Program : **B.Tech**

Subject Name: **Mathematics III**

Subject Code: **BT-401**

Semester: **4th**

Module 5:

Concept of Probability: Probability Mass Function, Probability density function. Discrete Distribution: Binomial, Poisson's. Continuous Distribution: Normal distribution, Exponential distribution.

Concept of Probability

In general:

Probability of an event happening $= \frac{\text{Number of ways it can happen}}{\text{Total number of outcomes}}$

Example: the chances of rolling a "4" with a die

Number of ways it can happen: 1 (there is only 1 face with a "4" on it)

Total number of outcomes: 6 (there are 6 faces altogether)

So the probability $= \frac{1}{6}$

Example: there are 5 marbles in a bag: 4 are blue, and 1 is red. What is the probability that a blue marble gets picked?

Number of ways it can happen: 4 (there are 4 blues)

Total number of outcomes: 5 (there are 5 marbles in total)

So the probability $= \frac{4}{5} = 0.8$

Words:-

Some words have special meaning in Probability:

Experiment or Trial: an action where the result is uncertain.

Tossing a coin, throwing dice, seeing what pizza people choose are all examples of experiments.

Sample Space: all the possible outcomes of an experiment .It is denoted by by capital letter such as S .

Example: choosing a card from a deck

There are 52 cards in a deck (not including Jokers)

So the **Sample Space is all 52 possible cards**: {Ace of Hearts, 2 of Hearts, etc... }

The Sample Space is made up of Sample Points:

Sample Point: just one of the possible outcomes

Example: Deck of Cards

- the 5 of Clubs is a sample point
- the King of Hearts is a sample point

"King" is not a sample point. As there are 4 Kings that is 4 different sample points.

Event: a single result of an experiment

Example Events:

- Getting a Tail when tossing a coin is an event
- Rolling a "5" is an event.

An event can include one or more possible outcomes:

- Choosing a "King" from a deck of cards (any of the 4 Kings) **is** an event

Rolling an "even number" (2, 4 or 6) is also an event

Random Variable:

A real valued function defined on a sample space is called a Random Variable or a Discrete Random Variable.

A Random Variable assumes only a set of real values & the values which variable takes depends on the chance.

For Example:

a) X takes only a set of discrete values 1,2,3,4,5,6.
b) The values which x takes depends on the chance.

The set values 1,2,3,4,5,6 with their probabilities 1/6 is called the **Probability Distribution** of the variate x.

Continuous Random Variable:

When we deal with variates like weights and temperature then we know that these variates can take an infinite number of values in a given interval. Such type of variates are known as **Continuous Random Variable.**

OR

A Variable which is not discrete i.e. which can take infinite number of values in a given interval a≤x≤b, is called **Continuous Random Variable**

Example: Sinx between (0,π) , x is a **Continuous Random Variable.**

Probability Mass Function:

Suppose that *X*: *S* → *A* is a discrete random variable defined on a sample space*S*. Then the probability mass function *p(x)*: *A* → *0, 1+ for *X* is defined as:

a) $P(x_i) \geq 0$,for every i=1,2,3..
b) $\sum_{i=1}^{\infty} p(x_i) = 1$

The sum of probabilities over all possible values of a discrete random variables must be equal to 1.

Thinking of probability as mass helps to avoid mistakes since the physical mass is conserved as is the total probability for all hypothetical outcomes *x*.

- The following exponentially declining distribution is an example of a distribution with an infinite number of possible outcomes—all the positive integers:

$$p(x_i) = \frac{1}{2^i}, \quad i = 1,2,3, \ldots$$

Despite the infinite number of possible outcomes, the total probability mass is 1/2 + 1/4 + 1/8 + ... = 1, satisfying the unit total probability requirement for a probability distribution.

Probability Density Function /Probability Function of Continuous Random Variable:

Let X be a continuous random variable and a function $f(x)$ is a continuous function of X and satisfies the following condition:

a) $f(x) \geq 0, \forall x \in R, -\infty < x < \infty$
b) $\int_{-\infty}^{\infty} f(x)\,dx = 1 \; if \; a \leq x \leq b$

Then the function $f(x)$ is called probability density function of the continuous random variable X.

Moreover, $P(a \leq X \leq b) = \int_a^b f(x)\,dx$ $\quad for\ any\ real\ constants\ a, b\ with\ a \leq b$

Continuous Probability Distribution:

The Probability distribution of continuous random variate is called the continuous probability distribution and it is expressed in terms of probability density function.

Cumulative Distribution Function /Distribution Function of Continuous Random Variable: The probability that the value of a random variate $X$is 'x or less than x' is called the Cumulative distribution function of X and is usually denoted by F(x) and the cumulative distribution functionof a continuous random variable is given by

$$F(x) = P(X \leq x) = P(-\infty < X \leq x) = \int_{-\infty}^{x} f(t)\,dt \qquad , -\infty < x < \infty$$

Some properties of Cumulative Distribution Function:

a) $F(-\infty) = 0 , F(\infty) = 1$
b) $F(x)$ is non-decreasing function
c) For a distribution variate

$$P(a < x < b) = F(b) - F(a)$$

d) F(x) is a discontinuous function for a discontinuous variate and F(x) is continuous function for a continuous variate.

e) If X continuous random variable , then $\frac{dF(x)}{dx} = f(x)$

$$where\ F(x)\ is\ cdf\ and\ f(x)\ is\ pdf.$$

Moreover,

Mean $=\mu'_1 = \int_{-\infty}^{\infty} x f(x)\,dx$, $\mu'_2 = \int_{-\infty}^{\infty} x^2 f(x)\,dx$

$$Variance = \mu'_2 - \mu'^2_1\ and\ S.D. = \sqrt{Variance}$$

Examples:

1) Let X be a random variable with PDF given by

$$f(x) = \begin{cases} cx^2 & |x| \le 1 \\ 0 & otherwise \end{cases}$$

a. Find the constant c

. b. Find *EX* and Var(*X*)

. c. Find *P*(*X*≥12).

Solution: **To find *c*, we can use** $\int_{-\infty}^{\infty} f(x)\,dx$=1:

$$1=\int_{-1}^{1} cx^2 dx$$

$$1=\frac{2}{3}c$$

ThereforeC=$\frac{3}{2}$

To find EX, we can write $\int_{-1}^{1} x f(x)\,dx$=o

In fact, we could have guessed *EX*=0 because the PDF is symmetric around *x*=0.
To find Var(*X*), we have

Var(*X*)

=EX2−(EX)2=EX2

$= \int_{-1}^{1} xf\,x\,dx$

=3/5

To Find P(X≥12):

P(X≥12)=$\frac{3}{2}\int_{\frac{1}{2}}^{1} x^2\,dx$

=7/16.

2) If f(x)=cx^2,0<x<1. Find the value of c and determine the probability that $\frac{1}{3} < x < \frac{1}{2}$

Solution: By property of p.d.f. we have, $\int_0^1 f(x)dx = 1$

So $\int_0^1 cx^2\,dx = 1$, or c $[\frac{x^3}{3}]_0^1 = 1$, so c = 3

Consequently f(x) = $3x^2$: $0 < x < 1$

Again P($\frac{1}{3} < X < \frac{1}{2}$) = $\int_{\frac{1}{3}}^{\frac{1}{2}} 3x^2\,dx = \frac{1}{8} - \frac{1}{27} = \frac{19}{216}$

3) For the distribution $dF = sinx\,dx, 0 \le x \le \pi/2.$ Find Mode and Median.

Solution: Here f(x) = sinx , $0 \le x \le \frac{\pi}{2}$

(a) For Mode: $f'(x) = 0$ & $f''(x) < 0$, $f'(x) = 0 \Rightarrow$ cosx = 0 $\Rightarrow$ x = $\frac{\pi}{2}$&

$[f''(x)]_{x=\frac{\pi}{2}} = -1 < 0$, Hence mode = $\frac{\pi}{2}$

Let M_d be median, then $\int_0^{M_d} sinx\,dx = \frac{1}{2} \Rightarrow M_d = \pi/3$

(b) Mean = $\mu_1' = \int_0^{\frac{\pi}{2}} (x-0)f(x)dx = \int_0^{\frac{\pi}{2}} xsinx\,dx = 1$ &

Variance = $\mu_2 = \int_0^{\frac{\pi}{2}} (x-1)^2 sinx\,dx = \pi - 3$

Continuous random variable – infinite number of values with no gaps between the values. [You might consider drawing a line, the sweeping hand on a clock, or the analog speedometer on a car.]

In this section, we restrict our discussion to discrete probability distributions. Each probability distribution must satisfy the following two conditions.

1 $\sum P(x) = 1$ where x assumes all possible values of the random variable

2 $0 \le P(x) \le 1$ for every value of x

As we found the mean and standard deviation with data in descriptive statistics, we can find the mean and standard deviation for probability distributions by using the following formulas.

1. $\mu = \sum [x \cdot P(x)]$ **mean** of probability distribution

2. $\sigma^2 = \sum [(x-\mu)^2 \cdot P(x)]$ **variance** of probability distribution

3. $\sigma^2 = \sum [x^2 \cdot P(x)] - \mu^2$ **variance** of probability distribution

4. $\sigma = \sqrt{\left[\sum [x^2 \cdot P(x)] - \mu^2\right]}$ **standard deviation** of probability distribution

Theoretical Distributions :
Definition : When frequency distribution of some universe are not based on actual observation or experiments , but can be derived mathematically from certain predetermined hypothesis , then such distribution are said to be theoretical distributions.

Types of Theoretical Distributions: Following two types of Theoretical Distributions are usually used in statistics:

1) Discrete Probability Distribution
   a) Binomial Distribution
   b) Poisson Distribution
2) Continuous Probability Distribution

Normal Distribution

Binomial Distribution:

1. The procedure has a **fixed number of trials**. [n trials]
2. The trials must be **independent**.
3. Each trial is in **one oftwo mutually exclusive categories**.
4. The **probabilities remain constant** for each trial.

Notations:

P(success) = P(S) = p probability of success in one of the n trials
P(failure) = P(F) = 1 – p = q probability of failure in one of the n trials

n = fixed number of trials; x = number of successes, where $0 \leq x \leq n$

P(x) = probability of getting exactly x successes among the n trials

P(x ≤ a) = probability of getting x-values less than or equal to the value of a.

P(x ≥ a) = probability of getting x-values greater than or equal to the value of a.

NOTE: Success (failure) does not necessarily mean good (bad).

Formula for Binomial Probabilities: $P(x) = \dfrac{n!}{(n-x)!x!} p^x q^{n-x} \quad for\ x = 0, 1, 2, \ldots, n$

Factorial definition: n! = n(n – 1)(n – 2)···2·1; 0! = 1; 1! = 1

Example (Formula): Find the probability of 2 successes of 5 trials when the probability of success is 0.3.

$$P(x=2) = \frac{5!}{(5-2)!2!} 0.3^2 0.7^{5-2} = \frac{5 \cdot 4 \cdot 3!}{3! \cdot 2!}(0.09)(0.343) = 10(0.03087) = 0.3087$$

Moment about the origin:

1) **First moment about the origin:**

$$\mu'_{1} = {}_{r=0}^{n} r.(nCr)p^{r} q^{n-r}$$

$$= np$$

2) Second moment about the origin:

$$\mu'_{2} = {}_{r=0}^{n} r^{2}.(nCr)p^{r} q^{n-r}$$

$$= npq + n^2p^2$$

Moment about the Mean:

1) First moment about the mean is 0.
2) Second moment about the mean or variance is given by=npq

Standard deviation= $\sqrt{npq}$

Mean and Variance of a binomial distribution:

Mean = $\mu = \sum x\, p(x)$

But for Binomial distribution

$\mu = \sum x\, p(x)$

$= \sum x\, nC_x\, p^{x}\, q^{n-x}$

$= 0.\, nC_0\, p^{0} q^{n-0} + 1.\, nC_1\, p^{1}\, q^{n-1} + 2.\, nC_2\, p^{2}\, q^{n-2} + \ldots.. + n.nC_n p^{n} q^{n-n}$

$= np^{1}\, q^{n-1} + n(n-1)p^{2}\, q^{n-2} + \ldots.. + n\, p^{n}$

$= np(q^{n-1} + (n-1)pq^{n-2} + \ldots.. + p^{n-1})$

$= np(\, q + p\,)^{n-1}$

$= np$

Mean = μ = np

Variance = Second moment about origin-(First moment about origin $)^2$

$= \sum x^{2}\, p(x) - \mu^{2}$

$= 0.\ {}^nC_0\, p^0 q^{n-0} + 1.\ {}^nC_1\, p^1 q^{n-1} + 4.\ {}^nC_2\, p^2 q^{n-2} + \ldots.. + n^2.{}^nC_n p^n q^{n-n} - (n^2p^2)$

$= np^1 q^{n-1} + 2n(n-1)p^2 q^{n-2} + 3/2n(n-1)(n-2)\, p^3q^{n-3} \ldots.. + n^2p^n - n^2p^2$

$= np\,(q^{n-1} + 2(n-1)pq^{n-2} + 3/2\,(n-1)(n-2)\, p^2q^{n-3} \ldots.. + n\,p^{n-1}) - n^2p^2$

$= np[\,(q^{n-1} + (n-1)pq^{n-2} + 1/2\,(n-1)(n-2)\, p^2q^{n-3} \ldots.. + p^{n-1})$

$+((n-1)pq^{n-2} + 1\,(n-1)(n-2)\, p^2q^{n-3} \ldots.. + (n-1)\, p^{n-1})] - n^2p^2$

$= np[(q+p)^{n-1} + (n-1)p\,(q^{n-2} + (n-2)\, pq^{n-3} \ldots.. + p^{n-2})] - n^2p^2$

$= np\,[\,(q+p)^{n-1} + (n-1)p(q+p)^{n-2}] - n^2p^2$

$= np\,[\,(q+p)^{n-1} + (n-1)p(q+p)^{n-2}] - n^2p^2$

$= np\,(1 + (n-1)p) - n^2p^2$

$= np + n^2p^2 - np^2 - n^2p^2$

$= np(1-p)$

$= npq$

Examples:

1) Six dice are thrown 729 times. How many times do you expect at least three dice to show a five or six?

Solution: We know that when a die is thrown , the probability to show a 5 or 6 = 2/6 = 2/3 = p (say)

q= 1-p = 1- (1/3) = 2/3

The probability to show a 5 or 6 in at least 3 dice

$= \sum_{x=3}^{6} p(x)$ = p(3) + p(4) + p(5) +p(6), where p(x) is the probability to show 5 or 6

$= {}^6C_3\, q^3p^3 + {}^6C_4\, p^4q^2 + {}^6C_5\, p^5q + {}^6C_6\, p^6 = \frac{233}{729}$ = p (say)

SO the required no. = np = 233

2) The mean and variance of a binomial variate are 16 & 8. Find i) P (X= 0)

ii) P(X ≥ 2)

Mean = np = 16

Variance = npq = 8

⇨ q = 8/16 = 1/2

p = 1 – q = ½

np = 16 ie, n = 32

i) $P (X = 0) = nC_0\, p^0 q^{n-0}$

$= \mathbf{(½)}^0 (1/2)^{32}$

$= (1/2)^{32}$

ii)P(X ≥ 2) = 1 – P (X < 2)

= 1 – P(X = 0 ,1)

= 1 – P(X = 0) – P(X = 1)

$= 1- 33\ (1/2)^{32}$

2) Six dice are thrown 729 times.Howmany times do you expect atleast 3 dice to show a 5 or 6 ?

Solution : Here n = 6 ,N = 729

$P(x \geq 3) = 6C_x p^x q^{n-x}$

Let p be the probability of getting 5 or 6 with 1 dice

ie, p = 2/6 = 1/3

$q = 1 - 1/3 = 2/3$

$P(x \ge 3) = P(x = 2,3,4,5,6)$

$= p(x=3)+p(x=4)+p(x=5)+p(x=6)$

$=0.3196$

number of times = 729*0.3196 = 233

3) A basket contains 20 good oranges and 80 bad oranges . 3 oranges are drawn at random from this basket . Find the probability that out of 3 i) exactly 2 ii)atleast 2 iii)atmost 2 are good oranges.

Solution: Let p be the probability of getting a good orange

ie, $p = \dfrac{80C_1}{100C_1}$

$= 0.8$

$q = 1- 0.8 = 0.2$

i) $p(x=2) = 3C2\,(0.8)^2(0.2)^1 = 0.384$

ii) $p(x\ge 2) = P(2) + p(3)= 0.896$

iii) $p(x\le 2) = p(0) + p(1) + p(2) = 0.488$

5) In a sampling a large number of parts manufactured by a machine , the mean number of defective in a sample of 20 is 2. Out of 1000 such samples howmany would expected to contain atleast 3 defective parts.

$n=20 \quad n\,p = 2$

ie , $p=1/10 \qquad q = 1-p = 9/10$

$p(x \ge 3) = 1 - p(x < 3)$

$= 1 - p(x = 0,1,2) = 0.323$

Number of samples having at least 3 defective parts = 0.323 * 1000

= 323

The process of determining the most appropriate values of the parameters from the given observations and writing down the probability distribution function is known as fitting of the binomial distribution.

Problems

1) Fit an appropriate binomial distribution and calculate the theoretical distribution

x :	0	1	2	3	4	5
f :	2	14	20	34	22	8

Here n = 5 , N = 100

Mean = $\frac{x_i f_i}{f_i}$ = 2.84

np = 2.84

p = 2.84/5 = 0.568

q = 0.432

$p(r) = 5C_r\,(0.568)^r\,(0.432)^{5-r}$, r = 0,1,2,3,4,5

Theoretical distributions are

r	p(r)	N* p(r)
0	0.0147	1.47 = 1

1	0.097	9.7 =10
2	0.258	25.8 =26
3	0.342	34.2 =34
4	0.226	22.6 =23
5	0.060	6 =6
		Total = 100

Poisson Distribution :.

The **Poisson distribution** is a discrete distribution. It is often used as a model for the number of events (such as the number of telephone calls at a business, number of customers in waiting lines, number of defects in a given surface area, airplane arrivals, or the number of accidents at an intersection) in a specific time period.

The mean is λ.The variance is λ.

Therefore the P.D. is given by

$P\,r = \frac{e^{-m} m^r}{r!}$ where r=0,1,2,3…

m is the parameter which indicates the average number of events in the given time interval.

Mean and Variance of a Poisson distribution:-

1. **Mean :-**

The mean of Poisson distribution is

$$\mu = \sum_{r=0}^{\infty} r\,\mathrm{P}\,\mathrm{r}, \lambda$$

$$= \sum_{r=0}^{\infty} r \frac{\lambda^r e^{-\lambda}}{r\,!}$$

$$= e^{-\lambda} \sum_{r=0}^{\infty} r \frac{\lambda^r}{r!}$$

$$= e^{-\lambda}\left[0 + \frac{1}{1!}\lambda + \frac{2}{2!}\lambda^2 + \cdots \ldots \ldots \ldots .. + \frac{r}{r!}\lambda^r + \cdots \ldots \ldots ..\right]$$

$$= \lambda\, e^{-\lambda}\, [1 + \frac{1}{1!}\lambda + \frac{1}{2!}\lambda^2 + \cdots \ldots \ldots \ldots .. + \frac{1}{(r-1)!}\lambda^r + \cdots \ldots \ldots ..]$$

$$\therefore \mu = \lambda\, e^{-\lambda}\, e^{\lambda} = \lambda = np$$

The mean and expected value of Poisson distribution are same.

2. Variance:-

$$\sigma^2 = \sum_{r=0}^{\infty} r - \mu^{\,2}\, \mathrm{P}\, \mathrm{r}, \lambda$$

$$= \sum_{r=0}^{\infty} r^2 - 2\mu r + \mu^2\, \mathrm{P}\, \mathrm{r}, \lambda$$

$$= \sum_{r=0}^{\infty} r^2\, \mathrm{P}\, \mathrm{r}, \lambda - 2\mu \sum_{r=0}^{\infty} r\, \mathrm{P}\, \mathrm{r}, \lambda + \mu^2 \sum_{r=0}^{\infty} \mathrm{P}\, \mathrm{r}, \lambda$$

$$= \sum_{r=0}^{\infty} r^2\, \mathrm{P}\, \mathrm{r}, \lambda - 2\mu\, .\, \mu + \mu^2 .\, 1$$

$$= \sum_{r=0}^{\infty} r^2\, \mathrm{P}\, \mathrm{r}, \lambda - \mu^2$$

$$= \sum_{r=0}^{\infty} r^2 \frac{\lambda^r\, e^{-\lambda}}{r!} - \lambda^2$$

$$= e^{-\lambda} \sum_{r=0}^{\infty} r^2 \frac{\lambda^r}{r!} - \lambda^2$$

$$= e^{-\lambda}\, 0 + \frac{1}{1!}\lambda + 2\frac{2}{2!}\lambda^2 + \frac{3^2}{3!}\lambda^3 + \cdots \ldots \ldots \ldots .. + \frac{r^2}{r!}\lambda^r + \cdots \ldots \ldots .. - \lambda^2$$

$$= \lambda e^{-\lambda}\, 1 + \frac{1}{1!}\lambda + \frac{\lambda^2}{2!} + \cdots . + \frac{1}{1!}\lambda + \frac{2\lambda^2}{2!} + \cdots . - \lambda^2$$

$$= \lambda e^{-\lambda}\, e^{\lambda} + \lambda e^{\lambda}\; - \lambda^2 = \lambda + \lambda^2 - \lambda^2 = \lambda = np$$

Note: Events, which are extremely rare but have a large number of independent opportunities, for occurrence are found to obey the law of Poisson probability distribution satisfactorily. Emission and disintegration of radioactive rays, number of occurrences of rare diseases etc. are phenomena of this nature**.**

Poisson distribution examples

1. The number of road construction projects that take place at any one time in a certain city follows a Poisson distribution with a mean of 3. Find the probability that exactly five road construction projects are currently taking place in this city. (0.100819)

2. The number of road construction projects that take place at any one time in a certain city follows a Poisson distribution with a mean of 7. Find the probability that more than four road construction projects are currently taking place in the city. (0.827008)

3. The number of traffic accidents that occur on a particular stretch of road during a month follows a Poisson distribution with a mean of 7.6. Find the probability that less than three accidents will occur next month on this stretch of road. (0.018757)

4. The number of traffic accidents that occur on a particular stretch of road during a month follows a Poisson distribution with a mean of 7. Find the probability of observing exactly three accidents on this stretch of road next month. (0.052129)

5. The number of traffic accidents that occur on a particular stretch of road during a month follows a Poisson distribution with a mean of 6.8. Find the probability that the next two months will both result in four accidents each occurring on this stretch of road. (0.009846)

Example-1:In a certain factory turning razor blades, there is a small chance (1/500) for any blade to be defective. The blades are in packets of 10. Use Poisson's distribution to calculate the approximate number of packets containing no defective, one defective and two defective blades respectively in

a consignment of 10,000 packets.
Solution: Here p = 1/500, n = 10 , N = 10,000 so m = np = 0.02

Now $e^{-m} = e^{-0.02} = 0.9802$

The respective frequencies containing no defective, 1 defective & 2 defective blades are given

As follows

Ne^{-m} , Ne^{-m} .m, N$e^{-m} \cdot \frac{1}{2} m^2$

i.e. 9802 ; 196; 2

Example 2: Fit a Poisson distribution to the set of observations

$x = r$	0	1	2	3	4
f	122	60	15	2	1

Solution: Let m be mean of Poisson distribution,

Then $m = \frac{\sum_{i=0}^{n} f_i r_i}{\sum_{i=0}^{n} f_i} = \frac{\sum_{i=0}^{4} f_i r_i}{\sum_{i=0}^{4} f_i} = \frac{0+60.1+15.2+2.3+1.4}{122+60+15+2+1} = 0.05$

And $N = \sum f = 200$

Hence ,the theoretical frequency distribution for r successes is

$N P(r, m) = N P(r) = N \frac{e^{-m} m^r}{r!} = 200 \frac{e^{-0.05} (0.05)^r}{r!}$ for $r = 0$ *to* 4

Therefore

$$N P(0) = 200 \frac{e^{-0.05}}{0!} (0.05)^0 = 200 \text{ X } 0.61 = 122$$

$$N P(1) = 200 \frac{e^{-0.05}}{1!} (0.05)^1 = 61$$

$$N P(2) = 200 \frac{e^{-0.05}}{2!} (0.05)^2 = 15$$

$$N P(3) = 200 \frac{e^{-0.05}}{3!} (0.05)^3 = 3$$

$$N P 4 = 200 \frac{e^{-0.05}}{4!}(0.05)^4 = 0$$

Hence ,the above theoretical frequency distribution is shown below

$x = r$	0	1	2	3	4
f	122	61	15	3	0

Normal Distribution:

The normal (or Gaussian) distribution is a continuous probability distribution that frequently occurs in nature and has many practical applications in statistics.

Characteristics of a normal distribution

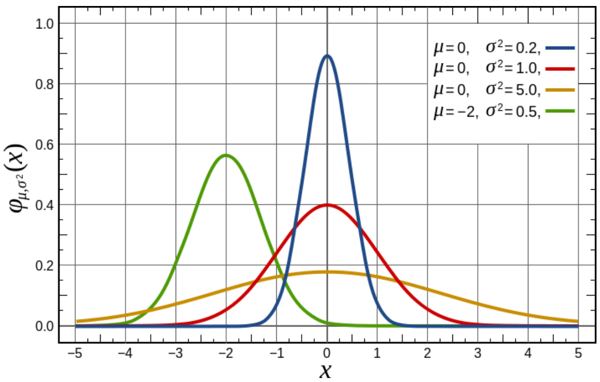
- Bell-shaped appearance
- Symmetrical
- Unimodal
- Mean = Median = Mode
- Described by two parameters: mean (μ_x) and standard deviation (σ_x)

- The normal distribution is described by the following formula:

$$f(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad -\infty < x < \infty$$

where the function f(x) defines the probability density associated with X = x. That is, the above formula is a probability density function

Because μ_x and $\sigma_x$can have infinitely many values, it follows there are infinitely many normal distributions:



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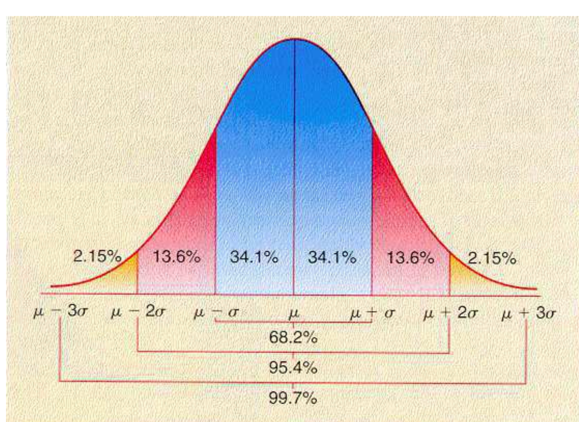
A standard normal distribution is a normal distribution rescaled to have μ_x = 0 and σ_x = 1. The *pdf* is:

$$f(z;0,1) = \frac{1}{\sqrt{2\pi}} e^{\frac{z^2}{2}} \quad \infty < x < \infty$$

The ordinate of the standard normal curve is no longer called *x*, but *z*.

For a normal curve, approximately 68.2%, 95.4%, and -99.7% of the observations fall within 1, 2, and 3 standard deviations of the mean, respectively.



Areas Under the Normal Curve

By standardizing a normal distribution, we eliminate the need to consider μ_x and σ_x; we have a standard frame of reference.

Areas Under the Standard Normal Curve

X (x values) of a normal distribution map into Z (z-values) of a standard normal distribution with a 1-to-1 correspondence.

If X is a normal random variable with mean μ_x and σ_x, then the standard normal variable (normal deviate) is obtained by:

$$z = \frac{x - \mu_x}{\sigma_x}$$

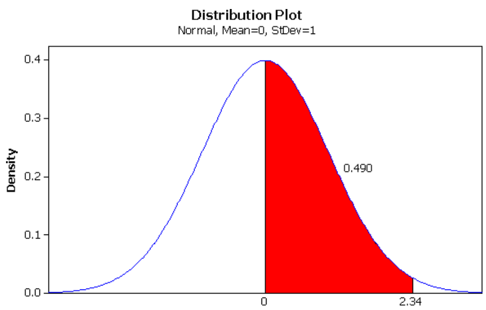
Example 1: What is the probability that Z falls z = 1.11 and z = 2.34?

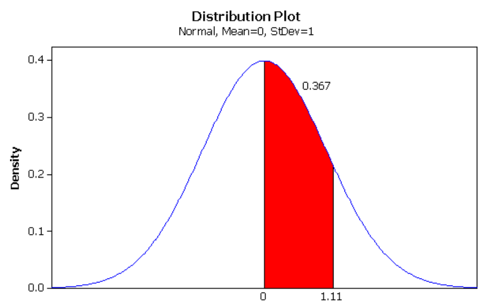
Pr(1.11 <z< 2.34)

= area from z = 2.34 to z = 1.11

= area from (– to z = 2.34) minus area from (- to z = 1.11)

= .9904 – .8665 = .1239





Note: figures above should also shade region from –inf to 0.

Table of the standard normal distribution values ($z \leq 0$)

$-z$	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.49601	0.49202	0.48803	0.48405	0.48006	0.47608	0.47210	0.46812	0.46414
0.1	0.46017	0.45621	0.45224	0.44828	0.44433	0.44038	0.43644	0.43251	0.42858	0.42466
0.2	0.42074	0.41683	0.41294	0.40905	0.40517	0.40129	0.39743	0.39358	0.38974	0.38591
0.3	0.38209	0.37828	0.37448	0.37070	0.36693	0.36317	0.35942	0.35569	0.35197	0.34827
0.4	0.34458	0.34090	0.33724	0.33360	0.32997	0.32636	0.32276	0.31918	0.31561	0.31207
0.5	0.30854	0.30503	0.30153	0.29806	0.29460	0.29116	0.28774	0.28434	0.28096	0.27760
0.6	0.27425	0.27093	0.26763	0.26435	0.26109	0.25785	0.25463	0.25143	0.24825	0.24510
0.7	0.24196	0.23885	0.23576	0.23270	0.22965	0.22663	0.22363	0.22065	0.21770	0.21476
0.8	0.21186	0.20897	0.20611	0.20327	0.20045	0.19766	0.19489	0.19215	0.18943	0.18673

0.9	0.18406	0.18141	0.17879	0.17619	0.17361
	0.17106	0.16853	0.16602	0.16354	0.16109
1.0	0.15866	0.15625	0.15386	0.15151	0.14917
	0.14686	0.14457	0.14231	0.14007	0.13786
1.1	0.13567	0.13350	0.13136	0.12924	0.12714
	0.12507	0.12302	0.12100	0.11900	0.11702
1.2	0.11507	0.11314	0.11123	0.10935	0.10749
	0.10565	0.10384	0.10204	0.10027	0.09853
1.3	0.09680	0.09510	0.09342	0.09176	0.09012
	0.08851	0.08692	0.08534	0.08379	0.08226
1.4	0.08076	0.07927	0.07780	0.07636	0.07493
	0.07353	0.07215	0.07078	0.06944	0.06811
1.5	0.06681	0.06552	0.06426	0.06301	0.06178
	0.06057	0.05938	0.05821	0.05705	0.05592
1.6	0.05480	0.05370	0.05262	0.05155	0.05050
	0.04947	0.04846	0.04746	0.04648	0.04551
1.7	0.04457	0.04363	0.04272	0.04182	0.04093
	0.04006	0.03920	0.03836	0.03754	0.03673
1.8	0.03593	0.03515	0.03438	0.03363	0.03288
	0.03216	0.03144	0.03074	0.03005	0.02938
1.9	0.02872	0.02807	0.02743	0.02680	0.02619
	0.02559	0.02500	0.02442	0.02385	0.02330

2.0	0.02275	0.02222	0.02169	0.02118	0.02068
	0.02018	0.01970	0.01923	0.01876	0.01831
2.1	0.01786	0.01743	0.01700	0.01659	0.01618
	0.01578	0.01539	0.01500	0.01463	0.01426
2.2	0.01390	0.01355	0.01321	0.01287	0.01255
	0.01222	0.01191	0.01160	0.01130	0.01101
2.3	0.01072	0.01044	0.01017	0.00990	0.00964
	0.00939	0.00914	0.00889	0.00866	0.00842
2.4	0.00820	0.00798	0.00776	0.00755	0.00734
	0.00714	0.00695	0.00676	0.00657	0.00639
2.5	0.00621	0.00604	0.00587	0.00570	0.00554
	0.00539	0.00523	0.00509	0.00494	0.00480
2.6	0.00466	0.00453	0.00440	0.00427	0.00415
	0.00403	0.00391	0.00379	0.00368	0.00357
2.7	0.00347	0.00336	0.00326	0.00317	0.00307
	0.00298	0.00289	0.00280	0.00272	0.00264
2.8	0.00256	0.00248	0.00240	0.00233	0.00226
	0.00219	0.00212	0.00205	0.00199	0.00193
2.9	0.00187	0.00181	0.00175	0.00170	0.00164
	0.00159	0.00154	0.00149	0.00144	0.00140
3.0	0.00135	0.00131	0.00126	0.00122	0.00118
	0.00114	0.00111	0.00107	0.00104	0.00100

3.1	0.00097	0.00094	0.00090	0.00087	0.00085
	0.00082	0.00079	0.00076	0.00074	0.00071
3.2	0.00069	0.00066	0.00064	0.00062	0.00060
	0.00058	0.00056	0.00054	0.00052	0.00050
3.3	0.00048	0.00047	0.00045	0.00043	0.00042
	0.00040	0.00039	0.00038	0.00036	0.00035
3.4	0.00034	0.00033	0.00031	0.00030	0.00029
	0.00028	0.00027	0.00026	0.00025	0.00024
3.5	0.00023	0.00022	0.00022	0.00021	
	0.00020	0.00019	0.00019	0.00018	
	0.00017	0.00017			

Table of the standard normal distribution values ($z \geq 0$)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327

0.9	0.81594	0.81859	0.82121	0.82381	0.82639
	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083
	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286
	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251
	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988
	0.91149	0.91308	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507
	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822
	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950
	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907
	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712
	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381
	0.97441	0.97500	0.97558	0.97615	0.97670

2.0	0.97725	0.97778	0.97831	0.97882	0.97932
	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382
	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745
	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036
	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266
	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446
	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585
	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693
	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774
	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836
	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882
	0.99886	0.99889	0.99893	0.99896	0.99900

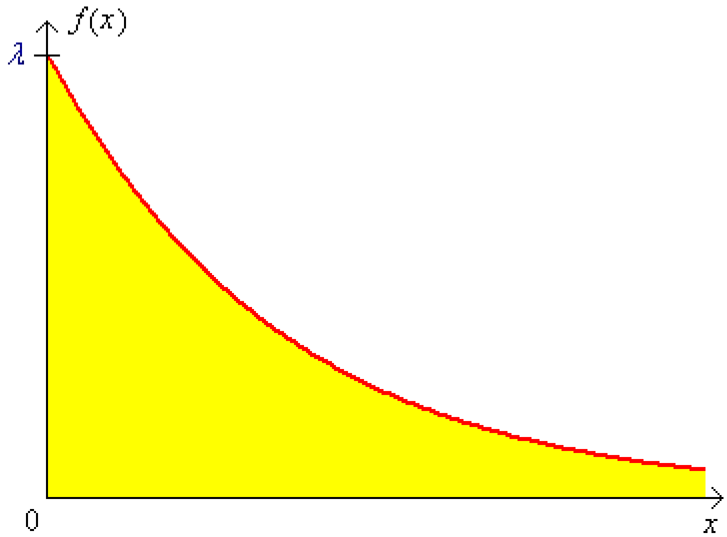
3.1	0.99903	0.99906	0.99910	0.99913	0.99916
	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940
	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958
	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971
	0.99972	0.99973	0.99974	0.99975	0.99976
3.5	0.99977	0.99978	0.99978	0.99979	
	0.99980	0.99981	0.99981	0.99982	
	0.99983	0.99983			

The Exponential Distribution:

This continuous probability distribution often arises in the consideration of lifetimes or waiting times and is a close relative of the discrete Poisson probability distribution.

The probability density function is

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & (x \geq 0) \\ 0 & (x < 0) \end{cases}$$



The cumulative distribution function is

$$F(x) = P[X \le x] = \int_{-\infty}^{x} f(t)\, dt = 0 + \left[-e^{-\lambda t}\right]_0^x = 1 - e^{-\lambda x} \quad (x \ge 0)$$

$$\Rightarrow \qquad P[X > x] = e^{-\lambda x} \quad (x \ge 0)$$

Also $\mu = E[X] = \dfrac{1}{\lambda}$ and $\sigma = \mu$

Reason:

$$\mu = 0 + \int_0^{\infty} x \cdot \lambda e^{-\lambda x}\, dx$$

$$= \left[\frac{-(\lambda x + 1) e^{-\lambda x}}{\lambda}\right]_0^{\infty} = \frac{1}{\lambda}$$

V[X] = E[X^2+ – (E*X])^2

$$\mathrm{E}\left[X^2\right] = \int_0^\infty x^2 \cdot \lambda\, e^{-\lambda x}\, dx = \ldots$$

OR

$$\sigma^2 = \int_0^\infty \left(x - \frac{1}{\lambda}\right)^2 \lambda\, e^{-\lambda x}\, dx$$

$$= \ldots = \frac{1}{\lambda^2}$$

Example:

The random quantity *X* follows an exponential distribution with parameter $\lambda = 0.25$.

Find μ , σ and P[*X*> 4] .

$$\mu = \sigma = \frac{1}{\lambda} = \frac{1}{.25} = \underline{\underline{\mathbf{4}}}$$

$$P[X > 4] = e^{-\lambda x} = e^{-\frac{1}{4}\times 4} = e^{-1} = .367879\ldots$$

$$\approx \underline{\underline{.368}}$$

Note: For *any* exponential distribution, P[$X>\mu$] ≈ .368 .

Example:

The waiting time *T* for the next customer follows an exponential distribution with a mean waiting time of five minutes. Find the probability that the next customer waits for at most ten minutes.

$$\lambda = \frac{1}{\mu} = \frac{1}{5} = .2$$

$$P[T \le 10] = F(10) = 1 - P[T > 10] = 1 - e^{-\frac{1}{5}\times 10}$$

$$= 1 - e^{-2} = 1 - .135335\ldots$$

∴ P[$T \le 10$+ ≈ $\underline{\underline{.865}}$

Note:

$$P[X>\mu + 2\sigma] = e^{-\lambda(\mu + 2\sigma)} = e^{-\lambda((1/\lambda)+(2/\lambda))} = e^{-3} = .049787$$

Therefore P[$X>\mu + 2\sigma$] ≈ 5.0% for <u>all</u> exponential distributions.

Also $\mu - \sigma = \frac{1}{\lambda} - \frac{1}{\lambda} = 0 \Rightarrow P[X < \mu - \sigma] = 0 = P[X < \mu - 2\sigma]$

Therefore $P[\,|X - \mu| > 2\sigma\,] \approx 5.0\%$, a result similar to the normal distribution, except that *all* of the probability is in the upper tail only.
