Conditional Probability

In this section, we examine how the information "an event B has occured" affects the probability assigned to A.

(*) For any two events A and B with $P(B) > 0$, the conditional probability of A given that B has occured is defined by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

(*) Multiplication rule for two events

$$P(A \cap B) = P(A|B) \cdot P(B).$$

Example. A fair coin is tossed two times. What is the probability that the second coin is a head if you know that at least one head appears.

The four outcomes are $\{HH, HT, TH, \cancel{TT}\}$.

Out of four, that satisfy the condition exactly two have heads for the second toss.

$\therefore P\left(\begin{array}{l}\text{second toss is a head given}\\ \text{at least one head}\end{array}\right) = \frac{2}{3}.$

Example. There are 100 people at a party
Forty are liars
Twenty-five are lawyers
15 of the lawyers are liar.

a. If a person is drawn at random at the party, what is the probability that he or she is a lawyer?

b. If a person is drawn at random at the party, what is the probability that he or she is a lawyer if you know that he or she is a liar?

(*) For 3 events

$$P(A \cap B \cap C) = P(A) \cdot P(B|A) \cdot P(C|A \cap B)$$

proof $\quad P(A) \cdot P(B|A) \cdot P(C|A \cap B)$.

$$= P(A) \cdot \frac{P(B \cap A)}{P(A)} \cdot \frac{P(C \cap A \cap B)}{P(A \cap B)}.$$

$$= P(A \cap B \cap C).$$

(*) For 'n' events.

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1) \cdot P(A_2|A_1) \cdot P(A_3|A_1 \cap A_2) \cdot \cdots P(A_n|A_1 \cap A_2 \cap \cdots \cap A_{n-1})$$

proof

$$R.H.S = P(A_1) \cdot \frac{P(A_2 \cap A_1)}{P(A_1)} \cdot \frac{P(A_1 \cap A_2 \cap A_3)}{P(A_2 \cap A_1)} \cdot \cdots \frac{P(A_1 \cap A_2 \cap \cdots \cap A_n)}{P(A_1 \cap \cdots \cap A_{n-1})} = L.H.S.$$

Independent Event.

Let A and B are two events associated with sample space, then, event A and B are said to be independent if and only if

$$P(A \cap B) = P(A) \cdot P(B).$$

Equivalently, two events A and B are independent if $P(A|B) = P(A)$. $- \dfrac{P(A \cap B)}{P(B)}$.

Example: The outcome of the second toss is in no way dependent on the outcome of the first toss.

Non-independent events

Two events are not independent if the probability of one event depends on the occurance or non occurance of the other event.

Ex. Two cards are drawn from a standard deck without replacing the first card.

(*) If A and B are independent, then A^c and B^c are also independent.

proof: $P(A \cap B) = P(A) \cdot P(B)$

claim: $P(A^c \cap B^c) = P(A^c) \cdot P(B^c)$

$\Rightarrow A^c \cap B^c = A^c \cup B^c = (A \cup B)^c$

$P(A^c \cap B^c) = P((A \cup B)^c)$

$= 1 - P(A \cup B)$ $[P(A^c) = 1 - P(A)]$

$= 1 - (P(A) + P(B) - P(A \cap B))$

$= 1 + P(A \cap B) - P(A) - P(B)$

$= 1 + P(A) \cdot P(B) - P(A) - P(B)$.

$= P(A^c) + P(B)(P(A) - 1)$

$= P(A^c) - P(A^c) \cdot P(B)$

$= P(A^c) \cdot (1 - P(B)) = P(A^c) \cdot P(B^c)$.

Total probability Theorem

Let $A_1, A_2, \ldots, A_n$ be n events that form a partitions of the sample space S, where all the events have a non-zero probability of occurance.

For any event B associated with S, then the total probability of B is given as

$$P(B) = \sum_{i=1}^{n} P(A_i) \cdot P(B|A_i).$$

proof Given $\bigcup_{i=1}^{n} A_i = S$

$$B = B \cap S$$
$$= B \cap \bigcup_{i=1}^{n} A_i$$

$$B = (B \cap A_1) \text{ or } (B \cap A_2) \text{ or } \cdots \text{ or } (B \cap A_n)$$
$$= (B \cap A_1) \cup (B \cap A_2) \cup \cdots \cup (B \cap A_n)$$

$$P(B) = P(B \cap A_1) + P(B \cap A_2) + \cdots + P(B \cap A_n)$$
$$= P(A_1) \cdot P(B|A_1) + P(A_2) \cdot P(B|A_2) + \cdots + P(A_n) \cdot P(B|A_n)$$
$$= \sum_{i=1}^{n} P(A_i)\, P(B|A_i).$$

Example:
A person has undertaken a mining job. The probabilities of completion of job on time with and without rain are 0.42 and 0.90 resp. If the probability that it will rain is 0.45, then determine the probability that the mining job will be completed on time.

Sol^n: Let A = event that the mining job will be completed.
B = event that it rains.

We have $P(B) = 0.45$

$$P(\text{no rain}) = 1 - 0.45 = 0.55.$$

$P(A|B) = 0.42$

$P(A|B') = 0.9$

Events B and B' form partitions of the sample space S.

By total probability theorem; we have

$$P(A) = P(B)\ P(A|B) + P(B')\ P(A|B')$$
$$= 0.45 \times 0.42 + 0.55 \times 0.9 = 0.684.$$

$\therefore$ P(the job will be completed on time) = 0.684.