Program : **B.Tech**

Subject Name: **Mathematics III**

Subject Code: **BT-401**

Semester: **4th**

Module -4

Transform Calculus

Contents

Laplace Transform, Laplace Transform of elementary function, Properties of Laplace Transform,Change of Scale property , First and Second Shifting Properties , Laplace Transform of periodic functions , Laplace Transform of Derivatives and Integrals, Inverse Laplace Transform and Its Properties, Convolution Theorem, Application of Laplace Transform in Solving the Ordinary Differential Equations. Fourier transforms.

1. Motivation: Laplace transform a very powerful technique is that it replaces operations of calculus by operations of algebra. Laplace transform is an integral transform method which is particularly useful in solving linear ordinary differential equations. It finds very wide applications in various areas of physics, electrical engineering, control engineering, optics, mathematics and signal processing .Laplace transforms help in solving complex problems with a very simple approach.

2. Prerequisite:

Function, the concept of limit, continuity, ordinary derivative of function, rules and formulae of differentiation and integration of function of one independent variable.

3. Objective: The Laplace transform method solves differential equationsand corresponding initial and boundary value problems. The Laplace transforms reduce the problem of solving a differential equation to an algebraic problem. It is also useful in problems where the mechanical or electrical driving force has discontinuities, is impulsive or is a complicated periodic function.

The Laplace transform also has the advantage that it solves problems directly,initial and boundary value problems without determining a general solution.

4. Key Notations:

(1) $L\{f(t)\}$:Laplace transform of a function

(2) $L^{-1}\{f(t)\}$:Inverse Laplace transform of a function

5. Key Definitions:

(1) LAPLACE TRANSFORM: Let $f(t)$ be a function defined for all positive values of t, then

$\phi(s) = \int_0^{\infty} e^{-st} f(t)\,dt$ provided the integral exists, is called the Laplace Transform of $f(t)$.

It is denoted as

$$L\{f(t)\} = \phi(s) = \int_0^{\infty} e^{-st} f(t)\,dt$$

(2) INVERSE LAPLACE TRANSFORM: If $L\{f(t)\} = \phi(s) = \int_0^{\infty} e^{-st} f(t)\,dt$ then $f(t)$ is called the

Inverse Laplace transform of $\phi(s)$.

It is denoted as $L^{-1}\left[\phi(s)\right] = f(t)$.

6. Important Formulae/ Theorems / Properties:

LAPLACE TRANSFORM:

STANDARD FORMULAE:

1) $L(e^{at}) = \dfrac{1}{s-a} \quad (s > a)$

2) $L(1) = \dfrac{1}{s} \quad (s > 0)$

3) $L(\sin at) = \dfrac{a}{s^2 + a^2}$

4) $L(\cos at) = \dfrac{s}{s^2 + a^2}$

5) $L(\sinh at) = \dfrac{a}{s^2 - a^2} \quad (s > |a|)$

6) $L(\cosh at) = \dfrac{s}{s^2 - a^2} \quad (s > |a|)$

7) $L(t^n) = \dfrac{\overline{\left\lfloor n+1 \right.}}{s^{n+1}}$

7. SAMPLE PROBLEMS:

I.Exercise can be solved based on following sample problem.

LAPLACE TRANSFORM BY DEFINITION:

Ex. Find the Laplace transform of

$$f(t) = \begin{cases} \cos t & for\ 0 < t < \pi \\ \sin t & for\ t > \pi \end{cases}$$

Solution: By the definition of Laplace transform we have,

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt = \int_0^\pi e^{-st} \cos t\, dt = + \int_\pi^\infty e^{-st} \sin t\, dt$$

$$But \int e^{ax} \cos bx\, dx = \frac{e^{ax}}{(a^2+b^2)}[a\cos bx + b\sin bx]$$

$$\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{(a^2+b^2)}[a\sin bx - b\cos bx]$$

$$L[f(t)] = \frac{1}{s^2+1}\left[e^{-st}(-s\cos t + \sin t)\right]_0^\pi + \frac{1}{s^2+1}\left[e^{-st}(-s\sin t - \cos t)\right]_\pi^\infty$$

$$= \frac{1}{s^2+1}\left[e^{-s\pi}(s) - (-s)\right] + \frac{1}{s^2+1}\left[-e^{-s\pi}\right]$$

$$= \frac{1}{s^2+1}\left[s + (s-1)e^{-s\pi}\right]$$

Unsolved Problem

Find the Laplace transform of following functions.

1) $f(t) = (t-2)^2 \quad t > 2, \quad f(t) = 0 \quad 0 < t < 2 \; Ans: \frac{2}{s^3}e^{-2s}$

2) $f(t) = t \quad 0 < t < a$

$= b \quad t > a \; Ans: \frac{1}{s^2} + \left[\frac{(b-a)}{s} - \frac{1}{s^2}\right]e^{-as}$

3) $f(t) = t,\ 0 < t < 3$
$= 6,\ t > 3 \quad Ans: \frac{1}{s^2} + \left[\frac{3}{s} - \frac{1}{s^2}\right]e^{-3s}$

II Exercise can be solved based on following sample problem.

LAPLACE TRANSFORM BY LINEARITY PROPERTY:

$$L\{k_1 f(t)+k_2 g(t)\}=k_1 L\{f(t)\}+k_2 L\{g(t)\}$$

Ex. Find the Laplace transform of $\sin(\omega t+\alpha)$

Solution: By linearity property, we have

$$\begin{aligned}L\left[\sin(\omega t+\alpha)\right]&=L\left[\sin\omega t\cos\alpha+\cos\omega t\sin\alpha\right]\\&=\cos\alpha\, L(\sin\omega t)+\sin\alpha\, L(\cos\omega t)\\&=\cos\alpha\frac{\omega}{s^2+\omega^2}+\sin\alpha\frac{s}{s^2+\omega^2}\end{aligned}$$

Unsolved Problem

Find the Laplace transform of following functions.

1) $t^2-e^{-2t}+\cosh^2 3t\quad Ans:\frac{2}{s^3}-\frac{1}{(s+2)}+\frac{1}{2}\left[\frac{1}{s}+\frac{s}{s^2-6^2}\right]$

2) $(\sin 2t-\cos 2t)^2\quad Ans:\frac{1}{s}-\frac{4}{s^2+4^2}$

3) $\cos(wt+b)\quad Ans:\frac{s}{s^2+w^2}\cos b-\frac{w}{s^2+w^2}\sin b$

4) $\sin(5t+3)\quad Ans:\frac{5}{s^2+5^2}\cos 3+\frac{s}{s^2+5^2}\sin 3$

5) $\cos t\cos 2t\cos 3t\quad Ans:\frac{1}{4}\left[\frac{1}{s}+\frac{s}{s^2+2^2}+\frac{s}{s^2+4^2}+\frac{s}{s^2+6^2}\right]$

6) $\sin^5 t\quad Ans:\frac{5!}{(s^2+1)(s^2+9)(s^2+25)}$

CHANGE OF SCALE PROPERTY:

If $L[f(t)] = \phi(s)$ then $L[f(at)] = \frac{1}{a}\phi\left(\frac{s}{a}\right)$

Ex. If $L[f(t)] = \log\left(\frac{s+3}{s+1}\right)$, find $L[f(2t)]$.

Solution: By change of scale property, we have

$$L[f(2t)] = \frac{1}{2}\phi\left(\frac{s}{2}\right)$$

$$L[f(2t)] = \frac{1}{2}\log\left[\frac{\left(\frac{s}{2}\right)+3}{\left(\frac{s}{2}\right)+1}\right]$$

$$= \frac{1}{2}\log\left(\frac{s+6}{s+2}\right)$$

Unsolved Problem

If $L[f(t)] = \phi(s)$ **then** $L[f(at)] = \frac{1}{a}\phi\left(\frac{s}{a}\right)$

1) Find $L[f(2t)]$ if $L[f(t)] = \log\left(\frac{s+3}{s+1}\right)$ $Ans: \frac{1}{2}\log\frac{(s+6)}{(s+2)}$

2) Find $L[(erf\, 2\sqrt{t})]$, If $L\left\{\text{ert}\sqrt{t} = \frac{1}{s\sqrt{s+1}}\right\}$ $Ans: \frac{2}{s\sqrt{s+4}}$

3) Find $L\{\cos 4t\}$, If $L\{\cos t\} = \frac{s}{s^2+1}$

FIRST SHIFTING THEOREM:

If $L[f(t)] = \phi(s)$ *then* $L[e^{-at}f(t)] = \phi(s+a)$ & $L[e^{at}f(t)] = \phi(s-a)$

Ex. Find the Laplace transform of $\sin 2t \cos t \cosh 2t$.

Solution: We know that

$$\sin 2t\cos t = \frac{1}{2}2\sin 2t\cos t = \frac{1}{2}\left[(\sin 3t + \sin t)\right]$$

$$\cosh 2t = \frac{e^{2t} + e^{-2t}}{2}$$

$$\therefore \sin 2t\cos t\cosh 2t = \frac{1}{2}\left(e^{2t} + e^{-2t}\right)(\sin 3t + \sin t)$$

$$\therefore \sin 3t = \frac{3}{s^2 + 9}$$

$$\therefore L\left[e^{2t}\sin 3t\right] = \frac{3}{(s-2)^2 + 9}, \quad L\left[e^{-2t}\sin 3t\right] = \frac{3}{(s+2)^2 + 9}$$

$$\therefore L\left(e^{2t}\sin 3t\right) + L\left(e^{-2t}\sin 3t\right) = 3\left[\frac{1}{(s-2)^2 + 9} + \frac{1}{(s+2)^2 + 9}\right]$$

$$= \frac{3.2\left(s^2 + 13\right)}{s^4 + 10s^2 + 13^2}$$

$$Now\ \sin t = \frac{1}{s^2 + 1}$$

$$\therefore L\left(e^{2t}\sin t\right) = \frac{1}{(s-2)^2 + 1}, \quad L\left(e^{-2t}\sin t\right) = \frac{1}{(s+2)^2 + 1}$$

$$L\left(e^{2t}\sin t\right) + L\left(e^{-2t}\sin t\right) = \frac{2\left(s^2 + 5\right)}{s^4 - 6s^2 + 5^2}$$

Form (1), (2) and (3), we get

$$L[\sin 2t\cos t\cosh 2t] = \frac{3\left(s^2 + 13\right)}{s^4 + 10s^2 + 13^2} + \frac{\left(s^2 + 5\right)}{s^4 - 6s^2 + 5^2}$$

Unsolved Problem

If $L[f(t)] = \phi(s) then\ L[e^{-at} f(t)] = \phi(s+a)\ \&\ L[e^{at} f(t)] = \phi(s-a)$

Find the Laplace transform of following functions.

1) $e^{-3t} t^4\ Ans: \dfrac{4!}{(s+3)^5}$

2) $\sinh(\frac{t}{2}) \sin(\frac{\sqrt{3}}{2} t)\ \ Ans: \dfrac{\sqrt{3}\, s}{2(s^4 + s^2 + 1)}$

3) $\dfrac{\cos 2t \sin t}{e^t}\ \ Ans: \dfrac{(s^2 + 2s - 2)}{(s^2 + 2s + 10)(s^2 + 2s + 2)}$

4) $e^{-4t} \sinh t \sin t\ \ Ans: \dfrac{2(s+4)}{(s^2 + 6s + 10)(s^2 + 10s + 26)}$

5) $e^t \sin 2t \sin 3t\ Ans: \dfrac{12s}{(s^2 - 2s + 2)(s^2 - 2s + 26)}$

6) $e^{-3t} \cosh 5t \sin 4t\ \ Ans: \dfrac{4(s^2 + 6s + 50)}{(s^2 - 4s + 20)(s^2 + 16s + 80)}$

7) $\sin 2t \cos t \cosh 2t\ Ans: \dfrac{3(s^2 + 13)}{(s^4 + 10s^2 + 13^2)} + \dfrac{(s^2 + 5)}{(s^4 - 6s^2 + 5^2)}$

8) $e^{-4t} \cosh t \sin t\ \ Ans: \dfrac{(s^2 + 8s + 18)}{(s^2 + 6s + 10)(s^2 + 10s + 26)}$

EFFECT OF MULTIPLICATION BY t^n:

$$If\ L[f(t)] = \phi(s)\ \ then\ L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} \phi(s)$$

Ex. Find the Laplace transform of $t\sqrt{1 + \sin t}$.

Solution:

$$\sqrt{1+\sin t} = \sqrt{\left[\sin^2\left(\frac{t}{2}\right)+\cos^2\left(\frac{t}{2}\right)+2\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)\right]}$$

$$= \sqrt{\left[\sin\left(\frac{t}{2}\right)+\cos\left(\frac{t}{2}\right)\right]^2} = \sin\left(\frac{t}{2}\right)+\cos\left(\frac{t}{2}\right)$$

$$\therefore\ L\sqrt{1+\sin t} = L\left[\sin\left(\frac{t}{2}\right)+\cos\left(\frac{t}{2}\right)\right]$$

$$= \frac{\left(\frac{1}{2}\right)}{s^2+\left(\frac{1}{2}\right)^2} + \frac{s}{s^2+\left(\frac{1}{2}\right)^2}$$

$$= \frac{1}{2}\frac{4}{\left(4s^2+1\right)} + \frac{4s}{\left(4s^2+1\right)}$$

$$= \frac{4s+2}{\left(4s^2+1\right)} = \frac{2\left(2s+1\right)}{\left(4s^2+1\right)}$$

$$L\left[\sqrt{1+\sin t}\right] = -\frac{d}{ds}\left[\frac{2\left(2s+1\right)}{4s^2+1}\right]$$

$$= -2\left[\frac{\left(4s^2+1\right)2-\left(2s+1\right)8s}{\left(4s^2+1\right)^2}\right]$$

$$= -2\left[\frac{-8s^2-8s+2}{\left(4s^2+1\right)^2}\right]$$

$$= 4\frac{\left(4s^2+4s-1\right)}{\left(4s^2+1\right)^2}$$

Unsolved Problem

$$If\ L[f(t)] = \phi(s)\ \ then\ L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}\phi(s)$$

Find the Laplace transform of following functions.

1) $t\sin^3 t\ \ Ans: \dfrac{24s(s+5)}{(s^2+1)^2(s^2+9)^2}$

2) $t\sin 2t\cosh t\ \ Ans: 2\left[\dfrac{(s-1)}{(s^2-2s+5)^2} + \dfrac{(s+1)}{(s^2+2s+5)^2}\right]$

3) $t\cos^2 t\ Ans: -\dfrac{1}{2s^2} + \dfrac{1}{2}\dfrac{s^2-2^2}{(s^2+2^2)^2}$

4) $te^{3t}\sin 4t\ \ Ans: \dfrac{8(s-3)}{(s^2-6s+2s)^2}$

5) $te^{3t}\sin t\ Ans: \dfrac{(2s-6)}{(s^2-6s+10)^2}$

6) $t\sqrt{1+\sin t}\ \ Ans: \dfrac{4(4s^2+4s-1)}{(4s^2+1)^2}$

7) $te^{3t}\sin 2t\ \ Ans: \dfrac{4(s-3)}{(s^2-6s+13)^2}$

8) $t^2\sin 3t\ \ Ans: -\ \ 18\dfrac{(s^2-3)}{(s^2+9)^3}$

EFFECT OF DIVISION BY t

If $L[f(t)] = \phi(s)\ then\ L\left[\dfrac{1}{t}f(t)\right] = \int_s^\infty \phi(s)\,ds$

Ex. Find $L\left[\dfrac{\sin^2 t}{t^2}\right]$

Solution: We know that

$$L\left(\sin^2 t\right)=L\left[\frac{1-\cos t}{2}\right]$$

$$=\frac{1}{2}\left[L(1)-L(\cos 2t)\right]$$

$$=\frac{1}{2}\left[\frac{1}{s}-\frac{s}{s^2+4}\right]$$

By effect of division, we have

$$L\left[\frac{\sin^2 t}{t}\right]=\frac{1}{2}\int_s^\infty\left[\frac{1}{s}-\frac{s}{s^2+4}\right]ds$$

$$=\frac{1}{2}\left[\log s-\frac{1}{2}\log\left(s^2+4\right)\right]_s^\infty$$

$$=\frac{1}{4}\left[\log\left(\frac{s^2}{s^2+4}\right)\right]_s^\infty$$

$$=-\frac{1}{4}\log\left(\frac{s^2}{s^2+4}\right)$$

$$=\frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right)$$

$$L\left[\frac{\sin^2 t}{t^2}\right]=\int_s^\infty\frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right)ds$$

Integrating by parts

$$L\left[\frac{\sin^2 t}{t^2}\right]=\frac{1}{4}\left[\log\left(\frac{s^2+4}{s^2}\right)s-\int s\frac{s^2}{s^2+4}\left(\frac{s^2\,2s-(s^2+4)2s}{s^4}\right)ds\right]_s^\infty$$

$$=\frac{1}{4}\left[s\log\left(\frac{s^2+4}{s^2}\right)+8\int\frac{ds}{s^2+4}\right]_s^\infty$$

$$=\frac{1}{4}\left[s\log\left(\frac{s^2+4}{s^2}\right)+2\tan^{-1}\left(\frac{s}{2}\right)\right]_s^\infty$$

Unsolved Problem :

If $L[f(t)]=\phi(s)$ *then* $L\left[\frac{1}{t}f(t)\right]=\int_s^\infty \phi(s)\,ds$

Find the Laplace transform of following functions.

1) $\frac{1}{t}(1-\cos t)$ $Ans:\frac{1}{2}\log\left(\frac{s^2+1}{s^2}\right)$

2) $\frac{1}{t}(e^{-at}-e^{-bt})$ $Ans:\log\left(\frac{s+b}{s+a}\right)$

3) $\frac{\sin^2 2t}{t}$ $Ans:\frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right)$

4) $\frac{e^{-2t}\sin 2t\cosh t}{t}$ $Ans:\frac{\pi}{2}-\frac{1}{2}\tan^{-1}\left(\frac{s+1}{2}\right)-\frac{1}{2}\tan^{-1}\left(\frac{s+3}{2}\right)$

5) $\frac{\sin^2 t}{t^2}$ $Ans:2\cot^{-1}\left(\frac{s}{2}\right)+s\log\frac{\sqrt{s^2+a^2}}{s}$

6) $\frac{1-\cos t}{t^2}$ $Ans:-\ \frac{\pi}{2}-\frac{s}{2}\log\left(\frac{s^2+1}{s^2}\right)-\tan^{-1}s$

LAPLACE TRANSFORM OF DERIVATIVE

If $L[f(t)] = \phi(s)$ then

$$L[f'(t)] = s\phi(s) - f(0)$$

$$L[f''(t)] = s^2\phi(s) - sf(0) - f'(0)$$

$$L[F'''(t)] = s^3\phi(s) - s^2 f(0) - sf'(0) - f''(0)$$

$$L\left\{ f^n(t) \right\} = s^n\phi(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) \cdots\cdots f^{(n-1)}(0)$$

Ex. Find $L\left[f(t)\right]$ and $L\left[f'(t)\right]$, where $f(t) = \frac{\sin t}{t}$

Solution: $\because$ $L\left[f(t)\right] = \overline{f}(s)$

$$L\left[\frac{\sin t}{t}\right] = \int_s^\infty L\left[\sin t\right] ds$$

$$= \int_s^\infty \frac{1}{s^2+1} ds$$

$$= \left[\tan^{-1} s\right]_s^\infty$$

$$= \cot^{-1} s$$

$\therefore$ $\overline{f}(s) = \cot^{-1} s$

$$L\left[f'(t)\right] = s\,\overline{f}(s) - f(0)$$

$$= s\cot^{-1} s - \lim_{t\to 0} \frac{\sin t}{t}$$

But $f(0) = \lim_{t\to 0} \frac{\sin t}{t}$ is an indeterminate form, which can be solved by L'Hospital's Rule

$$\therefore \lim_{t\to 0} \frac{\sin t}{t} = \lim_{t\to 0} \frac{\cos t}{1} \quad \left(\textit{By differentiating numerator and deonimnatorseparately}\right)$$

$$= 1$$

$$\therefore \quad L\left[f'(t)\right] = L\left[f(t)\right] - f(0)$$
$$= s\cot^{-1}s - 1$$

Unsolved Problem

1) Find $L\{f(t)\}$ *and* $L\{f'(t)\}$

i) If $f(t) = \dfrac{\sin t}{t}$ $\quad Ans: s\cot^{-1}s - 1$

ii) $f(t) = 3,\ 0 \le t < 5$
$\quad = 0,\ t > 5$ $\quad Ans: \dfrac{3}{s}\left(1 - e^{-5s}\right)$

iii) $f(t) = t,\ 0 \le t < 3$
$\quad = 6,\ t > 3$ $\quad Ans: \dfrac{1}{s^2} + e^{-3s}\left(\dfrac{3}{s} - \dfrac{1}{s^2}\right), \dfrac{1}{s} - \dfrac{1}{s^2} + e^{-3s}\left(\dfrac{3}{s} - \dfrac{1}{s}\right)$

2) If $L\left\{2\sqrt{\dfrac{t}{\pi}}\right\} = \dfrac{1}{s^{3/2}}$, Show that $L\left\{\dfrac{1}{\sqrt{\pi t}}\right\} = \dfrac{1}{\sqrt{s}}$

3) If $L\{t\sin\omega t\} = \dfrac{2\omega s}{\left(s^2+\omega^2\right)^2}$, evaluate i) $L\{\omega t\cos\omega t + \sin\omega t\}$ ii) $L\{2\cos\omega t - \omega t\sin\omega t\}$

Ans: i) $\dfrac{2\omega s}{\left(s^2+\omega^2\right)^2}$, ii) $\dfrac{2s^3}{\left(s^2+\omega^2\right)^2}$

LAPLACE TRANSFORM OF INTEGRAL

If $L\left[f(t)\right] = \phi(s)$ *then* $L\left[\int_0^t f(u)\,du\right] = \dfrac{\phi(s)}{s}$

Ex. Find the Laplace transform of $\int_0^t t\, e^{-4t} \sin 3t \, dt$

Solution:

$$L[t\sin 3t] = -\frac{d}{ds} L\,[\sin 3t]$$
$$= -\frac{d}{ds}\left(\frac{3}{s^2+9}\right)$$
$$= \frac{6s}{\left(s^2+9\right)^2}$$

$$L\left[t\, e^{-4t}\sin 3t\right] = \frac{\square 6(s+4)}{\left(s^2+8s+25\right)^2}$$

$$L\left[\int_0^t t\, e^{-4t}\sin 3t\right] = \frac{1}{s} L\left\{t\, e^{-4t}\sin 3t\right\}$$
$$= \frac{\square 6(s+4)}{s\left(s^2+8s+25\right)^2}$$

Unsolved Problem

Find the Laplace transform of following functions.

1) $\int_0^t t\cosh t\, dt$ $\quad Ans: \frac{-(s^2+a^2)}{s(s^2-a^2)^2}$

2) $\int_0^t t\cos^2 t\, dt$ $\quad Ans: \frac{1}{2s^3}+\frac{1}{2}\frac{s^2-2^2}{s(s^2+2^2)^2}$

3) $\int_0^t \frac{1-e^{-t}}{t} dt$ $\quad Ans: \frac{1}{s}\log\left(\frac{s+1}{s}\right)$

4) $\int_0^t \frac{\sin t}{t} dt$ $\quad Ans: \frac{1}{s}\cot^{-1} s$

5) $\int_0^t\int_0^t\int_0^t t\sin t\,dt\,dt\,dt$ $\quad Ans: \dfrac{2}{s^2\left(s^2+1\right)^2}$

EVALUATION OF INTEGRAL USING LAPLACE TRANSFORMS

Ex. Evaluate $\displaystyle\int_0^\infty e^{-t}\int_0^t \frac{\sin u}{u}\,du\,dt$

Solution: By comparing the given integral $\displaystyle\int_0^\infty e^{-t}\int_0^t \frac{\sin u}{u}\,du\,dt$ with the Definition of Laplace transform L [f (t)] = $\displaystyle\int_0^\infty e^{-st}\; f(t)\,dt$ we get,

s=1 and $\displaystyle f(t)=\int_0^t \frac{\sin u}{u}\,du$

$$L[\sin u]=\frac{1}{s^2+1}$$

$$L\left[\frac{\sin u}{u}\right]=\int_s^\infty L\,[\sin u]ds$$

Now,
$$=\int_s^\infty \frac{1}{s^2+1}ds$$

$$=\left[\tan^{-1}s\right]_s^\infty$$

$$=\frac{\pi}{2}-\tan^{-1}s$$

$$=\cot^{-1}s$$

$$L\left[\int_0^t \frac{\sin u}{u}\,du\right]=\frac{1}{s}L\left\{\frac{\sin u}{u}\right\}$$

$$=\frac{1}{s}\cot^{-1}s$$

Now, $\displaystyle\int_0^\infty e^{-st}\int_0^t \frac{\sin u}{u}\,du\,dt=\frac{1}{s}\cot^{-1}s$

Putting s=1, we get

$$\int_0^\infty e^{-t}\int_0^t \frac{\sin u}{u}\,du\,dt = \cot^{-1} 1 = \frac{\pi}{4}$$

Unsolved Problem

1) Show that $\int_0^\infty e^{-2t}\sin^3 t\,dt = \frac{6}{65}$

2) Show that $\int_0^\infty e^{-\sqrt{2}t}\frac{\sin t\sinh t}{t}dt = \frac{\pi}{8}$

4) Show that $\int_0^\infty \left(\frac{\sin 2t + \sin 3t}{te^t}\right)dt = \frac{3\pi}{4}$

5) Show that $\int_0^\infty \frac{e^{-t}\sin\sqrt{3}t}{t}dt = \frac{\pi}{3}$

6) Evaluate $\int_0^\infty t^3 e^{-t}\sin t\,dt$

7) Evaluate $\int_0^\infty\int_0^t e^{-t}\frac{\sin u}{u}\,du\,dt$

8) If $\int_0^\infty e^{-2t}\cos(t-\alpha)\sin(t+\alpha)\,dt = \frac{3}{8}$ find α Ans : $\alpha = \frac{\pi}{4}$

INVERSE LAPLACE TRANSFORM:

STANDARD FORMULAE:

1) $L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$

2) $L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$

3) $L^{-1}\left[\frac{1}{s}\right] = 1$

4) $L^{-1}\left[\frac{1}{s^n}\right]=\frac{t^{n-1}}{n}$

5) $L^{-1}\left[\frac{1}{s^2+a^2}\right]=\frac{\sin at}{a}$

6) $L^{-1}\left[\frac{s}{s^2+a^2}\right]=\cos at$

7) $L^{-1}\left[\frac{s}{s^2-a^2}\right]=\cosh at$

8) $L^{-1}\left[\frac{1}{s^2-a^2}\right]=\frac{\sinh at}{a}$

III Exercise can be solved based on following sample problem.

INVERSE BY DIRECT FORMULAE

Ex. Find the inverse Laplace transform of $\frac{3s+4}{s^2+16}$

Solution:

$$L^{-1}\left(\frac{3s+4}{s^2+16}\right)=3L^{-1}\left(\frac{s}{s^2+16}\right)+L^{-1}\left(\frac{4}{s^2+16}\right)$$
$$=3L^{-1}\left(\frac{s}{s^2+4^2}\right)+L^{-1}\left(\frac{4}{s^2+4^2}\right)$$
$$=3\cos 4t+\sin 4t$$

Unsolved Problem

Find the inverse Laplace transform of following function.

1) $\frac{1}{s^2+9}$ $\qquad$ $Ans: \frac{\sin 3t}{3}$

2)
$$\frac{s^2-3s+4}{s^3}$$
$$Ans: 2t^2-3t+1$$

3)
$$\frac{3s+4\sqrt{3}}{s^2+7}$$
$$Ans: \cos\sqrt{7}t+\frac{4\sqrt{3}}{\sqrt{7}}\sin\sqrt{7}t$$

INVERSE BY FIRST SHIFTING THEOREM

$$L^{-1}\left[\phi(s+a)\right]=e^{-at}L^{-1}\left[\phi(s)\right]$$

Ex. Find the inverse Laplace transform of $\frac{4s+12}{s^2+8s+12}$

Solution:

$$L^{-1}\left[\frac{4s+12}{s^2+8s+12}\right]=L^{-1}\left[\frac{4(s+4)-2^2}{(s+4)^2-2^2}\right]$$

$$=L^{-1}\left[\frac{4(s+4)}{(s+4)^2-2^2}\right]+L^{-1}\left[\frac{2^2}{(s+4)^2-2^2}\right]$$

By First shifting theorem, we have

$$=4e^{-4t}L^{-1}\left[\frac{s}{s^2-2^2}\right]-4e^{-4t}L^{-1}\left[\frac{2^2}{s^2-2^2}\right]$$

$$=4e^{-4t}\cosh 2t-4e^{-4t}\frac{1}{4}\sinh 2t$$

$$=e^{-4t}\left(4\cosh 2t-\sinh 2t\right)$$

Unsolved Problem

Find the inverse Laplace transform of following function.

1)
$$\frac{2s+2}{s^2+2s+10} \qquad Ans: 2e^{-t}\cos 3t$$

2)
$$\frac{s+2}{s^2+4s+7} \qquad Ans: e^{-2t}\cos\sqrt{3}t$$

3)
$$\frac{2s+3}{s^2+2s+2} \qquad Ans: 2e^{-t}\cos t + e^{-t}\sin t$$

INVERSE BY PARTIAL FRACTION

Ex. Find the inverse Laplace transform of $\dfrac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}$

Solution: $L^{-1}\left[\dfrac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}\right] = L^{-1}\left[\dfrac{(s+1)^2+2}{\left[(s+1)^2+2^2\right]\left[(s+1)^2+1^2\right]}\right]$

$$= e^{-t}L^{-1}\left[\frac{s^2+2}{(s^2+4)(s^2+1)}\right]$$

Let

$s^2 = x$

And hence

$$\left[\frac{s^2+2}{(s^2+4)(s^2+1)}\right]=\left[\frac{x+2}{(x+4)(x+1)}\right]$$

$$\left[\frac{x+2}{(x+4)(x+1)}\right]=\frac{a}{x+4}+\frac{b}{x+1}$$

$$\therefore \quad x+2=a(x+1)+b(x+4)$$

When x=-1, 1=3b; when x=-4, -2=-3a

$$\left[\frac{s^2+2}{(s^2+4)(s^2+1)}\right]=\frac{2}{3}\frac{1}{s^2+4}+\frac{1}{3}\frac{1}{s^2+1}$$

$$L^{-1}\left[\frac{s^2+2}{(s^2+4)(s^2+1)}\right]=\frac{2}{3}L^{-1}\left[\frac{1}{s^2+4}\right]+\frac{1}{3}L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$=\frac{2}{3}\frac{1}{2}\sin 2t+\frac{1}{3}\sin t$$

$$\therefore L^{-1}\left[\frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}\right]=\frac{e^{-t}}{3}(\sin 2t+\sin t)$$

Unsolved Problem

Find the inverse Laplace transform of following function.

1) $\dfrac{3s+1}{(s+1)(s^2+2)}$ $\quad Ans: -\dfrac{2}{3}e^{-t}+\dfrac{2}{3}\cos\sqrt{2}t+\dfrac{7}{3\sqrt{2}}\sin\sqrt{2}t$

2) $\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}$ $\quad Ans: \dfrac{1}{a^2-b^2}(a\sin at-b\sin bt)$

4) $\dfrac{s+2}{s^2(s+3)}$ $\quad Ans: \dfrac{1}{9}\left(1+6t-e^{-3t}\right)$

5) $\dfrac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}$ $\quad Ans: \dfrac{e^{-t}}{3}(\sin 2t+\sin t)$

6) $\dfrac{s}{(s^2+1)(s^2+4)}$ $\quad Ans: \dfrac{1}{3}(\cos t-\cos 2t)$

.

7) $\dfrac{s}{1+s^2+s^4}$ $\quad Ans: \dfrac{2}{\sqrt{3}}\sin\dfrac{\sqrt{3}t}{2}\sinh\dfrac{t}{2}$

8) $\dfrac{21s-33}{(s+1)(s-2)^3}$ $\quad Ans: 2e^{-t}-2e^{2t}+6te^{2t}+\dfrac{3}{2}t^2e^{2t}$

11) $\dfrac{s^2}{(s+a)^3}$ $\quad e^{-at}\left[1-2at+\dfrac{a^2t^2}{2}\right]$

12) $\dfrac{1}{s^2(s+1)}$ $\quad Ans:-\ \ -1+t+e^{-t}$

13) $\dfrac{s}{s^4+4a^4}$ $\quad Ans:-\ \ \dfrac{1}{2a^2}\sin at\sinh at.$

INVERSE BY CONVOLUTION THEOREM

Let $L\left[f_1(t)\right]=\phi_1(s)$ *and* $L[f_2(t)]=\phi_2(s)$ *then*

$$L^{-1}[\phi_1(s).\phi_2(s)]=\int_0^t f_1(u).f_2(t-u)\,du$$

$$where\ f_1(t)=L^{-1}[\phi_1(s)]\ \&\ f_2(t)=L^{-1}[\phi_2(s)]$$

Ex. Find the inverse Laplace transform of $\dfrac{(s+2)^2}{(s+2)^2+2^2}$

Solution: By convolution theorem, we have

$$L^{-1}\left[\frac{(s+2)^2}{(s^2+4s+8)^2}\right]=L^{-1}\left[\frac{(s+2)^2}{\left((s+2)^2+2^2\right)^2}\right]$$

$$=e^{-2t}L^{-1}\left[\frac{s^2}{\left(s^2+2^2\right)^2}\right]$$

$$L^{-1}\frac{s}{s^2+2^2}=\cos 2t$$

$$\therefore L^{-1}\left[\left(\frac{s}{s^2+2^2}\right)\left(\frac{s}{s^2+2^2}\right)\right]=\int_0^t \cos 2u\,\cos 2(t-u)\,du$$

$$=\frac{1}{2}\int_0^t \left[\cos 2t+\cos(4u-2t)\right]du$$

$$=\frac{1}{2}\left[u\cos 2t+\frac{1}{4}\sin(4u-2t)\right]_0^t$$

$$=\frac{1}{2}\left[t\cos 2t+\frac{1}{4}\sin 2t+\frac{1}{4}\sin 2t\right]$$

$$=\frac{1}{2}\left[t\cos 2t+\frac{1}{2}\sin 2t\right]$$

$$\therefore\quad L^{-1}\left[\frac{(s+2)^2}{(s^2+4s+8)^2}\right]=\frac{e^{-2t}}{2}\left[t\cos 2t+\frac{1}{2}\sin 2t\right]$$

$$=\frac{e^{-2t}}{4}\left[2t\cos 2t+\sin 2t\right]$$

Unsolved Problem

Find the inverse Laplace transform of following function.

1) $\dfrac{s^2}{(s^2+a^2)^2}$ $\qquad Ans\ \dfrac{1}{2}\left[\sinh at + at\cosh at\right]$

2) $\dfrac{1}{(s-2)^4(s+3)}$ $\qquad Ans: \dfrac{e^{-3t}}{625} - e^{2t}\left[\dfrac{1}{625} - \dfrac{t}{125} + \dfrac{t^2}{50} - \dfrac{t^3}{30}\right]$

3) $\dfrac{(s+2)^2}{(s^2+4s+8)^2}$ $\qquad Ans: \dfrac{e^{-2t}}{4}[\sin 2t + 2t\cos 2t]$

4) $\dfrac{1}{(s-2)(s+2)^2}$ $\qquad Ans: \dfrac{1}{16}[e^{2t} - e^{-2t} - 4te^{-2t}]$

5) $\dfrac{s^2}{\left(s^2+1\right)\left(s^2+4\right)}$ $\quad Ans: \dfrac{1}{3}(2\sin 2t - \sin t)$

6) $\dfrac{s}{(s^2+a^2)^2}$ $\quad Ans: \dfrac{t\sin at}{2a}$

7) $\dfrac{1}{\left(s^2+1\right)^3}$ $\quad Ans: \dfrac{1}{8}\left[\left(3-t^2\right)\sin t - 3t\cos t\right]$

8) $\dfrac{s^2+s}{\left(s^2+1\right)\left(s^2+2s+2\right)}$ $\quad Ans: \dfrac{1}{5}\left[3\cos t + \sin t - 3e^{-t}\cos t + e^{-t}\sin t\right]$

HEAVISIDE UNIT STEP FUNCTION

LAPLACE TRANSFORM OF HEAVISIDE UNIT STEP FUNCTION:

$$L\left[H(t-a)\right]=\frac{1}{s}e^{-as}$$

$$L\left[H(t)\right]=\frac{1}{s}$$

$$L\left[f(t)H(t-a)\right]=e^{-as}L\left[f(t+a)\right]$$

$$L\left[f(t)H(t)\right]=L\left[f(t)\right]$$

Ex 1. Express the function $f(t)=\begin{cases}\cos t & 0<t<\pi\\ \cos 2t & \pi<t<2\pi\\ \cos 3t & t>2\pi\end{cases}$ as Heaviside's unit step functions and find their Laplace transform.

Solution: By the formulae of Heaviside's unit step function, we have

$$\begin{aligned}f(t)&=\cos t\left[H(t)-H(t-\pi)\right]+\cos 2t\left[H(t-\pi)-H(t-2\pi)\right]+\cos 3t\,H(t-2\pi)\\&=\cos t\,H(t)+\left(\cos 2t-\cos t\right)H(t-\pi)+\left(\cos 3t-\cos 2t\right)H(t-2\pi)\end{aligned}$$

$$\begin{aligned}L\left[f(t)\right]&=L\left[\cos t\right]+e^{-\pi s}L\left[\cos 2(t+\pi)-\cos(t+\pi)\right]+e^{-2\pi s}L\left[\cos 3(t+2\pi)-\cos 2(t+2\pi)\right]\\&=L\left[\cos t\right]+e^{-\pi s}L\left[\cos 2t+\cos t\right]+e^{-2\pi s}L\left[\cos 3t-\cos 2t\right]\\&=\frac{s}{s^2+1}+e^{-\pi s}\left[\frac{s}{s^2+4}+\frac{s}{s^2+1}\right]+e^{-2\pi s}\left[\frac{s}{s^2+9}+\frac{s}{s^2+4}\right]\end{aligned}$$

Ex 2. Find the Laplace transform of $(1+2t-3t^2+4t^3)H(t-2)$.

Solution: Here

$$f(t) = 1 + 2t - 3t^2 + 4t^3 \; and \; a = 2$$

$$\therefore f(t+2) = 1 + 2(t+2) - 3(t+2)^2 + 4(t+2)^3$$

$$= 4t^3 + 21t^2 + 38t + 25$$

$$L[f(t+2)] = L\left[4t^3 + 21t^2 + 38t + 25\right]$$

$$= 4\frac{3!}{s^4} + 21\frac{2!}{s^3} + 38\frac{1}{s^2} + 25\frac{1}{s}$$

$$L[f(t)H(t-2)] = e^{-2s}\left[\frac{24}{s^4} + \frac{42}{s^3} + \frac{38}{s^2} + \frac{25}{s}\right]$$

Unsolved Problem

1) Prove that $L[H(t-a)] = \frac{e^{-as}}{s}$

2) Prove that $L[H(t-a)f(t-a)] = e^{-as}\phi(s)$

3) Evaluate

$$L\left[\sin t\, H\left(t - \frac{\pi}{2}\right) - H\left(t - \frac{3\pi}{2}\right)\right]$$

$$Ans: \frac{se^{-\frac{\pi s}{2}}}{s^2+1} - \frac{e^{-\frac{3\pi s}{2}}}{s}$$

4) Evaluate

$L\left[(1 + 2t - 3t^2 + 4t^3)H(t-2)\right]$ *and hence evaluate* $\int_0^\infty e^{-t}(1 + 2t - 3t^2 + 4t^3)H(t-2)\, dt$

$$Ans: e^{-2s}\left[\frac{25}{s} + \frac{38}{s^2} + \frac{42}{s^3} + \frac{24}{s^4}\right], \frac{129}{e^2}$$

6) $f(t) = t - 1,\ 1 < t < 2$ $\quad Ans: \frac{e^{-s}}{s^2} - \frac{2e^{-2s}}{s^2} + \frac{e^{-3s}}{s^2}$

$= 3 - t,\ 2 < t < 3$

7) $f(t) = \cos t,\ 0 < t < \pi$ $\quad Ans: \frac{1}{s^2+1}\left\{s + e^{-\pi s}(s-1) - e^{-2\pi s}\right\}$

$= \sin t, \quad \pi < t < 2\pi$

LAPLACE TRANSFORM OF DIRAC-DELTA (UNIT IMPULSE) FUNCTIONS

$$L\left[\delta(t-a)\right]=e^{-as}$$

$$L\left[\delta(t)\right]=1$$

Ex. Find the Laplace transform of $\sin 2t\ \delta(t-2)$.

Solution: By taking $f(t)=\sin at \text{ and } a=2$, we have

$$\begin{aligned} L\left[\sin 2t\,\delta(t-2)\right] &= L\left[f(t)\right]\delta(t-2) \\ &= e^{-as} f(a) \\ &= e^{-2s}\sin 4 \end{aligned}$$

Unsolved Problem

1) Prove that $L\left[\delta(t-a)\right]=e^{-as}$

2) Prove that $L\left[f(t)\delta(t-a)\right]=e^{-as}f(a)$

3) Find $L\left\{t^4 4(t-2)+t^2\delta(t-2)\right\}$ $Ans: e^{-2s}\left\{4+\frac{16}{s}+\frac{32}{s^2}+\frac{48}{s^3}+\frac{48}{s^4}+\frac{24}{s^5}\right\}$

4) Prove that $L\left\{t^2 H(t-2)-\cos t\,\delta(t-4)\right\}=\frac{2e^{-2s}}{s^3}\left[1+2s+2s^2\right]-e^{-4s}\cosh 4$

5) Prove that $\int_0^\infty t^2e^{-t}\sin t\,\delta(t-2)\,dt=4e^{-2}\sin 2$

LAPLACE TRANSFORM OF PERIODIC FUNCTION

If f (t) is a periodic function of period T then $L[f(t)]=\frac{1}{1-e^{-Ts}}\int_0^T e^{-st}f(t)dt$

Ex. Find the Laplace transform of $f(t)=\begin{cases}1 & 0\le t<a\\ -1 & a<t<2a\end{cases}$ and f (t) is periodic with period 2a.

Solution: Since f (t) is periodic with period 2a.

$$L[f(t)]=\frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st} f(t)\,dt$$

$$=\frac{1}{1-e^{-2as}}\left[\int_0^a e^{-st}(1)\,dt+\int_a^{2a} e^{-st}(-1)\,dt\right]$$

$$=\frac{1}{1-e^{-2as}}\left[\left\{\frac{-e^{-st}}{s}\right\}_0^a+\left\{\frac{e^{-st}}{s}\right\}_a^{2a}\right]$$

$$=\frac{1}{s}\frac{1}{1-e^{-2as}}\left(1-e^{-as}\right)^2=\frac{1}{s}\frac{1-e^{-as}}{1+e^{-as}}$$

$$=\frac{1}{s}\left[\frac{e^{as/2}-e^{-as/2}}{e^{as/2}+e^{-as/2}}\right]$$

$$=\frac{1}{s}\tanh\left\{\frac{as}{2}\right\}$$

Unsolved Problem

If $f(t)$ is a periodic function of period T then $L[f(t)]=\frac{1}{1-e^{-Ts}}\int_0^T e^{-st}f(t)dt$

1) Find Laplace Transform of $f(t)=kt,\ 0<t<1$ $Ans:\ \frac{k}{s^2}-\frac{ke^{-s}}{s\left(1-e^{-s}\right)}$

2) Find Laplace Transform of $f(t)=t\ ,0<t<1$

$=0\ ,1<t<2$

$-$ ________

3) Find Laplace Transform of

$$f(t) = a \sin pt \ for \ 0 < t < \frac{\pi}{p}$$
$$= 0 \qquad for \ \frac{\pi}{p} < t < \frac{2\pi}{p}$$

and $f(t)$ is periodic with the period $\frac{2\pi}{p}$

$$Ans: \frac{ap}{1 - e^{-\frac{s\pi}{p}}} \frac{1}{s^2 + p^2}$$

INVERSE LAPLACE TRANSFORM BY HEAVISIDE UNIT STEP FUNCTION:

$$L^{-1}\left[\frac{1}{s}\right] = H(t)$$
$$L^{-1}\left[\frac{1}{s} e^{-as}\right] = H(t-a)$$
$$L^{-1}\left[e^{-as}\phi(s)\right] = f(t-a)H(t-a)$$

Ex. Find the inverse Laplace transform of $\frac{e^{4-3s}}{(s+4)^{5/2}}$.

Solution: Here $\phi(s) = \frac{1}{(s+4)^{5/2}}$, We know that

$$f(t) = L^{-1}\phi(s) = L^{-1}\frac{1}{(s+4)^{5/2}}$$
$$= e^{-4t}L^{-1}\frac{1}{s^{5/2}} \quad (\text{By first shifting theorem})$$
$$= e^{-4t}\frac{t^{3/2}}{\Gamma 5/2} \qquad ,$$
$$= \frac{e^{-4t}t^{3/2}}{\left(3/2\right)\left(1/2\right)\Gamma 1/2}$$
$$= \frac{4e^{-4t}t^{3/2}}{3\sqrt{\pi}}$$

$$\therefore \quad L^{-1}\left[\frac{e^{4-3s}}{(s+4)^{5/2}}\right] = \frac{4e^{4}}{3\sqrt{\pi}} e^{-4t}\,(t-3)^{3/2}\; H\,(t-3)$$

Unsolved Problem

Evaluate

1)
$$L^{-1}\left[\frac{e^{4-3s}}{(s+4)^{\frac{5}{2}}}\right] \qquad Ans : \frac{4}{3\sqrt{\pi}}\, e^{-4(t-4)}\,(t-3)^{\frac{3}{2}} H\,(t-3)$$

2)
$$L^{-1}\left[\frac{(s+1)e^{-s}}{s^{2}+s+1}\right] \quad Ans : e^{-\frac{(t-1)}{2}}\left[\cos\sqrt{3}\,\frac{(t-1)}{2} + \frac{1}{\sqrt{3}}\sin\left(\sqrt{3}\,\frac{(t-1)}{2}\right)\right] H(t-1)$$

3)
$$L^{-1}\left[\frac{e^{-\pi s}}{s^{2}(s^{2}+1)}\right] \qquad Ans : (t-\pi) + \sin(t-\pi).H\,(t-\pi)$$

4)
$$L^{-1}\left[\frac{se^{-\frac{s}{2}} + \pi e^{-s}}{(s^{2}+\pi^{2})}\right] \qquad Ans : \sin \pi t\left[H\left(t-\frac{1}{2}\right) + H\,(t-1)\right]$$

IV Exercise can be solved based on following sample problem.

APPLICATIONS OF LAPLACE TRANSFORM : Application of Laplace transforms to solve ordinary differential equations :-

The following are steps to solve ordinary differential equations using the Laplace transform method

(A) Take the Laplace transform of both sides of ordinary differential equations.

(B) Express $Y(s)$ as a function of s .

(C) Take the inverse Laplace transform on both sides to get the solution.

Ex.1. Solve the following equation by using Laplace transform

$$\frac{dy}{dt}+2y+\int_0^t y\,dt=\sin t, \qquad given\ that \quad y(0)=1 .$$

Solution: Let $L(y)=\bar{y}$. Taking Laplace transform on both the sides, we get

$$L(y')+2L(y)+L\left[\int_0^t y\,dt\right]=L(\sin t)$$

But

$$L(y')=sL(y)-y(0)=s\bar{y}-1$$

$$L\left[\int_0^t y\,dt\right]=\frac{1}{s}L(y)=\frac{1}{s}\bar{y}$$

$$L(\sin t)=\frac{1}{s^2+1}$$

∴ The equation becomes

$$\therefore \quad \left(\frac{s^2+2s+1}{s}\right)\bar{y}=\frac{s^2+2}{s^2+1}$$

$$\therefore \quad \bar{y}=\frac{s(s^2+2)}{(s+1)^2(s^2+1)}$$

Let $$\frac{s(s^2+2)}{(s+1)^2(s^2+1)}=\frac{a}{s+1}+\frac{b}{(s+1)^2}+\frac{cs+d}{s^2+1}$$

$$\therefore \quad s(s^2+2)=a(s+1)(s^2+1)+b(s^2+1)+(cs+d)(s+1)^2$$

Putting $s=-1$, $\quad -3 = 2b \quad \therefore \quad b=-\frac{3}{2}$

Putting $s=0$, $\quad 0=a+b+d$

Equating the coefficients of s^2 and s^3, we get

$$0=a+b+2c+d \quad and \; 1=a=c$$
$$\because \; b=-\frac{3}{2}\,, \quad a+d=\frac{3}{2}$$
$$and \; a+2c+d=\frac{3}{2}$$
$$But \quad a+d=\frac{3}{2} \quad \therefore \; 2c=0 \quad \therefore \; c=0$$
$$\because \; 1=a+c \; and \; c=0 \quad \therefore \; a=1$$
$$a+d=\frac{3}{2} \quad and \; a=1 \quad \therefore \; d=\frac{1}{2}$$
$$\therefore \quad a=1, \; b=-\frac{3}{2}, \; c=0, \quad d=\frac{1}{2}$$

$$\therefore \quad \bar{y}=\frac{1}{s+1}-\frac{3}{2}\frac{1}{(s+1)^2}+\frac{1}{2}\frac{1}{(s^2+1)}$$
$$\therefore \quad y=L^{-1}\left(\frac{1}{s+1}\right)-\frac{3}{2}e^{-t}L^{-1}\frac{1}{s^2}+\frac{1}{2}L^{-1}\frac{1}{(s^2+1)}$$
$$\therefore \quad y=e^{-t}-\frac{3}{2}te^{-t}+\frac{1}{2}\sin t$$

Ex. 2. Solve by using Laplace transform

$$(D^2+2D+5)Y=e^{-t}\sin t, \quad when \; y(0)=0, y'(0)=1$$

Solution: Let $L(y)=\bar{y}$. Taking Laplace transform on both the sides, we get

$$L(y'')+2L(y')+5L(y)=L\left(e^{-t}\sin t\right)$$

But

$$L(y') = sL(y) - y(0) = s\bar{y}$$
$$L(y'') = s^2\bar{y} - sy(0) - y'(0) = s^2\bar{y} - 1$$
$$L\left(e^{-t}\sin t\right) = \frac{1}{(s+1)^2+1}$$

∴ The equation becomes

$$\therefore \quad (s^2\bar{y} - 1) + 2s\bar{y} + 5\bar{y} = \frac{1}{(s+1)^2+1}$$
$$(s^2 + 2s + 5)\,\bar{y} = 1 + \frac{1}{s^2+2s+2} = \frac{s^2+2s+3}{s^2+2s+2}$$
$$\therefore \quad \bar{y} = \frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}$$

Let $\bar{y} = \dfrac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)} = \dfrac{as+b}{(s^2+2s+5)} + \dfrac{cs+d}{(s^2+2s+2)}$

After simplication, we get

$$\therefore \quad \bar{y} = \frac{2}{3}\frac{1}{(s^2+2s+5)} + \frac{1}{3}\frac{1}{(s^2+2s+2)} = \frac{2}{3}\frac{1}{(s+1)^2+2^2} + \frac{1}{3}\frac{1}{(s+1)^2+1^2}$$

Taking inverse Laplacetransform

$$\therefore \quad y = \frac{2}{3}e^{-t}L^{-1}\left(\frac{1}{s^2+2^2}\right) + \frac{1}{3}e^{-t}L^{-1}\left(\frac{1}{s^2+1^2}\right)$$
$$y = \frac{2}{3}e^{-t}\cdot\frac{1}{2}\sin 2t + \frac{1}{3}e^{-t}\sin t = \frac{e^{-t}}{3}(\sin 2t + \sin t)$$

Ex.3. Solve $\quad 3\dfrac{dy}{dx} + 2y = e^{-x}, \; y(0) = 5$

Solution : Taking the Laplace transform of both sides, we get

$$L\left(3\frac{dy}{dx}+2y\right)=L\left(e^{-x}\right)$$

$$3[sY(s)-y(0)]+2Y(s)=\frac{1}{s+1}$$

Using the initial condition, $y(0)=5$ we get

$$3[sY(s)-5]+2Y(s)=\frac{1}{s+1}$$

$$(3s+2)Y(s)=\frac{1}{s+1}+15$$

$$(3s+2)Y(s)=\frac{15s+16}{s+1}$$

$$Y(s)=\frac{15s+16}{(s+1)(3s+2)}$$

Writing the expression for $Y(s)$ in terms of partial fractions

$$\frac{15s+16}{(s+1)(3s+2)}=\frac{A}{s+1}+\frac{B}{3s+2}$$

$$\frac{15s+16}{(s+1)(3s+2)}=\frac{3As+2A+Bs+B}{(s+1)(3s+2)}$$

$$15s+16=3As+2A+Bs+B$$

Equating coefficients of s^1 and s^0 gives

$$3A+B=15$$

$$2A+B=16$$

The solution to the above two simultaneous linear equations is

$$A=-1$$

$$B=18$$

$$Y(s) = \frac{-1}{s+1} + \frac{18}{3s+2}$$

$$= \frac{-1}{s+1} + \frac{6}{s+0.666667}$$

Taking the inverse Laplace transform on both sides

$$L^{-1}\{Y(s)\} = L^{-1}\left(\frac{-1}{s+1}\right) + L^{-1}\left(\frac{6}{s+0.666667}\right)$$

Since

$$L^{-1}\left(\frac{1}{s+a}\right) = e^{-at}$$

The solution is given by

$$y(x) = -e^{-x} + 6e^{-0.666667x}$$

Unsolved Problem :

1) Solve

$3\dfrac{dy}{dt} + 2y = e^{3t}, \quad y = 1 \quad at\ t = 0$ $\qquad Ans: \dfrac{10}{11} e^{-\frac{2}{3}t} + \dfrac{1}{11} e^{3t}$

2) Solve $\dfrac{dy}{dt} + 3y = 2 + e^{-t}, \quad y = 1\ at\ t = 0$ $\quad Ans: \dfrac{2}{3} + \dfrac{1}{2} e^{-t} - \dfrac{1}{6} e^{-3t}$

3) Solve $\dfrac{dx}{dt} + x = \sin wt,\ x(0) = 2$

$Ans: \dfrac{1}{1+w^2}[(2w^2 + w + 2)e^{-t} - w\cos wt + \sin wt]$

4) Solve $(D^2 - 3D + 2)\ y = 4e^{2t}$, $with\ y(0) = -3, y'(0) = 5$

$Ans: -\ y = -7e^{t} + 4e^{2t} + 4te^{2t}$

5) Solve $\frac{d^2y}{dt^2}+4\frac{dy}{dt}+8y=1, where\, y(0)=0, y'(0)=1$

$Ans: \frac{1}{8}-\frac{1}{8}e^{-2t}\cos 2t+\frac{3}{8}e^{-2t}\sin 2t$

6) Solve $\frac{d^2y}{dt^2}+y=t, \quad where\, y(0)=1, y'(0)=0 \quad Ans: t+\cos t-\sin t$

7) Solve $\frac{dy}{dt}+2y+\int_0^t y\,dt=\sin t$, given that $y(0)=1$.

$Ans:- \quad y=e^{-t}-\frac{3}{2}e^{-t}\cdot t+\frac{1}{2}\sin t$

8) Solve $\frac{d^2y}{dt^2}+9y=\cos 2t$ with $y(0)=1$ & $y\left(\frac{\pi}{2}\right)=-1$

$y=\frac{1}{5}\cos 2t-\frac{1}{5}\cos 3t+\frac{4}{5}\sin 3t$

9) Solve $\frac{d^2y}{dt^2}-4y=3e^t$ where $y(0)=0$ & $y'(0)=3$ $Ans:-e^t+\frac{3}{2}e^{2t}-\frac{1}{2}e^{-2t}$

Fourier Transform:

It is denoted by $F\,f(x)=F(s)$ and defined as

$$F\,f\,x=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f\,x\,e^{isx}\,dx$$

Inverse Fourier Transform:

It is given by

$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{-isx}\,ds$

Fourier Sine Transform & Inverse Fourier Sine Transform:

$$F_s\, f(x)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f\,x\,\text{Sinsx}\,dx$$

IFST $\quad f_s\,x=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\bar{F}(s)\text{Sinsx}\,ds$

Fourier Cosine Transform & Inverse Fourier Cosine Transform:

$$F_c\, f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\, dx$$

IFST $\quad f_c(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \bar{F}(s) \cos sx\, ds$

Q.1: Find Fourier Cosine Transform of the function f(x)= e^{-x} , $x \geq 0$

Solution: By definition of Fourier Cosine Transform

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx dx$$

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \cos sx dx$$

$$F_c(s) = \sqrt{\frac{2}{\pi}} \left[\frac{e^{-x}}{1+s^2}(-\cos sx + s \sin sx)\right]$$

$$F_c(s) = \sqrt{\frac{2}{\pi}} \frac{1}{1+s^2}$$

Which is the required transform.

Q.2 : Find Inverse Fourier Sine Transform of $F_s(s) = \frac{s}{1+s^2}$

Solution: By definition of Inverse Fourier Sine Transform

$$f_s(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F(s) \sin sx ds$$

$$f_s(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{s}{1+s^2} \sin sx ds$$

Multiply and divide by s

$$f_s(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{s^2}{s(1+s^2)} Sinsxds$$

Add and subtract 1

$$f_s x = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{s^2+1-1}{s(1+s^2)} Sinsxds$$

$$f_s x = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{s} Sinsxds - \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{s^2+1-1}{s(1+s^2)} Sinsxds$$

$$f_s x = \frac{\pi}{2} - \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{s(1+s^2)} Sinsxds$$

Differentiate bw.r.to x

$$\frac{df_s x}{dx} = -\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{s}{s(1+s^2)} \cos sx\, ds$$

Again Differentiate bw.r.to x

$$\frac{d^2 f_s x}{dx^2} = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{s}{(1+s^2)} \sin sx\, ds$$

$$\frac{d^2 f_s x}{dx^2} = f_s\ x$$

$$(D^2 - 1) f_s x = 0$$

$$f_s(x) = ae^x + be^{-x}$$

After solving , we get

$$f_s(x) = \frac{\pi}{2} e^{-x}$$

Q.3: Find the Fourier sine transform of $\frac{e^{-ax}}{x}$. Hence find Fourier sine transform of $\frac{1}{x}$

Solution: By definition of Fourier sine transform

$$F_s f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \operatorname{Sin} sx \, dx$$

$$F_s f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \operatorname{Sin} sx \, dx$$

Differentiating w.r.to s we get

$$\frac{dF_s(x)}{ds} = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} x \cos sx \, dx$$

$$\frac{dF_s(x)}{ds} = \sqrt{\frac{2}{\pi}} \frac{a}{a^2+s^2}$$

Now integrating w.r.to s,

We have

$$F_s(x) = \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{s}{a}\right) + c$$

But for s=0

F(0)=0

And therefore c=0

Hence,

$$\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \operatorname{Sin} sx \, dx = \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{s}{a}\right)$$

Taking a=0, both the sides

$$\int_0^\infty \frac{\sin sx}{x} dx = \frac{\pi}{2}$$

Application of Fourier Transform in solving the ordinary differential equation:

The method of Fouriertransform can be applied in solving some ordinary differential equation.

Procedure

Firstly take the Fourier transform of both sides of the given differential equation. Thus we shell get an algebraic equation. Simplify and get Fourier transform.

Now take the inverse Fourier transform. Apply initial condition and then required solution is obtained.

Remark: If we take the Fourier transform of $e^x \; or e^{-x}$ then we shell find that the integral diverges and hence Fourier transform does not exist.

Example: Solve $\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2}$

With Boundary conditions:1) V(0,t)=0

2) V(x,0)=e^{-x} , $x > 0$

3) V(x,t) is bounded , x>0,t>0

Solution: Taking Fourier Sine Transform of both the sides of given equation, we have

$$\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial V}{\partial t} \sin sx dx = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial^2 V}{\partial x^2} \sin sx dx$$

$$\Rightarrow \sqrt{\frac{2}{\pi}} \frac{d}{dt} \int_0^\infty V \sin sx dx = \sqrt{\frac{2}{\pi}} \{\sin sx \frac{\partial V}{\partial x} - s \int_0^\infty \frac{\partial V}{\partial x} \cos sx dx\}$$

$$\Rightarrow \frac{dV_s}{dt} = \sqrt{\frac{2}{\pi}} \{s[V \cos sx] - s^2 \int_0^\infty V \sin sx dx\}$$

$$\frac{d V_s}{dt} = \sqrt{\frac{2}{\pi}} \{s[V \cos sx] - s^2 \int_0^\infty V \sin sx dx\}$$

By using condition, we have

$$\Rightarrow \frac{dV_s}{dt} + 2s^2V_s = 0$$

$$\Rightarrow V_s = Ae^{-2s^2 t}$$

Now at t=0,

$$\Rightarrow V_s = A$$

$$\Rightarrow V_s(s,0) = \sqrt{\frac{2}{\pi}} \int_0^\infty V(x,0) \sin sx dx$$

$$\Rightarrow V_s(s,0) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \sin sx dx$$

$$\Rightarrow V_s(s,0) = \sqrt{\frac{2}{\pi}} \frac{s}{1+s^2}$$

Comparing both values of V_s, we get

$$\Rightarrow A = \sqrt{\frac{2}{\pi}} \frac{s}{1+s^2}$$

Therefore,

$$\Rightarrow V_s = \sqrt{\frac{2}{\pi}} \frac{s}{1+s^2} e^{-2s^2 t}$$

Now taking inverse fourier sine transform,

$$V(x,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{s}{1+s^2} e^{-2s^2 t} \sin sx ds$$

