Joint probability distributions and Random samples

Chapter 5

Let X and Y be two discrete rv's defined on the sample space S of an experiment. The joint probability mass function $p(x,y)$ is defined for each pair of numbers (x,y) by

$$p(x,y) = P(X=x \text{ and } Y=y).$$

It must satisfy the following conditions;

i) $p(x,y) \geq 0$

ii) $\sum_x \sum_y p(x,y) = 1$.

The marginal probability mass function of X, denoted by $p_X(x)$, is given by

$$p_X(x) = \sum_{y: p(x,y)>0} p(x,y) \quad \text{for each possible value of } x.$$

Similarly, the marginal probability mass function of Y, denoted by $p_Y(y)$, is given by

$$p_Y(y) = \sum_{x: p(x,y)>0} p(x,y) \quad \text{for each possible value } y.$$

E.g. A service station has both self-service and full-service islands. On each island, there is a single regular unleaded pump with two hoses. Let X denote the number of hoses being used on the self-service island at a particular time, and let Y denote the number of hoses on the full-service islands in use at that time. The joint pmf of X and Y appears in the accompanying tabulation.

	$p(x,y)$	0	1	2
	0	0.1	0.04	0.02
$\underline{x}$	1	0.08	0.2	0.06
	2.	0.06	0.14	0.3.

a) Find $P(X=1 \text{ and } Y=1) = p(1,1) = 0.2$

b) Find $P(X \leq 1 \text{ and } Y \leq 1)$

$\parallel$

$(X \leq 1 \text{ and } Y \leq 1) = (X=0, Y=0), (X=0, Y=1), (X=1, Y=0), (X=1, Y=1)$

$\parallel$

$p(0,0) + p(0,1) + p(1,0) + p(1,1).$

$0.1 + 0.04 + 0.08 + 0.2 = 0.42.$

c) Word description of the event $(X \neq 0 \text{ and } Y \neq 0)$ and compute the probability of this event.

$(X \neq 0 \text{ and } Y \neq 0)$ = At least one of the hosen is used on both Islands.

$P(X \neq 0 \text{ and } Y \neq 0) = p(1,1) + p(1,2) + p(2,1) + p(2,2)$

$= 0.2 + 0.06 + 0.14 + 0.3$

$= 0.7.$

d) Compute the marginal pmf of X and of Y. Using $P_X(x)$, what is $P(X \leq 1)$?

$$p_X(x) = \sum_{y \in \{0,1,2\}} p(x,y) = p(x,0) + p(x,1) + p(x,2)$$

$p(x,0) = 0.1 + 0.04 + 0.02 = 0.16.$

$p(x,1) = 0.08 + 0.2 + 0.06 = 0.34$

$p(x,2) = 0.06 + 0.14 + 0.3 = 0.5$

$\therefore p_X(x) = 0.16 + 0.34 + 0.5 = 1.0$

$$p_Y(y) = \sum_{x \in \{0,1,2\}} p(x,y) = p(0,y) + p(1,y) + p(2,y).$$
$$= 0.24 + 0.38 + 0.38 .$$

$P(X \leq 1) = p_X(0) + p_X(1) = 0.16 + 0.34 = 0.5.$

Def^n Two random variables X and Y are said to be independent if for every pair of x and y values

$$p(x,y) = p_X(x) \cdot p_Y(y) .$$

when X and Y are discrete.

If X and Y not satisfied the above condition for all (x,y), then X and Y are said to be dependent.

e) Are X and Y independent rv's.

We have, $p_X(2) = 0.5$, $p_Y(2) = 0.38$

$p_X(2) \times p_Y(2) = 0.5 \times 0.38 = 0.19.$

$p(2,2) = 0.3 .$

$\therefore p_X(2) \cdot p_Y(2) \neq p(2,2).$

$\therefore$ Therefore, X and Y are independent rvs.

Selected questions.

1, 2, 3, 6, 9, 11, 12, 19.