The Poisson Probability Distribution.

A discrete random variable X is said to have a Poisson distribution with parameter μ ($\mu > 0$) if the pmf of X is

$$p(x;\mu) = \frac{e^{-\mu} \cdot \mu^x}{x!} \quad \text{for } x = 0, 1, 2, 3, \ldots$$

$$\left(e^{\mu} = \sum_{x=0}^{\infty} \frac{\mu^x}{x!} \right)$$

Suppose that in the binomial pmf $b(x;n,p)$, we let $n \longrightarrow \infty$ and $p \longrightarrow 0$ in such a way that np approaches a value $\mu > 0$. Then $b(x;n,p) \longrightarrow p(x;\mu)$.

→ In any binomial experiment in which 'n' is large and p is small, $b(x;n,p) \approx p(x;\mu)$, where $\mu = np$.

(If $n > 50$ and $np < 5$)

(or)

When probability of success is very small and number of trials is very large, such that, np is some constant "μ", then instead of binomial distribution, it is suggested to use, poisson distribution.

proof For binomial distribution.

$$p_X(x) = \binom{n}{x} p^x q^{n-x}$$

$$= \frac{n!}{x!\,(n-x)!} \left(\frac{\mu}{n}\right)^x \left(1 - \frac{\mu}{n}\right)^{n-x}$$

$$= \frac{n(n-1)\cdots(n-x+1)}{x!} \left(\frac{\mu}{n}\right)^x \left(1 - \frac{\mu}{n}\right)^x$$

$$= \frac{n \cdot n\left(1-\frac{1}{n}\right)\cdot n\left(1-\frac{2}{n}\right)\cdots n\left(1-\frac{(x-1)}{n}\right)}{x!}\left(\frac{\mu}{n}\right)^x\left(1-\frac{\mu}{n}\right)^{n-x}$$

$$= \frac{\cancel{n^x}}{x!}\underbrace{\left[\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{(x-1)}{n}\right)\right]}_{\frac{n(n-1)(n-2)\cdots(n-x+1)}{n^x}}\frac{\mu^x}{\cancel{n^x}}\cdot\left(1-\frac{\mu}{n}\right)^n\left(1-\frac{\mu}{n}\right)^{-x}$$

Taking limits on both sides:

$$\lim_{n\to\infty}\left\{\frac{n(n-1)\cdots(n-x+1)}{n^x}\right\} = 1$$

$$\lim_{n\to\infty}\left(1-\frac{\mu}{n}\right)^n = e^{-\mu}$$

$$\lim_{n\to\infty}\left(1-\frac{\mu}{n}\right)^{-x} = 1$$

$$p(x;\mu) = \frac{\mu^x}{x!}e^{-\mu} = \lim_{n\to\infty} b(x;n,p) \quad //.$$

Ex. starting at a fixed time, each car entering an intersection is observed to see either it turn left (L), right (R) or goes straight ahead (A). The experinment terminates as soon as car is observed to turn left. Let X is the number of car observed. What are the possible values of X.

$\{L\} \to \{1\}$

$\{RL\} \to \{2\}$

$\{ARL\} \to \{3\}$

$\{RRRL\} \to \{4\}$

$\{ARARL\} \to \{5\}$

$\vdots \qquad \{\infty\}$

Q. I) Verify that it's a proper pmf.

II) Find expected value of X

III) Variance

Sol: I) $p_X(x) = e^{-\mu} \dfrac{\mu^x}{x!}$

So $\sum_{x=0}^{n\to\infty} p_X(x) = 1$

$$\therefore \sum_{x=0}^{n\to\infty} p_X(x) = \sum_{x=0}^{\infty} \left(e^{-\mu} \frac{\mu^x}{x!} \right)$$

$$= e^{-\mu} \sum_{x=0}^{\infty} \frac{\mu^x}{x!}$$

$$= e^{-\mu} \cdot e^{\mu} = 1. \text{ //}$$

II) $E(X) = \sum_{x=0}^{\infty} x_i \, p_X(x_i)$ for $x = 0, 1, 2, \cdots$
$\mu > 0$

$$= \sum_{x=0}^{\infty} x \cdot e^{-\mu} \frac{\mu^x}{x!}$$

$$= e^{-\mu} \sum_{x=1}^{\infty} \frac{\mu^x}{(x-1)!}$$

Let $(x-1) = t$ $\quad x = 1 \Rightarrow t = 0$

$x = \infty \Rightarrow t = \infty$

$$= e^{-\mu} \sum_{t=0}^{\infty} \frac{\mu^{t+1}}{t!}$$

$$= e^{-\mu} \cdot \mu \cdot \sum_{t=0}^{\infty} \frac{\mu^t}{t!} = e^{-\mu} \cdot \mu \cdot e^{\mu}$$

$$= \mu.$$

$$\boxed{\therefore E(X) = \mu}$$

iii) $V(x) = E(x^2) - (E(x))^2$

$$E(x^2) = E(x(x-1)+x) = E(x(x-1)) + E(x).$$

$$E(x(x-1)) = \sum_{x=0}^{\infty} (x(x-1)) \cdot \frac{e^{-\mu} \cdot \mu^x}{x!}$$

$$= \sum_{x=2}^{\infty} \frac{e^{-\mu} \cdot \mu^x}{(x-2)!}$$

Let $x-2=t$ $\quad x=2,\ t=0$

$x=\infty,\ t=\infty.$

$$= \sum_{t=0}^{\infty} \frac{e^{-\mu} \cdot \mu^{t+2}}{t!}$$

$$= e^{-\mu} \cdot \mu^2 \cdot \sum_{t=0}^{\infty} \frac{\mu^t}{t!}$$

$$= \mu^2 e^{-\mu} \cdot e^{\mu} = \mu^2.$$

So, $E(x^2) = \mu^2 + \mu.$

$$V(x) = E(x^2) - (E(x))^2$$

$$= \mu^2 + \mu - \mu^2$$

$$\boxed{V(X = \mu}$$

<u>Note</u>: Poisson's distribution is the only distribution in statistics, where the mean and variance of random variable is same.