mentioned in Sec. 5.6 and will be discussed here afresh.

Consider a two degrees of freedom system as shown in Fig. 5.7.1. It is different from the system of Fig. 5.6.1 in that it has no spring k_3. For reasons that will be obvious a little later, we will call the spring-mass system k_1-m_1 as the main system, and the spring-mass system k_2-m_2 as the absorber system.

The procedure for writing the differential equations and then deriving the expressions for the steady state amplitudes X_1 and X_2 is exactly the same as discussed in Sec. 5.6. It is not necessary to do that derivation again since we can write

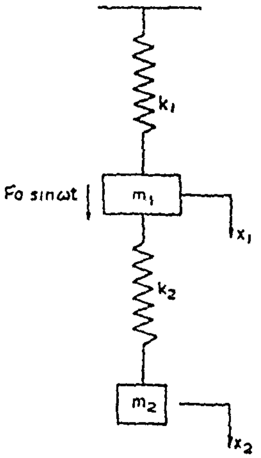


Fig. 5.7.1. Undamped dynamic vibration absorber.

the expressions for X_1 and X_2 from equations (5.6.4) and (5.6.5) by putting $k_3 = 0$ in those equations. Therefore, for the case of Fig. 5.7.1,

$$X_1 = \frac{(k_2 - m_2\omega^2)\,F_0}{[m_1 m_2\omega^4 - \{m_1 k_2 + m_2(k_1 + k_2)\}\omega^2 + k_1 k_2]} \tag{5.7.1}$$

$$X_2 = \frac{k_2 F_0}{[m_1 m_2\omega^4 - \{m_1 k_2 + m_2(k_1 + k_2)\}\omega^2 + k_1 k_2]} \tag{5.7.2}$$

To bring these equations into dimension-less forms, let us divide their numerators and denominators by $k_1\, k_2$ and introduce the following notations.

$X_{st} = F_0/k_1$ = zero frequency deflection of first mass,

$\omega_1 = \sqrt{k_1/m_1}$ = natural frequency of the a main system alone,

$\omega_2 = \sqrt{k_2/m_2}$ = natural frequency of the absorber system alone,

$\mu = m_2/m_1$ = ratio of the absorber mass to the main mass.

Equations (5.7.1) and (5.7.2) can then be written in the dimensionless form, as

$$\frac{X_1}{X_{st}} = \frac{\left(1 - \dfrac{\omega^2}{\omega_2^2}\right)}{\dfrac{\omega^4}{\omega_1^2\omega_2^2} - \left[(1+\mu)\dfrac{\omega^2}{\omega_1^2} + \dfrac{\omega^2}{\omega_2^2}\right] + 1} \tag{5.7.3}$$

$$\frac{X_2}{X_{st}} = \frac{1}{\dfrac{\omega^4}{\omega_1^2\omega_2^2} - \left[(1+\mu)\dfrac{\omega^2}{\omega_1^2} + \dfrac{\omega^2}{\omega_2^2}\right] + 1} \tag{5.7.4}$$

Equation (5.7.3) clearly shows that $X_1 = 0$ when $\omega = \omega_2$; that is when the excitation frequency is equal to the natural frequency of the absorber, the main system amplitude becomes zero even though it is excited by a harmonic force. This is the principle of an undamped dynamic vibration absorber, since, if a system (main system) has undesirable vibrations at the operating frequency, a secondary spring-mass system (absorber system) having its natural frequency equal to the operating frequency can be coupled to the main system to reduce its amplitude to zero. Let us see what happens to the amplitude X_2 of the absorber mass under these conditions. Substituting $\omega = \omega_2$ in equation (5.7.4), we get

$$X_2 = -\frac{X_{st}}{\mu\,\dfrac{\omega_2^2}{\omega_1^2}}$$

$$= -\frac{\dfrac{F_0}{k_1}}{\dfrac{m_2}{m_1}\cdot\dfrac{k_2}{m_2}\cdot\dfrac{m_1}{k_1}}$$

i.e., $X_2 = -F_0/k_2$
r $F_0 = -k_2X_2$ (5.7.5)

The above equation shows that the spring force k_2X_2 on he main mass due to the amplitude X_2 of the absorber mass s equal and opposite to the exciting force on the main mass esulting in no motion of the main system. The main system ibrations have been reduced to zero and these vibrations have een taken up by the absorber system. Hence the name *ibration absorber.*

The above drawn conclusion is of great importance and is he basis of the great many types of vibration absorbers.

The addition of a vibration absorber to a main system is not nuch meaningful unless the main system is operating at esonance or at least near it. Under these conditions we have $\omega = \omega_1$. But for the absorber to be effective we already have $\omega_2 = \omega$.

Therefore, for the effectiveness of the absorber at the operating frequency corresponding to the natural frequency of he main system alone, we have

$$\omega_2 = \omega_1, \text{ or } \quad \frac{k_2}{m_2} = \frac{k_1}{m_1} \tag{5.7.6}$$

When this condition is fulfilled, the absorber is known to be a *tuned absorber.*

For a tuned absorber, equations (5.7.3) and (5.7.4) now ecome

$$\frac{X_1}{X_{st}} = \frac{1-\dfrac{\omega^2}{\omega_2^2}}{\dfrac{\omega^4}{\omega_2^4}-(2+\mu)\dfrac{\omega^2}{\omega_2^2}+1} \tag{5.7.7}$$

$$\frac{X_2}{X_{st}} = \frac{1}{\dfrac{\omega^4}{\omega_2^4}-(2+\mu)\dfrac{\omega^2}{\omega_2^2}+1} \tag{5.7.8}$$

Equation (5.7.5) still holds good at $\omega = \omega_2$, at which point $X_1 = 0$.

To have a tuned absorber, we can have many combinations of k_2, m_2 as long as their ratio is equal to k_1/m_1 to satisfy equation (5.7.6). We can have a small spring k_2 and a small mass m_2 ; or k_2 large and m_2 large. In all these cases main system response will be zero at $\omega = \omega_2$. However, equation (5.7.5) shows that for the same exciting force the amplitude of the absorber mass is inversely proportional to its spring rate. In order to have small amplitude of absorber mass m_2, we must have a large k_2 and therefore large m_2 which may not be desirable from practical considerations. Small mass m_2 would be the best from practical considerations but that is associated with small k_2 and therefore large amplitude X_2 of vibration of the absorber mass. So a compromise is usually made between the amplitude and the mass ratio $\mu \left(= \dfrac{m_2}{m_1} \right)$. The mass ratio is usually kept between 0.05 and 0.25. A proper design of the absorber spring is also necessary which depends upon its amplitude.

The denominators of equations (5.7.7) and (5.7.8) are identical. At a value of ω when these denominators are zero the two masses have infinite amplitudes of vibration. The expression for the denominators is a quadratic in ω^2, and therefore there are two values of ω for which these expressions vanish. These two frequencies are the resonant frequencies or the natural frequencies of the system. When the excitation frequency equals any of the natural frequencies of the system, all the points in the system have infinite amplitudes of vibration, or the system is in resonance.

To find the two resonant frequencies of the system when $\omega_2 = \omega_1$, we equate the denominator of either of the equations (5.7.7) or (5.7.8) to zero,

or $$\left(\frac{\omega}{\omega_2}\right)^4 - (2+\mu)\left(\frac{\omega}{\omega_2}\right)^2 + 1 = 0 \tag{5.7.9}$$

Solving for $\left(\dfrac{\omega}{\omega_2}\right)$, we have

$$\left(\frac{\omega}{\omega_2}\right)^2 = \left(1 + \frac{\mu}{2}\right) \pm \sqrt{\mu + \frac{\mu^2}{4}} \tag{5.7.10}$$

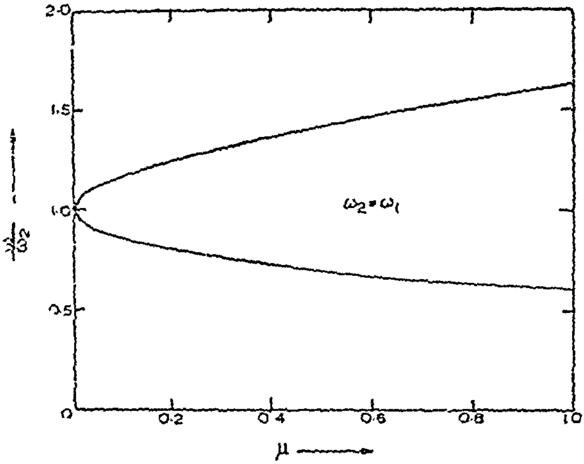
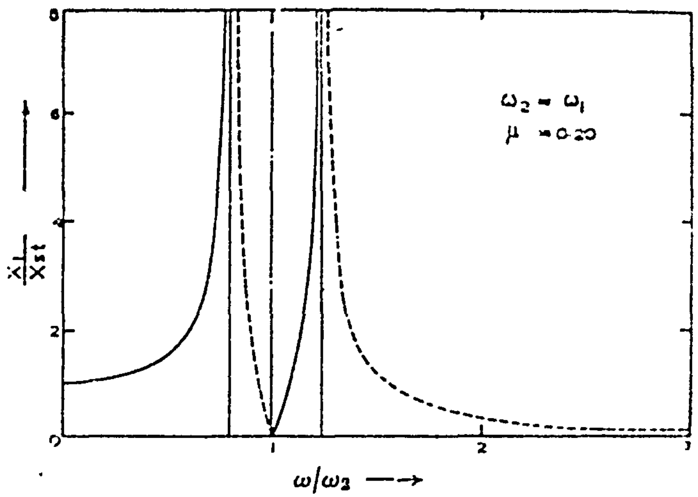


Fig. 5.7.2. Resonant speeds v/s mass ratio.

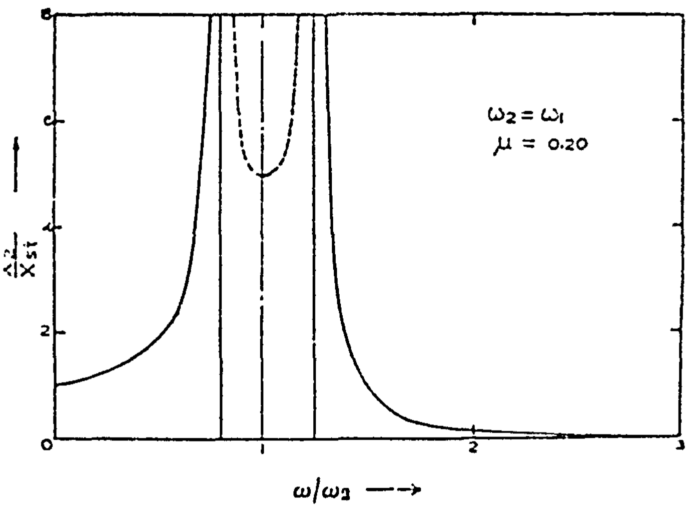
The relationship of equation (5.7.10) is plotted in Fig. 5.7.2. From this plot we see that greater the mass ratio, greater is the spread between the two resonant frequencies. The importance of this fact will be seen in the following paragraphs.

The dimensionless frequency response curves for the main system and for the absorber system given by equations (5.7.7) and (5.7.8) are shown in Fig. 5.7.3 (a) and (b) respectively for a value of $\mu = 0.2$. The dotted curves shown actually mean that the amplitude is negative or its phase difference with respect to the exciting force is 180°. These portions of the curves, however, are shown on the positive side of the ordinate. It can be seen from these curves that when $\omega/\omega_2 < 1$, the phase difference between the two masses is zero and when $\omega/\omega_2 > 1$, the phase difference between them is 180°.

For the main system alone without absorber we have only one resonant frequency at $\omega/\omega_1 = 1$. Imagine that for the main system alone the exciting frequency is very close to its natural frequency. To overcome this resonant condition we attach a vibration absorber ($\omega_2 = \omega_1$) to the main system, thereby reducing its vibrations to zero. Now if the exciting frequency is absolutely constant, the system will work fine; if the exciting frequency, which is governed by the speed of the



(a)



(b)

Fig. 5.7.3. Frequency-response curve for (a) main system, (b) absorber.

machine, is not constant but varies some what with the changes in load, then any changes in exciting frequency will shift the operating point from the optimum point and the main system response will no longer be zero. We also see that by adding the vibration absorber we have introduced two resonant points

instead of one in the original system. These two resonant points are spread on either side of the original resonant point corresponding to the main system alone. Now if the variation of the exciting frequency is such that the operating point shifts near one of the new resonant points then we are in trouble again. Thus, depending upon the variation of the execiting frequency, the spread between the two resonant frequencies has to be decided so that we do not come very near resonant point. After deciding the spreed between the resonant frequencies, the proper value of μ can be obtained from the curve of Fig. 5.7.2.

Undamped dynamic vibration absorbers are extemely effective for constant speed machines but they loose their effectivesness with any change in speed of the machines.

As in the case of rectilinear vibrations discussed so far, a torsional vibration absorber can be used to eliminate or reduce

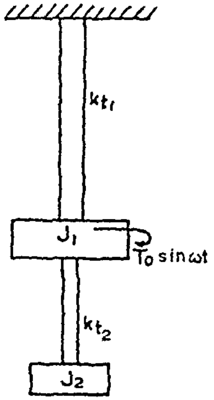


Fig. 5.7.4. Torsional absorber system.

the torsional oscillations of a system. The schematic arrangement is shown in Fig. 5.7.4 in which k_{t_1} and J_1 represent the main system and k_{t_2} and J_2 represent the absorber system. All the discussion for the rectilinear vibration absorber holds good for this case also.

5.7B. Centrifugal pendulum absorber. The undamped dynamic vibration absorber discussed in Sec. 5.7A is fully

effective only at a particular frequency for which it has been designed. In the case of a torsional system having torsional oscillations superimposed upon its rotation, it is possible to use a dynamic vibration absorber that will be effective at all rotating

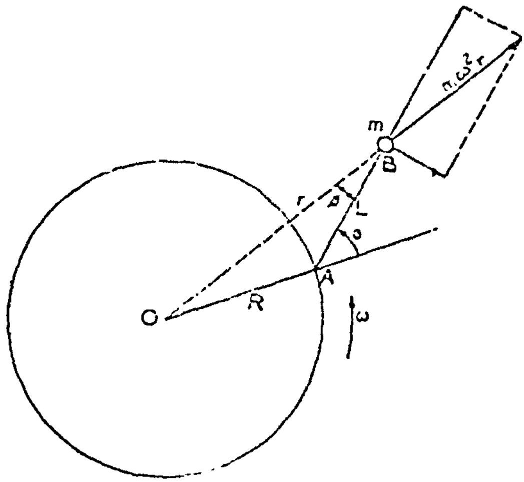


Fig. 5.7.5. Centrifugal pendulum.

speeds. This is a centrifugal pendulum absorber schematically shown in Fig. 5.7.5. It consists of a pendulum AB of length L having a mass m at one end and pivoted to the rotating body at the other end. The pivot point A is at a distance R from the centre O of the rotating body. Due to rotation of the system the pendulum bob is subjected to the centrifugal force which is large as compared to the gravitational force. At any instant, let the pendulum be displaced from the radial line through an angle θ. If ω is the speed of rotation of the body then the centrifugal force acting on the pendulum bob is $m\omega^2 r$, where r is the distance OB. This force is directed outward along OB. Let ϕ be the angle between OB and AB. This centrifugal force has one component along the pendulum line and the other perpendicular to it. The latter, $m\omega^2 r \sin\phi$, is the force that has to be taken into account in the differential equation of motion for the oscillation of the pendulum, which is written as

$$mL^2\ddot{\theta} = -(m\omega^2 r \sin\phi)\,L$$

$$\text{or} \quad \ddot{\theta} + \frac{r}{L}\,\omega^2 \sin\phi = 0 \tag{5.7.11}$$

From the traingle OAB,

$$\frac{R}{\sin\phi} = \frac{r}{\sin\,(180-\theta)}$$

$$\text{or} \qquad r\sin\phi = R\sin\theta \tag{5.7.12}$$

Substituting equation (5.7.12) in equation (5.7.11), we have

$$\ddot{\theta} + \frac{R\omega^2 \sin\theta}{L} = 0$$

or if θ is small,

$$\ddot{\theta} + \frac{R\omega^2}{L}\,\theta = 0 \tag{5.7.13}$$

The above equation for the pendulum is that of simple harmonic motion having its natural frequency

$$\omega_n = \omega\sqrt{R/L}$$

or the natural frequency in cycles per second is given by

$$f_n = \frac{\omega_n}{2\pi} = \frac{\omega}{2\pi}\sqrt{R/L} = n\sqrt{R/L} \tag{5.7.14}$$

where n = revolutions per second of the rotating body.

Thus the natural frequency of the pendulum absorber is always proportional to the speed of the rotating body.

The usual torsionally vibrating system receives a certain number of disturbing torques per revolution. The number of these disturbing torques per revolution is known as *Order Number* of the system. A two-cylinder engine working on four-stroke cycle has one disturbing torque per revolution and so its order number is *one*. A six-cylinder engine working on four-stroke cycle has its order number *three*.

For the pendulum absorber to be effective, its natural frequency f_n should be equal to the excitation frequency or the frequency of the disturbing torque. Therefore, from equation (5.7.14),

$$\sqrt{\frac{R}{L}} = \frac{f_n}{n}$$

$$= \frac{\text{Disturbing torque impluses/sec}}{\text{Revolutions/sec}}$$

or $$\sqrt{\frac{R}{L}} = \text{Disturbing torque impulses/revolution}$$

$$= \text{Order number.}$$

The procedure for the design of the centrifugal pendulum absorber is to equate the order number of the engine to $\sqrt{R/L}$; and choosing a value of R, solve for L. For a certain disturbing torque amplitude, greater the mass of the pendulum, the smaller is its amplitude of vibration. Thus, in order to keep the amplitude of vibration small, the pendulum mass is chosen as large as possible.

For reducing the torsional oscillations of an internal combustion engine, the pendulum is usually pivoted at the crank web. Thus R is about equal to the crank throw. Suppose for a six-cylinder engine working on four-stroke cycle (order number 3), the crank throw is equal to 6 cm. Take $R = 5$ cm. Then

$$3 = \sqrt{5/L}, \text{ or } L = 0{\cdot}555 \text{ cm.}$$

This is too small a length of the pendulum to fix an appreciable mass to. So, the following method is used to overcome this problem.

Bifilar suspension. It consists of a U-shaped counter weight suspended on two pins of diameter d_2. The holes in the counter weight and in the crankweb are slightly larger, of diameter d_1 as shown in fig. 5.7.6. Thus the counter weight fits loosely and rolls on the two pins which themselves roll in the bigger holes of the web. During vibrations all points on the counter weight have parallel motion, moving on circular paths of radius $(d_1 - d_2)$, which is therefore the length of the simple pendulum. Thus, by decreasing the clearance between the holes and the pins, the length L of the pendulum can be made as small as we like.

5.7C Dry friction damper. Fig. 5.7.7 shows a dry friction type of damper known as *Lanchester Damper.* It is very useful in reducing the torsional amplitudes of vibration near the resonance conditions. It consists of two flyw[illegible] mounted over a hub b rigidly fixed to the s[illegible]

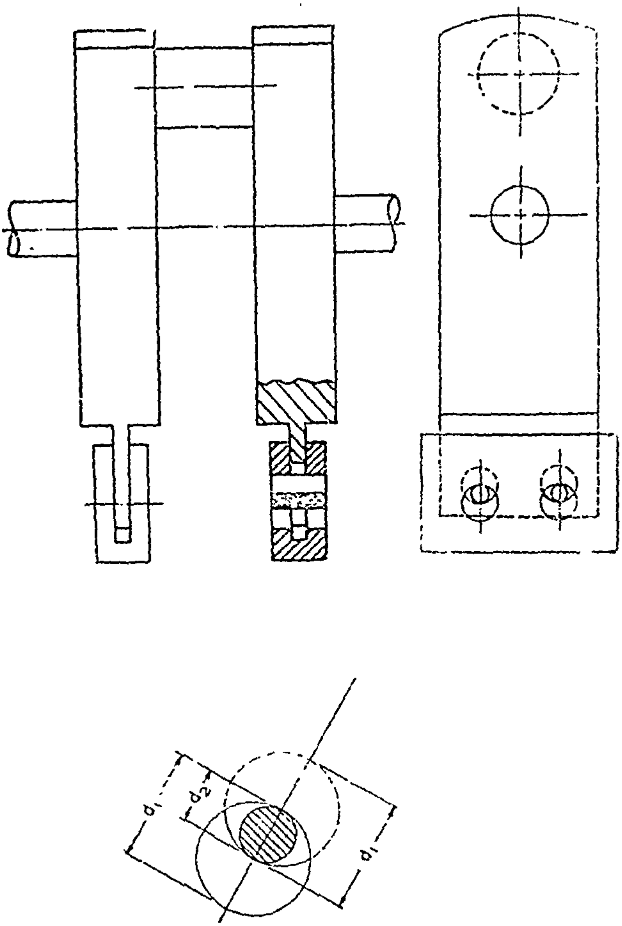


Fig. 5.7.6. Bifilar suspension.

vibrations. These flywheels are driven through the friction plates *c* fixed to the extension of the hub *b*, the pressure between the friction plates and the flywheels being adjustable through the spring loaded bolts *d*.

When the pressure between the friction plates and the flywheels is excessive which corresponds to a large friction torque, the flywheels become rigid with the shaft, have the

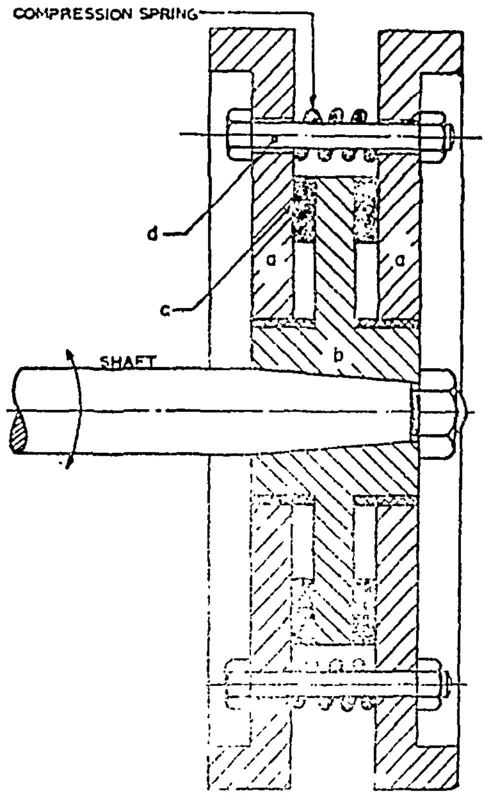


Fig. 5.7.7. Lanchester damper.

same oscillations as that of the shaft, and no energy is dissipated during vibrations since there is no relative rubbing. The energy dissipated is propotional to the friction torque times the relative velocity. When the pressure on the friction plates is zero, the relative velocity is maximum but the friction torque is zero. Again, no energy is dissipated. However, for an intermediate value of the pressure, there is friction torque as well as relative rubbing due to the inertia effect of the flywheels, and so, some energy is dissipated. This reduces the amplitude of torsional oscillations. Greater the energy dissipated, greater will be the amplitude reduction. The variation of energy dissipated against friction torque is shown in Fig. 5.7.8.

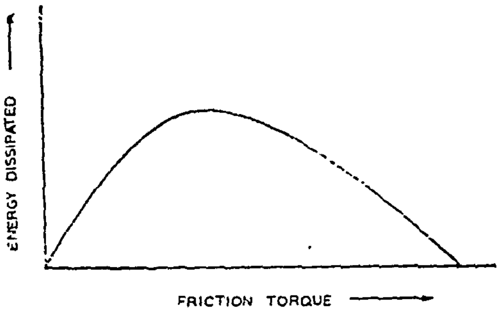


Fig. 5.7.8. Energy dissipated v/s friction torque for a Lanchester damper.

Let us consider the shaft to be oscillating about its mean speed as shown by the speed-time curve of Fig. 5.7.9. If the flywheels are continuously slipping over the shaft then they will be acted upon by a constant torque T which gives a constant angular acceleration T/J of the flywheels, J being their mass moment of inertia. This constant acceleration gives a constant slope of the velocity curve for the flywheels, as shown in Fig. 5.7.9. The velocity of the flywheels will continue to increase linearly as long as the shaft speed is more than that of the flywheels. When the shaft speed is less than that of the flywheels, the latter's speed will continue to decrease. The work done per cycle or the energy dissipated

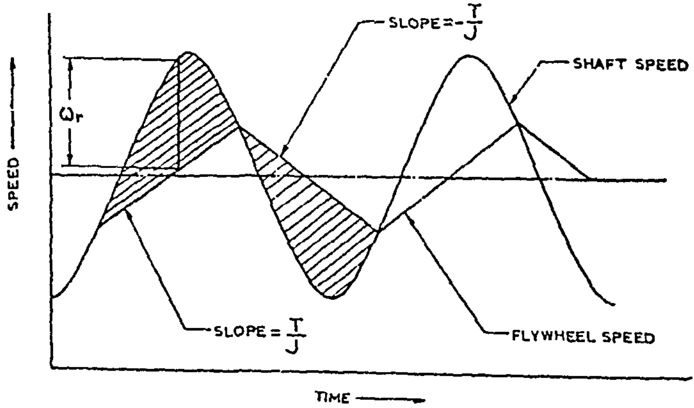


Fig. 5.7.9. Relative slipping in a Lanchester damper.

per cycle is then given by

$$E = \int_{\text{cycle}} T \times \omega_r \, dt$$

where ω_r is the relative velocity at any instant (see Fig. 5.7.9).

Or
$$E = T \times \int_{\text{cycle}} \omega_r \, dt$$
$$= T \times \text{shaded area.}$$

For decreasing pressure between the friction plates and the flywheels, the curve of flywheel speed becomes flatter and finally coincides with the mean line for zero pressure. For increasing pressure, the same curve becomes steeper and finally coincides with the shaft speed curve.

5.7D Untuned viscous damper. This type of damper is similar in principle to the one discussed in Sec. 5.7C except that instead of dry friction damping we have viscous damping. Such a damper is useful for damping out torsional oscillations and is commonly known as *Houdaille Damper*. It is schematically shown in Fig. 5.7.10

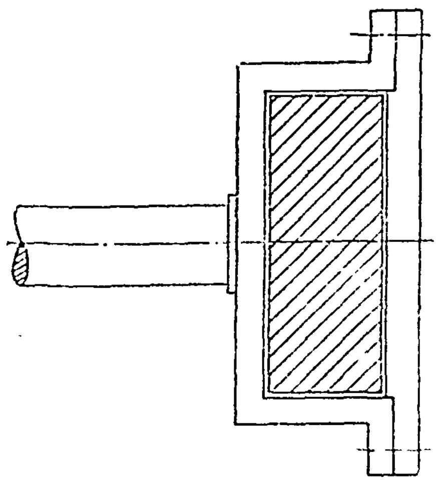


Fig. 5.7.10. Untuned viscous damper.

When the damping is zero in the damper, it is ineffective and we have a single degree of freedom system whose frequency-response curve is shown in Fig. 5.7.11. If the

damping is infinite, the damper mass becomes virtually integral with the main mass. It still remains a single degree of freedom system whose effective mass becomes larger. The frequency-response curve is of the same nature except that the peak now shifts towards the left by an amount depending upon the ratio of the damper inertia to the main system inertia. It can be shown that the point of intersection P of the above two curves is such that the response curves of different dampings pass through it. And a system having optimum damping has its response curve with P as the highest point. The idea is to introduce the optimum damping in the system in order that the maximum response over the entire frequency range does not go beyond a certain level.

The undamped dynamic vibration absorber has the disadvantage of having two resonance peaks in the frequency-response curve and so its use is limited only to fixed-speed

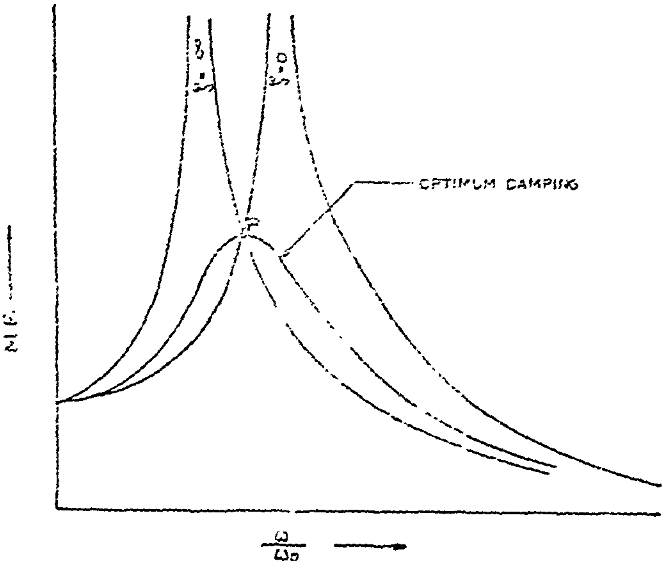


Fig. 5.7.11. Frequency-response curves for an untuned viscous damper.

machines. Houdaille damper can be used with variable speed machines, the maximum response being controlled by the ratio of the damper inertia to the main system inertia.

Illustrative Example 5.7.1

A torque $T_0 \sin \omega t$ is applied to J_1 of the torsional system shown in Fig. 5.7.12. If the moment of inertia of the main system $J_1 = 7.5$ kg-cm-sec^2, the torsional stiffness of the main system $k_{t1} = 7.5 \times 10^6$ kg-cm/rad, $T_0 = 3000$ kg-cm and $\omega = 10^3$ rad/sec; specify the minimum size J_2 of the absorber and k the stiffness of each of the four absorber springs such that the resonant frequencies are at least 20% from the excitation frequency. What will be the amplitude of vibration of this absorber ?

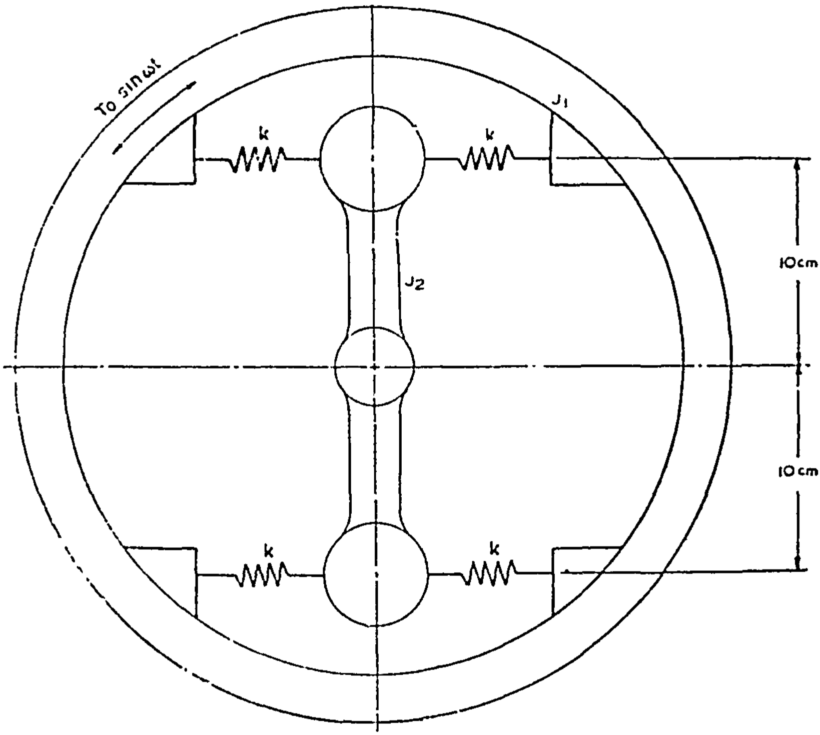


Fig. 5.7.12. Torsional vibration absorber.

Solution

$$\omega_1 = \sqrt{\frac{k_{t1}}{J_1}} = \sqrt{\frac{7.5 \times 10^6}{7.5}} = 10^3 \text{ rad/sec.}$$

For the undamped vibration absorber here, the excitation frequency is equal to the main system natural frequency of 10^3 rad/sec.

Taking $\omega_2 = \omega_1$, the two resonant frequencies of the main system are obtainable from equation (5.7.9), i.e.,

$$\left(\frac{\omega}{\omega_2}\right)^4 - (2+\mu)\left(\frac{\omega}{\omega_2}\right)^2 + 1 = 0$$

which gives

$$\mu = \frac{\left[\left(\frac{\omega}{\omega_2}\right)^2 - 1\right]^2}{\left(\frac{\omega}{\omega_2}\right)^2} \tag{5.7.15}$$

Now $(\omega/\omega_2) = 0.8$ or 1.2 for resonant frequencies to be at least 20% away from the original natural frequency of the main system.

If $\omega/\omega_2 = 0.8$, $\mu = 0.20$ from equation (5.7.15)

If $\omega/\omega_2 = 1.2$, $\mu = 0.13$ from equation (5.7.15)

Choosing the larger value of μ, we have $\mu = 0.20$

Therefore, $J_2 = 0.2 J_1 = 0.2 \times 7.5 = 1.5$ kg-cm-sec^2. **Ans.**

Since $\sqrt{\dfrac{k_{t1}}{J_1}} = \sqrt{\dfrac{k_{t2}}{J_2}}$

Therefore, $k_{t2} = 0.2\, k_{t1} = 0.2 \times 7.5 \times 10^6$

$$= 1.5 \times 10^6 \text{ kg-cm/rad.}$$

But $\quad k_{t2} = 4 \times k \times 10^2$

or $\quad k = \dfrac{k_{t2}}{400} = \dfrac{1.5 \times 10^6}{400} = 3750$ kg/cm. **Ans.**

The amplitude of vibration of the absorber at 10^3 rad/sec (exciting frequency) is given by the equation (5.7.5) after changing the translational quantities into torsional quantities,

or $\quad \beta_2 = -\dfrac{T_0}{k_{t2}}$

$$= -\frac{3000}{1.5 \times 10^6} = -2 \times 10^{-3} \text{ rad.}$$ **Ans.**

Illustrative Example 5.7.2.

A section of pipe pertaining to a certain machine vibrates with large amplitude at a compressor speed of 220 r.p.m. For analysing this system, a spring-mass system was suspended from

the pipe to act as an absorber. A 1 kg absorber weight tuned to 220 c.p.m. resulted in two resonant frequencies of 188 and 258 c.p.m. What must be the weight and the spring stiffness of the absorber if the resonant frequencies are to lie outside the range of 150 to 310 c.p.m. ?

Solution

$$\omega_1 = 220 \times \frac{2\pi}{60} = 23.0 \text{ rad/sec.}$$

$$\omega_2 = 23.0 \text{ rad/sec, also.}$$

$$m_2 = 1/980 \text{ kg-sec}^2/\text{cm}$$

$$k_2 = m_2\omega_2^2 = \frac{1}{980} \times 23^2 = 0.54 \text{ kg/cm}$$

$$\omega_{n1} = 180 \times \frac{2\pi}{60} = 19.7 \text{ rad/sec.}$$

$$\omega_{n2} = 258 \times \frac{2\pi}{60} = 27.0 \text{ rad/sec.}$$

For $\omega_2 = \omega_1$, we apply equation (5.7.15) to find μ by taking first $\omega = \omega_{n1}$ and then $\omega = \omega_{n2}$.

When $\omega = \omega_{n1} = 19.7$, $\mu = 0.100$.

When $\omega = \omega_{n2} = 27.0$, $\mu = 0.104$.

The average value of μ therefore is

$$\mu = 0.102$$

$$= \frac{W_2}{W_1}$$

Therefore, $W_1 = \dfrac{W_2}{0.102} = \dfrac{1}{0.102} = 9.8$ kg

Also $k_1 = \dfrac{k_2}{0.102} = \dfrac{0.54}{0.102} = 5.3$ kg/cm.

After finding the weight and the stiffness of the main system it is required first to find μ so that the new resonant frequencies will not be in the specified range.

$$\frac{\omega_{n1}}{\omega_2} = \frac{150}{220} = 0.681$$

$$\frac{\omega_{n2}}{\omega_2} = \frac{310}{220} = 1.41$$

Using equation (5.7.15), the two corresponding values of μ are
$\mu = 0.62$ and 0.493.
Choosing the higher value, we get
$\mu = 0.62$.
Therefore $W_2 = 0.62 \times 9.8 = 6.06$ kg
$k_2 = 0{\cdot}62 \times 5.3 = 3.29$ kg/cm. **Ans.**

PROBLEMS FOR PRACTICE

5.1 For the illustrative example 5.2.1 investigate the motion of the two masses for the following different cases.

(a) Both masses are given an initial velocity of 10 cm/sec simultaueously in the same direction.

(b) Both masses are given an initial velocity 10 cm/sec simultaneously in the opposite direction.

(c) Mass m_1 is given an initial velocity of 10 cm/sec and mass m_2 an initial velocity of 15 cm/sec, both in the same direction.

(d) Mass m_1 is given an initial velocity of 10 cm/sec and mass m_2 released from the position of rest simultaneously.

5.2 Write down the differential equations of motion for the system shown in Fig. P. 5.2. The quantities x_1 and x_2 are absolute displacements.

Find the two natural frequencies when $k_1 = 100$ kg/cm,

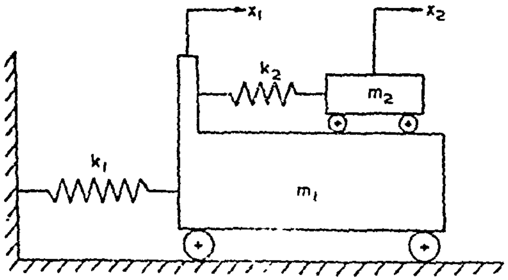


Fig. P. 5.2

$k_2 = 20$ kg/cm, $m_1 = 0.20$ kg-sec^2/cm and $m_2 = 0.05$ kg-sec^2/cm.

5.3 Determine the two natural frequencies and the modes of vibration of the system shown in Fig. P.5.3. The two equal masses are under tension T, which is large.

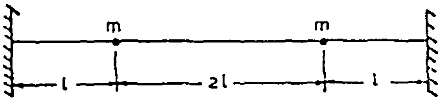


Fig. P.5.3.

5.4 Determine the two natural frequencies and the corresponding mode shapes for the system shown in Fig. P.5.4. The string is stretched with a large tension T.

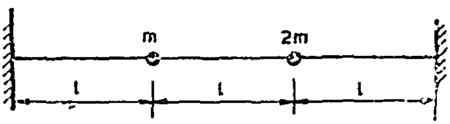
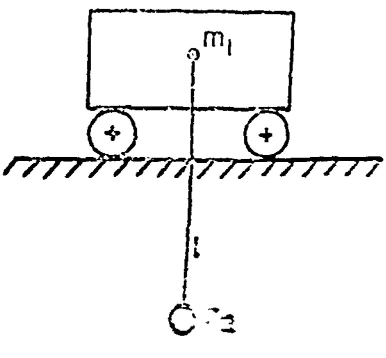


Fig. P.5.4.

5.5 Find the natural frequency of vibration for the system shown in Fig. P.5 5. Check your result from the solution of illustrative example 5.3.1 after putting $k = 0$.



[illegible]

5.6 Fig. P.5.6 shows [illegible] each. They are [illegible]

3000 kg/cm. How many natural frequencies does this system have ? Find their values.

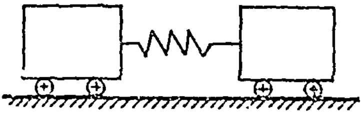


Fig. P.5.6.

5.7 Find out the two natural frequencies of vibration for the system shown in Fig. P.5.7. The light pendulum rod is pivoted at the centre of the roller. The spring acts through the centre of the roller.

What happens to the two natural frequencies when

(i) $k = 0$; and

(i i) $l = 0$.

What physical models do these systems pertain to ?

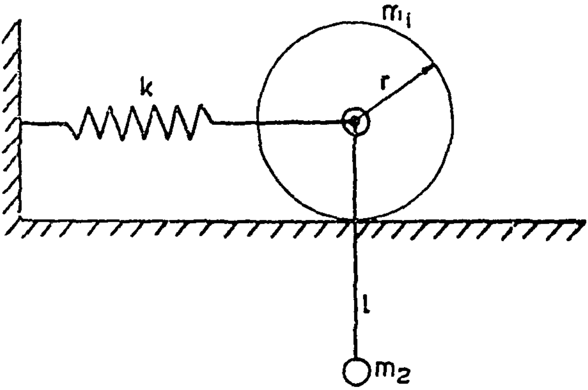


Fig. P.5.7.

5.8 Determine the expression for the two natural frequencies of the system shown in Fig. P.5.8. The cord is inextensible and there is no slippage between the cord and the pulley. The mass of the pulley is m_2

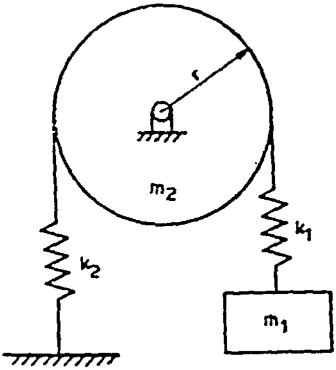


Fig. P.5.8.

5.9 Assuming the connecting rod AB shown in Fig. P.5.9. to be light and rigid, determine the natural frequency of oscillation of the system.

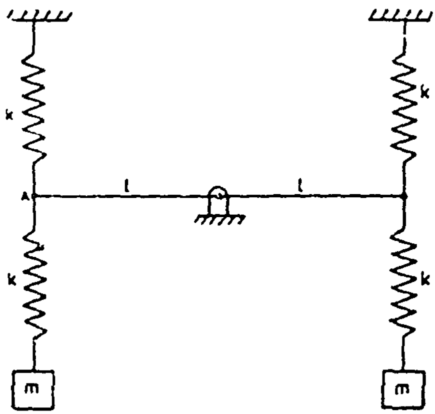


Fig. P.5.9.

5.10 Derive expressions for the two natural frequencies for

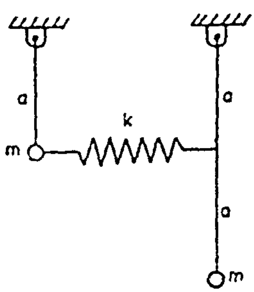


Fig. P.5.10.

small oscillations of the pendulums shown in Fig. P.5.10 in the plane of the paper. Assume the rods as weightless and rigid. Also obtain expressions for the angular amplitude ratios in the two modes.

5.11 Fig. P.5.11. shows two light and rigid rods pivoted at points O_1 and O_2 respectively, and both are horizontal under the action of three springs as shown. Obtain the frequency equation.

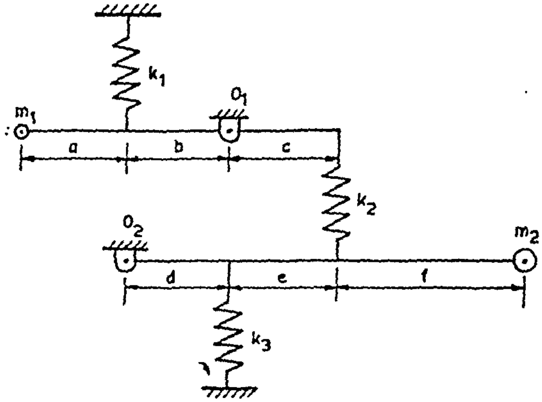


Fig. P.5.11.

Find the two natural frequencies

when $$a = b = c = d = e = l$$

$$f = 2\,l$$

$$m_1 = m_2 = m$$

$$k_1 = k_2 = k_3 = k$$

5.12 Determine the natural frequencies and the amplitude ratios of the system shown in Fig. P.5.12 in the two modes of vibration. The system consists of a mass m_1 with cross-sectional area A_1 floating in a fluid of specific weight W. The mass m_1 carries a second mass m_2 with volume V_2 and is connected to it by a spring of constant k.

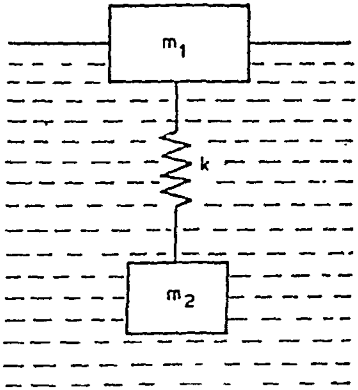


Fig P.5.12.

5.13 Derive an expression for the natural frequency of the torsional system shown in Fig. P.5.13 and draw the normal mode curve. Show that the nodal distance from J_2 is $L_2(1+k_{t2}/k_{t1})/(1+J_2/J_1)$.

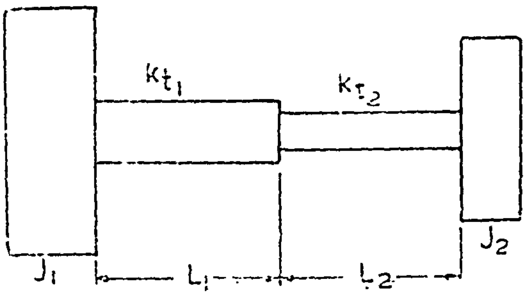


Fig. P.5.13.

5.14 If for the case of Prob. 5.13

$J_1 = 6$ kg-cm-sec^2
$J_2 = 4$ kg-cm-sec^2
$k_{t1} = 2\,0 \times 10^6$ kg-cm/rad.
$k_{t2} = 0.4 \times 16^6$ kg-cm/rad
$L_1 = 30$ cm
$L_2 = 15$ cm

determine the natural frequency and the mode shape for the system.

5.15 A mild steel shaft of 1 cm dia is built into walls at both ends. It carries two flywheels, each at 25 cm

from a wall and also 25 cm from each other, of steel 30 cm in diameter and 5 cm thick. Find the two natural frequencies in torsion.

5.16 In a two mass torsional system two wheels are mounted 15 cm apart on a shaft 4 cm diameter. If the moments of inertia of the two wheels are $J_1 = 1.2$ kg-m^2 and $J_2 = 2.0$ kg-m^2, find the position of the node and the frequency of free torsional oscillations.

If both the rotors are coupled to the shaft through similar torsional spring couplings, find the torsioal stiffness of these couplings so as to reduce to half the natural frequency of the complete system in torsions.

5.17 A thin rod of length l and mass m is supported horizontally on two unequal springs of stiffness k_1 and k_2, at points l_1 and l_1 from the centre, as shown in Fig. P.5.17. Derive the frequency equation and show that the two frequencies are equal if $k_1 = k_2$ and $l_1 = l/(2\sqrt{3})$.

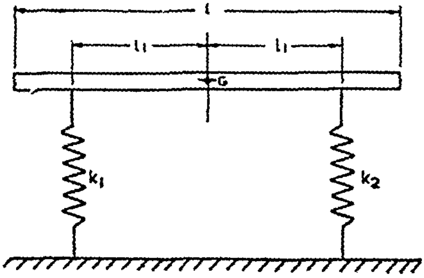


Fig. P.5.17.

5.18 The equations of motion representing the system of Fig. 5.4.1 (a) have been obtained to be as follows.

$$\ddot{x} + ax = b\theta$$
$$\ddot{\theta} + c\theta = (b/r^2)\, x.$$

If
$$a = c = 1000$$
$$b = 100$$
$$r = 1,$$

determine the resultant motion of the system when the mass is displaced one centimeter parallel to itself and released.

5.19 An automobile weighs 2000 kg and has a wheel base of 3.0 meters. Its centre of gravity is located 1.4 meters behind the front wheel axis and has a radius of gyration about its c.g. as 1.1 meter. The front springs have a combined stiffness of 6000 kg/cm and rear springs 6500 kg/cm. Find the principal modes of vibration of the automobile, and locate the nodal points for each mode.

5.20 Find the response of the system of Fig. P.5.20 to a sinusoidal excitation as shown in the figure.

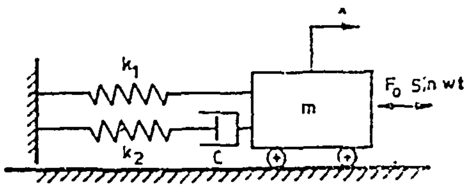


Fig. P.5.20.

5.21 For a two-degree of freedom damped system of Fig. P.5.21 derive expressions for the mode shapes and the characteristic equation. If the roots of the characteristic equation are

$$s_1 = -a + ib$$
$$s_2 = -a - ib$$
$$s_3 = -c$$
$$s_4 = -d,$$

write down the general equations of motion for m_1 and m_2.

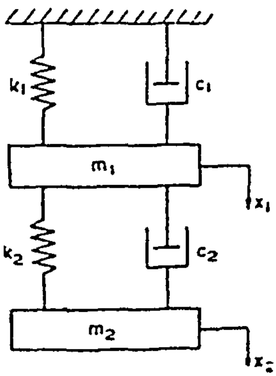


Fig. P.5.21.

5.22 Two concentrated masses are fixed upon a tight string stretched with tension T, as shown in Fig. P.5.22. Find the steady state amplitude of the mass on which the excitation acts. Find the frequencies corresponding to zero amplitude and infinite amplitude. Could you have predicted these frequencies before hand? If so, how?

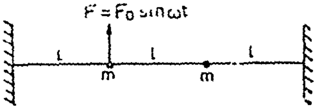


Fig. P.5.22.

5.23 Find the steady state amplitude of vibration for each of the two masses of Fig. P.5.23 when the support is excited with a displacement $A \sin \omega t$.

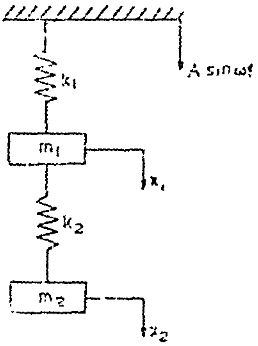


Fig. P.5.23.

5.24 A reciprocating engine weighs 40 kg and runs at a constant speed of 3000 rpm. After it was installed it vibrated with a large amplitude at the operating speed. What dynamic vibration absorber should be coupled to the system if the nearest resonant frequency of the combined system should be at least 25% away from the operating speed. Under these conditions what amplitude of the absorber will be obtained?

5.25 A torsional system has an inertia of 12 kg-cm-sec^2 and a torsional stiffness of 2400 kg-cm/rad. It is acted upon by a torsional excitation at 150 cpm. Determine the parameters of the absorber to be fixed to the main

system if it is desired to keep the natural frequencies at least 20% away from the impressed frequency.

5.26 For the system in Fig. 5.3.6. (a), an exciting force $F_0 \sin \omega t$ acts on mass m_1 in the horizontal direction. At what excitation frequency will the mass m_1 be stationary? What will be the amplitude of the pendulum at this frequency?

5.27 For a four cylinder engine working on four-stroke cycle, the crank throw is equal to 8.5 cm. What should be the length of the equivalent pendulum suspended at a radius of 7 cm to serve as a centrifugal absorber. If the disturbing torque on the main system is 1500 kg-cm what size of the pendulum should be taken so that its amplitude is limited to 15°. Discuss the practicability of the solution.

CHAPTER 6

MANY DEGREES OF FREEDOM SYSTEMS—EXACT ANALYSIS

6.1 Introduction.

Having understood the phenomenon of coupling in a two degrees of freedom system, we are in a better position to appreciate the problems involved in a multidegree of freedom system. Here the coupling between various coordinates makes the system a real complex. More the number of degrees of freedom, the more involved will be the equations; and the degree of complexity increases many folds with damping included in the system. Exact solutions are possible in certian cases only.

The procedure for analysing multi-degree of freedom system is only an extension of the method used for analysing two degrees of freedom systems. However, in the case of beams and strings, the use of influence coefficients makes the things somewhat simpler.

6.2 Undamped free vibrations.

Consider a system shown in Fig. 6.2.1 having n-degrees of freedom. In order to find the frequency equation for this system we first write down the differential equation for each mass separately from Newton's second law. If $x_1, x_2, x_3 \ldots x_n$ are the displacements from the equilibrium positions of the respective masses at any instant, then

$$\begin{aligned}
m_1\ddot{x}_1 &= -k_1x_1-k_2\,(x_1-x_2)\\
m_2\,\ddot{x}_2 &= +\,k_2(x_1-x_2)-k_3(x_2-x_3)\\
m_3\ddot{x}_3 &= +\,k_3(x_2-x_3)-k_4(x_3-x_4)\\
\ldots \quad &\ldots \quad \ldots \qquad \ldots \quad \ldots \quad \ldots \quad ..\\
m_n\,\ddot{x}_n &= +\,k_n(x_{n-1}-x_n)
\end{aligned}$$

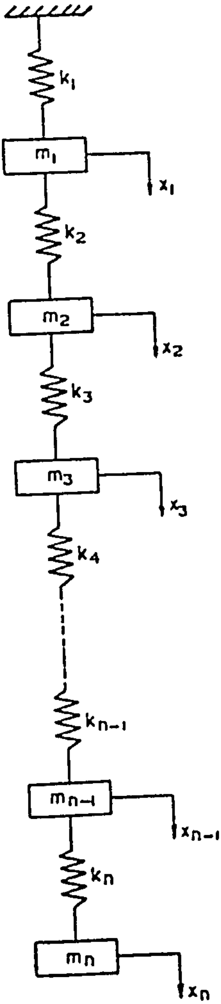


Fig. 6.2.1. Undamped free vibrations of a multi-degree of freedom system.

These equations can be arranged in the following forms.

$$\left.\begin{aligned}
&[m_1\ddot{x}_1+(k_1+k_2)x_1]-k_2x_2 = 0\\
&-k_2x_1+[m_2\ddot{x}_2+(k_2+k_3)x_2]-k_3x_3 = 0\\
&-k_3x_2+[m_3\ddot{x}_3+(k_3+k_4)x_3]-k_4x_4 = 0\\
&\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\\
&-k_nx_{n-1}+(m_n\ddot{x}_n+k_nx_n) = 0
\end{aligned}\right\} \qquad (6.2.1)$$

For the principal mode of vibration, let us assume the solution as,

$$\left.\begin{aligned} x_1 &= X_1 \sin \omega t \\ x_2 &= X_2 \sin \omega t \\ x_3 &= X_3 \sin \omega t \\ &\dots\dots\dots \\ x_n &= X_n \sin \omega t \end{aligned}\right] \tag{6.2.2}$$

Substituting equations (6.2.2) in equations (6.2.1) and cancelling out the common term $\sin \omega t$, we have

$$\left.\begin{aligned} &[(k_1+k_2)-m_1\omega^2]X_1-k_2X_2 = 0 \\ &-k_2+X_1[(k_2+k_3)-m_2\omega^2]X_2-k_3X_3 = 0 \\ &-k_3X_2+[(k_3+k_4)-m_3\omega^2]X_3-k_4X_4 = 0 \\ &\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ &-k_nX_{n-1}+(k_n-m_n\omega^2)X_n = 0 \end{aligned}\right] \tag{6.2 3}$$

For the above equations, the solution other than $X_1 = X_2 = X_3 = \quad = X_n = 0$ is possible only when the determinant composed of the coefficients of X's vanishes ; or

$$\begin{vmatrix} [(k_1+k_2)-m_1\omega^2] & -k_2 & \dots & 0 & 0 \\ -k_2 & [(k_2+k_3)-m_2\omega^2] & \dots & 0 & 0 \\ 0 & -k_3 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -k_n & (k_n-m_n\omega^2) \end{vmatrix} = 0 \tag{6.2.4}$$

This is the frequency equation, being of n^{th} degree in ω^2 and therefore gives n values of ω corresponding to n natural frequencies. The mode shapes can be obtained from equations (6.2.3) by using, one at a time, the various values of ω as obtained from equations (6.2.4).

Illustrative Example 6.2.1

A three degree of freedom system is schematically shown in Fig. 6.2.2. Calculate its three natural frequencies.

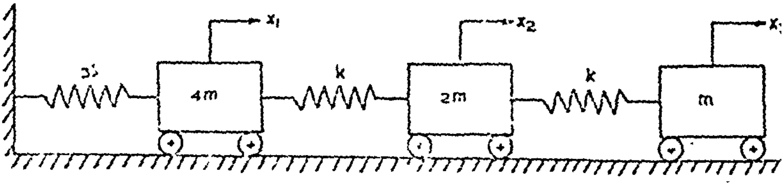


Fig. 6.2.2. A three degree of freedom system.

the system. Similarly δ_{nm} will be the deflection at point *n* due to a unit load applied at point *m* of the system. On the same principle, δ_{mm} (or δ_{nn}) will be the deflection at point *m* (or *n*) due to unit load applied at the same point of the system. All these δ's are called the *influence members* or the *influence coefficients*. δ_{mm}, δ_{nn}, δ_{pp}, etc. are called the *direct* influence numbers and δ_{mn}, δ_{nm}, δ_{np}, etc. as *cross* influence numbers.

Now, the Maxwell's reciprocal theorem states that the deflection at any point in the system due to a unit load acting at any other point of the same system is equal to the deflection at the second point due to a unit load acting at the first point; or $\delta_{mn} = \delta_{nm}$.

For the proof of this theorem consider a system as shown in Fig. 6.3.1. Let *m* and *n* be any two points in the system where

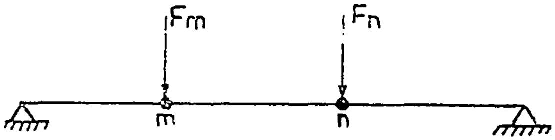


Fig. 6.3.1. A general system to prove Maxwell's reciprocal theorem.

we can apply loads F_m and F_n. Now imagine that loads F_m and F_n are applied to the system in the following two alternative ways.

(i) First apply F_m at point *m* gradually from zero to its full value; then apply F_n at point *n* gradually from zero to its full value with F_m there at point *m* all the time.

(ii) Reverse order; i.e. first apply F_n at point *n* gradually and then F_m at point *m* gradually with F_n there at point *n* all the time.

The total work done in applying F_m and F_n to the system in either of the above two manners will be the same since the final deflection curve would be the same and therefore the strain energy in the system the same.

Let us consider the work done in deforming the system as

described under (i) above. When load F_m is applied gradually at point m, the final deflection at that point is $\delta_{mm} F_m$, and since the load F_m is gradually applied the work done is $(\frac{1}{2} F_m).(\delta_{mm} F_m)$. Now when F_n is applied at point n, the work done by this force in the same way is $(\frac{1}{2} F_n).(\delta_{nn} F_n)$. This is not all. When the force F_n is applied at point n there is a further deflection $\delta_{mn} F_n$ at point m where a full force F_m is already acting. Therefore additional work done by force F_m in moving through a distance $\delta_{mn} F_n$ is equal to $F_m (\delta_{mn} F_n)$. Hence the total work done in the first mode is

$$(W.D.)_{(I)} = \tfrac{1}{2}\, \delta_{mm} F_m^2 + \tfrac{1}{2}\, \delta_{nn} F_n^2 + \delta_{mn} F_m F_n \tag{6.3.1}$$

Similarly the work done in the second mode of application of loading as described under (ii) above is

$$(W.D.)_{(II)} = \tfrac{1}{2}\, \delta_{mm} F_m^2 + \tfrac{1}{2} \delta_{nn} F_n^2 + \delta_{nm} F_m F_n \tag{6.3.2}$$

Equating the above two equations, we have

$$\delta_{mn} = \delta_{nm} \tag{6.3.3}$$

This proof is quite general since we did not put any restrictions on the system.

Let us use the influence numbers to analyse multi-degree of freedom systems. Consider a string stretched with a tension T between two points, and having a number of point masses $m_1, m_2 \dots m_n$ fixed along its length as shown in Fig. 6.3.2. When

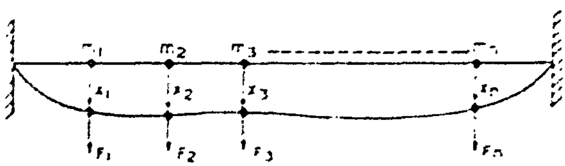


Fig. 6.3.2. Inertia loading on a multi-degree freedom system.

the system is vibrating in a principal mode, all the points in it have harmonic vibration. For any principal mode of vibration, let the amplitudes of various masses be $X_1, X_2 \dots X_n$. Then the assumed solutions are

$$\left.\begin{aligned} x_1 &= X_1 \sin \omega t \\ x_2 &= X_2 \sin \omega t \\ &\text{-------} \\ x_n &= X_n \sin \omega t \end{aligned}\right] \quad (6.3.4)$$

where ω is the frequency in that mode of vibration. In one of its positions at any instant, let us bring the system to static equilibrium by introducing inertia forces on the masses. If all x are taken positive in the downward direction, so will all $\ddot{x}$ be, and the inertia forces on various masses will be $m\ddot{x}$ in the upward direction, i.e. in a direction opposite to $\ddot{x}$. So, if F_1, F_2 F_n are the inertia forces as shown in the figure, then

$$\left.\begin{aligned} F_1 &= -m_1\ddot{x}_1 \\ F_2 &= -m_2\ddot{x}_2 \\ &\text{------} \\ F_n &= -m_n\ddot{x}_n \end{aligned}\right] \quad (6.3.5)$$

In a system with static equilibrium, if $F_1, F_2 \ldots F_n$ are the forces acting at various points of the system, then the deflection at any point is the sum of deflection at that point due to individual forces acting at their respective points; or mathematically speaking.

$$\left.\begin{aligned} x_1 &= \delta_{11} F_1 + \delta_{12} F_2 + \ldots \ldots \ldots \ldots + \delta_{1n} F_n \\ x_2 &= \delta_{21} F_1 + \delta_{22} F_2 + \ldots \ldots \ldots \ldots + \delta_{2n} F_n \\ &\text{------------} \\ x_n &= \delta_{n1} F_1 + \delta_{n2} F_2 + \ldots \ldots \ldots + \delta_{nn} F_n \end{aligned}\right] \quad (6.3.6)$$

Substituting in the above set of equations for F from equations (6.3.5) and for x from equations (6.3.4), and rearranging and cancelling out the common term $\sin \omega t$, we have

$$\left.\begin{aligned} &(\delta_{11}m_1\omega^2-1)X_1+\delta_{12}m_2\omega^2X_2+\ldots \ldots \ldots+\delta_{1n}m_n\omega^2X_n=0 \\ &\delta_{21}m_1\omega^2X_1+(\delta_{22}m_2\omega^2-1)X_2+\ldots \ldots+\delta_{2n}m_n\omega^2X_n=0 \\ &\text{-------------------} \\ &\delta_{n1}m_1\omega^2X_1+\delta_{n2}m_2\omega^2X_2+\ldots \ldots+(\delta_{nn}m_n\omega^2-1)X_n=0 \end{aligned}\right] \quad (6.3.7)$$

The determinant from the above set of expressions, equated to zero, gives the frequency equation for the system.

Special case. Let us consider a special case of the above general case by taking three equal masses attached to the string, dividing it in four equal parts as shown in Fig. 6.3.3 (a). In order to find the influence numbers, imagine a unit load applied at point 1. This load will deflect the system as shown in Fig.

6.3.3. (b). Let the tension T in the string be large and unaffected by this small deflection. At point 1 in the deflected position, the horizontal components of T in two parts of the string balance each other (deflection being small) and the vertical components balance the unit load. Or,

$$T\frac{\delta_{11}}{l} + T\frac{\delta_{11}}{3l} = 1$$

$T \cdot \theta + T\theta =$

which gives $\underbrace{\delta_{11} = \frac{3}{4}\frac{l}{T}}$

And from equation (6.3.3) and the proportional triangles in Fig. 6.3.3 (b), we have

$$\delta_{12} = \delta_{21} = \frac{2}{3}\delta_{11} = \frac{1}{2}\frac{l}{T}$$

$$\delta_{13} = \delta_{31} = \frac{1}{3}\delta_{11} = \frac{1}{4}\frac{l}{T}$$

The remaining coefficients can be obtained by considering the system loaded as in Fig. 6.3.3 (c), and these are

$$\delta_{22} = \frac{l}{T}$$

$$\delta_{23} = \delta_{32} = \frac{1}{2}\frac{l}{T}$$

and from symmetry

$$\delta_{33} = \delta_{11} = \frac{3}{4}\frac{l}{T}$$

Hence, the values of influence coefficients are

$$\delta_{11} = \frac{3}{4}\frac{l}{T} \qquad \delta_{12} = \frac{1}{2}\frac{l}{T} \qquad \delta_{13} = \frac{1}{4}\frac{l}{T}$$

$$\delta_{21} = \frac{1}{2}\frac{l}{T} \qquad \delta_{22} = \frac{l}{T} \qquad \delta_{23} = \frac{1}{2}\frac{l}{T}$$

$$\delta_{31} = \frac{1}{4}\frac{l}{T} \qquad \delta_{32} = \frac{1}{2}\frac{l}{T} \qquad \delta_{33} = \frac{3}{4}\frac{l}{T}$$

For this special case we have the following equations corresponding to equations (6.3.6).

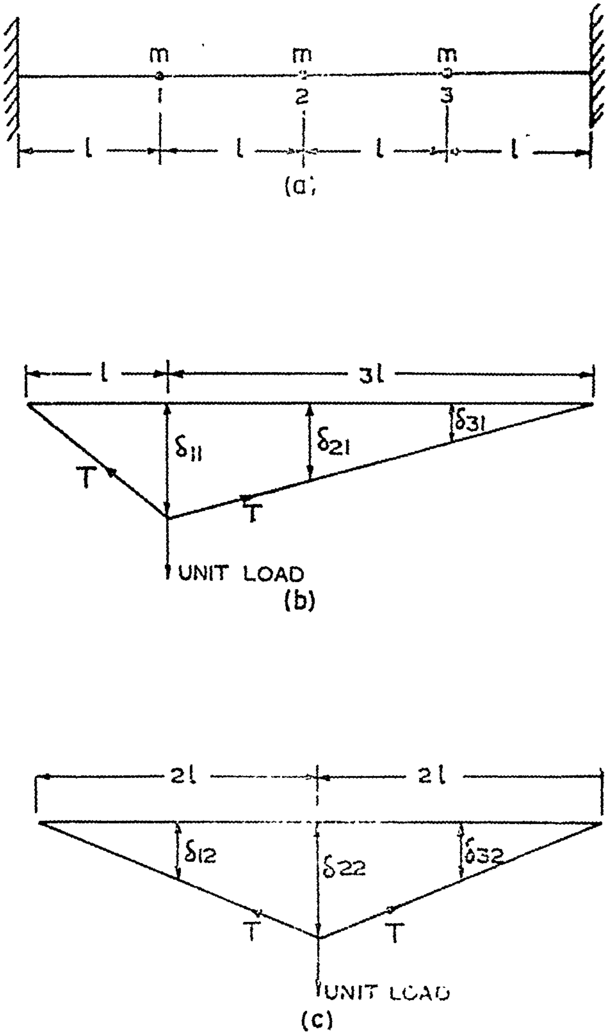


Fig. 6.3.3. Influence coefficients for a three degree of freedom system.

$$\left.\begin{aligned} x_1 &= \delta_{11}F_1+\delta_{12}F_2+\delta_{13}F_3 \\ x_2 &= \delta_{21}F_1+\delta_{22}F_2+\delta_{23}F_3 \\ x_3 &= \delta_{31}F_1+\delta_{32}F_2+\delta_{33}F_3 \end{aligned}\right] \tag{6.3 8}$$

The equations corresponding to equations(6.3.7) for this case, after substituting the values of influence numbers, now become

$$\left.\begin{aligned}
&\left(\frac{3}{4}\frac{l}{T}m\omega^2-1\right)X_1+\frac{1}{2}\frac{l}{T}m\omega^2X_2+\frac{1}{4}\frac{l}{T}m\omega^2X_3=0\\
&\frac{1}{2}\frac{l}{T}m\omega^2X_1+\left(\frac{l}{T}m\omega^2-1\right)X_2+\frac{1}{2}\frac{l}{T}m\omega^2X_3=0\\
&\frac{1}{4}\frac{l}{T}m\omega^2X_1+\frac{1}{2}\frac{l}{T}m\omega^2X_2+\left(\frac{3}{4}\frac{l}{T}m\omega^2-1\right)X_3=0
\end{aligned}\right]\tag{6.3.9}$$

Dividing all the above set of equations by $\frac{l}{T}m\omega^2$ and equating the determinant from the above expressions to zero, we get

$$\begin{vmatrix}
\left(\frac{3}{4}-\frac{T}{ml\omega^2}\right) & \frac{1}{2} & \frac{1}{4}\\
\frac{1}{2} & \left(1-\frac{T}{ml\omega^2}\right) & \frac{1}{2}\\
\frac{1}{4} & \frac{1}{2} & \left(\frac{3}{4}-\frac{T}{ml\omega^2}\right)
\end{vmatrix}=0$$

Expanding it out, gives

$$S^3-\frac{5}{2}S^2+\frac{3}{2}S-\frac{1}{4}=0\tag{6.3.10}$$

where $S=\dfrac{T}{ml\omega^2}$.

Equation (6.3.10) is the frequency equation and by simple trial and error, this equation can be broken up as

$$(S-\tfrac{1}{2})\,(S^2-2S+\tfrac{1}{2})=0$$

giving the three roots as

$$S_1=1+\frac{1}{\sqrt{2}},\ S_2=\frac{1}{2},\ S_3=1-\frac{1}{\sqrt{2}}$$

or
$$\left.\begin{aligned}
\omega_{n1}^2&=\frac{1}{S_1}\frac{T}{ml}=0.5858\,\frac{T}{ml}\\
\omega_{n2}^2&=\frac{1}{S_2}\frac{T}{ml}=2\frac{T}{ml}\\
\omega_{n3}^2&=\frac{1}{S_3}\frac{T}{ml}=3.4142\frac{T}{ml}
\end{aligned}\right]\tag{6.3.11}$$

Equations (6.3.11) give the square of the three natural frequencies for a three degree of freedom system.

To find the mode shapes, let us obtain the values of $\frac{X_2}{X_1}$ and $\frac{X_3}{X_1}$ from equations (6.3.9), first by dividing all the equations by X_1 and then solving for the two ratios from the first two equations. These can easily be obtained as

$$
\left.
\begin{aligned}
\frac{X_2}{X_1} &= 2 - \frac{l}{T}\, m\omega^2 \\
\frac{X_3}{X_1} &= \frac{4 - 7\frac{l}{T}\, m\omega^2 + 2\left(\frac{l}{T}\, m\omega^2\right)^2}{\frac{l}{T}\, m\omega^2}
\end{aligned}
\right] \tag{6.3.12}
$$

When $\omega = \omega_{n1}$ as found out in equation (6.3.11),

$$\left(\frac{X_2}{X_1}\right)_1 = 1.41 \quad \text{and} \quad \left(\frac{X_3}{X_1}\right)_1 = 1.$$

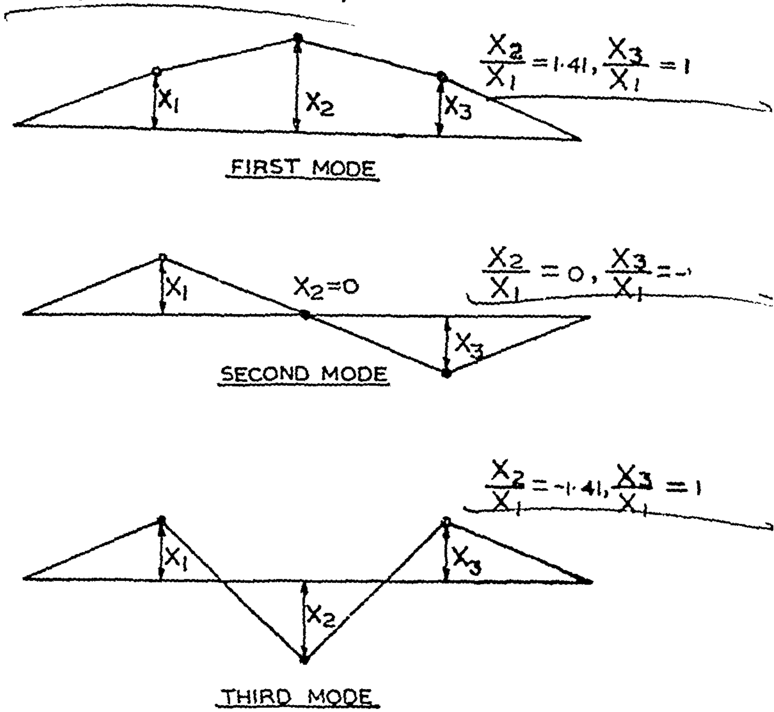


Fig. 6.3.4. Mode shapes.

When $\omega = \omega_{n2}$,

$$\left(\frac{X_2}{X_1}\right)_2 = 0 \quad \text{and} \quad \left(\frac{X_3}{X_1}\right)_2 = -1.$$

When $\omega = \omega_{n3}$,

$$\left(\frac{X_2}{X_1}\right)_3 = -1.41 \text{ and } \left(\frac{X_3}{X_1}\right)_3 = 1.$$

These mode shapes are shown in Fig. 6.3.4.

Illustrative Example 6.3.1

For the system shown in Fig. 6.2.2, write down the influence coefficients.

Find out the three natural frequencies by the method of influence coefficients.

Solution

Applying unit force at the first mass, the deflections of all the masses are $\frac{1}{3k}$.

Therefore, $\delta_{11} = \delta_{21} = \delta_{31} = \frac{1}{3k}$

$= \delta_{12} = \delta_{13}$ (by Maxwell's theorem).

Applying unit force at the second mass, the first two springs are in series and their equivalent spring constant is given by

$$\frac{1}{k_e} = \frac{1}{3k} + \frac{1}{k}, \text{ or } k_e = \frac{3}{4}k$$

Therefore, $\delta_{22} = \frac{1}{k_e} = \frac{4}{3k}$

Due to this unit force at the second mass, deflections of masses 2 and 3 are the same.

Therefore, $\delta_{22} = \delta_{32} = \frac{4}{3k}$

$= \delta_{23}$ (by Maxwell's theorem).

Applying a unit force at the third mass, the three springs are in series. The equivalent spring constant is given by

$$\frac{1}{k_e} = \frac{1}{3k} + \frac{1}{k} + \frac{1}{k}, \quad \text{or} \quad k_e = \frac{3}{7}k$$

Therefore, $\delta_{33} = \dfrac{7}{3k}$

Hence,
$$\left.\begin{aligned} \delta_{11} &= \frac{1}{3k}, \ \delta_{12} = \frac{1}{3k}, \ \delta_{13} = \frac{1}{3k} \\ \delta_{21} &= \frac{1}{3k}, \ \delta_{22} = \frac{4}{3k}, \ \delta_{23} = \frac{4}{3k} \\ \delta_{31} &= \frac{1}{3k}, \ \delta_{32} = \frac{4}{3k}, \ \delta_{33} = \frac{7}{3k} \end{aligned}\right] \quad \textbf{Ans.}$$

Equations for this case can be written similar to equations (6.3.6) after substituting for $F = -m\ddot{x}$. These equations are

$$x_1 = -\delta_{11}m_1\ddot{x}_1 - \delta_{12}m_2\ddot{x}_2 - \delta_{13}m_3\ddot{x}_3$$
$$x_2 = -\delta_{21}m_1\ddot{x}_1 - \delta_{22}m_2\ddot{x}_2 - \delta_{23}m_3\ddot{x}_3$$
$$x_3 = -\delta_{31}m_1\ddot{x}_1 - \delta_{32}m_2\ddot{x}_2 - \delta_{33}m_3\ddot{x}_3$$

Substituting in the above equations the values of various m and δ, and the values of x from equations (6.3 4), we have after rearranging and cancelling out the common term $\sin \omega t$, the following equations.

$$\left(\frac{4}{3}\ \frac{m\omega^2}{k} - 1\right)X_1 + \frac{2}{3}\ \frac{m\omega^2}{k}\ X_2 + \frac{1}{3}\ \frac{m\omega^2}{k}\ X_3 = 0$$
$$\frac{4}{3}\ \frac{m\omega^2}{k}\ X_1 + \left(\frac{8}{3}\ \frac{m\omega^2}{k} - 1\right)X_2 + \frac{4}{3}\ \frac{m\omega^2}{k}\ X_3 = 0$$
$$\frac{4}{3}\ \frac{m\omega^2}{k}\ X_1 + \frac{8}{3}\ \frac{m\omega^2}{k}\ X_2 + \left(\frac{7}{3}\ \frac{m\omega^2}{k} - 1\right)X_3 = 0$$

Putting $\dfrac{3k}{m\omega^2} = S$ in the above equations, the frequency equation, after simplification, is given by the following determinant equation.

$$\begin{vmatrix} (4-S) & 3 & 1 \\ 4 & (8-S) & 4 \\ 4 & 8 & (7-S) \end{vmatrix} = 0$$

Expanding the determinant, gives

$$S^3 - 19S^2 - 72S - 72 = 0,$$

the roots of which are

$$S_1 = 14.32, \ S_2 = 3, \ S_3 = 1.68$$

Since $\quad S = 3k/m\omega^2$, we have

$$\omega_{n1}^2 = 0.21\ k/m, \ \omega_{n2}^2 = k/m, \ \omega_{n3}^2 = 1.79\ k/m$$

or

$$\omega_{n1} = 0.46\sqrt{k/m}$$

$$\omega_{n2} = \sqrt{k/m}$$

$$\omega_{n3} = 1.34\sqrt{k/m}$$

These values are the same as obtained in Illustrative Example 6.2.1. **Ans.**

6.4 Undamped forced vibrations.

Consider the system of Fig. 6.3.3 (a) with a harmonic excitation $F_0 \sin \omega t$ acting on the first mass as shown in Fig. 6.4.1. The deflections at points 1, 2 and 3 due to this force would

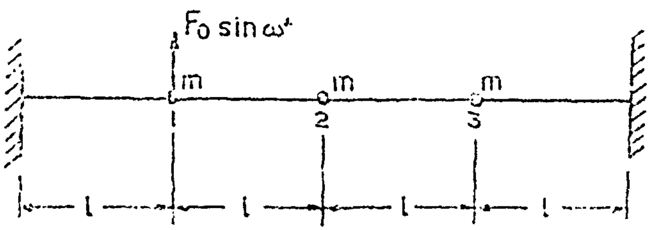


Fig. 6.4.1. Undamped forced vibrations of a three degree of freedom system.

respectively be $\delta_{11} F_0 \sin \omega t$, $\delta_{21} F_0 \sin \omega t$ and $\delta_{31} F_0 \sin \omega t$. The inertia forces continue to be there in this case also. Therefore, the equations of motion for the three masses of this system can be obtained by adding the above deflections respectively, to the right hand side of equations (6.3.8). Or,

$$\left.\begin{aligned} x_1 &= \delta_{11}F_1 + \delta_{12}F_2 + \delta_{13}F_3 + \delta_{11}F_0 \sin \omega t \\ x_2 &= \delta_{21}F_1 + \delta_{22}F_2 + \delta_{23}F_3 + \delta_{21}F_0 \sin \omega t \\ x_3 &= \delta_{31}F_1 + \delta_{32}F_2 + \delta_{33}F_3 + \delta_{31}F_0 \sin \omega t \end{aligned}\right\} \quad (6.4.1)$$

which equations can finally be written in the following forms

after substituting for F and x from equations (6.3.5) and (6.3.4) respectively, and the values of δ from p.257 which are for the same system.

$$\left.\begin{aligned}\left(\frac{3}{4}-S\right)X_1+\frac{1}{2}X_2+\frac{1}{4}X_3&=-\frac{3}{4}\frac{F_0S}{(T/l)}\\ \frac{1}{2}X_1+(1-S)X_2+\frac{1}{2}X_3&=-\frac{1}{2}\frac{F_0S}{(T/l)}\\ \frac{1}{4}X_1+\frac{1}{2}X_2+\left(\frac{3}{4}-S\right)X_3&=-\frac{1}{4}\frac{F_0S}{(T/l)}\end{aligned}\right] \quad (6.4.2)$$

where $S=\dfrac{T}{ml\omega^2}$, as before.

From the above three equations, the expressions for X_1, X_2 and X_3 can be obtained as follow.

$$\left.\begin{aligned}X_1&=\frac{F_0}{(T/l)}\,\frac{\frac{3}{4}S\,(S-0.333)\,(S-1)}{(S-1.707)\,(S-0.5)\,(S-0.293)}\\ X_2&=\frac{F_0}{(T/l)}\,\frac{\frac{1}{2}S^2\,(S-0.5)}{(S-1.707)\,(S-0.5)\,(S-0.293)}\\ X_3&=\frac{F_0}{(T/l)}\,\frac{\frac{1}{4}S^3}{(S-1.707)\,(S-0.5)\,(S-0.293)}\end{aligned}\right] \quad (6.4.3)$$

The denominator of each one of the above equations has been split up from the expression $(S^3-\frac{5}{2}S^2+\frac{3}{2}S-\frac{1}{4})$ and is the same as the expression for the frequency equation for free vibration of the system, i.e. equation (6.3.10), giving an infinite amplitude for each of the masses at a frequency equal to any of the three natural frequencies of the system. However, equation for X_2 above has a common factor $(S-0.5)$ in the numerator as well as denominator, and therefore, this mass has infinite amplitude at the two frequencies only corresponding to $S=1.707\left(=1+\dfrac{1}{\sqrt{2}}\right)$ and $S=0.293\left(=1-\dfrac{1}{\sqrt{2}}\right)$. These frequencies further correspond to the first mode and the third mode. The reason why the second mass does not have infinite amplitude and frequency corresponding to $S=0.5$ is that this frequency corresponds to the natural frequency for the second mode when mass m_2 becomes a node (see Fig. 6.3.4).

Amplitude X_1 of mass m_1 on which the forcing function acts, is zero at two frequencies corresponding to $S = 0{\cdot}333$ and

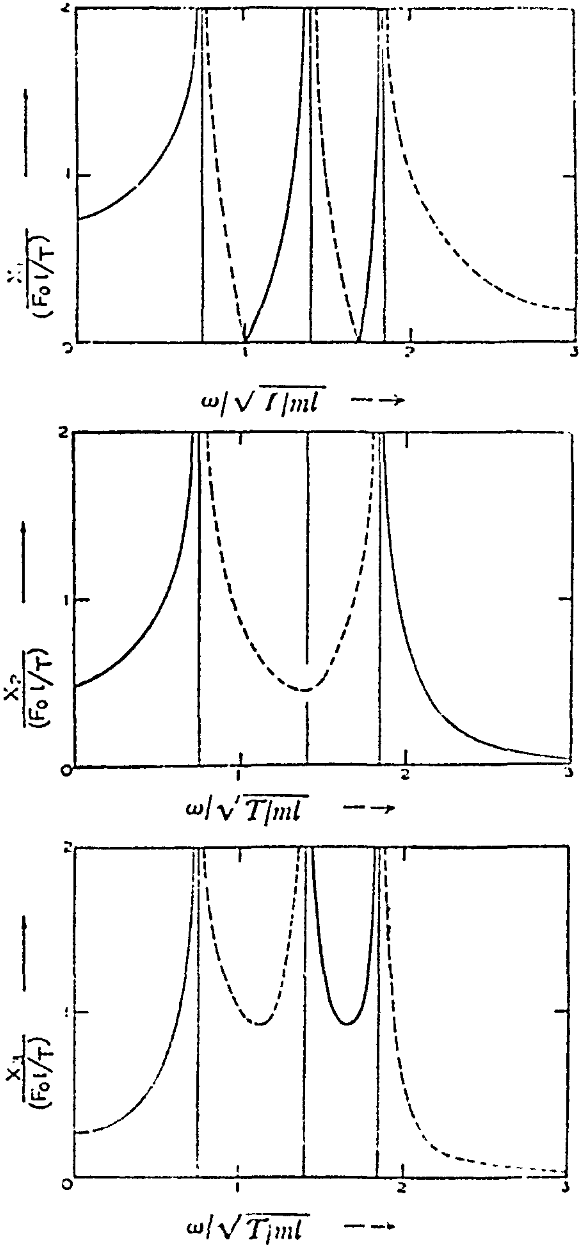


Fig. 6.4.2 Frequency response curves for the three degree of freedom system of Fig. 6.4.1.

$S = 1$. This may be explained as the dynamic absorber effect (see Problem 6.5 at the end of this chapter). The dimensionless frequency-response curves for the three masses are shown in Fig. 6.4.2. The abscissa for these graphs has been taken as

$$\frac{1}{\sqrt{S}}\left(=\frac{\omega}{\sqrt{T/ml}}, \text{ since } S = \frac{T}{ml\omega^2}\right).$$

The dashed portions of the curves indicate that the displacements are 180° out of phase with the exciting force.

6.5 Torsional vibrations of multi-rotor system.

Torsional vibrations of multi-rotor systems are quite common in internal combustion engines besides many other machinery.

Consider an n-rotor system having $(n-1)$ connecting shafts as shown in Fig. 6.5.1. The *moments of inertia of the rotors*

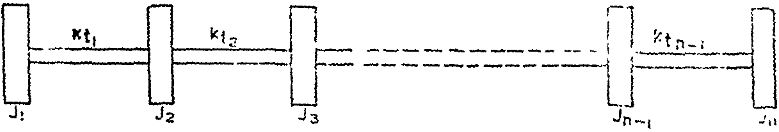


Fig. 6.5.1 Multi-rotor torsional system.

are represented by $J_1, J_2 \ldots J_n$ and the torsional stiffnesses of the shafts by $k_{t1}, k_{t2}, \cdot\, k_{tn-1}$. Let, at any instant, θ_1, θ_2 θ_n be the displacements of various rotors from their equilibrium positions when the system is having torsional vibrations. The twists of the respective shafts at this instant are $(\theta_1 - \theta_2)$, $(\theta_2 - \theta_3), \ldots (\theta_{n-1} - \theta_n)$. From Newton's second law of motion the differential equations of motion for differect rotors are :—

$$\left.\begin{aligned}
J_1 \ddot{\theta}_1 &= -k_{t1}(\theta_1 - \theta_2) \\
J_2 \ddot{\theta}_2 &= +k_{t1}(\theta_1 - \theta_2) - k_{t2}(\theta_2 - \theta_3) \\
J_3 \ddot{\theta}_3 &= +k_{t2}(\theta_2 - \theta_2) - k_{t3}(\theta_3 - \theta_4) \\
&- - - - - - - - - \\
J_{n-1} \ddot{\theta}_{n-1} &= k_{tn-2}(\theta_{n-2} - \theta_{n-1}) - k_{tn-1}(\theta_{n-1} - \theta_n) \\
J_n \ddot{\theta}_n &= k_{tn-1}(\theta_{n-1} - \theta_n)
\end{aligned}\right] \quad (6.5.1)$$

Adding all the above equations, we have

$$\sum_{i=1}^{n} J_i \ddot{\theta}_i = 0 \text{ (for free vibrations).} \qquad (6.5.2)$$

The left hand side of the above equation represents the sum of the inertia torques on all the discs and it is equal to zero since there is no external torque on the system. However, if there are external torques acting on the system at different points, then we must have

$$\sum_{i=1}^{n} J_i \theta_i = T_{ext} \text{ (for forced vibrations).} \qquad (6.5.3)$$

where T_{ext} is the sum of all external torques on the system.

Coming back to the case of free vibrations, let us assume for principal made of vibration, the solution to be

$$\left.\begin{aligned} \theta_1 &= \beta_1 \sin \omega t \\ \theta_2 &= \beta_2 \sin \omega t \\ &\text{— — — —} \\ \theta_n &= \beta_n \sin \omega t \end{aligned}\right] \qquad (6.5.4)$$

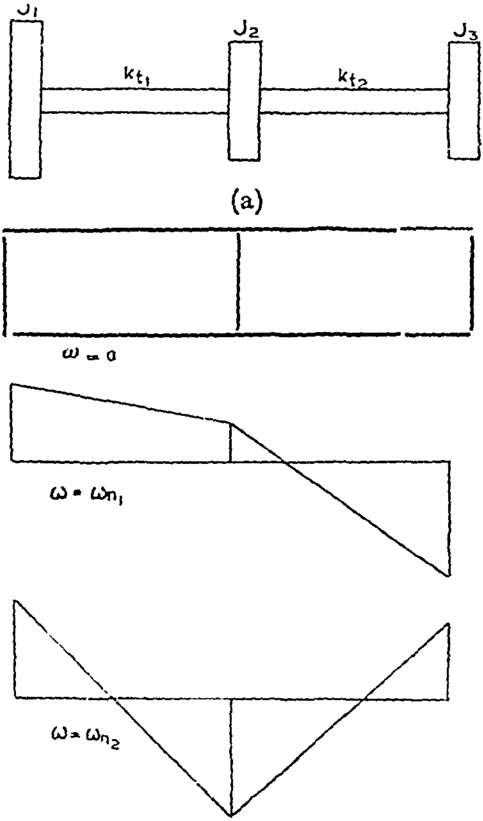
Substituting these in equations (6.5.1), we have

$$\left.\begin{aligned} &J_1\omega^2\beta_1 - k_{t1}(\beta_1 - \beta_2) = 0 \\ &J_2\omega^2\beta_2 + k_{t1}(\beta_1 - \beta_2) - k_{t2}(\beta_2 - \beta_3) = 0 \\ &J_3\omega^2\beta_2 + k_{t2}(\beta_2 - \beta_3) - k_{t3}(\beta_3 - \beta_4) = 0 \\ &\text{— — — — — — — — — —} \\ &J_n\omega^2\beta_n + k_{tn-1}(\beta_{n-1} - \beta_n) = 0 \end{aligned}\right] \qquad (6.5.5)$$

Eliminate $\beta_1, \beta_2, \beta_n$ from the above n homogenous set of equations in $\beta_1, \beta_2 \cdots \beta_n$, and the resulting n^{th} degree equation in ω^2 would then give n natural frequencies of the system.

Special case—Three rotor system.

For the case of a three rotor system having two ... shafts as shown in Fig. 6.5.2 (a), equations (6.5.6) ...



(b)

Fig. 6.5.2. Three rotor system and its mode shapes.

$$J_1\omega^2\beta_1 - k_{t1}(\beta_1 - \beta_2) = 0$$
$$J_2\omega^2\beta_2 + k_{t1}(\beta_1 - \beta_2) - k_{t2}(\beta_2 - \beta_3) = 0$$
$$J_3\omega^2\beta_3 + k_{t2}(\beta_2 - \beta_3) = 0$$

On rearranging, we have

$$\left.\begin{aligned} &(J_1\omega^2 - k_{t1})\,\beta_1 + k_{t1}\beta_2 = 0 \\ &k_{t1}\beta_1 + (J_2\omega^2 - k_{t1} - k_{t2})\,\beta_2 + k_{t2}\beta_3 = 0 \\ &k_{t2}\beta_2 + (J_3\omega^2 - k_{t2})\,\beta_3 = 0 \end{aligned}\right] \qquad (6.5.6)$$

This is a homogenous set of equations in β_1, β_2 and β_3, and can have a solution only if the determinant formed with their coefficients vanishes; or

$$\begin{vmatrix} (J_1\omega^2 - k_{t1}) & k_{t1} & 0 \\ k_{t1} & (J_2\omega^2 - k_{t1} - k_{t2}) & k_{t2} \\ 0 & k_{t2} & (J_3\omega^2 - k_{t2}) \end{vmatrix} = 0 \tag{6.5.7}$$

Expanding the above determinant, gives

$$\omega^2[J_1J_2J_3\omega^4 - \{(J_1J_2 + J_1J_3)k_{t2} + (J_2J_3 + J_1J_3)k_{t1}\}\omega^2 + k_{t1}k_{t2}\,(J_1 + J_2 + J_3)] = 0 \tag{6.5.8}$$

The above equation is cubic in ω^2 with one of the roots of $\omega^2 = 0$. This should have been expected as we are dealing with what is known as a semi-definite system. The same thing was obtained in the case of a two-rotor sytem also. The two definite frequencies in this three rotor system can be obtained from equation (6.5.8) as

$$\omega_{n1}^2, \omega_{n2}^2 = \frac{1}{2}\left[\left(\frac{k_{t1}}{J_1} + \frac{k_{t1} + k_{t2}}{J_2} + \frac{k_{t2}}{J_3}\right) \pm \sqrt{\left(\frac{k_{t1}}{J_1} + \frac{k_{t1} + k_{t2}}{J_2} + \frac{k_{t2}}{J_3}\right)^2 - \frac{4k_{t1}k_{t2}(J_1 + J_2 + J_3)}{J_1J_2J_3}}\,\right] \tag{6.5.9}$$

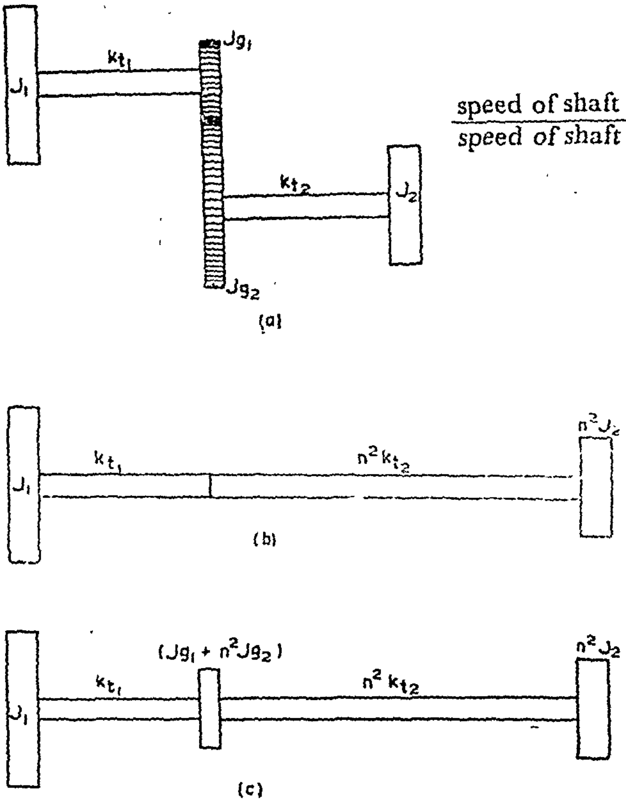
The mode shapes can be obtained from the first and third of equations (6.5.6) and these are:—

$$\left.\begin{aligned} \frac{\beta_1}{\beta_2} &= \frac{k_{t1}}{k_{t1} - J_1\omega^2} \\ \frac{\beta_3}{\beta_2} &= \frac{k_{t2}}{k_{t2} - J_3\omega^2} \end{aligned}\right\} \tag{6.5.10}$$

When $\omega = 0$, both the above ratios are unity, [illegible] that the whole system rotates rigidly. For value [illegible] (the smaller of the two natural frequencies), one of [illegible] equation (6.5 10) is positive while the other is [illegible] for $\omega = \omega_n^2$ both the ratios are negative. The mode [illegible] of the form shown in Fig. 6.5.2 (b).

6.5A Vibrations of geared systems [illegible] systems have geared connections in between.

such cases we take a system as shown in Fig. 6.5.3 (a). Let the gear ratio be n, that is, the speed of the second shaft is n times the speed of the first shaft. The first step in the analysis of this system is to convert the original system into an equivalent system. This we will do with respect to the first shaft although we can do it with respect to the second shaft also. The basis for this conversion is that the kinetic energy and potential energy for the equivalent system should be the same as that for the original system. Then if θ_1 and θ_2 are the angular displace-



$$\frac{\text{speed of shaft 2}}{\text{speed of shaft 1}} = n$$

Fig. 6.5.3. (a) Original geared system. (b) Equivalent system neglecting the inertia of gears. (c) Equivalent system taking into account the inertia of gears.

ments of the rotors J_1 and J_2 respectively, then, neglecting the inertias of the gears, the kinetic energy and the potential energy of the original system are given by

$$\text{K. E.} = \tfrac{1}{2} J_1\dot{\theta}_1^2 + \tfrac{1}{2} J_2\dot{\theta}_2^2$$
$$\text{P. E.} = \tfrac{1}{2} k_{t1}\theta_1^2 + \tfrac{1}{2} k_{t2}\theta_2^2$$

Since $\theta_2 = n\theta_1$, the above expressions become

$$\text{K. E.} = \tfrac{1}{2} J_1\dot{\theta}_1^2 + \tfrac{1}{2} J_2 (n\dot{\theta}_1)^2$$
$$\text{P. E.} = \tfrac{1}{2} k_{t1}\theta_1^2 + \tfrac{1}{2} k_{t2} (n\theta_1)^2$$

or
$$\left.\begin{aligned} \text{K. E.} &= \tfrac{1}{2} J_1\dot{\theta}_1^2 + \tfrac{1}{2} (n^2J_2)\dot{\theta}_1^2 \\ \text{P. E.} &= \tfrac{1}{2} k_{t1}\theta_1^2 + \tfrac{1}{2} (n^2k_{t2})\theta_1^2 \end{aligned}\right] \qquad (6.5.11)$$

The above equations show that the original system can be converted into an equivalent system with respect to the first shaft as shown in Fig 6 5.3 (b), by multiplying the inertia of the second rotor and the stiffness of the second shaft by n^2 and keeping this part of the system in series with the first part. The stiffness of this equivalent two rotor system is then

$$k_{te} = \frac{n^2\, k_{t1}\, k_{t2}}{k_{t1} + n^2k_{t2}} \qquad (6.5.12)$$

and therefore, the natural frequency is

$$\omega_n = \sqrt{\frac{k_{te}\,(J_1 + n^2J_2)}{n^2J_1J_2}} \qquad (6.5.13)$$

From Two rotor

If, however, the inertias of the gears are not negligible, then the equivalent system with respect to the first shaft can be obtained in the same way and finally we have a three rotor system as shown in Fig. 6.5.3 (c), which can be analysed in the usual way.

Illustrative Example 6.5.1.

Fig. 6.5.4 (a) represents two single rotor systems connected by a non-slipping belt. Convert the system to an equivalent torsinal system and find the natural frequency of the system.

lution:

Each of the rotor is connected by two shafts which are fixed the other ends. The system can be converted to that of g. 6.5.4 (b) where each rotor shaft now represents the stiffness the two original shafts in parallel. It may be noted that $_1'$ and k_{t1}'' are in parallel and not in series. The same is ture out the second set of shafts.

$$\text{Let} \quad k_{t1}' + k_{t1}'' = k_{t1}$$

$$\text{and} \quad k_{t2}' + k_{t2}'' = kt_2$$

ow $$n = \frac{\text{speed of shaft 2}}{\text{speed of shaft 1}} = \frac{D_1}{D_2}$$

The final equivalent system is shown in Fig. 6.5.4 (c), which ves the natural frequency of the system as

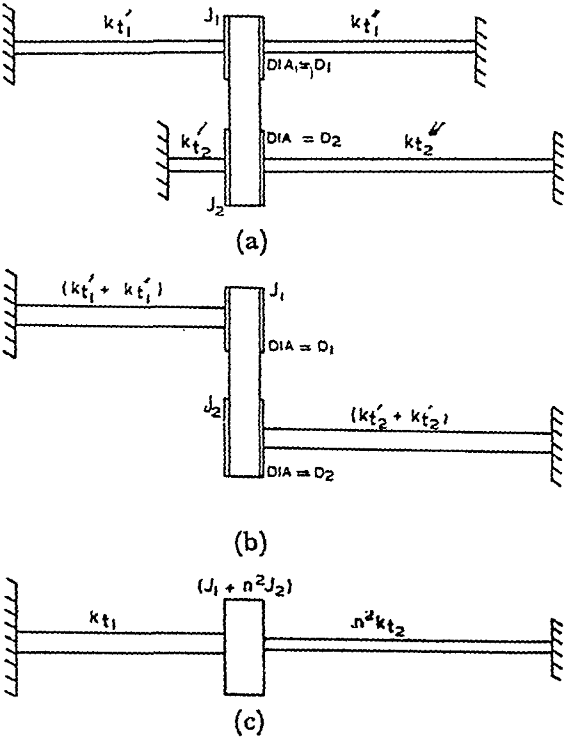


Fig. 6.5.4. Two single-rotor systems connected by belt.

$$\omega_{n1} = \sqrt{\frac{k_{11} + n^2 k_{12}}{J_1 + n^2 J_2}} \qquad \textbf{Ans.}$$

6.6 Generalized coordinates and coordinate coupling.

An n-degree of freedom system requires n independent coordinates to define the system completely at any intant. In

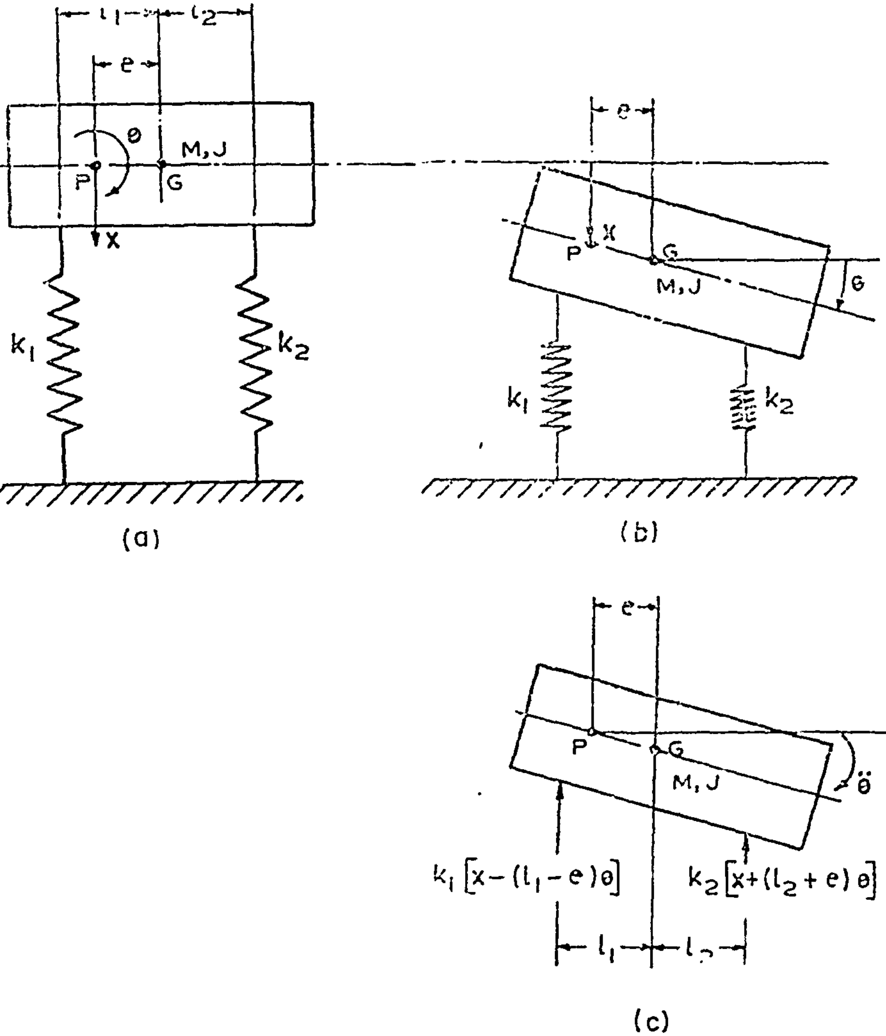


Fig. 6.6.1. Generalized coordinates for a two degree of freedom system.

most of the cases considered we have taken these coordinates from the equilibrium positions. However, we can have any other set of n independent coordinates to specify the configuration of the system. Any of these sets is called the *generalized coordinates*.

Let us go back to the system discussed in Sec. 5.4 and take the two coordinates as shown in Fig. 6.6.1 (a). Fig. 6.6.1 (b) shows the displacements of the two coordinates at any instant and Fig. 6.6.1 (c) gives the external forces acting on the body. The displacement of the CG of the body is $(x + e\theta)$ and therefore, the differential equation of rectilinear motion is

$$M(\ddot{x} + e\ddot{\theta}) = -k_1\,[x-(l_1-e)\theta]-k_2[x+(l_2+e)\theta] \qquad (6.6.1)$$

Also, the differential equation of angular motion is

$$J\ddot{\theta} = k_1\,[x-(l_1-e)\theta]\,l_1-k_2\,[x+(l_2+e)\theta]l_2 \qquad (6.6.2)$$

where J is the mass moment of inertia of the system about its CG. In order to write equation (6.6.2) in terms of J_P, the mass moment of inertia about the point P, add $Me^2\ddot{\theta}$ to both the sides of this equation, or multiply equation (6.6.1) by e and add it to equation (6.6.2). Doing the latter, we have

$$J\ddot{\theta} + Me\ddot{x} + Me^2\ddot{\theta} = k_1[x-(l_1-e)\theta]l_1-k_2[x+(l_2+e)\theta]l_2$$
$$-k_1[x-(l_1-e)\theta]e-k_2[x+(l_2+e)\theta]e = 0 \qquad (6.6.3)$$

Equations (6.6.1) and (6.6.3) can be arranged in the following forms after putting $(J+Me^2) = J_P$ in equation (6.6.3).

$$M\ddot{x} + Me\ddot{\theta} + (k_1+k_2)\,x+[k_2(l_2+e)-k_1(l_1-e)]\theta=0 \qquad (6.6.4)$$

$$J_P\ddot{\theta} + Me\ddot{x} + [k_1(l_1-e)^2+k_2\,(l_2+e)^2]\,\theta +$$
$$[k_2(l_2+e) - k_1(l_1-e)]\,x = 0 \qquad (6.6.5)$$

These are the two general differential equations for rectilinear motion and for angular motion respectively. We will study these equations for three different sets of generalized coordinates.

6.6 A Only static coupling. First of all let us take the point P at the CG of the body; that is, take $e=0$. The equation (6.6.4) and (6.6.5) reduce as below.

$$\left.\begin{aligned} M\ddot{x} + (k_1 + k_2)x + (k_2 l_2 - k_1 l_1)\,\theta &= 0 \\ J_p\ddot{\theta} + (k_1 l_1^2 + k_2 l_2^2)\theta + (k_2 l_2 - k_1 l_1)x &= 0 \end{aligned}\right] \quad (6.6.6)$$

This set of generalized coordinates is shown in Fig. 6.6.2 (a) and is the same as that taken in Sec. 5.4, Since the point P coincides with the CG, we have $J_p = J$. Then equations (6 6.6.) become the same as equations (5.4.1), which they should.

Now, equations of the type (6.6.6), both of which contain x and θ are said to be elastically or statically coupled equations or there is said to be elastic or static coupling between the coordinates. The physical concept of the static coupling is that if one coordinate is given a displacement, the other coordinate also undergoes some displacement. This can clearly be seen with respect to Fig. 6.6.2 (a) where, if a displacement x is given at the point G, the beam does not go down horizontally but is tilted i.e. there is displacement of coordinate θ also. And also if we give a displacement θ to the system, the point G does not stay in its position but undergoes displacement in the x—dire-tion also. Of course, there will be no static coupling if the first of equation (6.6.6) contains no θ term and the second equation contains no x term ; that is, if $k_1 l_1 = k_2 l_2$

6.6 B Only dynamic coupling. A system of generalized coordinates will have only inertia or dynamic coupling if both the equations (6.6.4) and (6.6.5) contain the terms $\ddot{x}$ and $\ddot{\theta}$, and the terms x and θ occur only in the respective equations. Now, the coefficient of θ in equation (6.6.4) and the coefficient of x in equation (6.6.5) are the same. If this coefficient is equated to zero, i.e.,

$$k_2\,(l_2 + e) - k_1\,(l_1 - e) = 0$$

then the coordinates will have only dynamic coupling and no static coupling. Under these conditions equations (6.6.4) and (6.6.5) reduce to

$$\left.\begin{aligned} M\ddot{x} + Me\ddot{\theta} + (k_1 + k_2)\,x &= 0 \\ J_p\ddot{\theta} + Me\ddot{x} + [k_1(l_1-e)^2 + k_2\,(l_2+e)^2]\theta &= 0 \end{aligned}\right] \quad (6.6.7)$$

Hence, for only dynamic coupling and no static coupling

$$k_1\,(l_1 - e) = k_2\,(l_2 + e)$$

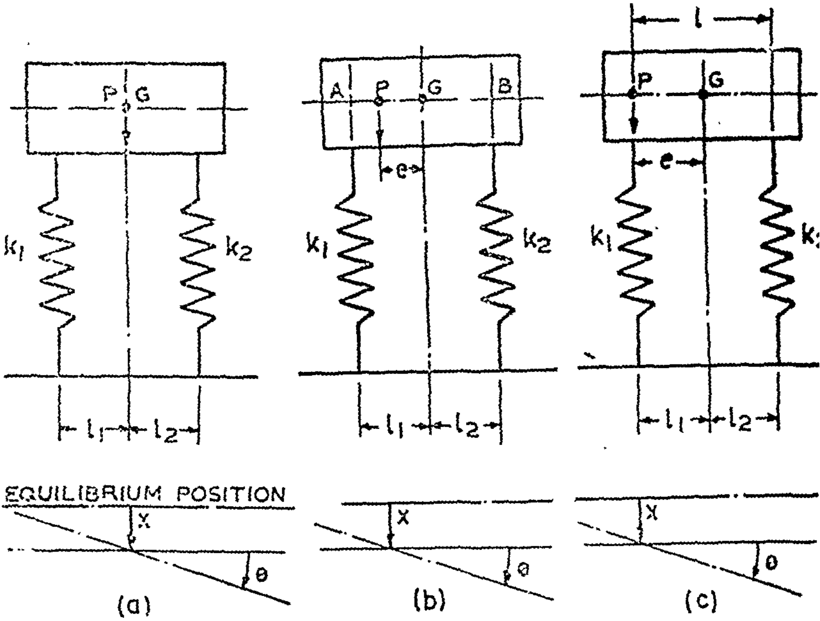


Fig. 6.6.2. Coordinate coupling. (a) Only static cupling. (b) Only dynamic coupling. (c) Static and dynamic coupling.

and $e \neq o$.

Refering to Fig. 6.6.2 (b), we have for the above equation

$$k_1.AP = k_2.BP$$

If the point P is chosen so as to satisfy the above relationship, then the coordinates shown in Fig. 6.6.2 (b) have only dynamic coupling and no static coupling.

The physical concept of dynamic coupling is that if an acceleration is given to one coordinate, the other coordinate also gets an acceleration. It can be seen with reference to Fig. 6.6.2 (b) that if an acceleration $\ddot{x}$ is given at point P, than there is an inertia force $M\ddot{x}$ at the centre of gravity G in opposite direction to $\ddot{x}$ giving a torque on the system which finally gives angular acceleration $\ddot{\theta}$ to it. It can also be seen in a similar manner that an angular acceleration $\ddot{\theta}$ given to the system

causes a translational acceleration at point P. There will be no static coupling in this case which can also be clearly visualized.

6.6C Static and dynamic coupling. If the point P is taken at any other point than the two points taken in the previous two cases, then we have both static and dynamic coupling. Let us take the point P immediately above k_1 giving $e = l_1$ as shown in Fig. 6.6.2 (c). Substituting for e in equations (6.6.4) and (6.6.5), we have

$$\left.\begin{aligned} M\ddot{x} + Ml_1\ddot{\theta} + (k_1 + k_2)x + k_2 l\theta &= 0 \\ J_0\ddot{\theta} + Ml_1\ddot{x} + k_2 l^2\theta + k_3 lx &= 0 \end{aligned}\right] \qquad (6.6.8)$$

Thus, for the coordinates chosen as in Fig. 6.6.2 (c), the differential equations are as obtained in equations (6.6.8). Since both the equations have both $\ddot{x}$ and $\ddot{\theta}$ terms, there is dynamic coupling between the coordinates. And since both the equations have both x and θ terms, there is static coupling between the coordinates. Thus, the system of coordinates chosen has both static and dynamic coupling.

Now, taking a general case of two degrees of freedom system having q_1, q_2 as the generalized coordinates, the equations of undamped free vibrations can be written as

$$\left.\begin{aligned} a_{11}\ddot{q}_1 + a_{12}\ddot{q}_2 + b_{11}q_1 + b_{12}q_2 &= 0 \\ a_{21}\ddot{q}_1 + a_{22}\ddot{q}_2 + b_{21}q_1 + b_{22}q_2 &= 0 \end{aligned}\right] \qquad (6.6.9)$$

in which a_{12} and a_{21} are the dynamic coupling coefficients and, b_{12} and b_{21} are the static coupling coefficients. These equations can be compared with equations (6.6.4) and (6.6.5) to determine the coupling coefficients between the two coordinates x and θ. The frequency equatian from equation (6.6.9) can be obtained by assuming the solution

$$q_1 = Q_1 \sin(\omega t - \phi)$$

and $$q_2 = Q_2 \sin(\omega t - \phi)$$

and substituting these in equations (6.6.9) and following the usual procedure.

For an n degree of freedom system, if the generalized coordinates are $q_1, q_2, \dots q_n$, then equations of undamped free vibrations can be written as

$$\left.\begin{array}{l} a_{11}\ddot{q}_1 + a_{12}\ddot{q}_2 + \quad a_{1n}\ddot{q}_n + b_{11}q_1 + b_{12}q_2 + \dots b_{1n}q_n = 0 \\ a_{21}\ddot{q}_1 + a_{22}\ddot{q}_2 + \dots a_{2n}\ddot{q}_n + b_{21}q_1 + b_{22}q_2 + \dots b_{2n}q_n = 0 \\ - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \\ a_{n1}\ddot{q}_1 + a_{n2}\ddot{q}_2 + \dots a_{nn}\ddot{q}_n + b_{n1}q_1 + b_{n2}q_2 + \dots b_{nn}q_n = 0 \end{array}\right] \tag{6.6.10}$$

The generalized coordinates may be lengths, angles or their combinations.

The existance of coupling can also be seen by writing the expressions for kinetic energy and potential energy of the system. If the kinetic energy expression contains the cross product of the coordinates, there is dynamic coupling between the coordinates. If the expression for potential energy contains the cross product of coordinates, there is static coupling between the coordinates.

Illustrative Example 6.6.1

Consider the coordinate system of Fig. 6.6.2. (b) for the case of a two degree of freedom system. Write down the expressions for the kinetic energy and potential energy of the system. Discuss the coordinate coupling for different values of e and compare with the values concluded for various couplings in Sec. 6.6 A, B and C.

Solution

The expressions for the kinetic energy and potential energy are written below.

$$\text{K.E.} = \tfrac{1}{2}M(\dot{x} + e\dot{\theta})^2 + \tfrac{1}{2}J\dot{\theta}^2$$

$$\text{P.E.} = \tfrac{1}{2}k_1[x - (l_1 - e)\theta]^2 + \tfrac{1}{2}k_2[x + (l_2 + e)\theta]^2$$

On rearranging, we have

$$\left.\begin{array}{rl} \text{K.E.} & = \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}(Me^2 + J)\dot{\theta}^2 + Me\dot{x}\dot{\theta} \\ \text{P.E.} & = \tfrac{1}{2}(k_1+k_2)x^2 + \tfrac{1}{2}[k_1(l_1-e)^2+k_2(l_2+e)^2]\theta^2 \\ & \quad + [k_2(l_2+e) - k_1(l_1-e)]x\theta \end{array}\right] \tag{6.6.11}$$

For no static coupling, the last term of the P. E. expression of equations (6.6.11) must vanish, i.e.,

$$k_2 (l_2 + e) = k_1 (l_1 - e). \tag{6.6.12}$$

For no dynamic coupling, the last term of the K.E. expression of equations (6.6.11) must vanish, i.e.,

$$e = 0. \tag{6.6.13}$$

Therefore, for only static coupling and no dynamic coupling, equation (6.6.13) only is the requirement which corresponds to the coordinate system of Fig. 6.6.2 (a), (see sec. 6.6 A).

For only dynamic coupling and no static coupling, equation (6.6.12) only is the requirement which corresponds to the coordinate system of Fig. 6.6.2 (b), with the condition that $e \neq 0$, (see Sec. 6.6 B).

For both static and dynamic coupling, neither of the equations (6.6.12) and (6.6.13) is satisfied. A particular value of $e = l_1$ which dissatisfies both these equations, gives both static and dynamic coupling. This coordinate system was chosen for Fig. 6.6.2 (c), (see Sec. 6.6 C).

To eliminate both static and dynamic couplings, equations (6.6.12) and (6.6.13) should both be satisfied, i.e.,

$$e = 0$$

and $\quad k_2 l_2 = k_1 l_1$ **Ans.**

6.7 Principal coordinates

For a particular system, as has been shown in the previous section, we can have more than one set of generalized coordinates. Depending upon the coordinates chosen we may have static, dynamic or both forms of coupling between them. By the proper selection of coordinates it is always possible to eliminate both static and dynamic coupling between them. Such coordinates are called the *Principal Coordinates.* When using principal coordinates, each equation of motion will have only one dependent variable and its solution will give its natural frequency and amplitude independently of other equations.

The existance of the principal coordinates can be seen with reference to the system discussed in Sec. 5.4.

For an n degree of freedom system, if the generalized coordinates are $q_1, q_2, \ldots q_n$, then equations of undamped free vibrations can be written as

$$\left.\begin{array}{l} a_{11}\ddot{q}_1 + a_{12}\ddot{q}_2 + \quad a_{1n}\ddot{q}_n + b_{11}q_1 + b_{12}q_2 + \ldots b_{1n}q_n = 0 \\ a_{21}\ddot{q}_1 + a_{22}\ddot{q}_2 + \ldots a_{2n}\ddot{q}_n + b_{21}q_1 + b_{22}q_2 + \ldots b_{2n}q_n = 0 \\ - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \\ a_{n1}\ddot{q}_1 + a_{n2}\ddot{q}_2 + \ldots a_{nn}\ddot{q}_n + b_{n1}q_1 + b_{n2}q_2 + \ldots b_{nn}q_n = 0 \end{array}\right] \tag{6.6.10}$$

The generalized coordinates may be lengths, angles or their combinations.

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Consider the coordinate system of Fig. 6.6.2. (b) for the case of a two degree of freedom system. Write down the expressions for the kinetic energy and potential energy of the system. Discuss the coordinate coupling for different values of e and compare with the values concluded for various couplings in Sec. 6.6 A, B and C.

Solution

The expressions for the kinetic energy and potential energy are written below.

$$\text{K.E.} = \tfrac{1}{2}M(\dot{x} + e\dot{\theta})^2 + \tfrac{1}{2}J\dot{\theta}^2$$

$$\text{P.E.} = \tfrac{1}{2}k_1[x - (l_1 - e)\theta]^2 + \tfrac{1}{2}k_2[x + (l_2 + e)\theta]^2$$

On rearranging, we have

$$\left.\begin{array}{l} \text{K.E.} = \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}(Me^2 + J)\dot{\theta}^2 + Me\dot{x}\dot{\theta} \\ \text{P.E.} = \tfrac{1}{2}(k_1+k_2)x^2 + \tfrac{1}{2}[k_1(l_1-e)^2+k_2(l_2+e)^2]\theta^2 \\ \qquad\qquad + [k_2(l_2+e) - k_1(l_1-e)]x\theta \end{array}\right] \tag{6.6.11}$$

For no static coupling, the last term of the P. E. expression of equations (6.6.11) must vanish, i.e.,

$$k_2 (l_2 + e) = k_1 (l_1 - e). \qquad (6.6.12)$$

For no dynamic coupling, the last term of the K.E. expression of equations (6.6.11) must vanish, i.e.,

$$e = 0. \qquad (6.6.13)$$

Therefore, for only static coupling and no dynamic coupling, equation (6.6.13) only is the requirement which corresponds to the coordinate system of Fig. 6.6.2 (a), (see sec. 6.6 A).

For only dynamic coupling and no static coupling, equation (6.6.12) only is the requirement which corresponds to the coordinate system of Fig. 6.6.2 (b), with the condition that $e \neq 0$, (see Sec. 6.6 B).

For both static and dynamic coupling, neither of the equations (6.6.12) and (6.6.13) is satisfied. A particular value of $e = l_1$ which dissatisfies both these equations, gives both static and dynamic coupling. This coordinate system was chosen for Fig. 6.6.2 (c), (see Sec. 6.6 C).

To eliminate both static and dynamic couplings, equations (6.6.12) and (6.6.13) should both be satisfied, i.e.,

$$e = 0$$

and $$k_2 l_2 = k_1 l_1$$ **Ans.**

6.7 Principal coordinates

For a particular system, as has been shown in the previous section, we can have more than one set of generalized coordinates. Depending upon the coordinates chosen we may have static, dynamic or both forms of coupling between them. By the proper selection of coordinates it is always possible to eliminate both static and dynamic coupling between them. Such coordinates are called the *Principal Coordinates.* When using principal coordinates, each equation of motion will have only one dependent variable and its solution will give its natural frequency and amplitude independently of other equations.

The existance of the principal coordinates can be seen with reference to the system discussed in Sec. 5.4.

The general solution for undamped free vibration of this ystem is

$$\left.\begin{aligned} x &= X' \sin(\omega_{n1}t - \phi_1) + X'' \sin(\omega_{n2}t - \phi_2) \\ \theta &= \beta' \sin(\omega_{n1}t - \phi_1) + \beta'' \sin(\omega_{n2}t - \phi_2) \end{aligned}\right] \qquad (6.7.1)$$

These equations are similar to equation (5.2 13) except that hese are more general. In the above equations ω_{n1} and ω_{n2} re the two natural frequencies of the system, and

$$\frac{X'}{\beta'} = \text{amplitude ratio in the first mode, and}$$

$$\frac{X''}{\beta''} = \text{amplitude ratio in the second mode.}$$

Let q_1, q_2 be a set of generalized coordinates such that x and are related with these as follow.

$$\left.\begin{aligned} x &= a_1 q_1 + a_2 q_2 \\ \theta &= \alpha_1 q_1 + \alpha_2 q_2 \end{aligned}\right] \qquad (6.7\ 2)$$

The choice of the coefficients a_1, a_2, α_1 and α_2 is arbitrary and ifferent values of these constants will give rise to different sets f generalized coordinates. Let us choose

$$\alpha_1 = 1$$

$$\alpha_2 = 1$$

$$a_1 = X'/\beta'$$

nd $\quad a_2 = X''/\beta''$

Then, solving for q_1 and q_2 from equations (6 7.2) and (6.7.1), nd substituting the above chosen values of the coefficients, we ave

$$\left.\begin{aligned} q_1 &= \beta' \sin(\omega_{n1}t - \phi_1) \\ q_2 &= \beta'' \sin(\omega_{n2}t - \phi_2) \end{aligned}\right] \qquad (6.7.3)$$

Equations (6.7.3) are the solutions for the following differenal equations,

$$\left.\begin{aligned} a_{11}\ddot{q}_1 + b_{11}\, q_1 &= 0 \\ a_{22}\ddot{q}_2 + b_{22}\, q_2 &= 0 \end{aligned}\right] \qquad (6.7.4)$$

the coordinates of which have neither static nor dynamic coupling and therefore, are the principal coordinates.

For the principal coordinates chosen, the expressions for kinetic energy and potential energy will be free of cross products of the coordinates.

Illustrative Example 6.7.1

For the two degree of freedom system shown in Fig. 6.7.1, find the principal coordinates for the case when $m_1 = m_2 = m$ and $k_1 = k_2 = k$.

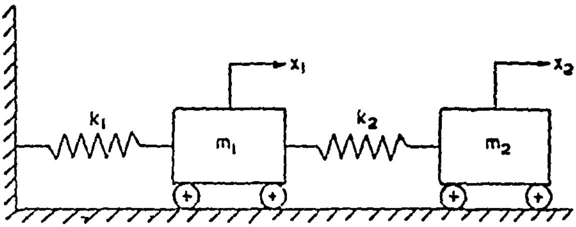


Fig. 6.7.1 Determination of principal coordinates.

Solution

It is first necessary to find the equation of motion in terms of generalized coordinates, say x_1 and x_2. The differential equations of motion after rearranging, are

$$m\ddot{x}_1 + 2kx_1 = kx_2$$

$$m\ddot{x}_2 + kx_2 = kx_1$$

Let $\quad x_1 = X_1 \sin \omega t$

and $\quad x_2 = X_2 \sin \omega t$

for principal mode of vibration.

After substituting these solutions in the equations of motion, the two equations can be written as

$$\frac{X_1}{X_2} = \frac{k}{2k - m\omega^2} = \frac{k - m\omega^2}{k} \tag{6.7.5}$$

The frequency equation, can be simplified to

$$m^2\omega^4 - 3km\omega^2 + k^2 = 0,$$

giving $\omega_{n1} = 0.616\sqrt{k/m}$

and $\omega_{n2} = 1.62\sqrt{k/m}$

Substituting these natural frequencies in equations (6.7.5), we have the following mode shapes.

$$\left(\frac{X_1}{X_2}\right)_1 = \frac{1}{1.62}$$

$$\left(\frac{X_1}{X_2}\right)_2 = -\frac{1}{0.62}$$

The equations of motion for the two masses can now be writtten as

$$\left.\begin{aligned} x_1 &= A_{11}\sin\left(0.616\sqrt{k/m}\,t - \phi_1\right) \\ &\qquad + A_{12}\sin\left(1.62\sqrt{k/m}\,t - \phi_2\right) \\ x_2 &= 1.62A_{11}\sin\left(0.616\sqrt{k/m}\,t - \phi_1\right) \\ &\qquad - 0.62\,A_{12}\sin\left(1.62\sqrt{k/m}\,t - \phi_2\right) \end{aligned}\right] \tag{6.7.6}$$

The constants ϕ_1 and ϕ_2 have been added to make the equations completely general to take care of any initial conditions.

Let us define a new set of generalized coordinates y_1 and y_2 such that

$$\left.\begin{aligned} y_1 &= A_{11}\sin\left(0.616\sqrt{k/m}\,t - \phi_1\right) \\ y_2 &= A_{12}\sin\left(1.62\sqrt{k/m}\,t - \phi_2\right) \end{aligned}\right] \tag{6.7.7}$$

Since y_1 and y_2 are harmonic motions, the corresponding differential equations can be written as

$$\ddot{y}_1 + \frac{0.38\,k}{m}\,y_1 = 0$$

$$\ddot{y}_2 + \frac{2.62\,k}{m}\,y_2 = 0$$

The above equations represent a two degree of freedom system with natural frequencies $\omega_{n1} = 0.616\sqrt{k/m}$ and $\omega_{n2} = 1.62\sqrt{k/m}$. These equations have neither static nor dynamic coupling between them. Therefore y_1 and y_2 are principal coordinates. Their relations with x_1 and x_2 can be obtained from equations (6.7.6) and (6.7.7).

Therefore,

$$x_1 = y_1 + y_2$$
$$x_2 = 1.62\,y_1 - 0.62\,y_2$$

giving

$$y_1 = 0.295\,x_1 + 0.447\,x_2$$
$$y_2 = 0.705\,x_1 - 0.447\,x_2 \qquad \textbf{Ans.}$$

6.8 Continuous systems.

The rest of the portion considered in this chapter deals with the continuous systems which have continuously distributed masses and stiffnesses. Such a system is equivalent to an infinite elements of masses concentrated at different points and hence is an infinite degree of freedom system. The equations for these systems are derived on the assumption that the bodies are homogenous and isotropic, and that they obey Hooke's law within the elastic limit.

These systems have infinite principal modes of vibration corresponding to the infinite natural frequencies of the system. In general, the vibration of these systems is the sum of all these principal modes. If the elastic curve of the body under which the vibration is started, is identical to any one of the principal mode shapes, then the system will vibrate only in that principal mode.

6.8A Vibrations of strings. A string stretched between two supports, as shown in Fig. 6.8.1 (a), is an infinite degrees of freedom system since it is equivalent to infinite elements of masses distributed along its entire length.

Let the tension S in the string be large and the amplitude of vibration small so that the tension remains appreciably constant throughout the string during its vibrations. To analyse this problem let us take an element of length dx at a distance x from the left end. Let, at any instant, this element of string be displaced through a distance y from the equilibrium position. Then, the tension at both ends of this element is S. If θ is the angle that the left end of the element makes with the X–axis, then angle at

The solutions of the above equations are

$$\left.\begin{aligned} X &= A \sin \frac{\omega}{c} x + B \cos \frac{\omega}{c} x \\ T &= C \sin \omega t + D \cos \omega t \end{aligned}\right] \tag{6.8.6}$$

But we have $y = X.T$

Therefore,

$$y = \left(A \sin \frac{\omega}{c} x + B \cos \frac{\omega}{c} x\right)\left(C \sin \omega t + D \cos \omega t\right.$$

Here, ω is seen to be the frequency of vibration and the above solution is for one value of ω. Since ω can have infinite values, being an infinite degrees of freedom system, the general solution can be written as

$$y = \sum_{i=1}^{\infty} \left(A_i \sin \frac{\omega_i x}{c} + B_i \cos \frac{\omega_i x}{c}\right)\left(C_i \sin \omega_i t + D_i \cos \omega_i t\right) \tag{6.8.7}$$

The constants in the above equation have to be determined from the boundary conditions and the initial conditions as will be shown in the example at the end of this main section.

6.8B Longitudinal vibrations of bars. To analyse this system it is assumed that the bar is of a constant cross-section A and that the plane sections normal to the axis remain plane and normal to the axis during vibrations.

Consider an element of length dx at a distance x from one end of the bar as shown in Fig. 6.8.2. At any instant, during vibrations, let P be the axial force at the left end of this element. Then the axial force at the other end of the element will be $\left(P + \frac{\partial P}{\partial x} \cdot dx\right)$. Further, if u is the displacement of the left end of the element (i.e. the cross-section of the bar at a distance x) at the same instant, then displacement of the section at the other end of element will be $\left(u + \frac{\partial u}{\partial x} \cdot dx\right)$. Therefore, the change in length of the element $= \frac{\partial u}{\partial x} \cdot dx$.

or, the unit strain $= \dfrac{\partial u}{\partial x}$.

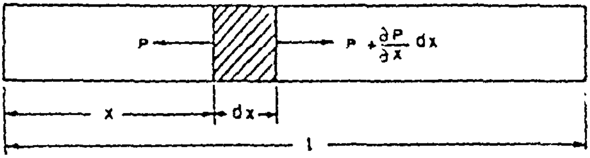


Fig. 6.8.2. Longitudinal vibrations of bars.

Also, the unit strain is the ratio of stress P/A to the modulus of elasticity E.

Therefore, $\dfrac{\partial u}{\partial x} = \dfrac{P/A}{E}$

or $$AE\frac{\partial u}{\partial x} = P \tag{6.8.8}$$

Considering the dynamic equilibrium of the element from the Newton's second law, we have

(mass of the element) × (acceleration)
= (resultant external force)

or $$\left(\frac{\gamma}{g}A\,dx\right) \times \left(\frac{\partial^2 u}{\partial t^2}\right) = \frac{\partial P}{\partial x}\cdot dx$$

where γ is the weight per unit volume of the bar.

Hence $$\frac{\partial P}{\partial x} = \frac{\gamma A}{g}\frac{\partial^2 u}{\partial t^2} \tag{6.8.9}$$

Also, from equation (6.8.8)

$$\frac{\partial P}{\partial x} = AE\frac{\partial^2 u}{\partial x^2} \tag{6.8.10}$$

Equating (6.8.9) and (6.8.10), we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{l}{c^2}\frac{\partial^2 u}{\partial t^2} \tag{6.8.11}$$

where $$c = \sqrt{\frac{Eg}{\gamma}} \tag{6.8.12}$$

is the volocity of wave propagation in the bar.

The equation (6.8.11) is similar to equation (6.8.1) for the case of the string and therefore the solution to equation (6.8.11) is exactly similar to equation (6.8.7), that is,

$$u = \sum_{i=1}^{\infty}\left(A_i \sin \frac{\omega_i x}{c} + \beta_i \cos \frac{\omega_i x}{c}\right)\left(C_i \sin \omega_i t + D_i \cos \omega_i t\right) \tag{6.8.13}$$

where c is given by equation (6.8.12).

6.8C Torsional vibrations of circular shafts. The equations of motion for the torsional vibrations of the circular uniform shafts are similar to the longitudinal vibrations of uniform bars discussed in Sec. 6.8 B. The method of derivation of these equations is also similar.

Let T and $\left(T + \frac{\partial T}{\partial x} . dx\right)$ be the torques at the two ends of the element of a circular shaft as shown in Fig. 6.8.3. If θ is the angular rotation of the shaft at the distance x, then

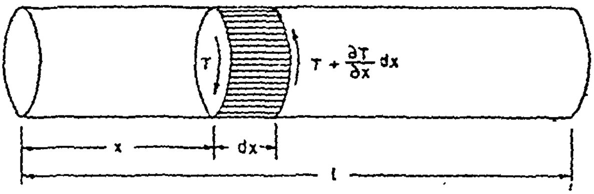


Fig. 6.8.3. Torsional vibrations of circular shafts.

$\left(\theta + \frac{\partial \theta}{\partial x} . dx\right)$ is the angular rotation of the shaft at the distance $(x + dx)$. Therefore, the angular twist of the element of length dx is $\frac{\partial x}{\partial x} . dx$. If I_p is the polar moment of inertia of the shaft and G the modulus of rigidity, then this angular twist is given by

$$\frac{\partial \theta}{\partial x} . dx = \frac{T dx}{G I_p}$$

or
$$G I_p \frac{\partial \theta}{\partial x} = T \tag{6.8.14}$$

Considering the dynamic equilibrium of the element, we have

$$\left(\frac{I_p \gamma \, dx}{g}\right)\left(\frac{\partial^2 \theta}{\partial t^2}\right) = \frac{\partial T}{\partial x} . dx$$

where $\left(\frac{I_p \gamma \, dx}{g}\right)$ is the mass moment of inertia of the element of the shaft, γ being the weight per unit volume of its material.

Or, $$\frac{I_p \gamma}{g} \frac{\partial^2\theta}{\partial t^2} = \frac{\partial T}{\partial x} \tag{6.8.15}$$

Also from equation (6.8.14)

$$GI_p \frac{\partial^2\theta}{\partial x^2} = \frac{\partial T}{\partial x} \tag{6.8.16}$$

From equations (6.8.15) and (6.8.16), we have

$$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2\theta}{\partial t^2} \tag{6.8.17}$$

where $$c = \sqrt{\frac{Gg}{\gamma}} \tag{6.8.18}$$

This solution is similar to equation (6.8.13) and is given by

$$\theta = \sum_{i=1}^{\infty}\left(A_i \sin\frac{\omega_i x}{c} + B_i \cos\frac{\omega_i x}{c}\right)\left(C_i \sin\omega_i t + D_i \cos\omega_i t\right) \tag{6.8.19}$$

where c is given by equation (6.8.18).

6.8D Lateral vibrations of beams. To derive the differential equation of motion for the lateral vibrations of beams, consider an element of beam of length dx with the forces and moments acting on it as shown in Fig. 6.8.4. Q and M are the shear force and bending moment respectively, and $p(x)$ is the distributed load per unit length.

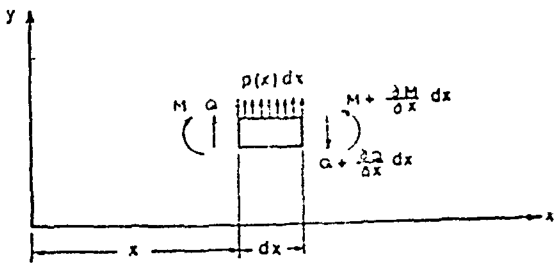


Fig. 6.8.4. Lateral vibrations of beams.

By equating to zero, the resultant vertical force on element, we have

$$p(x)\,dx - \frac{\partial Q}{\partial x}dx = 0$$

or

$$p(x) = \frac{\partial Q}{\partial x} \tag{6.8.20}$$

This states that the rate of change of shear along the length o the beam is equal to the load per unit length.

By equating the resultant moment about any point on the right end of the element to zero,

$$Q\,dx + p(x)\,dx\,\frac{dx}{2} - \frac{\partial M}{\partial x}\,dx = 0$$

which in the limiting case, becomes

$$Q = \frac{\partial M}{\partial x} \tag{6.8.21}$$

This states that the rate of change of bending moment along the length of the beam is equal to the shear force.

From equations (6.8.20) and (6.8.21), we have

$$p(x) = \frac{\partial^2 M}{\partial x^2} \tag{6.8.22}$$

The relation between the bending moment and the curvature of the beam is given by the flexture equation and, for the co-ordinate system of Fig. 6.8.4, is

$$EI\,\frac{\partial^2 y}{\partial x^2} = M \tag{6.8.23}$$

$$\frac{\partial^2}{\partial x^2}\left[EI\frac{\partial^2 y}{\partial x^2}\right] = p(x) \tag{6.8.24}$$

Now, for the dynamic case when the beam is having ansverse vibrations, the loading per unit length of the beam due to the inertia force because of its mass and acceleration. γ and A are the weight per unit volume and the cross-section ea of the beam respectively, then the inertia force per unit gth of the beam is $\frac{-\gamma A}{g}\frac{\partial^2 y}{\partial t^2}$. Replacing, therefore, $p(x)$ in

equation (6.8.24) by $-\dfrac{\gamma A}{g}\dfrac{\partial^2 y}{\partial t^2}$ we have

$$\frac{\partial^2}{\partial x^2}\left[E\,I\frac{\partial^2 y}{\partial x^2}\right] = -\frac{\gamma A}{g}\frac{\partial^2 y}{\partial t^2}$$

or
$$\frac{\partial^2}{\partial x^2}\left[E\,I\frac{\partial^2 y}{\partial x^2}\right] + \frac{\gamma A}{g}\frac{\partial^2 y}{\partial t^2} \quad 0 \qquad (6.8.25)$$

If EI is constant, then above equation reduces to

$$\frac{\partial^2 y}{\partial t^2} + a^2\frac{\partial^4 y}{\partial x^4} = 0 \qquad (6.8.26)$$

where
$$a = \sqrt{\frac{EIg}{\gamma A}} \qquad (6.8.27)$$

Solution of the differential equation (6.8.26) is obtained by considering y to be a product of two functions as shown below.

Let $y\,(x, t) = X\,(x)\;.\;T\,(t)$

For simplicity, let us write

$$y = X\;.\;T$$

Then
$$\frac{\partial^2 y}{\partial t^2} = X\;.\;\frac{d^2T}{dt^2}$$

and
$$\frac{\partial^4 y}{\partial x^4} = T\;.\;\frac{d^4X}{dx^4}$$

Substituting these in equation (6.8.26),

$$X\frac{d^2T}{dt^2} + a^2\,T\frac{d^4X}{dx^4} = 0$$

or
$$\frac{1}{T}\frac{d^2T}{dt^2} = -\frac{a^2}{X}\frac{d^4X}{dx^4} \qquad (6.8.29)$$

The left hand side of the above equation is a function of t alone and the right hand side a function of x alone. It can be shown that this is possible only if each side of this equation is equal to a negative constant, $-\omega^2$ (say), where ω is a real number.

Therefore,

$$\frac{1}{T}\frac{d^2T}{dt^2} = -\frac{a^2}{X}\frac{d^4X}{dx^4} = -\omega^2$$

which gives

$$\left.\begin{aligned}\frac{d^2T}{dt^2} + \omega^2 T &= 0\\ \frac{d^4X}{dx^4} - k^4 X &= 0\end{aligned}\right] \qquad (6.8.3$$

where $k = \sqrt{\omega/a}$ (6.8.1)

The solutions to the equations (6.8.30) are as below.

$$\left.\begin{aligned} T &= A \sin \omega t + B \cos \omega t \\ X &= C_1 \sin kx + C_2 \cos kx + C_3 \sinh kx + C_4 \cosh kx \end{aligned}\right] \quad (6.8.32)$$

X defines the shape of the normal mode of vibration and is termed as *Normal Function.* The constants $C_1 \ldots C_4$ can be determined from the four boundary conditions for the two ends of the vibrating beam, shown in Table 6.8.1

TABLE 6.8.1

Boundary conditions for different beam ends.

	Defl (x)	Slope $\left(\frac{dX}{dx}\right)$	B.M.$\left(\frac{d^2X}{dx^2}\right)$	S.F. $\left(\frac{d^3X}{dx^3}\right)$
Hinged End	zero	—	zero	—
Fixed End	zero	zero	—	—
Free End	—	—	zero	zero

The general solution can be written as

$$y = \sum_{i=1}^{\infty} X_i \,(A_i \sin \omega_i t + B_i \cos \omega_i t) \quad (6.8.33)$$

where

$$X_i = C_{1i} \sin k_i x + C_{2i} \cos k_i x + C_{3i} \sinh k_i x + C_{4i} \cosh k_i x \quad (6.8.34)$$

Illustrative Example 6.8.1.

A uniform string of length l and a large initial tension S, stretched between two supports, is displaced laterally through a distance a_o at the centre as shown in Fig. 6.8.5, and is released at $t = 0$. Find the equation of motion for the string.

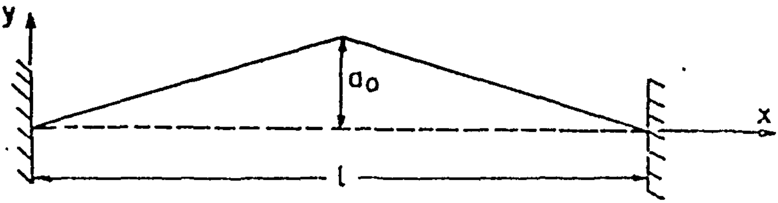


Fig. 6.8.5. Vibration of string with initial displacement at the centre.

Solution

The general equation of motion for the starting has been derived in Sec. 6.8A and is given in equation (6.8.7).

The boundary conditions for the string are

$$y = 0 \qquad \text{at} \quad x = 0,$$
$$y = 0 \qquad \text{at} \quad x = l.$$

Applying the first of these boundary conditions to equation (6.8.7), we have

$$B_i = 0$$

The equation, now, is modified to

$$y = \sum_{i=1}^{\infty} \sin \frac{\omega_i x}{c} \; (C_i \sin \omega_i t + D_i \cos \omega_i t)$$

the constant A_i having been included in C_i and D_i.

Applying the second boundary condition to the above equation, we have

$$\sin \frac{\omega_i l}{c} = 0$$

$$\therefore \quad \frac{\omega_i l}{c} = i\pi$$

or $$\omega_i = \frac{i\pi c}{l}$$

Therefore the equation for y now can be written as

$$y = \sum_{i=1}^{\infty} \sin \frac{i\pi x}{l} \left[C_i \sin \frac{i\pi c}{l} t + D_i \cos \frac{i\pi c}{l} t \right] \qquad (6.8.35)$$

The initial conditions are

$$\left(y \right)_{t=0} = \frac{2a_0 x}{l}, \qquad 0 \leqslant x \leqslant \frac{l}{2}$$

$$= 2a_0 \left(1 - \frac{x}{l}\right), \qquad \frac{l}{2} \leqslant x \leqslant l$$

$$\left(\dot{y} \right)_{t=0} = 0.$$

Applying these initial conditions to equation (6.8. 35) and its derivative, we have

$$\left.\begin{aligned} \left(y \right)_{t=0} &= \sum_{i=1}^{\infty} D_i \sin \frac{i\pi x}{l} \\ \left(\dot{y} \right)_{t=0} &= \sum_{i=1}^{\infty} C_i \frac{i\pi c}{l} \sin \frac{i\pi x}{l} \end{aligned}\right] \qquad (6.8.36)$$

The second of the equations (6.8.36) is zero which gives

$$C_i = 0$$

The first of equations (6.8.36) can be written as

$$\sum_{i=1}^{\infty} D_i \sin \frac{i\pi x}{l} = \frac{2a_0 x}{l}, \qquad 0 \leqslant x \leqslant \frac{l}{2}$$

$$= 2a_0 \left(1 - \frac{x}{l} \right), \quad \frac{l}{2} \leqslant x \leqslant l.$$

Multiplying both sides of the above equation by $\sin \frac{i\pi x}{l}$ and integrating between the limits $x = 0$ and $x = l$, we get

$$D_i \int_0^l \sin^2 \frac{i\pi x}{l}\, dx = \int_0^{l/2} \frac{2a_0 x}{l} \sin \frac{i\pi x}{l}\, dx$$

$$+ \int_{l/2}^{l} 2a_0 \left(1 - \frac{x}{l}\right) \sin \frac{i\pi x}{l}\, dx$$

Solving the above integrals we get

$$D_i = (-1)^{\frac{(i-1)}{2}} \frac{8a_0}{i^2\pi^2} \text{ when } i \text{ odd}$$

$$= 0, \qquad \text{when } i \text{ even.}$$

Equation (6·8.35) can now finally be written as

$$y = \sum_{i=1,3,5\ldots}^{\infty} (-1)^{\frac{(i-1)}{2}} \frac{8a_0}{i^2\pi^2} \sin \frac{i\pi x}{l} \cos \frac{i\pi c}{l} t$$

or $$y = \frac{8a_0}{\pi^2} \left[\sin \frac{\pi x}{l} \cos \frac{\pi c}{l} t - \frac{1}{9} \sin \frac{3\pi x}{l} \cos \frac{3\pi c}{l} t + \frac{1}{25} \sin \frac{5\pi x}{l} \cos \frac{5\pi c}{l} t \right]$$

Illustrative Example 6 8.2

Determine the normal functions for free longitudinal vibration of a bar of length l and uniform cross-section. One end of the bar is fixed and the other free.

Solution

The general solution has been obtained in Sec. 6.8B and is given by equation (6.8.13).

The boundary conditions are

$$u = 0 \quad \text{at} \quad x = 0,$$

$$\frac{\partial u}{\partial x} = 0 \quad \text{at} \quad x = l.$$

Substituting these boundary conditions in equation (6.8.13), gives

$$y = \sum_{i=1}^{\infty} \sin \frac{i\pi x}{l}\left[C_i \sin \frac{i\pi c}{l} t + D_i \cos \frac{i\pi c}{l} t\right] \tag{6.8.35}$$

The initial conditions are

$$\left(y\right)_{t=0} = \frac{2a_0 x}{l}, \qquad 0 \leqslant x \leqslant \frac{l}{2}$$

$$= 2a_0 \left(1 - \frac{x}{l}\right), \qquad \frac{l}{2} \leqslant x \leqslant l$$

$$\left(\dot{y}\right)_{t=0} = 0.$$

Applying these initial conditions to equation (6.8. 35) and its
erivative, we have

$$\left.\begin{aligned} \left(y\right)_{t=0} &= \sum_{i=1}^{\infty} D_i \sin \frac{i\pi x}{l} \\ \left(\dot{y}\right)_{t=0} &= \sum_{i=1}^{\infty} C_i \frac{i\pi c}{l} \sin \frac{i\pi x}{l} \end{aligned}\right] \tag{6.8.36}$$

The second of the equations (6.8.36) is zero which gives

$$C_i = 0$$

The first of equations (6.8.36) can be written as

$$\sum_{i=1}^{\infty} D_i \sin \frac{i\pi x}{l} = \frac{2a_0 x}{l}, \qquad 0 \leqslant x \leqslant \frac{l}{2}$$

$$= 2a_0 \left(1 - \frac{x}{l}\right), \quad \frac{l}{2} \leqslant x \leqslant l.$$

Multiplying both sides of the above equation by $\sin \frac{i\pi x}{l}$ and
ntegrating between the limits $x = 0$ and $x = l$, we get

$$D_i \int_0^l \sin^2 \frac{i\pi x}{l}\, dx = \int_0^{l/2} \frac{2a_0 x}{l} \sin \frac{i\pi x}{l}\, dx$$

$$+ \int_{l/2}^{l} 2a_0 \left(1 - \frac{x}{l}\right) \sin \frac{i\pi x}{l}\, dx$$

Solving the above integrals we get

$$D_i = (-1)^{\frac{(i-1)}{2}} \frac{8a_0}{i^2 \pi^2} \text{ when } i \text{ odd}$$

$$= 0, \qquad \text{when } i \text{ even.}$$

Equation (6·8.35) can now finally be written as

$$y = \sum_{i=1,3,5\ldots}^{\infty} (-1)^{\frac{(i-1)}{2}} \frac{8a_0}{i^2\pi^2} \sin \frac{i\pi x}{l} \cos \frac{i\pi c}{l} t$$

or $$y = \frac{8a_0}{\pi^2}\left[\sin \frac{\pi x}{l} \cos \frac{\pi c}{l} t - \frac{1}{9} \sin \frac{3\pi x}{l} \cos \frac{3\pi c}{l} t + \frac{1}{25} \sin \frac{5\pi x}{l} \cos \frac{5\pi c}{l} t \right]$$

Illustrative Example 6 8.2

Determine the normal functions for free longitudinal vibration of a bar of length l and uniform cross-section. One end of the bar is fixed and the other free.

Solution

The general solution has been obtained in Sec. 6.8B and is given by equation (6.8.13).

The boundary conditions are

$$u = 0 \quad \text{at} \quad x = 0,$$

$$\frac{\partial u}{\partial x} = 0 \quad \text{at} \quad x = l.$$

Substituting these boundary conditions in equation (6.8.13), gives

$$B_i = 0$$

and $\cos \dfrac{\omega_i l}{c} = 0.$

The latter equation has a solution

$$\frac{\omega_i l}{c} = \frac{i\pi}{2}, \quad i = 1,3,5\ldots$$

Thus the normal functions are

$$X_i = A_i \sin \frac{i\pi x}{2l}, \quad i = 1, 3, 5\ldots$$

Illustrative Example 6.8.3

A bar fixed at one end is pulled at the other end with a force P. The force is suddenly released. Investigate the vibration of the bar.

Solution

For longitudinal vibrations, a bar fixed at one end and free at the other has frequencies given by

$$\frac{\omega_i l}{c} = \frac{i\pi}{2}, \quad i = 1,3,5\ldots$$

as obtained in Illustrative Example 6.8 2.

Or $\omega_i = \dfrac{i\pi c}{2l}, \quad i = 1, 3, 5\ldots$

Also $B_i = 0$

Equation (6.8.13) then reduces to

$$u = \sum_{i=1,3\ldots}^{\infty} \sin \frac{i\pi x}{2l}\left[C_i \sin\frac{i\pi c}{2l}t + D_i \cos\frac{i\pi c}{2l}t\right] \tag{6.8.37}$$

the constant X_i having been included in constants C_i and D_i

The initial conditions are

$$\left(u\right)_{t=0} = \frac{P}{AE}x$$

$$\left(\dot{u}\right)_{t=0} = 0$$

Substituting these initial conditions in equation (6.8.37) and its derivative, we get

$$C_i = 0$$

and $$\sum_{i=1,3\cdots}^{\infty} D_i \sin\frac{i\pi x}{2l} = \frac{P}{AE}x.$$

Multiplying by $\sin\frac{i\pi x}{2l}$ on both sides and intergrating between 0 to l, we obtain

$$\int_0^l D_i \sin^2\frac{i\pi x}{2l}dx = \int_0^l \frac{P}{AE}x \sin\frac{i\pi x}{2l}dx.$$

Solving gives

$$D_i = \frac{8Pl}{AE\pi^2 i^2}\left(-1\right)^{\frac{(i-1)}{2}} \qquad i = 1,3,5$$

The equation of motion, therefore, becomes

$$u = \sum_{i=1,3\cdots}^{\infty} \frac{8Pl}{AE\pi^2 i^2}\left(-1\right)^{\frac{(i-1)}{2}} \sin\frac{i\pi x}{2l}\cos\frac{i\pi c}{2l}t$$

or $$u = \frac{8Pl}{AE\pi^2}\left[\sin\frac{\pi x}{2l}\cos\frac{\pi ct}{2l} - \frac{1}{9}\sin\frac{3\pi x}{2l}\cos\frac{3\pi ct}{2l} + \frac{1}{25}\sin\frac{5\pi x}{2l}\cos\frac{5c\pi t}{2l}\cdots\right]$$

Illustrative Example 6.8.4

Determine the normal functions in transverse vibration for a simply supported beam of length l and uniform cross-section.

Solution

The normal functions for transverse vibration are given by equation (6.8.34) which may be written as

$$X_i = C_{1i}(\sin k_i x + \sinh k_i x) + C_{2i}(\sin k_i x - \sinh k_i x) + C_{3i}(\cos k_i x + \cosh k_i x) + C_{4i}(\cos k_i x - \cosh k_i x) \qquad (6.8.38)$$

with the constants being different from those of equation(6.8.34).

Since the ends are simply supported, the deflection and bending moment are both zero for both the ends. Therefore, the boundary conditions are

$$(X_i)_{x=0} = 0 \quad --(1) \qquad (X_i)_{x=l} = 0 \quad --(3)$$

$$\left(\frac{d^2X_i}{dx^2}\right)_{x=0} = 0 \quad --(2) \qquad \left(\frac{d^2X_i}{dx^2}\right)_{x=l} = 0 \quad --(4)$$

The first of the boundary condition substituted in equation (6.8.38) gives

$$C_{3i} = 0$$

and the second boundary condition gives

$$C_{4i} = 0$$

Boundary conditions (3) and (4) give

$$C_{1i} = C_{2i}$$

and $\sin k_i l = 0$

This last equation is the frequency equation and its solution is

$$k_i l = i\pi$$

or $$k_i = \frac{i\pi}{l}$$

Equation (6.8.38) can, therefore, finally be written as

$$X_i = C_i \sin \frac{i\pi x}{l}$$

which are the normal functions, and can be written as

$$X_1 = C_1 \sin \frac{\pi x}{l}$$

$$X_2 = C_2 \sin \frac{2\pi x}{l}$$

$$X_3 = C_3 \sin \frac{3\pi x}{l}$$

— — — — — —

The above equations give the deflection curve during any principal mode of vibration. Suitable combination of the various constants can give the deflection curve corresponding to any initial deflection curve. The various mode shapes and

the correspoding natural frequencies ω_i obtained after

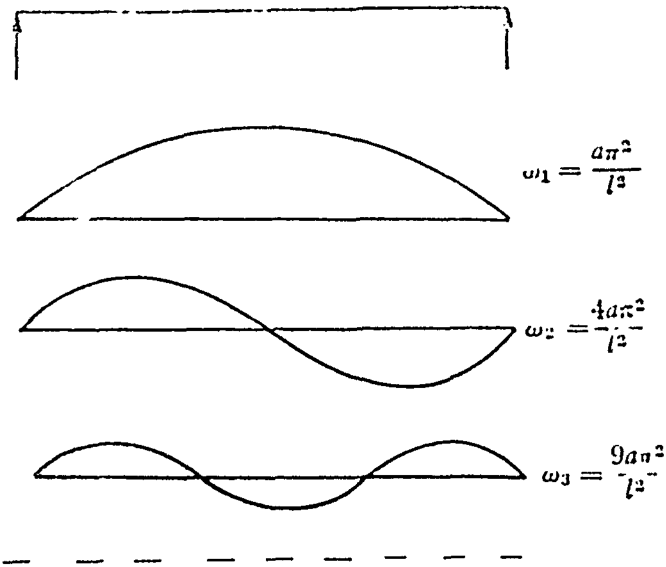


Fig. 6.8.6. Mode shapes for a simply supported beam.

substituting in equation (6.8.31) the value of k_i as found above, are shown in Fig. 6.8.6

Illustrative Example 6.8.5

Obtain the frequency equation for the lateral vibrations of a cantilever of uniform section having a length l.

Solution

The boundary conditions are that deflection and slope at the fixed end are equal to zero, and bending moment and shear force at the free end are zero.

Therefore

$$\left(X\right)_{x=o}=0 \qquad -(1) \qquad \left(\frac{d^2X}{dx^2}\right)_{x=l}=0 \quad -(3)$$

$$\left(\frac{dX}{dx}\right)_{x=o}=0 \qquad -(2) \qquad \left(\frac{d^3X}{dx^3}\right)_{x=l}=0 \quad -(4)$$

The first two conditions substituted in equation (6.8.38) give

$$C_{3i} = 0, \qquad C_{1i} = 0.$$

The other two conditions give

$$C_{2i}\,(-\sin k_i l - \sinh k_i l) + C_{4i}\,(-\cos k_i l - \cosh k_i l) = 0$$

and $$C_{2i}\,(-\cos k_i l - \cosh k_i l) + C_{4i}\,(\sin k_i l - \sinh k_i l) = 0$$

or $$\frac{C_{2i}}{C_{4i}} = \frac{-(\cos k_i l + \cosh k_i l)}{(\sin k_i l + \sinh k_i l)} = \frac{(\sin k_i l - \sinh k_i l)}{(\cos k_i l + \cosh k_i l)}$$

After cross multiplication and simplification, we obtain

$$\cos k_i l \,\cosh k_i l = -1$$

which is the frequency equation.

PROBLEMS FOR PRACTICE

6.1 Calculate the three frequencies of the system shown in Fig. P.6.1. When the pendulums are vertical, the coupling springs are unstressed.

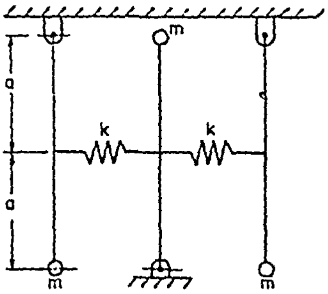


Fig. P.6.1.

6.2 Determine the frequency equation in the form of determinant for a general three degree of freedom spring-mass system shown in Fig. P.6.2.

6.3 By means of influence numbers determine the three natural frequencies for the system shown in Fig. P.6.3.

Could you have predicted these three natural frequencies?

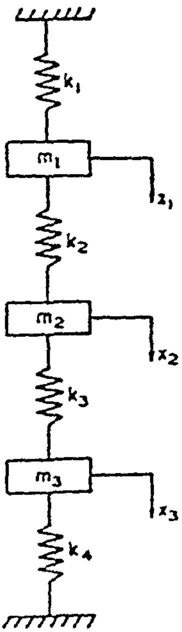


Fig. P. 6.2.

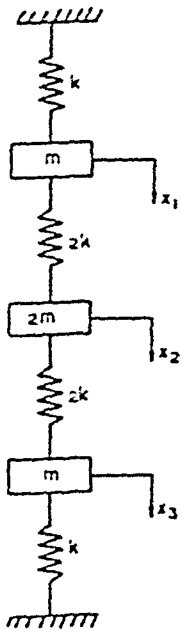


Fig. P. 6.3.

6.4 For a taut string having tension T and three concentrated masses as shown in Fig. P.6.4, use the method of influence numbers to find the three natural frequencies.

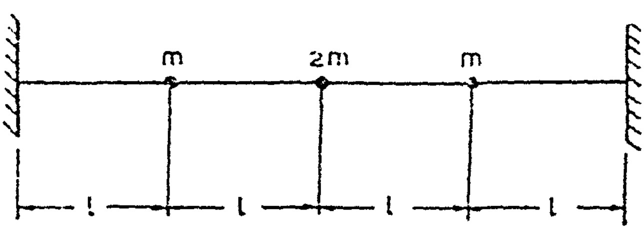


Fig. P.6.4.

6.5 For the system of Fig. 6.4.1., it has been shown by first of equations (6.4.3) that the amplitude of the mass on whicht he excitation acts becomes zero when $S=0.333$ or when $S = 1$, where $S = \dfrac{T}{ml\omega^2}$.

Show that for these two values of S the exciting frequencies become equal to the two natural frequencies of the system with the first mass considered to be stationary or fixed.

6.6 The system of Fig. 6.4.1. is excited by a sinusoidal force $F_0 \sin \omega t$ at the middle mass instead of the first mass. If T is the tension in the string find the expressions for the steady state amplitudes of each of the three masses. Discuss these expressions, explaining these from physical viewpoint.

6.7 A three rotor system of Fig. 6.5.2 (a) has the following physical constants.

$J_1 = 50$ kg-cm-sec^2
$J_2 = 100$ kg-cm-sec^2
$J_3 = 70$ kg-cm-sec^2
$k_{t1} = 2.2 \times 10^6$ kg-cm/rad
$k_{t2} = 0.8 \times 10^6$ kg-cm/rad.

Find the natural frequencies of the system and the corresponding mode shapes.

6.8 For the system shown in Fig. 6.5.3 (a)

$J_1 = 10$ kg-cm-sec^2
$J_2 = 20$ kg-cm-sec^2
$J_{g1} = 0.5$ kg-cm-sec^2
$J_{g2} = 2.0$ kg-cm-sec^2

Dia of gear 2 = Twice the dia of gear 1
$k_{t1} = 3.2 \times 10^5$ kg-cm/rad
$k_{t2} = 0.8 \times 10^5$ kg-cm/rad.

Find the natural frequency of torsional oscillations

(i) neglecting the inertias of the gears,

(i i) taking into account the inertias of the gears.

6.9 Study the type of couplings between the coordinates of the systems shown in Fig. P.6.9 (a), (b), (c) and (d). Use the differential equation method and the energy method.

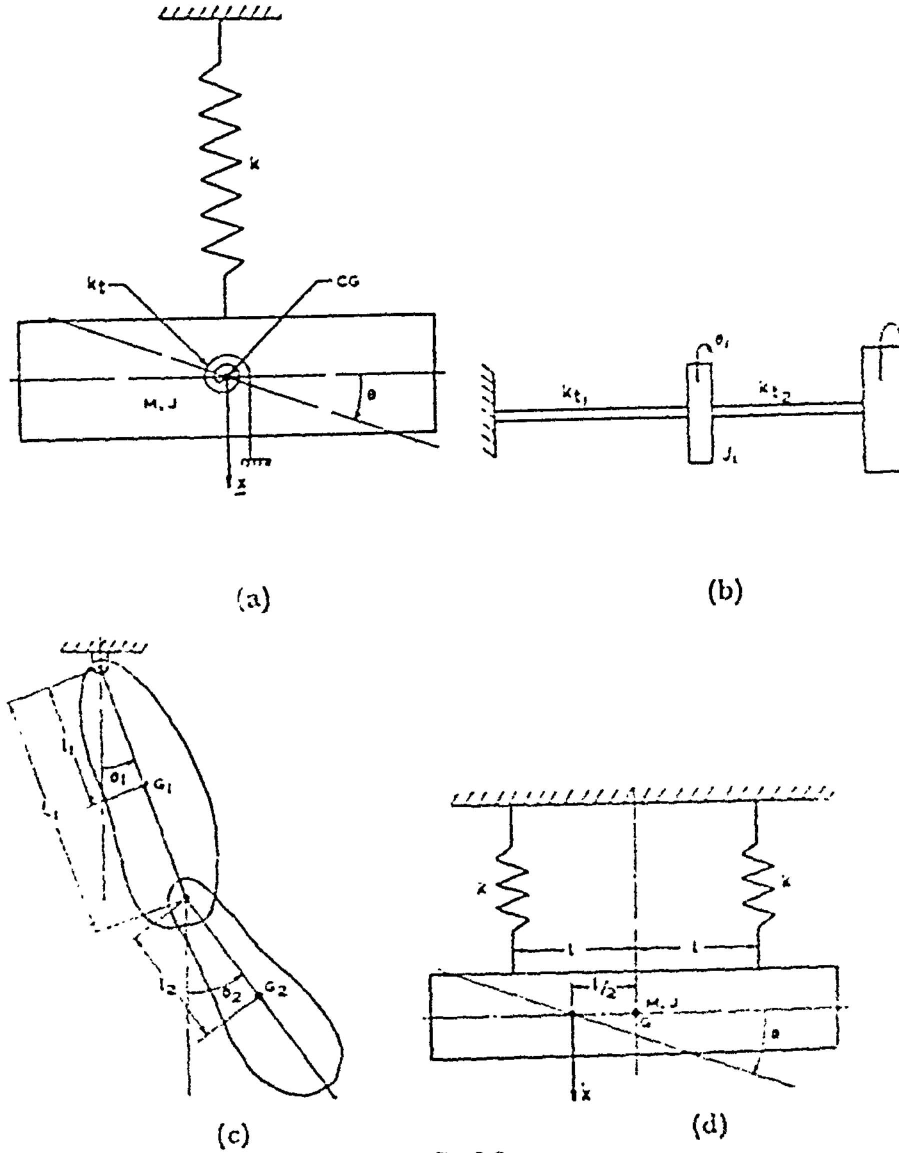


Fig. P. 6.9.

6.10 Find the principal coordinates for the system shown in Fig. P.6.10.

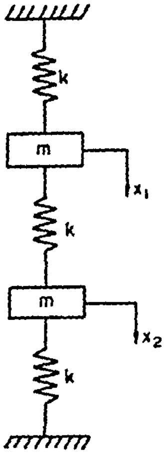


Fig. P.6.10.

6.11 Fig. P.6.11 represents a symmetrical two-degree of freedom system with tension T in the string. Determine the principal coordinates for the system.

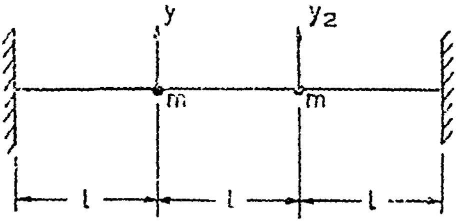


Fig. P. 6.11.

6.12 A uniform string of length l fixed at its end has a large initial tension. It is plucked at $x = l/3$ through a distance a_0 and released. Determine the subsequent motion.

6.13 A uniform taut string of length l fixed at both ends has a large initial tension. It is struck in such a manner as

to give an initial velocity to the string which varies linearly from zero at the ends to V_0 at the centre. Determine the subsequent motion.

6.14 A cantilever consists of uniform bar of length l. At mid point a force P which acts away from the fixed end is applied, and released at time $t=0$, suddenly. Find the ensuing motion.

6.15 A free-free bar of uniform section and length l is compressed on the two sides so as to give a total compression e. The compressive forces are released suddenly, simultaneously. Derive an expression for the resultant free vibrations.

6.16 A uniform circular shaft of length l is fixed at the two ends. At its middle point a torque T_0 is applied which twists it by θ_0 radians at the middle point. If the torque is released suddenly, find the subsequent motion.

6.17 A simply supported beam of length l is deflected by a force P applied at a point distant c from one end. Find the resulting transverse vibrations when the load is suddenly removed.

6.18 Determine the frequency equation in transverse vibration for a free-free beam of length l and having a uniform cross-section.

6.19 Determine the frequency equation in transverse vibration for a uniform beam of length l having one end fixed and the other simply supported.

CHAPTER 7

MANY DEGREES OF FREEDOM SYSTEMS—NUMERICAL METHODS

7.1 Introduction.

Chapter 6 gives the exact methods for finding the natural frequencies of multi-degree of freedom systems. The actual solution of the determinants of higher order becomes more and more difficult with increasing number of degrees of freedom. Numer cal methods are used to solve these problems. Discussed in this chapter are the methods based on Rayleigh, Dunkerley, Stodola, Holzer and the matrix iteration method. It may be pointed out here that these methods as well as those discussed in the previous chapter can be used for the systems which can be represented by mathematical models. In case of highly complex systems, solutions by these methods become extremely difficult if not impossible. Natural frequencies for such cases are determined experimentally by means of variable frequency exciters.

7.2 Rayleigh's method.

This method developed by Lord Rayleigh is very handy for finding the first natural frequency of a multi-degree of freedom system. This is a numerical method strictly speaking, but in the very first trial gives close enough fundamental natural frequency for all practical purposes. It is based upon equating the maximum kinetic energy of the vibrating system to the maximum potential energy as was discussed in Sec. 2.6. The only difference is that here we are dealing with a multi-degree of freedom system instead of a single degree of freedom system. And for multi-degree of freedom system there are many masses and thus many components of kinetic and potential energies, but all the masses will have simple harmonic motions passing

through their mean position at the same instant, for any principal mode of vibration.

The following procedure is adopted for finding the first natural frequency by Rayleigh's method.

(i) Assume a deflection curve of the system that is consistent with the boundary conditions.

(ii) Find the maximum kinetic energy and the maximum potential energy of the system for the configuration of (i), and equate the two to find the natural frequency.

That is, $\frac{1}{2}\sum\frac{W_i}{g}\, y_i^2\, \omega_n^2 = \frac{1}{2}\sum W_i\, y_i$

or $$\omega_n^2 = \frac{g\Sigma W_i\, y_i}{\Sigma W_i\, y_i^2} \qquad (7.2.1)$$

where W_i and y_i are the load and the deflection at point i.

In case the deflection curve assumed in (i) above is due to the loads considered as dead or static, as is generally done to start with, then the natural frequency as obtained from equation (7.2.1) will be some what higher than the actual value. The reason is that during the actual vibrations the deflection curve will be due to the inertia loads rather than static loads, and thus the assumed deflection will be different from the actual deflection ; and whenever assumed deflection is different from the actual one, a higher frequency will result. This is because any change of deflection curve from the actual is associated with the stiffening of the system which results in a higher natural frequency.

A more accurate value of the natural frequency can be obtained by considering the deflection curve due to inertia loading; this inertia loading being obtained from the frequency calculated in equation (7.2.1). The method converges very fast and a very accurate value of the natural frequency can be obtained. However, for most practical purposes the natural frequency as obtained from equation (7.2.1) serves the purpose.

Equation (7.2.1) can be further generalized to include the cases of distributed masses as in the case of beams. This, of course, will require the assumption of a reasonable deflection curve.

Consider a uniform beam of length l and of weight w per unit length. The maximum potential energy of this beam in bending is given by

$$\text{P.E.} = \frac{1}{2} \int_0^l M \, d\theta \tag{7.2.2}$$

where M is the bending moment and $d\theta$ is the change in slope over a distance dx.

From the beam theory we also know that

$$\frac{M}{EI} = \frac{1}{R} = \frac{d\theta}{dx} = \frac{d^2y}{dx^2}$$

Substituting from the above relation in equation (7.2.2), we have

$$\text{P.E.} = \frac{1}{2} \, \text{EI} \int_0^l \left(\frac{d^2y}{dx^2}\right)^2 dx \tag{7.2.3}$$

The maximum kinetic energy due to the mass of the beam is

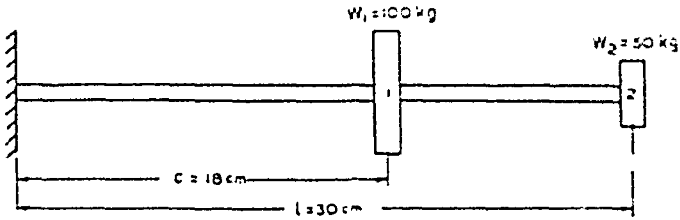
$$\text{K.E} = \frac{1}{2g} \int_0^l w \, (\omega_n \, y)^2 \, dx \tag{7.2.4}$$

where ω_n is the natural frequency of the system corresponding to the assumed deflection curve y. Equating the maximum potential energy to the maximum kinetic energy from equations (7.2.3) and (7.2.4), we have

$$\omega_n^2 = \frac{gEI}{w} \, \frac{\int_0^l \left(\frac{d^2y}{dx^2}\right)^2 dx}{\int_0^l y^2 \, dx} \tag{7.2.5}$$

Illustrative Example 7.2.1

Find the lower natural frequency of vibration for the system shown in Fig. 7.2.1 by Rayleigh's method.



$E = 2 \times 10^6$ kg/cm^2
$I = 40$ cm^4

Fig. 7.2.1. Fundamental frequency determination by Rayleigh's method.

Solution

Equation (7.2.1) gives

$$\omega_n^2 = \frac{g\Sigma W_i y_i}{\Sigma W_i y_i^2}$$

In order to find the deflections at the two points, we find the deflection at each point independently due to each load and then superimpose.

From the Strength of Materials, we know that

$$\delta_{11} = \frac{c^3}{3EI}$$

$$\delta_{21} = \delta_{12} = \frac{c^2}{6EI}(3l - c)$$

$$\delta_{22} = \frac{l^3}{3EI}$$

The total static deflections at the two points are then given by

$$y_1 = W_1\delta_{11} + W_2\delta_{12} \tag{7.2.6a}$$

$$= \frac{100 \times 18^3}{3 \times 2 \times 10^6 \times 40} + \frac{50 \times 18^2 \times 72}{6 \times 2 \times 10^6 \times 40} = 0{\cdot}00486 \text{ cm.}$$

$$y_2 = W_1\,\delta_{21} + W_2\,\delta_{22} \tag{7.2.6b}$$

$$= \frac{100 \times 18^2 \times 72}{6 \times 2 \times 10^6 \times 40} + \frac{50 \times 30^3}{3 \times 2 \times 10^6 \times 40} = 0.01048 \text{ cm.}$$

Therefore, $\omega_n^2 = \dfrac{980[100 \times .00486 + 50 \times .01048]}{[100 \times .00486^2 + 50 \times .01048^2]} = 126000$

or $\omega_n = 355$ rad/sec.

The above natural frequency has been found out by assuming the deflection curve as that due to static loading. For all practical purposes this value of the natural frequency is quite accurate. However, if still greater accuracy is desired, then the deflections at the two points should be obtained by considering inertia forces there instead of static loads.

Therefore the forces at points 1 and 2 may be considered as F_1 and F_2 instead of W_1 and W_2, where

$$F_1 = m_1 y_1 \omega_n^2 = \frac{100}{980} \times .00486 \times 126000 = 62.5 \text{ kg}$$

$$F_2 = m_2 y_2 \omega_n^2 = \frac{50}{980} \times .01048 \times 126000 = 67.4 \text{ kg}$$

The new deflections at points 1 and 2 are obtained by formulas given in equations (7.2.6a) and (7.2.6b) except that W will now be replaced by F. These values come out to be

$$y_1' = .00480 \text{ cm}$$

$$y_2' = .01058 \text{ cm}$$

and are very close to the previous values inspite of the fact that the loading has changed considerably.

For the case of inertia loading equation (7.2.1) is modified to

$$\omega_n'^2 = \frac{g \Sigma F_i x_i}{\Sigma W_i x_i^2}$$

or $$\omega'^2 = \frac{980[62.5 \times .00480 + 67.4 \times .01058]}{[100 \times .00480^2 + 50 \times .01058^2]}$$

$$= 125500$$

or $\omega_n' = 354$ rad/sec. **Ans.**

This value is lower than the previous value showing it to be more close to the actual natural frequency.

Illustrative Example 7.2.2

For a simply supported beam of length l and of uniform cross-section, find the first natural frequency of transverse vibration by Rayleigh's method.

Solution

(i) Let the deflection curve be assumed to be a sine curve.

Then $y = Y \sin \dfrac{\pi x}{l}$ (see Fig. 7.2.2)

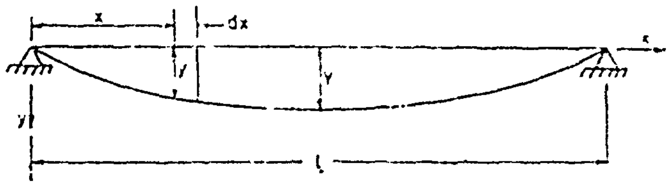


Fig. 7.2.2. Natural frequency of a simply supported beam by Rayleigh's method.

Substituting in equation (7.2.5),

$$\omega_n^2 = \frac{gEI}{w} \frac{\int_0^l \left(-\frac{\pi^2}{l^2} Y \sin \frac{\pi x}{l} \right)^2 dx}{\int_0^l \left(Y \sin \frac{\pi x}{l} \right)^2 dx}$$

$$= \frac{gEI}{w} \cdot \frac{\pi^4}{l^4}$$

or $$\omega_n = \sqrt{\frac{gEI}{w}} \cdot \frac{\pi^2}{l^2}$$

which is exactly the same as obtained in Illustratiue Example 6.8.4 for the first mode showing that the deflection curve assumed coresponds to the deflection curve for the first mode of vibration.

(ii) Let us assume a parabolic deflection curve, with coordinates at the centre of the beam,

i.e. $$y = Y\left[1 - \frac{4x^2}{l^2}\right]$$

Substituting in equation (7.2.5),

$$\omega_n^2 = \frac{gEI}{w} \frac{\int_{-l/2}^{l/2} \left(-\frac{8Y}{l^2}\right)^2 dx}{\int_{-l/2}^{l/2} \left[Y\left(1 - \frac{4x^2}{l^2}\right)\right]^2 dx}$$

$$= \frac{gEI}{w} \cdot \frac{120}{l^4}$$

or $$\omega_n = \sqrt{\frac{gEI}{w}} \cdot \frac{10.95}{l^2}$$

which is about 10% higher than the correct value. **Ans.**

Illustrative Example 7.2.3

A beam carrying uniformly distributed load can be represented by a light beam carrying equivalent concentrated load at a single point such that natural frequencies of the two are the same. Find out this equivalent concentrated load for different boundary conditions.

Solution

(a) Simply supported beam

It has been shown in Illustrative Example 7.2.2 (i) that the natural frequency of transverse vibrations of a simply supported beam is given by

$$\omega_n = \sqrt{\frac{gEI}{w}} \cdot \frac{\pi^2}{l^2}$$

$$= \sqrt{\frac{gEI\,\pi^4}{wl^4}}$$

or $$\omega_n = \sqrt{\frac{gEI\,\pi^4}{Wl^3}} \tag{7.2.7}$$

where $W = wl$ (the total beam load).

Now if W_{eq} is the equivalent load at the centre of the beam, then the natural frequency of this equivalent system is given by

$$\omega_n = \sqrt{\frac{k\,g}{W_{eq}}} = \sqrt{\frac{g.\,48\,EI}{W_{eq.}\;l^3}} \tag{7.2.8}$$

Equating the above two expressions for ω_n, we have

$$W_{eq} = 0.492\ W$$

Thus we see that the system can be replaced by 0.492 of the total beam load acting at the centre of the beam to give the same natural frequency as the original system.

Thus for simply supported beam, replace the total load of the beam by approximately 50% of its weight to act at the centre of the beam.

We could have started with a simpler deflection curve (e.g. a sine curve) without much effecting the final result. **Ans.**

(*b*) *Cantilever beam*

It can be shown in a similar fashion that the total uniformly distributed load of the cantilever can be replaced by 23.2% of its load to act at the free end. **Ans.**

(*c*) *Fixed-fixed beam*

For a fixed-fixed beam, the total uniformly distributed load can be shown to be replaced by 37.2% of its load to act at the centre. **Ans.**

7.3 Dunkerley's method.

Dunkerley's equation, which has many useful applications in case of multi-degree of freedom systems, is as follows :

$$\frac{1}{\omega_n^2} = \frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} + \frac{1}{\omega_3^2} + \ldots\ldots\ \ldots + \frac{1}{\omega_s^2} \tag{7.3.1}$$

where ω_n is the fundamental natural frequency of the system; $\omega_1, \omega_2, \omega_3 \ldots$ are the natural frequencies of the system with each mass acting separately at its point of application in the absence of other masses; and ω_s is the natural frequency of the shaft alone due to its distributed weight.

Illustrative Example 7.3.1

Use Dunkerley's method to find the fundamental natural frequency of transverse vibration for the system discussed in Illustrative Example 7 2.1 and shown in Fig. 7.2.1.

Solution

Consider only weight W_1 to be acting, then

$$y_1 = W_1.\delta_{11}$$

$$= \frac{100 \times 18^3}{3\times 2\times 10^6\times 40} \qquad \text{(from. Ill. Ex. 7.2.1)}$$

$$= .00243 \text{ cm.}$$

Similarly $y_2 = W_2\delta_{22}$

$$= \frac{50 \times 30^3}{3\times 2\times 10^6\times 40} \qquad \text{(from Ill. Ex. 7.2.1)}$$

$$= .00563 \text{ cm.}$$

If only weight W_1 is acting, then

$$\omega_1 = \sqrt{\frac{g}{y_1}} = \sqrt{\frac{980}{.00243}}$$

If only weight W_2 is acting, then

$$\omega_2 = \sqrt{\frac{g}{y^2}} = \sqrt{\frac{980}{.00563}}$$

Applying equation (7.3 1)

$$\frac{1}{\omega_n^2} = \frac{.00243}{980} + \frac{.00563}{980}$$

$$= \frac{.00806}{980}$$

or $\omega_n^2 = 122000$

which gives $\omega_n = 349$ rad/sec. **Ans.**

This frequency is slightly lower than the correct value.

In case the weight of the shaft has also to be considered then the natural frequency due to the weight of the shaft alone can be obtained by considering the shaft to be light and puutting a concentrated load at the end equal to 0.232 of the shaft load. This natural frequency ω_s has also to be included in equation (7.3.1) to find the first natural frequency of the complete system. If the shaft weight is negligible, ω_s need not be considered.

7.4 Stodola's method.

This method is a quickly converging iterative process used for the calculation of the fundamental natural frequency of undamped free vibrations for multi-degree of freedom systems. The procedure is as follows.

(i) Assume a reasonable deflection curve of the system. This may be taken as the static deflection curve as in Rayleigh's method.

(ii) Find out the inertia loading of the system for the deflection assumed in (i) above. This will be in terms of ω^2, where ω is the natural frequency of the fundamental mode.

(iii) Consider that the system is loaded with the inertia loads as found in (ii) above and find the corresponding new deflection curve. This will also be in terms of ω^2.

(iv) If the assumed deflection curve of (i) is similar to the derived deflection curve of (iii) along the system, then the shape of the assumed curve of (i) is correct. Simply equate the two expressions of (i) and (iii) and that gives the value of ω^2.

(v) If the deflection curve of (i) and (iii) are not similar, then the derived deflection curve of (iii) may be used as the next starting point and the process repeated till the assumed and the derived deflection curves are similar.

It can be proved that whatever deflection curve we originally start with, we will finally end up with the deflection curve corresponding to the fundamental mode. The frequency finally

d will be that of fundamental mode of vibration. It t mean that Stodola's method cannot be employed for g the natural frequencies of higher modes but the method within the scope of this text.

trative Example 7.4.1

lve the problem given in Illustrative Example 7.2.1 by dola's method.

ution

It is not necessary to start with the static deflection curve. ne method converges very fast and so let us start with mple values of x_1 and x_2.

st Trial

Let $x_1 = 1, x_2 = 1$

Therefore $\dfrac{x_2}{x_1} = 1$

$$F_1 = \frac{100}{980}\ \omega^2 = \cdot 102\omega^2$$

$$F_2 = \frac{50}{980}\ \omega^2 = \cdot 051\omega^2$$

$$x_1' = F_1\delta_{11} + F_2\delta_{12}$$

$$= \frac{.102\omega^2 \times 18^3}{3\times 2\times 10^6 \times 40} + \frac{.051\omega^2\times 18^2\times 72}{6\times 2\times 10^6\times 40} = 4.96\omega^2\times 10^{-6}$$

$$x_2' = F_1\delta_{21} + F_2\delta_{22}$$

$$= \frac{.102\omega^2\times 18^2\times 72}{6\times 2\times 10^6\times 40} + \frac{.051\omega^2\times 30^3}{3\times 2\times 10^6\times 40} = 1.071\omega^2\times 10^{-6}$$

$$\frac{x_2'}{x_1'} = \frac{1.071\omega^2\times 10^{-6}}{4.96\omega^2\times 10^{-6}} = 2.16$$

This ratio is much different from the staring value of t ratio.

2nd Trial

Let us start now with the ratio

$$\frac{x_2'}{x_1'} = 2.16$$

Let $x_1' = 1, x_2' = 2.16$

$$F_1' = \cdot 102\omega^2$$

$$F_2' = \frac{50}{980} \times 2.16 \times \omega^2 = \cdot 110\omega^2$$

The deflections for these inertia forces are obtained as in first trial and are as

$$x_1'' = 7.83\omega^2 \times 10^{-6}$$

$$x_2'' = 17.36\omega^2 \times 10^{-6}$$

giving $\dfrac{x_2''}{x_1''} = 2.21$

This ratio is still some what different from the starting ratio.

3rd Trial

Let now $\dfrac{x_2''}{x_1''} = 2.21$ be taken as the starting value.

Let $x_1'' = 1$, $x_2'' = 2.21$

The corresponding inertia forces are obtained as

$$F_1'' = .102\omega^2$$

$$F_2'' = .1125\omega^2$$

and the deflections as

$$x_1''' = 7.95\omega^2 \times 10^{-6}$$

$$x_2''' = 17.66\omega^2 \times 10^{-6}$$

giving $\dfrac{x_2'''}{x_1'''} = 2.22$

This value is quite close to the starting value for this trial and so we finish our trials here. The assumed and the derived values of the deflections are equated for the third trial and we have

$x_1'' = x_1'''$, or $1 = 7.95\omega^2 \times 10^{-6}$
giving $\omega^2 = 125500$,

and $x_2'' = x_2'''$, or $2.21 = 17.66\omega^2 \times 10^{-6}$
giving $\omega^2 = 125200$,

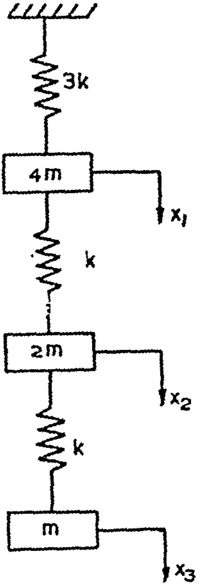
The mean value of $\omega^2 = 125350$ may be chosen, which gives

$$\omega = 353.6 \text{ rad/sec.}$$

This value is about the same as obtained in Ill. Ex. 7.2.1 **Ans.**

Illustrative Example 7.4.2

For the three degree of freedom system shown in Fig. 7.4.1, find the lowest natural frequency by Stodola's method.



$$\delta_{11} = \delta_{12} = \delta_{21} = \delta_{13} = \delta_{31} = \frac{1}{3k}$$

$$\delta_{22} = \delta_{32} = \delta_{23} = \frac{4}{3k}$$

$$\delta_{33} = \frac{7}{3k}$$

Fig. 7.4.1. Natural frequency determination by Stodola's method.

Solution

The values of the influence coefficients for this system are the same as those for the system of Fig. 6.2.2 derived in Illustrative Example 6.3.1. These are shown again by the side of Fig. 7.4.1.

We can start the trials by choosing any reasonable values of deflections of various masses or even start with simple values of unity each.

Ist Trial

Let $x_1 = 1$, $x_2 = 1$, $x_3 = 1$;
therefore $x_1 : x_2 : x_3 = 1 : 1 : 1$
$F_1 = 4m\omega^2$, $F^2 = 2m\omega^2$, $F_3 = m\omega^2$
$x'_1 = F_1\delta_{11} + F_2\delta_{12} + F_3\delta_{13}$

$$= 4m\omega^2.\frac{1}{3k} + 2m\omega^2.\ \frac{1}{3k} + m\omega^2\ \frac{1}{3k} = \frac{7}{3}\ \frac{m\omega^2}{k}$$

$$x_2' = F_1\delta_{21} + F_2\delta_{22} + F_3\delta_{23}$$

$$= 4m\omega^2.\frac{1}{3k} + 2m\omega^2.\ \frac{4}{3k} + m\omega^2.\ \frac{4}{3k} = \frac{16}{3}\ \frac{m\omega^2}{k}$$

$$x_3' = F_1\delta_{31} + F_2\delta_{32} + F_3\delta_{33}$$

$$= 4m\omega^2.\frac{1}{3k} + 2m\omega^2\ \frac{4}{3k} + m\omega^2\ \frac{7}{3k} = \frac{19}{3}\ \frac{m\omega^2}{k}$$

Therefore $x_1' : x_2' : x_3' = 1 : 2.3 : 2.7$, which doesnot agree with the assumed deflection shape.

2nd Trial

Let $x_1' = 1,\ x_2' = 2.3,\ x_3' = 2.7$
with $x_1' : x_2' : x_3' = 1 : 2.3 : 2.7$
$F_1' = 4m\omega^2,\ F_2' = 4.6\ m\omega^2,\ F_3' = 2.7\ m\omega^2$

Proceeding in the same manner as in Ist trial, we get

$$x_1'' = \frac{11.3}{3}\ \frac{m\omega^2}{k}$$

$$x_2'' = \frac{33.2}{3}\ \frac{m\omega^2}{k}$$

$$x_3'' = \frac{41.3}{3}\ \frac{m\omega^2}{k}$$

giving $x_1'' : x_2'' : x_3'' = 1 : 2.94 : 3.76$

This mode is also not similar to the starting mode in this trial.

3rd Trial

Let $x_1'' = 1,\ x_2'' = 2.94,\ x_3'' = 3.76$
with $x_1'' : x_2'' : x_3'' = 1 : 2.94 : 3.76$
$F_1'' = 4m\omega^2,\ F_2'' = 5.88\ m\omega^2,\ F_3'' = 3.76\ m\omega^2$

The values for deflections obtained are

$$x_1''' = \frac{13.64}{3}\ \frac{m\omega^2}{k}$$

$$x_2''' = \frac{42.56}{3}\ \frac{m\omega^2}{k}$$

$x_3'' = 53.84 \dfrac{m\omega^2}{k}$

$x_1'' : x_2'' : x_3'' = 1 : 3.12 \quad 3.95$

e ratios are still somewhat different from the starting
s in this trial.

Trial

Let $x_1''' = 1, x_2''' = 3.12, x_3''' = 3.95$

with $x''' : x_2''' : x_3''' = 1 : 3.12 : 3.95$

$F_1''' = 4m\omega^2, F_2''' = 6.24\, m\omega^2, F_3'''=3.95m\omega^2$

The deflections obtained are

$$x_1'''' = \frac{14\ 19}{3} \frac{m\omega^2}{k}$$

$$x_2'''' = \frac{44.76}{3} \frac{m\omega^2}{k}$$

$$x_3'''' = \frac{56.61}{3} \frac{m\omega^2}{k}$$

giving $x_1'''' : x_2'''' : x_3'''' = 1 : 3.16 : 3.99$

These ratios are approximately the same as the starting values in this trial. Therefore, equating the corresponding deflection, we have

$$x_1'''' = x_1'''$$

or $\quad \dfrac{14.19}{3} \dfrac{m\omega^2}{k} = 1$, which gives

$$\omega = 0.46 \sqrt{\frac{k}{m}} \text{ rad/sec.}$$

7.5 Rayleigh-Ritz method.

This is an extension of Rayleigh's method for findin natural frequencies of a vibrating system. For the l vibrations of a beam, equation (7.2.5) has been derive this equation, EI and w (the weight per unit length) wer as constants and thus were taken out of the integ [equation (7.2.3)]. If however, EI and w are not c then equation (7.2.5) becomes

$$\omega_n^2 = g\,\frac{\displaystyle\int_0^l EI\left(\frac{d^2y}{dx^2}\right)^2 dx}{\displaystyle\int_0^l wy^2 dx} \tag{7.5.1}$$

If the fundamental deflection curve y_1 is known, substituting it in equation (7.5.1), we will get the value of ω_{n1}^2, where ω_{n1} is the fundamental natural frequency of the system. If a deflection curve y_i is known and substituted in the above equation, the resulting value will be that of ω_{ni}^2 corresponding to i^{th} mode of vibration.

Now if y_1 is not known exactly and an approximate value that satisfies the boundary conditions is taken and substituted in equation (7.5.1), the resulting value of ω_n will always be higher than ω_{n1}, the true fundamental natural frequency. Equation (7.5.1) can therefore be considered to define ω_n (dependent variable) as a function of y (independent variable). Thus $\omega_n = \omega_n(y)$ is the relation which satisfies the required boundary conditions. The minimum of this function is equal to the fundamental natural frequency ω_{n1} and the corresponding value at which this function is attained is the deflection curve corresponding to the fundamental mode of vibration.

Coming to the procedure, it is as follows. Let $f_1(x)$, $f_2(x)$ ......$f_i(x)$ be a series of given functions satisfying the boundary conditions. The values $\omega_n(f_1)$, $\omega_n(f_2)$ ... $\omega_n(f_i)$ obtained from equation (7.5.1) are all greater than ω_{n1}. A linear combination of the functions f, that is,

$$C_1 f_1(x) + C_2 f_2(x) + \dots \dots\dots + C_i f(x) \tag{7.5.2}$$

also satisfies the given boundary conditions and the corresponding value of ω_n as obtained from equation (7.5.1), that is

$$\omega_n\,(C_1 f_1 + C_2 f_2 + \dots \dots\dots + C_i f_i) \tag{7.5.3}$$

will also be greater than ω_{n1} for all possible combinations of the coefficients C's. A proper combination of C's gives the lowest value of the expression (7.5.3) which will still be greater than ω_{n1} but will be the best possible approximation to ω_{n1}.

d the proper combination of the coefficients $C_1, C_2 ..$
substitute the expresion (7.5.2) in terms of C's for y,
e equation (7.5.1) and after performing the integration,
ve

$$\omega_n^2 = F(C_1, C_2 \ldots C_i) \tag{7.5.4}$$

e the right hand side is a function of $C_1, C_2 \ldots C_i$. For
imum value of ω_n^2, we must have

$$\left.\begin{aligned} \frac{\partial \omega_n^2}{\partial C_1} &= 0 \\ \frac{\partial \omega_n^2}{\partial C_2} &= 0 \\ - - &- - \\ \frac{\partial \omega_n^2}{\partial C_i} &= 0 \end{aligned}\right\} \tag{7.5.5}$$

Since the right hand side of equation (7.5.1) is a function of $C_1, C_2 \ldots C_i$, its partial derivatives with respect to $C_1, C_2, \ldots C_i$ will all be zero for minimum value of ω_n^2. Therefore, setting its derivative to zero, we have

$$\left[\int_0^l wy^2\, dx\right] \cdot \frac{\partial}{\partial C_i}\left[\int_0^l EI\left(\frac{d^2y}{dx^2}\right)^2 dx\right] - \left[\int_0^l EI\left(\frac{d^2y}{dx^2}\right)^2 dx\right] \cdot \frac{\partial}{\partial C_i}\left[\int_0^l wy^2 dx\right] = 0, \quad (i=1,2,\ldots i) \tag{7.5.6}$$

But from equation (7.5.1)

$$\left[\int_0^l EI\left(\frac{d^2y}{dx^2}\right)^2 dx\right] = \frac{\omega_n^2}{g}\left[\int_0^l wy^2\, dx\right]$$

Therefore, equation (7.5.6) simplifies to

$$\frac{\partial}{\partial C_i}\left[\int_0^l EI\left(\frac{d^2y}{dx^2}\right)^2 dx\right] - \frac{\omega_n^2}{g} \cdot \frac{\partial}{\partial C_i}\left[\int_0^l wy^2 dx\right] = 0, \quad (i=1,2,\ldots\ldots i)$$

Equations (7.5.7) are a set of i homogenous linear equations in i unknowns $C_1, C_2 \ldots C_i$. It will have a non-trivial solution if the determinant of its coefficients is equal to zero. This leads to an i^{th} degree equation in ω_n^2, the lowest root of which is a good approximation to ω_{n1}^2. The next higher root is somewhat poor approximation to ω_{n2}^2. For most practical purposes, only two terms in expression (7.5.2) give a very good approximation for the fundamental natural frequency. With increasing number of terms, the lowest root converges to ω_{n1}^2, and the next higher root converges to ω_{n2}^2 although not that fast.

The functions chosen in expression (7.5.2) should be simple enough to facilitate the integration process, and they must also satisfy the boundary conditions.

Illustrative Example 7.5.1

Find the lowest natural frequency of lateral vibration of a tapered bar fixed at its base as shown in Fig. 7.5.1. Take the width of this bar as unity.

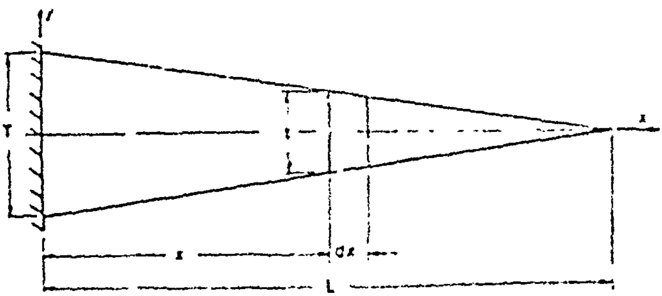


Fig. 7.5.1. Natural frequency of lateral vibration of a tapered bar.

Solution

Thickness of the bar at distance x from the base is given by

$$t = T\left(1 - \frac{x}{L}\right).$$

Moment of inertia of the section of the bar at distance x from the base is

$$I(x) = \frac{1}{12} bt^3 = \frac{1}{12} T^3\left(1 - \frac{x}{L}\right)^3$$

as b is given to be unity.

The weight per unit length $w(x)$ at distance x from the base is given by

$$w(x) = \frac{t.dx.\gamma}{dx} = \gamma t = \gamma T\left(1 - \frac{x}{L}\right)$$

where γ = weight per unit volume.

Let us assume for y the expression

$$y = C_1\frac{x^2}{L^2} + C_2\frac{x^3}{L^3} + C_3\frac{x^4}{L^4} + \ldots\ldots \tag{7.5.8}$$

in order that the following boundary conditions are satisfied.

$$y(0) = 0,\ \frac{dy}{dx}(0) = 0,\ EI(L).\frac{d^2y}{dx^2}(L) = 0,\ EI(L).\frac{d^3y}{dx^3}(L) = 0$$

Ist Trial. As a first approximation let us consider only the first term of equation (7.5.8), i.e.,

$$y = C_1\frac{x^2}{L^2}$$

Substituting the above quantities in equation (7.5.1), we have,

$$\omega_n^2 = g\,\frac{\displaystyle\int_0^L E\left[\frac{1}{12} T^3\left(1 - \frac{x}{L}\right)^3\right]\left[\frac{2C_1}{L^2}\right]^2 dx}{\displaystyle\int_0^L \left[\gamma\, T\left(1 - \frac{x}{L}\right)\right]\left(C_1\frac{x^2}{L^2}\right)^3 dx}$$

Solving and simplifying, gives

$$\omega_n = 1.58\,\frac{T}{L^2}\sqrt{\frac{Eg}{\gamma}}$$

The result is about 3.3% higher than the actual value.

2nd. Trial Let us now consider the deflection curve as that corresponding to the first two terms of equation (7.5.8), i.e.

$$y = C_1\frac{x^2}{L^2} + C_2\frac{x^3}{L^3}$$

Therefore, coming to equation (7.5.7), i.e.,

$$\frac{\partial}{\partial C_1}\left[\int_0^L EI(x)\left(\frac{d^2y}{dx^2}\right)^2 dx\right]-\frac{\omega_n^2}{g}\frac{\partial}{\partial C_1}\left[\int_0^L w(x)\,y^2 dx\right]=0$$

we have $\frac{\partial}{\partial C_1}\left[\int_0^L E\left[\frac{1}{12}T^3\left(1-\frac{x}{L}\right)^3\right]\left[\frac{2C_1}{L^2}+\frac{6C_2x}{L^3}\right]^2 dx\right]$

$$-\frac{\omega_n^2}{g}\frac{\partial}{\partial C_1}\left[\int_0^L \gamma T\left(1-\frac{x}{L}\right)\left(C_1\frac{x^2}{L^2}+C_2\frac{x^3}{L^3}\right)^2 dx\right]=0$$

Performing the integrations and then differentiating first with respect to C_1 and then with respect to C_2, we have the following two simulataneous equations after simplification.

$$(2-0.80s)C_1+(1.2-0.57s)C_2=0$$
$$(1.2-0.57s)C_1+(1.2-0.43s)C_2=0$$

where $s=\dfrac{\gamma L^4\omega_n^2}{T^2Eg}$

The above two simultaneous equations give a non-trivial solution if the determinant formed from the coefficients of C_1 and C_2 is zero.

i.e. $$\begin{vmatrix}(2-0.80s) & (1.2-0.57s)\\(1.2-0.57s) & (1.2-0.43s)\end{vmatrix}=0$$

This gives the lower root of s corresponding to the lowest natural frequency as

$$s=2.34$$

giving

$$\omega_n^2=\frac{2.34T^2Eg}{\gamma L^4}$$

or $$\omega_n=1.53\frac{T}{L^2}\sqrt{\frac{Eg}{\gamma}}$$

which value is less than 0.15% higher than the exact value of the lowest natural frequency. Ans.

$$I(x) = \frac{1}{12}\,bt^3 = \frac{1}{12}\,T^3\left(1-\frac{x}{L}\right)^3$$

as b is given to be unity.

The weight per unit length $w(x)$ at distance x from the base is given by

$$w(x) = \frac{t.dx.\gamma}{dx} = \gamma t = \gamma T\left(1-\frac{x}{L}\right)$$

where γ=weight per unit volume.

Let us assume for y the expression

$$y = C_1\frac{x^2}{L^2} + C_2\frac{x^3}{L^3} + C_3\frac{x^4}{L^4} + \ldots\ldots \tag{7.5.8}$$

in order that the following boundary conditions are satisfied.

$$y(0) = 0,\ \frac{dy}{dx}(0)=0,\ EI(L).\frac{d^2y}{dx^2}(L) = 0,\ EI(L).\frac{d^3y}{dx^3}(L)=0$$

Ist Trial. As a first approximation let us consider only the first term of equation (7.5.8), i.e.,

$$y = C_1\frac{x^2}{L^2}$$

Substituting the above quantities in equation (7.5.1), we have,

$$\omega_n^2 = g\,\frac{\int_0^L E\left[\frac{1}{12}\,T^3\left(1-\frac{x}{L}\right)^3\right]\left[\frac{2C_1}{L^2}\right]^2 dx}{\int_0^L\left[\gamma\,T\left(1-\frac{x}{L}\right)\right]\left(C_1\frac{x^2}{L^2}\right)^3 dx}$$

Solving and simplifying, gives

$$\omega_n = 1.58\,\frac{T}{L^2}\sqrt{\frac{Eg}{\gamma}}$$

The result is about 3.3% higher than the actual value.

2nd. Trial Let us now consider the deflection curve as that corresponding to the first two terms of equation (7.5.8), i.e.

$$y = C_1\frac{x^2}{L^2} + C_2\frac{x^3}{L^3}$$

Therefore, coming to equation (7.5.7), i.e.,

$$\frac{\partial}{\partial C_i}\left[\int_0^L EI(x)\left(\frac{d^2y}{dx^2}\right)^2 dx\right]-\frac{\omega_n^2}{g}\frac{\partial}{\partial C_i}\left[\int_0^L w(x)\,y^2 dx\right]=0$$

we have $\frac{\partial}{\partial C_i}\left[\int_0^L E\left[\frac{1}{12}T^3\left(1-\frac{x}{L}\right)^3\right]\left[\frac{2C_1}{L^2}+\frac{6C_2x}{L^3}\right]^2 dx\right]$

$$-\frac{\omega_n^2}{g}\frac{\partial}{\partial C_i}\left[\int_0^L \gamma T\left(1-\frac{x}{L}\right)\left(C_1\frac{x^2}{L^2}+C_2\frac{x^3}{L^3}\right)^2 dx\right]=0$$

Performing the integrations and then differentiating first with respect to C_1 and then with respect to C_2, we have the following two simulataneous equations after simplification.

$$(2-0.80s)C_1+(1.2-0.57s)C_2=0$$

$$(1.2-0.57s)C_1+(1.2-0.43s)C_2=0$$

where $s=\dfrac{\gamma L^4\omega_n^2}{T^2Eg}$

The above two simultaneous equations give a non-trivial solution if the determinant formed from the coefficients of C_1 and C_2 is zero.

i.e. $$\begin{vmatrix}(2-0.80s) & (1.2-0.57s)\\(1.2-0.57s) & (1.2-0.43s)\end{vmatrix}=0$$

This gives the lower root of s corresponding to the lowest natural frequency as

$$s=2.34$$

giving

$$\omega_n^2=\frac{2.34T^2Eg}{\gamma L^4}$$

or $$\omega_n=1.53\frac{T}{L^2}\sqrt{\frac{Eg}{\gamma}}$$

which value is less than 0.15% higher than the exact value of the lowest natural frequency. Ans.

6 Method of matrix iteration.

This is an iterative process which results in the principal nodes of vibration of a system and the corresponding natural frequencies. The equations of motion obtained by the method of influence coefficients, put in the matrix form, appear as

$$\begin{bmatrix} x_1 \\ x_2 \\ - \\ - \\ x_i \end{bmatrix} = \omega^2 \begin{bmatrix} a_{11} & a_{12} & --- & a_{1i} \\ a_{21} & a_{22} & --- & a_{2i} \\ - & - & --- & - \\ - & - & --- & - \\ a_{i1} & a_{i2} & --- & a_{ii} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ - \\ - \\ x_i \end{bmatrix} \tag{7.6.1}$$

The iterative process is started by estimating a set of deflections for the right column of equation (7.6.1) and then expanding the right hand side which results in a column o numbers. This is then normalized and the procedure repeate with the new estimate as the normalized column itself Th process is continued until the first mode repeats itself.

The iteration process as described above converges to lowest value of ω^2 so that the fundamental mode of vibra is obtained. For the next higher modes and the nat frequencies, the orthogonality principle is applied to obt modified matrix equation that does not contain the l modes. The iterative process is repeated as before.

Illustrative Example 7.6.1

Find the first natural frequency of the system shown 6.2.2 by the method of matrix iteration.

Solution

The first step is to write down the differential eq motion for the three masses in terms of the influence co This has been done in Ill. Ex. 6.3.1 and the eq rewritten below.

$$-x_1 = \delta_{11} m_1 \ddot{x}_1 + \delta_{12} m_2 \ddot{x}_2 + \delta_{13} m_3 \ddot{x}_3$$

$$-x_2 = \delta_{21} m_1 \ddot{x}_1 + \delta_{22} m_2 \ddot{x}_2 + \delta_{23} m_3 \ddot{x}_3$$

$$-x_3 = \delta_{31}\, m_1\ddot{x}_1 + \delta_{32}\, m_2\ddot{x}_2 + \delta_{33}\, m_3\ddot{x}_3$$

Replacing $\ddot{x}$ by $\omega^2 x$ we have

$$x_1 = \delta_{11}\, m_1\omega^2 x_1 + \delta_{12}\, m_2\omega^2 x_2 + \delta_{13}\, m_3\omega^2 x_3$$

$$x_2 = \delta_{21}\, m_1\omega^2 x_1 + \delta_{22}\, m_2\omega^2 x_2 + \delta_{23}\, m_3\omega^2 x_3$$

$$x_3 = \delta_{31}\, m_1\omega^2 x_1 + \delta_{32}\, m_2\omega^2 x_2 + \delta_{33}\, m_3\omega^2 x_3$$

The above equations can be written in matrix notations as follows.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \delta_{11}m_1\omega^2 & \delta_{12}m_2\omega^2 & \delta_{13}m_3\omega^2 \\ \delta_{21}m_1\omega^2 & \delta_{22}m_2\omega^2 & \delta_{23}m_3\omega^2 \\ \delta_{31}m_1\omega^2 & \delta_{32}m_2\omega^2 & \delta_{33}m_3\omega^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \\ x_3 \end{bmatrix}$$

Substituting the values of m and those of δ obtained in Ill Ex. 6.3.1, the above equation becomes

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{\omega^2 m}{3k} \begin{bmatrix} 4 & 2 & 1 \\ 4 & 2 & 4 \\ 4 & 8 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \\ x_3 \end{bmatrix}$$

The iterative process may be started by assuming any simple deflection shape.

First Iteration. Let $x_1 = 1, x_2 = 2, x_3 = 3$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{\omega^2 m}{3k} \begin{bmatrix} 4 & 2 & 1 \\ 4 & 8 & 4 \\ 4 & 8 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{\omega^2 m}{3k} \begin{bmatrix} 11 \\ 32 \\ 41 \end{bmatrix}$$

$$= \frac{w^2 m}{3k} \times 11 \begin{bmatrix} 1 \\ 2.9 \\ 3.7 \end{bmatrix}$$

Second Iteration. Let $x_1 = 1, x_2 = 2.9, x_3 = 3.7$

$$\begin{bmatrix} 1 \\ 2.9 \\ 3.7 \end{bmatrix} = \frac{\omega^2 m}{3k} \begin{bmatrix} 4 & 2 & 1 \\ 4 & 8 & 4 \\ 4 & 8 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 2.9 \\ 3.7 \end{bmatrix} = \frac{\omega^2 m}{3k} \begin{bmatrix} 13.5 \\ 42.0 \\ 53.1 \end{bmatrix}$$

$$= \frac{\omega^2 m}{3k} \times 13.5 \begin{bmatrix} 1 \\ 3.1 \\ 3.9 \end{bmatrix}$$

Third Iteration. Let $x_1 = 1$, $x_2 = 3.1$, $x_3 = 3.9$

$$\begin{bmatrix} 1 \\ 3.1 \\ 3.9 \end{bmatrix} = \frac{\omega^2 m}{3k} \begin{bmatrix} 4 & 2 & 1 \\ 4 & 8 & 4 \\ 4 & 8 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 3.1 \\ 3.9 \end{bmatrix} = \frac{\omega^2 m}{3k} \begin{bmatrix} 14.4 \\ 44.4 \\ 56.1 \end{bmatrix}$$

$$= \frac{\omega^2 m}{3k} \times 14.1 \begin{bmatrix} 1 \\ 3.15 \\ 3.98 \end{bmatrix}$$

Fourth Iteration. Let $x_1 = 1$, $x_2 = 3.15$, $x_3 = 3.98$

$$\begin{bmatrix} 1 \\ 3.15 \\ 3.98 \end{bmatrix} = \frac{\omega^2 m}{3k} \begin{bmatrix} 4 & 2 & 1 \\ 4 & 8 & 4 \\ 4 & 8 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 3.15 \\ 3.98 \end{bmatrix}$$

$$= \frac{\omega^2 m}{3k} \begin{bmatrix} 14.28 \\ 45.12 \\ 57.06 \end{bmatrix} = \frac{\omega^2 m}{3k} \times 14.28 \begin{bmatrix} 1 \\ 3.16 \\ 4.00 \end{bmatrix}$$

The modes have been repeated with sufficient accuracy in the fourth iteration.

Therefore $\quad \dfrac{\omega^2 m}{3k} \times 14.28 = 1$

which gives $\omega = 0.458 \sqrt{\dfrac{k}{m}}$

This value is about the same as obtained in Ill. Ex. 6.3.1

Although we can start the iterative process by any arbitrary deflection shape, but the number of iteretions can be reduced if the starting deflection shape is approximately close to the first mode shape.

7.7 Graphical method.

For the shafts of non uniform sections, or those containing a number of steps, the graphical method for finding the natural frequency is very well suited. The method consists in first dividing the whole length of the shaft into a number of elements, calculating the weight of each and assuming that these loads act at the mid-point of each element. The external weights, if any, are added at the corresponding points to the shaft weight. The direction of these loads should be taken as that of the centrifugal forces which are the actual loads causing the whirling of the shaft.

By the method of graphic statics the bending moment diagram is drawn over the entire beam length. After this, the values of the flexural rigidity, that is EI, are calculated for the various section lengths of the shaft and a M/EI diagram is drawn. A conjugate beam is assumed to be loaded with the area under M/EI diagram. These areas are calculated for each element of the shaft. The imaginary loads thus calculated will act at the centres of gravity of the areas which may be taken, with sufficient approximation, as the mid points of the elements. The B.M. diagram drawn with this imaginary loading gives actually the deflection curve of the beam. The deflection of each element is known now. The weight of each element was already found out in the begining. Equation (7.2.1) gives the fundamental natural frequency based on static deflection.

If further accuracy is required, the value of the natural frequency determined above can be used to find the inertia loads on the shaft, and the process repeated to obtain the modified natural frequency of the system.

Illustrative Example 7.7.1

A shaft of non uniform sections has two simple supports and

an overhang on one side. Two rotors are mounted on this shaft; one weighing 100 kg between the bearings and the other weighing 50 kg on the overhang. The schematic of this system is shown in Fig. 7.7.1 (a). Determine by graphical method, the critical speed of the shaft-rotor system.

Soultion

The scales mentioned in the following paragraphs refer to the original scales when plotting was done. This scale was further reduced for accomodating the complete diagram on one page of this text.

The shaft in Fig. 7.7.1 (a) has been drawn to a scale of 1 cm = 5 cm. There are two rotor weights which may be taken to be concentrated loads. The shaft weight is not negligible and the weights of different section length have been calculated by assuming the density of the material of the shaft to be 7.6 gm/cm^3.

Fig. 7.7.1 (b) shows a horizontal line corresponding to the shaft length. Each part of the shaft of uniform diameter has been divided into convenient number and the weight of these sub-sections including the rotor weights are shown by arrows above the horizontal line. The bearing reactions are shown by another two arrows below this line. The weights on the overhang side are shown directed upwards. This is so because the centrifugal forces will act in the upward direction for the overhang side and therefore the deflection curve, although considered for static loading, will conform to approximate dynamic shape with these directions of loading.

The forces on the shaft now are named by means of *Bow's notation* as ab, bc, cd . ... hi. These forces have been represented along a vertical line to a scale of 1 cm = 20 kg as shown in Fig. 7.7.1 (f). All the points are obtained on this line except i. Pole p is taken at any convenient distance from the vertical force lines (in this case 3 cm). Its location along the vertical is immaterial. Lines pa, pb ... ph are drawn and lines parallel to these are laid in spaces a, b ... h as shown in Fig. 7.7.1 (c). This polygon has been closed by a dotted line to give

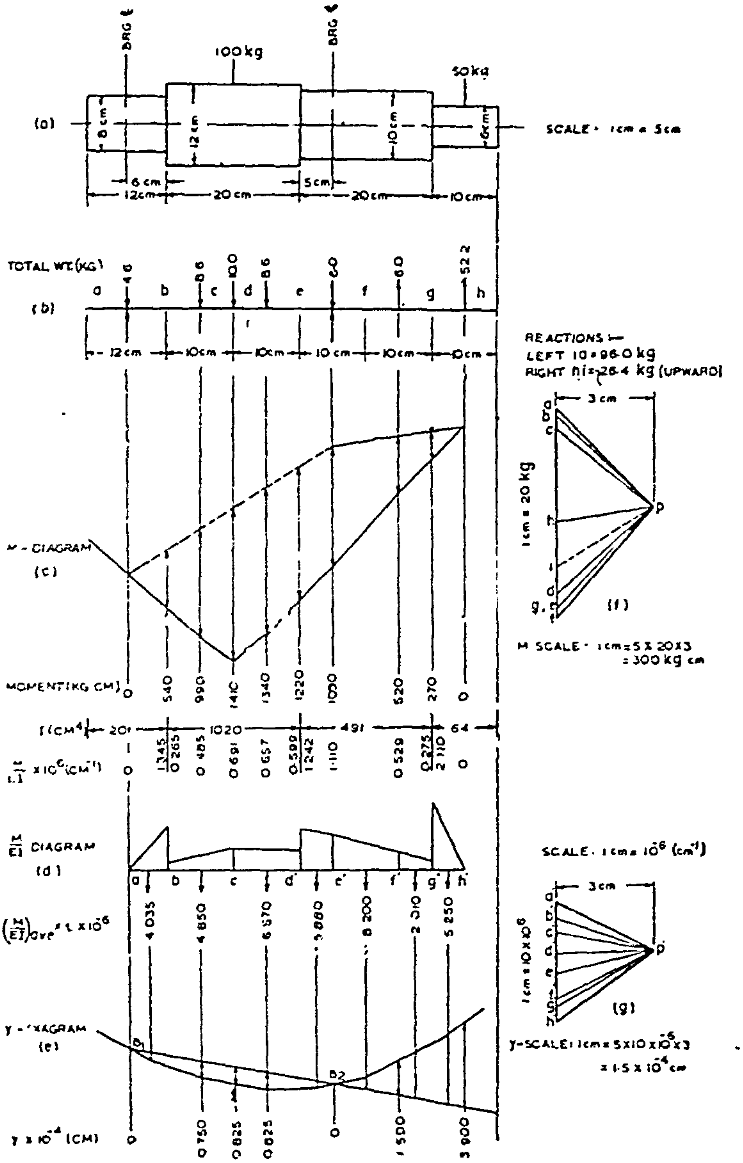


Fig. 7.7.1. Determination of the fundamental natural frequency of a non-uniform shaft with rotors fixed on it, by graphical method.

the complete moment diagram. The moments at various sections are measured and given below the moment diagram after multiplying with the moment scale which is (space scale × load scale × polar distance from the force lines)

$$1 \text{ cm} = 5 \times 20 \times 3 = 300 \text{ kg-cm.}$$

The value of I has been calculated for various sections and these are given below the moment values. Still below these are given the values of M/EI after taking $E = 2 \times 10^6$ kg/cm². It may be noted that at certain sections two values of M/EI are listed. These are the values immediately to the left and to the right of the section respectively and are due to the discontinuity in the shaft size or the sudden change in section. The final M/EI diagram is drawn with these values and is shown in Fig. 7.7.1 (d).

If a conjugate beam is considered with the area of the M/EI diagram as the loading on it, then the moment diagram with respect to this loading is nothing but the final deflection curve. First the conjugate beam load is divided into suitable sub-sections and the load equal to the area of each sub-section may be, with sufficient accuracy made to act at the middle of such sections. These area loads are shown below the M/EI diagram.

These imaginary loads are laid off along a vertical to a suitable scale (1 cm $= 10^{-6}$ in this case) as shown in Fig. 7.7.1 (g). A pole p′ is chosen at a convenient distance (3 cm in this case) from the vertical line. Lines p′a′, p′b′, p′h′ are joined and lines parallel to them are drawn in the corresponding spaces a′, b′ ... h′ as shown in in Fig 7.7.1 (e). This is the partial deflection curve. Since at the supports the deflection is zero, a straight line B_1B_2 is drawn and the vertical lines between B_1B_2 and the funicular polygon give the deflections at the corresponding points.

Now equation (7.2.1) can be applied for finding out the fundamental natural frequency. The deflections at various load points have been measured directly and multiplied by the x— scale which is (space scale × area scale × polar distance).

$$1 \text{ cm} = 5 \times 10 \times 10^{-6} \times 3 = 1.5 \times 10^{-4} \text{ cm.}$$

These values are shown below the deflection curve.

The following table has been constructed.

W	y	y^2	Wy	Wy^2
4.6	0	0	0	0
8.6	$.750\times10^{-4}$	$.56\times10^{-8}$	6.4×10^{-4}	4.81×10^{-8}
100.0	$.825\times10^{-4}$	$.68\times10^{-8}$	82.5×10^{-4}	68.0×10^{-8}
8.6	$.825\times10^{-4}$	$.68\times10^{-8}$	7.1×10^{-4}	5.85×10^{-8}
6.0	0	0	0	0
6.0	1.5×10^{-4}	2.25×10^{-8}	9.0×10^{-4}	13.5×10^{-8}
52.2	3.9×10^{-4}	15.20×10^{-8}	198×10^{-4}	793.0×10^{-8}

$$\Sigma Wy = 303\times10^{-4},\ \Sigma Wy^2 = 885\times10^{-8}$$

Now $$\omega_n^2 = \frac{g\Sigma Wy}{\Sigma Wy^2}$$

$$= \frac{980\times303\times10^{-4}}{885\times10^{-8}} = 335\times10^4$$

$$\therefore \quad \omega_n = 1830 \text{ rad/sec}$$

or $$N = \frac{1830\times60}{2\pi} = 17500 \text{ rpm.}$$ **Ans.**

7.8 Holzer's method.

In Sec. 6.5, a multi-rotor vibrating system was analysed. It is seen that the equations for obtaining the natural frequency of the system become increasingly complex with increase in the number of rotors in the system. Holzer's method, which is based on the same fundamental equations, is a trial and error method and is very effective in finding the natural frequencies of such systems.

Consider the system shown in Fig. 6.5.1 and the corresponding set of equations (6.5.5). Summation of these equations for the case of free vibrations, gives

$$\sum_{i=1}^{n} J_i\omega^2\beta_i = 0 \tag{7.8.1}$$

Torque acting on the shaft k_{11} due to inertia of the disc J_1

the complete moment diagram. The moments at various sections are measured and given below the moment diagram after multiplying with the moment scale which is (space scale × load scale × polar distance from the force lines)

$$1 \text{ cm} = 5 \times 20 \times 3 = 300 \text{ kg-cm}.$$

The value of I has been calculated for various sections and these are given below the moment values. Still below these are given the values of M/EI after taking $E = 2 \times 10^6$ kg/cm². It may be noted that at certain sections two values of M/EI are listed. These are the values immediately to the left and to the right of the section respectively and are due to the discontinuity in the shaft size or the sudden change in section. The final M/EI diagram is drawn with these values and is shown in Fig. 7.7.1 (d).

If a conjugate beam is considered with the area of the M/EI diagram as the loading on it, then the moment diagram with respect to this loading is nothing but the final deflection curve. First the conjugate beam load is divided into suitable sub-sections and the load equal to the area of each sub-section may be, with sufficient accuracy made to act at the middle of such sections. These area loads are shown below the M/EI diagram.

These imaginary loads are laid off along a vertical to a suitable scale (1 cm $= 10^{-6}$ in this case) as shown in Fig. 7.7.1 (g). A pole p′ is chosen at a convenient distance (3 cm in this case) from the vertical line. Lines p′a′, p′b′, p′h′ are joined and lines parallel to them are drawn in the corresponding spaces a′, b′ ... h′ as shown in in Fig 7.7.1 (e). This is the partial deflection curve. Since at the supports the deflection is zero, a straight line B_1B_2 is drawn and the vertical lines between B_1B_2 and the funicular polygon give the deflections at the corresponding points.

Now equation (7.2.1) can be applied for finding out the fundamental natural frequency. The deflections at various load points have been measured directly and multiplied by the x—scale which is (space scale × area scale × polar distance).

$$1 \text{ cm} = 5\times 10\times 10^{-6}\times 3 = 1.5\times 10^{-4} \text{ cm}.$$

These values are shown below the deflection curve.

The following table has been constructed.

W	y	y^2	Wy	Wy^2
4.6	0	0	0	0
8.6	$.750\times 10^{-4}$	$.56\times 10^{-8}$	$6\ 4\times 10^{-4}$	4.81×10^{-8}
100.0	$.825\times 10^{-4}$	$.68\times 10^{-8}$	82.5×10^{-4}	68.0×10^{-8}
8.6	$.825\times 10^{-4}$	$.68\times 10^{-8}$	7.1×10^{-4}	5.85×10^{-8}
6.0	0	0	0	0
6.0	1.5×10^{-4}	$2\ 25\times 10^{-8}$	9.0×10^{-4}	13.5×10^{-8}
52.2	3.9×10^{-4}	15.20×10^{-8}	198×10^{-4}	793.0×10^{-8}

$$\Sigma Wy = 303\times 10^{-4},\ \Sigma Wy^2 = 885\times 10^{-8}$$

$$\text{Now} \quad \omega_n^2 = \frac{g\Sigma Wy}{\Sigma Wy^2}$$

$$= \frac{980\times 303\times 10^{-4}}{885\times 10^{-8}} = 335\times 10^4$$

$$\therefore \quad \omega_n = 1830 \text{ rad/sec}$$

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Consider the system shown in Fig. 6.5.1 and the corresponding set of equations (6.5.5). Summation of these equations for the case of free vibrations, gives

$$\sum_{i=1}^{n} J_i\omega^2\beta_i = 0 \tag{7.8.1}$$

Torque acting on the shaft k_{12} due to inertia of the disc J_1

$$= -J_1 \ddot{\theta}_1 = J_1\omega^2\beta_1 \sin \omega t$$

Maximum value of this torque $= J_1\omega^2\beta_1$

Angular twist of the shaft $k_{t1} = \dfrac{J_1\omega^2\beta_1}{k_{t1}}$

$$\therefore \ \beta_2 = \beta_1 - \frac{J_1\omega^2\beta_1}{k_{t1}} = \beta_1\left[1 - \frac{J_1\omega^2}{k_{t1}}\right]$$

Maximum torque on the second disc

$$= J_1\omega^2\beta_1 + \beta_2\omega^2\beta_2$$

Angular twist of the second shaft

$$= \frac{J_1\omega^2\beta_1 + J_2\omega^2\beta_2}{k_{t2}}$$

$$\therefore \quad \beta_3 = \beta_2 - \frac{(J_1\beta_1 + J_2\beta_2)\omega^2}{k_{t2}}$$

In the same way, continuing upto the second last disc, we have

Maximum torque on the second last disc

$$= \sum_{i=1}^{n-1} J_i\omega^2\beta_i$$

Angular twist of the last shaft $k_{t(n-1)}$

$$= \frac{\sum_{i=1}^{n-1} J_i\omega^2\beta_i}{k_{t(n-1)}}$$

$$\beta_n = \beta_{n-1} - \frac{(J_1\beta_1 + J_2\beta_2 + \ldots\ldots\ldots J_{n-1}\,\beta_{n-1})\omega^2}{k_{t(n-1)}}$$

Maximum torque on the last disc

$$= \sum_{i=1}^{n} J_i\,\omega^2\beta_i$$

For the case of free vibrations, the value of β_1 is chosen to be any convenient value since the amplitude will not affect the natural frequency. An estimate is made for the natural frequency in any mode of vibration of interest. Starting from the inertia torque on the first disc with the above assumed value of ω^2 and shifting on to the angular twist of the first shaft and then the angular displacement β_2 of the second disc, the process is continued in the above mentioned steps to obtain the final maximum torque on the last disc. This should be zero in accordance with equation (7.8.1), provided the frequency ω chosen was one of the natural frequencies. If ω was different from the exact natural frequency (as will generally be the case), then the final inertia torque (remainder torque) on the last disc will not be zero. The remainder torque v/s frequency curve is of the type shown in Fig. 7.8.1, where the value ω_{n1}, ω_{n2} .. etc. correspond to the natural frequencies of the system. A few trials are necessary to get a good approximation of the natural frequency.

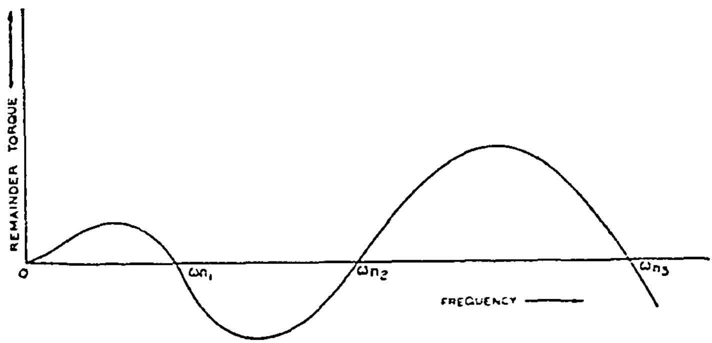


Fig. 7.8.1. Remainder torque v/s frequency for multi-degree freedom torsional system.

In case the other end of the shaft is fixed, then the angular displacement of this end should be zero and a few trials are again necessary to achieve this. All this is done in a tabular manner as is explained in the following examples.

The case of forced vibrations does not require any trial and error. The amplitude of the first disc is taken as β_1 and the rest of the expressions obtained in terms of β_1. The external

torque is added to the inertia torque of that rotor where the external torque is acting. The total torque at the last disc (which is the sum of inertia torques on all the discs plus the external torques) must again be zero. This is in terms of β_1, and equating it to zero gives the value of β_1. All other amplitudes which are in terms of β_1 can be obtained now.

Illustrated Example 7.8.1

A four cylinder engine whose shaft is coupled to a damper at one end and a generator at the other end has a flywheel mounted on the shaft between the engine and the generator. The schematic of the system is shown in Fig. 7.8.2 (a) with the values of the rotor inertias and the stiffnesses of the shafts. Estimate the two lowest natural frequencies and find these out by Holzer's method. Show also the corresponding mode shapes.

Solution

Since the first two natural frequencies are desired, the system will be converted into an approximate equivalent three rotor system. This can be done in many ways. In this case rotor numbers 1 to 5 are combined together and rotor number 6 and 7 are left as they are. The stiffness between the summed up rotors and rotor 6 is taken as the equivalent stiffness due to the shafts 1 to 5 in series. This is shown in Fig. 7.8.2 (b) with complete values of inertias and stiffnesses.

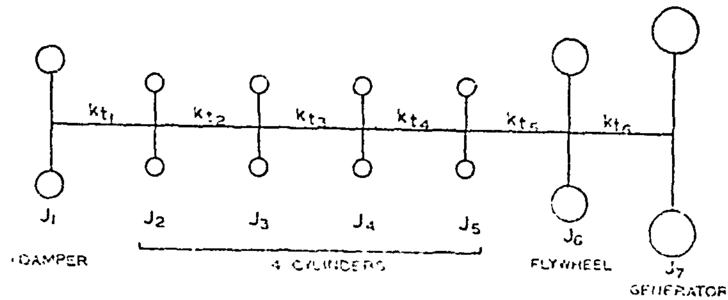
Now Fig. 7.8.2 (b) is a three rotor system and its natural frequencies are obtained from equation (6.5.9) as

$$\omega_{n1}^2 = 260{,}000$$

and $$\omega_{n2}^2 = 1{,}120{,}000$$

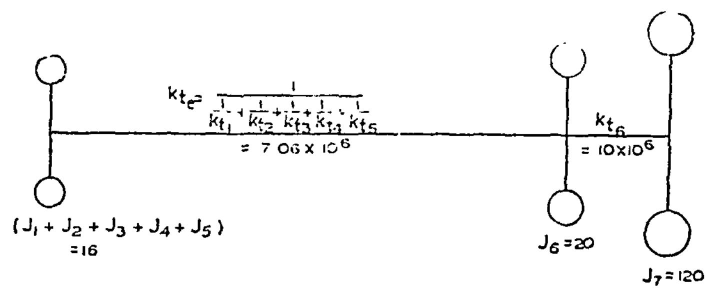
These are only approximate values and their degree of correctness depends upon how correct is the equivalent system to the actual system.

Determination of first natural frequency. The estimated value of the first natural frequency has been found above to be $\omega'^2 = 260{,}000$. This is taken as the starting trial value for the Holzer's table shown in Table 7.8.1 (a). The first column of this table gives the trial number. In the second column the serial



$J_1 = 10$ kg-cm-sec^2
$J_2 = J_3 = J_4 = J_5 = 1.5$
$J_6 = 20$
$J_7 = 120$
$k_{t1} = 40 \times 10^6$ kg-cm/rad
$k_{t2} = k_{t3} \quad k_{t4} = 30 \times 10^6$
$k_{t5} = 60 \times 10^6$
$k_{t6} = 10 \times 10^6$

(a)



(b)

Fig. 7.8 2 Natural frequency of a multi-rotor system by Holzer's method.

numbers are listed. These pertain to the rotors and the shafts. Col. No. 3 gives the moment of inertia of various rotors Col. No. 4 is $I\omega^2$, which is the product of figures in Col. No. 3 and the trial value of ω^2. This has been divided by 10^6 for convenience. Now let us come to Col. No. 8 which gives the stiffnesses of the various shafts. Since there are only six shafts in this case, the last space in this column will remain vacant.

All the columns described above (i.e. cols. No 1, 2, 3, 4 and 8) are filled up initially to work up the table. All these figures are shown in bold types.

TABLE 7.8.1

Holzer's tables for finding the first natural frequency of the system shown in Fig. 7.8.2 (a).

1	2	3	4	5	6	7	8	9
	S. No.	J (kg-cm -sec²)	$\frac{J\omega^2}{10^6}$ (kg-cm)	β (rad)	$\frac{J\omega^2\beta}{10^6}$ (kg-cm)	$\frac{\Sigma J\omega^2\beta}{10^6}$ (kg-cm)	$\frac{k_t}{10^6}$ (kg-cm/rad)	$\frac{\Sigma J\omega^2\beta}{k_t}$ (rad)
(a) FIRST TRIAL $\omega^2=260{,}000$	1	10	2.600	1.000	2.600	2.600	40	.065
	2	1.5	.390	.935	.354	2.954	30	.098
	3	1.5	.390	.837	.327	3.281	30	.109
	4	1.5	.390	.728	.284	3.565	30	.119
	5	1.5	.390	.609	.237	3.802	60	.063
	6	20	5.200	.546	2.840	6.652	10	.665
	7	120	31.200	−.119	−3.720	[3.068]	—	—
(b) SECOND TRIAL $\omega^2=290{,}000$	1	10	2.900	1.000	2.900	2.900	40	.073
	2	1.5	.435	.927	.403	3.303	30	.110
	3	1.5	.435	.817	.355	3.658	30	.122
	4	1.5	.435	.695	.302	3.960	30	.132
	5	1.5	.435	.563	.245	4.200	60	.070
	6	20	5.800	.493	2.860	7.060	10	.706
	7	120	34.800	−.213	−7.410	[−.350]	—	—
(c) THIRD TRIAL $\omega^2=288{,}000$	1	10	2.880	1.000	2.880	2.880	40	.072
	2	1.5	.432	.928	.400	3.280	30	.109
	3	1.5	.432	.819	.354	3.634	30	.121
	4	1.5	.432	.698	.302	3.936	30	.131
	5	1.5	.432	.567	.245	4.181	60	.070
	6	20	5.760	.497	2.860	7.041	10	.704
	7	120	34.560	−.207	−7.140	[−.099]	—	—

Col. No. 5 gives the amplitudes of vibration of various rotors. Col. No. 6 is the product of figures in col. No. 4 and 5, and represent the inertia torque on the particular rotor. Col. No. 7 shows the summed up inertia torques upto and including the particular rotor. Col. No. 9 gives the maximum twist of the particular shaft.

Filling up of the Cols. No. 5,6,7 and 9 is done by proceeding horizontally, line by line. In Col. No. 5, against the first rotor, an arbitrary amplitude is chosen since the system càn have free vibration in a particular mode with any amplitude of the first rotor. This value has been taken as unity for convenience. In Col. No. 6, against the same row, the product of the figures in the previous two columns is entered. This is the inertia torque on the first disc. In Col. No. 7. against the same row, we have the total inertia torque upto and including the first rotor, which in this case is just the same as figure in Col. No. 6. In Col. No. 9 we have the twist of the first shaft which is equal to the total inertia torque upto the first rotor (Col. No. 7) divided by the stiffness of the first shaft (Col. No. 8). The first row is complete now.

Coming to the second row, the amplitude of the second rotor is equal to the amplitude of the first rotor minus the twist of the first shaft. This is equal to the figure in Col. No. 5 minus the figure in Col. No. 9 of the first row, i.e., (1.000 —.065) = .935. The next space (Col. No 6) is the inertia torque on the second rotor and, as before, is the product of figures in Cols. No. 4 and 5 in the second row. For the Col. No. 7, we have the sum of the inertia torques upto and including the second rotor and is, therefore, the sum of the figures in Col. No. 7 of the previous row and Col. No. 6 of the present row, i.e., (2.600 + .354) = 2.954. Process is continued in the same manner till we reach the last row at Col. No. 7. This figure is the sum of inertia torques on all the rotors. This value should be zero if ω (starting trial value) is exactly equal to the one of the natural frequencies of the system, and there will always be a remainder torque if ω is not equal to any natural frequency. The remainder torque is positive in [illegible] case. If we look back to Fig. 7.3 1 (we already know [illegible]

re near the value $\omega=\omega_{n1}$), the positive remainder torque neans that $\omega<\omega_{n1}$, or the trial value of ω^2 taken in Table .8.1 (a) is too low.

The second trial [Table 7.8.1 (b)] is now done with omewhat higher value of ω^2 (taken = 290,000 for this case). The same procedure is repeated and the remainder torque now s -0.350. This value is much less than that in first trial and also has a negative sign with it. From Fig. 7.8.1 we can see that the negative remainder torque in the neighbourhood of the first natural frequency means that $\omega>\omega_{n1}$, and the low magnitude of this torque shows that we are only a little bit on the higher side.

The third trial is done with slightly reduced value of ω^2 (taken=288,000 for this case) as shown in Table 7.8.1 (c). The remainder torque now is -0.099 showing that we are very near the actual value on the higher side. The actual value of ω_n^2 will be slightly less than 288,000. However this is quite accurate for all practical purposes and therefore is taken as the required value.

$$\omega_{n1} = \sqrt{288{,}000} = 537 \text{ rad/sec.} \qquad \textbf{Ans.}$$

Determination of second natural frequency: The estimated value of the second natural frequency has been found to be $\omega^2=1{,}120{,}000$. This is taken as the starting trial value for the Holzer's table shown in Table 7.8.2 (a). The procedure is exactly the same till we get the remainder torque as -39.290.

If we look at Fig. 7.8.1, the negative remainder torque in neighbourhood of second natural frequency means that $\omega<\omega_{n2}$ and therefore larger value of ω^2 should be taken. Large magnitude of the remainder torque indicates that we are still quite far from the exact natural frequency. Therefore, the trial value taken for the second trial is $\omega^2=1{,}200{,}000$ as shown in table 7.8.2.(b) and the remainder torque obtained is -12.893. This is still negative indicating that we are still below the actual value of ω_{n2}, and the smaller magnitude of the resulting torque shows that we are nearer to the actual value than we were in the first trial.

The next trial value is taken as $\omega^2=1{,}240{,}000$ as shown in

TABLE 7.8.2

Holzer's tables for finding the second natural frequency of the system shown in Fig. 7.8.2 (a).

1	2	3	4	5	6	7	8	9
	S. No.	J (kg-cm -sec²)	$\frac{J\omega^2}{10^6}$ (kg-cm)	β (rad)	$\frac{J\omega^2\beta}{10^6}$ (kg-cm)	$\frac{\Sigma J\omega^2\beta}{10^6}$ (kg-cm)	$\frac{k_t}{10^6}$ (kg-cm/rad)	$\frac{\Sigma J\omega^2\beta}{k_t}$ (rad)
(a) FIRST TRIAL $\omega^2 = 1,120,000$	1	10	11.200	1.000	11.200	11.200	40	.280
	2	1.5	1.680	.720	1.208	12.408	30	.414
	3	1.5	1.680	.306	.514	12.922	30	.431
	4	1.5	1.680	−.125	−.210	12.712	30	.424
	5	1.5	1.680	−.549	−.923	11.789	60	.196
	6	20	22.400	−.745	−16.680	−4.891	10	−.489
	7	120	134.400	−.256	−34.400	[−39.290]	—	—
(b) SECOND TRIAL $\omega^2 = 1,200,000$	1	10	12.000	1.000	12.000	12.000	40	.300
	2	1.5	1.800	.700	1.260	13.260	30	.442
	3	1.5	1.800	.258	.465	13.725	30	.457
	4	1.5	1.800	−.199	−.358	13.367	30	.446
	5	1.5	1.800	−.645	−1.160	12.207	60	.203
	6	20	24.000	−.848	−20.400	−8.193	10	−.819
	7	120	144.000	−.029	−4.700	[−12.893]	—	—
(c) THIRD TRIAL $\omega^2 = 1,240,000$	1	10	12.400	1.000	12.400	12.400	40	.310
	2	1.5	1.860	.690	1.280	13.680	30	.456
	3	1.5	1.860	.234	.435	14.115	30	.471
	4	1.5	1.860	−.237	−.441	13.674	30	.456
	5	1.5	1.860	−.693	−1.290	12.384	60	.206
	6	20	24.800	−.899	−22.300	−9.916	10	−.992
	7	120	148.800	.093	13.820	[3.904]	—	—

Table 7.8.2 (c). The remainder torque obtained now is 3.904. The magnitude is smaller than in the previous cases and it is positive now showing that for this trial $\omega > \omega_{n2}$.

Now Fig. 7 8.3 is drawn for the remainder torque against ω^3 for the three values of ω^2 obtained in the above three trials. This curve cuts the abscissa at $\omega^3 = 1,230,000$ which is therefore equal to $\omega_{n2}{}^3$.

Or $\quad \omega_{n_2} = \sqrt{1,230,000} = 1110$ rad/sec. **Ans.**

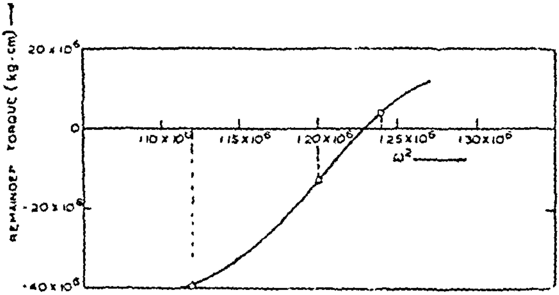


Fig. 7.8.3. Remainder torque v/s frequency square for system of Fig. 7.8.2 (a) when vibrating in the neighbourhood of second natural frequency.

Mode Shapes. For the first mode shape corresponding to the first natural frequency we take the values of the amplitudes for various rotors from Table 7.8.1 (c), Col. No. 5. Although these are not the exact values because in that case the remainder torque should be zero, but these are sufficiently close to the exact values. These are plotted in Fig. 7.8.4. (a).

Similarly the second mode shape plotted in Fig. 7.8.4 (b) is obtained from the amplitudes of various rotors as obtained in Table 7.8.2 (c), Col. No 5. **Ans.**

In case only the lowest natural frequency is required to be obtained, then the original system should be converted into an approximate equivalent two rotor system. This can be done, in this case, by combining the first five rotors on one side and the last two rotors on the other side. The stiffness of the connecting equivalent shaft may be taken as combination of all the shafts in series.

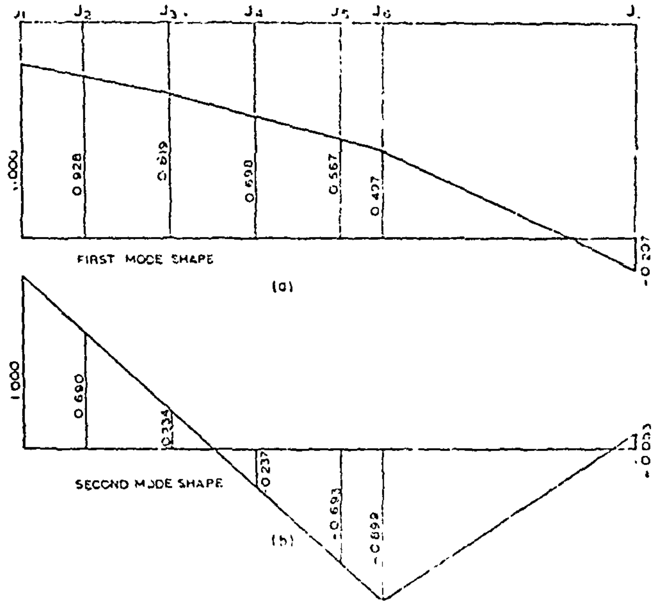


Fig. 7.8.4. First two mode shapes for the system of Fig. 7.8.2 (a)

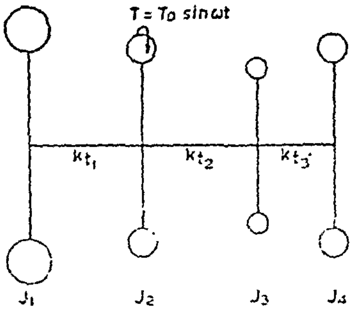
Illustrative Example 7.8.2

A four rotor system is represented schemetically in Fig. 7.8.5. The physical constants of the system are given alongside. A torque $T = T_0 \sin \omega t$ acts on the second rotor. Determine the amplitudes of vibration of various rotors. Find also the maximum twist in each section of the shaft and the corresponding twisting moment.

Solution

The procedure for making the Holzer's table for forced vibrations is similar to that of free vibrations except that there are some points of difference. In the case of forced vibrations the system vibrates with the exciting frequency which is given, and therefore no trial and error procedure is required. Only one table has to be constructed with this value of ω^2. Further, since each rotor has a definite amplitude (unlike free vibrations) the value of amplitude for the first rotor may be taken as β_1.

After filling Cols. No. 1,2,3,4 and 8 of Table 7.8.3 initially,



$$
\begin{aligned}
J_1 &= 100 \text{ kg-cm-sec}^2\\
J_2 &= 50\\
J_3 &= 10\\
J_4 &= 50\\
k_{t1} &= 1\times10^4 \text{ kg-cm/rad}\\
k_{t2} &= 1\times10^4\\
k_{t3} &= 2\times10^4\\
T_o &= 10{,}000 \text{ kg-cm}\\
\omega &= 5 \text{ rad/sec.}
\end{aligned}
$$

Fig. 7.8.5. Forced vibrations of a multi-rotor system.

we proceed to fill the table row-wise as in the case of free vibrations. All the quantities here will be in term of β_1.

When we come to Col. No. 7 for the second rotor (on which external torque is acting) it is necessary to include the etxernal torque here along with the summed up inertia torques. This expression $(2437\beta_1 + T_o)$, therefore gives the total torque (inertia plus external) upto and incuding the second rotor. The procedure is continued now till we get a final remainder torque in terms of β_1 and T_0. This is equated to zero since the remainder toque is the sum of external torque and inertia torques, and this is zero for forced vibrations.

Therefore $\quad 3825\beta_1 + .789\, T_0 = 0$

or $$\beta_1 = -\frac{0.789 T_0}{3825} = -\frac{0.789\times10^4}{3825} = -\,2.06 \text{ rad}$$

The amplitudes of various discs are obtained from Col. No. 5 after substituting for β_1 and T_0, and these are:—

TABLE 7.8.3.

Holzer's table for forced vibrations.

1	2	3	4	5	6	7	8	9
	S.No.	J (kg-cm-sec^2)	$J\omega^2$ (kg-cm)	β (rad)	$J\omega^2\beta$ (kg-cm)	$\Sigma J\omega^2\beta$ (kg-cm)	k_t (kg-cm/rad)	$\frac{\Sigma J\omega^2\beta}{k_t}$ (rad)
$\omega^2 = 25$	**1**	**100**	**2500**	β_1	$2500\beta_1$	$2500\beta_1$	$\mathbf{1\times10^4}$	$0.25\beta_1$
	2	**50**	**1250**	$0.75\beta_1$	$937\beta_1$	$3437\beta_1+T_o$	$\mathbf{1\times10^4}$	$0.3437\beta_1+1\times10^{-4}T_o$
	3	**10**	**250**	$0.4063\beta_1-1\times10^{-4}T_o$	$101\rho_1-0.025T_o$	$3538\beta_1+.975T_o$	$\mathbf{2\times10^4}$	$0.1769\beta_1+0.488\times10^{-4}T_o$
	4	**50**	**1250**	$0.2294\beta_1-1.488\times10^{-4}T_o$	$287\beta_1-0.186T_o$	$[3825\beta_1+0.789T_o]$	—	—

$\beta_1 = -\ 2.06$ rad

$\beta_2 = -\ 1.55$ rad

$\beta_3 = -\ 1.836$ rad

$\beta_4 = -\ 1.961$ rad.

The twists of the shafts are obtained from Col. No. 9 and these are

First shaft	$-\ 0{\cdot}515$ rad
Second shaft	$+\ 0.293$ rad
Third shaft	$+\ 0.123$ rad.

The twisting moments in the shafts are obtained from Col. No. 7 as

First shaft	$-\ 5150$ kg-cm
Second shaft	$+\ 2930$ kg-cm
Third shaft	$+\ 2460$ kg-cm.

From the above values of the twists and the twisting moments we can obtain the maximum stress in each section of the shaft provided its diameter is given. **Ans.**

PROBLEMS FOR PRACTICE

7.1 Find, by Rayleigh's method, the lowest natural frequency of the system shown in Fig. 7.2.1, when the weights W_1 and W_2 are interchanged.

7.2 Find the lowest natural frequency of transverse vibrations for the system shown schematically in Fig. P. 7.2 by Rayleigh's method. Take $E = 2 \times 10^6$ kg/cm^2 and $I = 100$ cm^4.

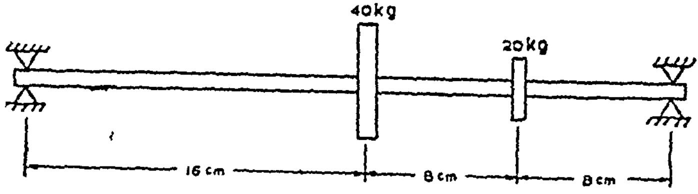


Fig. P. 7.2.

7.3 Same as Problem 7.2 except take into account the weight of the shaft which is given to be 4 kg. [Hint : Find the equivalent weight at the centre].

7.4 Using Rayleigh's method, estimate the fundamental frequency of the system shown in Fig. P. 7.4. The shaft weight $W_s = 10$ kg.

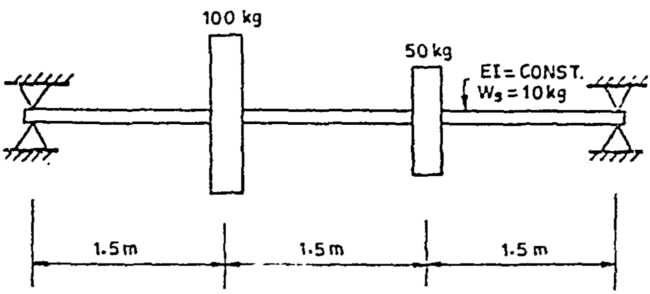


Fig. P. 7.4

7.5 Find the lowest natural frequency of transverse vibration for a cantilever carrying uniformly distributed load by assuming the deflection curve to be

(i) since curve,

(ii) static deflection curve,

(iii) parabola.

Which one of the above is the closest to the first mode shape ?

7.6 Solve parts (b) and (c) of Illustrative Example 7.2.3.

7.7 Solve Problem 7.2 by Dunkerly's method.

7.8 Solve Problem 7.2 by Dunkerly's method by taking into account the weight of the shaft also, which is 4 kg.

7.9 Solve Prob. 7.4 by Dunkerley's method

7.10 Solve Problem 7.2 by Stodola's method.

7.11 Solve Prob. 7.4 by Stodola's method.

7.12 Find by Stodola's the lowest natural frequency of the system shown in Fig. P. 7.12.

$\beta_1 = -\ 2.06$ rad
$\beta_2 = -\ 1.55$ rad
$\beta_3 = -\ 1.836$ rad
$\beta_4 = -\ 1.961$ rad.

The twists of the shafts are obtained from Col. No. 9 and these are

First shaft $-\ 0{\cdot}515$ rad
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PROBLEMS FOR PRACTICE

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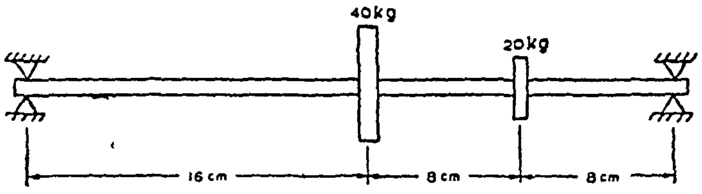

Fig. P. 7.2.

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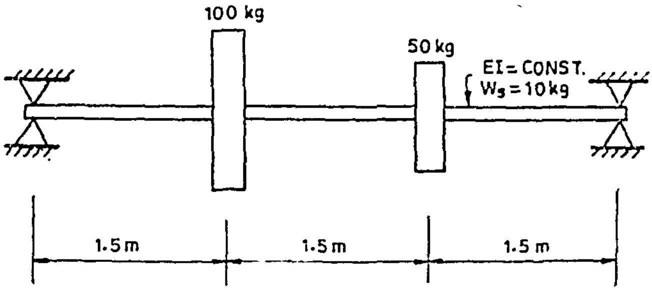

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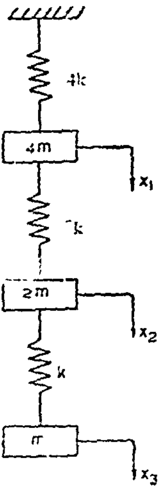


Fig. P. 7.12.

7.13 If in Fig. 7.5.1, the origin is taken at the tip of the cantilever and the x—axis towards left of this point, then at a distance x from the tip point

$$t = T\frac{x}{L}$$

$$I(x) = \frac{1}{12}\left(\frac{Tx}{L}\right)^3$$

and $\quad w(x) = \gamma\, T\,\frac{x}{L}.$

Assume the expression for the deflection curve as

$$y = C_1\left(1-\frac{x}{L}\right)^2 + C_2\,\frac{x}{L}\left(1-\frac{x}{L}\right)^2 + C_3\,\frac{x^2}{L^2}\left(1-\frac{x}{L}\right)^2 + \cdots\cdots$$

Show that this equations satisfies the boundary condition. Find the lowest natural frequency of the system in lateral vibrations. Take the width of the bar as unity.

7.14 A cantilever of the shape shown in Fig. P. 7.14 is of unit width and is fixed at its base. Find the lowest natural frequency of the bar in transverse vibrations.

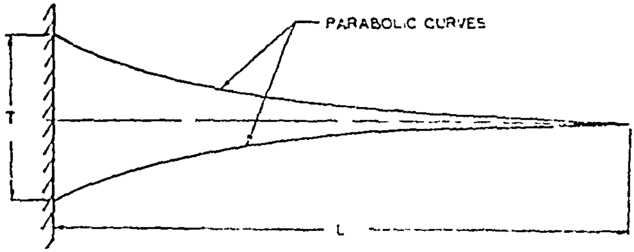


Fig. P. 7.14.

7.15 Find the first natural frequency of the triple pendulum shown in Fig. P. 7.15, by the method of matrix iteration.

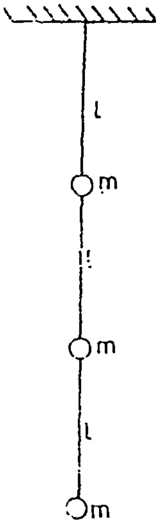


Fig. P. 7.15.

7.16 Find the first natural frequency of transverse vibration of three masses fixed on a string as shown in Fig. P. 7.16, by matrix iteration. The string is stretched with a a large tension T.

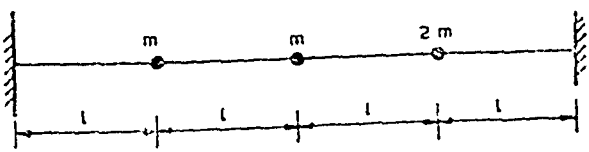


Fig. P. 7.16.

7·17 Find the first natural frequency of transverse vibrations of a tight string having four masses fixed on it as shown in Fig. P. 17.

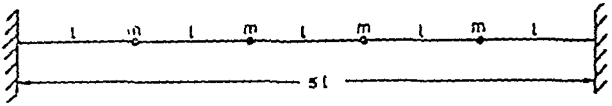


Fig. P. 7.17.

7.18 Find the fundamental natural frequency of transverse vibrations by the graphical method for the system of Problem 7.2.

7.19 Two rotors are mounted on a simply supported shaft of non-uniform sections as shown in Fig. P. 7.19. Find the fundamental natural frequency of the system.

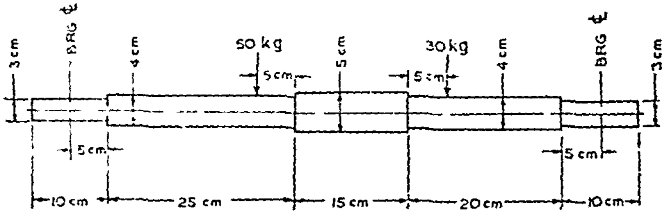


Fig. P. 7.19.

7.20 For the 5-rotor system shown in Fig. P. 7.20, calculate by Holzer's method the first two natural frequencies. The quantities mentioned on the figure are in the following units :

$J \rightarrow$ kg-cm-sec^2
$k_t \rightarrow$ kg-cm/rad.

7.21 Fig. P. 7.21 shows a torsional system consisting of three rotors. Make an estimate of the first natural frequency and starting with this value perform Holzer's calculations to determine the first natural frequency.

$J \rightarrow$ kg-cm-sec^2
$k_t \rightarrow$ kg-cm/rad.

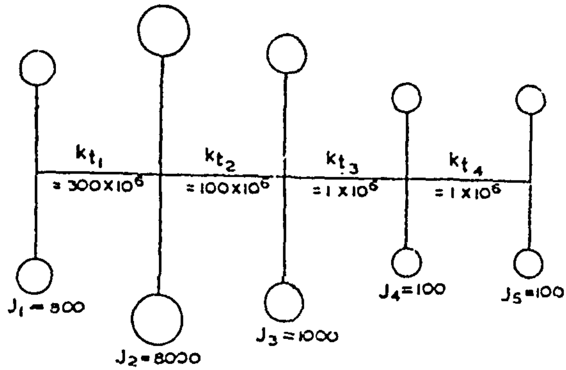


Fig. P. 7.20.

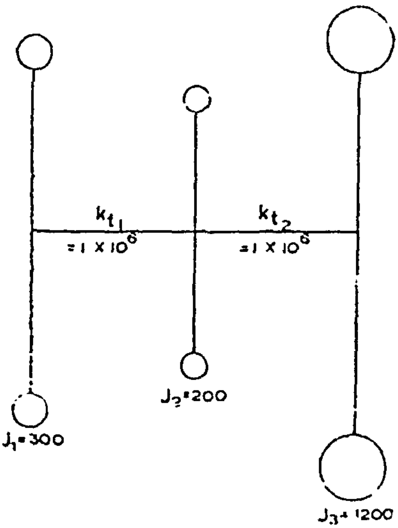


Fig. P. 7.21.

.22 Find by Holzer's method the natural frequency of the torsional system shown in Fig. P. 7.22 when the right end in fixed. The mass moments of inertia and the stiffcesses are in kg-cm-sec units.

.23 A four-rotor system schematically represented in Fig. P. 7.23 has the following physical quantities.

$J_1 = 817$ kg-cm-sec^2 $\qquad$ $J_3 = 100$ kg-cm-sec^2

$J_2 = 608$ kg-cm-sec^2 $\qquad$ $J_4 = 120$ kg-cm-sec^2

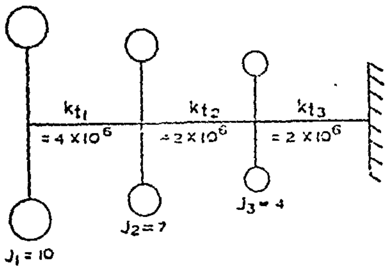


Fig. P. 7.22.

$k_{t1} = 30 \times 10^6$ kg-cm/rad $\qquad k_{t3} = 42 \times 10^6$ kg-cm/rad

$k_{t2} = 42 \times 10^6$ kg-cm/rad

$T_o = 12000$ kg-cm

$\omega = 200$ rad/sec.

Find the amplitudes of vibration when the external torque acts on the first rotor, as shown in the figure. If the diameter of each of the connecting shaft is 3 cm, find the maximum stress in each shaft section. Assume $G = 0.85 \times 10^6$ kg/cm^2.

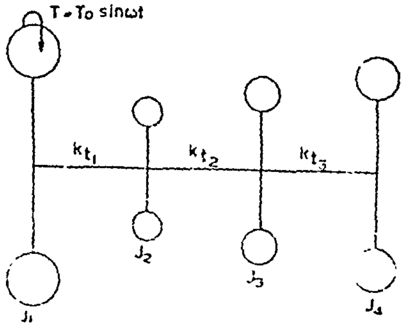


Fig. P. 7.23.

CHAPTER 8

CRITICAL SPEEDS OF SHAFTS

8.1 Introduction.

A critical speed of a rotating shaft is the speed at which the shaft starts to vibrate violently in the transverse direction. It is very dangerous to continue to run the shaft at its critical speed as the amplitude of vibrations will build up to such a level that the system may go to pieces. Critical speed is also, quite often, given the name of *whirling speed or whipping speed.*

The whirling of shafts result from various causes among which the most important and the one which will be discussed in detail in this chapter is the mass unbalance of the rotating system.

8.2 Critical speed of a light shaft having a single disc—without damping.

Consider a light vertical shaft in a deflected position with a single disc of mass m as shown in Fig. 8.2.1. Point S is the geometric centre of the disc through which the centre line of the shaft passes. Point G is the centre of gravity of the disc which is displaced from the geometric centre through a distance e because of the manufacturing inaccuracies or slight variation in the density of material of the disc. Point O is the intersection of the bearing centre line with the disc. In the deflected position the geometric centre S of the disc is deflected through a distance r from the undeflected position. Let k be the stiffness of the shaft in the lateral direction at the point where the disc is located.

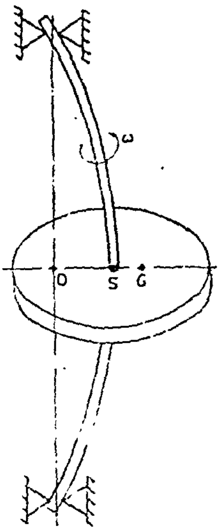


Fig. 8.2.1. Critical speed of rotating shaft.

Considering the equilibrium of the disc, there are two forces acting on it. The centrifiugal force at G acts radially outwards and the restoring force at S acts radially inwards. The friction forces have been neglected. For the equilibrium of the disc, these two forces must act along the same line and therefore the points O, S and G must lie on the same straight line. The centrifugal force is equal to $m\omega^2 (r+e)$ where ω is the angular velocity of the shaft. The restoring force is equal to kr. Equating the two, we have

$$m\omega^2(r+e) = kr \tag{8.2.1}$$

which gives

$$r = \frac{m\omega^2 e}{k - m\omega^2} = \frac{\left(\dfrac{\omega}{\omega_n}\right)^2 e}{\left[1-\left(\dfrac{\omega}{\omega_n}\right)^2\right]} \tag{8.2.2}$$

where $\omega_n = \sqrt{k/m}$, is the natural frequency of the lateral vibration of the shaft.

Equation (8.2.2) shows that the deflection r of the shaft tends to infinity when $\omega = \omega_n$. Thus the critical speed of the shaft is equal to the natural frequency of lateral vibration of the shaft. Also, it is seen that the deflection r is positive below the critical

speed and negative above the critical speed. This means that the disc rotates with heavy side outwards when $\omega < \omega_n$ and light side outwards when $\omega > \omega_n$. This corresponds to zero degree phase difference when $\omega < \omega_n$ and 180° phase difference when $\omega > \omega_n$. This is shown in Fig. 8.2.2.

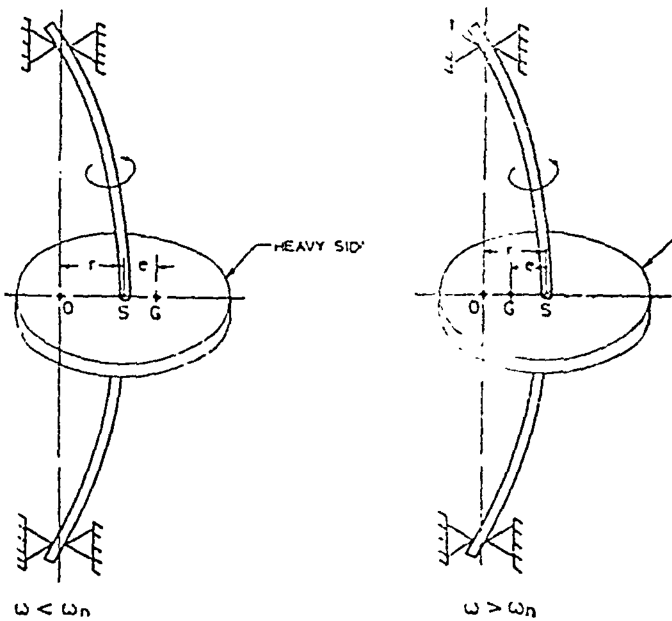


Fig. 8.2.2. Possible phase relationships—no damping.

The system has indifferent equilibrium when $\omega = \omega_n$. This is the point when the system undergoes violent vibrations. When $\omega >> \omega_n$, $r \rightarrow -e$, which means that the point G approaches O and the disc rotates about its centre of gravity.

Illustrative Example 8.2.1

A rotor weighing 5kg is mounted midway on a 1 cm dia shaft supported at the ends by two bearings. The bearing span is 40 cm. Because of certain manufacturing inaccuracies, the C.G. of the disc is 0.02 mm away from the geometric centre of the rotor. If the system rotates at 3000 r.p.m. find the amplitude of steady state vibrations and the dynamic force transmitted to the bearings. Neglect damping.

Solution

Assuming the ends to be simply supported, the stiffness of the shaft is given by

$$k = \frac{48EI}{l^3} = \frac{48 \times (2\times 10^6) \times [(\pi/64) \times 1^4]}{40^3}$$

$$= 73.5 \text{ kg/cm}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{73.5}{5/980}} = 85.6 \text{ rad/sec}$$

$$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 3000}{60} = 100\pi \text{ rad/sec.}$$

Applying equation (8.2.2), we get the amplitude of vibration as

$$r = \frac{\left(\frac{100\pi}{85.6}\right)^2 \times 0.002}{\left[1 - \left(\frac{100\pi}{85.6}\right)^2\right]} = -0.00216 \text{ cm.}$$

The negative sign shows that the displacement is out of phase with the centrifugal force. **Ans.**

The dynamic load on the bearings is due to this deflection of 0.00216 cm.

Therefore $R_B = kr$

$$= 73.5 \times 0.00216 = 0.16 \text{ kg.}$$

Hence, the dynamic load on each bearing is 0.08 kg. If the shaft is vertical this will be all the load on each bearing, but in case the shaft is horizontal there will be additional load due to the dead weight of the rotor. **Ans.**

8.3 Critical speed of a light shaft having a single disc-with damping.

When damping is present in the form of air resistance, etc., the analysis becomes slightly more involved. Now three forces act on the disc. These are

(i) the centrifugal force at G along OG produced,
(ii) the restoring force at S along SO, and
(iii) the damping force at S in a direction opposite to the

velocity of the point S.

Due to these forces the points O, S and G no longer lie on the straight line. These forces are shown in Fig. 8.3.1. G is the centre of gravity of the disc displaced from the geometric centre S of the shaft through a distance e which is fixed. O is the centre of rotation, being the point of intersection of the centreline of the bearings with the disc. Let r be the deflection of the shaft i.e. OS $= r$. Let OG $= a$, $\angle$GOS $= \alpha$ and $\angle$GSA $= \phi$. The centrifugal force is equal to $m\omega^2 a$ and acts at G. The restoring force is equal to kr and acts at S. Assuming viscous damping, the damping force is equal to $c\omega r$ and acts at S. Here c is the damping coefficient and ωr is the linear velocity of point S. The resultant of kr and $c\omega r$ at S must be equal to $m\omega^2 a$ and parallel to it to give a clockwise moment which has to be overcome by the driving torque in the anti-clockwise direction. This is because of the damping present in the system which requires the driving torque equal to $c\omega r.r = c\omega r^2$

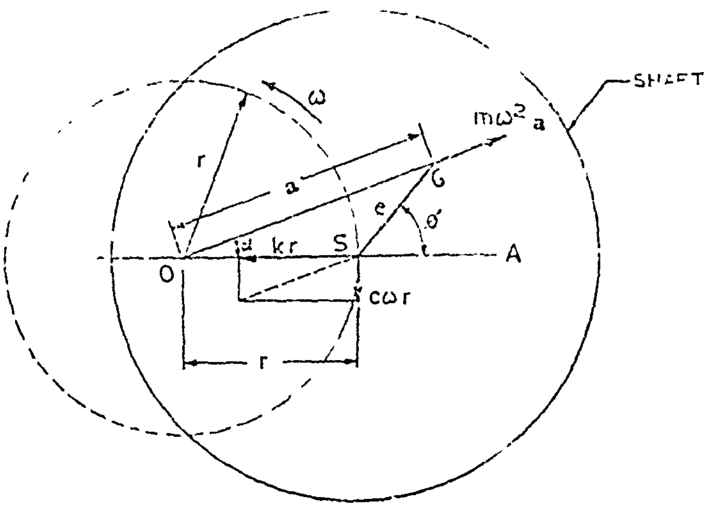


Fig. 8.3.1. Vector relationship for whirling shaft with single disc having damping.

From the geometry of the figure, we have

$$\left.\begin{aligned} a \sin \alpha &= e \sin \phi \\ a \cos \alpha &= r + e \cos \phi \end{aligned}\right] \qquad (8.3.1)$$

For the forces acting on the system, putting $\Sigma X=0$ and $\Sigma Y=0$, give

$$\left.\begin{aligned} -kr+m\omega^2 a\cos\alpha &= 0 \\ -c\omega r+m\omega^2 a\sin\alpha &= 0 \end{aligned}\right] \tag{8.3.2}$$

Eliminating a and α from equations (8.3.2) with the help of equations (8.3.1), we have

$$\left.\begin{aligned} -kr+m\omega^2(r+e\cos\phi) &= 0 \\ -c\omega r+m\omega^2(e\sin\phi) &= 0 \end{aligned}\right] \tag{8.3.3.}$$

The above equations give

$$\frac{r}{e}=\frac{m\omega^2}{\sqrt{(k-m\omega^2)^2+(c\omega)^2}}=\frac{\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\zeta\frac{\omega}{\omega_n}\right]^2}} \tag{8.3.4}$$

$$\text{and } \tan\phi=\left(\frac{c\omega}{k-m\omega^2}\right)=\left[\frac{2\zeta.\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right] \tag{8.3.5}$$

where $\zeta=\frac{c}{2\sqrt{km}}$

Equations (8.3.4) and (8.3.5) are exactly the same as equations (4.3.2) and (4.3.3) respectively and can be represented by the corresponding curves of Figs. 4.3 2 and 4.2.5. The ordinate in Fig. 4.3.2 now represents r/e for this case corresponding to equation (8.3.4).

From the curves of fig. 4.2.5 or equation (8.3.5), it is seen that

(i) $\phi \simeq 0$ when $\omega << \omega_n$ (heavy side out)

(ii) $0<\phi<90°$ when $\omega<\omega_n$ (heavy side out)

(iii) $\phi=90°$ when $\omega=\omega_n$

(iv) $90°<\phi<180°$ when $\omega>\omega_n$ (light side out)

(v) $\left.\begin{array}{l}\phi \simeq 180^\circ \\ r \approx -e\end{array}\right]$ when $\omega >> \omega_n$ $\left[\begin{array}{l}\text{light side out, disc} \\ \text{rotates about its CG}\end{array}\right]$

Some of the above phase relationships are shown in Fig. 8.3.2.

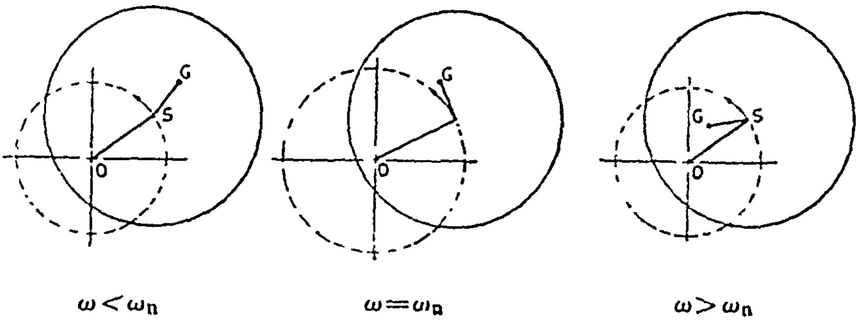


Fig. 8.3.2. Possible phase relationships with damping.

Illustrative Example 8.3.1

A disc weighing 4 kg is mounted midway between bearings which may be assumed to be simple supports. The bearing span is 48 cm. The steel shaft which is horizontal, is 0.9 cm in diameter. The CG of the disc is displaced 0.3 cm from the geometric centre. The equivalent viscous damping at the centre of the disc-shaft may be taken as 0.05 kg-sec/cm. If the shaft rotates at 760 rpm, find the maximum stress in the shaft and compare it with dead load stress in the shaft. Also find the power required to drive the shaft at this speed.

Solution

$$k = \frac{48\,EI}{l^3} = \frac{48 \times (2 \times 10^6) \times [(\pi/64) \times (0.9)^4]}{48^3}$$

$$= 28.0 \text{ kg/cm}$$

$$\omega_n = \sqrt{k/m} = \sqrt{\frac{28}{4/980}} = 82.8 \text{ rad/sec}$$

$$\omega = \frac{2\pi \times 760}{60} = 79.5$$

$$\frac{\omega}{\omega_n} = \frac{79.5}{82.8} = 0.96$$

$$\zeta = \frac{c}{2\sqrt{km}} = \frac{0.05}{2\times\sqrt{82.8\times4/980}} = 0.043$$

Substituting these values in equation (8.3.4), we have

$$\frac{r}{0.3} = \frac{(0.96)^2}{\sqrt{[1-(0.96)^2]^2+[2\times0.043\times0.96]^2}}$$

$$r = 2.47 \text{ cm}$$

The dynamic load on the bearings is equal to the centrifugal rce of the disc which is equal to the vector sum of the spring d damping forces (see Fig. 8.3.1),

or $F_d = \sqrt{(kr)^2+(c\omega r)^2}$

$$= r\sqrt{k^2+(c\omega)^2} = 2.47\sqrt{28^2+(0.05\times79.5)^2}$$

or $F_d = 70$ kg.

ıis is the effective dynamic force with which the shaft is de- cted under the operating conditions. Total maximum load the shaft under dynamic conditions is the sum of the above ad and the dead load, i.e.,

$$F_{max} = 70+4 = 74 \text{ kg.}$$

The load under static conditions is

$$F_s = 4 \text{ kg.}$$

For maximum stress due to a load acting at the centre of a nply supported shaft, we have

$$s = \frac{M}{I}\cdot\frac{d}{2} = \frac{(Fl/4)\times(d/2)}{[(\pi/64)\times(d^4)]}$$

$$\frac{(F\times48/4)\times(0.9/2)}{[(\pi/64)\times(0.9)^4]} = 168\,F$$

ıtal maximum stress under dynamic conditions

$$s_{max} = 168F_{max} = 168\times74 = 12{,}400 \text{ kg/cm}^2.$$ **Ans.**

aximum stress under dead load

$$s_s = 168\,F_s = 168\times4 = 670 \text{ kg/cm}^2.$$ **Ans.**

ımping force $= c\omega r$

$$= 0.05\times79.5\times2.47 = 9.85 \text{ kg.}$$

Damping torque $T = 9.85 \times 2.47 = 24.3$ kg-cm.

$$\text{H.P.} = \frac{2\pi N(T/100)}{4500} = \frac{2\pi \times 760 \times 24.3}{4500 \times 100} = 0.258 \quad \textbf{Ans.}$$

8.4 Critical speeds of a shaft having multiple discs.

So far we have dealt with the case of a light shaft with a single disc. For a shaft having more than one disc, there will be as many critical speeds as the number of discs. The procedure for finding these is as follows.

Fig. 8.4.1 is the schematic of a shaft with two discs, shown in deflected position while rotating. Let y_1 and y_2 be the deflections of the two discs from the centre line of the bearings. Let us assume that the two discs of weight W_1 and W_2 are having their centres of gravity displaced from the geometric centres through distances e_1 and e_2. If F_1 and F_2 are the centrifugal forces on these discs, then

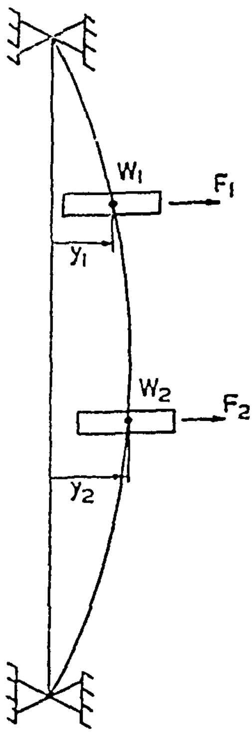


Fig. 8.4.1. Critical speeds of shaft with two discs.

$$\left.\begin{aligned} F_1 &= \frac{W_1}{g}\omega^2\,(y_1+e_1) \\ F_2 &= \frac{W_2}{g}\omega^2\,(y_2+e_2) \end{aligned}\right] \tag{8.4.1}$$

where ω is the angular velocity of the rotating shaft. It has been considered in equations (8.4.1) that e_1 and e_2 are along the same directions as y_1 and y_2 respectively. If it is not so, e_1 and e_2 can be treated as vectors with respect to y_1 and y_2. However, as will be seen in the following lines, it doesnot create any problems.

Using the notations of the influence coefficients, we may write

$$\left.\begin{aligned} y_1 &= \delta_{11}F_1 + \delta_{12}F_2 \\ y_2 &= \delta_{21}F_1 + \delta_{22}F_2 \end{aligned}\right] \tag{8.4.2}$$

Substituting for F_1 and F_2 in the above set of equations from equations (8.4.1), we have

$$\left.\begin{aligned} y_1 &= \delta_{11}\frac{W_1}{g}\omega^2\,(y_1+e_1)+\delta_{12}\frac{W_2}{g}\omega^2\,(y_2+e_2) \\ y_2 &= \delta_{21}\frac{W_1}{g}\omega^2\,(y_1+e_1)+\delta_{22}\frac{W_2}{g}\omega^2(y_2+e_2) \end{aligned}\right] \tag{8.4.3}$$

Equations (8.4.3) can be used to compute the values of y_1 and y_2. But more important than these values are the values of the critical speeds. It is understood that at the critical speeds, the deflections y_1 and y_2 will be very large. Therefore, neglecting e_1 and e_2 in comparison with y_1 and y_2 respectively, equations (8.4.3) become

$$\left.\begin{aligned} &\left(\delta_{11}\frac{W_1}{g}\omega^2-1\right)y_1+\left(\delta_{12}\frac{W_2}{g}\omega^2\right)y_2=0 \\ &\left(\delta_{21}\frac{W_1}{g}\omega^2\right)y_1+\left(\delta_{22}\frac{W_2}{g}\omega^2-1\right)y_2=0 \end{aligned}\right] \tag{8.4.4}$$

The solution of the above equations other than the trivial solution is possible only if

$$\begin{vmatrix} \left(\delta_{11}\dfrac{W_1}{g}\omega^2-1\right) & \delta_{12}\dfrac{W_2}{g}\omega^2 \\ \delta_{21}\dfrac{W_1}{g}\omega^2 & \left(\delta_{22}\dfrac{W_2}{g}\omega^2-1\right) \end{vmatrix}=0 \qquad (8.4.5)$$

And this gives

$$\omega^2 = g\left[\frac{p\pm\sqrt{p^2-4q}}{2q}\right] \qquad (8.4.6)$$

where

$$p = W_1\delta_{11}+W_2\delta_{22}$$
$$q = W_1W_2(\delta_{11}\delta_{22}-\delta_{12}{}^2)$$

The negative sign in equation (8.4.6) gives the lower or the first critical speed and the positive sign gives the higher or the second critical speed.

These critical speeds can also be calculated by Rayleigh's method by finding the natural frequency of lateral vibration of the shaft since the critical speed is numerically equal to the natural frequency of lateral vibrations.

In case the shaft is not light and its weight has also to be considered, then this can be taken into account very conveniently by Rayleigh's method or the method described above.

For the case of the shaft with more than two discs the solution becomes somewhat cumbersome, and since we are mostly interested in the lowest critical speed, it can be conveniently obtained by Rayleigh's method.

Illustrative Example 8.4.1.

Find the two critical speeds for the system shown in Fig. 7.2.1.

Solution

For the expressions given for the influence coefficients in Illustrative Example 7.2.1, we have

$$\delta_{11} = \frac{c^3}{3EI} = \frac{18^3}{3\times3\times10^6\times40} = 0.0243\times10^{-3}$$

$$\delta_{12} = \delta_{21} = \frac{c^2}{6EI}(3l-c) = \frac{18^2\times(90-18)}{6\times2\times10^6\times40} = 0.0486\times10^{-3}$$

$$\delta_{22} = \frac{l^3}{3EI} = \frac{30^3}{3\times2\times10^6\times40} = 0.1125\times10^{-3}$$

To apply equation (8.4.6.), we have

$$p = W_1\delta_{11}+W_2\delta_{22}$$

$$= (100\times.0243+50\times.1125)\times10^{-3}$$

or $\quad p = 8.05\times10^{-3}$

$$q = W_1W_2(\delta_{11}\delta_{22}-\delta_{12}^2)$$

$$= 100\times50\times(.0243\times.1125-.0486^2)\times10^{-6}$$

or $\quad q = 1.9\times10^{-6}$

Now using equation (8.4.6),

$$\omega^2 = \frac{980\,[8.05\pm\sqrt{8.05^2-4\times1.9}]\times10^{-3}}{2\times1.9\times10^{-6}}$$

we have

$$\omega_{n1}^2 = 126{,}500$$

$$\omega_{n2}^2 = 4{,}020{,}000.$$

or, $\quad \omega_{n1} = 355.5$ rad/sec

$$\omega_{n2} = 2005 \text{ rad/sec.}$$

The first of these values is about the same as obtained in Ill. Ex. 7.2.1.

The critical speeds, therefore, are finally given by

$$N_{c1} = \frac{355.5\times60}{2\pi} = 3395 \text{ rpm.}$$

$$N_{c2} = \frac{2005\times60}{2\pi} = 19150 \text{ rpm.}$$ **Ans.**

8.5 Secondary critical speed.

Apart from the main critical speed resulting from the centrifugal forces of the unbalanced masses, a good amount of vibration has been observed at half the critical speed. This effect has been noticed on horizontal shafts only and has been found to be totally absent in the vertical shafts, indicating that gravity

must be one of the causes of it. The importance and severity of this critical speed known as the secondary critical speed, is much less than of the main or primary critical speed.

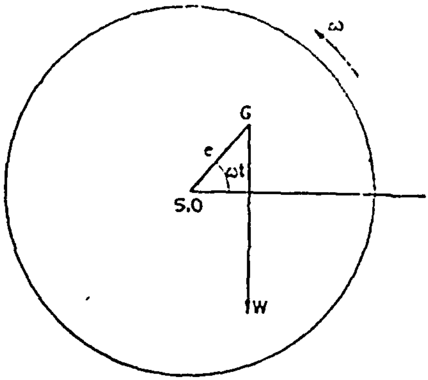


Fig. 8.5.1. Secondary critical speed.

Let Fig. 8 5.1 represent a rotating shaft having no vibrations, in which case, the geometric centre S of the disc coincides with O, the point where the bearing centre line intersects the plane of the disc. Let G be the of gravity of the disc at a distance e from the geometric centre. While the shaft is rotating in the anticlockwise direction, the torque on the disc due to the weight W accelerates the shaft when the point G lies on the left of S and retards the shaft when the point G lies on the right of S. The magnitude of this torque is $We \cos \omega t$. Let J be the mass moment of inertia of the disc about its geometric axis. Then, due to the varying torque $We \cos \omega t$, the angular acceleration of the shaft is $\frac{W}{J} e \cos \omega t$, and therefore the tangential acceleration of point G is $\frac{W}{J} e^2 \cos \omega t$. To have this tangential acceleration there must be an equivalent force $m \frac{W}{J} e^2 \cos \omega t$ where m is the mass of the disc. The component of this tangential force along the vertical is $\cos \omega t$ times as large.

Hence the vertical component of this force $= m \frac{W}{J} e^2 \cos^2 \omega t$

$$= \frac{m}{2} \frac{W}{J} e^2 \quad [1 + \cos 2\omega t] \qquad (8.5.1)$$

The first part of the force in the above expression is a constant and is taken up as a small additional deflection of the shaft, and is of no interest to us. The variable part has a frequency 2ω. If the shaft is running at half its critical speed, the variation of the vertical force occurs at the natural frequency so that large amount of vibrations occur.

8.6 Critical speeds of a light cantilever shaft with a large heavy disc at its end.

If a light shaft having two end supports has a central disc then the system has been shown to have one critical speed. Even if the disc is not central, the system will have one critical speed as long as we assume the mass of the disc to be concentrated. If, however, the disc has mass as well as moment of inertia, and is not central, then the system will have two critical speeds. The treatment given below is for a light cantilever shaft having a disc which has mass as well as moment of inertia. Since the critical speed is numerically equal to the natural frequency of lateral vibrations, we will find the latter for this system.

Consider the beam so as to be displaced from the equilibrium position as shown in Fig. 8.6.1. In this figure,

M = mass of the disc,

Mr^2 = moment of inertia of the disc about an axis passing through the CG of the disc and perpendicular to the plane of the paper,

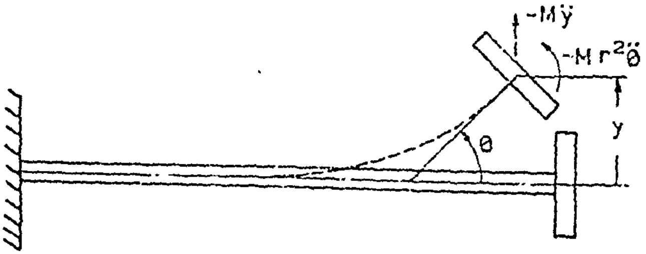


Fig. 8.6.1. Critical speeds of a light shaft having a large heavy disc at its end.

$y, \ddot{y}$ = displacement and acceleration of the CG of the disc,

$\theta, \ddot{\theta}$ = angular displacement and angular acceleration of the axis of the disc due to bending.

Further, let

δ_{11} = deflection of the CG of the disc per unit force acting on it in the lateral direction $= l^3/3EI$

δ_{22} = slope at the free end of the beam per unit moment acting on the CG of the disc, in the plane of the paper $= l/EI$

δ_{12} = deflection of the CG of the disc per unit moment acting on it = slope at the free end of the beam per unit force acting on it in a lateral direction $= l^2/2EI$

where

l = length of the beam,

I = moment of inertia of the section of the beam about the neutral axis,

E = modulus of elasticity of the material of the beam.

The inertia force and the inertia torque on the disc in the displaced position are shown in Fig. 8.6.1 along with their directions. These are as follows.

$$\text{Inertia force} = -M\ddot{y} = M\omega^2 y$$

$$\text{Inertia torque} = -Mr^2\ddot{\theta} = Mr^2\omega^2\theta$$

where ω is the natural frequency for the principal mode of vibration of the system.

Then the deflection at the CG of the disc and the rotation of the disc in the plane of the paper are given by

$$y = \delta_{11}\, M\omega^2 y + \delta_{12}\, Mr^2\omega^2\theta \tag{8.6 2}$$

$$\theta = \delta_{12}\, M\omega^2 y + \delta_{22}\, Mr^2\omega^2\theta \tag{8.6.3.}$$

Eliminating y and θ from the above two equations, and putting

$$\left.\begin{aligned} g &= \omega \sqrt{\frac{Ml^3}{3EI}} \\ h &= \frac{3r^2}{l^2} \end{aligned}\right] \tag{8.6.4}$$

we have

$$hg^4 - 4(h+1)\,g^2 + 4 = 0 \tag{8.6.5}$$

giving the two natural frequencies as

$$g_{1,2}^2 = \frac{2}{h}\left[(h+1) \pm \sqrt{(h+1)^2 - h}\right] \tag{8.6.6}$$

Fig. 8.6.2 is a plot of the above equation and shows the variation of the two natural frequencies of the system with the change in $h\left(=\frac{3r^2}{l^2}\right)$; it may be recalled that r is radius of gyration of the disc about an axis passing through its CG and perpendicular to the axis of the disc.

$h = 0$ corresponds to the concentrated mass.

$h \to \infty$ corresponds to the disc having large radius of gyration.

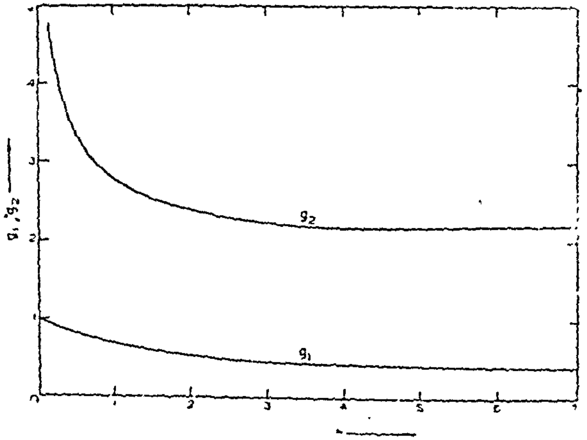


Fig. 8.6.2. Variation of the two natural frequencies with the change in $h\left(=\frac{3r^2}{l^2}\right)$.

Illustrative Example 8 6.1

A light cantilever steel shaft of 30 cm effective length has a heavy rotor fixed at its end. The weight of the rotor is 10 kg and has a radius of gyration of 12 cm about its axis. The thickness of the rotor is 6 cm. The moment of inertia of the section of the shaft about its neutral axis is $10 cm^4$. This shaft is to run at 10,000 rpm. Check if this operating speed is safe.

Solution

$$l = 30 \text{ cm}$$

$$W = 10 \text{ kg}, \quad \therefore M = 10/980 \text{ kg-sec}^2/\text{cm}$$

$$r_a = 12 \text{ cm}$$

$$b = 6 \text{ cm}$$

$$I = 10 \text{ cm}^4$$

$$E = 2 \times 10^6 \text{ kg/cm}^2 \text{ (assumed).}$$

If r is the radius of gyration of the rotor about a line through the CG and prependicular to the axis, then

$$r^2 = \frac{r_a^2}{2} + \frac{b^2}{12} \text{ (prove this relation),}$$

$$= \frac{12^2}{2} + \frac{6^2}{12} = 75 \text{ cm}^2.$$

Substituting the above values in the second of equations (8.6.4), we have

$$h = \frac{3 \times 75}{30^2} = 0.25$$

Substituting the value of h in equation (8.6.6), we have

$$g_{1,2}^2 = \frac{2}{0.25}\left[(0.25+1) \pm \sqrt{(0.25+1)^2 - 0.25}\right]$$

$$= 0.84, \ 19.16$$

Coming back to the first of equations (8.6.4), we have

$$g = \omega \sqrt{\frac{Ml^3}{3EI}}$$

or $$\omega = g \sqrt{\frac{3EI}{Ml^3}}$$

Now $\sqrt{\frac{3EI}{Ml^3}} = \sqrt{\frac{3\times(2\times10^6)\times10}{(10/980)\times30^3}} = 467$

$$\therefore \omega = 467g$$

Substituting for g above, the values already obtained,

$$\omega_1 = 467g_1 = 467\times\sqrt{0.84} = 428 \text{ rad/sec}$$

$$\omega_2 = 467g_2 = 467\times\sqrt{19.16} = 2040 \text{ rad/sec},$$

giving

$$N_{c1} = \frac{428\times60}{2\pi} = 4080 \text{ rqm}$$

$$N_{c2} = \frac{2040\times60}{2\pi} = 19500 \text{ rpm.}$$

The operating speed of 10,000 rpm is not near any of its critical speed. Hence the operating speed is safe. **Ans.**

PROBLEMS FOR PRACTICE

8.1 A variable speed machine has a diametral clearance of 2 mm between the stator and the rotor. The rotor weighs 37.5 kg and has an unbalance of 0.3 kg-cm. The rotor is mounted on a steel shaft midway between the two bearings. The operating speed of the machine varies from 500 to 6400 r.p.m. Specify the stiffness of the shaft so that the rotor is always clear of the stator at any operating speed within the range.

8.2 A single rotor weighing 7 kg is mounted midway between bearings on a steel shaft 1 cm dia. The bearing span is 40 cm. It is known that the CG of the rotor is .025 mm from its geometric axis. If the system rototes at 1000 r.p.m, find out the amplitude of vibration, the dynamic load transmitted to the bearings and the maximum stress in the shaft, when

(a) the shaft is vertically supported,

(b) the shaft is horizontally supported. Neglect the weight of the shaft and the damping in the system. Assume the shaft to be simply supported.

8.3 A vertical shaft 1.25 cm in diameter rotates in spherical bearings with a span of 90 cm and carries a disc of weight 10 kg midway between the two bearings. The mass centre of the disc is 0·025 cm away from the geometric axis. If the stress in the shaft is not to exceed 1050 kg/cm², determine the range of speed within which it is unsafe to run the shaft. Neglect the mass of the shaft and the damping in the system.

8.4 A disc made of solid steel with a diameter of 12 cm and a thickness of 2 cm is fixed in the centre of a 50 cm shaft 1.25 cm dia. The shaft may be considered to be simply supported at the two extreme ends. The bearings have equal flexibility in all the directions and the equivalent spring constant for each bearing is 20 kg/cm. Find the whirling speed of the shaft.

8.5 Calculate the whirling speed of the shaft supported by long bearings so as to give zero slope at both ends of the shaft.

8.6 A rotor having a weight of 9.5 kg is mounted on a 12mm horizontal steel shaft midway between bearings that are 60 cm apart. The centre of gravity of the disc is 6mm from its geometric centre. If the damping in the system is estimated at $\zeta = 0.1$, draw a diagram of the forces and displacements when the shaft rotates at 690 r.p.m. Compare the maximum dead load stress in the shaft with the maximum stress at the operating speed. Also determine the horsepower required to drive the shaft at this speed.

8.7 A rotor weighing 10 kg is mounted on a shaft of stiffness 5 kg/cm at the rotor. The equivalent damping coefficient of the system is 0.2 kg-sec/cm. When the

shaft rotates at 500 rpm, the power dissipated in damping is 0.01 H. P. Determine the eccentricity of rotor.

8.8 For the system shown in Fig. P. 7.2, find the two whirling speeds. Take $E=2\times10^6$ kg/cm² and $I=100$ cm⁴.

8.9 Find the two natural frequencies for the system shown in Fig. P.8.9. Take the shaft to be of uniform section with $I= 3.2$ cm⁴. Assume the modulus of elasticity as 2×10^6 kg/cm².

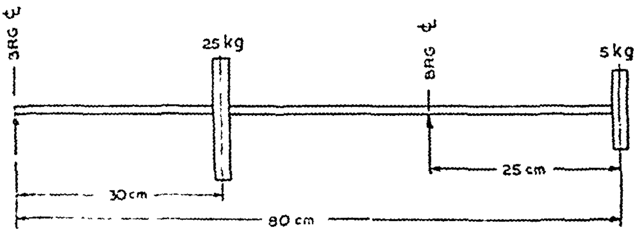


Fig. P. 8.9.

8.10 A light cantilever steel shaft 5 cm dia and 40 cm long has a heavy C.I. disc 30 cm dia and 5 cm thick fixed at its end as shown in Fig. P.8.10. Assuming the density of C.I. as 7.2 gm/cm³, determine the two critical speeds of the shaft.

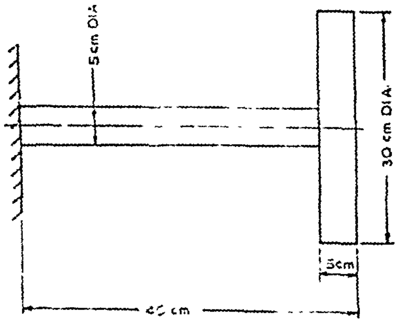


Fig. P. 8.10.

CHAPTER 9

TRANSIENT VIBRATIONS

9.1 Introduction.

A system subjected to periodic excitation has two components of motion, the transient and the steady state. In most of such cases the transient part is not important as it dies out soon and the steady state part is the one that persists. However, where the excitation is of aperiodic nature like a shock pulse or a transient excitation, the response of the system is purely transient. After the duration of the excitation, the system undergoes vibrations with its natural frequency with an amplitude depending upon the type and duration of the excitation. It is in such cases that the transient vibrations have importance. The practical examples of shock excited transient vibrations are rock explosions, gunfires, loading or unloading of packages by dropping them on hard floors, punching operations, automobiles at high speeds passing over pits or curbs on the road, etc.

The use of Laplace transform is introduced in this chapter for the analysis of systems subjected to shock pulses. The usual differential equations method or the so-called classical method becomes very lengthy and cumbersome with transient excitations of different shapes.

9.2 Laplace transformation.

Laplace transform is a powerful mathematical tool that is extremely useful in the solution of differential equations, and especially so, where transients are involved. It is that branch

of operational calculus wherein a function is transformed from t (time) domain to a new s domain. The original differential equation in t domain, by the use of Laplace transform, changes itself into an algebraic equation in s domain. The solution of an algebraic equation is very easy as compared to that of a differential equation. Once the solution in s domain is obtained, the process of inverse transformation gives the solution back in t domain. Manipulation with transformation and inverse transformation is facilitated by the use of table of transform pairs which is given later in this section.

Laplace transform $F(s)$ of a function $f(t)$ is defined as

$$F(s) = \int_0^\infty f(t)\, e^{-st}\, dt. \tag{9.2.1}$$

In shorthand it is generally written as

$$\mathrm{L}\,[\,f(t)\,] = F(s) \tag{9.2.2}$$

The use of the basic definition of Laplace transform [Eqn. (9.2.1)] is illustrated below by actually transforming a few common functions.

Illustrative Example 9.2.1

Find the Laplace transform of a step function (i) $A\,u(t)$, (ii) $A\,u(t-a)$. These functions are shown in Fig. 9.2.1 (a) and (b).

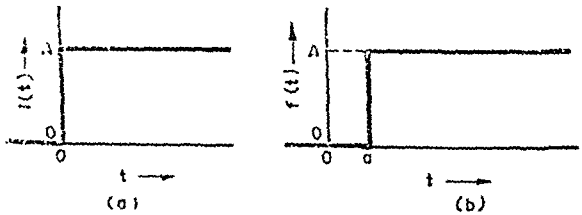


Fig. 9.2.1. Step functions.

Note : $u(t)$ is called a unit step function starting at time $t = 0$ and $u(t-a)$ is a unit step function starting at time $t = a$. Before these respective times the function is zero.

Solution

$$\text{(i)} \quad L\,[A\,u(t)] = \int_0^\infty A\,e^{-st}\,dt$$

$$= A \left| \frac{e^{-st}}{-s} \right|_0^\infty = \frac{A}{s} \qquad \textbf{Ans.}$$

$$\text{(ii)} \quad L\,[A\,u(t-a)] = \int_a^\infty A\,e^{-st}\,dt$$

$$= A \left| \frac{e^{-st}}{-s} \right|_a^\infty = \frac{A\,e^{-sa}}{s} \qquad \textbf{Ans.}$$

Illustrative Example 9.2.2

Find the Laplace transform of a pulse of height A and duration τ as shown in Fig. 9.2.2. Deduce the Laplace transform of a unit impluse.

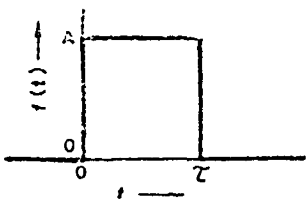


Fig. 9.2.2. Rectangular pulse.

Solution

$$F(s) = \int_0^\tau A\,e^{-st}\,dt = A \left| \frac{e^{-st}}{-s} \right|_0^\tau$$

$$= \frac{A}{s}\left[1 - e^{-s\tau}\right] \qquad \textbf{Ans.}$$

If in the above pulse of area $\tau A = a$ (say), $\tau \to 0$ and $A \to \infty$ with the condition that the area a of the pulse remains constant, then this pulse of infinitermally small duration and area a is known as an impluse of strength a. A unit impluse will have a unit area in the limiting case.

Thus the Laplace transform of an impluse of strength a can be obtained from the Laplace transform of a pulse of area $\tau A = a$ (found above)

$$F(s) = \underset{\substack{\tau \to 0 \\ A \to \infty \\ \tau A \to a}}{Lt} \frac{A}{s}\left[1 - e^{-s\tau}\right] = \underset{\substack{\tau \to 0 \\ A \to \infty \\ \tau A \to a}}{Lt} \frac{\tau A}{\tau s}\left[1 - e^{-s\tau}\right]$$

or $$F(s) = \underset{\tau \to 0}{Lt} \frac{a}{s}\left[\frac{1 - e^{-s\tau}}{\tau}\right] = \frac{a}{s} . s = a$$

That is, the Laplace transform of an impulse of strength a is equal to a itself. Hence the Laplece transform of a unit impluse is unity. A unit impulse is written as $\delta(t)$.

Therefore

$$L\,[\delta(t)] = 1$$ **Ans.**

Illustrative Example 9.2.3

If $f(t) = \sin \omega t$, find $L\,[f(t)]$.

Solution

$$L\,[f(t)] = L\,[\sin \omega t] = \int_0^\infty e^{-st} \sin \omega t \, dt$$

or $$L\,[f(t)] = \left|\frac{e^{-st}}{-s} \sin \omega t\right|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s} \omega \cos \omega t \, dt$$

$$= \frac{\omega}{s} \int_0^\infty e^{-st} \cos \omega t \, dt$$

or $$L\,[f(t)] = \frac{\omega}{s}\left[\left|\frac{e^{-st}}{-s}\cos\omega t\right|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}\cdot -\omega\sin\omega t\,dt\right]$$

$$= \frac{\omega}{s}\left[\frac{1}{s} - \frac{\omega}{s}\int_0^\infty e^{-st}\sin\omega t\,dt\right]$$

$$= \frac{\omega}{s^2} - \frac{\omega^2}{s^2}\,L\,[f(t)]$$

giving

$$L\,[f(t)] = \frac{\omega}{s^2+\omega^2}$$

or $$L\,[\sin\omega t] = \frac{\omega}{s^2+\omega^2}$$ **Ans.**

Illustrative Example 9.2.4

Find the Laplace transform of the derivative of a function.

Solution

Let $$f(t) = \frac{dx(t)}{dt}$$

and let the initial value of $x(t)$ at $t = 0$ be $x(0)$.

$$L\left[\frac{dx(t)}{dt}\right] = \int_0^\infty \frac{dx(t)}{dt}\,e^{-st}\,dt$$

$$= \left|x(t).\,e^{-st}\right|_0^\infty + s\int_0^\infty x(t)\,e^{-st}\,dt$$

$$= -\,x(0) \qquad + s\,X(s)$$

Therefore

$$L\left[\frac{dx(t)}{dt}\right] = s\,X(s) - x(0)$$ **Ans.**

Table 9.2.1 gives various Laplace transform pairs. For any function in the left column the corresponding transform in the right column can be derived in a manner similar to that illustrated in the above four examples.

Table 9.2.1

Laplace Transform Pairs

	$f(t)$	$L[f(t)]$
1.	$f(t)$	$\int_0^\infty f(t)\, e^{-st}\, dt = F(s)$
2.	$x(t) + y(t)$	$X(s) + Y(s)$
3.	$K\, f(t)$	$K\, F(s)$
4.	$u(t)$ or 1	$\frac{1}{s}$
5.	$\delta(t)$	1
6.	t	$\frac{1}{s^2}$
7.	t^n	$\frac{n!}{s^{n+1}}$
8.	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
9.	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
10.	e^{-at}	$\frac{1}{s+a}$
11.	$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
12.	$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$
13.	$e^{-at} f(t)$	$F(s+a)$
14.	$u(t-a)$	$\frac{e^{-as}}{s}$
15.	$\delta(t-a)$	e^{-as}
16.	$\begin{bmatrix} 0 & \text{when } t < a \\ f(t-a) & \text{when } t > a \end{bmatrix} = f(t-a)\, u(t-a)$	$e^{-as}\ F(s)$
17.	$\frac{d\, f(t)}{dt}$	$s\, F(s) - f(0)$
18.	$\frac{d^2 f(t)}{dt}$	$s^2 F(s) - sf(0) - \frac{df(0)}{dt}$
19.	$\int_0^t f(t)\, dt$	$\frac{F(s)}{s}$

9.3 Response to an impulsive input.

Consider a damped spring mass system as shown in Fig. 9.3.1 subjected to an impulse $\hat{F}\,\delta(t)$, the strength of the impulse

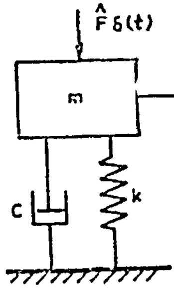


Fig 9.3.1. A damped system subjected to an impulse.

being $\hat{F}$. Since the impulse acts for an extremely small duration, its effect is to give an initial velocity to the mass, given by

$$\hat{F} = m\,dv$$

where dv is the change in velocity of the mass due to the impulse $\hat{F}$. If the system is initially at rest, the impulse gives it a starting velocity of

$$dv = \frac{\hat{F}}{m}$$

The initial displacement of the mass from the equilibrium position is zero because of the extremely small duration of the impulse.

Thus the initial conditions for the mass are

$$\left.\begin{aligned} x(0) &= 0 \\ \dot{x}(0) &= \frac{\hat{F}}{m} \end{aligned}\right\} \qquad (9.3.1)$$

The differential equation for the system can now be written as

$$m\ddot{x} + c\dot{x} + kx = 0 \qquad (9.3.2)$$

The forcing function on the right has been taken to be zero since the impulse effectively gives only the initial conditions obtained in eqn. (9.3.1).

Dividing eqn. (9.3.2) by m throughout, it can be written as

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = 0 \tag{9.3.3}$$

Taking the Laplace transform of the above equation, we have

$$[s^2 X(s) - s\,x(0) - \dot{x}(0)]$$
$$+ 2\zeta\omega_n [s\,X(s) - x(0)] + \omega_n^2 X(s) = 0$$

Substituting the initial conditions of eqn. (9.3.1), and rearranging, gives

$$X(s) = \frac{\hat{F}}{m} \cdot \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{9.3.4}$$

In order to obtain the inverse transformation for the above equation, the expression on the right has to be rearranged in one of the forms corresponding to the transform pairs of Table 9.2.1, for direct inversion. If $\zeta < 1$, the above equation is re-written in the following form.

$$X(s) = \frac{\hat{F}}{m\sqrt{1-\zeta^2}.\,\omega_n}\left[\frac{\sqrt{1-\zeta^2}.\,\omega_n}{(s+\zeta\omega_n)^2 + (\sqrt{1-\zeta^2}.\,\omega_n)^2}\right] \tag{9.3.5}$$

The inverse transform of the above equation can be obtained by comparing the bracketed expression with item 11 in Table 9.2.1. Therefore

$$x(t) = \frac{\hat{F}}{m\sqrt{1-\zeta^2}.\,\omega_n} e^{-\zeta\omega_n t} \sin\sqrt{1-\zeta^2}\,\omega_n t \tag{9.3.6}$$

which is the response of the system to an impulsive input. The same equation could have been obtained by the classical method of solution of the differential equations, but as we advance further in this chapter, the classical approach becomes more and more difficult whereas Laplace transform approach lends itself to comparatively easier solutions.

The response solution as obtained in eqn. (9.3.6) is true only for the case when $\zeta < 1$ (i.e. underdamped system), since for the case of $\zeta \geqslant 1$, eqn. (9.3.4) cannot be written as eqn. (9.3.5). For the latter case, eqn. (9.3.4) has to be split up

into partial fractions and then compared with standard transform pairs in the table.

Eqn. (9.3.6) is plotted in Fig. 9.3.2 in dimensionless form for various values of the damping factor.

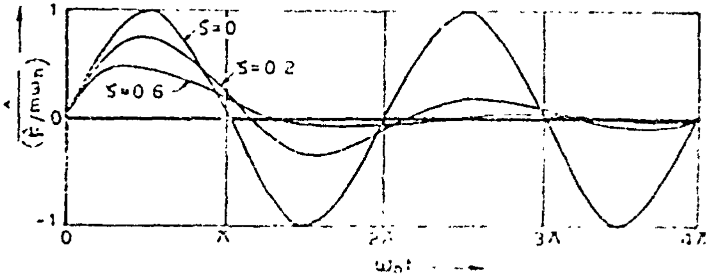


Fig. 9.3.2. System response to an impulsive input for different amounts of damping.

In case the impulse $\hat{F}$ occurs at $t = a$, that is the impulse is $\hat{F}\,\delta(t-a)$, the response eqn. (9.3.6) is modified to

$$x(t) = \frac{\hat{F}}{m\sqrt{1-\zeta^2}\,\omega_n}\,e^{-\zeta\omega_n(t-a)}\sin\sqrt{1-\zeta^2}\,\omega_n(t-a)\,u(t-a)$$

The last factor $u(t-a)$ being included to mean that the complete response is zero unful $t = a$. This is because

$u\,(t-a) = 0$ for $t < 0$

$u\,(t-a) = 1$ for $t \geqslant 0$.

Illustrative Example 9.3.1

A container having an apparatus suitably packaged inside it is schematically represented in Fig. 9.3.3. When the package

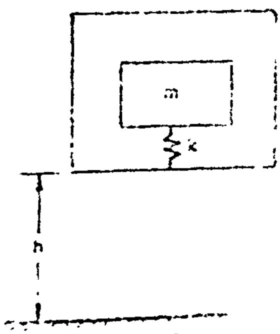


Fig. 9.3.3. Package dropping on a hard surface.

is dropped on a hard surface during loading or unloading, analyse the system for its response.

Solution

At the instant the container hits the ground, its velocity along with its content will be $\sqrt{2gh}$.*

The differential equation of motion of the mass m during the period the container is in contact with the ground is

$$m\ddot{x} + kx = 0$$

with the initial conditions

$$x(0) = 0$$

$$\dot{x}(0) = \sqrt{2gh}$$

the positive directions of x and $\dot{x}$ being taken in the downward direction. The solution can be directly obtained from eqn (9.3.6) by substituting $\sqrt{2gh}$ for $\frac{\hat{F}}{m}$ and putting $\zeta = 0$. Therefore

$$x(t) = \frac{\sqrt{2gh}}{\omega_n} \sin \omega_n t \tag{9.3.7}$$

the maximum acceleration of the mass is given by

$$\ddot{x}_{max} = \omega_n^2 \cdot \frac{\sqrt{2gh}}{\omega_n} = \omega_n \sqrt{2gh} \tag{9.3.8}$$

If the maximum acceleration is to be limited in order to prevent damage to the apparatus m, then the natural frequency ω_n of the spring mass system has to be controlled to a suitable low value. This is seen from eqn (9.3.8). Further the container will rebound up if the inertia force $m\ddot{x}$ increases the gravitational force on the combined container and its content. **Ans.**

*Actually the velocity of the container just before hitting will be slightly greater than $\sqrt{2gh}$ and that of the mass m slightly less than $\sqrt{2gh}$. This is because the strained spring before the fall relaxes during the fall, thus helping the container down and opposing the mass m in its downward motion. But this effect is neglected.

9.4 Response to a step input.

Fig. 9.4.1. shows a spring-mass-dashpot system subjected to a step force $F_0\, u(t)$. The magnitude of the force is constant at a value F_0 for all time greater than or equal or zero. The force is zero for $t < 0$. The differential equation of motion can be written as

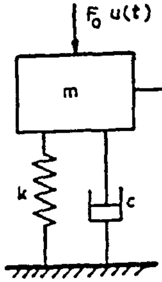


Fig. 9.4.1. A damped system subjected to a step forcing function.

$$m\ddot{x} + c\dot{x} + kx = F_0\, u(t)$$

or $$\ddot{x} + 2\,\zeta\,\omega_n\,\dot{x} + \omega_n^2\,x = \frac{F_0}{m}\,u(t) \qquad (9.4.1)$$

Taking the Laplace transform of the above equation, we have

$$[s^2X(s) - s\,x(0) - \dot{x}(0)]$$
$$+\,2\zeta\,\omega_n\,[sX(s) - x(0)] + \omega_n^2X(s) = \frac{F_0}{m}\cdot\frac{1}{s}$$

A second order system subjected to a finite step cannot have any initial velocity or displacement. So, putting all initial conditions zero in the above equation, and rearranging, we have

$$X(s) = \frac{F_0}{m}\cdot\frac{1}{s\,(s^2 + 2\,\zeta\omega_n s - \omega_n^2)} \qquad (9.4.2)$$

The inverse transform of the above equation cannot be obtained straightaway from the tables. Hence splitting the right hand side into partial fractions, we have

$$X(s) = \frac{F_0}{m}\cdot\frac{1}{\omega_n^2}\left[\frac{1}{s} - \frac{s - 2\,\zeta\omega_n}{s^2 + 2\zeta\omega_n s - \omega_n^2}\right]$$

The right hand expression in the bracket above is still not invertable directly. Assuming an underdamped system, *i.e.* $\zeta < 1$, the above equation is written as follows :

$$X(s) = \frac{F_0}{m} \cdot \frac{1}{\omega_n^2} \left[\frac{1}{s} - \frac{(s+\zeta\omega_n)}{(s+\zeta\omega_n)^2 + (\sqrt{1-\zeta^2}\,\omega_n)^2} - \frac{\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right) \cdot \sqrt{1-\zeta^2}\,\omega_n}{(s+\zeta\omega_n)^2 + (\sqrt{1-\zeta^2}\,\omega_n)^2} \right] \quad (9.4.3)$$

Inverse transform of eqn. (9.4.3) can now be obtained directly from the table and is given below

$$x(t) = \frac{F_0}{m\omega_n^2} \left[1 - e^{-\zeta\omega_n t} \cos\sqrt{1-\zeta^2}\,\omega_n t - \frac{\zeta}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\sqrt{1-\zeta^2}\,\omega_n t \right]$$

Putting $m\omega_n^2 = k$, we have finally

$$x(t) = \frac{F_0}{k} \left[1 - e^{-\zeta\omega_n t} \left(\cos\sqrt{1-\zeta^2}\,\omega_n t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin\sqrt{1-\zeta^2}\,\omega_n t \right) \right] \quad (9.4.4)$$

For an undamped case, the response equation can be written from the above equation by putting $\zeta = 0$, or

$$x(t) = \frac{F_0}{k} \left[1 - \cos\omega_n t \right] \quad (9.4.5)$$

Eqn. (9.4.4) is plotted in Fig. 9.4.2 for various values of the damping factor.

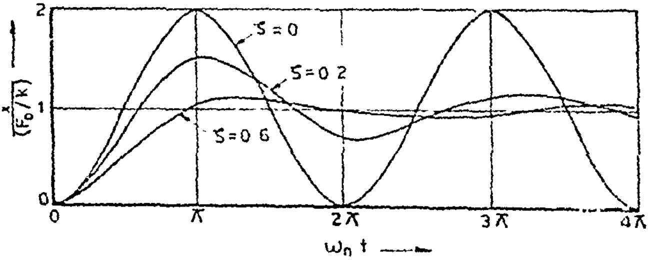


Fig. 9.4.2. System response to step input for different amounts of damping.

In case of a multi-step input to the system at different times as shown in Fig. 9.4.3 (a), this type of input may be considered to be the superposition of three steps as shown in Fig. 9.4.3 (b). Considering the system to be an undamped second order system, the differential equation of motion is written as

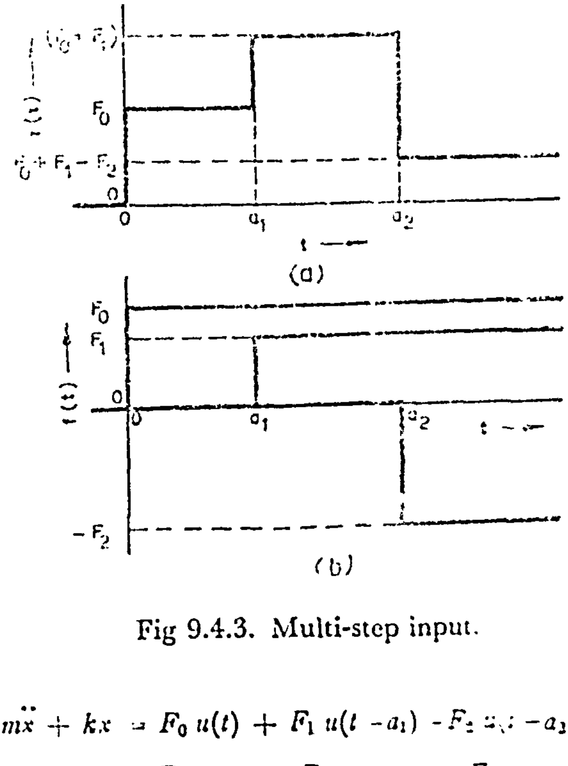


Fig 9.4.3. Multi-step input.

$$m\ddot{x} + kx = F_0\, u(t) + F_1\, u(t-a_1) - F_2\, u(t-a_2)$$

or $$\ddot{x} + \omega_n^2 x = \frac{F_0}{m}\, u(t) + \frac{F_1}{m}\, u(t-a_1) - \frac{F_2}{m}\, u(t-a_2)$$

Taking the Laplace transform and putting all initial conditions to be zero, we have

$$(s^2+\omega_n^2)\, X(s) = \frac{F_0}{m}\,\frac{1}{s} + \frac{F_1}{m}\,\frac{e^{-a_1 s}}{s} - \frac{F_2}{m}\,\frac{e^{-a_2 s}}{s}$$

or $$X(s) = \frac{F_0}{m}\,\frac{1}{s(s^2+\omega_n^2)} + \frac{F_1}{m}\,\frac{e^{-a_1 s}}{s(s^2+\omega_n^2)} - \frac{F_2}{m}\,\frac{e^{-a_2 s}}{s(s^2+\omega_n^2)}$$

In order to take the inverse transform, we take the three terms on the right one by one.

$$\frac{F_0}{m}\frac{1}{s(s^2+\omega_n^2)}=\frac{F_0}{m\omega_n^2}\left[\frac{1}{s}-\frac{s}{s^2+\omega_n^2}\right]$$

Thus the inverse transform of the first term is

$$=\frac{F_0}{m\omega_n^2}\left[1-\cos\omega_n t\right]$$

$$\text{or}\quad =\frac{F_0}{k}\left[1-\cos\omega_n t\right]$$

The inverse transform of the second term is similar to that of the first except that it is delayed by a time a_1. From the transform pair No. 16 of the table, its inverse transform is

$$\frac{F_1}{k}\left[1-\cos\omega_n(t-a_1)\right]u(t-a_1)$$

and similarly for the third term.

Thus the final inverse transform of eqn. (9.4.6) is given by

$$x(t)=\frac{F_0}{k}\left[1-\cos\omega_n t\right]u(t)+\frac{F_1}{k}\left[1-\cos\omega_n(t-a_1)\right]$$
$$\times u(t-a_1)-\frac{F_2}{k}\left[1-\cos\omega_n(t-a_2)\right]u(t-a_2) \qquad (9.4.7)$$

The inclusion of the terms $u(t)$, $u(t-a_1)$ and $u(t-a_2)$ mean that when $t \leqslant a_1$, only the first term would be considered and when $a_1 < t \leqslant a_2$, the first two terms would be considered and so on.

Illustrative Example 9.4.1

A trailer being pulled at a high speed, hits a h cm high curb. Considering the trailer to be a single degree freedom spring-mass system, analyse the system for its response.

Solution

Consider the system as shown in Fig. 9.4.4. Hitting the curb while travelling at a high speed is equivalent to a

displacement step to the base of the system. The differential equation of motion for the system may be written as

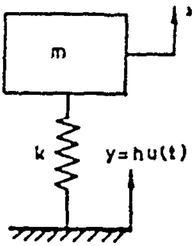


Fig. 9.4.4. A displacement step input to the base of a spring mass system.

$$m\ddot{x} + kx = ky$$
$$= k\,h\,u(t)$$

or $\ddot{x} + \omega_n{}^2 x = \omega_n{}^2 h\, u(t)$

The solution of the above equation can be easily obtained by comparing it with eqn. (9.4.1) and its solution eqn. (9.4.5) for zero damping. Hence

$$x(t) = h\,(1 - \cos \omega_n t)$$

The maximum acceleration to which the trailer is subjected is

$$\ddot{x}_{max} = \omega_n{}^2 h$$

The trailer as well as any instruments rigidly fixed to it must be strong enough to stand this maximum acceleration.

Ans.

9.5 Response to a pulse input.

Pulse applications in engineering practice are very common. An explosion occuring on a system with a comparatively larger natural period will be an impulse while the same explosion occuring on a system with a smaller natural period will be a pulse. In this section two important types of pulses, rectangular and half sinusoidal, are treated. The method lends itself to the analysis of any type of pulse for which a mathematical equation can be written.

he vibratory systems considered in this section have been en as undamped systems to make the response equations npler. Further, since most physical systems are lightly amped and in most cases we are interested in maximum splacements and accelerations, we will be slightly erring on ne safer side in neglecting the small amount of damping.

9.5A Rectangular pulse. Consider a spring-mass syster subjected to a rectangular pulse of height F_0 and duration as shown in Fig. 9.5.1. The response equation can be written directly by comparing the response of the system to a multi-step

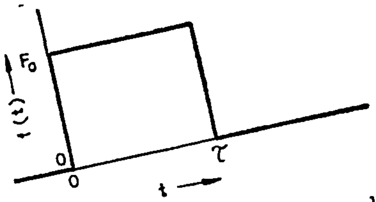


Fig. 9.5.1. A rectangular pulse.

input as given in eqn. (9.4.7) by considering in this cas two equal and opposite steps, one at $t = 0$ and the other $t = \tau$. Therefore

$$x(t) = \frac{F_0}{k}\Big[1 - \cos \omega_n t\Big]u(t) - \frac{F_0}{k}\Big[1 - \cos\omega_n(t-\tau)\Big]u$$

(9

The above equation can be written as the followi equations

$$x(t) = \frac{F_0}{k}\,[1-\cos\omega_n t] \quad \text{for } 0<t \leqslant \tau$$

$$(t) = \frac{F_0}{k}\,[\cos\omega_n(t-\tau)-\cos\omega_n t] \quad \text{for } t>\tau$$

Fig. 9.5.2 shows the response of systems with different natural period $\frac{2\pi}{\omega_n}$, to a rectangular pulse of duration τ.

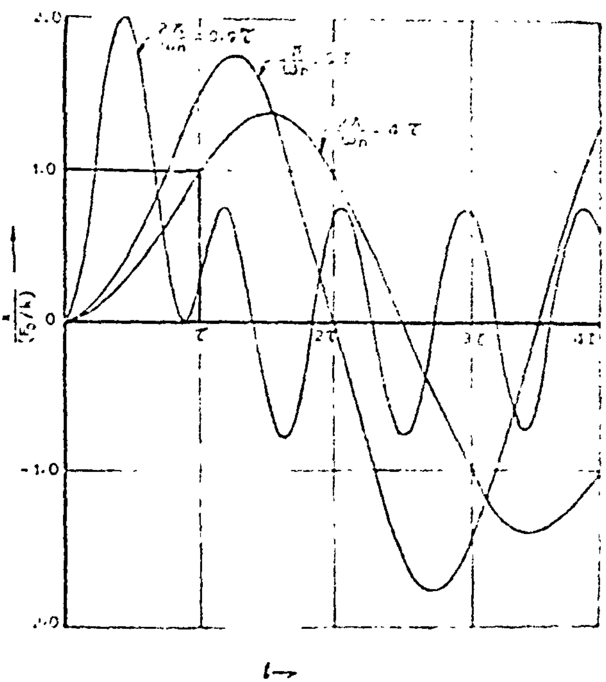


Fig. 9.5.2. Response of systems with different natural periods to a rectanguler pulse.

9.5 B Half sinusoidal pulse. A half sine pulse of duration τ shown in Fig. 9.5.3 (a) can be made from two identical sine waves one starting at zero time and the other starting at time $t = \tau$. The function representing the half sine pulse, therefore, is

$$F_0 \sin \frac{\pi t}{\tau} \ u(t) + F_0 \sin \frac{\pi(t-\tau)}{\tau} \ u(t-\tau).$$

If a spring-mass system is subjected to this pulse, the differential equation of motion of the mass is given by

$$m\ddot{x} + kx = F_0 \sin \frac{\pi t}{\tau} \ u(t) + F_0 \sin \frac{\pi(t-\tau)}{\tau} \ u(t-\tau)$$

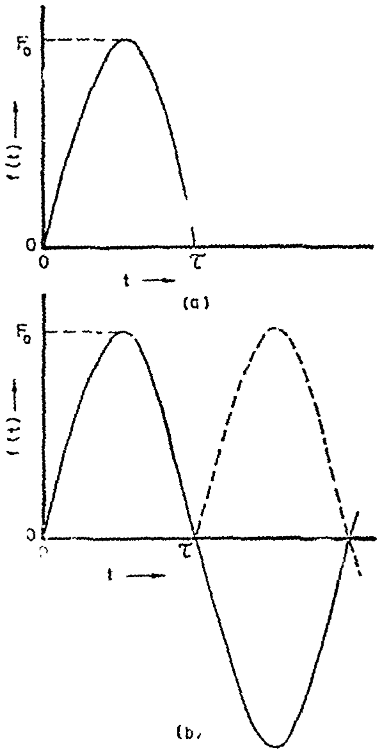


Fig. 9.5.3. A half sine pulse made from superposition of two identical sine waves.

Dividing by m throughout, taking the Laplace transform and putting all initial conditions zero, we have

$$(s^2+\omega_n^2)\ X(s) = \frac{F_0}{m}\left[\frac{(\pi/\tau)}{s^2+(\pi/\tau)^2} + \frac{(\pi/\tau)e^{-\tau s}}{s^2+(\pi/\tau)^2}\right]$$

or

$$X(s) = \frac{F_0\pi}{m\tau}\left[\frac{1}{(s^2+\omega_n^2)\ [s^2+(\pi/\tau)^2]} + \frac{e^{-\tau s}}{(s^2+\omega_n^2)\ [s^2+(\pi/\tau)^2]}\right] \tag{9.5.3}$$

The first term in the bracket on the right of the above equation can be written as

$$\frac{1}{(s^2+\omega_n^2)\ [s^2+(\pi/\tau)^2]} = \frac{1}{[(\pi/\tau)^2-\omega_n^2]}\left[\frac{1}{s^2+\omega_n^2} - \frac{1}{s^2+(\pi/\tau)^2}\right]$$

Its inverse transform is

$$\frac{1}{[(\pi/\tau)^2-\omega_n^2]}\left[\frac{1}{\omega_n}\sin\omega_n t-\frac{\tau}{\pi}\sin\frac{\pi}{\tau}t\right]$$

The inverse transform of the second term in the bracket on the right of eqn (9.5.3) can be written directly now and is

$$\frac{1}{[(\pi/\tau)^2-\omega_n^2]}\left[\frac{1}{\omega_n}\sin\omega_n(t-\tau)-\frac{\tau}{\pi}\sin\frac{\pi}{\tau}(t-\tau)\right]u(t-\tau)$$

Therefore the complete inverse transform of eqn. (9.5.3) is

$$x(t)=\frac{F_0}{m}\cdot\frac{\pi}{\tau}\cdot\frac{1}{[(\pi/\tau)^2-\omega_n^2]}\left\{\left[\frac{1}{\omega_n}\sin\omega_n t-\frac{\tau}{\pi}\sin\frac{\pi}{\tau}t\right]u(t)+\left[\frac{1}{\omega_n}\sin\omega_n(t-\tau)-\frac{\tau}{\pi}\sin\frac{\pi}{\tau}(t-\tau)\right]u(t-\tau)\right\}$$

Rearranging, the above equation can be written as

$$x(t)=\frac{F_0}{k}\cdot\frac{1}{(\pi/\omega_n\tau)-(\omega_n\tau/\pi)}\left\{\left[\sin\omega_n t-\frac{\omega_n\tau}{\pi}\sin\frac{\pi}{\tau}t\right]u(t)+\left[\sin\omega_n(t-\tau)-\frac{\omega_n\tau}{\pi}\sin\frac{\pi}{\tau}(t-\tau)\right]u(t-\tau)\right\} \tag{9.5.4}$$

It is left as an exercise for the student to plot the response of systems with different natural periods $2\pi/\omega_n$, to a half sine pulse of duration τ.

Illustrative Example 9.5.1

Find the response of a second order undamped system to a trianguler pulse force as shown in Fig. 9.5.4 (a).

Solution

The triangular pulse of Fig. 9.5.4 (a) can be broken up into

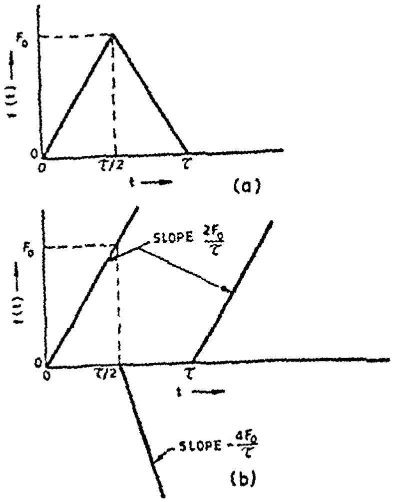


Fig. 9.5.4. A triangular pulse made from superposition of three ramp functions.

three ramp functions of slopes $\frac{2F_0}{\tau}$, $\frac{-4F_0}{\tau}$ and $\frac{2F_0}{\tau}$ starting at times 0, $\tau/2$ and τ respectively.

Therefore, the expression for the given pulse can mathematically be put down as

$$f(t) = \frac{2F_0}{\tau} \; t\,u(t) - \frac{4F_0}{\tau}\,(t-\tau/2)\;\; u(t-\tau/2)$$

$$+\frac{2F_0}{\tau}\;(t-\tau)\;u(t-\tau).$$

The differential equation of motion can now be written as

$$m\ddot{x} + kx = \frac{2F_0}{\tau}\Big[t\,u(t) - 2\,(t-\tau/2)\,u(t-\tau/2)$$

$$+ (t-\tau)\,u(t-\tau)\Big] \tag{9.5.5}$$

Dividing the above equation throughout by m, taking the Laplace transform and putting all initial conditions zero, we have

$$(s^2+\omega_n^2)\,X(s) = \frac{2F_0}{m\tau}\left[\frac{1}{s^2} - \frac{2e^{-(\tau/2)s}}{s^2} + \frac{e^{-\tau s}}{s^2}\right]$$

or
$$X(s) = \frac{2F_0}{m\tau}\left[\frac{1}{s^2(s^2+\omega_n^2)} - \frac{2e^{-(\tau/2)s}}{s^2(s^2+\omega_n^2)}\right.$$
$$\left. + \frac{e^{-\tau s}}{s^2(s^2+\omega_n^2)}\right] \qquad (9.5.6)$$

The first term in the bracket on the right of the above equation can be written as

$$\frac{1}{s^2(s^2+\omega_n^2)} = \frac{1}{\omega_n^2}\left[\frac{1}{s^2} - \frac{1}{s^2+\omega_n^2}\right]$$

Its inverse transform is

$$\frac{1}{\omega_n^2}\left[t - \frac{1}{\omega_n}\sin\omega_n t\right]$$

Similarly for the second term

$$\frac{-2e^{-(\tau/2)s}}{s^2(s^2+\omega_n^2)} = -\frac{2}{\omega_n^2}\left[\frac{e^{-(\tau/2)s}}{s^2} - \frac{e^{-(\tau/2)s}}{s^2+\omega_n^2}\right]$$

Its inverse transform is

$$-\frac{2}{\omega_n^2}\left[(t-\tau/2) - \frac{1}{\omega_n}\sin\omega_n(t-\tau/2)\right]u(t-\tau/2)$$

and so on for the third term.

Thus the complete inverse transform of eqn. (9.5.6) is

$$x(t) = \frac{2F_0}{k\tau}\left\{\left[t - \frac{1}{\omega_n}\sin\omega_n t\right]u(t)\right.$$
$$-2\left[(t-\tau/2) - \frac{1}{\omega_n}\sin\omega_n(t-\tau/2)\right]u(t-\tau/2)$$
$$\left.+\left[(t-\tau) - \frac{1}{\omega_n}\sin\omega_n(t-\tau)\right]u(t-\tau)\right\}$$

which is the required response equation. **Ans.**

Illustrative Example 9.5.2

Obtain the response of a second order undamped system to to an arbitrary forcing function by the extension of the method used for response to a step input.

Solution

We may consider the effect of an extremely small change in the force between times $t = \xi$ and $t = \xi + d\xi$ as that of a constant force $f'(\xi)d\xi$ suddenly applied as a step force at time $t = \xi$ (see Fig. 9.5.5). From the response eqn. (9.4.5) to a step

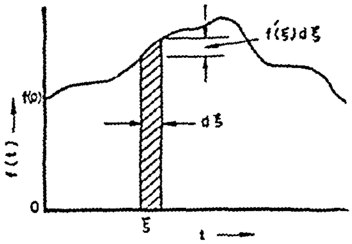


Fig. 9.5.5—An arbitrary forcing function.

input, we can write the response equation for the present case as

$$x(t) = \frac{f(0)}{k}\left[1 - \cos\omega_n t\right] + \int_0^t \frac{f'(\xi)\,d\xi}{k}[1 - \cos\omega_n(t-\xi)]$$

$$\text{or} \quad x(t) = \frac{f(0)}{k}[1 - \cos\omega_n t] + \frac{1}{k}\int_0^t f'(\xi)[1 - \cos\omega_n(t-\xi)]d\xi$$

Integrating by parts the expression within the integral sign in the above equation and simplifying, we have

$$x(t) = \frac{\omega_n}{k}\left[\sin\omega_n t\int_0^t f(\xi)\cos\omega_n\xi\,d\xi - \cos\omega_n t\int_0^t f(\xi)\sin\omega_n\xi\,d\xi\right] \tag{9.5.7}$$

Equation (9.5.7) is a general response equation to any type of forcing function. The solutions to all the types of pulses discussed earlier in this section can also be obtained from the response equation (9.5.7).

9.6 Phase plane method.

A spring mass system with initial conditions X_0 and V_0, has its differential equation for free vibrations written as

$$\ddot{x} + \omega_n^2 x = 0$$

Its solution may be written as

$$x = A \sin(\omega_n t + \phi) \tag{9.6.1}$$

where $$A = \sqrt{X_0^2 + \frac{V_0^2}{\omega_n^2}}$$

and $$\phi = \tan^{-1}\left(\frac{\omega_n X_0}{V_0}\right)$$

Differentiating equation (9.6.1) for velocity, we have

$$\dot{x} = A\omega_n \cos(\omega_n t + \phi)$$

or $$\frac{\dot{x}}{\omega_n} = A \cos(\omega_n t + \phi) \tag{9 6.2}$$

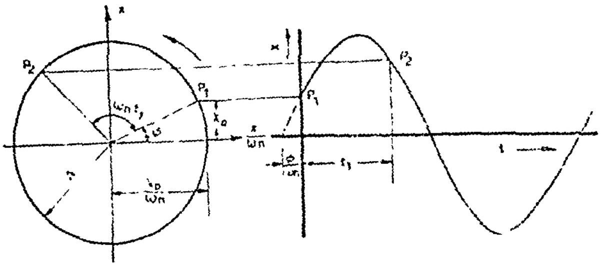
Squaring and adding eqns. (9.6.1) and (9.6.2), we have

$$x^2 + \left(\frac{\dot{x}}{\omega_n}\right)^2 = A^2 \tag{9.6.3}$$

The above equation is a circle in a plane with the coordinate axes x and $\frac{\dot{x}}{\omega_n}$. Its radius is A and centre at the origin. This is shown in Fig. 9.6.1 (a). The starting point on this displacement velocity plot is marked P_1. At t_1 seconds later the displacement and velocity of the system are represented by point P_2 where $\angle P_1 O P_2 = \omega_n t_1$ radians. From this diagram, the displacement and velocity phase of the motion are available from the single point which corresponds to a particular time. This is the phase plane plot. The horizontal projection of the phase trajectory on a time base gives the displacement-time plot of the motion and is shown in Fig. 9.6.1 (b). Similarly the vertical projection on the time base will give the velocity-time plot of the motion.

It may be noted that the centre of the phase trajectory always lies on the x-axis at a distance equal to the static equilibrium

displacement of the system. In the case discussed the static equilibrium displacement was zero and therefore the centre of the circle was located at the origin. In case of a step force input F_0, the static equilibrium position suddenly changes through a distance F_0/k. Thus the phase plane plot for such a motion will be a circle whose centre lies F_0/k above the centre. The radius of this circle will be $-F_0/k$ so that the trajectory starts from the origin corresponding to zero initial conditions.



(a) (b)

Fig. 9.6.1. Phase plane plot and displacement time plot.

The use of the phase plane method is illustrated by the following examples for systems subjected to multiple steps.

Illustrative Example 9.6.1

Draw the phase plane plot and the displecement-time plot for a spring-mass system subjected to a rectangular pluse of duration τ.

Solution

Since the initial conditions are zero, the phase trajectory starts from the origin. For the duration of the rectangular pulse force from $t = 0$ to $t = \tau$, the static equilibrium position of the mass is at O (see Fig. 9.6.2 (a), such that $P_1O = F_0/k$, where F_0 is the pulse height of the forcing function and k the spring constant. With centre O and radius $= OP_1$, draw an arc P_1P_3 locating it within an angular region of $\omega_n\tau$ radians, where τ is the duration of the pulse. The phase trajectory therefore in the pulse duration is P_1P_2. At the end of this duration the forcing pulse finishes. The force acting on the system beyond this time is zero. The static equilibrium position of the mass at the end of the pulse shifts to the origin

suddenly. Therefore the centre of the phase trajectory at the end of the pulse also suddenly shifts to the origin. The final conditions of motion at the end of the pulse, represented by P_2 become the initial conditions for the next era. Since the force is zero indefinitely after the pulse finishes, the phase trajectory in this era starting from P_2, is a circle with centre at the origin and radius equal to P_1P_2.

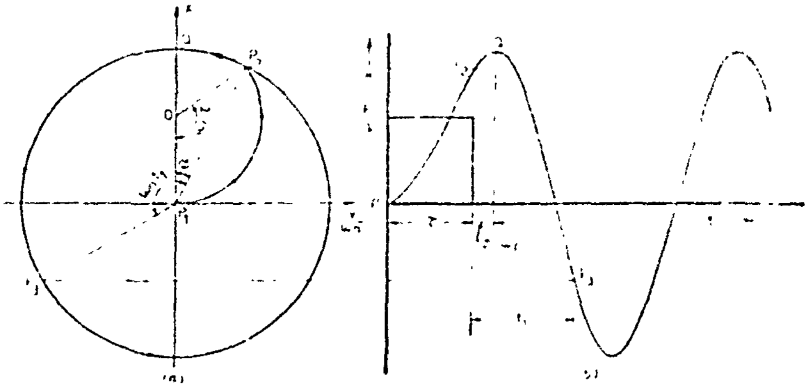


Fig. 9.6.2. Phase plane plot and displacement-time plot for a system subjected to a pulse.

At any time t_1 after the duration of the pulse, the system coordinates are represented by the point P_3 where $\angle P_2P_1P_3 = \omega_n t_1$ radians.

The displacement-time plot has been shown in Fig. 9.6.2 (b), on the right of the phase plane plot. This has been done by horizontal projection of the phase plane plot on the time base.

The maximum displacement point Q in the phase plane plot occurs at an angular distance of ϕ radians from the end of the pulse. Correspondingly, in the displacement-time plot, the equivalent time has been shown as ϕ/ω_n. **Ans.**

Illustrative Example 9.6.2.

The natural period of a spring-mass system is T sec. It is subjected to an irregular pulse shown in Fig. 9.6.3. Taking the dotted steps as an approximate equivalent of the original pulse, plot the response of the system. Also find the maximum displacement that the system experiences.

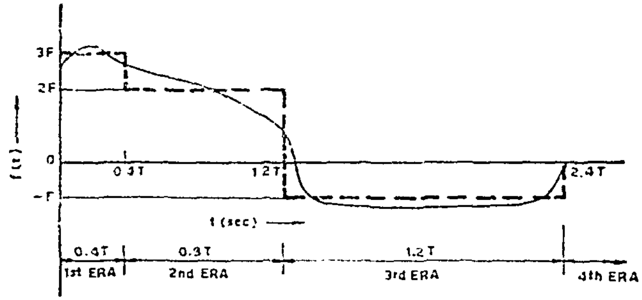


Fig. 9.6.3. An irregular pulse changed into approximate equivalent steps.

Solution

Plot the coordinate axes of the phase plane plot and the displacement—time plot, as shown in Fig. 9.6.4 (a). For the step forces shown in Fig. 9.6.3, plot the corresponding static equilibrium positions for the different durations of the steps as shown in Fig. 9.6.4 (b). The static equilibrium positions for the four steps respectively in the four eras are $3F/k$, $2F/k$, $-F/k$, and zero from the origin. These are projected on the x—axis of the phase plane as points A, B, C and O.

The phase trajectory for the first era is the arc of a circle included in an angle

$$\omega_n t_1 = \omega_n \times 0.4T = \frac{2\pi}{T} \times 0.4T = 0\ 8\pi \text{ rad} = 144°$$

The centre of this arc lies at A and its radius is AO. Thus, the arc OP_1 is the trajectory for the first era. The corresponding displacement-time plot is projected as OP_1 in Fig. 9.6.4 (b).

The motion conditions represnted by point P_1 are the end conditions for the first era and the initial conditions for the second era. The centre of the arc for trajectory in the second era lies at B and the radius of this arc is BP_1. This arc is included in an angle

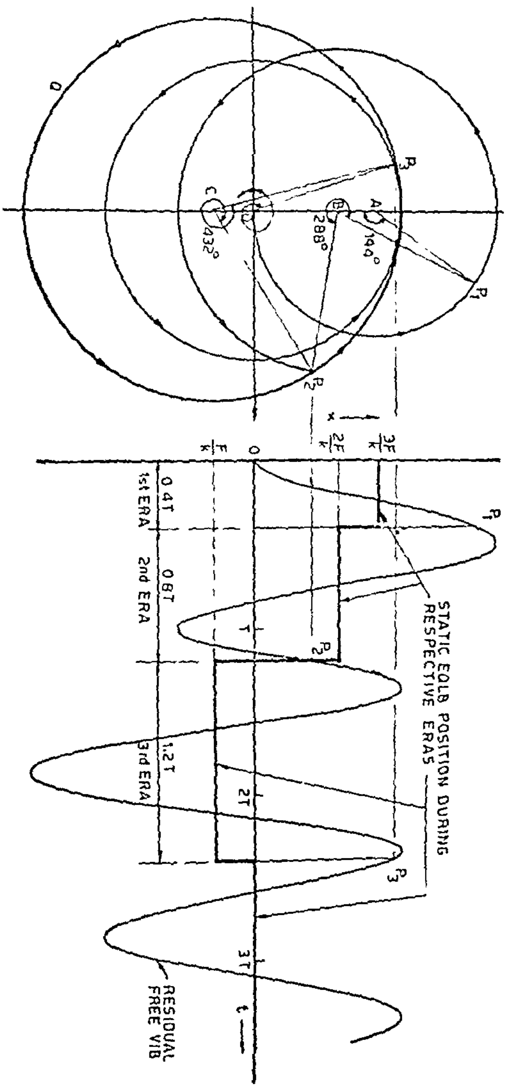
$$\omega_n t_2 = \omega_n \times 0.8T = \frac{2\pi}{T} \times 0.8T = 1.6\pi = 288°$$

Consequently this is shown as arc P_1P_2 in the phase plane plot and projected as displacement-time curve P_1P_3 in the second era.

The point P_2 in the phase plane represents the end conditions for the second era and the starting conditions for the third era for which the trajectory centre is C. The third arc is therefore drawn with C as centre and radius $= CP_2$. The angle included by this arc is

$$\omega_n t_3 = \omega_n \times 1.2T = 432°$$

This arc is a little more than a full circle and is represented



(a) (b)

Fig. 9 6.4. Phase plane plot and displacement-time plot for a system subjected to multi-step forcing function.

by $P_2P_3QP_2P_3$ in the phase plane. The correspoding curve in the displacement-time plot is shown as P_3P_3.

The point P_3 in the phase plane represents the end conditions for the third era and starting conditions for the fourth era. The centre of the trajectory for the fourth era is O, and therefore its radius $= OP_3$. This trajectory is a full circle repeating over and over again.

The maximum displacement the system experiences is at the first peak and is

$$x_{max} = 5.9 \frac{F}{k} \qquad \textbf{Ans.}$$

9.7 Shock Spectrum

The response of a spring-mass system to a particular pulse depends upon the natural frequency of the system. The plot of the maximum response of the system against the natural frequency of the system is called the *shock spectrum of the* particular disturbance. The shock spectrum shows at a glance the natural frequencies which cause large response amplitudes for the the particular disturbance.

Illustrative Example 9.7.1

Determine the shock spectrum of a rectangular pulse.

Solution

The response of the undamped spring mass system to a rectangular pulse has been dealt with in Sec. 9.5 A. We will obtain the maximas here in two stages; one, during the duration of the pulse, and the other after the pulse finishes; and call the respective spectra as the initial and the residual shock spectrum.

Initial shock spectrum. Eqn. (9.5.2) gives

$$x(t) = \frac{F_0}{k}\left[1-\cos\omega_n t\right] \qquad \text{for } 0 < t \leqslant \tau$$

The maximum value of $x(t)$ is straightaway known to be $\frac{2F_0}{k}$ at $t = \pi/\omega_n$

But since $t \leqslant \tau$, we have

$$\pi/\omega_n \leqslant \tau$$

or $$\omega_n \geq \frac{\pi}{\tau}$$

That is, for natural frequency given by

$$\omega_n \tau \geq \pi \tag{9.7.1}$$

the maximum value of the response is

$$\frac{x_{max}}{(F_0/k)} = 2 \tag{9.7.2}$$

For $\omega_n \tau < \pi$, there is no maximain the duration of the pulse. This is shown by the dotted line in Fig. 9.7.1.

Residual shock spectrum. The response after the duration of the pulse is given by eqn. (9.5·2), as

$$x(t) = \frac{F_0}{k}\left[\cos \omega_n (t-\tau) - \cos \omega_n t\right] \qquad \text{for } t > \tau$$

The maximum value of $x(t)$ is obtained by differentiating the above expression and equating to zero. Accordingly,

$$-\sin \omega_n (t-\tau) + \sin \omega_n t = 0$$

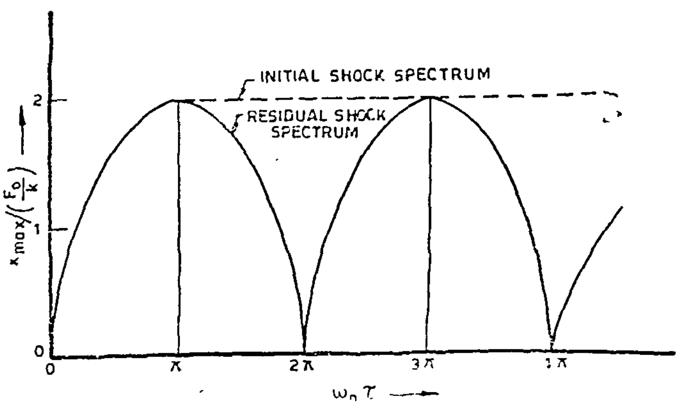


Fig. 9.7.1. Shock spectrum for a rectangular pulse.

The solution of the above equation is obtained as

$$\omega_n(t-\tau) + \omega_n t = (2n+1)\pi \qquad \text{for } n=0, 1,2,\ldots\ldots$$ (9.7.3)

or $$t = \frac{(2n+1)\pi + \omega_n \tau}{2\omega_n}$$

Substituting for $\omega_n(t-\tau)$ from eqn. (9.7.3a) in the response equation, we have

$$x(t) = \frac{F_0}{k}\left\{\cos\left[(2n+1)\pi-\omega_n t\right]-\cos\omega_n t\right\}$$

or $x(t) = -\dfrac{2F_0}{k}\cos\omega_n t$

Substituting the value of t from eqn. (9.7.3b) in the above equation, we have

$$x_{max} = \left| -\frac{2F_0}{k}\cos\left[\frac{(2n+1)\pi+\omega_n\tau}{2}\right]\right|$$

$$= \left| 2\frac{F_0}{k}\cos\left(\frac{\pi+\omega_n\tau}{2}\right)\right|$$

or $$\frac{x_{max}}{(F_0/k)} = \left| 2\sin\frac{\omega_n\tau}{2}\right| \tag{9.7.4}$$

Eqn. 9.7.4 is plotted as full line curve in Fig. 9.7.1.

Depending upon the application of the information obtained, either the residual shock spectrum or the complete shock spectrum may be used.

PROBLEMS FOR PRACTICE

9.1 Show that Laplace transform of $f(t) = t\sin\omega t$ is given by

$$F(s) = \frac{2\omega s}{(s^2+\omega^2)^2}$$

9.2 Obtain the Laplace transform of a pulse shown in Fig. P. 9.2, and show that it is given by

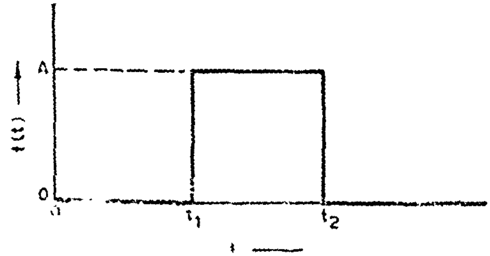


Fig. P. 9.2.

$$F(s) = \frac{A}{s}\left[e^{-a_1 s} - e^{-a_2 s}\right]$$

9.3 If $F(s) = \dfrac{10s + 24}{s^2 + 4s + 8}$, find $f(t)$.

9.4 Show that the inverse transform

of $F(s) = \dfrac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ for $(\zeta < 1)$

is $f(t) = \dfrac{1}{\omega_n\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_n t} \sin \omega_n\sqrt{1-\zeta^2}\, t$

9.5 For a vibratory system subjected to an impulse, plot the maximum peak displacement against the damping ratio.

9.6 A mass m_1 is dropped on to a vibratory system from a height h as shown in Fig. P.9.6. Assuming that the two masses adhere together after the impact, obtain the response of the combined system after the impact. Plot it against time.

9.7 The light base of the spring-mass-dashpot sysytem is dropped on to a hard floor through a height h = 1.6 metres as shown in Fig. P. 9.7. If W = 18 kg, k = 2 kg/cm and c = 0·1 kg-sec/cm, find the subsequent motion of the mass. What is its maximum acceleration ?

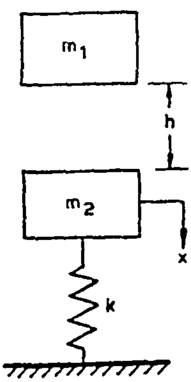


Fig. P. 9.6.

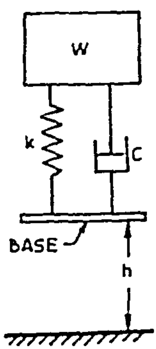


Fig. P. 9.7.

9.8 An undamped spring mass system is subjected to a saw-tooth pulse shown in Fig. P. 9.8. Obtain the response equation.

9.9 Repeat problem 9.8 for the case when the pulse is as shown in Fig. P. 9.9.

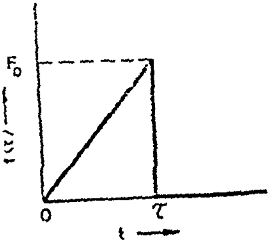


Fig. P. 9.8.

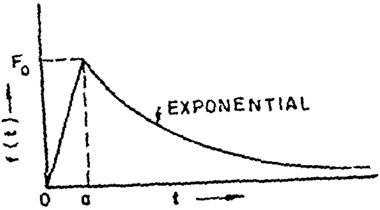


Fig. P. 9.9.

9.10 Obtain the response equation for an undamped system subjected to

(i) rectangular pulse,

(ii) half sine pulse,

by the integral method of Illustrative Example 9.5.2.

9.11 A system having a natural frequency of 10 c.p.s. is subjected to an explosive type of input which has been changed to equivalent approximate steps shown in Fig. P. 9.11.

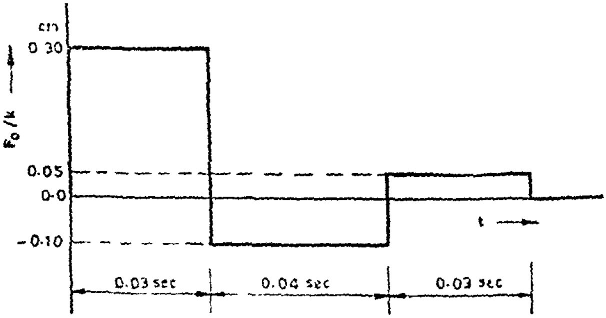


Fig. P. 9.11.

Determine the phase plane plot and the displacement-time plot. What is the maximum displacement the system experiences.

9.12 An undamped vibratory system having $W = 50$ kg and $k = 30$ kg/cm is acted upon by a forcing function

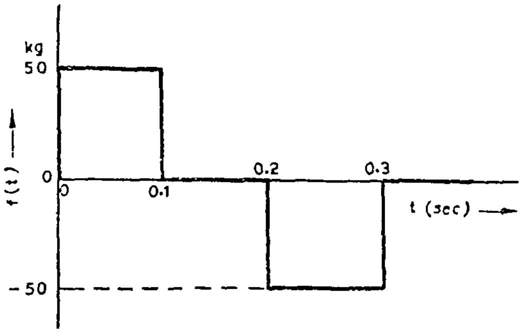


Fig. P. 9.12.

shown in Fig. P. 9.12. Obtain the phase plane plot and displacement-time plot of the motion of the system.

9.13 A rectangular pulse of constant area $\triangle$ but variable duration acts on an undamped spring-mass system. Plot the amplitude of vibration of the system after the expiry of the pulse against the duration of the pulse.

9.14 A ramp-step forcing function shown in Fig. P. 9.14 acts on a spring mass system. Obtain the shock spectrum of this forcing function.

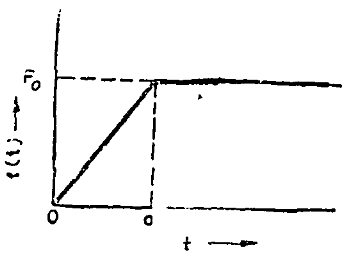


Fig. P. 9.14

9.15 Obtain the initial and residual shock spectra of a half sine pulse. What is the total shock spectrum ?

CHAPTER 10

NON-LINEAR VIBRATIONS

10.1 Introduction.

Most physical systems can be represented by linear differential equations, the types of which have been dealt with in the previous chapters. A general equation of this type is

$$m\ddot{x} + c\dot{x} + kx = F(t) \qquad (10.1.1)$$

In this equation which is for a linear system, the inertia force, the damping force and the spring force are linear functions of $\ddot{x}$, $\dot{x}$ and x respectively. This is not so in the case of non-linear systems. A general equation for a non linear system is

$$m\ddot{x} + \phi(\dot{x}) + f(x) = F(t) \qquad (10.1.2)$$

in which the damping force and the spring force are not linear functions of $\dot{x}$ and x. There are quite some physical systems which have non-linear spring and damping characteritics. Rubber springs and other similar isolators have spring stiffness which increases with amplitude. Cast iron and concrete have spring stiffness which decreases with amplitude. Examples of nonlinear damping are dry friction damping and material damping. Even socalled linear systems tend to become non linear with larger amplitudes of vibration. The anaiysis of non linear systems is comperatively difficult. In certain cases there is no exact solution.

One major difference between the linear and non linear systems is that the *law of superposition* does not hold good for non-linear systems. Mathematically speaking, if x_1 is a solution of

$$m\ddot{x} + c\dot{x} + kx = F_1(t)$$

and x_2 is a solution of

$$m\ddot{x} + c\dot{x} + kx = F_2(t)$$

then $(x_1 + x_2)$ is a solution of

$$m\ddot{x} + c\dot{x} + kx = F_1(t) + F_2(t).$$

This is not so in the case of non iinear systems. Even for the case of free vibration any two known solutions of the non linear system cannot be superimposed to obtain a general solution.

10.2 Examples of non-linear systems.

Fig. 10.2.1 (a) shows a system with an abrupt nonlinearity in spring. As long as the amplitude of vibration of the amplitude of vibration of the mass is less than or equal to d, the system behaves in a linear manner. When the amplitude exceeds d, there is an abrupt change in spring stiffness. The spring force versus displacement characteristic of the system is given in Fig. 10.2.1 (b). The system differential equation is

$$m\ddot{x} + f(x) = 0$$

where
$$\begin{aligned} f(x) &= k_1 x, & |x| &\leqslant d \\ &= k_1 x + k_2(x-d), & |x| &> d \end{aligned}$$

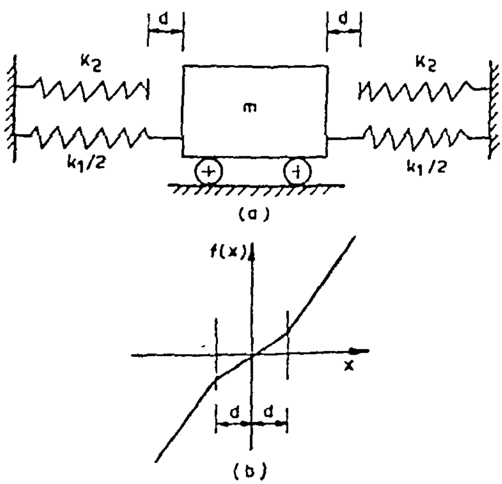


Fig. 10.2.1. An abrupt non-linearity.

second example may be considered with respect to the shown in Fig. 10.2.2 (a). A point mass m is attached mid-point of a stretched string having an initial tension

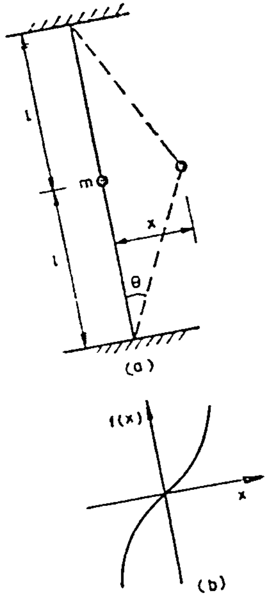


Fig. 10.2.2. Hard spring.

At any instant when the mass is displaced through a distance x, each half of the string has been extended through a distance δl, such that

$$\delta l = \sqrt{l^2 + x^2} - l \qquad (10.2$$

$$= \frac{1}{2} \frac{x^2}{l}$$

The increase in tension δT of the string at this instant du extension δl is

$$\delta T = AE \frac{\delta l}{l} = \frac{AE}{2} \frac{x^2}{l^2} \qquad (1$$

where A is the cross-sectional area of the wire and E Young's Modulus of the wire material. The total ter

the wire at this instant is $(T+\delta T)$ and the restoring spring force is given by

$$f(x) = 2(T + \delta T)\sin\theta$$
$$= (T + \delta T)\frac{x}{\sqrt{l^2+x^2}}$$

Substituting for δT from eqn (10.2.2), we have

$$f(x) = 2\left(T + \frac{AE}{2}\frac{x^2}{l^2}\right)\frac{x}{\sqrt{l^2+x^2}}$$
$$= 2\left(T + \frac{AE}{2}\frac{x^2}{l^2}\right)\frac{x}{l}\left(1 + \frac{x^2}{l^2}\right)^{-1/2}$$
$$= 2\left(T + \frac{AE}{2}\frac{x^2}{l^2}\right)\frac{x}{l}\left(1 - \frac{x^2}{2l^2}\cdots\cdots\right)$$
$$= \frac{2x}{l}\left[T + \tfrac{1}{2}(AE - T)\frac{x^2}{l^2}\cdots\cdots\right]$$

Now T is very small as compared to AE and so, is neglected. Also neglecting higher powers of x, we have

$$f(x) = \frac{2x}{l}\left[T + \tfrac{1}{2}AE\,\frac{x^2}{l}\right]$$

or $$f(x) = \frac{2T}{l}x + \frac{AE}{l^3}x^3 \qquad (10.2.3)$$

The above equation shows the hardening spring nonlinearity. The spring stiffness in this case increases with the displacement. A sketch of eqn (10.2.3) is shown in Fig. 10.2.2(b). If, however, the displacement x is very small, then the term containing x^3 can be neglected and the system reduces to a linear system.

The third example of the non linear spring will be taken for a simple pendulum which has soft spring characteristic, i.e. the stiffness decreases with displacement.

The differential equation for the simple pendulum of Fig. 10.2.3 (a) is given by

$$ml^2\ddot{\theta} + mgl\sin\theta = 0$$

The restoring torque, therefore, is given by

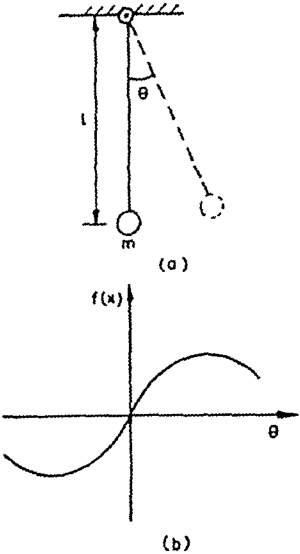


Fig. 10.2.3. Soft spring.

$$
\begin{aligned}
f(x) &= mgl \sin \theta \\
&= mgl \left(\theta - \frac{\theta^3}{6} + \cdots \right)
\end{aligned}
$$

Neglecting the higher powers of θ, we have

$$f(x) = mgl\,\theta - \frac{mgl}{6}\,\theta^3 \qquad (10.2.4)$$

The above equation shows the softening spring non-linearity. The spring stiffness in this case decreases with displacement. A sketch of equation (10.2.4) is shown in Fig. 10.2.3 (b). If, again, the displacement θ is taken small, then the term containing θ^3 can be neglected and the system reduces to a linear one.

It may be mentioned here, that unlike linear systems, the frequency of a non linear system is not constant. For a hard spring, the frequency increases with displacement since the effective stiffness increases ; and similarly for a soft spring, the frequency decreases with displacement since the effective stiffness decreases.

Illustrative Example 10.2.1

For the system shown in Fig. 10.2.1, find the time period per cycle as a function of amplitude of vibration. Show this in the form of a graph.

Solution

Refer to Fig. 10.2.4.

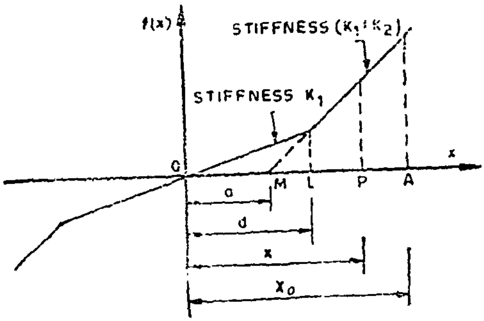


Fig. 10.2.4. Analysis of a system with abrupt non linearity.

Let the amplitude of vibration be $X_0 = OA$. Starting from A, the system moves towards left with a simple harmonic motion about point M with a natural frequency

$$\omega_1 = \sqrt{(k_1 + k_2)/m}.$$

It moves with this motion upto point L. The end conditions at point L become the initial conditions for the second era from L to O. This part of the motion is again a simple harmonic motion, but this time about point O and with a natural frequency $\omega_2 = \sqrt{k_1/m}$. If the time taken for the system from $A = L$ and from L to O respectively are t_1 and t_2, then the total period of one cycle will be $4\,(t_1 + t_2)$.

At any instant the displacement x of the mass is given by

$$x = a + (X_0 - a) \cos \omega_1 t, \qquad (x \geqslant d) \qquad (10.2.5)$$

if the time is counted from the extreme point A.

Or, $$t = \frac{1}{\omega_1} \cos^{-1} \left(\frac{x - a}{X_0 - a}\right)$$

When $x = d$, we have

$$t_1 = \frac{1}{\omega_1} \cos^{-1}\left(\frac{d-a}{X_0-a}\right) \tag{10.2.6}$$

The velocity at point L is obtained by differentiating equation (10.2.5) and substituting $t = t_1$ from eqn. (10.2.6). This is obtained as

$$\left.\begin{aligned} (\dot{x})_{x=d} &= -\omega_1 \sqrt{(X_0-a)^2-(d-a)^2} \\ \text{Also} \quad x &= d \end{aligned}\right] \tag{10.2.7}$$

Eqns. (10.2.7) give the initial conditions for the second era for which the natural frequency is ω_2.

Now, for the era L to O, starting from the point L (i.e. $t = 0$ at $x = d$), with the help of the above initial conditions the equation of motion can be obtained as

$$x = d \cos \omega_2 t - \frac{\omega_1}{\omega_2} \sqrt{(X_0-a)^2-(d-a)^2} \sin \omega_2 t \qquad (x \leqslant d)$$

The time t_2 from L to O can be obtained from the above equation by substituting zero for x. Or,

$$t_2 = \frac{1}{\omega^2} \tan^{-1}\left[\frac{d\,(\omega_2/\omega_1)}{\sqrt{(X_0-a)^2-(d-a)^2}}\right] \tag{10.2.8}$$

Hence, the time period per cycle is given by

$$\begin{aligned} \tau &= 4\,(t_1+t_2) \\ &= 4\left[\frac{1}{\omega_1} \cos^{-1}\left(\frac{d-a}{X_0-a}\right) + \right. \\ &\qquad \left. \frac{1}{\omega_2} \tan^{-1}\left\{\frac{d\,(\omega_2/\omega_1)}{\sqrt{(X_0-a)^2-(d-a)^2}}\right\}\right] \end{aligned} \tag{10.2.9}$$

The above equation is applicable for $X_0 \geqslant d$. For $X_0 \leqslant d$ the time period is constant and equal to $2\pi/\omega_2$. A sketch of the variation of τ against amplitude X_0 is given in Fig. 10.2.5.

10.3 Phase Plane.

Phase plane was introduced in Sec. 9.6 for the case of linear systems. Here we extend it for the case of non linear systems.

Consider the differential equation

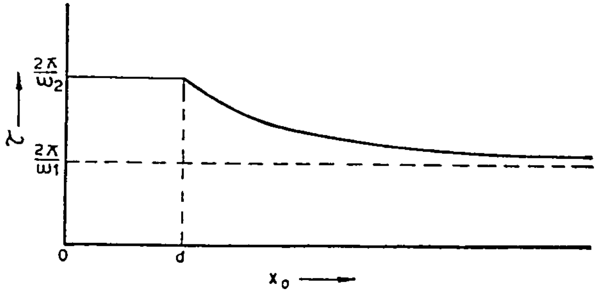


Fig. 10.2.5. Time period v/s amplitude for a system with abrupt nonlinearity.

$$m\ddot{x} + f(x) = 0 \tag{10.3.1}$$

The acceleration $\ddot{x}$ can also be written as

$$\ddot{x} = v\frac{dv}{dx}$$

where v is the velocity of the particle. Substituting it in eqn. (10.3.1), we have

$$mv\frac{dv}{dx} + f(x) = 0$$

or $$mv\,dv = -f(x)\,dx \tag{10.3.2}$$

The above equation is integrable directly. If $v = V_0$ when $x = X_0$, then the integration of eqn. (10.3.2) gives

$$\int_{V_0}^{v} mv\,dv = -\int_{X_0}^{x} f(x)\,dx \tag{10.3.2a}$$

or $$\frac{mv^2}{2} + \frac{mV_0^2}{2} = -[F(x) - F(X_0)]$$

The above equation is in accordance with the Law of of Conservation of Energy. The left hand side is the increase in kinetic energy of the system and the right hand side is the decrease in potential energy. This equation can also be written as

$$\frac{mv^2}{2} + F(x) = \frac{mV_0^2}{2} + F(X_0) \tag{10.3.3}$$

which states that the total energy of the system at any instant is equal to the total initial energy of the system. Curve in

$x-v$ plane can be drawn from eqn. (10.3.3) and this will be a curve of constant energy. A set of such curves can be drawn, each for different total energy. These curves are known as *Energy Curves* or *Integral Curves* in phase plane.

In Sec. 9.6 we had taken the phase plane with x along the ordinate and v along the abscissa, and the trajectory was always counter-clock-wise. Here, for convenience we will take x along the abscissa and v along the ordinate. The trajectories here will be clockwise.

Systems which have periodic motion would be represented on the phase plane by means of a set of closed curves, each curve for different energy of the system.

Consider the linear case when $f(x) = kx$. Eqn. (10.3.2a) then integrates to

$$\frac{mv^2}{2} - \frac{mV_0^2}{2} = -\left(\frac{kx^2}{2} - \frac{kX_0^2}{2}\right)$$

or
$$\frac{mv^2}{2} + \frac{kx^2}{2} = \frac{mV_0^2}{2} + \frac{kX_0^2}{2}$$
$$= E(\text{say}) \tag{10.3.4}$$

The phase plane trajectories are clearly a set of ellipses with the origin as the centre. The right hand side of eqn. (10.3.4) is the total initial energy of the system. As the value of this initial energy (depending upon initial conditions) increases, the size of the ellipse also increases. This set of trajectories is shown in Fig. 10.3.1.

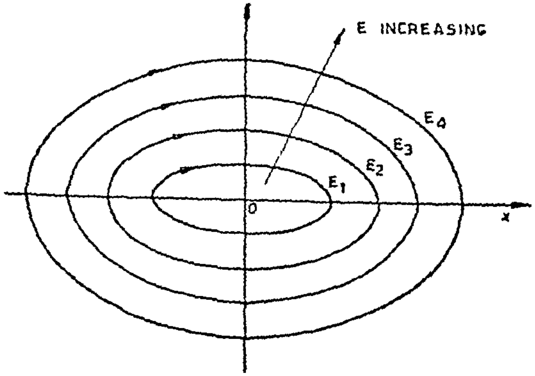


Fig. 10.3.1. Phase plane trajectories for a linear system.

In general a point $P(x, v)$ in the phase plane, called the representative point of the system, represents the state of the system at any instant t and the trajectory traced gives the history of the system. Through any and every point of the phase plane passes one and only one trajectory and thus the trajectories in the plane do not intersect one another.

With the passage of time the representative point moves along the trajectory in a clockwise direction with what is known as the *phase velocity* given by

$$u = \sqrt{\dot{x}^2 + \dot{v}^2} \tag{10.3.5}$$

The phase velocity is always non-zero except at points of equilibrium. Origin is a point of stable equilibrium.

Most non-linear equations cannot be solved explicitly. However, their phase-plane plots can be drawn graphically and these diagrams give several important conclusions regarding motions of the systems.

Consider the general case of a system with non-linear damping and non linear spring. Let the differential equation of motion be

$$m\ddot{x} + \phi(\dot{x}) + f(x) = 0 \tag{10.3.6}$$

Letting $\dot{x} = v$, the above equation can be written down in the form of the following two equations

$$\left.\begin{aligned} \frac{dx}{dt} &= v \\ \frac{dv}{dt} &= -\frac{\phi(v) + f(x)}{m} \end{aligned}\right] \tag{10.3.7}$$

Differential dt can be eliminated from eqns (10.3.7) to give

$$\frac{dv}{dx} = -\frac{\phi(v) + f(x)}{mv} \tag{10.3.8}$$

Eqn. (10.3.8) gives the slope of the trajectory at any point (x, v) in the phase plane and is useful for constructing the plots. The slope of the trajectory is directly obtainable at all points except the ones where $v = 0$ and $\phi(v) + f(x) = 0$. At these points the slope becomes indeterminate and these points are called

singular points. From eqn (10.3.7), these points correspond to the conditions $v = 0$ and $\frac{dv}{dt} = 0$, *i.e.* the points of equilibrium.

Singular points exist on x-axis ($v = 0$) where $\frac{dv}{dt} = 0$. Origin is always a singular point. Other singular points may or may not be there for the system. At singular points the phase velocity is zero.

10.3 A Method of isoclines. Consider eqn (10.3.8) giving the slope of the trajectory at any point (x, v). Curves of constant slopes given by

$$-\frac{\phi(v) + f(x)}{mv} = p \tag{10.3.9}$$

for different slopes p_1, p_2.........form a family of curves known as isoclines. All the trajectories for a particular system cut any single isocline at constant slope. Or, in other words, an isocline is the locus of points in the phase plane where the trajectories pass them with constant slope. The method of isoclines is used for constructing the trajectory by first filling the phase plane with isoclines and marking small slope lines on them, and then drawing the trajectory with the help of these directional guide lines.

For the case of a linear system, eqn. (10.3.9) is modified to

$$-\frac{cv + kx}{mv} = p$$

or $$\frac{c}{m} + \frac{k}{m}\frac{x}{v} = -p$$

or $$2\,\zeta\,\omega_n + \omega_n^2\frac{x}{v} = -p$$

For a simple case where $\zeta = 0.5$ and $\omega_n = 1$, the above equation reduces to

$$v = -\frac{x}{p+1}$$

When $p = -1$, $x = 0$ (v — axis); *i.e.* the trajectories cross the v — axis at an inclination of $-45°$ (see Fig. 10.3.2).

When $p = \infty$, $v = 0$ (x – axis); *i.e.* the trajector iescross the x – axis at an inclination of 90°. And so on for other slopes and corresponding isoclines. Fig. 10.3.2 shows a number of isoclines drawn with small lengths, of corresponding slope lines marked on them. Then, the slope lines serving as guide lines, the trajectory can be drawn from any point.

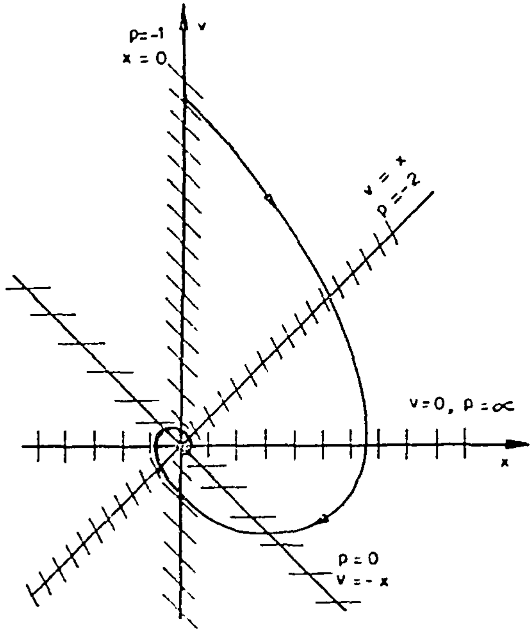


Fig. 10.3.2. Trajectory of a linear system with $\omega_n = 1$ and $\zeta = 0.5$.

Illustrative Example 10.3.1

A system with dry friction damping has its differential equation of motion given by

$$\ddot{x} + \phi(\dot{x}) + x = 0$$

where $\qquad \phi(\dot{x}) = F \qquad$ when $\dot{x}$ is $+ve$.

$\qquad\qquad = -F \qquad$ when $\dot{x}$ is $-ve$.

Obtain the trajectory of motion when the system is given an initial displacement and released.

Solution

Write the differential equation as

$$\frac{dv}{dx} = -\frac{\phi(v) + x}{v} = p \text{ (say)}$$

where $\phi(v) = F$ when v is $+ve$

$= -F$ when v is $-ve$

or,
$$v = -\frac{x}{p} - \frac{\phi(v)}{p}$$

The isoclines are straight lines. When the slope of the trajectory is p, the slope of the corresponding isocline from the above equation is $-\frac{1}{p}$. This means that the trajectories always out the isoclines at right angles. Further, the isoclines are two sets of straight lines as given below

$$v + \frac{x}{p} = \frac{-F}{p} \qquad \text{when } v \text{ is } +ve$$

$$v + \frac{x}{p} = \frac{+F}{p} \qquad \text{when } v \text{ is } -ve$$

The first set lies above the x – axis and emerge from point A where $OA = -F$ (see Fig. 10.3.3). The other set lies below the x – axis and emerge from point B where $OB = F$. The slopes marked on these isoclines are always at right angles to them. Hence in the lower half plane the trajectory is a semicircle with B as centre and in the upper half plane the trajectory is again a semicircle with A as centre. With the starting point on the x – axis the complete trajectory is shown in the figure.

10.4. Undamped free vibration with non-linear spring forces.

Let the system be represented by

$$m\ddot{x} + f(x) = 0 \qquad (10.4.1)$$

the above equation can be written as

$$mv\frac{dv}{dx} + f(x) = 0 \qquad (10.4.2)$$

Integrating eqn. (10.4.2) we have

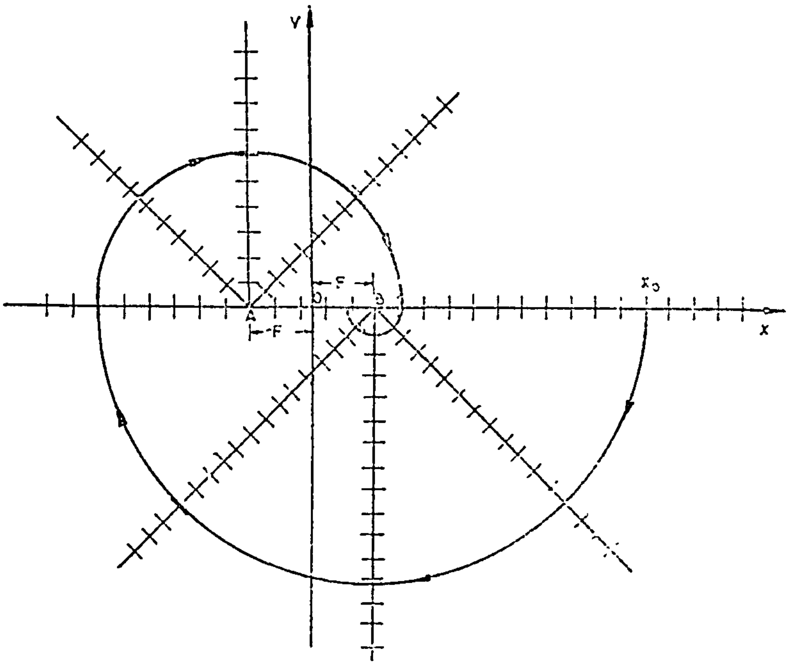


Fig. 10.3.3. Trajectory of a system having dry friction damping.

$$\frac{mv^2}{2} + F(x) = E \tag{10.4.3}$$

where $F(x)$ is the integral of $f(x)$ and so represents the potential energy of the system; and E is the total energy of the system and depends upon the initial conditions.

From eqn (10.4.3), we have

$$v = \sqrt{\frac{2}{m}\left[E - F(x)\right]} \tag{10.4.4}$$

Let the system have cubic nonlinearity represented by

$$f(x) = \alpha x + \beta x^3 \qquad (\alpha > 0) \tag{10.4.5}$$

then, $$F(x) = \frac{\alpha x^2}{2} + \frac{\beta x^4}{4} \tag{10.4.6}$$

From eqns (10.4.3) and (10.4.6), we get

$$\frac{mv^2}{2} + \frac{\alpha x^2}{2} + \frac{\beta x^4}{4} = E$$

The above equation gives the plot in the phase plane for different values of E. These are closed curves when $\beta > 0$ for any amplitude. For the case when $\beta < 1$, the phase plane plots are closed curves upto a certain amplitude and beyond that they are unstable. For the case of closed curves, the system has periodic motion. Let $x_{max} = a$ be the amplitude of vibration. When $x = x_{max} = a$, $v = 0$; eqn (10.4.7) becomes

$$\frac{\beta a^4}{4} + \frac{\alpha a^2}{2} - E = 0 \tag{10.4.8}$$

giving
$$a^2 = \frac{-\alpha + \sqrt{\alpha^2 + 4\beta E}}{\beta} \tag{10.4.9}$$

In the above equation only positive sign before the radical is applicable whether $\beta >$ or < 0.

Writing v as $\frac{dx}{dt}$, eqn (10.4.4) is written as

$$dt = \frac{dx}{\sqrt{\frac{2}{m}\left[E - F(x)\right]}} = \sqrt{\frac{m}{2}} \cdot \frac{dx}{\sqrt{E - F(x)}} \tag{10.4.10}$$

For the case of periodic motion of amplitude a, the time period per cycle of vibration is given by integrating the above equation over a quarter of a cycle and multiplying it by 4. Thus,

$$\tau = 4\sqrt{\frac{m}{2}} \int_0^a \frac{dx}{\sqrt{E - F(x)}} \tag{10.4.11}$$

Substituting for $F(x)$ from eqn (10.4.6) in the above equation, we have

$$\tau = 4\sqrt{\frac{m}{2}} \int_0^a \frac{dx}{\sqrt{E - \frac{\alpha x^2}{2} - \frac{\beta x^4}{4}}} \tag{10.4.12}$$

The quadratic expression in the radical sign of the above equation can easily be factorized since comparing it with eqn (10.4.8) it shows that $(a^2 - x^2)$ is one of the factors of this quadratic function.

Therefore, let

$$\left(E - \frac{\alpha x^2}{2} - \frac{\beta x^4}{4}\right) \equiv \left(a^2 - x^2\right)\left(b^2 + \frac{\beta x^2}{4}\right) \qquad (10.4.13)$$

Comparing the coefficients of x^2 and the constant terms in the above equation, we have

$$\left.\begin{aligned} E &= a^2 b^2 \\ -\frac{\alpha}{2} &= \frac{\beta a^2}{4} - b^2 \end{aligned}\right] \qquad (10.4.14)$$

The second of the above equations gives

$$b^2 = \frac{\beta a^2}{4} + \frac{\alpha}{2} \qquad (10.4.15)$$

Eqn. (10.4.12) can now be written with the help of eqns (10.4.13) and (10.4.15) as

$$\tau = 4\sqrt{\frac{m}{2}} \int_0^a \frac{dx}{\sqrt{(a^2 - x^2)\left(\frac{\beta a^2}{4} + \frac{\alpha}{2} + \frac{\beta x^2}{4}\right)}}$$

or $$\tau = 8\sqrt{\frac{m}{2}} \int_0^a \frac{dx}{\sqrt{(a^2 - x^2)(\beta a^2 + 2\alpha + \beta x^2)}} \qquad (10.4.16)$$

It is possible to convert the above complex integral into complete elliptic integral of the first kind the value of which can be obtained from tables of Elliptic Integrals. Disscussed below are the two cases falling under this type of non-linearity.

10.4A Hard spring ($\beta > 0$). Making a substitution

$$x = a \cos \phi$$

in eqn (10.4.16), we have

$$\tau = 8\sqrt{\frac{m}{2}} \int_{\pi/2}^{0} \frac{-a \sin\phi \, d\phi}{\sqrt{(a^2 - a^2 \cos^2 \phi)(\beta a^2 + 2\alpha + \beta a^2 \cos^2 \phi)}}$$

$$= 8\sqrt{\frac{m}{2}} \int_0^{\pi/2} \frac{d\phi}{\sqrt{\beta a^2 + 2\alpha + \beta a^2 \cos^2\phi}}$$

$$= 8\sqrt{\frac{m}{2}}\int_0^{\pi/2}\frac{d\phi}{\sqrt{2\beta a^2 + 2a - \beta a^2\sin^2\phi}}$$

or $$\tau = \frac{4}{\omega_n\sqrt{1+\dfrac{\beta a^2}{a}}}\int_0^{\pi/2}\frac{d\phi}{\sqrt{1-\left[\dfrac{\beta a^2}{2(a+\beta a^2)}\right]\sin^2\phi}} \tag{10.4.17}$$

where $\omega_n = \sqrt{\dfrac{a}{m}}$ = natural frequency of the linear system (when $\beta = 0$).

The integral in eqn (10.4.17) is of the type

$$K(k) = \int_0^{\pi/2}\frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}} \quad (0<k^2<1) \tag{10.4.18}$$

where $k^2 = \dfrac{\beta a^2}{2(a+\beta a^2)}$

This is a complete elliotic integral of the first kind and its value for different values of k^2 can be seen from the tables of Elliptic Integrals.

The time period for the linear system ($\beta=0$) can be obtained easily from eqn (10.4.17) as

$$\tau_l = \frac{2\pi}{\omega_n}$$

The phase plane plot for a hard spring system discussed above can be drawn easily as follows.

Draw the $F(x)$ (potential energy) variation against x. For $F(x)$ given in eqn. (10.4.6) for $\beta > 0$, this curve is shown on top in Fig. 10.4.1. The origin for the phase plane is taken below the origin of the energy curve. For the total energy $E = E_1$ of the system, draw a horizontal line indicating a total energy E_1 of the system. Where this straight line cuts the potential energy curve, there will be zero velocity at this displacement since when the potential energy is equal to total energy, the kinetic energy is zero. Also in accordance with

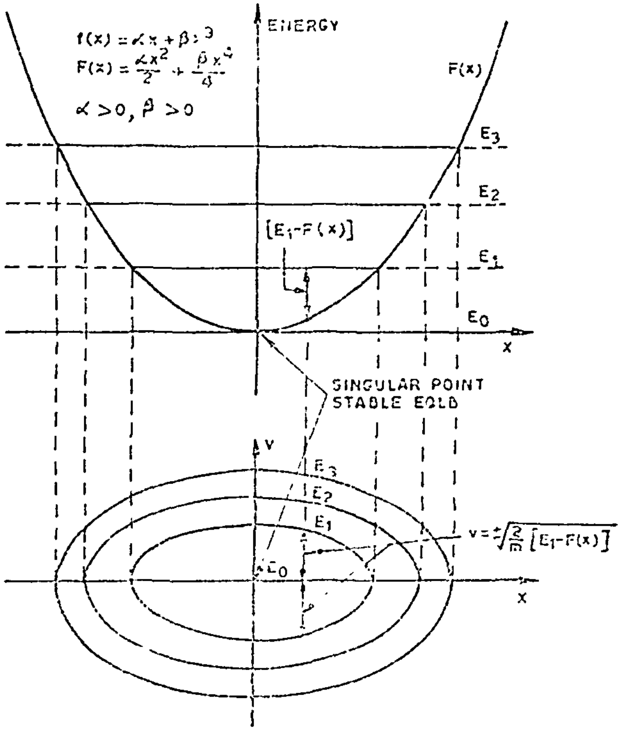


Fig. 10.4.1. Phase plane plot of a hard spring system.

eqn (10.4.4) since $[E_1 - F(x)]$ is zero at these points, v will be zero. Project these points on the x-axis of the phase plane where they lie on the trajectory of total energy E_1. At any other point between these extreme points, measure $[E_1 - F(x)]$. On this value of x on the phase plane the velocity then is given by

$$v = \pm\sqrt{\frac{2}{m}\left[E_1 - F(x)\right]}$$

in accordance with eqn. (10.4.4). Plotting these lengths above and below the x-axis on the phase plane, two more points are obtained on the trajectory with total energy E_1. It can be seen that this trajectory is a closed curve symmetrical about the x-axis. Similar trajectories can be drawn for different values of E, and are shown in the phase plane. It may be noted that the trajectory for a particular value of E lies only within the

x-range for which $E > F(x)$. When $E < F(x)$, eqn. (10.4.4) becomes absurd. Even otherwise it would mean that the total energy is less than the potential energy at that point, which is not possible.

One more point to note is that where the total energy line becomes a tangent to the $F(x)$ curve, that point in the phase plane correponds to a singular point. The reason is that at the point where E line touches $F(x)$ curve, $F(x) = E$ and therefore $v = 0$. Also at that point $\frac{d}{dx}[F(x)] = f(x) = 0$. These are the conditions for the singular point. In this particular case, origin is the singular point. In case the $F(x)$ curve is concave upwards, the minima is a singular point of stable equilibrium. It will be seen in the next sub-section that if $F(x)$ curve is convex upwards, the maxima corresponds to a singular point of unstable equilibrium.

10.4 B. Soft spring ($\beta<0$). Making a substitution

$$x = a \sin \phi$$

in eqn. (10.4.16), the time period can be obtained as

$$\tau = \frac{4}{\omega_n \sqrt{1+\frac{\beta a^2}{2a}}} \int_0^{\pi/2} \frac{d\phi}{\sqrt{1+\left[\frac{\beta a^2}{2a+\beta a^2}\right]\sin^2\phi}} \tag{10.4.19}$$

The integral in eqn. (10.4.19) is again of the type

$$K(k) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} \qquad (0 < k^2 < 1)$$

where $k^2 = -\dfrac{\beta a^2}{2a+\beta a^2}$

This is again a complete elliptic integral of first kind and its value for different values of k^2 can be obtained from the tables.

The phase plane plot for a soft spring system is shown in the lower part of Fig. 10.4.2. The upper part shows the plot of the $F(x)$ (potential energy) curve. The minima of the $F(x)$

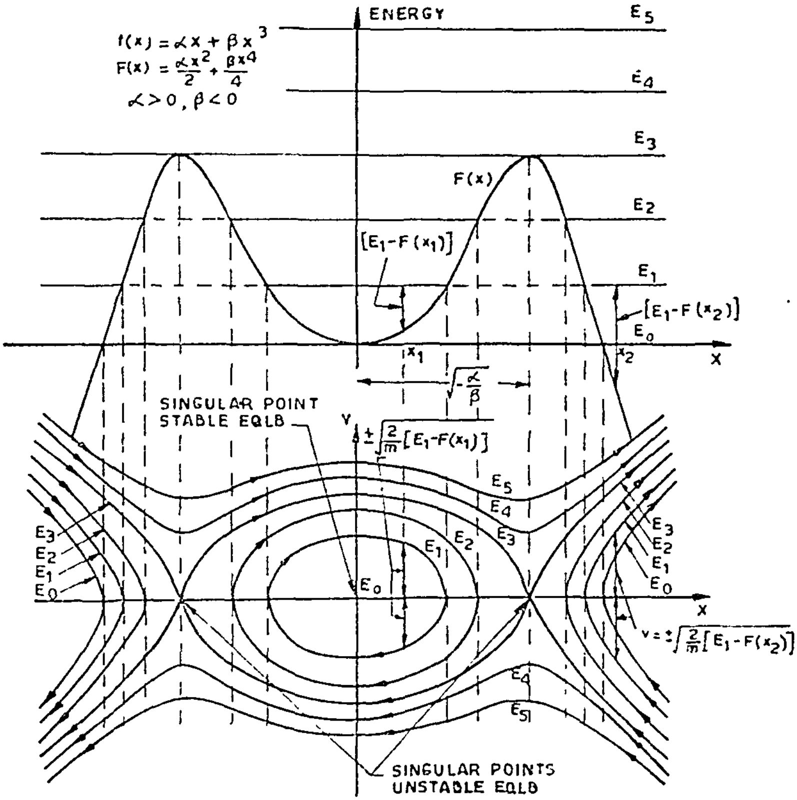


Fig. 10.4.2. Phase plane plot of a soft spring system.

curve occurs at the origin which therefore is the singular point of stable equilibrium. Two maximas occur at the points where $\frac{d}{dx} F(x) = f(x) = 0$, i.e. $x = \pm \sqrt{\frac{-\alpha}{\beta}}$ $(\beta < 0)$. Since at these points the potential energy is maximum and on either side the P.E. decreases, these points are singular points of unstable equilibrium. For any value of total energy E the phase plane plot can be drawn in a manner similar to that discussed under the hard spring system. For displacements less than $x = \pm \sqrt{\frac{-\alpha}{\beta}}$ i.e. the total energy E less than the maximum P. E., the trajectories are closed curves. The trajectories for the same value of total energy become unstable

curves at values of x on the other sides of the two humps. For values of E greater than the maximum potential energy, since the total energy lines never cut the P.E. curve, the corresponding trajectory never has zero velocity and so does not cut the x-axis. For $E = E_2$ the trajectory is in a state of transition.

Illustrative Example 10.4.1

For the system represented by the differential equation

$$m\ddot{x} + x + \beta x^3 = 0,$$

obtain a plot of the time period against amplitude of vibration when

(i) $\beta = +1/3$ (hard spring).

(ii) $\beta = -1/3$ (soft spring).

Solution

Here $\alpha = 1$.

(i) Hard spring ($\beta = +1/3$). Substituting the values of α and β in eqn. (10.4.17), we have

$$\tau = \frac{4}{\omega_n\sqrt{1+\frac{a^2}{3}}}K(k)$$

where $k^2 = \dfrac{a^2}{6+2a^2}$

Comparing it with the time period of the linear system

$$\tau_l = \frac{2\pi}{\omega_n}$$

We obtain the dimensionless time period

$$\frac{\tau}{\tau_l} = \frac{2}{\pi\sqrt{1+\frac{a^2}{3}}}K(k)$$

where $k^2 = \dfrac{a^2}{6+2a^2}$

k^2 has been calculated for different values of amplitude a, and the corresponding values of $K(k)$ seen from the tables. The variation of τ/τ_l against amplitude is shown in Fig. 10.4.3.

(ii) Soft spring ($\beta = -1/3$). Substituting the values of α and β in eqn. (10.4.19) and dividing it by τ_l, we obtain

$$\frac{\tau}{\tau_l} = \frac{2}{\pi\sqrt{1-\frac{a^2}{6}}} K(k)$$

where $k^2 = \dfrac{a^2}{6 - a^2}$

The variation of time period against amplitude is shown in Fig. 10.4.3.

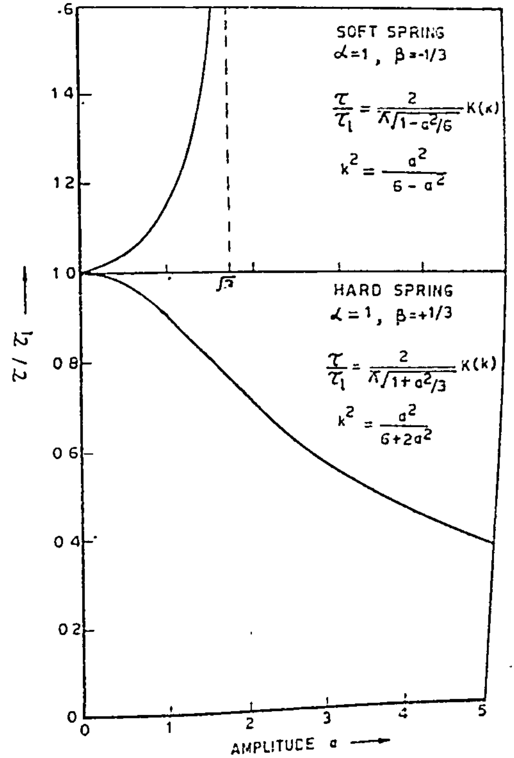


Fig. 10.4.3. Time period variation against amplitude for non-linear systems.

It may be noted that the time period of a hard spring system decreases with the increase in amplitude. For the case of a soft spring system the time period increases with amplitude. It becomes infinite for the amplitude $a = \sqrt{3}$ which corresponds to the amplitude $\sqrt{\dfrac{-\alpha}{\beta}}$ of the transition curve. Amplitudes greater than this do not give closed cu hence no question of time periods.

10.5 Perturbation method.

This is a very useful method for obtaining solutions of non-linear systems to any degree of accuracy by successive approximations. Consider the system to be represented by the differential equation

$$\ddot{x} + \omega_0^2 x + \beta x^3 = 0 \tag{10.5.1}$$

where ω_0 is the natural frequency of the linear system. Assuming that the solution can be written in the form of a Taylor series in terms of the parameter β (known as perturbation parameter), we can write

$$x = x_0 + \beta x_1 + \beta^2 x_2 + - - - - \tag{10.5.2}$$

where all x's are functions of time t. The only restriction in the above way of writing is that β be a small quantity.

Since the frequency of vibration ω which is dependent upon amplitude of vibration, is also unknown, we can write

$$\omega = \omega_0 + \beta \omega_1 + \beta^2 \omega_2 + - - - -$$

But as only ω_0^2 appears in the differential equation, it is more convenient to write

$$\omega^2 = \omega_0^2 + \beta \mu_1 + \beta^2 \mu_2 + - - - - \tag{10.5.3}$$

Substituting equations (10.5.2) and (10.5.3) in eqn. (10.5.1), we have

$$(\ddot{x}_0 + \beta \ddot{x}_1 + \beta^2 \ddot{x}_2 + \,.\,) + (\omega^2 - \beta\mu_1 - \beta^2\mu_2 \ldots)(x_0 + \beta x_1 + \beta^2 x_2 + \,..\,) + \beta(x_0 + \beta x_1 + \beta^2 x_2 + \ldots)^3 = 0 \tag{10.5.4}$$

The above equation after expanding can be written as

$$(\ddot{x}_0 + \omega^2 x_0) + (\ddot{x}_1 + \omega^2 x_1 - \mu_1 x_0 + x_0^3)\beta + (\ddot{x}_2 + \omega^2 x_2 - \mu_1 x_1 - \mu_2 x_0 + 3 x_0^2 x_1)\beta^2 + \ldots \tag{10.5.5}$$

Since the above equation must hold good for any small value of β, it means that each of the terms in the paranthesis must individually be zero therefore,

$$\left.\begin{aligned}
&\ddot{x}_0 + \omega^2 x_0 = 0 \\
&\ddot{x}_1 + \omega^2 x_1 = \mu_1 x_0 - x_0^3 \\
&\ddot{x}_2 + \omega^2 x_2 = \mu_1 x_1 + \mu_2 x_0 - 3 x_0^2 x_1 \\
&- \quad - \quad - \quad - \quad - \quad -
\end{aligned}\right] \tag{10.5.6}$$

Now let the initial conditions be,

$$\left.\begin{aligned} x &= a \\ \dot{x} &= 0 \end{aligned}\right] \text{at } t = 0$$

Substituting these in eqn. (10.5.2) and its derivative, we have

$$a = x_0(0) + \beta\, x_1(0) + \beta^2\, x_2(0) + \ldots$$

$$0 = \dot{x}_0(0) + \beta\, \dot{x}_1(0) + \beta^2\, \dot{x}_2(0) + \ldots$$

Again since these equations must be satisfied for any small value of β, we have

$$\left.\begin{aligned} x_0(0) &= a & \dot{x}_0(0) &= 0 \\ x_1(0) &= 0 & \dot{x}_1(0) &= 0 \\ x_2(0) &= 0 & \dot{x}_2(0) &= 0 \\ &---- & &---- \end{aligned}\right] \qquad (10.5.7)$$

With the first set of initial conditions in (10.5.7), the solution of the first differential equation in (10.5.6) is

$$x_0 = a \cos \omega t \qquad (10.5.8)$$

Substituting the above in the right side of the second differential equation in (10.5.6), we get

$$\ddot{x}_1 + \omega^2 x_1 = \mu_1 a \cos \omega t - a^3 \cos^3 \omega t \qquad (10.5.9)$$

Applying the relation

$$\cos^3 \omega t = \tfrac{3}{4} \cos \omega t + \tfrac{1}{4} \cos 3\omega t$$

to eqn. (10.5.9), it becomes

$$\ddot{x}_1 + \omega^2 x_1 = (\mu_1 a - \tfrac{3}{4} a^3) \cos \omega t - \tfrac{1}{4} a^3 \cos 3\omega t \quad (10.5.10)$$

In the above equation the forcing function $(\mu_1 a - \frac{3}{4} a^3) \cos \omega t$ will cause resonance to the system since the left hand side shows the natural frequency of the system as ω, the same as that of the first part of excitation. In order to avoid this absurdity, we must have

$$\mu_1 a - \tfrac{3}{4} a^3 = 0 \qquad (10.5.11)$$

Therefore

$$\ddot{x}_1 + \omega^2 x_1 = -\tfrac{1}{4} a^3 \cos 3\omega t$$

the solution of which is

$$x_1 = A_1 \cos \omega t + A_2 \sin \omega t + \frac{a^3}{32\omega^2} \cos 3\omega t \qquad (10.5.12)$$

Applying the zero initial conditions from (10.5.7), we get the constants as

$$A_1 = -\frac{a^3}{32\omega^2}$$

$$A_2 = 0$$

Eqn. (10.5.12) becomes

$$x_1 = -\frac{a^3}{32\omega^2} [\cos \omega t - \cos 3\omega t] \qquad (10.5.13)$$

Substituting eqns. (10.5.8) and (10.5 13) in the first two terms of eqn. (10.5.2), the solution upto first order correction is obtained as

$$x = x_0 + \beta x_1$$

$$= a \cos \omega t - \frac{\beta a^2}{32\omega^2} [\cos \omega t - \cos 3\omega t] \qquad (10.5.14)$$

with ω^2 given by eqn. (10.5.3) upto first order correction as

$$\omega^2 = \omega_0^2 + \beta\, \mu_1$$

$$= \omega_0 + \beta \left(\frac{3}{4} a^2\right) \text{ [from eqn (10.5.11)]} \qquad (10.5.15)$$

The above process of successive approximations can be continued to include the higher order corrections.

10.6 Forced vibration with non-linear spring forces.

Consider a system represented by the differential equation

$$\ddot{x} + \omega_n^2 x + \beta\, x^3 = F_0 \cos \omega t \qquad (10.6.1)$$

This equation is known as *Duffings Equation* after the name of the mathematician who made an exhaustive study of this equation. Rewrite eqn (10.6.1) as

$$\ddot{x} = -\omega_n^2 x - \beta x^3 + F_0 \cos \omega t \qquad (10.6.2)$$

Considering only small values of β and F_0, it is known from experience that the frequency of steady state vibration will be the same as that of excitation plus some higher harmonics. So,

the first aproximate solution can be written as

$$x_1 = a \cos \omega t \tag{10.6.3}$$

Substituting the above approximate solution in the right hand side of eqn (10.6.2), we have

$$\ddot{x}_2 = -\omega_n^2 a \cos \omega t - \beta a^3 \cos^3 \omega t + F_0 \cos \omega t \tag{10.6.4}$$

The double integration of the above equation will give x_2 which will be a better approximation than x_1 given in eqn (10.6.3). Using the relationship

$$\cos^3 \omega t = \tfrac{3}{4} \cos \omega t + \tfrac{1}{4} \cos 3\omega t$$

eqn (10.6.4) is rewritten as

$$\ddot{x}_2 = (F_0 - \omega_n^2 a - \tfrac{3}{4} \beta a^3) \cos \omega t - \tfrac{1}{4} \beta a^3 \cos 3\omega t \tag{10.6.5}$$

Integrating the above equation twice and dropping out the constants of integration to ensure periodic motion, we have

$$x_2 = -\frac{1}{\omega^2}\left(F_0 - \omega_n^2 a - \tfrac{3}{4} \beta a^3\right) \cos \omega t + \frac{\beta a^3}{36\omega^2} \cos 3\omega t \tag{10.6.6}$$

If x_1 were a good first approximation of the system, then the coefficient of cos ωt in eqn (10.6.3) should be approximately the same as the coefficient of cos ωt in eqn (10.6.6). Equating the two

$$a = \frac{1}{\omega^2} (\omega_n^2 a + \tfrac{3}{4} \beta a^3 - F_o) \tag{10.6.7}$$

or $$\omega^2 = \omega_n^3 + \tfrac{3}{4} \beta a^2 - \frac{F_o}{a} \tag{10.6.7a}$$

Then the solution $x(t)$ can be written as

$$x(t) \approx x_2 = a \cos \omega t + \frac{\beta a^3}{36\omega^2} \text{ co } 3\omega t \tag{10.6.8}$$

The variation of the amplitude of vibration against frequency as governed by eqn (10.6.7) is shown in Fig. 10.6.1 (a), (b) and (c) for a linear, hard spring and soft spring systems respectively. Eqn (10.6.7) is a cubic in a and therefore for any value of ω there are in general three values of a; one root is always rea' the other two may be real or complex conjugate. This the reflected in figures (b) and (c) *i.e.* for non linear systems only

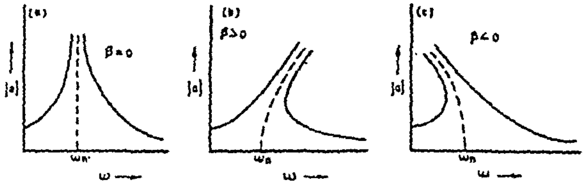


Fig. 10.6.1. Frequency-response curves for linear, hard spring and soft spring systems.

In the non linear systems there is no resonance like we have in linear systems. The amplitude never becomes infinity. The dotted lines in these figures show the relationship between the amplitude of vibration and the natural frequency. This is obtained from eqn (10.6.7a) by setting F_o to zero. These lines show that the natural frequency of a hard spring system increases withe amplitude and for a soft spring system the natural frequenc decreases with the increase in amplitude. Since the natural frequency is different at different amplitudes, the resonance can not build up.

When damping is present in the nonlinear systems, the skewed peaks wind up at a certain stage. The frequency response curves then are as shown in Figs. 10.6.2 (a) and (b).

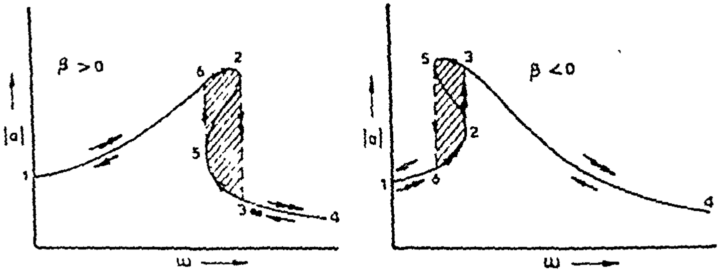


Fig. 10.6.2. Jump phenomenon.

When the frequency of excitation is gradually increasing from zero, the response varies along the points 1, 6, 2, 3, 4, there being a sudden change in amplitude from 2 to 3 at the corresponding frequency. The portion of the response curve 2, 5, 3 is never traced.

When the frequency of excitation is gradually decreasing from a large value, the response varies along the points 4, 3 5, 6, 1, there being a sudden change in amplitude from 5 to 6 at the corresponding frequency. The portion of the response curve 5, 2, 6 is never traced.

This phenomenon of sudden change in amplitude from 2 to 3 while the frequency is gradually increasing and the sudden change from 5 to 6 when the frequency is gradually decreasing is known as *Jump Phenomenon.*

10.7 Self excited vibrations.

The self excited vibrations differ from forced vibrations in that the fluctuating force that sustains the motion is controlled by some part of the motion itself. The exciting force may be a function of displacement, velocity or acceleration of the motion. When motion is stopped by some means, the fluctuating force also disappears. The forcing function is thus dependent upon motion itself unlike forced vibrations. Tool chatter and aeroplane wing flutter are some of the common examples of self excited vibration.

It is better here to define what is known as *Stability of Oscillations.* If the system is such that when disturbed from its equilibrium position it comes back there after the transients die out, the system is known as dynamically stable. In case any disturbance cause the amplitude to built up *with time*, the system is said to be dynamically unstable. Fig. 10.7.1 shows a stable and unstable performance of a system.

Effectively, the system becomes unstable when negative damping appears in its differential equation of motion. A

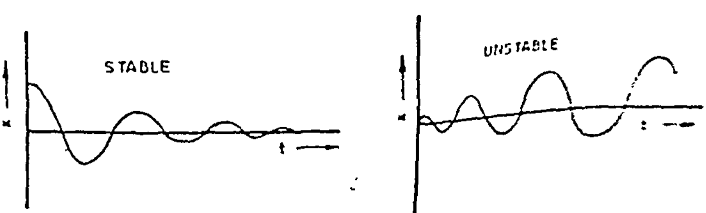


Fig. 10 7.1. Stable and unstable oscillations.

more general difinition of stability is that the roots of the characteristic equation of the system should either be negative real numbers or complex with negative real parts. Consider a third order system with its roots of the characteristic equation, as

$$s_1 = a_1$$

$$s_2 = a_2 + jb_2$$

$$s_3 = a_2 - jb_2$$

The solution of the equation would then be given by

$$x = C_1 e^{a_1 t} + C_2 e^{a_2 t} \cos (b_2 t + \phi_2)$$

If either a_1 or a_2 is positive, the response would build up with time, giving instability to the system. If on the other hand both a_1 and a_2 are negative, the response would die out with time giving stability to the system.

Consider the system of Fig. 10.7.2 (a) where a spring-mass system is supported on a horizontal belt moving with a constant velocity V. The coefficient of friction between the mass and the belt material is such that it decreases slightly with the increase in relative velocity. This variation is shown in Fig. 10.7.2 (b) by an approximate straight line of slope $-\beta$ wl er β is a small positive number.

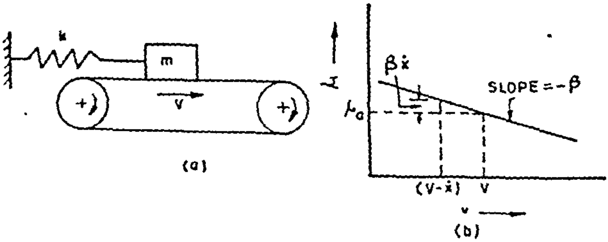


Fig. 10.7.2. Self excited vibrations caused by dry friction.

When mass is stationary the friction coefficient between the mass and the belt is μ_0. When mass is moving towards the right, the relative velocity decreases and the coefficient of friction increases. On the other hand when the mass is moving towards the left, the relative velocity increases and

therefore the coefficient of friction decreases. Since the friction force on the mass is always towards right, the helping friction force when mass moves towards right is always greater than the opposing friction force when mass moves towards left. That means a certain net energy is put into the system in each cycle. The amplitude continues to increase. If, however, the mass is brought to rest in the equilibrium position, it stays in that position. The least disturbance will set it vibrating with increasing amplitude. This is a case of self excited vibration. The frequency of vibration in the case of self excited systems is approximately equal to the natural frequency of the system provided damping is not large.

This problem can also be tackled analytically. At any instant when the displacement of the mass is x and its velocity $\dot{x}$, the relative velocity between the mass and the belt is $(V-\dot{x})$. At this instant coefficient of f iction is $(\mu_a+\beta\dot{x})$. The normal reaction on the mass is mg. The differential equation of motion is then written as

$$m\ddot{x} + kx = mg(\mu_a + \beta\dot{x})$$

$$m\ddot{x} - mg\beta\dot{x} + kx = mg\,\mu_a \tag{10.7.1}$$

Eqn (10.7.1) gives an effective negative damping to the system which sends it into large increasing amplitudes. The static displacement is $mg\mu_a/k$.

In case there is viscous damping between the mass and the belt the slope of the equivalent friction line is no longer negative and there will not be any self excited vibrations. There are numerous other examples of self excited vibrations caused by dry friction, a few of which are :—

(i) Excitation of violin string by a bow.

(ii) Screeching of door joints when dry.

(iii) Shaft whirl due to dry friction.

PROBLEMS FOR PRACTICE

10.1 Prove that the principle of superposition doesnot hold good for non-linear differential equations. Take a specific differential equation.

10.2 The mass shown in Fig.P. 10.2 is given an initial velocity v. Find the ime period τ per cycle of oscillation as a function of v. Plot τ versus v.

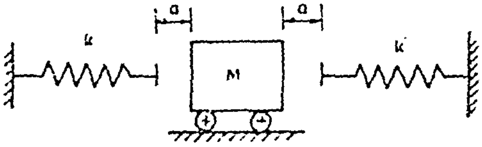


Fig. P. 10 2.

10.3 Investigate the system shown in Fig. P. 10.3 for its period of vibration being independent of amplitude. A point in the pendulum is fixed at the middle of a horizontally stretched wire.

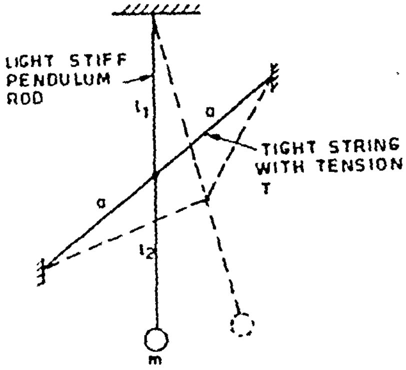


Fig. P. 10.3.

10.4 In problem 10.2 when $k = 1$, $m = 1$, plot the phase plane trajectory for the motion of the system.

10.5 Sketch the phase plane plot of a simple pendulum.

10.6 Obtain the solution for the vibration of a simple pendulum in the form of an elliptic integral when the

amplitude of vibration is ϕ_0.

[Hint : After writing the expression for time period let $\sin \phi_0/2 = k$, and $\sin \phi/2 = k \sin \theta$; where ϕ = displacement at any time]

10.7 Find the time period in problem 10.6 when

(i) $\phi_0 = 90°$

(ii) $\phi_0 = 180°$

10.8 Find the singular points for the following differential equation and say whether they are stable or unstable :—

$$m\ddot{x} + \alpha x + \beta x^3 = 0 \quad \alpha < o,\ \beta < o$$

10.9 Using the first two terms of the expansion of $\sin \theta$, determine by the method of perturbation the time period of simple pendulum as a function of amplitude.

10.10 For the system in Fig. 10 7.2 include viscous damping c also between the mass and the side support. Obtain the condition of stability for this system.

If $m = 1.0$ kg-sec²/cm.

$k = 1.0$ kg/cm

$c = 0.6$ kg-sec/cm.

$\mu_a = 1/98$

$\beta = 1/980$ sec/cm.

check up if the system is stable or unstable. Find the amplitude reduction or amplitude gain per cycle.

10.11 Show that a pendulum suspended from a rotating shaft (Fig. P. 10.11) with dry friction between the mating parts will have self excited vibrations.

10.12 A spring-mass-dashpot system is excited by a force $F_0\, x$. Write down the differential equation of motion and obtain the stability condition.

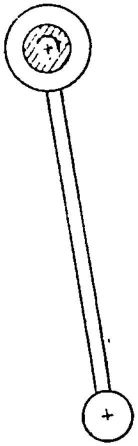


Fig. P. 10.11.

10.13 In case of the problem 10.10, if the mass is placed very near the static equilibrium position, plot the phase plane trajectory of the ensuing motion.

CHAPTER 11

ELECTRICAL ANALOGY

11.1 Introduction.

It is possible to study the characteristics of a mechanical vibratory system by representing the mechanical system by its equivalent eletrical circuit. These equivalent electrical circuits can be easily constructed than the corresponding mechanical models and besides, the effect of changing different parameters can be conveniently studied by varying electrical quantities. The electrical quantities to be used, varied and measured are inductance, resistance, capacitance, voltage, current and frequency.

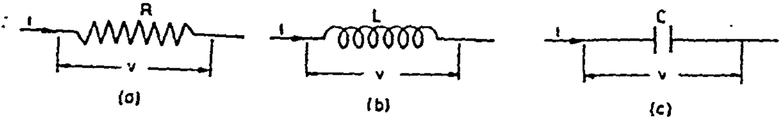
It is rather cumbersome to vary mechanical parameters. Of course, a mechanical system can be represented by a mathematical model and can be studied by solving the mathematical equations, but where the system becomes a little complex, experimental study is easier and it is here that equivalent electrical circuits can be experimentally studied much more easily. Qualitative as well as quantitative results can be obtained by designing a proper circuit.

The basis for the equivalence of the two systems is that the differential equations of motion for the two systems should be similar. The two systems are analogous if their differential equations of motion are mathematically the same. Under these conditions the corresponding terms in differential equations are analogous to one another.

There are two electrical analogies that can possibly be used for mechanical systems. The first is the *voltage-force analogy* and the other is *current-force analogy*. The former is very useful for most systems and is more commonly employed, so it will be used in this text.

11.2 Principles of electrical analogue.

The electrical circuits to be constructed are based on the Kirchhoff's second low, viz., in any network, the algebraic sum of the potential differences around any closed circuit is zero. Further, the voltage drop across a resistance R, inductance L, and a capacitance C are respectively given as shown in Fig. 11.2.1 (a), (b) and (c), where i is the current passing through the circuit and q is the charge.



$$v = Ri = R\frac{dq}{dt} \qquad v = L\frac{di}{dt} = L\frac{d^2q}{dt^2} \qquad v = \frac{1}{C}\int i\,dt = \frac{q}{C}$$

Fig. 11.2.1. Voltage drop across a resistance, an inductance and a capacitance.

If now we write down the differential equations for the two systems shown in Fig. 11.2.2 (a) and (b), they are as follow.

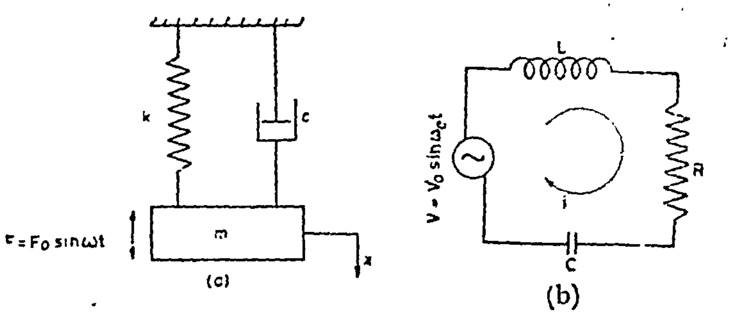


Fig. 11.2.2. Equivalent electrical circuit for a parallel mechanical circuit.

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F_0 \sin \omega t \tag{11.2.1}$$

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = V_0 \sin \omega_e t \tag{11.2.2}$$

Equation (11.2.2) can also be written as

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt^2} + \frac{1}{C}q = V_0 \sin \omega_e t \tag{11.2.3}$$

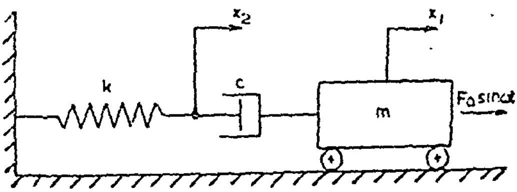
Equations (11.2.1) and (11.2.3) are similar in all respects and the corresponding analogous quantities as seen from these equations are shown in Table 11.2.1.

TABLE 11.2.1

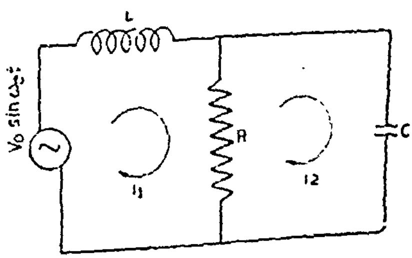
Mechanical-Electrical equivalents.

Mechanical System		Electrical System	
F	Force (kg)	V	Voltage (volts)
m	Mass (kg-sec²/cm)	L	Inductance (henrys)
c	Damping coefficient (kg-sec/cm)	R	Resistance (ohms)
k	Spring stiffness (kg/cm)	$\frac{1}{C}$	$\frac{1}{\text{Capacitance}}$ $\left(\frac{1}{\text{farad}}\right)$
x	Displacement (cm)	q	Charge (coulombs)
$\frac{dx}{dt}$	Velocity (cm/sec)	i	Current (amperes)
ω	Frequency (rad/sec)	ω_e	Frequency (rad/sec)

In Fig. 11.2.2, we had taken the case of spring and dashpot in parallel. The case when spring and dashpot are in series needs also to be discussed. Consider Fig. 11.2.3 (a) and (b) to



(a)



(b)

Fig. 11.2.3. Equivalent electrical circuit for a series mechanical circuit.

represent a mechanical system and an equivalent electrical system. The equivalence is checked by writing their differential equations.

The differential equations for the mechanical system of Fig. 11.2.3 (a) are

$$\left.\begin{aligned} m\frac{d^2x_1}{dt^2} + c\left(\frac{dx_1}{dt} - \frac{dx_2}{dt}\right) &= F_0 \sin \omega t \\ c\left(\frac{dx_1}{dt} - \frac{dx_2}{dt}\right) - kx_2 &= 0 \end{aligned}\right] \tag{11.2.4}$$

The differential equations for the electrical system of Fig. 11.2.3 (b) are

$$\left.\begin{aligned} L\frac{di_1}{dt} + R(i_1 - i_2) &= V_0 \sin \omega_e t \\ R(i_1 - i_2) - \frac{1}{C}\int i_2\, dt &= 0 \end{aligned}\right] \tag{11.2.5}$$

Equations (11.2.5) can be re-written as

$$\left.\begin{aligned} L\frac{d^2q_1}{dt^2} + R\left(\frac{dq_1}{dt} - \frac{dq_2}{dt}\right) &= V_0 \sin \omega_e t \\ R\left(\frac{dq_1}{dt} - \frac{dq_2}{dt}\right) - \frac{1}{C}\, q_2 &= 0 \end{aligned}\right] \tag{11.2.6}$$

Equations (11.2.4) and (11.2.6) are exactly similar, hence the two circuits are analogous.

If we examine back carefully the two examples of similar systems discussed above, we may conclude that if the mechanical system is such that the forces on the mass act in *parallel*, then the equivalent electrical circuit will have its components in *series* (Fig. 11.2.2). If, on the other hand, the mechanical system has its components in *series*, then the equivalent electrical circuit will have its components in *para lel* (Fig. 11.2.3).

Illustrative Example 11.2.1

A two degree of freedom spring-mass-dashpot system is shown in Fig. 11.2.4 (a). Make an equivalent electrical circuit for this mechanical system.

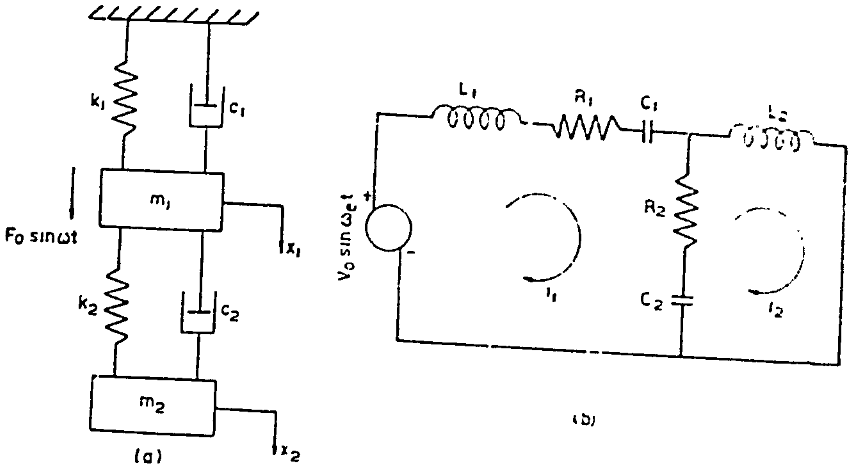


Fig. 11.2.4. Equivalent electrical circuit for a two degree of freedom mechanical system.

Solution

The equations of motion for the mechanical system can be written as below.

$$\left.\begin{aligned} m_1 \frac{d^2x_1}{dt^2} + c_1 \frac{dx_1}{dt} + k_1x_1 + c_2\left(\frac{dx_1}{dt} - \frac{dx_2}{dt}\right) \\ + k_2(x_1 - x_2) = F_0 \sin \omega t \\ m_2 \frac{d^2x_2}{dt^2} + c_2\left(\frac{dx_2}{dt} - \frac{dx_1}{dt}\right) + k_2(x_2 - x_1) = 0 \end{aligned}\right] \quad (11.2.7)$$

From the above equations we see that m_1, c_1 and k_1 are associated with x_1 only and m_2 is associated with x_2 only, where as c_2 and k_2 are associated with $(x_1 - x_2)$. The corresponding electrical circuit will, therefore, have two loops with i_1 and i_2 as the two respective loop currents. The first loop will have quantities L_1, R_1 and C_1 (corresponding to m_1, c_1 and k_1) and the second loop will have L_2 (corresponding to m_2). Besides, R_2 and C_2 (corresponding to c_2 and k_2) will be common elements between the two loops to be associated with $(i_1 - i_2)$ [corresponding to $(x_1 - x_2)$]. The electrical forcing function $V_0 \sin \omega_e t$ (corresponding to $F_0 \sin \omega t$) will be put only in the first loop. The trial elecrical circuit is shown in Fig. 11.2.4 (b). The differential equations are

$$\left.\begin{aligned} L_1 \frac{di_1}{dt} + R_1 i_1 + \frac{1}{C_1} \int i_1 dt + R_2 (i_1 - i_2) & \\ + \frac{1}{C_2} \int (i_1 - i_2)\, dt &= V_0 \sin \omega_e t \\ L_2 \frac{di_2}{dt} + R_2 (i_2 - i_1) + \frac{1}{C_2} \int (i_2 - i_1)\, dt &= 0 \end{aligned}\right] \tag{11.2.8}$$

The above equations can be written as

$$\left.\begin{aligned} L_1 \frac{d^2q_1}{dt^2} + R_1 \frac{dq_1}{dt} + \frac{1}{C_1} q_1 + R_2 \left(\frac{dq_1}{dt} - \frac{dq_2}{dt}\right) & \\ + \frac{1}{C_2} (q_1 - q_2) &= V_0 \sin \omega_e t \\ L_2 \frac{d^2q_2}{dt^2} + R_2 \left(\frac{dq_2}{dt} - \frac{dq_1}{dt}\right) + \frac{1}{C_2} (q_2 - q_1) &= 0 \end{aligned}\right] \tag{11.2.9}$$

Equations (11 2.7) and (11.2.9) are absolutely similar in all respects. Hence the electrical circuit of Fig. 11.2.4 (b) is equivalent circuit to the given mechanical system. **Ans.**

Illustrative Example 11.2.2.

For the mechanical system shown in Fig. 11.2.5 (a), draw an

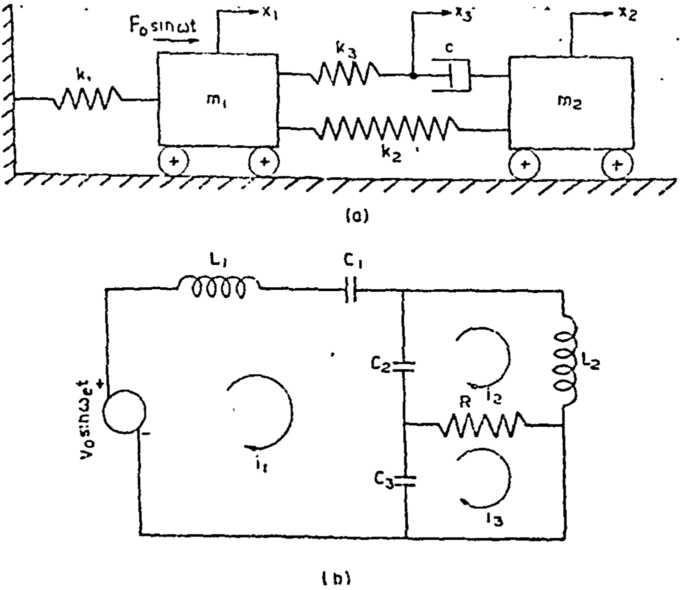


Fig. 11.2.5. Equivalent electrical circuit for a three degree of freedom mechanical system.

equivalent electrical circuit.

Solution

The differential equations of motion for the mechanical system are

$$\left.\begin{aligned} m_1 \frac{d^2x_1}{dt^2} + k_1x_1 + k_2(x_1 - x_2) + k_3(x_1 - x_3) &= F_0 \sin \omega t \\ m_2 \frac{d^2x_2}{dt^2} + c\left(\frac{dx_2}{dt} - \frac{dx_3}{dt}\right) + k_2(x_2 - x_1) &= 0 \\ c\left(\frac{dx_3}{dt} - \frac{dx_2}{dt}\right) + k_3(x_3 - x_1) &= 0 \end{aligned}\right\} \quad (11.2.10)$$

From the above equations, we see that the electrical circuit will consist of three loops with L_1, C_1 (corresponding to m_1, k_1) in the first loop ; L_2 (corresponding to m_2) in the second loop ; C_2 (corresponding to k_2) common to first and second loop ; C_3 (corresponding to k_3) common to first and third loop ; and R (corresponding to c) common to second and third loop. The trial electrical circuit is shown in Fig. 11.2.5 (b), with the forcing function included in the first loop. The differential equations for the electrical circuit are as below.

$$\left.\begin{aligned} L_1 \frac{di_1}{dt} + \frac{1}{C_1}\int i_1 dt + \frac{1}{C_2}\int (i_1 - i_2)\, dt & \\ + \frac{1}{C_3}\int (i_1 - i_3)\, dt &= V_0 \sin \omega_e t \\ L_2 \frac{di_2}{dt} + R(i_2 - i_3) + \frac{1}{C_2}\int (i_2 - i_1)\, dt &= 0 \\ R(i_3 - i_2) + \frac{1}{C_3}\int (i_3 - i_1)\, dt &= 0 \end{aligned}\right\} \quad (11.2.11)$$

The above equations can be re-written as

$$\left.\begin{aligned} L_1 \frac{d^2q_1}{dt^2} + \frac{1}{C_1} q_1 + \frac{1}{C_2}(q_1 - q_2) + \frac{1}{C_3}(q_1 - q_3) &= V_0 \sin \omega_e t \\ L_2 \frac{d^2q_2}{dt^2} + R\left(\frac{dq_2}{dt} - \frac{dq_3}{dt}\right) + \frac{1}{C_2}(q_2 - q_1) &= 0 \\ R\left(\frac{dq_3}{dt} - \frac{dq_2}{dt}\right) + \frac{1}{C_3}(q_3 - q_1) &= 0 \end{aligned}\right\} \quad (11.2.12)$$

The above equations are similar to equations (11.2.10) in all respects. Hence the electrical circuit of Fig. 11.2.5 (b) is equivalent of the given mechanical system.

11.3 Truly analogous circuits.

After putting down the electrical circuits as described in the preceeding section, the next step to obtain a truly analogous circuit is to determine the values of the electrical components. It is not necessary that each electrical component individually be numerically equal to the corresponding mechanical counterpart, but a number of dimensionless groups can be formed and the values of these dimensionless groups should be equal for the two systems.

Buckingham's pi theorm states that, if there are x quantities to be considered and y fundamental dimensions, the number of independent dimensionless groups, designated as $\pi_1, \pi_2, \pi_3 \ldots$etc., that will be formed is $(x-y)$.

For the mechanical system we have three fundamental dimensions of *force* F, *length* L and *time* T. Each quantity appearing in the system can be represented in terms of these fundamental units. Dimensionless groups can be formed for the system as illustrated in the following example.

Illustrative Example 11.8 1

Fig. 11.3.1 (a) shows the schematic of a damped dynamic vibration absorber with the following component and excitation values.

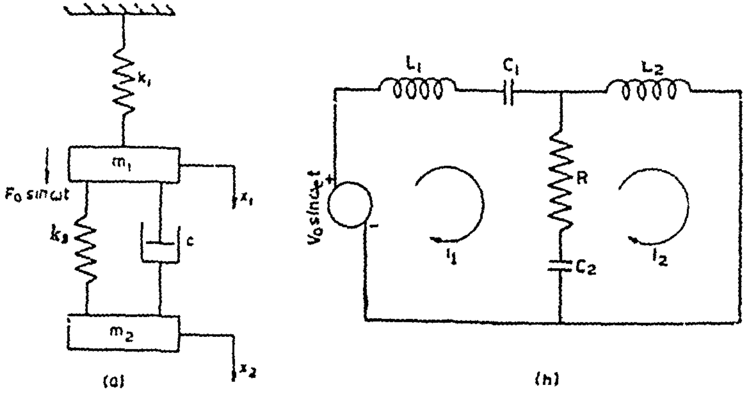


Fig. 11.3.1 Determination of component values of the equivalent electrical circuit for a damped dynamic vibration absorber.

$m_1 = 0.10$ kg-sec²/cm
$m_2 = 0.02$ kg-sec²/cm
$k_1 = 8.0$ kg/cm
$k_2 = 1.2$ kg/cm
$c_1 = 0{\cdot}2$ kg-sec/cm
$F_o = 3{\cdot}0$ kg
ω varies between 1C0 and 400 rad/sec.

(a) Draw an equivalent electrical circuit.

(b) A coil having an inductance of 0·15 henrys is available. It is decided to represent this inductance as mass m_2. The alternating frequency in the electrical circuit can be varied from 250 to 1000 rad/sec. Find the remaining component values for a truly analogous electrical circuit.

(c) How would you find the amplitudes of vibration of the two masses from the equivalent electrical circuit ?

Solution

(a) The equivalent electrical circuit is given in Fig. 11.3. 1(b). It can be shown that the differential equations for both the systems are similar in all respects. **Ans.**

(a) The parameters in the mechanical system are m_1, m_2, k_1, k_2, c, F_o, ω and x_1 (or x_2). These are 8 in number.

The fundamental dimensions are F, L and T, which are 3 in number. Hence we expect (8−3)=5 dimensionless groups from Buckingham's pi theorem. We will pick up 3 parameters m_1, k_1 and x from the original 8 parameters, give them exponents a, b and d respectively, and combine with them one more parameter at a time to obtain 5 dimensionless groups. We may choose any other set of 3 parameters but these should contain all the fundamental units amongst them.

The fundamental dimensions of the mechanical system parameters are

m_1, m_2	$FL^{-1}T^2$
k_1, k_2	FL^{-1}
c	$FL^{-1}T$
F_o	F
ω	T^{-1}
x_1, x_2	L

11.3 Truly analogous circuits.

After putting down the electrical circuits as described in the preceeding section, the next step to obtain a truly analogous circuit is to determine the values of the electrical components. It is not necessary that each electrical component individually be numerically equal to the corresponding mechanical counterpart, but a number of dimensionless groups can be formed and the values of these dimensionless groups should be equal for the two systems.

Buckingham's pi theorm states that, if there are x quantities to be considered and y fundamental dimensions, the number of independent dimensionless groups, designated as π_1, π_2, π_3...etc., that will be formed is $(x-y)$.

For the mechanical system we have three fundamental dimensions of *force* F, *length* L and *time* T. Each quantity appearing in the system can be represented in terms of these fundamental units. Dimensionless groups can be formed for the system as illustrated in the following example.

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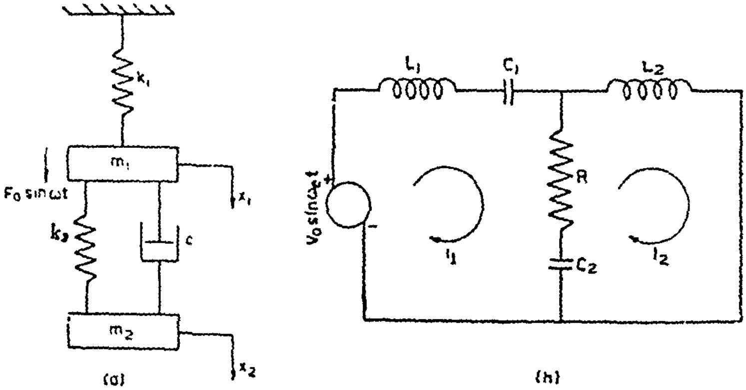

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$k_1 = 8.0$ kg/cm
$k_2 = 1.2$ kg/cm
$c_1 = 0{\cdot}2$ kg-sec/cm
$F_o = 3{\cdot}0$ kg
ω varies between 100 and 400 rad/sec.

(a) Draw an equivalent electrical circuit.

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The fundamental dimensions of the mechanical system parameters are

m_1, m_2	$FL^{-1}T^2$
k_1, k_2	FL^{-1}
c	$FL^{-1}T$
F_o	F
ω	T^{-1}
x_1, x_2	L

Let us therefore write the dimensionless *pi* groups as

$$\pi_1 = m_1^a k_1^b x^d m_2 = (FL^{-1}T^2)^a\,(FL^{-1})^b\,(L)^d\,(FL^{-1}T^2)$$
$$\pi_2 = m_1^a k_1^b x^d\,\omega = (FL^{-1}T^2)^a\,(FL^{-1})^b\,(L)^d\,(T^{-1})$$
$$\pi_3 = m_1^a k_1^b x^d\,k_2 = (FL^{-1}T^2)^a\,(FL^{-1})^b\,(L)^d\,(FL^{-1})$$
$$\pi_4 = m_1^a k_1^b x^d\,c = (FL^{-1}T^2)^a\,(FL^{-1})^b\,(L)^d\,(FL^{-1}T)$$
$$\pi_5 = m_1^a k_1^b x^d\,F_o = (FL^{-2}T^2)^a\,(FL^{-1})^b\,(L)^d\,(F)$$

Since all the above groups are dimensionless, the resultant powers of F, L and T in each group must be zero. Considering π_1 dimensionless group, the exponent for

F is $b+a-1=0$

L is $-a-b+d-1=0.$

T is $2a+2=0$

These give $a=-1,\ b=0,\ d=0$

Therefore $\pi_1 = \dfrac{m_2}{m_1}$

The rest of the dimensionless groups are also obtained in the same manner to give us complete list of these groups. The equivalent electrical circuit component groups are also shown along with these groups for mechanical system.

$$\pi_1 = \frac{m_2}{m_1} = \frac{L_2}{L_1}$$

$$\pi_2 = \omega\sqrt{\frac{m_1}{k_1}} = \omega_e\sqrt{L_1C_1}$$

$$\pi_3 = \frac{k_2}{k_1} = \frac{C_1}{C_2}$$

$$\pi_4 = \frac{c}{\sqrt{m_1k_1}} = R\sqrt{\frac{C_1}{L_1}}$$

$$\pi_5 = \frac{F_o}{k_1x} = \frac{V_o\,C_1}{q}$$

The electrical circuit components can now be obtained as below.

$$\pi_1 = \frac{m_2}{m_1} = \frac{L_2}{L_1}$$

or, $$L_1 = \frac{m_1}{m_2}\,L_2$$

$$= \frac{0{\cdot}10}{0{\cdot}02} \times 0{\cdot}15 \text{ [because } L_2 = 0{\cdot}15 \text{ henrys]}$$

$$= 0.75 \text{ henrys.}$$

$$\pi_2 = \omega \sqrt{\frac{m_1}{k_1}} = \omega_e \sqrt{L_1 C_1}$$

$$\text{or,}\quad C_1 = \left(\frac{\omega}{\omega_e}\right)^2 \cdot \frac{m_1}{k_1} \cdot \frac{1}{L_1}$$

$$= \left(\frac{100 \text{ to } 400}{250 \text{ to } 1000}\right)^2 \times \frac{0.10}{8.0} \times \frac{1}{0.75} = 0.00267 \text{ farads.}$$

$$\pi_3 = \frac{k_2}{k_1} = \frac{C_1}{C_2}$$

$$\text{or,}\quad C_2 = \frac{k_1}{k_2} \cdot C_1$$

$$= \frac{8.0}{1.2} \times 0.00267 = 0.01778 \text{ farads}$$

$$\pi_4 = \frac{c}{\sqrt{m_1 k_1}} = R \sqrt{\frac{C_1}{L_1}}$$

$$\text{or, } R = \frac{c}{\sqrt{m_1 k_1}} \sqrt{\frac{L_1}{C_1}}$$

$$= \frac{0.2}{\sqrt{0.1 \times 8.0}} \times \sqrt{\frac{0.75}{0.00267}} = 3.75 \text{ ohms.}$$

The component values of the electrical circuit as obtained above are grouped together as below

$L_2 = 0.15$ henrys (given)
$L_1 = 0.75$ henrys
$C_1 = 0.00267$ farads
$C_2 = 0.01778$ farads
$R = 3.75$ ohms. **Ans.**

$$\text{(c)}\quad \pi_5 = \frac{F_0}{k_1 x} = \frac{V_0 C_1}{q}$$

$$\text{Or,}\quad \frac{3}{8x} = \frac{V_0 \times 0.00267}{q}$$

$$\text{or, } x = 140.6 \frac{q}{V_0}$$

But $q = C.\ V_c$

where V_c is the voltage drop across the particular capacitor.

$\therefore \quad q_1 = C_1 V_{c1}$

and $\quad q_1 - q_2 = C_2.\ V_{c2} \qquad$ or $\quad q_2 = C_1 V_{c1} - C_2 V_{c2}$

$$\text{Hence } x_1 = \frac{140.6}{V_0} C_1 V_{c1} = \frac{140.6}{V_0} \times 0.00267 \times V_{c1}$$

$$\text{and } x_2 = \frac{140 \cdot 6}{V_0}(C_1 V_{c1} - C_2 V_{c2})$$

$$= \frac{140.6}{V_0}(0 \cdot 00267\, V_{c1} - 0 \cdot 01778\, V_{c2})$$

The above two equations are simplified to

$$\left.\begin{aligned} x_1 &= 0.375 \frac{V_{c1}}{V_0} \\ x_2 &= 0 \cdot 375 \frac{V_{c1}}{V_0} - 2.5 \frac{V_{c2}}{V_0} \end{aligned}\right] \tag{11.3.1}$$

We will measure the voltage drop across the particular capacitor, find its ratio with the inspressed voltage and multiply it with the particular constants in equations (11.3.1) above to get x_1 or x_2. **Ans.**

PROBLEMS FOR PRACTICE

11.1 Draw the equivalent electrical circuit for the mechanical systems shown in Fig. P. 11.1 (a), (b) and (c).

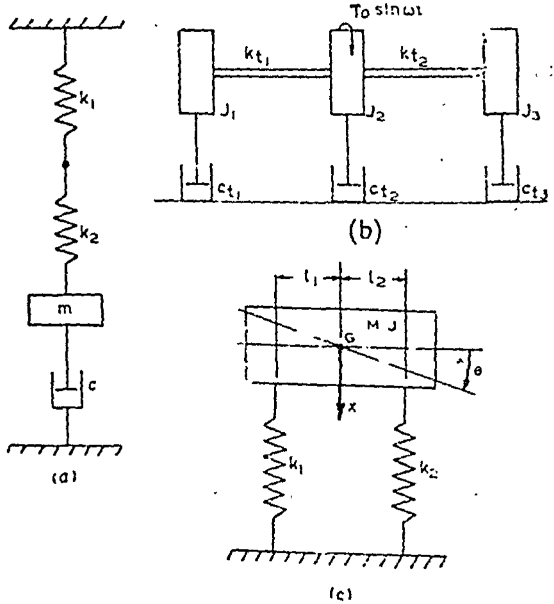
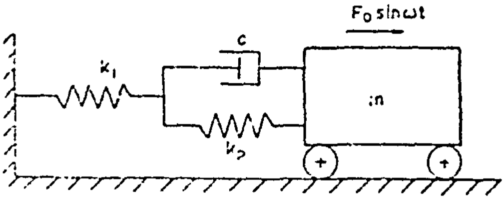
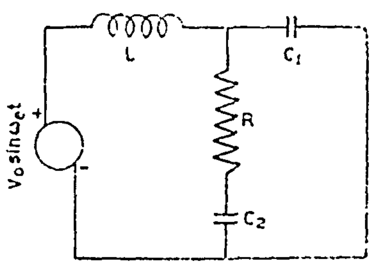


Fig. P. 11.1

11.2 Show that the electrical circuit of Fig. P. 11.2 (b) is equivalent to the mechanical system of Fig. P. 11.2 (a).



(a)



(b)

Fig. P. 11.2

11.3 A single degree of freedom spring-mass-dashpot system shown in Fig. P. 11.3 has following component values.

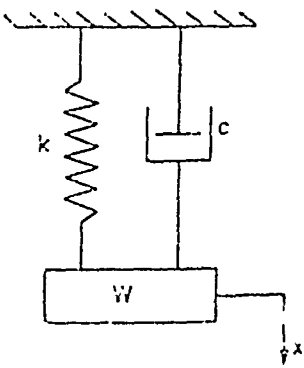


Fig. P. 11.3

$W = 5$ kg

$k = 20$ kg/cm

$c = 0.01$ kg-sec/cm.

If inductance L for the equivalent electrical circuit is 0.01 henrys and the electrical frequencies are 100 times the mechanical frequencies, determine the values for the remaining electrical circuit components.

11.4 Draw an equivalent electrical circuit for the mechanical system shown in Fig. 5.6.1 having the following physical quantities.

$m_1 = 0.010$ kg-sec^2/cm

$m_2 = 0.050$ kg-sec^3/cm

$k_1 = 20$ kg/cm

$k_2 = 14$ kg/cm

$k_3 = 7$ kg/cm

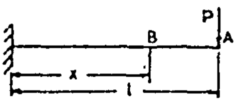
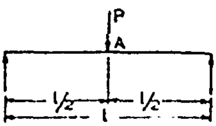
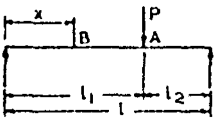
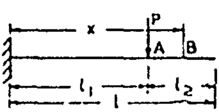
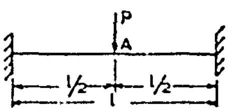
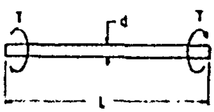
$F_o = 2$ kg

ω varies from 5 to 50 rad/sec.

Find the sizes of electrical components if the capacitance C_1 is 123 microfarads and the speed of the 24 volts generator may vary between 500 and 5000 cycles per minnte. Find also the expressions for steady state amplitudes x_1 and x_2 in terms of V_{c1} and V_{c2}.

APPENDIX—I

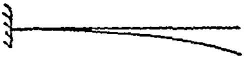
Deflection formulae

	Coil spring	$x = \frac{8PND^3}{Gd^4}$
	Cantlever with end load.	$y_A = \frac{Pl^3}{3EI}$ $y_B = \frac{Px^2(3l-x)}{6EI}$
	Simply supported beam with a central load.	$y_A = \frac{Pl^3}{48EI}$
	Simply supported beam with an off-centre load.	$y_A = \frac{Pl_1^2 l_2^2}{3EIl}$ $y_B = \frac{Pl_2 x(l^2 - x^2 - l_2^2)}{6EIl}$
	Cantilever with a non-end load.	$y_A = \frac{Pl_1^3}{3EI}$ $y_B = \frac{Pl_1^2(3x - l_1)}{6EI}$
	Fixed-fixed beam with a centre load.	$y_A = \frac{Pl^3}{192EI}$
	Shaft.	$\theta = \frac{Tl}{GI_p} = \frac{Tl}{\left(G\frac{\pi}{32}d^4\right)}$

APPENDIX—II

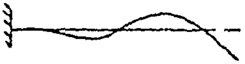
Natural frequencies of transverse vibrations of uniform beams

Cantilever



$$\omega_1 = \frac{3{\cdot}52\,a}{l^2}$$

$$\omega_2 = \frac{22{\cdot}0\,a}{l^2}$$



$$\omega_3 = \frac{61{\cdot}7\,a}{l^2}$$

Simply supported

$$\omega_1 = \frac{\pi^2 a}{l^2}$$

$$\omega_2 = \frac{4\,\pi^2\,a}{l^2}$$

$$\omega_3 = \frac{9\,\pi^2\,a}{l^2}$$

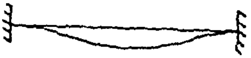
Free-free

$$\omega_1 = \frac{22{\cdot}0\,a}{l^2}$$

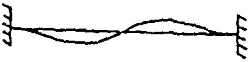
$$\omega_2 = \frac{61{\cdot}7\,a}{l^2}$$

$$\omega_3 = \frac{121{\cdot}0\,a}{l^2}$$

Fixed-fixed



$$\omega_1 = \frac{22{\cdot}0\,a}{l^2}$$



$$\omega_2 = \frac{61{\cdot}7\,a}{l^2}$$

$$\omega_3 = \frac{121{\cdot}0\,a}{l^2}$$

where $a = \sqrt{\dfrac{EIg}{\gamma A}}$

ANSWERS TO PRACTICE PROBLEMS

Chapter 1

1.1 .04 sec ; 786 cm/sec ; 123,000 cm/sec^2.

1.2 0.254 cm.

1.5 1152 π^2 cm^2/sec.

1.6 9.34 sin $(\omega t + 75.7°)$.

1.7 $x = 12.78 \sin(\omega t - 21.3°)$

1.8 6.72

1.10 13 sin $(\omega t + 242\ 2°)$

1.11 3.66 sin ωt ; 7.07 sin $(\omega t + 45°)$.

1.12 10 sin ωt ; 7.13 sin $(\omega t + 127.7°)$

1.13 2.39 cos ωt ; 6.54 cos $(\omega t + \pi/3)$

1.15 (i) periodic, 4 sec ; (ii) periodic, π sec ;
(iii) non-periodic ; (iv) non-periodic.

1.16 7 ; 1 ; $1/2\pi$ cps.

1.19 (i) $5e^{j0\ 926}$; (ii) $5e^{-j0.926}$ (or $5e^{j5.358}$)
(iii) $5e^{j2.216}$; (iv) $5e^{j4.068}$

1.20 (i) $8.6 + j\,2\ 66$; (ii) $-2.5 + j\,4.33$;
(iii) $-13.2 - j\,4.79$; (iv) $4.54 - j\,8.9$

1.21 (i) 54.5 kg-cm ; (ii) 54.5 kg-cm ; (iii) 18.6 kg.cm

1.22 (i) $x = \dfrac{4}{\pi}\sum_{n=1}^{\infty} \dfrac{1}{n} \sin 10n\pi t$

$$\text{(ii) } x = 3+\sum_{n=1}^{\infty}\left[\frac{10}{\pi n}\sin 1.2\,\pi n + \frac{8.33}{\pi^2 n^2}\,(\cos\ 1.2\pi n-1)\ \right]\cos 4n\pi t$$

$$+\sum_{n=1}^{\infty}\left[-\frac{10}{\pi n}\cos 1.2\,\pi n + \frac{8.33}{\pi^2 n^2}\ \sin\ 12\,\pi n\ \right]\sin 4n\pi t$$

$$\text{(iii) } x = \frac{2A_o}{\pi} - \frac{4A_o}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos 100\,n\pi t$$

Chapter 2

2.1 4.98 cps.

2.2 2.49 cps.

2.3 4.55 kg ; 26.4 kg/cm.

2.4 32.4 kg.

2.5 11.65 cps ; 0.134 cm ; 2205 kg/cm^2.

2.6 2 : 1

2.7 3.66 kg-cm-sec^2.

2.8 $\sqrt{2k/3m}$ rad/sec.

2.9 $\omega_n = \sqrt{\dfrac{W\ b\ \cos\alpha}{J}}$

2.10 $\sqrt{(mgl+ka^2)/ml^2}$ rad/sec ; (i) $\sqrt{g/l}$ rad/sec;
(ii) $\sqrt{g/l}$ rad/sec ; (iii) $\sqrt{k/m}$ rad/sec.

2.11 $\dfrac{1}{2\pi}\sqrt{\dfrac{ka^2}{mb^2}}$ cps.

2.12 (a) $2\pi\sqrt{\dfrac{ml^2}{ka^2-mgl}}$ sec ; $k > mgl/a^2$;

(b) $2\pi\sqrt{\dfrac{ml^2}{ka^2}}$ sec.

2.13 (i) 3.86 cps ; (ii) 1313 kg-cm.

2.14 $\sqrt{\dfrac{(k_1+k_2)g}{W_1+W_2+W_3}}$ rad/sec.

2.15 79.6 rad/sec.

2.16 1.35 cm.

2.18 $2\pi\sqrt{\dfrac{L^2\, l}{3\, g\, a^2}}$ sec.

2.19 1.94 cps.

2.20 401.56 cm.

2.21 $\sqrt{\dfrac{3\, EI\, kg}{(3\, EI + kl^3)\, W}}$ rad/sec.

2.22 $\sqrt{\dfrac{k}{m+M/2}}$ rad/sec.

2.23 $\omega_n = \sqrt{\dfrac{5g}{7(R-r)}}$

2.24 $2\pi\sqrt{\dfrac{4W}{\pi\rho g D^2}}$ rad/sec.

2.25 $2\omega\sqrt{\dfrac{8D}{3g}}$ sec ; 3.27 sec.

2.28 $\sqrt{\dfrac{12\, Rg}{l^2}}$ rad/sec.

Chapter 3

3.1 (i) $x = 2.16e^{-2.68t} - 0.155e^{-37.3t}$

(ii) $x = 2(1 + 10t)e^{-10t}$

(iii) $x = 2.04e^{-2t} \sin(9.8t + 78.5^\circ)$

3.3 0.035 cm.

3.7 0.408 kg-sec/cm.

3.8 $mL^2\, \ddot{\theta} + cb^2\, \dot{\theta} + ka^2\, \theta = 0;$

$\omega_d = \dfrac{a}{l}\sqrt{\dfrac{k}{m} - \dfrac{1}{4}\,\dfrac{b^4}{L^2 a^2}\,\dfrac{c^2}{m^2}}\,; \quad c_c = \dfrac{2\, aL}{b^2}\sqrt{km}$

3 9 $x = X_o e^{-(k/c)t}$

3.10 99 rad/sec ; 0.1875 ; 0.172 cm.

3.11 0.00545 kg-sec/cm.

3.12 0.00192 kg-sec/cm.

3.13 $X_0 / X_5 = 1.368$

3.15 5/12 kg-sec/cm; oscillatory.

3.18 9 97 cps ; 0.15 cm ; same as equilibrium position.

3.19 2.5 kg/cm; 11.14 cps ; 5/41 kg; 2/41 cm from equilibrium position.

3.20 5 full cycles; 2.27 sec ; will stop at the unstressed position.

3.21 58.3 kg/cm ; 10.2 kg.

3.22 8.03 cm ; 11 + fraction cycle ; + 0.03 cm.

3.23 $\omega_{n1} / \omega_{n2} = 2$; first system,

Chapter 4

4.1 (a) 14 rad/sec ; (b) 2.86 cm ; (c) 90°

4.2 safe.

4.6 (a) 1000 cpm ; (b) 0.0062 ; (c) 178.6°

4.9 1.02 cm ; 105.9°

4.8 0.0017 cm.

4.10 2.06 cm.

4.12 96.2 km/hr.

4.13 It will lose contact with the ground as the inertia force will be much higher than the dead weight ; 5.4 km/hr.

4.14 (a) 25.1 km/hr ; (b) 5.66 cm ; (c) 2.09 cm.

4.15 (a) $X = \dfrac{Y}{\sqrt{1 + \left(\dfrac{c\omega}{k}\right)^2}}$; $\phi = \tan^{-1} \dfrac{c\omega}{k}$.

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4.16 0.061

4.18 4.11

4.19 (i) 0.394 cm. ; (ii) 0.612 cm.

4.21 $x = 2.5 - 2.5e^{-1.816t}$ [0.204 sin 23.8t + cos 23.8t]; 2.5 cm.

4.24 (a) 0.00427 cm; (b) 0.1483; (c) 5.94 kg.

4.25 81.2 kg : 0.0121 cm; $\phi = 12.5°$

4.26 279 kg.

4.27 38.95 kg/cm.

4.28 Max Stiffness = 112.4 kg/cm.

4.29 26.6 kg/cm; 12.78 kg.

4.30 32.8 kg/cm.

4.33 0.0487 cm.

4.34 (i) 0.0297 cm; (ii) 0.0225 cm.

4.35 25.4 cps to infinity.

4.36 50700 cm/sec^2; 1.43 cm.

4.37 (i) 50.5 cm/sec^2; (ii) 59.0 cm/sec^2.

4.38 (a) 0.0056 cm; (b) 0 0734 cm/sec; (c) 0.9615 cm/sec^2.

4.39 0.045 cm; 0.565 cm/sec; 7.1 cm/sec^2.

4.40 6.62 kg-cm/rad ; 0.0662 kg-cm.

4.41 1.094 cps.

4.42 1.143 kg; 0.0715 kg.

Chapter 5

5.1 (a) $x_1 = \frac{1}{3} \sin 30\,t$; $x_2 = \frac{1}{3} \sin 30\,t$;

(b) $x_1 = \frac{1}{4} \sin 40t$; $x_2 = -\frac{1}{4} \sin 40t$;

(c) $x_1 = \frac{5}{12} \sin 30t - \frac{1}{16} \sin 40t$;

$x_2 = \frac{5}{12} \sin 30t + \frac{1}{16} \sin 40\,t$;

3.9 $x = X_0 e^{-(k/c)t}$

3.10 99 rad/sec ; 0.1875 ; 0.172 cm.

3.11 0.00545 kg-sec/cm.

3.12 0.00192 kg-sec/cm.

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(b) $x_1 = \frac{1}{4} \sin 40\,t$; $x_2 = -\frac{1}{4} \sin 40t$;

(c) $x_1 = \frac{5}{12} \sin 30t - \frac{1}{16} \sin 40t$;

$x_2 = \frac{5}{12} \sin 30t + \frac{1}{16} \sin 40\,t$;

6.4 $S = \frac{3+\sqrt{5}}{2}, \frac{1}{2}, \frac{3-\sqrt{5}}{2}$

where $S = T/ml\omega^2$.

6.6 $X_1 = X_3 = \frac{F_o}{T/l} \cdot \frac{\frac{1}{2}S^2(S-0.5)}{(S-1.707)(S-0.5)(S-0.293)}$

$$X_2 = \frac{F_o}{T/l} \cdot \frac{S(S-0.5)^2}{(S-1.707)(S-0.5)(S-0.293)}$$

6.7 $\omega_{n1} = 125.6$ rad/sec ; $\omega_{n2} = 264$ rad/sec ;

$$\left(\frac{\beta_1}{\beta_2}\right)_1 = 1.56, \left(\frac{\beta_3}{\beta_2}\right)_1 = -2.62 ;$$

$$\left(\frac{\beta_1}{\beta_2}\right)_2 = -1.72, \left(\frac{\beta_3}{\beta_2}\right)_2 = -0.196.$$

6.8 (i) 75 rad/sec; (ii) 75.4, 608 rad/sec.

6.10 $q_1 = \frac{1}{2}(x_1+x_2)$; $q_2 = \frac{1}{2}(x_1-x_2)$.

6.11 $q_1 = \frac{1}{2}(y_1+y_2)$; $q_2 = \frac{1}{2}(y_1-y_2)$.

6.12 $y = \frac{9\sqrt{3}}{2}\frac{a_o}{\pi^2}\Big[\sin\frac{\pi x}{l}\cos\frac{\pi ct}{l} + \frac{1}{2^2}\sin\frac{2\pi x}{l}\cos\frac{2\pi ct}{l}$

$$-\frac{1}{4^2}\sin\frac{4\pi x}{l}\cos\frac{4\pi ct}{l} - \frac{1}{5^2}\sin\frac{5\pi x}{l}\cos\frac{5\pi ct}{l} + \dots\Big]$$

6.13 $y = \frac{8V_o l}{\pi^3 c}\Big[\sin\frac{\pi x}{l}\sin\frac{\pi ct}{l} - \frac{1}{27}\sin\frac{3\pi x}{l}\sin\frac{3\pi ct}{l}$

$$+\frac{1}{125}\sin\frac{5\pi x}{l}\sin\frac{5\pi ct}{l} - \dots\Big]$$

6.15 $u = \frac{4e}{\pi^2}\sum_{i=1,3\dots}^{\infty}\frac{1}{i^2}\cos\frac{i\pi x}{l}\cos\frac{i\pi ct}{l}$

6.16 $\theta = \frac{8\theta_o}{\pi^2}\sum_{i=1,3\dots}^{\infty}(-1)^{\frac{i-1}{2}}\frac{1}{i^2}\sin\frac{i\pi x}{l}\cos\frac{i\pi ct}{l}$

6.17 $y = \frac{2Pl^2}{\pi^4 EI}\sum_{i=1}^{\infty}\frac{1}{i^4}\sin\frac{i\pi c}{l}\cos\frac{i^2\pi^2 at}{l^2}$

6.18 $\cos k_t l \cosh k_t l = 1$

6.19 $\tan k_t l = \tanh k_t l$

Chapter 7

7.1 282 rad/sec.

7.2 2400 rad/sec.

7.4 $\omega_n = 21 \times 10^{-4} \sqrt{EI}$

7.12 $0.574 \sqrt{k/m}$ rad/sec.

7.15 $0.65 \sqrt{g/l}$ rad/sec.

7.16 $\omega_n = 0.67 \sqrt{T/ml}$

7.17 $0.618 \sqrt{T/ml}$ rad/sec.

7.21 44 rad/sec.

7.22 226 rad/sec.

7.23 $\beta_1 = 0.000087$ radians.

Chapter 8

8.1 18,550 kg/cm.

8 2 -0.0442 cm; dyn load on each brg = 1.62 kg;
f_{max} (vertical) = 331 kg/cm^2 ; f_{max} (horizontal) = 1045 kg/cm^2

8.6 $f_{st} = 840$ kg/cm^2 ; $f_{d\,(max)} = 12{,}040$ kg/cm^2; 0 695.

8 8 2390 rad/sec ; 142000 rad/sec.

8.9 210 rad/sec ; 456 rad/sec.

8.10 3330 rpm ; 20820 rpm.

Chapter 11

11.3 $C = 2.55\ \mu$F ;
$R = 1.96$ ohms.

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