Modern Quantum Chemistry, Szabo & Ostlund

HW

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Contents

2 Many-electron Wave Functions and Operators

2.1 The Electronic Problem

2.1.1 Atomic Units

2.1.2 The B-O Approximation

2.1.3 The Antisymmetry or Pauli Exclusion Principle

2.2 Orbitals, Slater Determinants, and Basis Functions

2.2.1 Spin Orbitals and Spatial Orbitals

Ex 2.1 Consider $\langle \chi_k \,|\, \chi_m \rangle$. If $k = m$,

$$\langle \chi_{2i-1} \,|\, \chi_{2i-1} \rangle = \langle \psi_i^\alpha \,|\, \psi_i^\alpha \rangle \langle \alpha \,|\, \alpha \rangle = 1 \tag{2.2.1}$$

$$\langle \chi_{2i} \,|\, \chi_{2i} \rangle = \left\langle \psi_i^\beta \,\middle|\, \psi_i^\beta \right\rangle \langle \alpha \,|\, \alpha \rangle = 1 \tag{2.2.2}$$

thus

$$\langle \chi_k \,|\, \chi_k \rangle = 1 \tag{2.2.3}$$

If $k \neq m$, three cases may occur as below

$$\langle \chi_{2i-1} \,|\, \chi_{2j-1} \rangle = \langle \psi_i^\alpha \,|\, \psi_j^\alpha \rangle \langle \alpha \,|\, \alpha \rangle = 0 \cdot 1 = 0 \qquad (i \neq j) \tag{2.2.4}$$

$$\langle \chi_{2i-1} \,|\, \chi_{2j} \rangle = \left\langle \psi_i^\alpha \,\middle|\, \psi_j^\beta \right\rangle \langle \alpha \,|\, \beta \rangle = S_{ij} \cdot 0 = 0 \tag{2.2.5}$$

$$\langle \chi_{2i} \,|\, \chi_{2j} \rangle = \left\langle \psi_i^\beta \,\middle|\, \psi_j^\beta \right\rangle \langle \beta \,|\, \beta \rangle = 0 \cdot 1 = 0 \qquad (i \neq j) \tag{2.2.6}$$

thus

$$\langle \chi_k \,|\, \chi_m \rangle = 0 \qquad (k \neq m) \tag{2.2.7}$$

Overall,

$$\langle \chi_k \,|\, \chi_m \rangle = \delta_{km} \tag{2.2.8}$$

2.2.2 Hartree Products

Ex 2.2

$$\begin{aligned}
\mathscr{H}\Psi^{HP} &= \sum_{i=1}^{N} h(i)\chi_i(\mathbf{x}_1)\chi_j(\mathbf{x}_2)\cdots\chi_k(\mathbf{x}_N) \\
&= \varepsilon_i\chi_i(\mathbf{x}_1)\chi_j(\mathbf{x}_2)\cdots\chi_k(\mathbf{x}_N) + \chi_i(\mathbf{x}_1)[\varepsilon_j\chi_j(\mathbf{x}_2)]\cdots\chi_k(\mathbf{x}_N) + \cdots + \chi_i(\mathbf{x}_1)\chi_j(\mathbf{x}_2)\cdots[\varepsilon_k\chi_k(\mathbf{x}_N)] \\
&= (\varepsilon_i + \varepsilon_j + \cdots + \varepsilon_k)\Psi^{HP}
\end{aligned} \tag{2.2.9}$$

2.2.3 Slater Determinants

Ex 2.3

$$\begin{aligned}
\langle \Psi \,|\, \Psi \rangle &= \frac{1}{2}(\langle \chi_i \,|\, \chi_i \rangle \langle \chi_j \,|\, \chi_j \rangle - \langle \chi_i \,|\, \chi_j \rangle \langle \chi_j \,|\, \chi_i \rangle - \langle \chi_j \,|\, \chi_i \rangle \langle \chi_i \,|\, \chi_j \rangle + \langle \chi_j \,|\, \chi_j \rangle \langle \chi_i \,|\, \chi_i \rangle) \\
&= \frac{1}{2}(1 + 0 + 0 + 1) = 1
\end{aligned} \tag{2.2.10}$$

Ex 2.4 According to Ex. 2.2, we know that $\chi_i(\mathbf{x}_1)\chi_j(\mathbf{x}_2)$ are an eigenfunction of $\mathscr{H}$ and has the eigenvalue $\varepsilon_i\varepsilon_j$. Similarly, we have the same conclusion for $\chi_i(\mathbf{x}_2)\chi_j(\mathbf{x}_1)$.
For the antisymmetrized wave function,

$$\begin{aligned}
\langle \Psi \,|\, \mathscr{H} \,|\, \Psi \rangle &= \frac{1}{2}\left(\langle \chi_i(\mathbf{x}_1)\chi_j(\mathbf{x}_2) \,|\, \mathscr{H} \,|\, \chi_i(\mathbf{x}_1)\chi_j(\mathbf{x}_2) \rangle - \langle \chi_i(\mathbf{x}_1)\chi_j(\mathbf{x}_2) \,|\, \mathscr{H} \,|\, \chi_j(\mathbf{x}_1)\chi_i(\mathbf{x}_2) \rangle\right. \\
&\quad \left. - \langle \chi_j(\mathbf{x}_1)\chi_i(\mathbf{x}_2) \,|\, \mathscr{H} \,|\, \chi_i(\mathbf{x}_1)\chi_j(\mathbf{x}_2) \rangle + \langle \chi_j(\mathbf{x}_1)\chi_i(\mathbf{x}_2) \,|\, \mathscr{H} \,|\, \chi_j(\mathbf{x}_1)\chi_i(\mathbf{x}_2) \rangle\right) \\
&= \frac{1}{2}(\varepsilon_i + \varepsilon_j - 0 - 0 + \varepsilon_i + \varepsilon_j) \\
&= \varepsilon_i + \varepsilon_j
\end{aligned} \tag{2.2.11}$$

Ex 2.5

$$
\begin{aligned}
\langle K \mid L\rangle &= \frac{1}{2}\left\langle \chi_i(\mathbf{x}_1)\chi_j(\mathbf{x}_2) - \chi_j(\mathbf{x}_1)\chi_i(\mathbf{x}_2) \mid \chi_k(\mathbf{x}_1)\chi_l(\mathbf{x}_2) - \chi_l(\mathbf{x}_1)\chi_k(\mathbf{x}_2)\right\rangle \\
&= \frac{1}{2}(\langle \chi_i \mid \chi_k\rangle \langle \chi_j \mid \chi_l\rangle - \langle \chi_i \mid \chi_l\rangle \langle \chi_j \mid \chi_k\rangle - \langle \chi_j \mid \chi_k\rangle \langle \chi_i \mid \chi_l\rangle + \langle \chi_j \mid \chi_l\rangle \langle \chi_i \mid \chi_k\rangle) \\
&= \frac{1}{2}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} - \delta_{jk}\delta_{il} + \delta_{jl}\delta_{ik}) \\
&= \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}
\end{aligned} \tag{2.2.12}
$$

2.2.4 The Hartree-Fock Approximation

2.2.5 The Minimal Basis H_2 Model

Ex 2.6

$$\langle \psi_1 \mid \psi_1\rangle = \frac{1}{2(1+S_{12})}(\langle \phi_1 \mid \phi_1\rangle + 2\langle \phi_1 \mid \phi_2\rangle + \langle \phi_2 \mid \phi_2\rangle) = \frac{2+2S_{12}}{2(1+S_{12})} = 1 \tag{2.2.13}$$

$$\langle \psi_2 \mid \psi_2\rangle = \frac{1}{2(1-S_{12})}(\langle \phi_1 \mid \phi_1\rangle - 2\langle \phi_1 \mid \phi_2\rangle + \langle \phi_2 \mid \phi_2\rangle) = \frac{2-2S_{12}}{2(1-S_{12})} = 1 \tag{2.2.14}$$

$$\langle \psi_1 \mid \psi_2\rangle = \frac{1}{2\sqrt{1+S_{12}}\sqrt{1-S_{12}}}(\langle \phi_1 \mid \phi_1\rangle - \langle \phi_2 \mid \phi_2\rangle) = 0 \tag{2.2.15}$$

2.2.6 Excited Determinants

2.2.7 Form of the Exact Wfn and CI

Ex 2.7 Size of full CI matrix

$$C_{72}^{42} = 164307576757973059488 \approx 1.64 \times 10^{20} \tag{2.2.16}$$

The number of singly excited determinants

$$42 \times 30 = 1260 \tag{2.2.17}$$

The number of doubly excited determinants

$$C_{42}^{2}C_{30}^{2} = 374535 \tag{2.2.18}$$

2.3 Operators and Matrix Elements

2.3.1 Minimal Basis H_2 Matrix Elements

Ex 2.8

$$
\begin{aligned}
\left\langle \Psi_{12}^{34} \middle| h(1) \middle| \Psi_{12}^{34}\right\rangle &= \frac{1}{2}\left\langle \chi_3(\mathbf{x}_1)\chi_4(\mathbf{x}_2) - \chi_3(\mathbf{x}_2)\chi_4(\mathbf{x}_1) \mid h(1) \mid \chi_3(\mathbf{x}_1)\chi_4(\mathbf{x}_2) - \chi_3(\mathbf{x}_2)\chi_4(\mathbf{x}_1)\right\rangle \\
&= \frac{1}{2}(\langle \chi_3 \mid h(1) \mid \chi_3\rangle - 0 - 0 + \langle \chi_4 \mid h(1) \mid \chi_4\rangle) \\
&= \frac{1}{2}(\langle \chi_3 \mid h(1) \mid \chi_3\rangle + \langle \chi_4 \mid h(1) \mid \chi_4\rangle)
\end{aligned} \tag{2.3.1}
$$

thus

$$\left\langle \Psi_{12}^{34} \middle| \mathcal{O}_1 \middle| \Psi_{12}^{34}\right\rangle = \langle 3 \mid h \mid 3\rangle + \langle 4 \mid h \mid 4\rangle \tag{2.3.2}$$

$$
\begin{aligned}
\left\langle \Psi_0 \middle| h(1) \middle| \Psi_{12}^{34}\right\rangle &= \frac{1}{2}\left\langle \chi_1(\mathbf{x}_1)\chi_2(\mathbf{x}_2) - \chi_2(\mathbf{x}_2)\chi_1(\mathbf{x}_1) \mid h(1) \mid \chi_3(\mathbf{x}_1)\chi_4(\mathbf{x}_2) - \chi_3(\mathbf{x}_2)\chi_4(\mathbf{x}_1)\right\rangle \\
&= \frac{1}{2}(0 - 0 - 0 + 0) \\
&= 0
\end{aligned} \tag{2.3.3}
$$

thus

$$\left\langle \Psi_0 \middle| \mathcal{O}_1 \middle| \Psi_{12}^{34}\right\rangle = 0 \tag{2.3.4}$$

Similarly, we get

$$\left\langle \Psi_{12}^{34} \middle| \mathcal{O}_1 \middle| \Psi_0\right\rangle = 0 \tag{2.3.5}$$

Ex 2.9 From Eq. (2.92) in textbook, we get

$$\langle \Psi_0 \,|\, \mathscr{H} \,|\, \Psi_0 \rangle = \langle 1 \,|\, h \,|\, 1 \rangle + \langle 2 \,|\, h \,|\, 2 \rangle + \langle 12 \,|\, 12 \rangle - \langle 12 \,|\, 21 \rangle \tag{2.3.6}$$

From Ex 2.8, we get

$$\left\langle \Psi_0 \,\middle|\, \mathcal{O}_1 \,\middle|\, \Psi_{12}^{34} \right\rangle = \left\langle \Psi_{12}^{34} \,\middle|\, \mathcal{O}_1 \,\middle|\, \Psi_0 \right\rangle = 0 \tag{2.3.7}$$

thus

$$\begin{aligned}\left\langle \Psi_0 \,\middle|\, \mathscr{H} \,\middle|\, \Psi_{12}^{34} \right\rangle &= \left\langle \Psi_0 \,\middle|\, \mathcal{O}_2 \,\middle|\, \Psi_{12}^{34} \right\rangle \\ &= \frac{1}{2} \left\langle \chi_1(\mathbf{x}_1)\chi_2(\mathbf{x}_2) - \chi_1(\mathbf{x}_2)\chi_2(\mathbf{x}_1) \,\middle|\, \frac{1}{r_{12}} \,\middle|\, \chi_3(\mathbf{x}_1)\chi_4(\mathbf{x}_2) - \chi_3(\mathbf{x}_2)\chi_4(\mathbf{x}_1) \right\rangle \\ &= \langle 12 \,|\, 34 \rangle - \langle 12 \,|\, 43 \rangle\end{aligned} \tag{2.3.8}$$

$$\begin{aligned}\left\langle \Psi_{12}^{34} \,\middle|\, \mathscr{H} \,\middle|\, \Psi_0 \right\rangle &= \left\langle \Psi_{12}^{34} \,\middle|\, \mathcal{O}_2 \,\middle|\, \Psi_0 \right\rangle \\ &= \frac{1}{2} \left\langle \chi_3(\mathbf{x}_1)\chi_4(\mathbf{x}_2) - \chi_3(\mathbf{x}_2)\chi_4(\mathbf{x}_1) \,\middle|\, \frac{1}{r_{12}} \,\middle|\, \chi_1(\mathbf{x}_1)\chi_2(\mathbf{x}_2) - \chi_2(\mathbf{x}_2)\chi_1(\mathbf{x}_1) \right\rangle \\ &= \langle 34 \,|\, 12 \rangle - \langle 34 \,|\, 21 \rangle\end{aligned} \tag{2.3.9}$$

$$\begin{aligned}\left\langle \Psi_{12}^{34} \,\middle|\, \mathscr{H} \,\middle|\, \Psi_{12}^{34} \right\rangle &= \left\langle \Psi_{12}^{34} \,\middle|\, h(1) + h(2) + \frac{1}{r_{12}} \,\middle|\, \Psi_{12}^{34} \right\rangle \\ &= 2 \times \frac{1}{2} \left\langle \chi_3(\mathbf{x}_1)\chi_4(\mathbf{x}_2) - \chi_3(\mathbf{x}_2)\chi_4(\mathbf{x}_1) \,|\, h(1) \,|\, \chi_3(\mathbf{x}_1)\chi_4(\mathbf{x}_2) - \chi_3(\mathbf{x}_2)\chi_4(\mathbf{x}_1) \right\rangle \\ &+ \frac{1}{2} \left\langle \chi_3(\mathbf{x}_1)\chi_4(\mathbf{x}_2) - \chi_3(\mathbf{x}_2)\chi_4(\mathbf{x}_1) \,\middle|\, \frac{1}{r_{12}} \,\middle|\, \chi_3(\mathbf{x}_1)\chi_4(\mathbf{x}_2) - \chi_3(\mathbf{x}_2)\chi_4(\mathbf{x}_1) \right\rangle \\ &= \langle 3 \,|\, h \,|\, 3 \rangle + \langle 4 \,|\, h \,|\, 4 \rangle + \langle 34 \,|\, 34 \rangle - \langle 34 \,|\, 43 \rangle\end{aligned} \tag{2.3.10}$$

2.3.2 Notations for 1- and 2-Electron Integrals

2.3.3 General Rules for Matrix Elements

Ex 2.10

$$\langle K \,|\, \mathscr{H} \,|\, K \rangle = \sum_m^N [m|h|m] + \frac{1}{2} \sum_m^N \sum_n^N \langle mn \,\|\, mn \rangle = \sum_m^N [m|h|m] + \frac{1}{2} \sum_m^N \sum_n^N ([mm|nn] - [mn|nm]) \tag{2.3.11}$$

When $m = n$,

$$[mm|mm] - [mm|mm] = 0 \tag{2.3.12}$$

thus

$$\langle K \,|\, \mathscr{H} \,|\, K \rangle = \sum_m^N [m|h|m] + \frac{1}{2} \sum_m^N \sum_{n \neq m}^N ([mm|nn] - [mn|nm]) = \sum_m^N [m|h|m] + \sum_m^N \sum_{n > m}^N ([mm|nn] - [mn|nm]) \tag{2.3.13}$$

Ex 2.11

$$\begin{aligned}\langle K \,|\, \mathscr{H} \,|\, K \rangle = \langle K \,|\, \mathcal{O}_1 + \mathcal{O}_2 \,|\, K \rangle &= \sum_m^N [m|h|m] + \sum_m^N \sum_{n > m}^N \langle mn \,\|\, mn \rangle \\ &= \langle 1 \,|\, h \,|\, 1 \rangle + \langle 2 \,|\, h \,|\, 2 \rangle + \langle 3 \,|\, h \,|\, 3 \rangle + \langle 12 \,\|\, 12 \rangle + \langle 13 \,\|\, 13 \rangle + \langle 23 \,\|\, 23 \rangle\end{aligned} \tag{2.3.14}$$

Ex 2.12

$$\begin{aligned}\langle \Psi_0 \,|\, \mathscr{H} \,|\, \Psi_0 \rangle &= \langle 1 \,|\, h \,|\, 1 \rangle + \langle 2 \,|\, h \,|\, 2 \rangle + \langle 12 \,\|\, 12 \rangle \\ &= \langle 1 \,|\, h \,|\, 1 \rangle + \langle 2 \,|\, h \,|\, 2 \rangle + \langle 12 \,|\, 12 \rangle - \langle 12 \,|\, 21 \rangle\end{aligned} \tag{2.3.15}$$

$$\left\langle \Psi_0 \,\middle|\, \mathscr{H} \,\middle|\, \Psi_{12}^{34} \right\rangle = \langle 12 \,\|\, 34 \rangle = \langle 12 \,|\, 34 \rangle - \langle 12 \,|\, 43 \rangle \tag{2.3.16}$$

$$\left\langle \Psi_{12}^{34} \,\middle|\, \mathscr{H} \,\middle|\, \Psi_0 \right\rangle = \langle 34 \,\|\, 12 \rangle = \langle 34 \,|\, 12 \rangle - \langle 34 \,|\, 21 \rangle \tag{2.3.17}$$

$$\begin{aligned}\left\langle \Psi_{12}^{34} \,\middle|\, \mathscr{H} \,\middle|\, \Psi_{12}^{34} \right\rangle &= \langle 3 \,|\, h \,|\, 3 \rangle + \langle 4 \,|\, h \,|\, 4 \rangle + \langle 34 \,\|\, 34 \rangle \\ &= \langle 3 \,|\, h \,|\, 3 \rangle + \langle 4 \,|\, h \,|\, 4 \rangle + \langle 34 \,|\, 34 \rangle - \langle 34 \,|\, 43 \rangle\end{aligned} \tag{2.3.18}$$

Which are exactly the same with Ex 2.9.

Ex 2.13 if $a=b$, $r=s$

$$\langle \Psi_a^r \,|\, \mathcal{O} \,|\, \Psi_b^s \rangle = \langle \Psi_a^r \,|\, \mathcal{O}_1 \,|\, \Psi_a^r \rangle = \sum_c^N \langle c \,|\, h \,|\, c \rangle - \langle a \,|\, h \,|\, a \rangle + \langle r \,|\, h \,|\, r \rangle \tag{2.3.19}$$

if $a=b$, $r\neq s$

$$\langle \Psi_a^r \,|\, \mathcal{O} \,|\, \Psi_b^s \rangle = \langle \Psi_a^r \,|\, \mathcal{O}_1 \,|\, \Psi_a^s \rangle = \langle r \,|\, h \,|\, s \rangle \tag{2.3.20}$$

if $a\neq b$, $r=s$

$$\langle \Psi_a^r \,|\, \mathcal{O} \,|\, \Psi_b^s \rangle = \langle \Psi_a^r \,|\, \mathcal{O}_1 \,|\, \Psi_b^r \rangle = \langle \Psi_a^r \,|\, \mathcal{O}_1 \,|\, -(\Psi_a^r)_b^a \rangle = -\langle b \,|\, h \,|\, a \rangle \tag{2.3.21}$$

if $a\neq b$, $r\neq s$

$$\langle \Psi_a^r \,|\, \mathcal{O} \,|\, \Psi_b^s \rangle = \langle \Psi_a^r \,|\, \mathcal{O}_1 \,|\, (\Psi_a^r)_{rb}^{as} \rangle = 0 \tag{2.3.22}$$

Ex 2.14

$${}^N E_0 = \sum_m^N \langle m \,|\, h \,|\, m \rangle + \sum_m^M \sum_{n>m}^M \langle mn \,\|\, mn \rangle \tag{2.3.23}$$

$${}^{N-1} E_0 = \sum_{m\neq a}^N \langle m \,|\, h \,|\, m \rangle + \sum_{m\neq a}^M \sum_{n>m,n\neq a}^M \langle mn \,\|\, mn \rangle \tag{2.3.24}$$

$${}^N E_0 - {}^{N-1} E_0 = \langle a \,|\, h \,|\, a \rangle + \sum_{b\neq a}^N \langle ab \,\|\, ab \rangle \tag{2.3.25}$$

2.3.4 Derivation of the Rules for Matrix Elements

Ex 2.15

$$\begin{aligned}\langle \Psi \,|\, \mathscr{H} \,|\, \Psi \rangle &= \frac{1}{N!} \left\langle \sum_{n=1}^{N!} (-1)^{p_n} \mathscr{P}_n \{\chi_i(1)\chi_j(2)\cdots\chi_k(N)\} \,\middle|\, \sum_{c=1}^N h(c) \,\middle|\, \sum_{m=1}^{N!} (-1)^{p_m} \mathscr{P}_m \{\chi_i(1)\chi_j(2)\cdots\chi_k(N)\} \right\rangle \\ &= \frac{1}{N!} \sum_{n=1}^{N!} \sum_{m=1}^{N!} (-1)^{p_n+p_m} \sum_{c=1}^N \langle \mathscr{P}_n \{\chi_i(1)\chi_j(2)\cdots\chi_k(N)\} \,|\, h(c) \,|\, \mathscr{P}_m \{\chi_i(1)\chi_j(2)\cdots\chi_k(N)\} \rangle\end{aligned} \tag{2.3.26}$$

Since the integral inside equals 0 when $\mathscr{P}_n \neq \mathscr{P}_m$,

$$\langle \Psi \,|\, \mathscr{H} \,|\, \Psi \rangle = \frac{1}{N!} \sum_{n=1}^{N!} (-1)^{p_n+p_n} (\varepsilon_i + \varepsilon_j + \cdots + \varepsilon_k) = \varepsilon_i + \varepsilon_j + \cdots + \varepsilon_k \tag{2.3.27}$$

Ex 2.16 Suppose

$$c = \langle K^{HP} \,|\, \mathscr{H} \,|\, L \rangle = \left\langle K^{HP} \,\middle|\, \mathscr{H} \,\middle|\, \sum_{m=1}^{N!} (-1)^{p_m} \mathscr{P}_m L^{HP} \right\rangle \tag{2.3.28}$$

thus

$$\langle K \,|\, \mathscr{H} \,|\, L \rangle = \sum_{n=1}^{N!} (-1)^{p_n} \left\langle \mathscr{P}_n K^{HP} \,\middle|\, \mathscr{H} \,\middle|\, \sum_{m=1}^{N!} (-1)^{p_m} \mathscr{P}_m L^{HP} \right\rangle \tag{2.3.29}$$

2.3.5 Transition from Spin Orbitals to Spatial Orbitals

Ex 2.17

$$\begin{aligned} |1\rangle &= |\psi_1 \alpha\rangle \quad & |2\rangle &= |\psi_1 \beta\rangle \\ |3\rangle &= |\psi_2 \alpha\rangle \quad & |4\rangle &= |\psi_2 \beta\rangle \end{aligned} \tag{2.3.30}$$

thus

$$\begin{aligned}\mathbf{H} &= \begin{pmatrix} \langle 1 \,|\, h \,|\, 1 \rangle + \langle 2 \,|\, h \,|\, 2 \rangle + \langle 12 \,|\, 12 \rangle - \langle 12 \,|\, 21 \rangle & \langle 12 \,|\, 34 \rangle - \langle 12 \,|\, 43 \rangle \\ \langle 34 \,|\, 12 \rangle - \langle 34 \,|\, 21 \rangle & \langle 3 \,|\, h \,|\, 3 \rangle + \langle 4 \,|\, h \,|\, 4 \rangle + \langle 34 \,|\, 34 \rangle - \langle 34 \,|\, 43 \rangle \end{pmatrix} \\ &= \begin{pmatrix} 2(1|h|1) + (11|11) & (12|12) \\ (21|21) & 2(2|h|2) + (22|22) \end{pmatrix}\end{aligned} \tag{2.3.31}$$

Ex 2.18

$$\begin{aligned}|\langle ab \| rs\rangle|^2 &= (\langle ab\,|\,rs\rangle - \langle ab\,|\,sr\rangle)^*(\langle ab\,|\,rs\rangle - \langle ab\,|\,sr\rangle)\\ &= \langle rs\,|\,ab\rangle\langle ab\,|\,rs\rangle - \langle rs\,|\,ab\rangle\langle ab\,|\,sr\rangle - \langle sr\,|\,ab\rangle\langle ab\,|\,rs\rangle + \langle sr\,|\,ab\rangle\langle ab\,|\,sr\rangle\\ &= [ra|sb][ar|bs] - [ra|sb][as|br] - [sa|rb][ar|bs] + [sa|rb][as|br]\\ &= [ar|bs]^2 - 2[ar|bs][as|br] + [as|br]^2\end{aligned} \tag{2.3.32}$$

Let's calculate $E_0^{(2)}$ term by term.

$$\begin{aligned}\left(E_0^{(2)}\right)_1 &= \frac{1}{4}\sum_{abrs}\frac{[ar|bs]^2}{\varepsilon_a+\varepsilon_b-\varepsilon_r-\varepsilon_s}\\ &= \frac{1}{4}\sum_{a,b}^{N/2}\sum_{r,s=N/2+1}^{K}\frac{[ar|bs]^2+[\bar{a}\bar{r}|bs]^2+[ar|\bar{b}\bar{s}]^2+[\bar{a}\bar{r}|\bar{b}\bar{s}]^2}{\varepsilon_a+\varepsilon_b-\varepsilon_r-\varepsilon_s}\\ &= \sum_{a,b}^{N/2}\sum_{r,s=N/2+1}^{K}\frac{[ar|bs]^2}{\varepsilon_a+\varepsilon_b-\varepsilon_r-\varepsilon_s}\\ &= \sum_{a,b}^{N/2}\sum_{r,s=N/2+1}^{K}\frac{\langle ab\,|\,rs\rangle\langle rs\,|\,ab\rangle}{\varepsilon_a+\varepsilon_b-\varepsilon_r-\varepsilon_s}\end{aligned} \tag{2.3.33}$$

$$\begin{aligned}\left(E_0^{(2)}\right)_2 &= \frac{1}{4}\sum_{abrs}\frac{-2[ar|bs][as|br]}{\varepsilon_a+\varepsilon_b-\varepsilon_r-\varepsilon_s}\\ &= -\frac{1}{2}\sum_{a,b}^{N/2}\sum_{r,s=N/2+1}^{K}\frac{[ar|bs][as|br]+[\bar{a}\bar{r}|\bar{b}\bar{s}][\bar{a}\bar{s}|\bar{b}\bar{r}]}{\varepsilon_a+\varepsilon_b-\varepsilon_r-\varepsilon_s}\\ &= -\sum_{a,b}^{N/2}\sum_{r,s=N/2+1}^{K}\frac{[ar|bs][as|br]}{\varepsilon_a+\varepsilon_b-\varepsilon_r-\varepsilon_s}\\ &= -\sum_{a,b}^{N/2}\sum_{r,s=N/2+1}^{K}\frac{\langle ab\,|\,rs\rangle\langle rs\,|\,ba\rangle}{\varepsilon_a+\varepsilon_b-\varepsilon_r-\varepsilon_s}\end{aligned} \tag{2.3.34}$$

$$\begin{aligned}\left(E_0^{(2)}\right)_3 &= \frac{1}{4}\sum_{abrs}\frac{[as|br]^2}{\varepsilon_a+\varepsilon_b-\varepsilon_r-\varepsilon_s} = \frac{1}{4}\sum_{absr}\frac{[ar|bs]^2}{\varepsilon_a+\varepsilon_b-\varepsilon_s-\varepsilon_r}\\ &= \sum_{a,b}^{N/2}\sum_{r,s=N/2+1}^{K}\frac{\langle ab\,|\,rs\rangle\langle rs\,|\,ab\rangle}{\varepsilon_a+\varepsilon_b-\varepsilon_r-\varepsilon_s}\end{aligned} \tag{2.3.35}$$

thus,

$$E_0^{(2)} = \sum_{a,b}^{N/2}\sum_{r,s=N/2+1}^{K}\frac{\langle ab\,|\,rs\rangle\left(2\langle rs\,|\,ab\rangle - \langle rs\,|\,ba\rangle\right)}{\varepsilon_a+\varepsilon_b-\varepsilon_r-\varepsilon_s} \tag{2.3.36}$$

2.3.6 Coulomb and Exchange Integrals

Ex 2.19

$$J_{ii} = (ii|ii) = K_{ii} \tag{2.3.37}$$

$$J_{ij}^* = \langle ij\,|\,ij\rangle^* = \langle ij\,|\,ij\rangle = J_{ij} \tag{2.3.38}$$

$$K_{ij}^* = \langle ij\,|\,ji\rangle^* = \langle ji\,|\,ij\rangle = \langle ij\,|\,ji\rangle = K_{ij} \tag{2.3.39}$$

$$J_{ij} = (ii|jj) = (jj|ii) = J_{ji} \tag{2.3.40}$$

$$K_{ij} = (ij|ji) = (ji|ij) = K_{ji} \tag{2.3.41}$$

Ex 2.20 For real spatial orbitals

$$K_{ij} = (ij|ji) = (ij|ij) = (ji|ji) \tag{2.3.42}$$

$$K_{ij} = \langle ij\,|\,ji\rangle = \langle ii\,|\,jj\rangle = \langle jj\,|\,ii\rangle \tag{2.3.43}$$

Ex 2.21

$$\mathbf{H} = \begin{pmatrix} 2(1|h|1) + (11|11) & (12|12) \\ (21|21) & 2(2|h|2) + (22|22) \end{pmatrix} = \begin{pmatrix} 2h_{11} + J_{11} & K_{12} \\ K_{12} & 2h_{22} + J_{22} \end{pmatrix} \tag{2.3.44}$$

Ex 2.22

$$E^{HP}_{\uparrow\downarrow} = \left\langle \Psi^{HP}_{\uparrow\downarrow} \middle| h(1) + h(2) + \frac{1}{r_{12}} \middle| \Psi^{HP}_{\uparrow\downarrow} \right\rangle = (1|h|1) + (2|h|2) + (11|22) = h_{11} + h_{22} + J_{12} \tag{2.3.45}$$

$$E^{HP}_{\downarrow\downarrow} = \left\langle \Psi^{HP}_{\downarrow\downarrow} \middle| h(1) + h(2) + \frac{1}{r_{12}} \middle| \Psi^{HP}_{\downarrow\downarrow} \right\rangle = (1|h|1) + (2|h|2) + (11|22) = h_{11} + h_{22} + J_{12} \tag{2.3.46}$$

2.3.7 Pseudo-Classical Interpretation of Determinantal Energies

Ex 2.23 a.-g. can be obtained immediately with definition.

2.4 Second Quantization

2.4.1 Creation and Annihilation Operators and Their Anticommutation Relations

Ex 2.24 Since $a_i^\dagger a_j^\dagger + a_j^\dagger a_i^\dagger = 0$, we have

$$(a_1^\dagger a_2^\dagger + a_2^\dagger a_1^\dagger)\,|K\rangle = 0 \tag{2.4.1}$$

for any $|K\rangle$.

Ex 2.25 Since $a_i a_j^\dagger + a_j^\dagger a_i = \delta_{ij}$, we have

$$(a_1 a_2^\dagger + a_2^\dagger a_1)\,|K\rangle = 0 \tag{2.4.2}$$

$$(a_1 a_1^\dagger + a_1^\dagger a_1)\,|K\rangle = |K\rangle \tag{2.4.3}$$

for any $|K\rangle$.

Ex 2.26

$$\langle \chi_i\,|\,\chi_j\rangle = \left\langle 0 \middle| a_i a_j^\dagger \middle| 0 \right\rangle = \left\langle 0 \middle| \delta_{ij} - a_j^\dagger a_i \middle| 0 \right\rangle = \delta_{ij} \tag{2.4.4}$$

where $|0\rangle$ is the vacuum state.

Ex 2.27 First, if $i \notin \{1, 2, \cdots, N\}$ or $j \notin \{1, 2, \cdots, N\}$,

$$\left\langle K \middle| a_i^\dagger a_j \middle| K \right\rangle = 0 \tag{2.4.5}$$

because inexistent electron cannot be annihilated.
Thus, $i, j \in \{1, 2, \cdots, N\}$, and

$$\left\langle K \middle| a_i^\dagger a_j \middle| K \right\rangle = \delta_{ij}\,\langle K\,|\,K\rangle - \left\langle K \middle| a_j a_i^\dagger \middle| K \right\rangle \tag{2.4.6}$$

$\left\langle K \middle| a_j a_i^\dagger \middle| K \right\rangle$ would be 0 because χ_i is created twice. Thus,

$$\left\langle K \middle| a_i^\dagger a_j \middle| K \right\rangle = \delta_{ij} \tag{2.4.7}$$

Overall, $\left\langle K \middle| a_i^\dagger a_j \middle| K \right\rangle = 1$ when $i = j$ and $i \in \{1, 2, \cdots, N\}$, but is 0 otherwise.

Ex 2.28

a. That's obvious since inexistent electron cannot be annihilated.

b. That's obvious since an electron cannot be created twice.

c.

$$\begin{aligned} a_r^\dagger a_a \left|\Psi_0\right\rangle &= a_r^\dagger a_a(-\left|\chi_a \cdots \chi_1\chi_b \cdots \chi_N\right\rangle) \\ &= -a_r^\dagger \left|\cdots \chi_1\chi_b \cdots \chi_N\right\rangle \\ &= -\left|\chi_r \cdots \chi_1\chi_b \cdots \chi_N\right\rangle \\ &= \left|\chi_1 \cdots \chi_r\chi_b \cdots \chi_N\right\rangle \\ &= \left|\Psi_a^r\right\rangle \end{aligned} \tag{2.4.8}$$

d. That's similar to 2.28.c.

e.

$$\begin{aligned} a_s^\dagger a_b a_r^\dagger a_a \left|\Psi_0\right\rangle &= a_s^\dagger a_b a_r^\dagger(-\left|\chi_2 \cdots \chi_1\chi_b \cdots \chi_N\right\rangle) \\ &= -a_s^\dagger a_b \left|\chi_r\chi_2 \cdots \chi_1\chi_b \cdots \chi_N\right\rangle \\ &= -a_s^\dagger(-\left|\chi_2 \cdots \chi_1\chi_r \cdots \chi_N\right\rangle) \\ &= \left|\chi_s\chi_2 \cdots \chi_1\chi_r \cdots \chi_N\right\rangle \\ &= \left|\chi_1 \cdots \chi_r\chi_s \cdots \chi_N\right\rangle \\ &= \left|\Psi_{ab}^{rs}\right\rangle \end{aligned} \tag{2.4.9}$$

$\therefore$

$$\left|\Psi_{ab}^{rs}\right\rangle = a_s^\dagger a_b a_r^\dagger a_a \left|\Psi_0\right\rangle = a_s^\dagger(-a_r^\dagger a_b)a_a \left|\Psi_0\right\rangle = a_r^\dagger a_s^\dagger a_b a_a \left|\Psi_0\right\rangle \tag{2.4.10}$$

f. That's similar to 2.28.e.

2.4.2 Second-Quantized Operators and Their Matrix Elements

Ex 2.29

$$\begin{aligned} \left\langle \Psi_0 \,|\, \mathcal{O}_1 \,|\, \Psi_0 \right\rangle &= \sum_{ij} \left\langle i \,|\, h \,|\, j \right\rangle \left\langle 0\right| a_2 a_1 a_i^\dagger a_j a_1^\dagger a_2^\dagger \left|0\right\rangle \\ &= \sum_{ij} \left\langle i \,|\, h \,|\, j \right\rangle \left\langle 0\right| a_2 a_1 (\delta_{ij} - a_j^\dagger a_i) a_1^\dagger a_2^\dagger \left|0\right\rangle \\ &= \sum_{i} \left\langle i \,|\, h \,|\, i \right\rangle \left\langle 0\right| a_2 a_1 a_1^\dagger a_2^\dagger \left|0\right\rangle - \sum_{ij} \left\langle i \,|\, h \,|\, j \right\rangle \left\langle 0\right| a_2 a_1 a_j a_i^\dagger a_1^\dagger a_2^\dagger \left|0\right\rangle \end{aligned} \tag{2.4.11}$$

The second terms must be 0 since $i \in 1, 2$.
Thus,

$$\left\langle \Psi_0 \,|\, \mathcal{O}_1 \,|\, \Psi_0 \right\rangle = \sum_{i} \left\langle i \,|\, h \,|\, i \right\rangle \left\langle 0\right| a_2 a_1 a_1^\dagger a_2^\dagger \left|0\right\rangle = \left\langle 1 \,|\, h \,|\, 1 \right\rangle + \left\langle 2 \,|\, h \,|\, 2 \right\rangle \tag{2.4.12}$$

Ex 2.30

$$\begin{aligned} \left\langle \Psi_a^r \,|\, \mathcal{O}_1 \,|\, \Psi_0 \right\rangle &= \sum_{ij} \left\langle i \,|\, h \,|\, j \right\rangle \left\langle \Psi_0 \,\middle|\, a_a^\dagger a_r a_i^\dagger a_j \,\middle|\, \Psi_0 \right\rangle = \sum_{ij} \left\langle i \,|\, h \,|\, j \right\rangle \left\langle \Psi_0 \,\middle|\, a_a^\dagger(\delta_{ri} - a_i^\dagger a_r) a_j \,\middle|\, \Psi_0 \right\rangle \\ &= \sum_{j} \left\langle r \,|\, h \,|\, j \right\rangle \left\langle \Psi_0 \,|\, a_a^\dagger a_j \,|\, \Psi_0 \right\rangle - \sum_{ij} \left\langle i \,|\, h \,|\, j \right\rangle \left\langle \Psi_0 \,\middle|\, a_a^\dagger a_i^\dagger a_r a_j \,\middle|\, \Psi_0 \right\rangle \\ &= \sum_{j} \left\langle r \,|\, h \,|\, j \right\rangle \left\langle \Psi_0 \,|\, (\delta_{aj} - a_j a_a^\dagger) \,|\, \Psi_0 \right\rangle \\ &= \left\langle r \,|\, h \,|\, a \right\rangle \left\langle \Psi_0 \,|\, \Psi_0 \right\rangle - \sum_{j} \left\langle r \,|\, h \,|\, j \right\rangle \left\langle \Psi_0 \,|\, a_j a_a^\dagger \,|\, \Psi_0 \right\rangle \\ &= \left\langle r \,|\, h \,|\, a \right\rangle \end{aligned} \tag{2.4.13}$$

Ex 2.31

$$\langle \Psi_a^r \,|\, \mathcal{O}_2 \,|\, \Psi_0 \rangle = \frac{1}{2} \sum_{ijkl} \langle ij \,|\, kl \rangle \left\langle \Psi_0 \,\middle|\, a_a^\dagger a_r a_i^\dagger a_j^\dagger a_l a_k \,\middle|\, \Psi_0 \right\rangle \tag{2.4.14}$$

while

$$\begin{aligned}
\left\langle \Psi_0 \,\middle|\, a_a^\dagger a_r a_i^\dagger a_j^\dagger a_l a_k \,\middle|\, \Psi_0 \right\rangle &= \left\langle \Psi_0 \,\middle|\, a_a^\dagger \delta_{ri} a_j^\dagger a_l a_k \,\middle|\, \Psi_0 \right\rangle - \left\langle \Psi_0 \,\middle|\, a_a^\dagger a_i^\dagger a_r a_j^\dagger a_l a_k \,\middle|\, \Psi_0 \right\rangle \\
&= \delta_{ri} \Big(\left\langle \Psi_0 \,\middle|\, a_j^\dagger \delta_{ak} a_l \,\middle|\, \Psi_0 \right\rangle - \left\langle \Psi_0 \,\middle|\, a_j^\dagger a_k a_a^\dagger a_l \,\middle|\, \Psi_0 \right\rangle \Big) \\
&\quad - \Big(\left\langle \Psi_0 \,\middle|\, a_a^\dagger a_i^\dagger \delta_{rj} a_l a_k \,\middle|\, \Psi_0 \right\rangle - \left\langle \Psi_0 \,\middle|\, a_a^\dagger a_i^\dagger a_j^\dagger a_r a_l a_k \,\middle|\, \Psi_0 \right\rangle \Big) \\
&= \delta_{ri} \delta_{ak} \left\langle \Psi_0 \,\middle|\, a_j^\dagger a_l \,\middle|\, \Psi_0 \right\rangle - \delta_{ri} \delta_{al} \left\langle \Psi_0 \,\middle|\, a_j^\dagger a_k \,\middle|\, \Psi_0 \right\rangle \\
&\quad - \delta_{rj} \Big(\left\langle \Psi_0 \,\middle|\, a_i^\dagger \delta_{ak} a_l \,\middle|\, \Psi_0 \right\rangle - \left\langle \Psi_0 \,\middle|\, a_i^\dagger a_k a_a^\dagger a_l \,\middle|\, \Psi_0 \right\rangle \Big) + 0 \\
&= \delta_{ri} \delta_{ak} \left\langle \Psi_0 \,\middle|\, a_j^\dagger a_l \,\middle|\, \Psi_0 \right\rangle - \delta_{ri} \delta_{al} \left\langle \Psi_0 \,\middle|\, a_j^\dagger a_k \,\middle|\, \Psi_0 \right\rangle \\
&\quad - \delta_{rj} \delta_{ak} \left\langle \Psi_0 \,\middle|\, a_i^\dagger a_l \,\middle|\, \Psi_0 \right\rangle + \delta_{rj} \delta_{al} \left\langle \Psi_0 \,\middle|\, a_i^\dagger a_k \,\middle|\, \Psi_0 \right\rangle
\end{aligned} \tag{2.4.15}$$

According to Ex. 2.27, we have

$$\begin{aligned}
\langle \Psi_a^r \,|\, \mathcal{O}_2 \,|\, \Psi_0 \rangle &= \frac{1}{2} \Bigg(\sum_{jl} \langle rj \,|\, al \rangle \left\langle \Psi_0 \,\middle|\, a_j^\dagger a_l \,\middle|\, \Psi_0 \right\rangle - \sum_{jk} \langle rj \,|\, ka \rangle \left\langle \Psi_0 \,\middle|\, a_j^\dagger a_k \,\middle|\, \Psi_0 \right\rangle \\
&\qquad - \sum_{il} \langle ir \,|\, al \rangle \left\langle \Psi_0 \,\middle|\, a_i^\dagger a_l \,\middle|\, \Psi_0 \right\rangle + \sum_{ik} \langle ir \,|\, ka \rangle \left\langle \Psi_0 \,\middle|\, a_i^\dagger a_k \,\middle|\, \Psi_0 \right\rangle \Bigg) \\
&= \frac{1}{2} \left(\sum_j^N \langle rj \,|\, aj \rangle - \sum_j^N \langle rj \,|\, ja \rangle - \sum_i^N \langle ir \,|\, ai \rangle + \sum_i^N \langle ir \,|\, ia \rangle \right) \\
&= \sum_j^N \langle rj \,|\, aj \rangle - \sum_j^N \langle rj \,|\, ja \rangle \\
&= \sum_j^N \langle rj \,\|\, aj \rangle
\end{aligned} \tag{2.4.16}$$

2.5 Spin-Adapted Configurations

2.5.1 Spin Operators

Ex 2.32

a)

$$\widehat{\mathbf{s}}_+ \,|\alpha\rangle = (\widehat{\mathbf{s}}_x + \mathrm{i}\,\widehat{\mathbf{s}}_y)\,|\alpha\rangle = \left(\frac{1}{2} + \mathrm{i}\,\frac{\mathrm{i}}{2} \right) |\beta\rangle = 0 \tag{2.5.1}$$

$$\widehat{\mathbf{s}}_+ \,|\beta\rangle = (\widehat{\mathbf{s}}_x + \mathrm{i}\,\widehat{\mathbf{s}}_y)\,|\beta\rangle = \left(\frac{1}{2} - \mathrm{i}\,\frac{\mathrm{i}}{2} \right) |\alpha\rangle = |\alpha\rangle \tag{2.5.2}$$

$$\widehat{\mathbf{s}}_- \,|\alpha\rangle = (\widehat{\mathbf{s}}_x - \mathrm{i}\,\widehat{\mathbf{s}}_y)\,|\alpha\rangle = \left(\frac{1}{2} - \mathrm{i}\,\frac{\mathrm{i}}{2} \right) |\beta\rangle = |\beta\rangle \tag{2.5.3}$$

$$\widehat{\mathbf{s}}_- \,|\beta\rangle = (\widehat{\mathbf{s}}_x - \mathrm{i}\,\widehat{\mathbf{s}}_y)\,|\beta\rangle = \left(\frac{1}{2} + \mathrm{i}\,\frac{\mathrm{i}}{2} \right) |\alpha\rangle = 0 \tag{2.5.4}$$

b)

$$\widehat{\mathbf{s}}_+ \,\widehat{\mathbf{s}}_- = (\widehat{\mathbf{s}}_x + \mathrm{i}\,\widehat{\mathbf{s}}_y)(\widehat{\mathbf{s}}_x - \mathrm{i}\,\widehat{\mathbf{s}}_y) = \widehat{\mathbf{s}}_x^2 + \widehat{\mathbf{s}}_y^2 + \mathrm{i}(\widehat{\mathbf{s}}_y\,\widehat{\mathbf{s}}_x - \widehat{\mathbf{s}}_x\,\widehat{\mathbf{s}}_y) = \widehat{\mathbf{s}}_x^2 + \widehat{\mathbf{s}}_y^2 + \widehat{\mathbf{s}}_z \tag{2.5.5}$$

$$\widehat{\mathbf{s}}_-\,\widehat{\mathbf{s}}_+ = (\widehat{\mathbf{s}}_x - \mathrm{i}\,\widehat{\mathbf{s}}_y)(\widehat{\mathbf{s}}_x + \mathrm{i}\,\widehat{\mathbf{s}}_y) = \widehat{\mathbf{s}}_x^2 + \widehat{\mathbf{s}}_y^2 + \mathrm{i}(\widehat{\mathbf{s}}_x\,\widehat{\mathbf{s}}_y - \widehat{\mathbf{s}}_y\,\widehat{\mathbf{s}}_x) = \widehat{\mathbf{s}}_x^2 + \widehat{\mathbf{s}}_y^2 - \widehat{\mathbf{s}}_z \tag{2.5.6}$$

thus,

$$\widehat{\mathbf{s}}^2 = \widehat{\mathbf{s}}_x^2 + \widehat{\mathbf{s}}_y^2 + \widehat{\mathbf{s}}_z^2 = \widehat{\mathbf{s}}_+\,\widehat{\mathbf{s}}_- - \widehat{\mathbf{s}}_z + \widehat{\mathbf{s}}_z^2 \tag{2.5.7}$$

$$= \widehat{\mathbf{s}}_-\,\widehat{\mathbf{s}}_+ + \widehat{\mathbf{s}}_z + \widehat{\mathbf{s}}_z^2 \tag{2.5.8}$$

Ex 2.33

$$\widehat{\mathbf{s}}^2 = \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{3}{4} \end{pmatrix} \quad \widehat{\mathbf{s}}_z = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \quad \widehat{\mathbf{s}}_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \widehat{\mathbf{s}}_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{2.5.9}$$

thus

$$\widehat{\mathbf{s}}_+\,\widehat{\mathbf{s}}_- - \widehat{\mathbf{s}}_z + \widehat{\mathbf{s}}_z^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{3}{4} \end{pmatrix} = \widehat{\mathbf{s}}^2 \tag{2.5.10}$$

$$\widehat{\mathbf{s}}_-\,\widehat{\mathbf{s}}_+ + \widehat{\mathbf{s}}_z + \widehat{\mathbf{s}}_z^2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{3}{4} \end{pmatrix} = \widehat{\mathbf{s}}^2 \tag{2.5.11}$$

Ex 2.34

$$\begin{aligned}
[\widehat{\mathbf{s}}^2, \widehat{\mathbf{s}}_z] &= [\widehat{\mathbf{s}}_+\,\widehat{\mathbf{s}}_- - \widehat{\mathbf{s}}_z + \widehat{\mathbf{s}}_z^2, \widehat{\mathbf{s}}_z] \\
&= \widehat{\mathbf{s}}_+[\widehat{\mathbf{s}}_-, \widehat{\mathbf{s}}_z] + [\widehat{\mathbf{s}}_+, \widehat{\mathbf{s}}_z]\,\widehat{\mathbf{s}}_- - 0 + 0 \\
&= \widehat{\mathbf{s}}_+[\widehat{\mathbf{s}}_x - \mathrm{i}\,\widehat{\mathbf{s}}_y, \widehat{\mathbf{s}}_z] + [\widehat{\mathbf{s}}_x + \mathrm{i}\,\widehat{\mathbf{s}}_y, \widehat{\mathbf{s}}_z]\,\widehat{\mathbf{s}}_- \\
&= \widehat{\mathbf{s}}_+(-\,\mathrm{i}\,\widehat{\mathbf{s}}_y - \mathrm{i}\cdot\mathrm{i}\,\widehat{\mathbf{s}}_x) + (-\,\mathrm{i}\,\widehat{\mathbf{s}}_y + \mathrm{i}\cdot\mathrm{i}\,\widehat{\mathbf{s}}_x)\,\widehat{\mathbf{s}}_- \\
&= \widehat{\mathbf{s}}_+\,\widehat{\mathbf{s}}_- - \widehat{\mathbf{s}}_+\,\widehat{\mathbf{s}}_- \\
&= 0
\end{aligned} \tag{2.5.12}$$

Ex 2.35

$$\mathscr{H}\mathscr{A}\,|\Phi\rangle = \mathscr{A}\mathscr{H}\,|\Phi\rangle = \mathscr{A}E\,|\Phi\rangle = E\mathscr{A}\,|\Phi\rangle \tag{2.5.13}$$

thus $\mathscr{A}\,|\Phi\rangle$ is also an eigenfunction of $\mathscr{H}$ with eigenvalue E.

Ex 2.36

$$\langle \Psi_1 \,|\, \mathscr{H}\mathscr{A} \,|\, \Psi_2 \rangle = a_2 \langle \Psi_1 \,|\, \mathscr{H} \,|\, \Psi_2 \rangle \tag{2.5.14}$$

Since $[\mathscr{A}, \mathscr{H}] = 0$ and $\mathscr{A}$ is Hermitian,

$$\langle \Psi_1 \,|\, \mathscr{H}\mathscr{A} \,|\, \Psi_2 \rangle = \langle \Psi_1 \,|\, \mathscr{A}\mathscr{H} \,|\, \Psi_2 \rangle = \left\langle \Psi_1 \,\middle|\, \mathscr{A}^\dagger\mathscr{H} \,\middle|\, \Psi_2 \right\rangle = a_1 \langle \Psi_1 \,|\, \mathscr{H} \,|\, \Psi_2 \rangle \tag{2.5.15}$$

thus

$$(a_1 - a_2) \langle \Psi_1 \,|\, \mathscr{H} \,|\, \Psi_2 \rangle = 0 \tag{2.5.16}$$

Since $a_1 \neq a_2$,

$$\langle \Psi_1 \,|\, \mathscr{H} \,|\, \Psi_2 \rangle = 0 \tag{2.5.17}$$

Ex 2.37

$$\begin{aligned}
\hat{\mathscr{S}}_z\,|\chi_i\chi_j\cdots\chi_k\rangle &= \hat{\mathscr{S}}_z \frac{1}{\sqrt{N!}} \sum_{n=1}^{N!} (-1)^{p_n}\,\hat{\mathscr{P}}_n\{\chi_i(1)\chi_j(2)\cdots\chi_k(N)\} \\
&= \frac{1}{\sqrt{N!}} \sum_{n=1}^{N!} (-1)^{p_n}\,\hat{\mathscr{P}}_n\{\hat{\mathscr{S}}_z\chi_i(1)\chi_j(2)\cdots\chi_k(N)\} \\
&= \frac{1}{\sqrt{N!}} \sum_{n=1}^{N!} (-1)^{p_n}\,\hat{\mathscr{P}}_n\left\{\sum_{i=1}^{N} \widehat{\mathbf{s}}_z(i)\chi_i(1)\chi_j(2)\cdots\chi_k(N)\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{N!}} \sum_{n=1}^{N!} (-1)^{p_n} \hat{\mathscr{P}}_n \left\{ \left(\frac{1}{2} N^\alpha - \frac{1}{2} N^\beta \right) \chi_i(1) \chi_j(2) \cdots \chi_k(N) \right\} \\
&= \frac{1}{2} (N^\alpha - N^\beta) \left| \chi_i \chi_j \cdots \chi_k \right\rangle
\end{aligned} \tag{2.5.18}
$$

2.5.2 Restricted Determinants and Spin-Adapted Configurations

Ex 2.38 From Ex 2.37, we have

$$\hat{\mathscr{S}}_z \left| \psi_i \bar{\psi}_i \psi_j \bar{\psi}_j \cdots \right\rangle = 0 \tag{2.5.19}$$

thus

$$\hat{\mathscr{S}}_z^2 \left| \psi_i \bar{\psi}_i \psi_j \bar{\psi}_j \cdots \right\rangle = 0 \tag{2.5.20}$$

While

$$
\begin{aligned}
\hat{\mathscr{S}}_+ \left| \psi_i \bar{\psi}_i \cdots \psi_k \bar{\psi}_k \cdots \right\rangle &= \hat{\mathscr{S}}_+ \frac{1}{\sqrt{N!}} \sum_{n=1}^{N!} (-1)^{p_n} \hat{\mathscr{P}}_n \{ \psi_i \bar{\psi}_i \cdots \psi_k \bar{\psi}_k \cdots \} \\
&= \frac{1}{\sqrt{N!}} \sum_{n=1}^{N!} (-1)^{p_n} \hat{\mathscr{P}}_n \{ \hat{\mathscr{S}}_+ \psi_i \bar{\psi}_i \cdots \psi_k \bar{\psi}_k \cdots \} \\
&= \frac{1}{\sqrt{N!}} \sum_{n=1}^{N!} (-1)^{p_n} \hat{\mathscr{P}}_n \left\{ \sum_a^N \widehat{\mathbf{s}}_+(a) \psi_i \bar{\psi}_i \cdots \psi_k \bar{\psi}_k \cdots \right\} \\
&= \sum_a^N \frac{1}{\sqrt{N!}} \sum_{n=1}^{N!} (-1)^{p_n} \hat{\mathscr{P}}_n \{ \widehat{\mathbf{s}}_+(a) \psi_i \bar{\psi}_i \cdots \psi_k \bar{\psi}_k \cdots \}
\end{aligned} \tag{2.5.21}
$$

Since

$$\widehat{\mathbf{s}}_+(a) \psi_k(a) = 0 \quad \widehat{\mathbf{s}}_+(a) \bar{\psi}_k(a) = \psi_k(a) \tag{2.5.22}$$

$$\hat{\mathscr{S}}_+ \left| \psi_i \bar{\psi}_i \cdots \psi_k \bar{\psi}_k \cdots \right\rangle = \sum_a^N 0 = 0 \tag{2.5.23}$$

thus

$$\hat{\mathscr{S}}_- \hat{\mathscr{S}}_+ \left| \psi_i \bar{\psi}_i \psi_j \bar{\psi}_j \cdots \right\rangle = 0 \tag{2.5.24}$$

Therefore,

$$\hat{\mathscr{S}}^2 \left| \psi_i \bar{\psi}_i \psi_j \bar{\psi}_j \cdots \right\rangle = (\hat{\mathscr{S}}_- \hat{\mathscr{S}}_+ + \hat{\mathscr{S}}_z + \hat{\mathscr{S}}_z^2) \left| \psi_i \bar{\psi}_i \psi_j \bar{\psi}_j \cdots \right\rangle = 0 \tag{2.5.25}$$