

Modern Quantum Chemistry, Szabo & Ostlund

HW

WSF

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1 Mathematical Review

1.1 Linear Algebra

1.1.1 3-D Vector Algebra

Ex 1.1

a)

$$\mathcal{O}\mathbf{e}_j = \sum_{i=1}^3 \mathbf{e}_i O_{ij} \quad (1.1.1)$$

$$\mathbf{e}_i \cdot \mathcal{O}\mathbf{e}_j = \mathbf{e}_i \cdot \sum_{i=1}^3 \mathbf{e}_i O_{ij} = O_{ij} \quad (1.1.2)$$

b)

$$\begin{aligned} \mathbf{b} = \mathcal{O}\mathbf{a} &= \sum_{i=1}^3 a_i \sum_{j=1}^3 \mathbf{e}_j O_{ji} \\ &= \sum_{j=1}^3 a_j \sum_{i=1}^3 \mathbf{e}_i O_{ij} = \sum_{i=1}^3 \mathbf{e}_i \sum_{j=1}^3 a_j O_{ij} \end{aligned} \quad (1.1.3)$$

thus

$$\mathbf{b}_i = \sum_{j=1}^3 a_j O_{ij} \quad (1.1.4)$$

Ex 1.2

$$[\mathbf{A}, \mathbf{B}] = \begin{bmatrix} 0 & -2 & 4 \\ 2 & 0 & 3 \\ -4 & -3 & 0 \end{bmatrix} \quad (1.1.5)$$

$$\{\mathbf{A}, \mathbf{B}\} = \begin{bmatrix} 0 & 0 & -2 \\ 0 & -2 & 3 \\ -2 & 3 & -2 \end{bmatrix} \quad (1.1.6)$$

1.1.2 Matrices

Ex 1.3

$$(AB)_{nk} = \sum_m^M A_{nm} B_{mk} \quad (1.1.7)$$

$$(AB)_{kn}^\dagger = (AB)_{nk}^* = \sum_m^M A_{nm}^* B_{mk}^* = \sum_m^M B_{km}^\dagger A_{mn}^\dagger = (B^\dagger A^\dagger)_{kn} \quad (1.1.8)$$

thus

$$(\mathbf{AB})^\dagger = \mathbf{B}^\dagger \mathbf{A}^\dagger \quad (1.1.9)$$

Ex 1.4

a. suppose \mathbf{A} is $N \times M$ and \mathbf{B} is $M \times N$

$$\text{tr } \mathbf{AB} = \sum_n^N (AB)_{nn} = \sum_n^N \sum_m^M A_{nm} B_{mn} = \sum_m^M \sum_n^N B_{mn} A_{nm} = \sum_m^M (BA)_{mm} = \text{tr } \mathbf{BA} \quad (1.1.10)$$

b.

$$\mathbf{AB}(\mathbf{AB})^{-1} = \mathbf{1} \quad (1.1.11)$$

$$\mathbf{B}^{-1}\mathbf{A}^{-1}\mathbf{AB}(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}\mathbf{1} \quad (1.1.12)$$

$$\mathbf{B}^{-1}(\mathbf{A}^{-1}\mathbf{A})\mathbf{B}(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} \quad (1.1.13)$$

$$\mathbf{B}^{-1}\mathbf{1B}(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} \quad (1.1.14)$$

thus

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} \quad (1.1.15)$$

c.

$$\mathbf{B} = \mathbf{U}^\dagger \mathbf{A} \mathbf{U} \quad (1.1.16)$$

huhhj

$$\mathbf{UBU}^\dagger = \mathbf{UU}^\dagger \mathbf{AUU}^\dagger = \mathbf{1A1} = \mathbf{A} \quad (1.1.17)$$

d. $\because \mathbf{C}$ is Hermitian, \therefore

$$\mathbf{C} = \mathbf{C}^\dagger \quad (1.1.18)$$

$$\mathbf{AB} = (\mathbf{AB})^\dagger = \mathbf{B}^\dagger \mathbf{A}^\dagger \quad (1.1.19)$$

Since \mathbf{A}, \mathbf{B} are Hermitian,

$$\mathbf{AB} = \mathbf{B}^\dagger \mathbf{A}^\dagger = \mathbf{BA} \quad (1.1.20)$$

\therefore

$$[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA} = 0 \quad (1.1.21)$$

i.e. \mathbf{A}, \mathbf{B} commute

e. Since \mathbf{A} is Hermitian,

$$\mathbf{A} = \mathbf{A}^\dagger \quad (1.1.22)$$

thus

$$(\mathbf{A}^{1-})^\dagger \mathbf{A} = (\mathbf{A}^{1-})^\dagger \mathbf{A}^\dagger = (\mathbf{AA}^{-1})^\dagger = \mathbf{1}^\dagger = \mathbf{1} \quad (1.1.23)$$

thus

$$(\mathbf{A}^{1-})^\dagger \mathbf{AA}^{-1} = \mathbf{A}^{-1} \quad (1.1.24)$$

$$(\mathbf{A}^{1-})^\dagger = \mathbf{A}^{-1} \quad (1.1.25)$$

i.e. \mathbf{A}^{-1} , if it exists, is Hermitian.

f. Suppose

$$\mathbf{A}^{-1} = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \quad (1.1.26)$$

thus

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (1.1.27)$$

the solution is

$$\begin{aligned} x &= \frac{A_{22}}{A_{11}A_{22} - A_{12}A_{21}} \\ y &= \frac{-A_{12}}{A_{11}A_{22} - A_{12}A_{21}} \\ z &= \frac{-A_{21}}{A_{11}A_{22} - A_{12}A_{21}} \\ w &= \frac{A_{11}}{A_{11}A_{22} - A_{12}A_{21}} \end{aligned} \quad (1.1.28)$$

thus

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix} \quad (1.1.29)$$

1.1.3 Determinants

Ex 1.5 Suppose

$$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad (1.1.30)$$

1.

$$\begin{vmatrix} 0 & 0 \\ A_{21} & A_{22} \end{vmatrix} = 0 \cdot A_{22} - 0 \cdot A_{21} = 0 \quad (1.1.31)$$

$$\begin{vmatrix} 0 & A_{12} \\ 0 & A_{22} \end{vmatrix} = 0 \cdot A_{22} - 0 \cdot A_{12} = 0 \quad (1.1.32)$$

2.

$$\det(\mathbf{A}) = A_{11}A_{22} - 0 \cdot 0 = A_{11}A_{22} \quad (1.1.33)$$

3.

$$\det(\mathbf{A}) = A_{11}A_{22} - A_{12}A_{21} \quad (1.1.34)$$

$$\begin{vmatrix} A_{21} & A_{22} \\ A_{11} & A_{12} \end{vmatrix} = A_{21}A_{12} - A_{22}A_{11} = -\det(\mathbf{A}) \quad (1.1.35)$$

4.

$$\det(\mathbf{A}^\dagger)^* = \begin{vmatrix} A_{11}^* & A_{21}^* \\ A_{12}^* & A_{22}^* \end{vmatrix}^* = (A_{11}^*A_{22}^* - A_{21}^*A_{12}^*)^* = A_{11}A_{22} - A_{12}A_{21} = \det(\mathbf{A}) \quad (1.1.36)$$

5. Suppose $\mathbf{B} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$

$$\begin{aligned} \det(\mathbf{AB}) &= \begin{vmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{vmatrix} \\ &= (A_{11}B_{11} + A_{12}B_{21})(A_{21}B_{12} + A_{22}B_{22}) - (A_{11}B_{12} + A_{12}B_{22})(A_{21}B_{11} + A_{22}B_{21}) \\ &= A_{11}B_{11}A_{21}B_{12} + A_{11}B_{11}A_{22}B_{22} + A_{12}B_{21}A_{21}B_{12} + A_{12}B_{21}A_{22}B_{22} \\ &\quad - (A_{11}B_{12}A_{21}B_{11} + A_{11}B_{12}A_{22}B_{21} + A_{12}B_{22}A_{21}B_{11} + A_{12}B_{22}A_{22}B_{21}) \\ &= A_{11}B_{11}A_{22}B_{22} + A_{12}B_{21}A_{21}B_{12} - A_{11}B_{12}A_{22}B_{21} - A_{12}B_{22}A_{21}B_{11} \end{aligned} \quad (1.1.37)$$

$$\begin{aligned} \det(\mathbf{A})\det(\mathbf{B}) &= (A_{11}A_{22} - A_{12}A_{21})(B_{11}B_{22} - B_{12}B_{21}) \\ &= A_{11}A_{22}B_{11}B_{22} - A_{11}A_{22}B_{12}B_{21} - A_{12}A_{21}B_{11}B_{22} + A_{12}A_{21}B_{12}B_{21} \\ &= A_{11}B_{11}A_{22}B_{22} + A_{12}B_{21}A_{21}B_{12} - A_{11}B_{12}A_{22}B_{21} - A_{12}B_{22}A_{21}B_{11} \end{aligned} \quad (1.1.38)$$

\therefore

$$\det(\mathbf{A})\det(\mathbf{B}) = \det(\mathbf{AB}) \quad (1.1.39)$$

Ex 1.6

6. If two rows (e.g. i th and j th) are equal

$$\det(\mathbf{A}) = \begin{vmatrix} \dots & \dots & \dots & \dots \\ A_{i1} & A_{i2} & \dots & A_{in} \\ \dots & \dots & \dots & \dots \\ A_{j1} & A_{j2} & \dots & A_{jn} \\ \dots & \dots & \dots & \dots \end{vmatrix} \xrightarrow{1.5.3} - \begin{vmatrix} \dots & \dots & \dots & \dots \\ A_{j1} & A_{j2} & \dots & A_{jn} \\ \dots & \dots & \dots & \dots \\ A_{i1} & A_{i2} & \dots & A_{in} \\ \dots & \dots & \dots & \dots \end{vmatrix} = - \begin{vmatrix} \dots & \dots & \dots & \dots \\ A_{i1} & A_{i2} & \dots & A_{in} \\ \dots & \dots & \dots & \dots \\ A_{j1} & A_{j2} & \dots & A_{jn} \\ \dots & \dots & \dots & \dots \end{vmatrix} \quad (1.1.40)$$

i.e.

$$\det(\mathbf{A}) = -\det(\mathbf{A}) \quad (1.1.41)$$

thus

$$\det(\mathbf{A}) = 0 \quad (1.1.42)$$

7. From Ex 1.5.5, we have

$$\det(\mathbf{A}) \det(\mathbf{A}^{-1}) = \det(\mathbf{1}) = 1 \quad (1.1.43)$$

thus

$$\det(\mathbf{A}^{-1}) = \det(\mathbf{A})^{-1} \quad (1.1.44)$$

8.

$$\mathbf{A}\mathbf{A}^\dagger = \mathbf{1} \Rightarrow \det(\mathbf{A}) \det(\mathbf{A}^\dagger) = \det(\mathbf{1}) = 1 \quad (1.1.45)$$

From Ex 1.5.4, we have

$$\det(\mathbf{A}) \det(\mathbf{A})^* = 1 \quad (1.1.46)$$

9. From Ex 1.5.5, we get

$$\det(\mathbf{U}^\dagger) \det(\mathbf{O}) \det(\mathbf{U}) = \det(\mathbf{\Omega}) \quad (1.1.47)$$

and

$$\det(\mathbf{U}^\dagger) \det(\mathbf{U}) = \det(\mathbf{1}) = 1 \quad (1.1.48)$$

\therefore

$$\det(\mathbf{O}) = \det(\mathbf{\Omega}) \quad (1.1.49)$$

Ex 1.7 If $\det(\mathbf{A}) \neq 0$, thus \mathbf{A}^{-1} exists, we have

$$\mathbf{A}^{-1}\mathbf{A}\mathbf{c} = \mathbf{0} \Rightarrow \mathbf{c} = \mathbf{0} \quad (1.1.50)$$

□

1.1.4 N-D Complex Vector Spaces

1.1.5 Change of Basis

Ex 1.8

$$\Omega_{\alpha\beta} = \sum_{ij} U_{\alpha i}^\dagger O_{ij} U_{j\beta} \quad (1.1.51)$$

gives

$$\begin{aligned} \text{tr } \mathbf{\Omega} &= \sum_{\alpha} \Omega_{\alpha\alpha} = \sum_{\alpha} \sum_{ij} U_{\alpha i}^\dagger O_{ij} U_{j\alpha} \\ &= \sum_{ij} O_{ij} \sum_{\alpha} U_{j\alpha} U_{\alpha i}^\dagger = \sum_{ij} O_{ij} \delta_{ji} = \text{tr } \mathbf{O} \end{aligned} \quad (1.1.52)$$

1.1.6 The Eigenvalue Problem

Ex 1.9

$$\mathbf{O}\mathbf{U} = \mathbf{U}\mathbf{\omega} \Rightarrow \mathbf{O}(\mathbf{c}^1 \quad \mathbf{c}^2 \quad \cdots \quad \mathbf{c}^N) = (\omega_1 \mathbf{c}_1 \quad \omega_2 \mathbf{c}_2 \quad \cdots \quad \omega_N \mathbf{c}_N) \quad (1.1.53)$$

thus

$$\mathbf{O}\mathbf{c}^\alpha = \omega_\alpha \mathbf{c}^\alpha \quad (1.1.54)$$

Ex 1.10

$$\begin{cases} O_{11} - \omega + O_{12}c = 0 \\ O_{21} + (O_{22} - \omega)c = 0 \end{cases} \quad (1.1.55)$$

$$(O_{11} - \omega)(O_{22} - \omega) - O_{21}O_{12} = 0 \quad (1.1.56)$$

$$\omega^2 - (O_{11} + O_{22})\omega + O_{11}O_{22} - O_{21}O_{12} = 0 \quad (1.1.57)$$

$$\begin{cases} \omega_1 = \frac{1}{2} \left(O_{11} + O_{22} - \sqrt{(O_{11} - O_{22})^2 + 4O_{21}O_{12}} \right) \\ \omega_2 = \frac{1}{2} \left(O_{11} + O_{22} + \sqrt{(O_{11} - O_{22})^2 + 4O_{21}O_{12}} \right) \end{cases} \quad (1.1.58)$$

Ex 1.11

a)

$$\begin{vmatrix} 3-\omega & 1 \\ 1 & 3-\omega \end{vmatrix} = 0 \Rightarrow (3-\omega)^2 - 1 = 0 \quad (1.1.59)$$

Eigenvalues

$$\omega_1 = 2 \quad \omega_2 = 4 \quad (1.1.60)$$

Eigenvectors

$$\mathbf{c}^1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \mathbf{c}^2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (1.1.61)$$

$$\begin{vmatrix} 3-\omega & 1 \\ 1 & 2-\omega \end{vmatrix} = 0 \Rightarrow (3-\omega)(2-\omega) - 1 = 0 \quad (1.1.62)$$

Eigenvalues

$$\omega_1 = \frac{5+\sqrt{5}}{2} \quad \omega_2 = \frac{5-\sqrt{5}}{2} \quad (1.1.63)$$

Eigenvectors

$$\mathbf{c}^1 = \begin{pmatrix} \frac{1}{2}(1+\sqrt{5}) \\ 1 \end{pmatrix} \quad \mathbf{c}^2 = \begin{pmatrix} \frac{1}{2}(1-\sqrt{5}) \\ 1 \end{pmatrix} \quad (1.1.64)$$

b)

$$\theta_0 = \frac{1}{2} \tan^{-1} \frac{2O_{12}}{O_{11} - O_{12}} \quad (1.1.65)$$

for **A**

$$\theta_0 = \frac{1}{2} \tan^{-1} \frac{2 \times 1}{3-3} = \frac{\pi}{4} \quad (1.1.66)$$

Eigenvalues

$$\omega_1 = 2 \quad \omega_2 = 4 \quad (1.1.67)$$

Eigenvectors

$$\mathbf{c}^1 = \begin{pmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \end{pmatrix} \quad \mathbf{c}^2 = \begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix} \quad (1.1.68)$$

for **B**

$$\theta_0 = \frac{1}{2} \tan^{-1} \frac{2 \times 1}{3-2} = \frac{1}{2} \tan^{-1} 2 \quad (1.1.69)$$

Eigenvalues

$$\omega_1 = \frac{10}{5+\sqrt{5}} = \frac{5-\sqrt{5}}{2} \quad \omega_2 = \frac{10}{5-\sqrt{5}} = \frac{5+\sqrt{5}}{2} \quad (1.1.70)$$

Eigenvectors

$$\mathbf{c}^1 = \begin{pmatrix} \sqrt{\frac{\sqrt{5}+5}{10}} \\ \sqrt{\frac{2}{\sqrt{5}+5}} \end{pmatrix} = \sqrt{\frac{2}{\sqrt{5}+5}} \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix} \quad (1.1.71)$$

$$\mathbf{c}^2 = \begin{pmatrix} \sqrt{\frac{2}{\sqrt{5}+5}} \\ -\sqrt{\frac{\sqrt{5}+5}{10}} \end{pmatrix} = -\sqrt{\frac{\sqrt{5}+5}{10}} \begin{pmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{pmatrix} \quad (1.1.72)$$

Details are in **1-1.nb**

1.1.7 Functions of Matrices

Ex 1.12

a.

$$\mathbf{A}^n = \mathbf{U} \mathbf{a}^n \mathbf{U}^\dagger \quad (1.1.73)$$

$$\det(\mathbf{A}^n) = \det(\mathbf{U}) \det(\mathbf{a}^n) \det(\mathbf{U}^\dagger) = \det(\mathbf{U}) \det(\mathbf{U}^\dagger) \begin{vmatrix} a_1^n & & \\ & a_2^n & \\ & & \ddots \\ & & & a_N^n \end{vmatrix} = a_1^n a_2^n \cdots a_N^n \quad (1.1.74)$$

b. From 1.4.a, we have

$$\text{tr } \mathbf{A}^n = \text{tr}(\mathbf{U} \mathbf{a}^n \mathbf{U}^\dagger) = \text{tr}(\mathbf{U} \mathbf{U}^\dagger \mathbf{a}^n) = \text{tr}(\mathbf{a}^n) = \sum_{\alpha=1}^N a_\alpha^n \quad (1.1.75)$$

c.

$$\mathbf{U}^\dagger (\omega \mathbf{1} - \mathbf{A}) \mathbf{U} = \omega \mathbf{1} - \mathbf{a} \quad (1.1.76)$$

$$(\omega \mathbf{1} - \mathbf{A})^{-1} = [(\mathbf{U}(\omega \mathbf{1} - \mathbf{a}) \mathbf{U}^\dagger)^{-1}] = \mathbf{U}(\omega \mathbf{1} - \mathbf{a})^{-1} \mathbf{U}^\dagger \quad (1.1.77)$$

while

$$(\omega \mathbf{1} - \mathbf{a})^{-1} = \begin{pmatrix} \omega - a_1 & & & \\ & \omega - a_2 & & \\ & & \ddots & \\ & & & \omega - a_N \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{\omega - a_1} & & & \\ & \frac{1}{\omega - a_2} & & \\ & & \ddots & \\ & & & \frac{1}{\omega - a_N} \end{pmatrix} \quad (1.1.78)$$

thus

$$\mathbf{G}(\omega) = (\omega \mathbf{1} - \mathbf{A})^{-1} = \mathbf{U} \begin{pmatrix} \frac{1}{\omega - a_1} & & & \\ & \frac{1}{\omega - a_2} & & \\ & & \ddots & \\ & & & \frac{1}{\omega - a_N} \end{pmatrix} \mathbf{U}^\dagger \quad (1.1.79)$$

$$\mathbf{G}(\omega)_{ij} = \sum_{\alpha} U_{i\alpha} \frac{1}{\omega - a_{\alpha}} U_{\alpha j}^\dagger = \sum_{\alpha} \frac{U_{i\alpha} U_{j\alpha}^*}{\omega - a_{\alpha}} \quad (1.1.80)$$

Since $U_{i\alpha} = \langle i | \alpha \rangle$, $U_{\alpha j}^\dagger = U_{j\alpha}^* = \langle \alpha | j \rangle$

$$\mathbf{G}(\omega)_{ij} = \sum_{\alpha} \frac{\langle i | \alpha \rangle \langle \alpha | j \rangle}{\omega - a_{\alpha}} \quad (1.1.81)$$

Ex 1.13 The eigenvalues and eigenvectors of \mathbf{A} are

$$\omega_1 = a - b \quad \omega_2 = a + b \quad (1.1.82)$$

$$\mathbf{c}^1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \mathbf{c}^2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (1.1.83)$$

$$\mathbf{A} = \mathbf{U} \mathbf{a} \mathbf{U}^\dagger = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a+b & 0 \\ 0 & a-b \end{pmatrix} \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad (1.1.84)$$

$$\begin{aligned} f(\mathbf{A}) &= \mathbf{U} f(\mathbf{a}) \mathbf{U}^\dagger = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} f(a+b) & 0 \\ 0 & f(a-b) \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} f(a+b) & f(a-b) \\ f(a+b) & -f(a-b) \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} f(a+b) + f(a-b) & f(a+b) - f(a-b) \\ f(a+b) - f(a-b) & f(a+b) + f(a-b) \end{pmatrix} \end{aligned} \quad (1.1.85)$$

1.2 Orthogonal Functions, Eigenfunctions, and Operators

Ex 1.14

$$\int_{-\infty}^{\infty} dx a(x) \delta(x) = \lim_{\varepsilon \rightarrow 0} \int_{-\varepsilon}^{\varepsilon} dx a(x) \frac{1}{2\varepsilon} = \lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} dx a(x) \stackrel{\text{L'Hôpital}}{=} \lim_{\varepsilon \rightarrow 0} \frac{a(\varepsilon) - [-a(-\varepsilon)]}{2} = a(0) \quad (1.2.1)$$

Ex 1.15

$$\begin{aligned} \int dx \psi_j^*(x) \mathcal{O} \psi_i(x) &= \int dx \psi_j^*(x) \sum_k \psi_k(x) O_{ki} = \sum_k O_{ki} \int dx \psi_j^*(x) \psi_k(x) \\ &= \sum_k O_{ki} \delta_{jk} = O_{ji} \end{aligned} \quad (1.2.2)$$

In bra-ket notation, (1) becomes

$$\mathcal{O} |i\rangle = \sum_j |j\rangle \langle j | \mathcal{O} | i \rangle \quad (1.2.3)$$

which is identical to Eq.(1.55) in the textbook.

Ex 1.16 With bra-ket notation,

$$\mathcal{O} \sum_{i=1}^{\infty} c_i |i\rangle = \omega \sum_{i=1}^{\infty} c_i |i\rangle \quad (1.2.4)$$

Multiply by $\langle j |$

$$\sum_{i=1}^{\infty} c_i \langle j | \mathcal{O} | i \rangle = \omega \sum_{i=1}^{\infty} c_i \langle j | i \rangle = \omega c_j \quad (1.2.5)$$

i.e.

$$\sum_{i=1}^{\infty} O_{ji} c_i = \omega c_j \quad (1.2.6)$$

$$\mathbf{O} \mathbf{c} = \omega \mathbf{c} \quad (1.2.7)$$

It's similar to prove that without bra-ket notation.

Ex 1.17

a.

$$\int dx \langle i | x \rangle \langle x | j \rangle = \langle i | j \rangle = \delta_{ij} \quad (1.2.8)$$

i.e.

$$\int dx \psi_i^*(x) \Psi_j(x) = \delta_{ij} \quad (1.2.9)$$

b.

$$\sum_{i=1}^{\infty} \langle x | i \rangle \langle i | x' \rangle = \langle x | x' \rangle = \delta(x - x') \quad (1.2.10)$$

thus

$$\sum_{i=1}^{\infty} \psi_i^*(x) \psi_i(x') = \sum_{i=1}^{\infty} \langle x | i \rangle \langle i | x' \rangle = \delta(x - x') \quad (1.2.11)$$

c.

$$\int dx \langle x' | x \rangle \langle x | a \rangle = \langle x' | a \rangle \quad (1.2.12)$$

thus

$$\int dx \delta(x' - x) a(x) = a(x') \quad (1.2.13)$$

i.e.

$$\int dx' \delta(x - x') a(x') = a(x) \quad (1.2.14)$$

d.

$$\langle x' | \mathcal{O} | a \rangle = \int dx \langle x' | \mathcal{O} | x \rangle \langle x | a \rangle = \langle x' | b \rangle \quad (1.2.15)$$

∴

$$\mathcal{O}a(x') = \int dx \mathcal{O}(x', x)a(x) = b(x') \quad (1.2.16)$$

i.e.

$$b(x) = \mathcal{O}a(x) = \int dx' \mathcal{O}(x, x')a(x') \quad (1.2.17)$$

e.

$$\begin{aligned} \mathcal{O}(x, x') &= \langle x | \mathcal{O} | x' \rangle = \langle x | \left(\sum_i |i\rangle \langle i| \right) \mathcal{O} \left(\sum_j |j\rangle \langle j| \right) | x' \rangle \\ &= \sum_{ij} \langle x | i \rangle \langle i | \mathcal{O} | j \rangle \langle j | x' \rangle \\ &= \sum_{ij} \psi_i(x) \mathcal{O}_{ij} \psi_j^*(x') \end{aligned} \quad (1.2.18)$$

1.3 The Variation Method

1.3.1 The Variation Principle

Ex 1.18

$$\begin{aligned} \mathcal{E} &= \frac{\left\langle \tilde{\Phi} \left| -\frac{1}{2} \frac{d^2}{dx^2} - \delta(x) \right| \tilde{\Phi} \right\rangle}{\left\langle \tilde{\Phi} | \tilde{\Phi} \right\rangle} = \frac{N^2 \int_{-\infty}^{\infty} dx e^{-\alpha x^2} \left[-\frac{1}{2}(-2\alpha + 4\alpha^2 x^2) - \delta(x) \right] e^{-\alpha x^2}}{N^2 \int_{-\infty}^{\infty} dx e^{-2\alpha x^2}} \\ &= \frac{\alpha \frac{\pi^{1/2}}{(2\alpha)^{1/2}} - 2\alpha^2 \frac{2\pi^{1/2}}{4(2\alpha)^{3/2}} - 1}{\frac{\pi^{1/2}}{(2\alpha)^{1/2}}} \\ &= \frac{\alpha \pi^{1/2} - \alpha^2 \frac{\pi^{1/2}}{(2\alpha)} - (2\alpha)^{1/2}}{\pi^{1/2}} \\ &= \alpha - \frac{1}{2}\alpha - \frac{(2\alpha)^{1/2}}{\pi^{1/2}} \\ &= \frac{1}{2}\alpha - \frac{(2\alpha)^{1/2}}{\pi^{1/2}} \end{aligned} \quad (1.3.1)$$

Let $\frac{d\mathcal{E}}{d\alpha} = 0$, we have

$$\frac{1}{2} - \frac{1}{(2\pi\alpha)^{1/2}} = 0 \Rightarrow \alpha = \frac{2}{\pi} \quad (1.3.2)$$

thus

$$\mathcal{E}_{min} = -\frac{1}{\pi} \quad (1.3.3)$$

Ex 1.19

$$\begin{aligned}
\mathcal{E} &= \frac{\left\langle \tilde{\Phi} \left| -\frac{1}{2}\nabla^2 - \frac{1}{r} \right| \tilde{\Phi} \right\rangle}{\left\langle \tilde{\Phi} \left| \tilde{\Phi} \right\rangle} = \frac{N^2 \cdot 4\pi \int_{-\infty}^{\infty} r^2 dr e^{-\alpha r^2} \left[-\frac{1}{2}(4\alpha^2 r^2 - 6\alpha) - \frac{1}{r} \right] e^{-\alpha r^2}}{N^2 \cdot 4\pi \int_{-\infty}^{\infty} r^2 dr e^{-2\alpha r^2}} \\
&= \frac{-2\alpha^2 \frac{24\pi^{1/2}}{64(2\alpha)^{5/2}} + 3\alpha \frac{2\pi^{1/2}}{8(2\alpha)^{3/2}} - \frac{1}{2(2\alpha)}}{\frac{2\pi^{1/2}}{8(2\alpha)^{3/2}}} \\
&= -2\alpha^2 \frac{12}{8(2\alpha)} + 3\alpha - \frac{2(2\alpha)^{1/2}}{\pi^{1/2}} \\
&= \frac{3}{2}\alpha - \frac{2(2\alpha)^{1/2}}{\pi^{1/2}}
\end{aligned} \tag{1.3.4}$$

Let $\frac{d\mathcal{E}}{d\alpha} = 0$,

$$\frac{3}{2} - \frac{2}{\sqrt{2\pi\alpha}} = 0 \Rightarrow \alpha = \frac{8}{9\pi} \tag{1.3.5}$$

$$\mathcal{E}_{min} = \frac{4}{3\pi} - \frac{8}{3\pi} = -\frac{4}{3\pi} \tag{1.3.6}$$

Ex 1.20

$$\begin{aligned}
\omega(\theta) &= \mathbf{c}^\dagger \mathbf{O} \mathbf{c} = \begin{pmatrix} \cos \theta & \sin \theta \end{pmatrix} \begin{pmatrix} O_{11} \cos \theta + O_{12} \sin \theta \\ O_{12} \cos \theta + O_{22} \sin \theta \end{pmatrix} \\
&= O_{11} \cos^2 \theta + 2O_{12} \cos \theta \sin \theta + O_{22} \sin^2 \theta
\end{aligned} \tag{1.3.7}$$

Let $\frac{d\omega}{d\theta} = 0$, thus

$$O_{11}(-2 \cos \theta \sin \theta) + O_{12} \cdot 2 \cos 2\theta + O_{22} \cdot 2 \sin \theta \cos \theta = 0 \tag{1.3.8}$$

$$(O_{22} - O_{11}) \sin 2\theta + 2O_{12} \cos 2\theta = 0 \tag{1.3.9}$$

$$\theta = \frac{1}{2} \arctan \frac{2O_{12}}{O_{11} - O_{22}} \tag{1.3.10}$$

$$\omega = O_{11} \cos^2 \theta + O_{12} \sin 2\theta + O_{22} \sin^2 \theta \tag{1.3.11}$$

which are exactly the results in Eq. (1.105) and Eq. (1.106a) in the textbook. We get the result because the trial vector \mathbf{c} is the exact eigenvector of \mathbf{O} .

1.3.2 The Linear Variational Problem

Ex 1.21

a.

$$\left\langle \tilde{\Phi}' \left| \tilde{\Phi}' \right\rangle = 1 = \sum_{\alpha\beta} \left\langle \tilde{\Phi}' \left| \Phi_\alpha \right\rangle \left\langle \Phi_\alpha \left| \Phi_\beta \right\rangle \left\langle \Phi_\beta \left| \tilde{\Phi}' \right\rangle \right. \tag{1.3.12}$$

Since $\left\langle \tilde{\Phi}' \left| \Phi_0 \right\rangle = 0$, we have

$$\sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} \left\langle \tilde{\Phi}' \left| \Phi_\alpha \right\rangle \left\langle \Phi_\alpha \left| \Phi_\beta \right\rangle \left\langle \Phi_\beta \left| \tilde{\Phi}' \right\rangle = 1 \tag{1.3.13}$$

thus

$$\sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} \left\langle \tilde{\Phi}' \left| \Phi_\alpha \right\rangle \delta_{\alpha\beta} \left\langle \Phi_\beta \left| \tilde{\Phi}' \right\rangle = 1 \tag{1.3.14}$$

$$\sum_{\alpha=1}^{\infty} \left\langle \tilde{\Phi}' \left| \Phi_\alpha \right\rangle \left\langle \Phi_\alpha \left| \tilde{\Phi}' \right\rangle = 1 \tag{1.3.15}$$

$$\sum_{\alpha=1}^{\infty} \left| \langle \Phi_{\alpha} | \tilde{\Phi}' \rangle \right|^2 = 1 \quad (1.3.16)$$

Similarly,

$$\langle \tilde{\Phi}' | \mathcal{H} | \tilde{\Phi}' \rangle = \sum_{\alpha\beta} \langle \tilde{\Phi}' | \Phi_{\alpha} \rangle \langle \Phi_{\alpha} | \mathcal{H} | \Phi_{\beta} \rangle \langle \Phi_{\beta} | \tilde{\Phi}' \rangle = \sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} \langle \tilde{\Phi}' | \Phi_{\alpha} \rangle \langle \Phi_{\alpha} | \mathcal{H} | \Phi_{\beta} \rangle \langle \Phi_{\beta} | \tilde{\Phi}' \rangle \quad (1.3.17)$$

From Eq. (1.170) from the textbook, we get

$$\langle \Phi_{\alpha} | \mathcal{H} | \Phi_{\beta} \rangle = \mathcal{E}_{\alpha} \delta_{\alpha\beta} \quad (1.3.18)$$

thus

$$\langle \tilde{\Phi}' | \mathcal{H} | \tilde{\Phi}' \rangle = \sum_{\alpha=1}^{\infty} \left| \langle \Phi_{\alpha} | \tilde{\Phi}' \rangle \right|^2 \mathcal{E}_{\alpha} \geq \sum_{\alpha=1}^{\infty} \left| \langle \Phi_{\alpha} | \tilde{\Phi}' \rangle \right|^2 \mathcal{E}_1 = \mathcal{E}_1 \quad (1.3.19)$$

b.

$$\langle \tilde{\Phi}' | \tilde{\Phi}' \rangle = 1 = \left(x^* \langle \tilde{\Phi}_0 | + y^* \langle \tilde{\Phi}_1 | \right) \left(x | \tilde{\Phi}_0 \rangle + y | \tilde{\Phi}_1 \rangle \right) = |x|^2 + |y|^2 \quad (1.3.20)$$

c.

$$\begin{aligned} \langle \tilde{\Phi}' | \mathcal{H} | \tilde{\Phi}' \rangle &= |x|^2 \langle \tilde{\Phi}_0 | \mathcal{H} | \tilde{\Phi}_0 \rangle + |y|^2 \langle \tilde{\Phi}_1 | \mathcal{H} | \tilde{\Phi}_1 \rangle + x^* y \langle \tilde{\Phi}_0 | \mathcal{H} | \tilde{\Phi}_1 \rangle + x y^* \langle \tilde{\Phi}_1 | \mathcal{H} | \tilde{\Phi}_0 \rangle \\ &= E_1 - |x|^2 (E_1 - E_0) \end{aligned} \quad (1.3.21)$$

thus

$$\mathcal{E}_1 \leq \langle \tilde{\Phi}' | \mathcal{H} | \tilde{\Phi}' \rangle \leq E_1 - |x|^2 (E_1 - E_0) = E_1 \quad (1.3.22)$$

Ex 1.22

$$\begin{aligned} H_{11} &= \langle 1s | \mathcal{H} | 1s \rangle = -\frac{1}{2} + F \langle 1s | r \cos \theta | 1s \rangle = -\frac{1}{2} \\ H_{12} &= H_{21} = \langle 1s | \mathcal{H} | 2p_z \rangle = 0 + F \langle 1s | r \cos \theta | 2p_z \rangle = \frac{512\sqrt{2}\pi}{243} F \\ H_{22} &= \langle 2p_z | \mathcal{H} | 2p_z \rangle = -\frac{1}{8} + F \langle 2p_z | r \cos \theta | 2p_z \rangle = -\frac{1}{8} \end{aligned} \quad (1.3.23)$$

Suppose $\mathbf{c} = \begin{pmatrix} \cos p \\ \sin p \end{pmatrix}$,

$$p = \frac{1}{2} \arctan \frac{2H_{12}}{H_{11} - H_{22}} = -\frac{1}{2} \tan^{-1} \left(\frac{2048\sqrt{2}F}{729} \right) \quad (1.3.24)$$

thus

$$\mathcal{E}(F) = H_{11} \cos^2 p + H_{12} \sin 2p + H_{22} \sin^2 p = -\frac{1}{2} - \frac{262144}{177147} F^2 + \mathcal{O}(F^3) \quad (1.3.25)$$

\therefore

$$\alpha = 2 \times \frac{262144}{177147} = 2.96 \quad (1.3.26)$$