Program : **B.Tech**

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Module 1: Numerical Methods-1

Contents : Solution of polynomial and transcendental equations Bisection method, Newton Raphson method and Regula Falsi method. Finite differences, Relation between operators, Interpolation using Newton's forward and backward Difference formulae. Interpolation with unequal intervals: Newton's divided difference and Lagrange's formulae.

Introduction

In this unit we will discuss one of the most basic problems in numerical analysis. The problem is called a root-finding problem and consists of finding values of the variable x (real) that satisfy the equation f(x) = 0, for a given function f. Let f be a real-value function of a real variable. Any real number α for which $f(\alpha) = 0$ is called a root of that equation or a zero of f. We shall confine our discussion to locating only the real roots of f(x), that is, locating non-real complex roots of f(x) = 0 will not be discussed. This is one of the oldest numerical approximation problems. The procedures we will discuss range from the classical Newton-Raphson method developed primarily by Isaac Newton over 300 years ago to methods that were established in the recent past.

We shall consider the problem of numerical computation of the real roots of a given equation f(x) = 0. Which may be algebraic or transcendental. It will be assumed that the function f(x) is continuously differentiable a sufficient number of times. Mostly, we shall confine to simple roots and indicate the iteration function for multiple roots in case of Newton Raphson method.

All the methods for numerical solution of equations discussed here will consist of two steps. First step is about the location of the roots, that is, rough approximate value of the roots are obtained as initial approximation to a root. Second step consists of methods, which improve the rough value of each root.

A method for improvement of the value of a root at a second step usually involves a process of successive approximation of iteration. In such a process of successive approximation a sequence $\{X_n\}$ n = 0, 1, 2, … is generated by the method used starting with the initial approximation x_o of the root α obtained in the first step such that the sequence $\{X_n\}$ converges to α as $n \to \infty$. This x_n is called the nth approximation of nth iterate and it gives a sufficiently accurate value of the root α.

For the first step we need the following theorem:

Theorem 1: If f(x) is continuous in the closed internal [a, b] and f(a) are of opposite signs, then there is at least one real root α of the equation f(x) = 0 such that a <α< b.

Bisection Method (Bolzano Method) :

In this method we find an interval in which the root lies and that there is no other root in that interval. Then we keep on narrowing down the interval to half at each successive iteration. We proceed as follows:

(1) Find interval $I = (x_1, x_2)$ in which the root of f(x) = 0 lies and that there is no other root in I.

(2) Bisect the interval at $x = \frac{x_1 + x_2}{2}$ and compute f(x). If | f(x) | is less than the desired accuracy then it is the root of f(x) = 0.

(3) Otherwise check sign of f(x). If sign {f(x)} = sign {f(x_2)} then root lies in the interval[x_1, x] and if they are of opposite signs then the root lies in the interval [x, x_2]. Changex to x_2 or x_1 accordingly. We may test sign of f(x) $\times$ f(x_2) for same sign or opposite signs.

(4) Check the length of interval | $x_1 - x_2$ |. If an accuracy of say, two decimal places is required then stop the process when the length of the interval is 0.005 or less. We may take the midvalue $x = \frac{x_1 + x_2}{2}$ as the root of f(x) = 0. The convergence of this method is very slow in the beginning.

Example

Find the positive root of the equation $x^3 + 4x^2 - 10 = 0$ by bisection method correct upto two places of decimal.

Solution

$$f(x) \equiv x^3 + 4x^2 - 10 = 0$$

Let us find location of the + ive roots.

x	0	1	2	> 2

f(x)	– 10	– 5	14	
Sign f(x)	–	–	+	+

There is only one + ive root and it lies between 1 and 2. Let $x_1 = 1$ and $x_2 = 2$; at x = 1, f(x) is– ive and at x = 2, f(x) is + ive. We examine the sign of f(x) at $x = \frac{x_1 + x_2}{2} = 1.5$ and check whether the root lies in the interval

(1, 1.5) or (1.5, 2). Let us show the computations in the table below :

Iteration No.	$x = \frac{x_1 + x_2}{2}$	Sign f(x)	Sign f(x) × f(x₂)	x_1	x_2
1	1.5	+ 2.375	+	1	1.5
2	1.25	– 1.797	–	1.25	1.5
3	1.375	+ 0.162	+	1.25	1.375
4	1.3125	– 0.8484	–	1.3125	1.375
5	1.3438	– 0.3502	–	1.3438	1.375
6	1.3594	– 0.0960	–	1.3594	1.375
7	1.367	– 0.0471	–	1.367	1.375
8	1.371	+ 0.0956	+	1.367	1.371

We see that $|x_1 - x_2| = 0.004$.

We can choose the root as $x = \frac{1.367 + 1.371}{2} = 1.369$.

Regula-Falsi Method (or Method of False Position)

In this method also we find two values of x say x_1 and x_2 where function f(x) has opposite signs and there is only one root in the interval (x_1, x_2). Let us express the function of y = f(x) and we are interested in finding the value of x where curve y = f(x) intersects x-axis i.e. y = 0. We identify two points (x_1, y_1) and (x_2, y_2) on the curve. Then we approximate the curve by a straight line joining these two points. We find the point on the x-axis where this line cuts the x-axis. The equation of the straight line passing through (x_1, y_1) and (x_2, y_2) is given by

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

The point on x-axis where y = 0 is given by

$$x = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1}$$

Now we check the sign of f(x) and proceed like in the bisection method. That is, if f(x) has same sign as $f(x_2)$ then root lies in the interval (x_1, x) and if they have opposite signs, then it lies in the interval (x, x_2). See Figure 1.

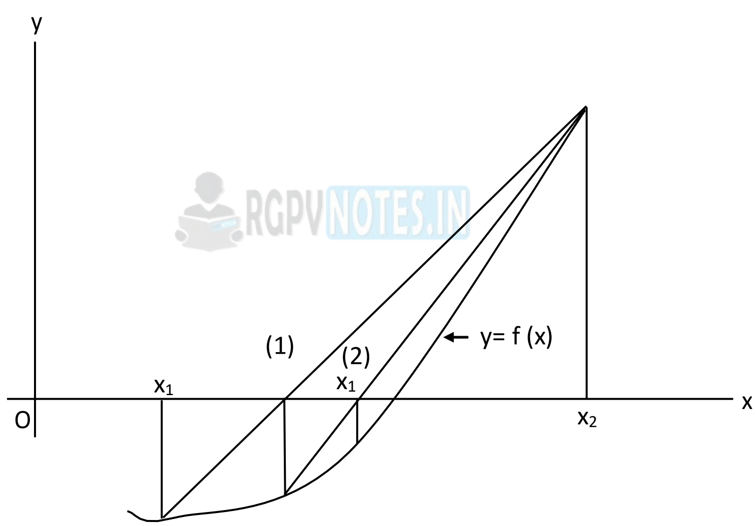

Figure 1 : Regula-Falsi Method, Superscript Shows Iteration Number

Example

Find positive root of $x^3 + 4x^2 - 10 = 0$ by Regula-Falsi method. Compute upto the two decimal places only.

Solution

It is the same problem as given in the previous example. We start by taking $x_1 = 1$ and $x_2 = 2$. We have $y = x^3 + 4x^2 - 10$; $y_1 = -5$ and $y_2 = 14$. The point on the curve are (1, – 5) and(2, 14). The points on the x-axis where the line joining these two pints cuts it, is given by

I-Iteration

$$x = \frac{1 \times 14 - 2 \times (-5)}{14 + 5} = \frac{24}{19} = 1.26$$

$$y = f(x) = -1.65$$

II-Iteration

Take points (1.26, – 1.65) and (2, 14)

$$x = \frac{1.26 \times 14 - 2 \times (-1.65)}{14 + 1.65} = 1.34$$

$$y = f(x) = -0.41$$

III-Iteration

Take two points (1.34, – 0.41) and (2, 14)

$$x = \frac{1.34 \times 14 - 2 \times (-0.41)}{14 + 0.41} = 1.36$$

$$y = f(x) = -0.086$$

IV-Iteration

Take two points (1.36, – 0.086) and (2, 14)

$$x = \frac{1.36 \times 14 - 2 \times (-0.086)}{14 + 0.086} = 1.36$$

Since value of x repeats we take the root as x = 1.36.

Example: Find the real root of the equation $x\log_{10} x - 1.2 = 0$ by method of False- position, correct to four decimal places.

Solution: $f(x) = x\log_{10} x - 1.2 = 0$

f(2)=-0.5979, f(3)=0.2313

Therefore root lies between 2 & 3

by method of False position

$$x_1 = \frac{af\ b - bf(a)}{f\ b - f(a)}$$

$$x_1 = 2.7210, f\ x_1 = -0.01709$$

Now the roots lie between $x_1 = 2.7210$ & 3, because the functions are of opposite sign.

$$x_2 = 2.7402, f\ x_2 = 0.00038$$

Root lies between 2.7402 & 3,

$x_3 = 2.7406$ hence the root 2.7406

Newton-Raphson (N-R) Method

The Newton-Raphson's method or commonly known as N-R method is most popular for finding the roots of an equation. Its approach is different from all the methods discussed earlier in the sense that it uses only one value of x in the neighbourhood of the root instead of two. We can explain the method geometrically as follows :

Let us suppose we want to find out the root of an equation f(x) = 0 while y = f(x) represents a curve and we are interested to find the point where it cuts the x-axis. Let $x = x_0$ be an initial approximate value of the root close to the actual root. We evaluate $y(x_0) = f(x_0) = y_0$ (say). Then point (x_0, y_0) lies on the curve y = f(x). We find $\frac{dy}{dx} = f'(x)$ for $x = x_0$, say $f'(x_0)$.

Then we may draw a tangent at (x_0, y_0) given as,

$$y - y_0 = f'(x_0)(x - x_0)$$

The point where the tangent cuts the x-axis (y = 0) is taken as the next estimate $x = x_1$ for the root, i.e.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

In general $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, see Figure 2

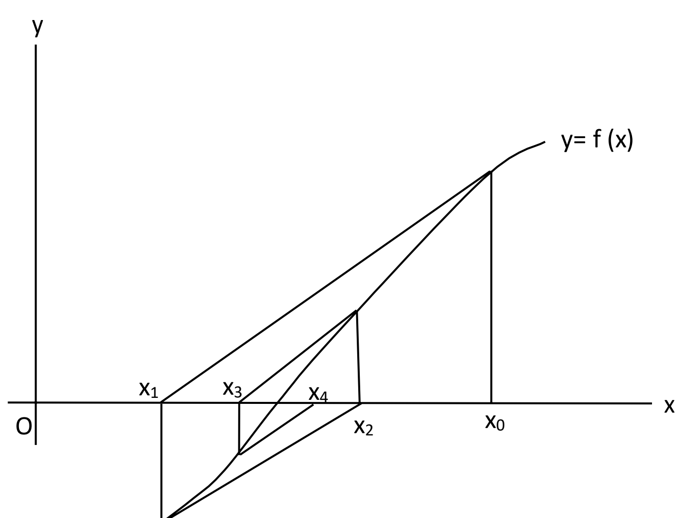

Figure 2 : Newton-Raphson Method

Theoretically, the N-R method may be explained as follows :

Let α be the exact root of f(x) = 0 and let $\alpha = x_0 + h$ where h is a small number to be determined. From Taylor's series as have,

$$f(\alpha) = f(x_0 + h) = f(x_0) + h f'(x_0) + \frac{h^2}{2} f''(x_0) + \ldots = 0$$

Neglecting h^2 and higher powers we get an approximate value of h, as $h = -\frac{f(x_0)}{f'(x_0)}$. Hence, an approximation for the exact root α may be written as,

$$x_1 = x_0 + h = x_0 - \frac{f(x_0)}{f'(x_0)}$$

In general the N-R formula may be written as,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad n = 0, 1, 2, \ldots$$

It is same as derived above geometrically. It may be stated that the root of convergence of N-R method is faster as compared to other methods. Further, comparing the N-R method with method of successive substitution, it can be seen as iterative scheme for

$$x = x - \frac{f(x)}{f'(x)}$$

where $\phi(x) = x - \frac{f(x)}{f'(x)}$

The condition for convergence $|\phi'(\alpha)| < 1$ in this case would be

$$\phi'(x) = 1 - \frac{\{f'(x)\}^2 - f(x) f''(x)}{\{f'(x)\}^2} = \frac{f(x) f''(x)}{\{f'(x)\}^2}, \text{ at } x = \alpha.$$

This implies that $f'(\alpha) \neq 0$.

Example: Write N-R iterative scheme to find inverse of an integer number N. Hence, find inverse of 17 correct upto 4 places of decimal starting with 0.05.

Solution : Let inverse of N be x, so that we the equation to solve as,

$$x = N^{-1} \qquad \text{or} \qquad x^{-1} - N = 0$$

$$f(x) = x^{-1} - N \; ; \; f'(x) = -\frac{1}{x^2}$$

N-R scheme is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n + x_n^2 \left(\frac{1}{x_n} - N \right)$$

$$= x_n (2 - N x_n) = x_n (2 - 17 x_n) \qquad \because N = 17.$$

We take $x_0 = 0.05$.

Substituting in the formula, we get

$x_1 = 0.0575$; $x_2 = 0.0588$; $x_3 = 0.0588$

Hence, $\frac{1}{17} = 0.0588$.

Example: Write down N-R iterative scheme for finding q^{th} root of a positive number N. Hence, find cuberoot of 10 correct up to 3 places of decimal taking initial estimate as 2.0.

Solution: We have to solve $x = N^{\frac{1}{q}}$ or $x^q - N = 0$

$$f(x) = x^q - N \; ; \; f'(x) = q x^{q-1}$$

The N-R iterative scheme may be written

$$x_{n+1} = x_n - \frac{x_n^q - N}{q\, x_n^{q-1}} = \frac{(q-1)\, x_n^q + N}{q\, x_n^{q-1}}$$

For cuberoot of 10 we have N = 10, q = 3.

Hence, $x_{n+1} = \frac{2x_n^3 + 10}{3x_n^2}$

Taking $x_0 = 2.0$ we get the following iterated values

$$x_1 = \frac{16}{2} = 2.167 \; ; \; x_2 = \frac{30.3520}{14.0877} = 2.154 \; ; \; x_3 = \frac{29.9879}{13.9191} = 2.154$$

Hence, we get $10^{\frac{1}{3}} = 2.154$.

Example: Using N-R method find the root of the equation x – cos x = 0 correct upto two places of decimal only. Take the starting value as $\frac{\pi}{4}$ (π = 3.1416, π radian = 180°).

Solution : $f(x) = x - \cos x \quad ; \quad f'(x) = 1 + \sin x$

N-R scheme is given by

$$x_{n+1} = x_n - \frac{x_n - \cos x_n}{1 + \sin x_n} = \frac{x_n \sin x_n + \cos x_n}{1 + \sin x_n}$$

Taking $x_0 = \frac{\pi}{4}$

$$x_1 = \frac{\frac{\pi}{4}\sin\frac{\pi}{4} + \cos\frac{\pi}{4}}{1 + \sin\frac{\pi}{4}} = \frac{\frac{\pi}{4} \times \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}}{1 + \frac{1}{\sqrt{2}}}$$

$$= \frac{\frac{\pi}{4} + 1}{\sqrt{2} + 1} = \frac{1.7854}{2.4142} = 0.7395$$

$$x_2 = \frac{0.7395 \sin(0.7395) + \cos(0.7395)}{1 + \sin(0.7395)}$$

$$= \frac{0.7395 \times 0.6724 + 0.7449}{1 + 0.6724} = 0.7427$$

Up to two places of decimal the root is 0.74.

Note : If starting value is not given, we can plot graphs of y = x and y = cos x and locate their point of intersection which will be root of x – cos x = 0. See Figure 3

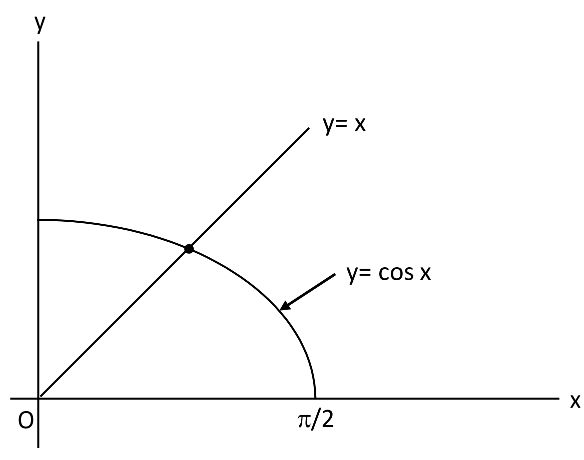

Figure 3 : Intersection of y = x and y = cos x

Example: Find one real root of $3x = \cos x + 1$ by Newton Raphson method.
Solution : Let $f(x) = 3x - cosx - 1$

$$f'(x) = 3 + \sin x$$

By Newton Raphson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f(0) = -2, f(1) = 2.540$$

Since f(0)=-2 is nearer to 0,therefore $x_0 = 0$ is our first approximation.

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$=0.6

$$x_2 = 0.6071$$

$$x_3 = 0.6071$$

The real root of equation correct to four decimal places 0.6071.

Finite Differences and Interpolation

Suppose we are given the following values of y = f(x) for a set of values of x

X :	x_0	x_1	x_2	……	x_n
y :	y_0	y_1	y_2	……	y_n

The process of finding the values of y corresponding to any value of $x=x_i$ between $x_0$and x_n is called interpolation.

(1) The technique of estimating the value of a function for any intermediate value of the independent variable is called interpolation.

(2) The technique of estimating the value of a function outside the given range is called extrapolation.

(3) The study of interpolation is based on the concept of differences of a function.

(4) Suppose that the function y=f(x) is tabulated for the equally spaced values $x = x_0$, $x_1=x_0+h$, $x_2=x_0+2h$, …, $x_n=x_0+nh$ giving $y = y_0, y_1, y_2, …, y_n$. To determine the values of f(x) and f '(x) for some intermediate values of x, we use the following two types of differences

1.Forward difference

2.Backward difference

Forward difference: The first order forward differences are defined and denoted by $\Delta f(x)=f(x+h)-f(x)$,

$$\Delta y_0 = y_1 - y_0$$
$$\Delta y_1 = y_2 - y_1$$
$$\Delta y_2 = y_3 - y_2$$
$$\ldots\ldots\ldots\ldots$$
$$\Delta y_r = y_{r+1} - y_r$$
$$\ldots\ldots\ldots\ldots$$
$$\Delta y_{n-1} = y_n - y_{n-1}$$

These are called the first forward differences and Δ is the forward difference operator.

Similarly the second forward differences are defined by

$$\Delta^2 y_r = \Delta y_{r+1} - \Delta y_r.$$

In general

$$\Delta^p y_r = \Delta^{p-1} y_{r+1} - \Delta^{p-1} y_r ,$$

p^{th} forward differences.

The forward differences systematically set out in a table called forward difference table.

Value of x	Value of y	1^{st} diff. Δ	2^{nd} diff. Δ^2	3^{rd} diff. Δ^3	4^{th} diff. Δ^4	5^{th}diff. Δ^5
x_0	y_0					
		Δy_0				
x_1	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$	
		Δy_3		$\Delta^3 y_2$		
x_4	y_4		$\Delta^2 y_3$			
		Δy_4				

x_5	y_5					

Backward Differences: Thefirst orderbackward differences are defined and denoted by$\nabla f(x)= f(x)-f(x-h)$,

$$\nabla y_1 = y_1 - y_0$$
$$\nabla y_2 = y_2 - y_1$$
$$\nabla y_3 = y_3 - y_2$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$\nabla y_r = y_r - y_{r-1}$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$\nabla y_n = y_n - y_{n-1}.$$

These are called the first backward differences and $\nabla$is the backward difference operator.
Similarly the second backward differences are defined by

$$\nabla^2 y_r = \nabla y_r - \nabla y_{r-1}.$$

In general

$$\nabla^p y_r = \nabla^{p-1} y_r - \nabla^{p-1} y_{r-1} ,$$

p^{th} backward differences. The backward differences systematically set out in a table called backward difference table.

Value of x	**Value of y**	**1st diff. ∇**	**2nd diff. ∇^2**	**3rd diff. ∇^3**	**4th diff. ∇^4**	**5th diff. ∇^5**
x_0	y_0					
		∇y_1				
x_1	y_1		$\nabla^2 y_2$			
		∇y_2		$\nabla^3 y_3$		
x_2	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$	
		∇y_3		$\nabla^3 y_4$		$\nabla^5 y_5$
x_3	y_3		$\nabla^2 y_4$		$\nabla^4 y_5$	

		∇y_4		$\nabla^3 y_5$		
x_4	y_4		$\nabla^2 y_5$			
		∇y_5				
x_5	y_5					

Example1: Evaluate

(i) $\Delta \tan^{-1}x$

(ii) $\Delta (e^x \log 2x)$

(iii) $\Delta^2 \cos 2x$

Sol. From the definition of forward differences $\Delta f(x) = f(x+h) - f(x)$.

(i) Let $f(x) = \tan^{-1}x$, then

$\Delta\tan^{-1}x = \tan^{-1}(x+h) - \tan^{-1}x$

$$= \tan^{-1}\left(\frac{x+h-x}{1+(x+h)x}\right) = \tan^{-1}\left(\frac{h}{1+hx+x^2}\right).$$

(ii)

$$\Delta(e^x \log 2x) = e^{x+h}\log 2(x+h) - e^x \log 2x$$

$$= e^{x+h}\log 2(x+h) - e^{x+h}\log 2x + e^{x+h}\log 2x - e^x \log 2x$$

$$= e^{x+h}\log\left(\frac{x+h}{x}\right) + (e^{x+h} - e^x)\log 2x$$

$$= e^x\left[e^h\left(\log(1+\frac{h}{x})\right) + (e^h - 1)\log 2x\right].$$

(iii) $\Delta^2 \cos 2x = \Delta[\Delta\cos 2x]$

$= \Delta[\cos 2(x+h) - \cos 2x]$

$= \Delta\cos 2(x+h) - \Delta\cos 2x$

$= \cos 2(x+2h) - \cos 2(x+h) - [\cos 2(x+h)) - \cos 2x]$

$= -2 \cos (2x+3h) \sin h + 2 \sin(2x+h) \sin h$

$= -2 \sin h [\sin(2x+3h) - \sin(2x+h)]$

$= -2 \sin h [2 \cos(2x+2h)\sin h]$

$= -2 \sin^2 h \cos 2(x+h).$

Example2: Evaluate the following, with interval of difference being unity

(i) $\Delta^2 (ab^x)$ (ii) $\Delta^n e^x$

Sol. From the definition of forward differences $\Delta f(x) = f(x+h) - f(x)$.

(i) $\Delta(ab^x) = a\,\Delta b^x = a(b^{x+1} - b^x) = ab^x(b - 1)$

$\Delta^2 (ab^x) = \Delta [\Delta (ab^x)]$
$= \Delta ab^x(b - 1) = a(b - 1) \Delta(b^x)$
$= a(b - 1)(b^{x+1} - b^x)$
$= a(b - 1)^2 b^x.$

(ii) $\Delta e^x = e^{x+1} - e^x = e^x(e - 1)$
$\Delta^2 e^x = \Delta[\Delta e^x] = \Delta [e^{x+1} - e^x]= (e - 1)\Delta e^x$
$= (e - 1)e^x(e - 1) = (e - 1)^2 e^x.$

Similarly $\Delta^2 e^x= (e - 1)^2 e^x$, $\Delta^3 e^x= (e - 1)^3 e^x$, … and $\Delta^n e^x= (e - 1)^n e^x$.

Factorial Notation: A product of the form x(x – 1) (x – 2) …(x – r +1) is denoted by $[x]^r$ and is called a factorial. In particular,

$[x] = x$, $[x]^2 = x(x - 1)$, $[x]^3 = x(x - 1) (x - 2)$, …*x+^{n} = x(x – 1) (x – 2) …(x – n +1).

If the interval of difference is h, then

$$[x]^n = x(x - h) (x - 2h) \ldots(x - (n - 1)h).$$

The factorial notation is of special utility in the theory of finite differences. It helps in finding the successive differences of a polynomial directly by simple rule of differentiation ($[x]^r$ as x^r).

To express a polynomial of nth degree in the factorial notation, we use the following two steps

1. Arrange the coefficients of the powers of x in descending order, replacing missing powers by zeros.
2. Using detached coefficients divide by x, x – 1, x – 2, … x – (n – 1) successively.

Example3. Express $f(x) = 2x^3 - 3x^2 + 3x - 10$ in a factorial notation and hence find all differences.

Sol. Let $f(x) = A[x]^3 + B[x]^2 + C[x] + D$. Then

	x^3	x^2	x	
1	2	-3	3	-10 = D
	_	2	-1	
2	2	-1	2 = C	
	_	4		
3	2	3 = B		

$$\frac{-}{2 = A}$$

Hence $f(x) = 2[x]^3 + 3[x]^2 + 2[x] - 10$. Therefore,

$$\Delta f(x) = 6[x]^2 + 6[x] + 2$$
$$\Delta^2 f(x) = 12[x] + 6$$
$$\Delta^3 f(x) = 12.$$

Other Difference Operators:

(1) Shift Operator: Shift operator E is the operation of increasing the argument x by h so that

$$E f(x) = f(x+h),\ E^2 f(x) = f(x+2h), \dots$$
$$E^n f(x) = f(x+nh).$$

The inverse operator E^{-1} is defined by

$$E^{-1} f(x) = f(x-h).$$

Similarly

$$E^{-n} f(x) = f(x-nh).$$

(2) Averaging Operator: Averaging operator μ is defined by the equation

$$\mu f(x) = \tfrac{1}{2}[f(x + h/2) + f(x - h/2)].$$

In the difference calculus, Δ and E are regarded as the fundamental operators and,δ and μ can be expressed in terms of these.

Relations Between the Operators:

1. $\Delta = E - 1$
2. $\nabla = 1 - E^{-1}$
3. $\delta = E^{1/2} - E^{-1/2}$
4. $\mu = \tfrac{1}{2}[E^{1/2} + E^{-1/2}]$
5. $\Delta = E\nabla = \nabla E = \delta E^{1/2}$
6. $E = e^{hD}$.

Example1. Determine the missing values in the following table:

x	45	50	55	60	65
y	3	?	2	?	-2.4

Sol. Let p and q be the missing values in the given table, then the difference table is as follows:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
45	3			
		$p - 3$		
50	p		$5 - 2p$	
		$2 - p$		$3p + q - 9$
55	2		$p + q - 4$	
		$q - 2$		$3.6 - p - 3q$
60	q		$-0.4 - 2q$	
		$-2.4 - q$		
65	-2.4			

Since three entries are given, the function y can be represented by a second degree polynomial.
Therefore, $\Delta^3 y_0 = 0$ and $\Delta^3 y_1 = 0$. Thus $3p + q - 9 = 0$ and $3.6 - p - 3q = 0$.
Solving these equations, we get $p = 2.925$ and $q = 0.225$.

Example2. Determine the missing values in the following table without using difference table.

x	45	50	55	60	65
y	3	?	2	?	-2.4

Sol. Given that $y_0 = 3$, $y_2 = 2$ and $y_4 = -2.4$ and missing values be taken as $y_1= p$ and $y_3 = q$. Since three entries are given, the function y can be represented by a second degree polynomial.
Therefore, $\Delta^3 y_0 = 0$ and $\Delta^3 y_1 = 0$.

$$(E - 1)^3 y_0 = 0 \qquad (E - 1)^3 y_1 = 0$$
$$(E^3 - 3E^2 + 3E - 1)y_0 = 0 \qquad (E^3 - 3E^2 + 3E - 1)y_1 = 0$$
$$y_3 - 3y_2 + 3y_1 - y_0 = 0 \qquad y_4 - 3y_3 + 3y_2 - y_1 = 0$$
$$q - 3(2)+ 3p - 3 = 0 \qquad -2.4 - 3q + 3(2) - p = 0$$
$$3p + q - 9 = 0 \qquad 3.6 - p - 3q = 0.$$

Solving these equations, we get $p = 2.925$ and $q = 0.225$.

Newton's Forward Interpolation Formulae:

Let the function $y=f(x)$ take the values $y_0, y_1, y_2, \ldots$ corresponding to the values $x_0, x_1, x_2, \ldots$ of x. Suppose it is required to evaluate f(x) for $x=x_0+ph$, p is any real number.

For any real number p, we have defined E such that

$E^p f(x) = f(x_0+ph)$

$y_p = f(x_0+ph) = E^p f(x_0)= (1+\Delta)^p y_0$

$= *1+p\Delta+p(p-1)/2!\ \Delta^2 + p(p-1)(p-2)/3!\ \Delta^3 +\ldots+ y_0$

$= y_0 + p\,\Delta y_0 + p(p-1)/2!\,\Delta^2 y_0 + p(p-1)(p-2)/3!\,\Delta^3 y_0 + \ldots$

It is called Newton's forward interpolation formulae.

Newton's Backward Interpolation Formulae:

Suppose it is required to evaluate f(x) for $x=x_n+ph$, where p is any real number.

$E^p f(x) = f(x_n+ph)$

$y_p = f(x_n+ph) = E^p f(x_n) = (1-\nabla)^{-p} y_n$

$= [1+p\nabla + p(p+1)/2!\,\nabla^2 + p(p+1)(p+2)/3!\,\nabla^3 + \ldots + y_n$

$= y_n + p\nabla y_n + p(p+1)/2!\,\nabla^2 y_n + p(p+1)(p+2)/3!\,\nabla^3 y_n + \ldots$

It is called Newton's backward interpolation formulae.

Choice of Newton's Interpolation formulae:

- Newton's forward interpolation formulae is used for interpolating the values of y near the beginning of a set of tabulated values and extrapolating values of y a little backward of y_0.
- Newton's backward interpolation formulae is used for interpolating the values of y near the end of a set of tabulated values and also extrapolating values of y a little ahead of y_n.

Example1. The table gives the distances in nautical miles of the visible horizon for the given heights in feet above the earth's surface :

x=height	100	150	200	250	300	350	400
y=distance	10.63	13.03	15.04	16.81	18.42	19.90	21.27

Find the values of y when (i) x= 218 ft. (ii) x= 410 ft.

Sol. The difference table is

x	**y**	**Δ**	**Δ^2**	**Δ^3**	**Δ^4**
100	10.63				
		2.4			
150	13.03		-0.39		
		2.01		0.15	
x_0=200	**15.04**		-0.24		-0.07
		1.77		0.08	
250	16.81		**-0.16**		-0.05

		1.61		**0.03**	
300	18.42		-0.13		**-0.01**
		1.48		**0.02**	
350	19.90		**-0.11**		
		1.37			
x_n=400	**21.27**				

(i) If we take x_0=200, then y_0=15.04, Δy_0=1.77, $\Delta^2 y_0$=-0.16, $\Delta^3 y_0$=0.03, $\Delta^4 y_0$=-0.01.

Since x=218, step length h=50 and p=(x-x_0)/h =18/50 = 0.36.

By Newton's forward interpolation formula, we have

$y(218) = y_0 + p\,\Delta y_0 + p(p-1)/2!\,\Delta^2 y_0 + p(p-1)(p-2)/3!\,\Delta^3 y_0 + p(p-1)(p-2)(p-3)/4!\,\Delta^4 y_0$

= 15.04 + 0.36 (1.77) + 0.36(0.36-1)/2 (-0.16)+ 0.36(0.36-1)(0.36-2)/6 (0.03)

+ 0.36(0.36-1)(0.36-2)(0.36-3)/24 (-0.01)

=15.04+0.6372+0.0184+0.0018+ 0.00041 = 15.69741

≈ 15.7 nautical miles.

(ii) If we take x_n=400, then y_n=21.27, ∇y_n=1.37, $\nabla^2 y_n$=-0.11, $\nabla^3 y_n$=0.02,

$\nabla^4 y_n$=-0.01.

Since x=410, step length h=50 and p=(x-x_n)/h =10/50 = 0.2.

By Newton's backward interpolation formula, we have

$y(410) = y_n + p\nabla y_n + p(p+1)/2!\,\nabla^2 y_n + p(p+1)(p+2)/3!\,\nabla^3 y_n + p(p+1)(p+2)(p+3)/4!\,\nabla^4 y_n$

= 21.27 + 0.2 (1.37) + 0.2(0.2+1)/2 (-0.11) + 0.2(0.2+1)(0.2+2)/6 (0.02)

$+ 0.2(0.2+1)(0.2+2)(0.2+3)/24\ (-0.01)$

$=21.27+0.274-0.0132+0.0017-0.0007 = 21.5318$

≈ 21.53 nautical miles.

Interpolation with unequal intervals:

The disadvantage for the previous interpolation formulas is that, they are used only for equal intervals. The following are the interpolation with unequal intervals;

1) Lagrange's formula for unequal intervals,
2) Newton's divided difference formula.

Lagrange's interpolation formula: If $y = f(x)$ takes the values $y_0, y_1, y_2, \dots, y_n$ corresponding to $x_0, x_1, x_2, \dots, x_n$, then

$$f(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}y_0 + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)}y_1 + \cdots + \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})}y_n,$$

Which is known as Lagrange's formula.

Divided Differences: If $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$ are given points, then the first divided differences for the argument x_0, x_1 is defined by

$$[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0}.$$

Similarly

$$[x_1, x_2] = \frac{y_2 - y_1}{x_2 - x_1}, [x_2, x_3] = \frac{y_3 - y_2}{x_3 - x_2}, \dots, [x_{n-1}, x_n] = \frac{y_n - y_{n-1}}{x_n - x_{n-1}}.$$

The second divided differences for x_0, x_1, x_2 is

$$[x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0}.$$

The third divided differences for x_0, x_1, x_2, x_3 is

$$[x_0, x_1, x_2, x_3] = \frac{[x_1, x_2, x_3] - [x_0, x_1, x_2]}{x_3 - x_0}.$$

And so on, the nth divided differences for $x_0, x_1, x_2, \dots, x_n$ is

$$[x_0, x_1, x_2, \dots, x_n] = \frac{[x_1, x_2, \dots, x_n] - [x_0, x_1, \dots, x_{n-1}]}{x_n - x_0}.$$

All the divided differences systematically set out in a table called divided difference table.

Value of x	Value of y	1st divided difference	2nd divided difference	3rd divided difference	4th divided difference	5th divided difference
x_0	y_0					
		$[x_0,x_1]$				
x_1	y_1		$[x_0,x_1,x_2]$			
		$[x_1,x_2]$		$[x_0,x_1,x_2,x_3]$		
x_2	y_2		$[x_1,x_2,x_3]$		$[x_0,x_1,x_2,x_3,x_4]$	
		$[x_2,x_3]$		$[x_1,x_2,x_3,x_4]$		$[x_0,x_1,x_2,x_3,x_4,x_5]$
x_3	y_3		$[x_2,x_3,x_4]$		$[x_1,x_2,x_3,x_4,x_5]$	
		$[x_3,x_4]$		$[x_2,x_3,x_4,x_5]$		
x_4	y_4		$[x_3,x_4,x_5]$			
		$[x_4,x_5]$				
x_5	y_5					

Newton's divided difference formula: If y = f(x) takes the values $y_0, y_1, y_2, \dots, y_n$ corresponding to $x_0, x_1, x_2, \dots, x_n$, then

$$f(x) = y_0+(x-x_0)[x_0, x_1] + (x-x_0)(x-x_1)[x_0, x_1, x_2+ + \dots\dots$$
$$+ (x-x_0)(x-x_1)\dots(x-x_{n-1})[x_0, x_1, x_2, \dots, x_n],$$

Which is known as Newton's general interpolation formula with divided differences.

Example1. Given the values

x :	5	7	11	13	17
f(x):	150	392	1452	2366	5202

Evaluate f(9), using

(i) Lagranges formula

(ii) Newton's divided difference formula.

Sol. Let y = f(x), then from the given data, we have

$x_0 = 5$, $x_1 = 7$, $x_2 = 11$, $x_3 = 13$, $x_4 = 17$ and $y_0 = 150$, $y_1 = 392$, $y_2 = 1452$, $y_3 = 2366$, $y_4 = 5202$.

(i) By Lagrnge's interpolation formula

$$f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_n)}y_0+\frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_n)}y_1$$
$$+\frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)}y_2+\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)}y_3$$
$$+\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)}y_4.$$

$$f(9)=\frac{(9-7)(9-11)(9-13)(9-17)}{(5-7)(5-11)(5-13)(5-17)}\times 150+\frac{(9-5)(9-11)(9-13)(9-17)}{(7-5)(7-11)(7-13)(7-17)}\times 392$$
$$+\frac{(9-5)(9-7)(9-13)(9-17)}{(11-8)(11-7)(11-13)(11-17)}\times 1452+\frac{(9-5)(9-7)(9-11)(9-17)}{(13-5)(13-7)(13-11)(13-17)}\times 2366$$
$$+\frac{(9-5)(9-7)(9-11)(9-13)}{(17-5)(17-7)(17-11)(17-13)}\times 5202=-\frac{50}{3}+\frac{3136}{15}+\frac{3872}{3}-\frac{2366}{3}+\frac{578}{5}=810.$$

(iii) The divided difference table is

Value of x	Value of y	1st divided difference	2nd divided difference	3rd divided difference	4th divided difference
5	150				
		121			
7	392		**24**		
		265		**1**	
11	1452		32		**0**
		457		1	
13	2366		42		
		709			
17	5202				

By Newton divided difference formula

$f(x) = y_0+(x-x_0)[x_0, x_1] + (x-x_0)(x-x_1)[x_0, x_1, x_2] +(x-x_0)(x-x_1)(x-x_2)[x_0, x_1, x_2,x_3]$
$+ (x-x_0)(x-x_1)(x-x_2)(x-x_3)[x_0, x_1, x_2,x_3,x_4].$

$$f(9) = 150 + (9-5)\times 121 + (9-5)\,(9-7)\times 24 + (9-5)(9-7)(9-11)\times 1 + (9-5)(9-7)(9-11)(9-13)\times 0$$

$$= 150 + 484 + 192 - 16 + 0$$

$$= 810.$$