

# Modern Quantum Chemistry, Szabo & Ostlund

## HW

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## 3 The Hartree-Fock Approximation

### 3.1 The HF Equations

#### 3.1.1 The Coulomb and Exchange Operators

#### 3.1.2 The Fock Operator

**Ex 3.1**

$$\begin{aligned}
 \langle \chi_i | \hat{f} | \chi_j \rangle &= \left\langle \chi_i(1) \left| h(1) + \sum_b [\mathcal{J}_b(1) - \mathcal{K}_b(1)] \right| \chi_j(1) \right\rangle \\
 &= [i|h|j] + \sum_{b \neq j} \left[ \left\langle \chi_i(1) \chi_b(2) \left| \frac{1}{r_{12}} \right| \chi_b(2) \chi_j(1) \right\rangle - \left\langle \chi_i(1) \chi_b(2) \left| \frac{1}{r_{12}} \right| \chi_b(1) \chi_j(2) \right\rangle \right] \\
 &= [i|h|j] + \sum_{b \neq j} ([ij|bb] - [ib|bj])
 \end{aligned} \tag{3.1.1}$$

Since

$$[ij|jj] - [ij|jj] = 0 \tag{3.1.2}$$

we have

$$\begin{aligned}
 \langle \chi_i | \hat{f} | \chi_j \rangle &= \langle i | h | j \rangle + \sum_b (\langle ib | jb \rangle - \langle ib | bj \rangle) \\
 &= \langle i | h | j \rangle + \sum_b \langle ib || jb \rangle
 \end{aligned} \tag{3.1.3}$$

### 3.2 Derivation of the HF Equations

#### 3.2.1 Functional Variation

#### 3.2.2 Minimization of the Energy of a Single Determinant

**Ex 3.2** Take the complex conjugate of

$$\mathcal{L}[\{\chi_\alpha\}] = E_0[\{\chi_\alpha\}] - \sum_a^N \sum_b^N \varepsilon_{ba}([a|b] - \delta_{ab}) \tag{3.2.1}$$

we have

$$\mathcal{L}[\{\chi_\alpha\}]^* = E_0[\{\chi_\alpha\}]^* - \sum_a^N \sum_b^N \varepsilon_{ba}^*([a|b]^* - \delta_{ab}^*) \tag{3.2.2}$$

i.e.

$$\mathcal{L}[\{\chi_\alpha\}] = E_0[\{\chi_\alpha\}] - \sum_a^N \sum_b^N \varepsilon_{ba}^*([b|a] - \delta_{ab}) \tag{3.2.3}$$

thus

$$\sum_a^N \sum_b^N \varepsilon_{ba}([a|b] - \delta_{ab}) = \sum_a^N \sum_b^N \varepsilon_{ba}^*([b|a] - \delta_{ab}) = \sum_b^N \sum_a^N \varepsilon_{ab}^*([a|b] - \delta_{ba}) \tag{3.2.4}$$

$\therefore$

$$\varepsilon_{ba} = \varepsilon_{ab}^* \tag{3.2.5}$$

**Ex 3.3**  $\therefore$

$$[\delta \chi_a | h | \chi_a] = [\chi_a | h | \delta \chi_a]^* \tag{3.2.6}$$

$$[\chi_a \delta \chi_a | \chi_b \chi_b] = [\delta \chi_a \chi_a | \chi_b \chi_b]^* \tag{3.2.7}$$

$$[\chi_a \chi_a | \chi_b \delta \chi_b] = [\chi_a \chi_a | \delta \chi_b \chi_b]^* \tag{3.2.8}$$

$$[\chi_a \chi_b | \chi_b \delta \chi_a] = [\chi_b \delta \chi_a | \chi_a \chi_b] = [\delta \chi_a \chi_b | \chi_b \chi_a]^* \tag{3.2.9}$$

$$[\chi_a \chi_b | \delta \chi_b \chi_a] = [\delta \chi_b \chi_a | \chi_a \chi_b] = [\chi_a \delta \chi_b | \chi_b \chi_a]^* \tag{3.2.10}$$

∴

$$\begin{aligned}\delta E_0 &= \sum_a^N [\delta \chi_a | h | \chi_a] + \frac{1}{2} \sum_a^N \sum_b^N ([\delta \chi_a \chi_a | \chi_b \chi_b] + [\chi_a \chi_a | \delta \chi_b \chi_b]) \\ &\quad - \frac{1}{2} \sum_a^N \sum_b^N ([\delta \chi_a \chi_b | \chi_b \chi_a] + [\chi_a \chi_b | \delta \chi_b \chi_a]) + \text{complex conjugates}\end{aligned}\quad (3.2.11)$$

while

$$\sum_a^N \sum_b^N [\chi_a \chi_a | \delta \chi_b \chi_b] = \sum_b^N \sum_a^N [\chi_b \chi_b | \delta \chi_a \chi_a] = \sum_a^N \sum_b^N [\delta \chi_a \chi_a | \chi_b \chi_b] \quad (3.2.12)$$

$$\sum_a^N \sum_b^N [\chi_a \chi_b | \delta \chi_b \chi_a] = \sum_b^N \sum_a^N [\chi_b \chi_a | \delta \chi_a \chi_b] = \sum_a^N \sum_b^N [\delta \chi_a \chi_b | \chi_b \chi_a] \quad (3.2.13)$$

thus

$$\delta E_0 = \sum_a^N [\delta \chi_a | h | \chi_a] + \sum_a^N \sum_b^N ([\delta \chi_a \chi_a | \chi_b \chi_b] - [\delta \chi_a \chi_b | \chi_b \chi_a]) + \text{complex conjugates} \quad (3.2.14)$$

### 3.2.3 The Canonical HF Equations

## 3.3 Interpretation of Solutions to the HF Equations

### 3.3.1 Orbital Energies and Koopmans' Theorem

**Ex 3.4**

$$f_{ij} = \langle \chi_i | f | \chi_j \rangle = \langle i | h | j \rangle + \sum_b \langle ib || jb \rangle \quad (3.3.1)$$

$$\begin{aligned}f_{ji}^* &= \langle \chi_j | f | \chi_i \rangle^* = \langle j | h | i \rangle^* + \sum_b \langle jb || ib \rangle^* \\ &= \langle i | h | j \rangle + \sum_b \langle ib || jb \rangle \\ &= f_{ij}\end{aligned}\quad (3.3.2)$$

thus the Fock operator is Hermitian.

**Ex 3.5**

$$\text{IP} = {}^{N-2} E - E_0$$

$$\begin{aligned}&= \sum_{a \neq c, d} \langle a | h | a \rangle + \frac{1}{2} \sum_{a \neq c, d} \sum_{b \neq c, d} \langle ab || ab \rangle - \left[ \sum_a \langle a | h | a \rangle + \frac{1}{2} \sum_a \sum_b \langle ab || ab \rangle \right] \\ &= -\langle c | h | c \rangle - \langle d | h | d \rangle - \frac{1}{2} \sum_{a \neq c, d} \langle ac || ac \rangle - \frac{1}{2} \sum_{a \neq c, d} \langle ad || ad \rangle - \frac{1}{2} \sum_{b \neq c, d} \langle cb || cb \rangle - \frac{1}{2} \sum_{b \neq c, d} \langle db || db \rangle - \langle cd || cd \rangle \\ &= -\langle c | h | c \rangle - \langle d | h | d \rangle - \sum_{a \neq c, d} \langle ac || ac \rangle - \sum_{a \neq c, d} \langle ad || ad \rangle - \langle cd || cd \rangle \\ &= -\langle c | h | c \rangle - \langle d | h | d \rangle - \left( \sum_{a \neq c} \langle ac || ac \rangle - \langle dc || dc \rangle \right) - \left( \sum_{a \neq d} \langle ad || ad \rangle - \langle cd || cd \rangle \right) - \langle cd || cd \rangle \\ &= -\varepsilon_c - \varepsilon_d + \langle cd || cd \rangle - \langle cd || dc \rangle\end{aligned}\quad (3.3.3)$$

**Ex 3.6**

$$\begin{aligned}
{}^N E_0 - {}^{N+1} E^r &= \sum_a \langle a | h | a \rangle + \frac{1}{2} \sum_a \sum_b \langle ab || ab \rangle \\
&\quad - \left[ \sum_a \langle a | h | a \rangle + \langle r | h | r \rangle + \frac{1}{2} \sum_a \sum_b \langle ab || ab \rangle + \frac{1}{2} \sum_b \langle rb || rb \rangle + \frac{1}{2} \sum_a \langle ar || ar \rangle \right] \\
&= - \langle r | h | r \rangle - \frac{1}{2} \sum_b \langle rb || rb \rangle - \frac{1}{2} \sum_b \langle br || br \rangle \\
&= - \langle r | h | r \rangle - \sum_b \langle rb || rb \rangle
\end{aligned} \tag{3.3.4}$$

**3.3.2 Brillouin's Theorem**

**3.3.3 The HF Hamiltonian**

**Ex 3.7** Suppose  $\mathcal{H}_0$  commutes with  $\mathcal{P}_n$ ,

$$\begin{aligned}
\mathcal{H}_0 |\Psi_0\rangle &= \mathcal{H}_0 \frac{1}{\sqrt{N!}} \sum_n (-1)^{p_n} \mathcal{P}_n \left\{ \sum_i^N f(i) \chi_j(1) \cdots \chi_k(N) \right\} \\
&= \frac{1}{\sqrt{N!}} \sum_n (-1)^{p_n} \mathcal{P}_n \{ (\varepsilon_j + \cdots + \varepsilon_k) \chi_j(1) \cdots \chi_k(N) \} \\
&= \sum_a \varepsilon_a
\end{aligned} \tag{3.3.5}$$

Now we show  $\mathcal{H}_0$  commutes with  $\mathcal{P}_n$ , for example,  $\mathcal{P}_{ab}$

$$\mathcal{P}_{ab} \mathcal{H}_0 = \mathcal{P}_{ab} (\cdots + f(a) + \cdots + f(b) + \cdots) = (\cdots + f(b) + \cdots + f(a) + \cdots) \mathcal{P}_{ab} = \mathcal{H}_0 \mathcal{P}_{ab} \tag{3.3.6}$$

**Ex 3.8**

$$\mathcal{V} = \sum_i^N \sum_{j>i}^N \mathcal{O}_2 - \sum_i^N \sum_b^N [\mathcal{G}_b(i) - \mathcal{K}_b(i)] \tag{3.3.7}$$

thus

$$\begin{aligned}
\langle \Psi_0 | \mathcal{V} | \Psi_0 \rangle &= \sum_i^N \sum_{j>i}^N \langle \Psi_0 | \mathcal{O}_2 | \Psi_0 \rangle - \sum_i^N \sum_b^N [\langle \Psi_0 | \mathcal{G}_b(i) - \mathcal{K}_b(i) | \Psi_0 \rangle] \\
&= \frac{1}{2} \sum_a^N \sum_b^N \langle ab || ab \rangle - \sum_i^N \sum_b^N [\langle ib | ib \rangle - \langle ib | bi \rangle] \\
&= -\frac{1}{2} \sum_a^N \sum_b^N \langle ab || ab \rangle
\end{aligned} \tag{3.3.8}$$

**3.4 Restricted Closed-shell HF: The Roothaan Equations**

**3.4.1 Closed-shell HF: Restricted Spin Orbitals**

**Ex 3.9**

$$\begin{aligned}
\varepsilon_i &= (i|h|i) + \sum_b^N (\langle ib | ib \rangle - \langle ib | bi \rangle) \\
&= (i|h|i) + \sum_c^{N/2} (\langle ic | ic \rangle - \langle ic | ci \rangle) + \sum_{\bar{c}}^{N/2} (\langle i\bar{c} | i\bar{c} \rangle - \langle i\bar{c} | \bar{c}i \rangle)
\end{aligned} \tag{3.4.1}$$

Assume  $\chi_j$  has  $\alpha$  spin, since assuming  $\alpha$  or  $\beta$  is identical

$$\begin{aligned}
\varepsilon_i &= (i|h|i) + \sum_c^{N/2} [(ic|ic) \langle \alpha | \alpha \rangle \langle \alpha | \alpha \rangle - (ic|ci) \langle \alpha | \alpha \rangle \langle \alpha | \alpha \rangle] + \sum_c^{N/2} [(ic|ic) \langle \alpha | \alpha \rangle \langle \beta | \beta \rangle - (ic|ci) \langle \alpha | \beta \rangle \langle \beta | \alpha \rangle] \\
&= (i|h|i) + \sum_c^{N/2} [2(ic|ic) - (ic|ci)] \\
&= (i|h|i) + \sum_n^{N/2} (2J_{ib} - K_{ib})
\end{aligned} \tag{3.4.2}$$

### 3.4.2 Introduction of a Basis: The Roothaan Equations

**Ex 3.10**

$$\begin{aligned}
(\mathbf{C}^\dagger \mathbf{S} \mathbf{C})_{\mu\nu} &= \sum_i \sum_j C_{\mu i}^\dagger S_{ij} C_{j\nu} \\
&= \sum_i \sum_j C_{i\mu}^* \langle \phi_i | \phi_j \rangle C_{j\nu} \\
&= \langle \phi_\mu | \phi_\nu \rangle \\
&= \delta_{\mu\nu}
\end{aligned} \tag{3.4.3}$$

thus

$$\mathbf{C}^\dagger \mathbf{S} \mathbf{C} = \mathbf{1} \tag{3.4.4}$$

### 3.4.3 The Charge Density

**Ex 3.11**

$$\begin{aligned}
\rho(\mathbf{r}) &= \langle \Psi_0 | \hat{\rho}(\mathbf{r}) | \Psi_0 \rangle \\
&= \sum_i^N \frac{1}{N!} \sum_I^{N!} \sum_J^{N!} (-1)^{p_I} (-1)^{p_J} \int d\mathbf{x}_1 \cdots d\mathbf{x}_N \hat{\mathcal{P}}_I \{ \chi_1(1) \cdots \chi_N(N) \}^* \delta(\mathbf{r}_i - \mathbf{r}) \hat{\mathcal{P}}_J \{ \chi_1(1) \cdots \chi_N(N) \}
\end{aligned} \tag{3.4.5}$$

Since  $\{\chi_m\}$  are orthogonal,

$$\begin{aligned}
\rho(\mathbf{r}) &= \sum_i^N \frac{1}{N!} \sum_I^{N!} \int d\mathbf{x}_1 \cdots d\mathbf{x}_N \hat{\mathcal{P}}_I \{ \chi_1(1) \cdots \chi_N(N) \}^* \delta(\mathbf{r}_i - \mathbf{r}) \hat{\mathcal{P}}_I \{ \chi_1(1) \cdots \chi_N(N) \} \\
&= \sum_i^N \frac{1}{N!} (N-1)! \sum_s^N \int d\mathbf{x}_i \chi_s^*(\mathbf{x}_i) \delta(\mathbf{r}_i - \mathbf{r}) \chi_s(\mathbf{x}_i) \\
&= \sum_i^N \frac{1}{N} \cdot 2 \sum_s^{N/2} \int d\mathbf{r}_i \phi_s(\mathbf{r}_i) \delta(\mathbf{r}_i - \mathbf{r}) \phi_s(\mathbf{r}_i) \\
&= \sum_i^N \frac{2}{N} \sum_s^{N/2} \phi_s(\mathbf{r}) \phi_s(\mathbf{r}) \\
&= N \frac{2}{N} \sum_s^{N/2} \phi_s(\mathbf{r}) \phi_s(\mathbf{r}) \\
&= 2 \sum_s^{N/2} \phi_s(\mathbf{r}) \phi_s(\mathbf{r})
\end{aligned} \tag{3.4.6}$$

**Ex 3.12** From Ex 3.10, we have

$$\mathbf{C}^\dagger \mathbf{S} \mathbf{C} = \mathbf{1} \tag{3.4.7}$$

i.e.

$$\sum_i^K \sum_j^K C_{i\mu}^* S_{ij} C_{j\nu} = \delta_{\mu\nu} \quad (3.4.8)$$

thus

$$\begin{aligned} (\mathbf{PSP})_{\mu\sigma} &= \sum_{\nu}^K \sum_{\lambda}^K P_{\mu\nu} S_{\nu\lambda} P_{\lambda\sigma} \\ &= 4 \sum_{\nu}^K \sum_{\lambda}^K \sum_a^{N/2} C_{\mu a} C_{\nu a}^* S_{\nu\lambda} \sum_b^{N/2} C_{\lambda b} C_{\sigma b}^* \\ &= 4 \sum_a^{N/2} \sum_b^{N/2} C_{\mu a} \left( \sum_{\nu}^K \sum_{\lambda}^K C_{\nu a}^* S_{\nu\lambda} C_{\lambda b} \right) C_{\sigma b}^* \\ &= 4 \sum_a^{N/2} \sum_b^{N/2} C_{\mu a} \delta_{ab} C_{\sigma b}^* \\ &= 4 \sum_a^{N/2} C_{\mu a} C_{\sigma a}^* \\ &= 2P_{\mu\sigma} \end{aligned} \quad (3.4.9)$$

thus

$$\mathbf{PSP} = 2\mathbf{P} \quad (3.4.10)$$

**Ex 3.13** Eq. 3.122 shows

$$f(\mathbf{r}_1) = h(\mathbf{r}_1) + \sum_a^{N/2} \int d\mathbf{r}_2 \psi_a^*(\mathbf{r}_2) (2 - \hat{\mathcal{P}}_{12}) r_{12}^{-1} \psi_a(\mathbf{r}_2) \quad (3.4.11)$$

thus

$$\begin{aligned} f(\mathbf{r}_1) &= h(\mathbf{r}_1) + \sum_a^{N/2} \int d\mathbf{r}_2 \sum_{\sigma} C_{\sigma a}^* \phi_{\sigma}^*(\mathbf{r}_2) (2 - \hat{\mathcal{P}}_{12}) r_{12}^{-1} \sum_{\lambda} C_{\lambda a} \phi_{\lambda}(\mathbf{r}_2) \\ &= h(\mathbf{r}_1) + \sum_{\sigma} \sum_{\lambda} \left( \sum_a^{N/2} C_{\sigma a}^* C_{\lambda a} \right) \int d\mathbf{r}_2 \phi_{\sigma}^*(\mathbf{r}_2) (2 - \hat{\mathcal{P}}_{12}) r_{12}^{-1} \phi_{\lambda}(\mathbf{r}_2) \\ &= h(\mathbf{r}_1) + \frac{1}{2} \sum_{\sigma, \lambda} P_{\lambda\sigma} \int d\mathbf{r}_2 \phi_{\sigma}^*(\mathbf{r}_2) (2 - \hat{\mathcal{P}}_{12}) r_{12}^{-1} \phi_{\lambda}(\mathbf{r}_2) \end{aligned} \quad (3.4.12)$$

### 3.4.4 Expression for the Fock Matrix

**Ex 3.14** In expression  $(\mu\nu|\lambda\sigma)$ , there are three interchangeable pairs, i.e.  $\mu \leftrightarrow \nu$ ,  $\lambda \leftrightarrow \sigma$ , and  $\mu\nu \leftrightarrow \lambda\sigma$ . Thus  $(\mu\nu|\lambda\sigma)$  has an 8-fold symmetry. Similarly,  $(\mu\mu|\lambda\sigma)$ ,  $(\mu\nu|\mu\lambda)$ ,  $(\mu\nu|\mu\nu)$ ,  $(\mu\mu|\sigma\sigma)$  has 2-fold symmetry, and  $(\mu\mu|\mu\nu)$ ,  $(\mu\mu|\mu\mu)$  has 1-fold symmetry.

Therefore, the number of unique 2e integrals is

expression	number	$K = 100$
$(\mu\nu \lambda\sigma)$	$K(K-1)(K-2)(K-3)/8$	11763675
$(\mu\mu \lambda\sigma)$	$K(K-1)(K-2)/2$	485100
$(\mu\nu \mu\lambda)$	$K(K-1)(K-2)/2$	485100
$(\mu\nu \mu\nu)$	$K(K-1)/2$	4950
$(\mu\mu \sigma\sigma)$	$K(K-1)/2$	4950
$(\mu\mu \mu\nu)$	$K(K-1)$	9900
$(\mu\mu \mu\mu)$	$K$	100

thus the total number is 12 753 775.

### 3.4.5 Orthogonalization of the Basis

**Ex 3.15**  $\therefore$

$$\mathbf{U}^\dagger \mathbf{S} \mathbf{U} = \mathbf{s} \quad (3.4.13)$$

$\therefore$

$$\mathbf{S} \mathbf{U} = \mathbf{U} \mathbf{s} \quad (3.4.14)$$

i.e.

$$\sum_{\nu} S_{\mu\nu} U_{\nu i} = U_{\mu i} s_i \quad (3.4.15)$$

thus

$$\sum_{\mu} U_{\mu i}^* \sum_{\nu} S_{\mu\nu} U_{\nu i} = \sum_{\mu} U_{\mu i}^* U_{\mu i} s_i \quad (3.4.16)$$

$$\sum_{\mu} \sum_{\nu} U_{\mu i}^* \langle \phi_{\mu} | \phi_{\nu} \rangle U_{\nu i} = s_i \sum_{\mu} |U_{\mu i}|^2 \quad (3.4.17)$$

Suppose

$$\phi'_i = \sum_{\nu} U_{\nu i} \phi_{\nu} \quad (3.4.18)$$

thus

$$\langle \phi'_i | \phi'_i \rangle = s_i \sum_{\mu} |U_{\mu i}|^2 \quad (3.4.19)$$

$\therefore$

$$\langle \phi'_i | \phi'_i \rangle > 0 \quad |U_{\mu i}|^2 > 0 \quad (3.4.20)$$

$\therefore$

$$s_i > 0 \quad (3.4.21)$$

**Ex 3.16**

- (3.174)

Since  $(\phi, \phi', \psi)$  are row vectors

$$\psi = \phi \mathbf{C} \quad (3.4.22)$$

$$\psi = \phi' \mathbf{C}' = \phi \mathbf{X} \mathbf{C}' \quad (3.4.23)$$

we have

$$\mathbf{C} = \mathbf{X} \mathbf{C}' \quad (3.4.24)$$

i.e.

$$\mathbf{C}' = \mathbf{X}^{-1} \mathbf{C} \quad (3.4.25)$$

- (3.177)

$$\begin{aligned} F'_{\mu\nu} &= \langle \phi'_{\mu} | f | \phi'_{\nu} \rangle \\ &= \left\langle \sum_i \phi_i X_{i\mu} \left| f \right| \sum_j \phi_j X_{j\nu} \right\rangle \\ &= \sum_i \sum_j X_{i\mu}^* X_{j\nu} \langle \phi_i | f | \phi_j \rangle \\ &= \sum_i \sum_j X_{i\mu}^* F_{ij} X_{j\nu} \end{aligned} \quad (3.4.26)$$

i.e.

$$\mathbf{F}' = \mathbf{X}^\dagger \mathbf{F} \mathbf{X} \quad (3.4.27)$$



### 3.4.6 The SCF Procedure

### 3.4.7 Expectation Values and Population Analysis

**Ex 3.17** From (3.148) in the textbook, we get

$$F_{\mu\nu} = H_{\mu\nu}^{\text{core}} + G_{\mu\nu} = H_{\mu\nu}^{\text{core}} + \sum_a^{N/2} [2(\mu\nu|aa) - (\mu a|a\nu)] \quad (3.4.28)$$

thus

$$\begin{aligned} E_0 &= \sum_a^{N/2} [2h_{aa} + \sum_b^{N/2} (2J_{ab} - K_{ab})] \\ &= 2 \sum_a^{N/2} (a|h|a) + \sum_a^{N/2} \sum_b^{N/2} [2(aa|bb) - (ab|ba)] \\ &= 2 \sum_a^{N/2} \sum_\mu \sum_\nu C_{\mu a}^* C_{\nu a} (\mu|h|\nu) + \sum_a^{N/2} \sum_b^{N/2} \left[ 2 \sum_\mu \sum_\nu C_{\mu a}^* C_{\nu a} (\mu\nu|bb) - \sum_\mu \sum_\nu C_{\mu a}^* C_{\nu a} (\mu b|b\nu) \right] \\ &= \sum_\mu \sum_\nu P_{\nu\mu} H_{\mu\nu}^{\text{core}} + \frac{1}{2} \sum_b^{N/2} \sum_\mu \sum_\nu [2P_{\nu\mu} (\mu\nu|bb) - P_{\nu\mu} (\mu b|b\nu)] \\ &= \sum_\mu \sum_\nu P_{\nu\mu} [H_{\mu\nu}^{\text{core}} + \frac{1}{2} G_{\mu\nu}] \\ &= \frac{1}{2} \sum_\mu \sum_\nu P_{\nu\mu} [H_{\mu\nu}^{\text{core}} + F_{\mu\nu}] \end{aligned} \quad (3.4.29)$$

**Ex 3.18** For symmetrically orthogonalized basis,

$$\mathbf{C}' = \mathbf{S}^{1/2} \mathbf{C} \quad (3.4.30)$$

thus

$$\begin{aligned} P'_{\mu\nu} &= 2 \sum_a^{N/2} C'_{\mu a} C'^*_{\nu a} \\ &= 2 \sum_a^{N/2} \sum_i S_{\mu i}^{1/2} C_{ia} \sum_j S_{\nu j}^{1/2*} C_{ja}^* \\ &= \sum_i \sum_j S_{\mu i}^{1/2} \left( 2 \sum_a^{N/2} C_{ia} C_{ja}^* \right) S_{\nu j}^{1/2*} \\ &= \sum_i \sum_j S_{\mu i}^{1/2} P_{ij} S_{\nu j}^{1/2*} \\ &= \sum_i \sum_j S_{\mu i}^{1/2} P_{ij} S_{j\nu}^{1/2} \end{aligned} \quad (3.4.31)$$

i.e.

$$\mathbf{P}' = \mathbf{S}^{1/2} \mathbf{P} \mathbf{S}^{1/2} \quad (3.4.32)$$

thus

$$\sum_\mu (\mathbf{S}^{1/2} \mathbf{P} \mathbf{S}^{1/2})_{\mu\mu} = \sum_\mu \mathbf{P}'_{\mu\mu} \quad (3.4.33)$$

### 3.5 Model Calculations on H<sub>2</sub> and HeH<sup>+</sup>

#### 3.5.1 The 1s Minimal STO-3G Basis Set

**Ex 3.19**

$$\begin{aligned}
\phi_{1s}^{\text{GF}}(\alpha, \mathbf{r} - \mathbf{R}_A) \phi_{1s}^{\text{GF}}(\alpha, \mathbf{r} - \mathbf{R}_B) &= \left(\frac{2\alpha}{\pi}\right)^{3/4} e^{-\alpha|\mathbf{r}-\mathbf{R}_A|^2} \left(\frac{2\beta}{\pi}\right)^{3/4} e^{-\beta|\mathbf{r}-\mathbf{R}_B|^2} \\
&= \left(\frac{2\alpha}{\pi}\right)^{3/4} \left(\frac{2\beta}{\pi}\right)^{3/4} e^{-\alpha|\mathbf{r}-\mathbf{R}_A|^2 - \beta|\mathbf{r}-\mathbf{R}_B|^2} \\
&= \left(\frac{2\alpha}{\pi}\right)^{3/4} \left(\frac{2\beta}{\pi}\right)^{3/4} \exp\left(-\left[(\alpha + \beta)|\mathbf{r}|^2 - 2\mathbf{r} \cdot (\alpha\mathbf{R}_A + \beta\mathbf{R}_B) + \alpha|\mathbf{R}_A|^2 + \beta|\mathbf{R}_B|^2\right]\right)
\end{aligned} \tag{3.5.1}$$

Let

$$p = \alpha + \beta \quad \mathbf{R}_P = \frac{\alpha\mathbf{R}_A + \beta\mathbf{R}_B}{\alpha + \beta} \tag{3.5.2}$$

we have

$$\begin{aligned}
\phi_{1s}^{\text{GF}}(\alpha, \mathbf{r} - \mathbf{R}_A) \phi_{1s}^{\text{GF}}(\alpha, \mathbf{r} - \mathbf{R}_B) &= \left(\frac{2\alpha}{\pi}\right)^{3/4} \left(\frac{2\beta}{\pi}\right)^{3/4} \exp\left(-\left[p|\mathbf{r}|^2 - 2\mathbf{r} \cdot (p\mathbf{R}_P) + \alpha|\mathbf{R}_A|^2 + \beta|\mathbf{R}_B|^2\right]\right) \\
&= \left(\frac{2\alpha}{\pi}\right)^{3/4} \left(\frac{2\beta}{\pi}\right)^{3/4} \exp\left(-\left[p|\mathbf{r} - \mathbf{R}_P|^2 - p|\mathbf{R}_P|^2 + \alpha|\mathbf{R}_A|^2 + \beta|\mathbf{R}_B|^2\right]\right) \\
&= \left(\frac{2\alpha\beta/p}{\pi}\right)^{3/4} \left(\frac{2p}{\pi}\right)^{3/4} e^{-p|\mathbf{r}-\mathbf{R}_P|^2} \exp\left(p|\mathbf{R}_P|^2 - \alpha|\mathbf{R}_A|^2 - \beta|\mathbf{R}_B|^2\right)
\end{aligned} \tag{3.5.3}$$

Let

$$\phi_{1s}^{\text{GF}}(\alpha, \mathbf{r} - \mathbf{R}_A) \phi_{1s}^{\text{GF}}(\alpha, \mathbf{r} - \mathbf{R}_B) = K_{AB} \left(\frac{2p}{\pi}\right)^{3/4} e^{-p|\mathbf{r}-\mathbf{R}_P|^2} \tag{3.5.4}$$

thus

$$\begin{aligned}
K_{AB} &= \left(\frac{2\alpha\beta/p}{\pi}\right)^{3/4} \exp\left(p|\mathbf{R}_P|^2 - \alpha|\mathbf{R}_A|^2 - \beta|\mathbf{R}_B|^2\right) \\
&= \left(\frac{2\alpha\beta/p}{\pi}\right)^{3/4} \exp\left(\frac{1}{p}(\alpha^2|\mathbf{R}_A|^2 + \beta^2|\mathbf{R}_B|^2 + 2\alpha\beta\mathbf{R}_A \cdot \mathbf{R}_B) - \alpha|\mathbf{R}_A|^2 - \beta|\mathbf{R}_B|^2\right) \\
&= \left(\frac{2\alpha\beta/p}{\pi}\right)^{3/4} \exp\left(\frac{1}{p}(\alpha^2|\mathbf{R}_A|^2 + \beta^2|\mathbf{R}_B|^2 + 2\alpha\beta\mathbf{R}_A \cdot \mathbf{R}_B - p\alpha|\mathbf{R}_A|^2 - p\beta|\mathbf{R}_B|^2)\right) \\
&= \left(\frac{2\alpha\beta/p}{\pi}\right)^{3/4} \exp\left(\frac{1}{p}(-\alpha\beta|\mathbf{R}_A|^2 - \alpha\beta|\mathbf{R}_B|^2 + 2\alpha\beta\mathbf{R}_A \cdot \mathbf{R}_B)\right) \\
&= \left(\frac{2\alpha\beta}{p\pi}\right)^{3/4} \exp\left(-\frac{\alpha\beta}{p}|\mathbf{R}_A - \mathbf{R}_B|^2\right)
\end{aligned} \tag{3.5.5}$$

**Ex 3.20** At  $r = 0$ ,

$$\phi_{1s}^{\text{CGF}}(\zeta = 1.0, \text{STO-1G}) = 0.267\,656 \tag{3.5.6}$$

$$\phi_{1s}^{\text{CGF}}(\zeta = 1.0, \text{STO-2G}) = 0.389\,383 \tag{3.5.7}$$

$$\phi_{1s}^{\text{CGF}}(\zeta = 1.0, \text{STO-3G}) = 0.454\,986 \tag{3.5.8}$$

while

$$\phi_{1s}^{\text{SF}}(\zeta = 1.0) = \frac{1}{\sqrt{\pi}} = 0.564\,19 \tag{3.5.9}$$

### 3.5.2 STO-3G H<sub>2</sub>

**Ex 3.21**

$$\phi_{1s}^{\text{CGF}}(\zeta = 1.0, \text{STO-1G}) = \phi_{1s}^{\text{GF}}(0.270\,950) \quad (3.5.10)$$

Since  $\alpha = \alpha_{(\zeta=1.0)} \times \zeta^2$ ,

$$\phi_{1s}^{\text{CGF}}(\zeta = 1.24, \text{STO-1G}) = \phi_{1s}^{\text{GF}}(0.416\,613) \quad (3.5.11)$$

thus

$$\begin{aligned} S_{12} &= K_{AB} \left( \frac{2 \cdot 2\alpha}{\pi} \right)^{3/4} \int d\mathbf{r} e^{-2\alpha|\mathbf{r}-\mathbf{R}_P|^2} \\ &= \left( \frac{2\alpha}{2\pi} \right)^{3/4} e^{-\frac{\alpha}{2}R^2} \left( \frac{2 \cdot 2\alpha}{\pi} \right)^{3/4} \int d\mathbf{r} e^{-2\alpha|\mathbf{r}-\mathbf{R}_A|^2} \\ &= \left( \frac{2\alpha}{\pi} \right)^{3/2} e^{-\frac{\alpha}{2}R^2} 4\pi \int dr r^2 e^{-2\alpha r^2} \\ &= \left( \frac{2\alpha}{\pi} \right)^{3/2} e^{-\frac{\alpha}{2}R^2} 4\pi \frac{\sqrt{\pi}}{8\sqrt{2}\alpha^{3/2}} \\ &= e^{-\frac{\alpha}{2}R^2} \end{aligned} \quad (3.5.12)$$

At  $R = 1.4, \alpha = 0.416\,613$ ,

$$S_{12} = 0.6648 \quad (3.5.13)$$

**Ex 3.22** Let

$$\psi_1 = c_1(\phi_1 + \phi_2) \quad \psi_2 = c_2(\phi_1 - \phi_2) \quad (3.5.14)$$

$$\begin{aligned} 1 &= \langle \phi_1 | \psi_1 \rangle = c_1^2(S_{11} + S_{12} + S_{21} + S_{22}) \\ &= c_1^2(2 + 2S_{12}) \end{aligned} \quad (3.5.15)$$

$\therefore$

$$c_1 = [2(1 + S_{12})]^{-1/2} \quad (3.5.16)$$

$$\begin{aligned} 1 &= \langle \phi_2 | \psi_2 \rangle = c_2^2(S_{11} - S_{12} - S_{21} + S_{22}) \\ &= c_2^2(2 - 2S_{12}) \end{aligned} \quad (3.5.17)$$

$\therefore$

$$c_2 = [2(1 - S_{12})]^{-1/2} \quad (3.5.18)$$

**Ex 3.23** Suppose

$$\psi_1 = c_1(\phi_1 + \phi_2) \quad \psi_2 = c_2(\phi_1 - \phi_2) \quad (3.5.19)$$

thus

$$\mathbf{H}^{\text{core}} \mathbf{C} = \mathbf{S} \mathbf{C} \boldsymbol{\varepsilon} \quad (3.5.20)$$

$$\begin{pmatrix} H_{11}^{\text{core}} & H_{12}^{\text{core}} \\ H_{21}^{\text{core}} & H_{22}^{\text{core}} \end{pmatrix} \begin{pmatrix} c_1 & c_2 \\ c_1 & -c_2 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} c_1 & c_2 \\ c_1 & -c_2 \end{pmatrix} \begin{pmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{pmatrix} \quad (3.5.21)$$

$$\begin{pmatrix} (H_{11}^{\text{core}} + H_{12}^{\text{core}})c_1 & (H_{11}^{\text{core}} - H_{12}^{\text{core}})c_2 \\ (H_{21}^{\text{core}} + H_{22}^{\text{core}})c_1 & (H_{21}^{\text{core}} - H_{22}^{\text{core}})c_2 \end{pmatrix} = \begin{pmatrix} (S_{11} + S_{12})c_1\varepsilon_1 & (S_{11} - S_{12})c_2\varepsilon_2 \\ (S_{21} + S_{22})c_1\varepsilon_1 & (S_{21} - S_{22})c_2\varepsilon_2 \end{pmatrix} \quad (3.5.22)$$

$\therefore$

$$\begin{cases} \varepsilon_1 = (H_{11}^{\text{core}} + H_{12}^{\text{core}})/(1 + S_{12}) \\ \varepsilon_2 = (H_{11}^{\text{core}} - H_{12}^{\text{core}})/(1 - S_{12}) \end{cases} \quad (3.5.23)$$

$$\varepsilon_1 = (-1.1204 - 0.9584)/(1 + 0.6593) = -1.2528 \quad (3.5.24)$$

$$\varepsilon_2 = (-1.1204 + 0.9584)/(1 - 0.6593) = -0.4755 \quad (3.5.25)$$

**Ex 3.24**

**Ex 3.25**

**Ex 3.26**

**Ex 3.27**

### 3.5.3 An SCF Calculation on STO-3G HeH<sup>+</sup>

**Ex 3.28**

$$\begin{aligned}
\mathbf{X}_{\text{Schmidt}}^\dagger \mathbf{S} \mathbf{X}_{\text{Schmidt}} &= \begin{pmatrix} 1 & 0 \\ -S_{12}/\sqrt{1-S_{12}^2} & 1/\sqrt{1-S_{12}^2} \end{pmatrix} \begin{pmatrix} 1 & S_{12} \\ S_{12} & 1 \end{pmatrix} \begin{pmatrix} 1 & -S_{12}/\sqrt{1-S_{12}^2} \\ 0 & 1/\sqrt{1-S_{12}^2} \end{pmatrix} \\
&= \begin{pmatrix} 1 & S_{12} \\ 0 & \sqrt{1-S_{12}^2} \end{pmatrix} \begin{pmatrix} 1 & -S_{12}/\sqrt{1-S_{12}^2} \\ 0 & 1/\sqrt{1-S_{12}^2} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned} \tag{3.5.26}$$

thus the Schmidt transformation produces orthonormal basis.

**Ex 3.29**

$$E_0(R \rightarrow \infty) = \frac{1}{2} \sum_{\mu} \sum_{\nu} P_{\nu\mu}(R \rightarrow \infty) [2H_{\mu\nu}^{\text{core}} + G_{\mu\nu}] \tag{3.5.27}$$

where

$$P_{\nu\mu}(R \rightarrow \infty) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \tag{3.5.28}$$

$$\begin{aligned}
G_{\mu\nu} &= \sum_{\lambda} \sum_{\sigma} P_{\lambda\sigma}(R \rightarrow \infty) \left[ (\mu\nu|\sigma\lambda) - \frac{1}{2}(\mu\lambda|\sigma\nu) \right] \\
&= 2 \left[ (\mu\nu|\phi_1\phi_1) - \frac{1}{2}(\mu\phi_1|\phi_1\nu) \right]
\end{aligned} \tag{3.5.29}$$

thus

$$\begin{aligned}
E_0(R \rightarrow \infty) &= \frac{1}{2} \sum_{\mu} \sum_{\nu} P_{\nu\mu}(R \rightarrow \infty) [2H_{\mu\nu}^{\text{core}} + G_{\mu\nu}] \\
&= \frac{1}{2} \times 2 [2H_{11}^{\text{core}} + G_{11}] \\
&= 2(T_{11} + V_{11}^1) + 2 \left[ (\phi_1\phi_1|\phi_1\phi_1) - \frac{1}{2}(\phi_1\phi_1|\phi_1\phi_1) \right] \\
&= 2T_{11} + 2V_{11}^1 + (\phi_1\phi_1|\phi_1\phi_1)
\end{aligned} \tag{3.5.30}$$

## 3.6 Polyatomic Basis Sets

### 3.6.1 Contracted Gaussian Functions

### 3.6.2 Minimal Basis Sets: STO-3G

### 3.6.3 Double Zeta Basis Sets: 4-31G

**Ex 3.30** The outer basis function

$$\phi_{1s}''(\mathbf{r}) = g_{1s}(0.298073, \mathbf{r}) \tag{3.6.1}$$

The inner basis function

$$\phi_{1s}'(\mathbf{r}) = N[0.46954g_{1s}(1.242567, \mathbf{r}) + 0.15457g_{1s}(5.782948, \mathbf{r}) + 0.02373g_{1s}(38.47497, \mathbf{r})] \tag{3.6.2}$$

Renormalize it, we get

$$N = 1.689 \tag{3.6.3}$$

thus

$$\phi_{1s}'(\mathbf{r}) = 0.79330g_{1s}(1.242567, \mathbf{r}) + 0.26115g_{1s}(5.782948, \mathbf{r}) + 0.04009g_{1s}(38.47497, \mathbf{r}) \tag{3.6.4}$$

### 3.6.4 Polarized Basis Sets: 6-31G\* and 6-31G\*\*

#### Ex 3.31

	C	H	total
STO-3G	5	1	36
4-31G	9	2	66
6-31G* (Cartesian)	15	2	102
6-31G** (Cartesian)	15	5	120

## 3.7 Some Illustrative Closed-shell Calculations

### 3.7.1 Total Energies

#### Ex 3.32 Reaction I

basis	$\Delta E$ / a.u.	$\Delta E$ /(kcal/mol)	
STO-3G	-0.061	-38.28	exoergic
4-31G	-0.069	-43.30	
6-31G*	-0.045	-28.24	
6-31G**	-0.055	-34.51	
HF-limit	-0.051	-32.00	

#### Reaction II

basis	$\Delta E$ / a.u.	$\Delta E$ /(kcal/mol)	
STO-3G	0.186	116.72	endoergic
4-31G	-0.114	-71.54	exoergic
6-31G*	-0.088	-55.22	
6-31G**	-0.095	-59.61	
HF-limit	-0.097	-60.87	