Note The important application of poisson's distribution is found in poisson's process.

Poisson's process:
Occurance of event occuring over time is called Poisson's process. The parameter 'α' is specified as rate of process.

Number of events by the time interval of length T, can be modelled using Poisson random variable with parameter $\mu = \alpha \cdot T$. This indicates that,

$$p_x(t) = \frac{e^{-(\alpha t)} (\alpha t)^x}{x!}.$$

The expected number of event during any such time interval is, ~~there~~

$$E(t) = \alpha t \quad \text{and} \quad V(t) = \alpha t.$$

Example: The number of requests for assistance ~~request~~ received by a towing service, is a poisson's dist. with rate $\alpha = 4$/hour.

i) Compute the probability that, exactly 10 request are received during a particular 2 hrs. period.

ii) How many calls do you expect during this time interval.

iii) If the operators of the towing service take a 30-min. break for lunch, what is the prob that they do not miss any calls for assistance?

Sol^n a) For a two hour period the parameter of the distribution is

$$\mu = \alpha t = 4 \times 2 = 8.$$

$$P(X=10) = \frac{e^{-(8)} \cdot 8^{10}}{10!} = 0.099$$

b) $E(t) = 2 \times 4 = 8.$

c) Y = probability that they don't miss any calls during lunch break of half an hour.

$\alpha = 4, \; t = 0.5$

$\mu = 4 \times 0.5 = 2$

$$P(Y=0) = \frac{e^{-2} \times (2)^0}{0!} = 0.135$$

Moment Generating Function

Let X be a discrete random variable, then moment generating function of X is denoted as,

$$M_X(t) = E(e^{tX}), \; t \in \mathbb{R}. \qquad \boxed{\left.\frac{d^n}{dt^n} M_X(t)\right|_{t=0} = E(X^n)}$$

(1) Moment generating function of Binomial distribution

$$p_X(x) = {}^nC_x \, p^x q^{n-x}$$

$$M_X(t) = E(e^{tx})$$

$$= \sum_{x=0}^{n} e^{tx} p_X(x) = \sum_{x=0}^{n} e^{tx} \, {}^nC_x \, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} {}^nC_x (pe^t)^x q^{n-x}$$

$$= (pe^t + q)^n$$

$$\therefore M_X(t) = (pe^t + q)^n.$$

For first order,

$$\left.\frac{d M_x(t)}{dt}\right|_{t=0} = \left. n(pe^t + q)^{n-1} pe^t \right|_{t=0}$$

$$= pn(\underbrace{p+q}_{1})^{n-1}$$

$$= np.$$

Exercise i) Find Moment generating func. of Poisson's distribution.

ii) Given an example where $E(X)$ doesn't exist.