## Bernoulli trial.

In the theory of probability and statistics, a Bernoulli trial (binomial trial) is a random-experiment with exactly two possible outcomes, "success" and "failure", in which the probability of success is the same every time the experiment is conducted.

Generally, we denote success as 'p' and failure as 'q' $= 1-p$ $(p+q=1)$.

## Binomial Distribution.

A discrete random variable X is said to have a binomial distribution, if it's pmf is given as.

$$b(x; n, p) = {}^{n}C_{x}\, p^{x}\, q^{n-x}$$

(X depends on the parameters n and p.)

$$= {}^{n}C_{x}\, p^{x}\, (1-p)^{n-x} \quad \text{for } x = 0, 1, \dots, n$$

Here n: number of trials (fixed)

X: number of success in 'n' trials such that $p+q=1$.

$\boxed{X \sim b(n,p)}$ ($\sim$: as distributed as)

Q. 1) Check if it's a proper pmf.

2) Find expected values of X.

3) Find variance of X.

(1) We know $p_X(x) > 0$, and $\sum_{x \in R_X} p_X(x) = 1$ for $x = 0, 1, 2, \cdots, n$

Now, RHS
$$\sum_{x \in R_X} p_X(x) = \sum_{x \in 0,1,\cdots,n} {}^{n}C_x \, p^x q^{n-x}$$

$$= {}^{n}C_0 \, p^0 q^n + {}^{n}C_1 \, p \, q^{n-1} + {}^{n}C_2 \, p^2 q^{n-2} + \cdots + {}^{n}C_n \, p^n q^0$$

$$= q^n + n p q^{n-1} + \frac{n(n-1)}{2} p^2 q^{n-2} + \cdots + p^n$$

$$= (p+q)^n$$

$$= 1^n = 1. \;//$$

(2) $E(x) = \sum_{x \in R_X} x \, p_X(x)$ for $x \in 0, 1, 2, \cdots, n.$

$$= \sum_{x=0}^{n} x \cdot {}^{n}C_x \, p^x q^{n-x}$$

$$= \sum_{x=1}^{n} \left( \frac{n(n-1)!}{(n-x)!(x-1)!} \right) p^x q^{n-x}$$

Let $x - 1 = t$ $\quad t \in \{0, 1, \cdots, n-1$

So, $$\sum_{t=0}^{n-1} \left( \frac{n(n-1)!}{t!(n-t-1)!} \right) p^{t+1} q^{n-t-1}$$

$$= np \sum_{t=0}^{n-1} \left( \frac{(n-1)!}{t!(n-(t+1))!} \right) p^t q^{n-t-1}.$$

$$= np \underbrace{\sum_{t=0}^{n-1} {}^{n-1}C_t \, p^t q^{n-t-1}}_{\|}$$

$$= np \, (p+q)^{n-1}$$

$$= np.$$

$$\boxed{\therefore E(x) = np}$$

3) $V(X) = E(X^2) - (E(X))^2$

$$\therefore E(X^2) = E(X(X-1)+X)$$
$$= E(X(X-1)) + E(X)$$
$$= E(X(X-1)) + np.$$

Now, $E(X(X-1)) = \sum_{x=0}^{n} x(x-1)\, {}^nC_x\, p^x q^{n-x}.$

$$= \sum_{x=0}^{n} x(x-1) \frac{n!}{(n-x)!\, x!} p^x q^{n-x}$$

Let $x$

$$= \sum_{x=2}^{n} \frac{n!}{(n-x)!\,(x-2)!} p^x q^{n-x}$$

Let $x-2 = t \Rightarrow t \in 0, 1, \cdots, n-2$.

$$= \sum_{t=0}^{n-2} \frac{n!}{t!\,(n-t-2)!} p^{t+2} q^{n-t-2}$$

$$= n(n-1)p^2 \sum_{t=0}^{n-2} \frac{(n-2)!}{(n-2-t)!\,t!} p^t q^{(n-2)-t}$$

$$= n(n-1)p^2 (p+q)^{n-2}$$

$$= n(n-1)p^2.$$

$$\therefore E(X^2) = n(n-1)p^2 + np.$$

$$\therefore V(X) = n(n-1)p^2 + np - n^2p^2$$
$$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$$
$$= np(1-p).$$
$$= npq$$

$$\boxed{\therefore V(X) = npq} \qquad \therefore \sigma_X = \sqrt{npq}$$

Ex. Compute the following binomial probabilities directly from the formula for $b(x;n,p)$

a) $b(3;8,0.35)$

b) $P(3 \leq X \leq 5)$ when $n=7$ and $p=0.6$.

Solⁿ: a) $\underset{\substack{\| \\ x}}{b(3}; \underset{\substack{\| \\ n}}{8}, \underset{\substack{\| \\ p}}{0.35}) = {}^{8}C_{3}\ 0.53^{3}\ (1-0.53)^{8-3}$

$= {}^{8}C_{3}\ 0.53^{3} \cdot 0.65^{5} = 0.2786.$

b) For $n=7$ and $p=0.6$

$P(3 \leq X \leq 5) = b(3;7,0.6) + b(4;7,0.6) + b(5;7,0.6)$

$= {}^{7}C_{3}\ 0.6^{3}(1-0.6)^{7-3} + {}^{7}C_{4}\ 0.6^{4}(1-0.6)^{7-4}$

$+ {}^{7}C_{5}\ 0.6^{5}(1-0.6)^{7-5}$

$= 0.7451.$