

# Notes of Modern Quantum Chemistry, Szabo & Ostlund

hebrewsnabla

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# 0

spatial mol orb –  $\psi - i, j, k, \dots$   
 spatial basis fcn –  $\phi - \mu, \nu, \lambda, \dots$   
 spin orb –  $\chi$   
 occ mol orb –  $a, b, c, \dots$   
 vir mol orb –  $r, s, t, \dots$   
 exact many-elec wfn –  $\Phi$   
 approx many-elec wfn –  $\Psi$   
 exact energy –  $\mathcal{E}$   
 approx energy –  $E$

# 1

## 1.1

## 1.2

## 1.3

## 1.4 N-D Complex Vector Spaces

Suppose

$$\mathcal{O} |a\rangle = |b\rangle \quad (1.1)$$

$$\langle i | \mathcal{O} | j \rangle = O_{ij} \quad (1.2)$$

def the **adjoint** of  $\mathcal{O}$  as  $\mathcal{O}^\dagger$

$$\langle a | \mathcal{O}^\dagger = \langle b | \quad (1.3)$$

$$\langle i | \mathcal{O}^\dagger | j \rangle = O_{ji}^* \quad (1.4)$$

### 1.4.1 Change of Basis

$$|\alpha\rangle = \sum_i |i\rangle \langle i | \alpha \rangle = \sum_i |i\rangle U_{i\alpha} \quad (1.5)$$

$$|i\rangle = \sum_\alpha |\alpha\rangle \langle i | \alpha \rangle = \sum_\alpha |\alpha\rangle U_{i\alpha}^* \quad (1.6)$$

If  $i, \alpha$  are all orthonormal,  $\mathbf{U}$  must be unitary.

$$\Omega_{\alpha\beta} = \langle \alpha | \mathcal{O} | \beta \rangle = \dots \sum_{ij} U_{\alpha i}^* O_{ij} U_{j\beta} \quad (1.7)$$

or

$$\mathbf{\Omega} = \mathbf{U}^\dagger \mathbf{O} \mathbf{U} \quad (1.8)$$

## 2

### 2.1 The Electronic Problem

#### 2.1.1 Atomic Units

#### 2.1.2 The B-O Approximation

#### 2.1.3 The Antisymmetry or Pauli Exclusion Principle

### 2.2 Orbitals, Slater Determinants, and Basis Functions

#### 2.2.1 Spin Orbitals and Spatial Orbitals

#### 2.2.2 Hartree Products

#### 2.2.3 Slater Determinants

def

$$|\chi_i(\mathbf{x}_1)\chi_j(\mathbf{x}_2)\cdots\chi_k(\mathbf{x}_N)\rangle \equiv \frac{1}{\sqrt{N!}} \begin{vmatrix} \chi_i(\mathbf{x}_1) & \chi_j(\mathbf{x}_1) & \cdots & \chi_k(\mathbf{x}_1) \\ \chi_i(\mathbf{x}_2) & \chi_j(\mathbf{x}_2) & \cdots & \chi_k(\mathbf{x}_2) \\ \vdots & \vdots & & \vdots \\ \chi_i(\mathbf{x}_N) & \chi_j(\mathbf{x}_N) & \cdots & \chi_k(\mathbf{x}_N) \end{vmatrix} \quad (2.1)$$

It can be further shortened to

$$|\chi_i\chi_j\cdots\chi_k\rangle \quad (2.2)$$

#### 2.2.4 The Hartree-Fock Approximation

#### 2.2.5 The Minimal Basis H<sub>2</sub> Model

gerade, ungerade

#### 2.2.6 Excited Determinants

Suppose the ground state det

$$|\psi_0\rangle = |\chi_1\cdots\chi_a\cdots\chi_b\cdots\chi_N\rangle \quad (2.3)$$

thus, singly excited det

$$|\psi_a^r\rangle = |\chi_1\cdots\chi_r\cdots\chi_b\cdots\chi_N\rangle \quad (2.4)$$

$$|\psi_{ab}^{rs}\rangle = |\chi_1\cdots\chi_r\cdots\chi_s\cdots\chi_N\rangle \quad (2.5)$$

How does program determine what dets can exist? by gerade/ungerade?

### 2.3 Operators and Matrix Elements

#### 2.3.1 Minimal Basis H<sub>2</sub> Matrix Elements

#### 2.3.2 Notations for 1- and 2-Electron Integrals

For spin orb,  
physicists'

$$\langle ij | kl \rangle = \left\langle \chi_i(1)\chi_j(2) \left| \frac{1}{r_{12}} \right| \chi_k(1)\chi_l(2) \right\rangle \quad (2.6)$$

$$\langle ij \parallel kl \rangle = \langle ij \mid kl \rangle - \langle ij \mid lk \rangle \quad (2.7)$$

chemists'

$$[ij \mid kl] = \left\langle \chi_i(1) \chi_j(1) \left| \frac{1}{r_{12}} \right| \chi_k(2) \chi_l(2) \right\rangle \quad (2.8)$$

For spatial orb

$$(ij \mid kl) = \left\langle \psi_i(1) \psi_j(1) \left| \frac{1}{r_{12}} \right| \psi_k(2) \psi_l(2) \right\rangle \quad (2.9)$$

### 2.3.3 General Rules for Matrix Elements

$$1. \quad |K\rangle = |\cdots mn \cdots\rangle$$

$$\langle K \mid \mathcal{H} \mid K \rangle = \sum_m^N [m|h|m] + \frac{1}{2} \sum_m^N \sum_n^N ([mm|nn] - [mn|nm]) \quad (2.10)$$

or (Since  $[mm|mm] - [mm|mm] = 0$ )

$$\langle K \mid \mathcal{H} \mid K \rangle = \sum_m^N [m|h|m] + \sum_m^N \sum_{n>m}^N ([mm|nn] - [mn|nm]) \quad (2.11)$$

$$2. \quad |K\rangle = |\cdots mn \cdots\rangle, |L\rangle = |\cdots pn \cdots\rangle$$

$$\langle K \mid \mathcal{H} \mid L \rangle = [m|h|p] + \sum_n^N ([mp|nn] - [mn|np]) \quad (2.12)$$

$$3. \quad |K\rangle = |\cdots mn \cdots\rangle, |L\rangle = |\cdots pq \cdots\rangle$$

$$\langle K \mid \mathcal{H} \mid L \rangle = [mp|nq] - [mq|np] \quad (2.13)$$

## 2.4 Second Quantization

### 2.4.1 Creation and Annihilation Operators and Their Anticommutation Relations

$$a_i^\dagger a_j^\dagger + a_j^\dagger a_i^\dagger = 0 \quad a_i a_j + a_j a_i = 0 \quad (2.14)$$

$$\{a_i, a_j^\dagger\} \equiv a_i a_j^\dagger + a_j^\dagger a_i = \delta_{ij} \quad (2.15)$$

## 2.5 Spin-Adapted Configurations

### 2.5.1 Spin Operators

total spin

$$\hat{\mathcal{S}} = \sum_i^N \hat{\mathbf{s}}(i) \quad (2.16)$$

$$\hat{\mathcal{S}}_x = \sum_i^N \hat{\mathbf{s}}_x(i) \quad (2.17)$$

$$\hat{\mathcal{S}}_+ = \sum_i^N \hat{\mathbf{s}}_+(i) \quad (2.18)$$

$$\hat{\mathcal{S}}^2 = \hat{\mathcal{S}}_+ \hat{\mathcal{S}}_- - \hat{\mathcal{S}}_z + \hat{\mathcal{S}}_z^2 \quad (2.19)$$

$$\hat{\mathcal{S}}^2 |\Phi\rangle = S(S+1) |\Phi\rangle \quad (2.20)$$

$$\hat{\mathcal{S}}_z |\Phi\rangle = M_S |\Phi\rangle \quad (2.21)$$

$$\hat{\mathcal{S}}_z |ij \cdots k\rangle = \frac{1}{2}(N^\alpha - N^\beta) |ij \cdots k\rangle \quad (2.22)$$

### 2.5.2 Restricted Determinants and Spin-Adapted Configurations

$$|^1\Psi_1^2\rangle = \frac{1}{\sqrt{2}}(|1\bar{2}\rangle + |\bar{1}2\rangle) = \frac{1}{\sqrt{2}}(\psi_1(1)\psi_2(2)\alpha(1)\beta(2) - \psi_2(1)\psi_1(2)\alpha(1)\beta(2)) \quad (2.23)$$

## 3 The Hartree-Fock Approximation

### 3.1 The HF Equations

#### 3.1.1 The Coulomb and Exchange Operators

#### 3.1.2 The Fock Operator

### 3.2 Derivation of the HF Equations

#### 3.2.1 Functional Variation

#### 3.2.2 Minimization of the Energy of a Single Determinant

#### 3.2.3 The Canonical HF Equations

### 3.3 Interpretation of Solutions to the HF Equations

#### 3.3.1 Orbital Energies and Koopmans' Theorem

$$\varepsilon_i = \langle i | h | i \rangle + \sum_b \langle ib | ib \rangle \quad (3.1)$$

$$= \langle i | h | i \rangle + \sum_b (\langle ib | ib \rangle - \langle ib | bi \rangle) \quad (3.2)$$

#### Koopmans' Theorem

$$\text{IP} = -\varepsilon_a \quad \text{EA} = -\varepsilon_r \quad (3.3)$$

Koopmans' EA is often bad.

#### 3.3.2 Brillouin's Theorem

$$\langle \Psi_0 | \mathcal{H} | \Psi_a^r \rangle = 0 \quad (3.4)$$

#### 3.3.3 The HF Hamiltonian

$$\mathcal{H}_0 = \sum_i^N f(i) \quad (3.5)$$

### 3.4 Restricted Closed-shell HF: The Roothaan Equations

#### 3.4.1 Closed-shell HF: Restricted Spin Orbitals

$$E_0 = 2 \sum_a h_{aa} + \sum_a \sum_b (2J_{ab} - K_{ab}) \quad (3.6)$$

#### 3.4.2 Introduction of a Basis: The Roothaan Equations

#### 3.4.3 The Charge Density

#### 3.4.4 Expression for the Fock Matrix

#### 3.4.5 Orthogonalization of the Basis

$$\mathbf{X}^\dagger \mathbf{S} \mathbf{X} = \mathbf{1} \quad (3.7)$$

$\mathbf{S}$  can be diagonalized by unitary matrix  $\mathbf{U}$ :

$$\mathbf{U}^\dagger \mathbf{S} \mathbf{U} = \mathbf{s} \quad (3.8)$$

**Symmetric Orthogonalization**

$$\mathbf{X} = \mathbf{S}^{-1/2} = \mathbf{U} \mathbf{s}^{-1/2} \mathbf{U}^\dagger \quad (3.9)$$

(linear dependence must be removed)

**Canonical Orthogonalization**

$$\mathbf{X} = \mathbf{U} \mathbf{S}^{-1/2} = \mathbf{U} \mathbf{s}^{-1/2} \mathbf{U}^\dagger \quad (3.10)$$

Suppose  $\mathbf{s}$  has  $m$  small values, we make a truncated  $K \times (K - m)$  matrix

$$\tilde{\mathbf{X}} = \dots \quad (3.11)$$

thus

$$\phi'_\mu = \sum_\nu^K \phi_\nu \tilde{X}_{\nu\mu} \quad \mu = 1, 2, \dots, K - m \quad (3.12)$$

However, calculate 2e integrals in transformed matrix is very time-consuming.  
Since

$$\phi' = \phi \mathbf{X} \quad (3.13)$$

$$\psi = \phi \mathbf{C} \quad (3.14)$$

we have

$$\psi = \phi' \mathbf{X}^{-1} \mathbf{C} \quad (3.15)$$

Let

$$\mathbf{C}' = \mathbf{X}^{-1} \mathbf{C} \quad \text{or} \quad \mathbf{C} = \mathbf{X} \mathbf{C}' \quad (3.16)$$

thus

$$\mathbf{F} \mathbf{X} \mathbf{C}' = \mathbf{S} \mathbf{X} \mathbf{C}' \epsilon \quad (3.17)$$

$$(\mathbf{X}^\dagger \mathbf{F} \mathbf{X}) \mathbf{C}' = (\mathbf{X}^\dagger \mathbf{S} \mathbf{X}) \mathbf{C}' \epsilon = \mathbf{C}' \epsilon \quad (3.18)$$

def:

$$\mathbf{F}' = \mathbf{X}^\dagger \mathbf{F} \mathbf{X} \quad (3.19)$$