

# Modern Quantum Chemistry, Szabo & Ostlund

## HW

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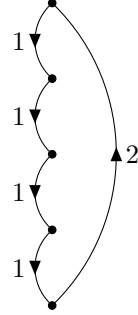
## 6 Many-body Perturbation Theory

### 6.1 RS Perturbation Theory

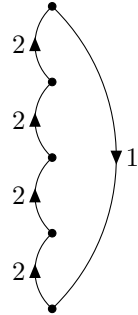
### 6.2 Diagrammatic Representation of RS Perturbation Theory

#### 6.2.1 Diagrammatic Perturbation Theory for Two States

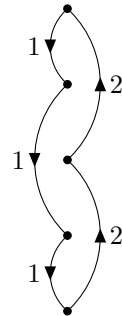
Ex 6.1



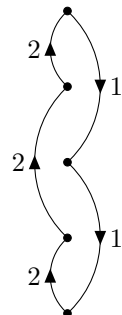
$$= (-1)^5 \frac{V_{12}V_{21}V_{11}^3}{(E_1^{(0)} - E_2^{(0)})^4} = -\frac{V_{12}V_{21}V_{11}^3}{(E_1^{(0)} - E_2^{(0)})^4}$$



$$= (-1)^2 \frac{V_{12}V_{21}V_{22}^3}{(E_1^{(0)} - E_2^{(0)})^4} = \frac{V_{12}V_{21}V_{22}^3}{(E_1^{(0)} - E_2^{(0)})^4}$$



$$= (-1)^4 \frac{V_{12}V_{21}V_{11}^2V_{22}}{(E_1^{(0)} - E_2^{(0)})^4} = \frac{V_{12}V_{21}V_{11}^2V_{22}}{(E_1^{(0)} - E_2^{(0)})^4}$$



$$= (-1)^3 \frac{V_{12}V_{21}V_{11}V_{22}^2}{(E_1^{(0)} - E_2^{(0)})^4} = -\frac{V_{12}V_{21}V_{11}V_{22}^2}{(E_1^{(0)} - E_2^{(0)})^4}$$

Similarly,

$$\begin{aligned}
& \text{Diagram 1} + \text{Diagram 2} = \frac{V_{12}V_{21}V_{11}^2V_{22}}{(E_1^{(0)} - E_2^{(0)})^4} \\
& \text{Diagram 3} + \text{Diagram 4} = -\frac{V_{12}V_{21}V_{11}V_{22}^2}{(E_1^{(0)} - E_2^{(0)})^4}
\end{aligned}$$

thus, the sum of above terms is

$$\frac{V_{12}V_{21}(V_{22}^3 - V_{11}^3)}{(E_1^{(0)} - E_2^{(0)})^4} + 3 \times \frac{V_{12}V_{21}(V_{11}^2V_{22} - V_{11}V_{22}^2)}{(E_1^{(0)} - E_2^{(0)})^4} = \frac{V_{12}V_{21}(V_{22} - V_{11})^3}{(E_1^{(0)} - E_2^{(0)})^4} \quad (6.2.1)$$

### 6.2.2 Diagrammatic Perturbation Theory for $N$ States

**Ex 6.2** The 4th-order perturbation energy of state  $i$  can be expressed as

$$\begin{aligned}
& \sum_{k,n,m \neq i} \frac{V_{ki}V_{nk}V_{mn}V_{im}}{(E_i^{(0)} - E_k^{(0)})(E_i^{(0)} - E_n^{(0)})(E_i^{(0)} - E_m^{(0)})} + \sum_{n \neq i} \frac{V_{ii}^2V_{ni}V_{in}}{(E_i^{(0)} - E_n^{(0)})^3} - \sum_{m,n \neq i} \frac{V_{ii}V_{mi}V_{in}V_{nm}}{(E_i^{(0)} - E_m^{(0)})^2(E_i^{(0)} - E_n^{(0)})} \\
& - \sum_{m,n \neq i} \frac{V_{ii}V_{ni}V_{im}V_{mn}}{(E_i^{(0)} - E_m^{(0)})^2(E_i^{(0)} - E_n^{(0)})} - \sum_{m,n \neq i} \frac{V_{mi}V_{im}V_{in}V_{ni}}{(E_i^{(0)} - E_m^{(0)})(E_i^{(0)} - E_n^{(0)})(2E_i^{(0)} - E_n^{(0)} - E_m^{(0)})} \\
& - \sum_{m,n \neq i} \frac{V_{mi}V_{im}V_{in}V_{ni}}{(E_i^{(0)} - E_n^{(0)})^2(2E_i^{(0)} - E_n^{(0)} - E_m^{(0)})} \\
& = \sum_{k,n,m \neq i} \frac{V_{ki}V_{nk}V_{mn}V_{im}}{(E_i^{(0)} - E_k^{(0)})(E_i^{(0)} - E_n^{(0)})(E_i^{(0)} - E_m^{(0)})} + \sum_{n \neq i} \frac{V_{ii}^2V_{ni}V_{in}}{(E_i^{(0)} - E_n^{(0)})^3} - 2 \sum_{m,n \neq i} \frac{V_{ii}V_{mi}V_{in}V_{nm}}{(E_i^{(0)} - E_m^{(0)})^2(E_i^{(0)} - E_n^{(0)})} \\
& - \sum_{m,n \neq i} \frac{V_{mi}V_{im}V_{in}V_{ni}}{(E_i^{(0)} - E_m^{(0)})(E_i^{(0)} - E_n^{(0)})^2} \quad (6.2.2)
\end{aligned}$$

while

$$\langle n | \mathcal{H} | \Psi_i^{(3)} \rangle + \langle n | \mathcal{V} | \Psi_i^{(2)} \rangle = E_i^{(0)} \langle n | \Psi_i^{(3)} \rangle + E_i^{(1)} \langle n | \Psi_i^{(2)} \rangle + E_i^{(2)} \langle n | \Psi_i^{(1)} \rangle \quad (6.2.3)$$

$$\begin{aligned}
(E_i^{(0)} - E_n^{(0)}) \langle n | \Psi_i^{(3)} \rangle &= \langle n | \mathcal{V} | \Psi_i^{(2)} \rangle - E_i^{(1)} \langle n | \Psi_i^{(2)} \rangle - E_i^{(2)} \langle n | \Psi_i^{(1)} \rangle \\
&= \langle n | \mathcal{V} | \Psi_i^{(2)} \rangle - E_i^{(1)} \frac{\langle n | \mathcal{V} | \Psi_i^{(1)} \rangle - E_i^{(1)} \langle n | \Psi_i^{(1)} \rangle}{E_i^{(0)} - E_n^{(0)}} - E_i^{(2)} \langle n | \Psi_i^{(1)} \rangle \\
&= \langle n | \mathcal{V} | \Psi_i^{(2)} \rangle - E_i^{(1)} \frac{\langle n | \mathcal{V} | \Psi_i^{(1)} \rangle}{E_i^{(0)} - E_n^{(0)}} + [E_i^{(1)}]^2 \frac{\langle n | \mathcal{V} | i \rangle}{[E_i^{(0)} - E_n^{(0)}]^2} - E_i^{(2)} \frac{\langle n | \mathcal{V} | i \rangle}{E_i^{(0)} - E_n^{(0)}} \quad (6.2.4)
\end{aligned}$$

$$\begin{aligned}
E_i^{(4)} &= \langle i | \mathcal{V} | \Psi_i^{(3)} \rangle \\
&= \sum_{n \neq i} \frac{\langle i | \mathcal{V} | n \rangle}{E_i^{(0)} - E_n^{(0)}} \left\{ \langle n | \mathcal{V} | \Psi_i^{(2)} \rangle - E_i^{(1)} \frac{\langle n | \mathcal{V} | \Psi_i^{(1)} \rangle}{E_i^{(0)} - E_n^{(0)}} + [E_i^{(1)}]^2 \frac{\langle n | \mathcal{V} | i \rangle}{[E_i^{(0)} - E_n^{(0)}]^2} - E_i^{(2)} \frac{\langle n | \mathcal{V} | i \rangle}{E_i^{(0)} - E_n^{(0)}} \right\} \\
&= \sum_{n \neq i} \frac{\langle i | \mathcal{V} | n \rangle}{E_i^{(0)} - E_n^{(0)}} \langle n | \mathcal{V} | \Psi_i^{(2)} \rangle - E_i^{(1)} \sum_{n \neq i} \frac{\langle i | \mathcal{V} | n \rangle}{[E_i^{(0)} - E_n^{(0)}]^2} \langle n | \mathcal{V} | \Psi_i^{(1)} \rangle \\
&\quad + [E_i^{(1)}]^2 \sum_{n \neq i} \frac{V_{in} V_{ni}}{[E_i^{(0)} - E_n^{(0)}]^3} - E_i^{(2)} \sum_{n \neq i} \frac{V_{in} V_{ni}}{[E_i^{(0)} - E_n^{(0)}]^2} \\
&= \sum_{n, m \neq i} \frac{\langle i | \mathcal{V} | n \rangle}{E_i^{(0)} - E_n^{(0)}} \langle n | \mathcal{V} | m \rangle \langle m | \Psi_i^{(2)} \rangle - E_i^{(1)} \sum_{n, m \neq i} \frac{\langle i | \mathcal{V} | n \rangle \langle n | \mathcal{V} | m \rangle \langle m | \mathcal{V} | i \rangle}{[E_i^{(0)} - E_n^{(0)}]^2 [E_i^{(0)} - E_m^{(0)}]} \\
&\quad + [E_i^{(1)}]^2 \sum_{n \neq i} \frac{V_{in} V_{ni}}{[E_i^{(0)} - E_n^{(0)}]^3} - E_i^{(2)} \sum_{n \neq i} \frac{V_{in} V_{ni}}{[E_i^{(0)} - E_n^{(0)}]^2} \\
&= \sum_{n, m \neq i} \frac{V_{in} V_{nm}}{E_i^{(0)} - E_n^{(0)}} \frac{\langle m | \mathcal{V} | \Psi_i^{(1)} \rangle - E_i^{(1)} \langle m | \Psi_i^{(1)} \rangle}{E_i^{(0)} - E_m^{(0)}} - E_i^{(1)} \sum_{n, m \neq i} \frac{V_{in} V_{nm} V_{mi}}{[E_i^{(0)} - E_n^{(0)}]^2 [E_i^{(0)} - E_m^{(0)}]} \\
&\quad + [E_i^{(1)}]^2 \sum_{n \neq i} \frac{V_{in} V_{ni}}{[E_i^{(0)} - E_n^{(0)}]^3} - E_i^{(2)} \sum_{n \neq i} \frac{V_{in} V_{ni}}{[E_i^{(0)} - E_n^{(0)}]^2} \\
&= \sum_{n, m, k \neq i} \frac{V_{in} V_{nm}}{E_i^{(0)} - E_n^{(0)}} \frac{\langle m | \mathcal{V} | k \rangle \langle k | \mathcal{V} | i \rangle}{[E_i^{(0)} - E_m^{(0)}] [E_i^{(0)} - E_k^{(0)}]} - E_i^{(1)} \sum_{n, m \neq i} \frac{V_{in} V_{nm}}{E_i^{(0)} - E_n^{(0)}} \frac{\langle m | \mathcal{V} | i \rangle}{[E_i^{(0)} - E_m^{(0)}]^2} \\
&\quad - E_i^{(1)} \sum_{n, m \neq i} \frac{V_{in} V_{nm} V_{mi}}{[E_i^{(0)} - E_n^{(0)}]^2 [E_i^{(0)} - E_m^{(0)}]} + [E_i^{(1)}]^2 \sum_{n \neq i} \frac{V_{in} V_{ni}}{[E_i^{(0)} - E_n^{(0)}]^3} - E_i^{(2)} \sum_{n \neq i} \frac{V_{in} V_{ni}}{[E_i^{(0)} - E_n^{(0)}]^2} \\
&= \sum_{n, m, k \neq i} \frac{V_{in} V_{nm} V_{mk} V_{ki}}{[E_i^{(0)} - E_n^{(0)}] [E_i^{(0)} - E_m^{(0)}] [E_i^{(0)} - E_k^{(0)}]} - 2V_{ii} \sum_{n, m \neq i} \frac{V_{in} V_{nm} V_{mi}}{[E_i^{(0)} - E_n^{(0)}] [E_i^{(0)} - E_m^{(0)}]^2} \\
&\quad + V_{ii}^2 \sum_{n \neq i} \frac{V_{in} V_{ni}}{[E_i^{(0)} - E_n^{(0)}]^3} - \sum_{m \neq i} \frac{V_{mi} V_{im}}{[E_i^{(0)} - E_m^{(0)}]} \sum_{n \neq i} \frac{V_{in} V_{ni}}{[E_i^{(0)} - E_n^{(0)}]^2} \tag{6.2.5}
\end{aligned}$$

which agrees with diagrammatic results above.

### 6.2.3 Summation of Diagrams

## 6.3 Orbital Perturbation Theory: One-Particle Perturbations

**Ex 6.3** Since  $n \neq 0$  and  $v(i)$  is one-particle operator,  $n$  must be single-excited, i.e.  $|\Psi_a^r\rangle$ . Thus,

$$\begin{aligned}
E_0^{(2)} &= \sum_{a,r} \frac{|\langle \Psi_0 | \sum_i v(i) | \Psi_a^r \rangle|^2}{\langle \Psi_0 | \mathcal{H} | \Psi_0 \rangle - \langle \Psi_a^r | \mathcal{H} | \Psi_a^r \rangle} \\
&= \sum_{a,r} \frac{v_{ar} v_{ra}}{\sum_b \varepsilon_b^{(0)} - (\sum_{b \neq a} \varepsilon_b^{(0)} + \varepsilon_r^{(0)})} \\
&= \sum_{a,r} \frac{v_{ar} v_{ra}}{\varepsilon_a^{(0)} - \varepsilon_r^{(0)}} \tag{6.3.1}
\end{aligned}$$

**Ex 6.4** Eq 6.15 in textbook gives

$$\begin{aligned}
E_i^{(3)} &= \sum_{n, m \neq i} \frac{\langle i | \mathcal{V} | n \rangle \langle n | \mathcal{V} | m \rangle \langle m | \mathcal{V} | i \rangle}{(E_i^{(0)} - E_n^{(0)})(E_i^{(0)} - E_m^{(0)})} - E_i^{(1)} \sum_{n \neq i} \frac{|\langle i | \mathcal{V} | n \rangle|^2}{(E_i^{(0)} - E_n^{(0)})^2} \\
&= A_i^{(3)} + B_i^{(3)} \tag{6.3.2}
\end{aligned}$$

a.

$$\begin{aligned}
B_0^{(3)} &= -E_0^{(1)} \sum_{n \neq 0} \frac{|\langle \Psi_0 | \mathcal{V} | n \rangle|^2}{(E_0^{(0)} - E_n^{(0)})^2} \\
&= -\sum_b v_{bb} \sum_{a,r} \frac{v_{ar} v_{ra}}{(\varepsilon_a^{(0)} - \varepsilon_r^{(0)})^2} \\
&= -\sum_{a,b,r} \frac{v_{aa} v_{br} v_{rb}}{(\varepsilon_b^{(0)} - \varepsilon_r^{(0)})^2}
\end{aligned} \tag{6.3.3}$$

b.

$$\begin{aligned}
A_0^{(3)} &= \sum_{n,m \neq 0} \frac{\langle \Psi_0 | \mathcal{V} | n \rangle \langle n | \mathcal{V} | m \rangle \langle m | \mathcal{V} | \Psi_0 \rangle}{(E_0^{(0)} - E_n^{(0)})(E_0^{(0)} - E_m^{(0)})} \\
&= \sum_{a,r,b,s} \frac{\langle \Psi_0 | \mathcal{V} | \Psi_a^r \rangle \langle \Psi_a^r | \mathcal{V} | \Psi_b^s \rangle \langle \Psi_b^s | \mathcal{V} | \Psi_0 \rangle}{(\varepsilon_a^{(0)} - \varepsilon_r^{(0)})(\varepsilon_b^{(0)} - \varepsilon_s^{(0)})} \\
&= \sum_{a,r,b,s} \frac{v_{ar} v_{sb} \langle \Psi_a^r | \mathcal{V} | \Psi_b^s \rangle}{(\varepsilon_a^{(0)} - \varepsilon_r^{(0)})(\varepsilon_b^{(0)} - \varepsilon_s^{(0)})}
\end{aligned} \tag{6.3.4}$$

c. Clearly, if  $a \neq b, r \neq s$

$$\langle \Psi_a^r | \mathcal{V} | \Psi_b^s \rangle = 0 \tag{6.3.5}$$

If  $a = b, r \neq s$ ,

$$\begin{aligned}
\langle \Psi_a^r | \mathcal{V} | \Psi_b^s \rangle &= \langle r | v | s \rangle \\
&= v_{rs}
\end{aligned} \tag{6.3.6}$$

If  $a \neq b, r = s$ ,

$$\begin{aligned}
\langle \Psi_a^r | \mathcal{V} | \Psi_b^s \rangle &= \langle \Psi_a^r | \mathcal{V} | \Psi_b^r \rangle \\
&= \langle \Psi_a^r | \mathcal{V} | -\Psi_{ab}^r \rangle \\
&= -\langle b | v | a \rangle \\
&= -v_{ba}
\end{aligned} \tag{6.3.7}$$

If  $a = b, r = s$ ,

$$\begin{aligned}
\langle \Psi_a^r | \mathcal{V} | \Psi_b^s \rangle &= \langle \Psi_a^r | \mathcal{V} | \Psi_a^r \rangle \\
&= \sum_c v_{cc} - v_{aa} + v_{rr}
\end{aligned} \tag{6.3.8}$$

d.

$$\begin{aligned}
E_0^{(3)} &= A_0^{(3)} + B_0^{(3)} \\
&= \sum_{a,r,b,s} \frac{v_{ar} v_{sb} \langle \Psi_a^r | \mathcal{V} | \Psi_b^s \rangle}{(\varepsilon_a^{(0)} - \varepsilon_r^{(0)})(\varepsilon_b^{(0)} - \varepsilon_s^{(0)})} - \sum_{a,b,r} \frac{v_{aa} v_{br} v_{rb}}{(\varepsilon_b - \varepsilon_r)^2} \\
&= \sum_{a,r \neq s} \frac{v_{ar} v_{sa} v_{rs}}{(\varepsilon_a^{(0)} - \varepsilon_r^{(0)})(\varepsilon_a^{(0)} - \varepsilon_s^{(0)})} + \sum_{a \neq b, r} \frac{v_{ar} v_{rb} (-v_{ba})}{(\varepsilon_a^{(0)} - \varepsilon_r^{(0)})(\varepsilon_b^{(0)} - \varepsilon_r^{(0)})} \\
&\quad + \sum_{a,r} \frac{v_{ar} v_{ra} (\sum_c v_{cc} - v_{aa} + v_{rr})}{(\varepsilon_a^{(0)} - \varepsilon_r^{(0)})^2} - \sum_{a,b,r} \frac{v_{aa} v_{br} v_{rb}}{(\varepsilon_b^{(0)} - \varepsilon_r^{(0)})^2} \\
&= \sum_{a,r \neq s} \frac{v_{ar} v_{sa} v_{rs}}{(\varepsilon_a^{(0)} - \varepsilon_r^{(0)})(\varepsilon_a^{(0)} - \varepsilon_s^{(0)})} + \sum_{a \neq b, r} \frac{v_{ar} v_{rb} (-v_{ba})}{(\varepsilon_a^{(0)} - \varepsilon_r^{(0)})(\varepsilon_b^{(0)} - \varepsilon_r^{(0)})} \\
&\quad + \sum_{a,r} \frac{v_{ar} v_{ra} (\sum_c v_{cc} - v_{aa} + v_{rr})}{(\varepsilon_a^{(0)} - \varepsilon_r^{(0)})^2} - \sum_{a,r} \frac{\sum_c v_{cc} v_{ar} v_{ra}}{(\varepsilon_a^{(0)} - \varepsilon_r^{(0)})^2}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{a,r \neq s} \frac{v_{ar}v_{sa}v_{rs}}{(\varepsilon_a^{(0)} - \varepsilon_r^{(0)})(\varepsilon_a^{(0)} - \varepsilon_s^{(0)})} + \sum_{a \neq b,r} \frac{v_{ar}v_{rb}(-v_{ba})}{(\varepsilon_a^{(0)} - \varepsilon_r^{(0)})(\varepsilon_b^{(0)} - \varepsilon_r^{(0)})} + \sum_{a,r} \frac{v_{ar}v_{ra}(-v_{aa} + v_{rr})}{(\varepsilon_a^{(0)} - \varepsilon_r^{(0)})^2} \\
&= \sum_{a,r,s} \frac{v_{ar}v_{sa}v_{rs}}{(\varepsilon_a^{(0)} - \varepsilon_r^{(0)})(\varepsilon_a^{(0)} - \varepsilon_s^{(0)})} - \sum_{a,b,r} \frac{v_{ar}v_{rb}v_{ba}}{(\varepsilon_a^{(0)} - \varepsilon_r^{(0)})(\varepsilon_b^{(0)} - \varepsilon_r^{(0)})}
\end{aligned} \tag{6.3.9}$$

e. That's obvious.

**Ex 6.5** Since  $a, b$  run over all  $n$  occupied orbitals  $i, j$  and  $r$  runs over all  $n$  unoccupied orbitals  $k^*$ , we have

$$\begin{aligned}
-2 \sum_{a,b,r}^{N/2} \frac{v_{ra}v_{ab}v_{br}}{(\varepsilon_a^{(0)} - \varepsilon_r^{(0)})(\varepsilon_b^{(0)} - \varepsilon_r^{(0)})} &= -\frac{2}{(2\beta)^2} \sum_i^n \sum_j^n \sum_k^n \langle i | v | j \rangle \langle j | v | k^* \rangle \langle k^* | v | i \rangle \\
&= -\frac{2}{(2\beta)^2} \sum_i^3 \left[ \langle i | v | i+1 \rangle \langle i+1 | v | (i+2)^* \rangle \langle (i+2)^* | v | i \rangle \right. \\
&\quad \left. + \langle i | v | i+2 \rangle \langle i+2 | v | (i+1)^* \rangle \langle (i+1)^* | v | i \rangle \right] \\
&= -\frac{2}{(2\beta)^2} \sum_i^3 [(\beta/2)(\beta/2)(-\beta/2) + (\beta/2)(-\beta/2)(\beta/2)] \\
&= -\frac{2}{(2\beta)^2} \times 3 \times (-\beta^3/4) \\
&= 3\beta/8
\end{aligned} \tag{6.3.10}$$

**Ex 6.6**

a. Using the general expression, we get

$$\begin{aligned}
\mathcal{E}_0 &= 6\alpha - 2 \sum_{j=-1}^1 (\beta_1^2 + \beta_2^2 + 2\beta_1\beta_2 \cos \frac{2j\pi}{3})^{1/2} \\
&= 6\alpha - 2(\beta_1^2 + \beta_2^2 + 2\beta_1\beta_2 \cos \frac{-2\pi}{3})^{1/2} - 2(\beta_1^2 + \beta_2^2 + 2\beta_1\beta_2 \cos 0)^{1/2} - 2(\beta_1^2 + \beta_2^2 + 2\beta_1\beta_2 \cos \frac{2\pi}{3})^{1/2} \\
&= 6\alpha - 2(\beta_1^2 + \beta_2^2 - \beta_1\beta_2)^{1/2} - 2(\beta_1^2 + \beta_2^2 + 2\beta_1\beta_2)^{1/2} - 2(\beta_1^2 + \beta_2^2 - \beta_1\beta_2)^{1/2} \\
&= 6\alpha - 2|\beta_1 + \beta_2| - 4(\beta_1^2 + \beta_2^2 - \beta_1\beta_2)^{1/2} \\
&= 6\alpha + 2(\beta_1 + \beta_2) - 4(\beta_1^2 + \beta_2^2 - \beta_1\beta_2)^{1/2}
\end{aligned} \tag{6.3.11}$$

Using Hückel matrix:

$$\mathbf{H} = \begin{pmatrix} \alpha & \beta_1 & 0 & 0 & 0 & \beta_2 \\ \beta_1 & \alpha & \beta_2 & 0 & 0 & 0 \\ 0 & \beta_2 & \alpha & \beta_1 & 0 & 0 \\ 0 & 0 & \beta_1 & \alpha & \beta_2 & 0 \\ 0 & 0 & 0 & \beta_2 & \alpha & \beta_1 \\ \beta_2 & 0 & 0 & 0 & \beta_1 & \alpha \end{pmatrix} \tag{6.3.12}$$

Eigenvalues of  $\mathbf{H}$  are

$$\begin{aligned}
&\alpha + (\beta_1 + \beta_2), \\
&\alpha - \sqrt{\beta_1^2 + \beta_2^2 - \beta_1\beta_2} \quad (2\text{-fold}), \\
&\alpha + \sqrt{\beta_1^2 + \beta_2^2 - \beta_1\beta_2} \quad (2\text{-fold}), \\
&\alpha - (\beta_1 + \beta_2),
\end{aligned} \tag{6.3.13}$$

thus

$$\begin{aligned}
\mathcal{E}_0 &= 2[\alpha + (\beta_1 + \beta_2)] + 4 \left[ \alpha - \sqrt{\beta_1^2 + \beta_2^2 - \beta_1\beta_2} \right] \\
&= 6\alpha + 2(\beta_1 + \beta_2) - 4\sqrt{\beta_1^2 + \beta_2^2 - \beta_1\beta_2}
\end{aligned} \tag{6.3.14}$$

**b.**

$$\begin{aligned}
E_R &= \mathcal{E}_0 - (N\alpha + N\beta) \\
&= 6\alpha + 2(\beta_1 + \beta_2) - 4\sqrt{\beta_1^2 + \beta_2^2 - \beta_1\beta_2} - (6\alpha + 6\beta) \\
&= -4\beta_1 + 2\beta_2 - 4\sqrt{\beta_1^2 + \beta_2^2 - \beta_1\beta_2} \\
&= 4\beta \left( -1 + \frac{1}{2}x + \sqrt{1 + x^2 - x} \right)
\end{aligned} \tag{6.3.15}$$

**c.**

$$\begin{aligned}
E_R &= 4\beta \left( -1 + \frac{1}{2}x + \sqrt{1 + x^2 - x} \right) \\
&= 4\beta \left[ -1 + \frac{1}{2}x + 1 + \frac{1}{2}(x^2 - x) - \frac{1}{8}(x^2 - x)^2 + \frac{1}{16}(x^2 - x)^3 - \frac{5}{128}(x^2 - x)^4 \right] \\
&= 4\beta \left[ \frac{1}{2}x^2 - \frac{1}{8}(x^4 + x^2 - 2x^3) + \frac{1}{16}(-x^3 + 3x^4) - \frac{5}{128}x^4 + \dots \right] \\
&= 4\beta \left[ \frac{3}{8}x^2 + \frac{3}{16}x^3 + \frac{3}{128}x^4 + \dots \right] \\
&= \beta \left[ \frac{3}{2}x^2 + \frac{3}{4}x^3 + \frac{3}{32}x^4 + \dots \right]
\end{aligned} \tag{6.3.16}$$