Expectation Value :

Let X be a continious random variable, then $E(x)$ is defined as

$$E(x) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

$$V(x) = E(x^2) - (E(x))^2, \text{ where}$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx.$$

Uniform distribution :

A continuous rv X is said to have a uniform distribution on the interval $[A, B]$ if the pdf of X is

$$f(x) = \begin{cases} \frac{1}{B-A} & \text{if } A \leq x \leq B \\ 0 & \text{otherwise.} \end{cases}$$

Q. a) Verify if it's proper pdf.

b) Find $E(x)$ and $V(x)$.

Soln a)

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

$$\| $$

$$\cancel{\int_{-\infty}^{A} f(x)\, dx}^{\,0} + \int_{A}^{B} f(x)\, dx + \cancel{\int_{B}^{\infty} f(x)\, dx}^{\,0}$$

$$\| $$

$$\int_{A}^{B} \frac{1}{B-A}\, dx = \frac{1}{B-A}\, x \Big|_{A}^{B} = \frac{B-A}{B-A} = 1 \text{ (R.H.S)}$$

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$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

$$= \int_A^B \frac{x}{B-A}\, dx = \left.\frac{x^2}{2}\right|_A^B \times \frac{1}{B-A}$$

$$= \frac{1}{B-A} \times \frac{B^2-A^2}{2} = \frac{B+A}{2}$$

$$V(X) = E(X^2) - (E(X))^2$$

$$E(X^2) = \int_A^B x^2 \cdot \frac{1}{B-A}\, dx = \frac{1}{B-A} \left.\frac{x^3}{3}\right|_A^B$$

$$= \frac{1}{B-A} \times \frac{B^3-A^3}{3}$$

$$= \frac{B^2+A^2+AB}{3}$$

$$\therefore V(X) = \frac{B^2+A^2+AB}{3} - \frac{(B+A)^2}{4}$$

Q2. Suppose the reaction temp X in a certain chemical process has a uniform dist. with $A=-5$ and $B=5$.

a) Compute $P(X<10)$. (b) $P(-2.5 < X < 2.5)$

c) $P(-2 \leq X \leq 3)$

d) For 'K' satisfying $-5 < K \leq K+4 \leq 5$. Compute $P(K < X < K+4)$.

Soln:

Given $f(x)$ is uniform in $[-5, 5]$ (with $A = -5$, $B = 5$)

$$\therefore f_X(x) = \frac{1}{B-A} = \frac{1}{5-(-5)} = \frac{1}{10} = 0.1$$

Hence, $f(x) = \begin{cases} 0.1 & -5 \leq x \leq 5 \\ 0 & \text{otherwise.} \end{cases}$

a) $P(X<10) = P(-5 \leq X \leq 5)$

$\underset{\text{lower limit}}{-5}$, $\underset{\text{upper limit}}{5}$

$$P(a \leq X \leq b) = \int_a^b 0.1\, dx = 0.1 \int_a^b dx = 0.1(b-a)$$

$$\therefore P(\underset{a}{-5} \leq X \leq \underset{b}{5}) = 0.1(5-(-5)) = 1$$

b) $P(\underset{a}{-2.5} \leq X \leq \underset{b}{2.5}) = 0.1(2.5-(-2.5)) = 0.5$

c) $P(\underset{a}{-2} \leq X \leq \underset{b}{3}) = 0.1(3-(-2)) = 0.5$

d) $P(\underset{a}{K} < X < \underset{b}{K+4}) = 0.1((K+4)-K) = 0.4$.

Q5. Let X = the time that elapses betn the end of the hour and the end of the lecture.

pdf of X is $f(x) = \begin{cases} Kx^2 & 0 \leq x \leq 2 \\ 0 & \text{otherwise.} \end{cases}$

a) Find the value of 'K'.

We know $\int_{-\infty}^{\infty} f(x)\, dx = 1$

$$\int_0^2 Kx^2\, dx = \left.\frac{Kx^3}{3}\right|_0^2 = \frac{K\, 2^3}{3} = 1$$

$$\Rightarrow K = \frac{3}{8}.$$

$$\therefore f(x) = \begin{cases} \frac{3}{8} x^2 & \text{for } 1 \leq x \leq 2 \\ 0 & \text{otherwise.} \end{cases}$$

b) P(the lecture ends within 1 min of the end of the hour)

$$P(X<1) = \int_0^1 \frac{3x^2}{8}\,dx = \left.\frac{x^3}{8}\right|_0^1 = 0.125$$

c) P(the lecture continues beyond the hour for betn 60 and 90 sec.)

$$= P(1<X<1.5) = \int_1^{1.5} \frac{3x^2}{8}\,dx = \left.\frac{x^3}{8}\right|_1^{1.5} = 0.2969.$$

d) P(the lecture cont. for atleast 90 sec. beyond the end of the hour.)

$$= P(X \geq 1.5) = \int_{1.5}^{2} \frac{3x^2}{8}\,dx$$

$$= \left.\frac{x^3}{8}\right|_{1.5}^{2} = 0.5781.$$