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Module3:Numerical Methods–3

Contents

Ordinary differential equations: Many problems in Engineering and Science can be formulated into ordinary differentional equations satisfying certain given conditions. If these conditions are prescribed for one point only, then the differential equation together with the condition is known as an initial value problem. If the conditions are prescribed for two or more points, then the problem is termed as boundary value problem. A limited number of these differentional equations can be solved by analytical methods. Hence numerical methods play very important role in the solution of differential equations.

We shall discuss some of the following methods for obtaining numerical solution of first order and first degree Ordinary Differential Equation.

[I] Picard'smethod of successive approximation**.**

[II] Taylor's Seriesmethod.

[III] Euler's method.

*IV+ Modified Euler's method.

*V+ Milne'smethod / Milne'sPredictor-Corrector method.

[VI Runge-Kutta method.

*VII+ Adam's Bash forth Method.

Method-I :Picard's Method of successive approximations for first order and first degree differential equation :--

Consider the differential equation

$$\frac{dy}{dx} = f(x, y) \text{ --------------------------(1)}$$

Subject to $y(x_0) = y_0$. This equation can be written as

$$dy = f(x, y)\,dx$$

Integrating between the limits,

We get $\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x, y)\,dx$

$$y - y_0 = \int_{x_0}^{x} f(x, y)\,dx$$

$$y = y_0 + \int_{x_0}^{x} f(x, y)\,dx \quad \text{----------------------------} \quad (2)$$

Which is an integral equation because the unknown function "y" is present under the integral sign.Such an equation can be solved by successive approximation as follows;

For first approximation y_1,we replace y by y_0 in $f(x, y)$ in the R.H.S. of equation (2),

i.e. $y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$--------------------------------- (3)

Now, for second approximation y_2,we replace y by y_1 in $f(x, y)$ in the R.H.S. of equation (2),

i.e. $y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx$--------------------------------- (4)

Proceeding in this way, we get $y_3, y_4, y_5, -------$

The n^{th} approximation is given by

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\,dx, n=1,2,3\text{----------}$$

The process is to be stopped when the two values of y, viz y_n , y_{n-1} ,are same to the desired degree of accuracy.

Example-: Find the Picard approximations y_1, y_2, y_3 to the solution of the initial value problem $y'=y$, $y(0)=2$ .Use y_3 to estimate the value of y (0.8) and compare it with the exact solution.

Solution: The n^{th} approximation is given by

$$y_n = y_0 + \int_{x_o}^{x} f(x, y_{n-1})\, dx, n=1,2,3 \text{ ----------(1)}$$

For First approximation y_1 , we replace y by y_0 in $f(x, y)$ in the R.H.S. of equation (1)

Therefore, $y_1 = y_0 + \int_{x_o}^{x} f(x, y_0)\, dx$

Here, $y_0 = 2$, the value of y_1 is

$y_1 = 2 + \int_0^x 2\, dx = 2 + 2x$ where $f(x, y_0) = y_0 = 2$

Similarly,

$y_2 = 2 + \int_0^x (2+2x)\, dx = 2 + 2x + x^2$

also

$y_3 = 2 + \int_0^x (2+2x+x^2)dx = 2 + 2x + x^2 + \frac{x^3}{3}$

At x= 0.8

$y_3 = 2 + 2(0.8) + (0.8)^2 + \frac{1}{3}(0.8)^3$

=4.41

By Direct Method: The solution of the initial-value problem, found by separation of variables, is $y=2e^x$. At x=0.8

$y=2e^{0.8}=4.45$

Method-II :Numerical Solution to Ordinary Differential Equation By Taylor's Series method

Consider the differential equation

$$\frac{dy}{dx} = f(x, y) \qquad \text{--------------------------} \qquad (1)$$

Subject to $y(x_0) = y_0.$

Then ,we by Taylor's

$y = y(x) = y(x_0 + x - x_0)$

$$= y(x_0) + \frac{x - x_0}{1\,!} y'(x_0) + \frac{(x - x_0)^2}{2\,!} y''(x_0) + \frac{(x - x_0)^3}{3\,!} y'''(x_0) + \cdots \ldots \ldots \ldots \ldots$$

If $x_0 = 0$ then

$$y = y(0) + \frac{x}{1\,!} y'(0) + \frac{x^2}{2\,!} y''(0) + \frac{x^3}{3\,!} y'''(0) + \cdots \ldots \ldots \ldots \ldots$$

Example- Using Taylor's Series method, find the solution of

$\frac{dy}{dx} = xy - 1$ With $y(1) = 2$ correct to five decimal places at $x = 1.02$

Solution: Given that $\frac{dy}{dx} = xy - 1$ with $y(1) = 2$ i.e. $x_0 = 1$ and $y_0 = 2$

i.e. $y' = xy - 1 \qquad \therefore y'(0) = x_0y_0 - 1 = 1.2 - 1 = 1 = y'(x_0)$

Differentiate with respect to x

$y'' = y + xy' \therefore y''(0) = y_0 + x_0y'(0) = 2 + 1.1 = 3 = y''(x_0)$

Again Differentiate with respect to x

$$y''' = 2y' + xy'' \therefore y'''(0) = 2y'(0) + x_0 y''(0) = 2.1 + 1.3 = 5 = y'''(x_0)$$

Again Differentiate with respect to x

$$y'''' = 3y'' + xy''' \therefore y''''(0) = 3y''(0) + x_0 y'''(0) = 3.3 + 1.5 = 14 = y''''(x_0)$$

andso on

Now by Taylor's Series method,

$$y = y(x_0) + \frac{(x - x_0)}{1!} y'(x_0) + \frac{(x - x_0)^2}{2!} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \ldots \ldots \ldots \ldots$$

$$\Rightarrow y = 2 + \frac{(x-1)}{1!}.1 + \frac{(x-1)^2}{2!}.3 + \frac{(x-1)^3}{3!}.5 + \frac{(x-1)^4}{4!}.14 + \cdots \ldots \ldots \ldots \ldots$$

At $x = 1.02$

$$\Rightarrow y = 2 + \frac{(1.02-1)}{1!}.1 + \frac{(1.02-1)^2}{2!}.3 + \frac{(1.02-1)^3}{3!}.5 + \frac{(1.02-1)^4}{4!}.14 + \cdots \ldots \ldots \ldots \ldots$$

$\Rightarrow y = 2.020606.$

Method-III :Numerical Solution to Ordinary Differential Equation By Euler'smethod

Euler'smethod,as such, is of very little practical importance, but illustrates in simple form ,the basic idea of those numerical methods,which seek to determine the changeΔy in y corresponding to small increase in the argument x.

Let us consider the equation

$$\frac{dy}{dx} = f(x, y) \qquad \text{-------------------------} \qquad (1)$$

Subject to $y(x_0) = y_0.$

Let $y = g\ x$ be the solution of (1) and let x_0 , $x_1 = x_0 + \mathrm{h}$, $x_2 = x_0 + 2\mathrm{h}$,……… be equidistant values of x.

In an small interval, a curve is nearly a straight line. This is the property used in **Euler's method.**

The n^{th} approximation is given by

y_{n+1} =y_n + h f $\mathrm{x_n}$, y_n where $n = 0.1{,}2{,}3\ \ldots\ \ldots\ \ldots$

Example- UsingEuler'smethod, find an approximate value of$y$ corresponding to $x = 1, givent h at$

$\frac{dy}{dx} = x + y$With$y\ 0 = 1$

Solution: Given that$\frac{dy}{dx} = x + y$With$y\ 0 = 1$

We have ,then^{th} approximation is given by

y_{n+1} =y_n + h f $\mathrm{x_n}$, y_n where $n = 0.1{,}2{,}3\ \ldots\ \ldots\ \ldots$ (1). We take $n = 10\ and\ h = 0.1, which is sufficiently small. The various calculations are arranged as followes by$ (1)

x	y	$\frac{dy}{dx} = x + y = f\ x, y$	Old y +0.1($\frac{dy}{dx}$) $= new\ y$
0.0	1.0	1.0	1.0 +0.1 (1.0) = 1.10
0.1	1.10	1.20	1.10 +0.1 (1.20) = 1.22
0.2	1.22	1.42	1.22+0.1 (1.42) = 1.36
0.3	1.36	1.66	1.36+0.1 (1.66) = 1.53
0.4	1.53	1.93	1.53+0.1 (1.93) = 1.72
0.5	1.72	2.22	1.72+0.1 (2.22) = 1.94
0.6	1.94	2.54	1.94+0.1 (2.54) = 2.19
0.7	2.19	2.89	2.19+0.1 (2.89) = 2.48
0.8	2.48	3.29	2.48+0.1 (3.29) = 2.81
0.9	2.81	3.71	2.81+0.1 (3.71) = 3.18
1.0	3.18		

Thus the required approximate value is y =3.18.

Method-IV :Numerical Solution to Ordinary Differential Equation By Euler's Modified method

Let us consider the equation

$$\frac{dy}{dx} = f(x, y) \qquad \text{--------------------------} \qquad (1)$$

Subject to $y(x_0) = y_0$.

Let $y = g(x)$ be the solution of (1) and let x_0 , $x_1 = x_0 + \mathrm{h}$, $x_2 = x_0 + 2\mathrm{h}$,......be equidistant values of x.

In each step of this method ,we first compute the auxiliary value y_{n+1} *by*

$y_{n+1} = y_n + \mathrm{h}\, f(x_n, y_n)$ where $n = 0.1,2,3 \ldots \ldots \ldots$ (2)

andthenthe new value

$y^{(r)}_{n+1} = y_n + \frac{\mathrm{h}}{2}\left[f(x_n, y_n) + f\left(x_{n+1}, y^{(r-1)}_{n+1}\right)\right]$..........(3)

where$r$ *denotesnumberofiteration* $1,2,3 \ldots \ldots ..$

This Modified method is a Predictor-Corrector method; because in each step, we first predict a value by (2) and then correct it by (3).

Example: Using Euler's modified method, solve numerically the equation

$$\frac{dy}{dx} = x + y$$

Subject to $y(0) = 1$ *for* $0 \le x \le 0.6$ in the steps of 0.2

Solution: Given that

$$\frac{dy}{dx} = x + y$$

Subject to $y(0) = 1$ i.e.$x_0 = 0$, $y_0 = 1$ *and* $\mathrm{h} = 0.2$

$\therefore$ we have ,

$\boldsymbol{x_0 = 0, x_1 = 0.2, x_2 = 0.4, x_3 = 0.6}$

$\boldsymbol{y_{0.}} = \mathbf{1}, \boldsymbol{y_{1.}} = ?, \boldsymbol{y_{2.}} = ?, \quad \boldsymbol{y_{3.}} = ?$

Now for this, we have by Euler's modified method ,

Predictor formula

$y_{n+1} = y_n + h\, f(x_n, y_n)$ where $n = 0.1,2,3 \ldots \ldots \ldots$ (1)

and then the new value by Corrector formula

$y_{n+1}^{r} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, y_{n+1}^{r-1})]$ ....(2)

Where r denotes number of iteration $1,2,3 \ldots \ldots ..$

Now by (1), for finding the value of $y_{1.}$ put $n = 0$
$\therefore$ we have ,

$y_1 = y_0 + h\, f(x_0, y_0) = 1 + 0.2[x_0 + y_0] = 1.2$

Now by (2)this value of
y_1, *thus obtained is improved or modified as followes*

For First iteration put $r = 1$ and $n = 0$ in (2)

$y_1^1 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^0)]$

$y_1^1 = 1 + \frac{0.2}{2}[(x_0 + y_0) + (x_1 + y_1^0)]$

$=> y_1^1 = 1 + \frac{0.2}{2}[0 + 1 + 0.2 + 1.2$ where $y_1^0 = y_1 = 1.2$

Now for second iteration put $r = 2$ and $n = 0$ in 2 , we get

$y_1^2 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^1)] = 1 + \frac{0.2}{2}[(x_0 + y_0) + (x_1 + y_1^1)]$
$= 1.2309$

Similarly third iteration put $r = 3$ and $n = 0$ in 2 , we get

$y_1^3 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^2)] = 1 + \frac{0.2}{2}[(x_0 + y_0) + (x_1 + y_1^2)]$
$= 1.2309$

Hence ,from above ,Since $y_1^2 \approx y_1^3 = 1.2309$

$\therefore$ **we have** , $\boldsymbol{At x_{1.} = 0.2 , y_1 = 1.2309}$

Again apply **Euler's** Modified **method for more accurate approximations ,**

Hence , by (1), for finding the value of $y_{2.} put n = 1$
$\therefore$ we have ,

$y_2 = y_1 + h\, f\, x_1 , y_1 = 1.2309 + 0.2[x_1 + y_1 = 1.49279$

Now by (2) this value of
y_2, *thus obtained is improved or modified as followes*

For First iteration put $r = 1$ $and n = 1$ in (2)

$$y_2^{\ 1} = y_1 + \frac{h}{2}[f\ x_1 , y_1 + f\ x_2 , y_2^{\ 0}$$

$$y_2^{\ 1} = 1.2309 + \frac{0.2}{2}[x_1 + y_1 + x_2 + y_2^{\ 0}$$

$$=> y_2^{\ 1} = 1.2309 + \frac{0.2}{2}[0.2 + 1.2309 + 0.4 + 1.49279 = 1.52402$$

where $y_2^{\ 0} = y_2 = 1.49279$

Now for second iteration put $r = 2$ $and n = 1$ in 2 , $we get$

$$y_2^{\ 2} = y_1 + \frac{h}{2}[f\ x_1 , y_1 + f\ x_2 , y_2^{\ 1}$$
$$= 1.2309 + \frac{0.2}{2}[x_1 + y_1 + x_2 + y_2^{\ 1} = 1.525297$$

Similarly third iteration put $r = 3$ $and n = 1$ in 2 , $we get$

$$y_2^{\ 3} = y_1 + \frac{h}{2}[f\ x_1 , y_1 + f\ x_2 , y_2^{\ 2}$$
$$= 1.2309 + \frac{0.2}{2}[x_1 + y_1 + x_2 + y_2^{\ 2} = 1.52535$$

Also,similarly fourth iteration put $r = 4$ $and n = 1$ in 2 , $we get$

$y_2^3 = 1.52535$

$\therefore$ we have , $At x_{2.} = 0.4$, $\boldsymbol{y_2} = 1.52535$

Hence ,by again (1), for finding the value of $y_{3.} put n = 2$
$\therefore$ we have ,

$y_3 = y_2 + \mathrm{h\ f\ x_2}, y_2 = 1.52535 + 0.2[x_2 + y_2 = 1.85236$

Now by (2) this value of
y_3, *thus obtained is improved or modified as followes*

For First iteration put $r = 1$ $and n = 2$ in (2)

$$y_3^{\ 1} = y_2 + \frac{h}{2}[f\ x_2, y_2 + f\ x_3, y_3^{\ 0}$$

$y_3^{\ 1} = 1.52535 + \frac{0.2}{2}[x_2 + y_2 + x_3 + y_3^{\ 0}$ where $y_3^{\ 0} = y_3 = 1.85236$

$=> y_3^{\ 1} = 1.52535 + \frac{0.2}{2}[0.4 + 1.52535 + 0.6 + 1.85236 = 1.88496$

Similarly by putting $r = 2, ,3,4$ $and n = 2$ in 2 , $we get$

$=> y_3^{\ 2} = 1.88615, y_3^{\ 3} = 1.88619, y_3^{\ 4} = 1.88619$

$\therefore$ we have , $At x_{3.} = 0.6$, $\boldsymbol{y_3} = 1.88619$

Method-V :Numerical Solution to Ordinary Differential Equation By Milne's method / Milne's Predictor-Corrector method

As the name suggests,predictor-corrector method is the method in which,we first predict a value of y_{n+1} by using a certain formula and then correct this value by using a more accurate formula. i.e.

Let us consider the equation

$$\frac{dy}{dx} = f\ x, y \qquad \text{--------------------------} \qquad (1)$$

Subject to $y\ x_0 = y_{0.}$

Then the predictor formula for finding y_{n+1} is

$$y_{n+1,p} = y_{n-3} + \frac{4h}{3}[2y'_{n-2} - y'_{n-1} + 2y'_n] \ldots\ldots\ldots\ldots\ldots.. (2)$$

And the Corrector formula for finding y_{n+1}

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}[2\,y'_{n-1} + 4\,y'_{n} + y'_{n+1}] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Now from above , it is clear that only $n = 3,4,5, \ldots\ldots\ldots$ possible i.e. we shall find $\boldsymbol{y_4, y_5, y_6} \ldots\ldots\ldots..$ **,we must required four prior values** $\boldsymbol{y_0, y_1, y_2\, and\, y_3}$ **of** y .

Example: Using Milne's method, solve numerically the equation

$$\frac{dy}{dx} = x^2(1 + y)$$

Subject to $y(1) = 1$, $y(1.1) = 1.233$, $y(1.2) = 1.548$, $y(1.3) = 1.979$ *and evalute* $y(1.4)$

Solution: Given that

$$\frac{dy}{dx} = x^2(1 + y)$$

$\therefore$ we have ,

$\boldsymbol{x_0 = 1, x_1 = 1.1, x_2 = 1.2, x_3 = 1.3, x_4 = 1.4}$

$\boldsymbol{y_0 = 1, y_1 = 1.233, y_2 = 1.548, y_3 = 1.979, y_4} = ?$

We have by Milne's method,

$$y_{n+1,p} = y_{n-3} + \frac{4h}{3}[2\,y'_{n-2} - y'_{n-1} + 2\,y'_{n}] \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

And the Corrector formula for finding y_{n+1}

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}[2\,y'_{n-1} + 4\,y'_{n} + y'_{n+1}] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Now to find $\boldsymbol{y_4}$ **,** put $n = 3,$ *in* (1)

$$y_{4,p} = y_0 + \frac{4h}{3}[2\,y'_1 - y'_2 + 2\,y'_3] \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

For this , we have $\boldsymbol{y_0} = \boldsymbol{1}$,

$$y'_1 = x^2_1(1 + \boldsymbol{y_1}) = 2.7019$$

$$y'_2 = x^2_2(1 + \boldsymbol{y_2}) = 3.6691$$

$y'_3 = x^2_3\, 1 + y_{3.} = 5.0345$

Hence ,on putting all the above values in (3)

We get ,$y_{4,p} = 1 + \frac{4.0.1}{3}[2 . 2.7019 - 3.6691 + 2. 5.0345] = 2.5738$

$$y'_4 = x^2_4\, 1 + y_{4.} = 7.0046$$

Now ,we shall correct this value of $y_{4.}$ **of** y by the corrector formula (2) as

put$n = 3, in\ 2$

$y_{4,c} = y_2 + \frac{h}{3}[2 y'_2 + 4 y'_3 + y'_4] = 2.5750$

i.e.$y\ 1.4 = 2.5750$

Method-VI: Numerical Solution to Ordinary Differential Equation By Runge-Kutta method [i.e. Fourth order Runge-Kutta method]

Runge-Kutta method is more accurate method of great practical importance. The working procedure is as follows,

To solve $\frac{dy}{dx} = f\ x, y$ -------------------------- (1)

Subject to $y\ x_0 = y_{0.}$ by Runge-Kutta method, compute $y_{1.}$ as followes

$k_{1.} = h\, f\ x_0, y_{0.}$

$k_{2.} = h\, f\ x_0 + \frac{h}{2}, y_0 + \frac{k_{1.}}{2}$

$k_3 = h\, f\ x_0 + \frac{h}{2}, y_0 + \frac{k_{2.}}{2}$

$k_4 = h\, f\ x_0 + \boxed{?}, y_0 + k_{3.}$

$\therefore$ **wehave**

$$K = \frac{k_{1.} + 2\ \ k_{2.} + \ \ k_3 + \ \ k_4}{6}$$

Hence, $y_{1.} = y_{0.} + K = y\ x_1$

Similarly, we can compute $y\ x_2 = y_{2.}$

$$k_1 = h\, f\, x_1\,, y_1$$

$$k_2 = h\, f\ \ x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}$$

$$k_3 = h\, f\ \ x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}$$

$$k_4 = h\, f\ x_1 + ⍰, y_1 + k_3$$

$\therefore$ **wehave**

$$K = \frac{k_1 + 2\ \ k_2 + \ \ k_3\ + \ \ k_4}{6}$$

Hence, $y_2 = y_1 + K = y\ x_2$ and so on for succeeding intervals.

Therefore , we can notice that the only change in the formulae for succeeding intervals is in the values of $x\ and\ y$.This refinement was carried out by two German mathematicians C.D.T. Runge (1856-1927) and M.W.Kutta (1867-1944).

Example 1. UsingRunge-Kutta method to find $yw⍰enx = 1.2\ i.e. y\ \ 1.2\ \ instepsof\ 0.1$

$$\frac{dy}{dx} = x^2 + y^2 \text{ Subject to } y\ 1 = 1.5$$

Solution: Given that $\frac{dy}{dx} = x^2 + y^2$ Subject to $y\ 1 = 1.5$

$$\therefore \quad \text{we have , } ⍰ = 0.1$$

$\boldsymbol{x_0 = 1, x_1 = 1.1, x_2 = 1.2,}$

$\boldsymbol{y_0 = 1.5, y_1} = ?\,, \boldsymbol{y_2} = ?\,,$

First , we will compute $y\ 1.1\ i.e.\ \boldsymbol{y_1}$

For this , we have

$$k_1 = h\, f\, x_0\,, y_0 = 0.1\ 1^2 + 1.5\ ^2 = 0.325$$

$$k_2 = h\, f\ \ x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2} = 0.1\ 1.05\ ^2 + 1.6625\ ^2 = 0.3866$$

$$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_{2.}}{2}\right) = 0.1\left[(1.05)^2 + (1.6933)^2\right] = 0.3969$$

$$k_4 = h f\left(x_0 + ?, y_0 + k_{3.}\right) = 0.1\left[(1.1)^2 + (1.8969)^2\right] = 0.4808$$

$\therefore$ **wehave**

$$K = \frac{k_{1.} + 2(k_{2.} + k_3) + k_4}{6} = 0.3954$$

Hence, $\mathbf{y_{1.} = y_{0.} + K = y(x_1) = 1.5 + 0.3954 = 1.8954}$

Similarly, we can compute $y(x_2) = y_{2.}$

For this, we have

$$k_{1.} = h f(x_1, y_{1.}) = 0.1\left[(1.1)^2 + (1.8954)^2\right] = 0.4802$$

$$k_{2.} = h f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_{1.}}{2}\right) = 0.1\left[(1.1 + 0.05)^2 + (1.8954 + 0.2401)^2\right]$$
$$= 0.5882$$

$$k_3 = h f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_{2.}}{2}\right) = 0.6116$$

$$k_4 = h f\left(x_1 + ?, y_1 + k_{3.}\right) = 0.7725$$

$\therefore$ we have

$$K = \frac{k_{1.} + 2(k_{2.} + k_3) + k_4}{6} = 0.60815$$

Hence, $y_{2.} = y_{1.} + K = y(x_2) = 2.5035$

Method-VII:Numerical Solution to Ordinary Differential Equation By Adam's Bash forth method

It is thepredictor-corrector method ,in which, we first predict a value of y_1 by using a certain formula and then correct this value by using a more accurate formula. i.e.

Let us consider the equation

$$\frac{dy}{dx} = f(x, y) \qquad \text{--------------------------} \qquad (1)$$

Subject to $y\ x_0 = y_0.$

Then the predictor formula for finding y_1 is

$$y_{1,p} = y_0 + \frac{h}{24}[55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3}] \ldots \ldots \ldots \ldots \ldots . . (2)$$

It is called **Adam's Bashforth** predictor formula.

And the Corrector formula for finding y_1

$$y_{1,c} = y_0 + \frac{h}{24}[9f_1 + 19f_0 - 5f_{-1} + f_{-2}] \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (3)$$

Now from above, it is clear that for applying this method , we require four starting values of y.

Note :-In practice ,the Adam's Bash forth method together with fourth order Runge-Kutta method have been found to be most useful.

Example: UsingAdam'sBashforthmethod, solve the equation

$$\frac{dy}{dx} = x^2\ 1 + y$$

Subject to $y\ 1 = 1$, $y\ 1.1 = 1.233$, $y\ 1.2 = 1.548$, $y\ 1.3 = 1.979\ and\ evalute\ y\ 1.4$

Solution: Given that

$$\frac{dy}{dx} = x^2\ 1 + y$$

$\therefore$ we have ,

$\boldsymbol{x_0 = 1.0, x_{1.} = 1.1, x_{2.} = 1.2, x_{3.} = 1.3, x_{4.} = 1.4}$

$\boldsymbol{y_{-3.} = 1.000, y_{-2.} = 1.233, y_{-1.} = 1.548\ , y_{0.} = 1.979, y_{4.} = i.e.}\ y_{1.} = \ ?$

Hence , $f_{-3} = 1.0^{\ 2}\ 1 + 1.000 = 2.000$

Similarly $f_{-2} = 1.1^{\ 2}\ 1 + 1.233 = 2.702$, $f_{-1} = 1.2^{\ 2}\ 1 + 1.548 = 3.669$

, $f_0 = 1.3^{\ 2}\ 1 + 1.979 = 5.035$

Now, the predictor formula for finding y_1 is

$$y_{1,p} = y_0 + \frac{h}{24}[55 f_0 - 59 f_{-1} + 37 f_{-2} - 9 f_{-3}] = 2.573$$

and $f_1 = 1.4^2 (1 + 2.573) = 7.004$

And the Corrector formula for finding y_1

$$y_{1,c} = y_0 + \frac{h}{24}[9 f_1 + 19 f_0 - 5 f_{-1} + f_{-2}]$$

$$= 1.979 + \frac{0.1}{24}[9 \times 7.004 + 19 \times 5.035 - 5 \times 3.669 + 2.702] = 2.575$$

i.e. $y(1.4) = 2.575$.

Numerical Solution of Partial differential equations:**[**Finite difference solution two dimensional Laplace equation and Poission equation, Implicit and explicit methods for one dimensional heat equation (Bender-Schmidt and Crank- Nicholson methods), Finite difference explicit method for wave equation]:Many physical problems are mathematically modeled as boundary value problems ,associated with second order Partial differential equations. These second order Partial differential equations are classified into three distinct types. They are elliptic, parabolic and hyperbolic. A general second order linear Partial differential equation is of the form

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G \text{ ....(1)}$$

Where A, B, C, D, E, F, G are all functions of x, y.

[I]If $B^2 - 4AC < 0$ at a point in the x, y plane, then the equation (1) is called Elliptic

The standard examples are

(1) Two dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

(2) Two dimensional Poission equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$$

[II]If $B^2 - 4\,AC = 0$ at a point in the x, y plane, then the equation (1) is called parabolic.

The standard example is one dimensional heat equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial u}{\partial t}$$

[III]If $B^2 - 4\,AC > 0$ at a point in the x, y plane, then the equation (1) is called hyperbolic.

The standard example is one dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

Numerical methods for solving boundary value problems for each three types are slightly different from one another.Most commonly used numerical method to solve such problems is termed as finite difference method or net method. In this method, the derivatives appearing in the equation and the boundary conditions are replaced by their finite difference approximations. Then the given equation is changed into a system of linear equations, which are solved by iterative procedures.This process is slow but produces good resultsin many boundary value problems.

Geometrical representation of Partial Difference Quotients:

The x, y plane is divided into a series of rectangles of sides $\Delta x =$? and $\Delta y = k$ by equidistant lines drawn parallel to the axis of coordinates. As shown in above figure (1).

			(i, j+2)		
K=1			(i, j+1)		
	(i-2, j)	(i-1, j)	(i, j)	(i+1, j)	(i+2, j)
			(i, j-1)		
			(i, j-2)		

h=1

Now ,we can interpret the above idea in a different notation by drawing two sets of parallel lines $x = i\boxed{?}$ and $y = jk$, $i = j = 0,1,2, \ldots \ldots \ldots$

The point of intersection of these family of lines are called mesh points or lattice points. The point i, j is called the grid point and is surrounded by the neighboring points as shown in below figure (2).

If u is a function of two variables x, y then the values of $u\ x, y$ at the point i, j is denoted by $u_{i,j}$

We can write

$$\frac{\partial u}{\partial x} = u_x = \frac{u_{i+1,j} - u_{i,j}}{\boxed{?}}$$
$$= \frac{u_{i,j} - u_{i-1,j}}{\boxed{?}}$$
$$= \frac{u_{i+1,j} - u_{i-1,j}}{2\boxed{?}},$$

$$\frac{\partial u}{\partial y} = u_y = \frac{u_{i,j+1} - u_{i,j}}{k}$$
$$= \frac{u_{i,j} - u_{i,j-1}}{k}$$
$$= \frac{u_{i,j+1} - u_{i,j-1}}{2k}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{\boxed{?}^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2}$$

Now replacing the derivatives in any partial differential equation by their corresponding difference approximations, we obtain the finite difference analogues of the given equations.

[I] **Numerical Solution ofElliptic equations by Finite difference method**:-

An important equation of the **Elliptic type is** Two dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ ....(1)}$$

This type of equation arises in potential and steady state flow problems. The solution u x, y of (1) is satisfied at every point of the region subject to given boundary conditions on the closed curve. Consider a rectangular region R for which u x, y is known at the boundary. Divide this region$R$ for which u x, y is Known at the boundary.Divide this region into a network of square mesh of side $\boxed{?}$.

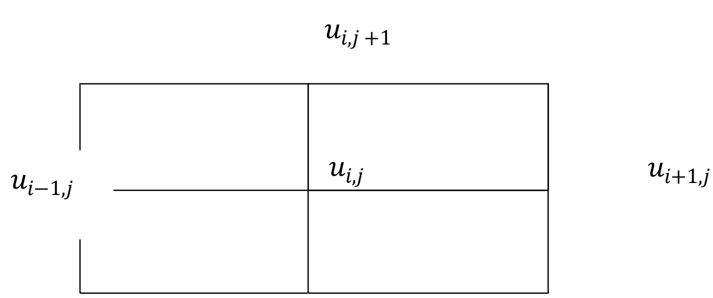
Now, replacing the derivatives in (1) by their difference approximations,

$$\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{\boxed{?}^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2} = 0$$

Since $k = \boxed{?}$,

$$\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{\boxed{?}^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{\boxed{?}^2} = 0$$

$$\Rightarrow u_{i,j} = \frac{u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1}}{4} \quad \text{.......................... (2)}$$

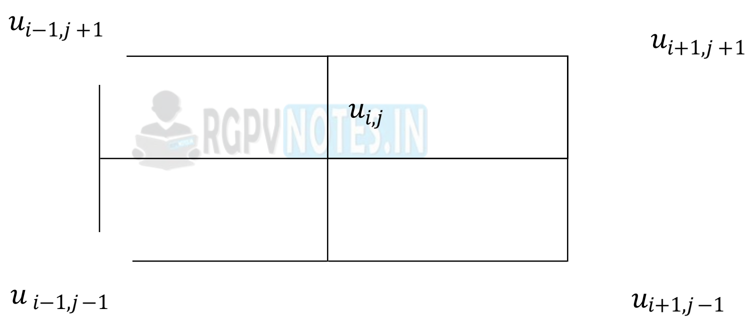


$u_{i,j-1}$

This shows that the value of $u_{i,j} = \mathrm{u}\ x, y$ at any interior mesh point is the average of it's values at four neighboring points to left, right, above and below.This is called the Standard five point formula (SFPF) or as Lineman's averaging procedure.

Sometimes a formula similar to (2) is used which is given by

$$=> u_{i,j} = \frac{u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j-1}}{4} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$



This shows that the value of $u_{i,j} = \mathrm{u}\ x, y$ is the average (or Arithmetic mean) of it's values at the four neighboring diagonal mesh points.This is called the Diagonal five point formula (DFPF).
Although (3) is less accurate than (2), yet it serves as a reasonably good approximation for obtaining the starting values at the mesh points. In the iteration procedure, but wheneverpossible formula (2) is preferred in comparison to formula (3).
Now to solve a Laplace equation by finite difference method,we adapt the following set procedure

Suppose that the given boundary values are a_1, a_2 ,a_3…………..,a_{16}.Now ,we are to determine initial values of $u_{i,j} = \mathrm{u}\ x, y$ at the interior mesh points region R. Since the value u_5 at the center an therefore the values u_1, u_3, u_7, u_9, are computed by using Diagonal five point formula (DFPF), therefore ,we have

$u_5 = \frac{1}{4}[a_1 + a_9 + a_{13} + a_5]$; $u_1 = \frac{1}{4}[a_{15} + a_{11} + a_{13} + u_5]$; $u_3 = \frac{1}{4}[u_5 + a_9 + a_{11} + a_7]$;

$u_7 = \frac{1}{4}[a_1 + u_5 + a_{15} + a_3]$; $u_9 = \frac{1}{4}[a_3 + u_5 + a_7 + a_5]$

The values at the remaining interior mesh points i.e. the values of u_2, u_4, u_6, u_8, are computed by using, Standard five point formula (SFPF) therefore, we have

$u_2 = \frac{1}{4}[u_1 + u_3 + a_{11} + u_5]$; $u_4 = \frac{1}{4}[a_{15} + u_5 + u_1 + u_7]$; $u_6 = \frac{1}{4}[u_5 + u_7 + u_3 + u_9]$;

$$u_8 = \frac{1}{4}[u_7 + u_9 + u_5 + a_3]$$

Thus ,we have computed all values $u_1, u_2, u_3, \ldots, u_9$ once. The accuracy of $u_1, u_2, u_3, \ldots, u_9$ are improved by the repeated application of the any one of the following iterative formulae.

(i) Gauss sJacobi's iterative method

$$u_{i,j}^{(n+1)} = \frac{1}{4}[u_{i-1,j}^{n} + u_{i+1,j}^{n} + u_{i,j+1}^{n} + u_{i,j-1}^{n}]$$

Where $n = 0,1,2, \ldots \ldots$ be the no. of iterations.

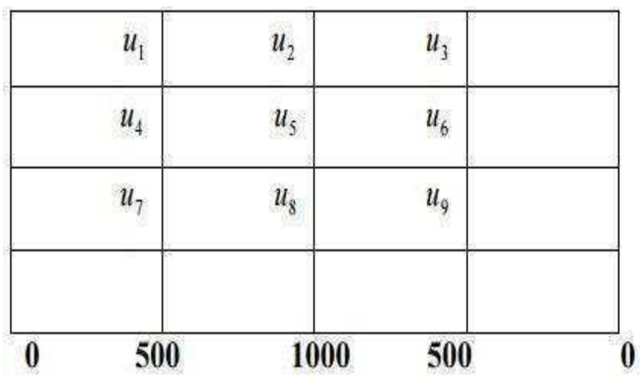
(ii) Gauss s Seidal 's iterative method

$$u_{i,j}^{(n+1)} = \frac{1}{4}[u_{i-1,j}^{n+1} + u_{i+1,j}^{n} + u_{i,j+1}^{n+1} + u_{i,j-1}^{n}]$$

Where $n = 0,1,2, \ldots \ldots$ be the no. of iterations.

This method of finite difference is well explained by Example.

1.Solve the equation $\nabla^2 u = 0$ for the following mesh,with boundary vaues as shown using Leibmann's iteration process.



Sol:

Let u_1 , u_2 ……u_9 be the values of u at the interior mesh points of the given region.By symmetry about the lines AB and the line CD,we observe

$$u_1 = u_3 \qquad u_1 = u_7$$

$$u_2 = u_8 \qquad u_4 = u_6$$

$$u_3 = u_9 \qquad u_7 = u_9$$

$$u_1 = u_3 = u_7 = u_9, u_2 = u_8, u_4 = u_6$$

Hence it is enough to find u_1 , u_2 , u_4 , u_5

Calculation of rough values

u_5 =1500

u_1 =1125

u_2 =1187.5

u_4 =1437.5

Gauss-seidel Scheme

$$u_1 = \frac{1}{4}[1500 + u_2 + u_4]$$

$$u_2 = \frac{1}{4}[2u_1 + u_5 + 1000]$$

$$u_4 = \frac{1}{4}[2000 + u_5 + u_4]$$

$$u_5 = \frac{1}{4}[2u_2 + 2u_4]$$

.

The iteration values are tabulated as follows

Iteration No k	u_1	u_2	u_4	u_5
0	**1500**	**1125**	**1187.5**	**1437.5**
1	**1031.25**	**1125**	**1375**	**1250**
2	**1000**	**1062.5**	**1312.5**	**1187.5**
3	**968.75**	**1031.25**	**1281.25**	**1156.25**
4	**953.1**	**1015.3**	**1265.6**	**1140.6**
5	**945.3**	**1007.8**	**1257.8**	**1132.8**
6	**941.4**	**1003.9**	**1253.9**	**1128.9**
7	**939.4**	**1001.9**	**1251.9**	**1126.9**
8	**938.4**	**1000.9**	**1250.9**	**1125.9**
9	**937.9**	**1000.4**	**1250.4**	**1125.4**
10	**937.7**	**1000.2**	**1250.2**	**1125.2**
11	**937.6**	**1000.1**	**1250.1**	**1125.1**
12	**937.6**	**1000.1**	**1250.1**	**1125.1**

$u_1 = u_3 = u_7 = u_9 = 937.6, u_2 = u_8 = 1000.1, u_4 = u_6 = 1250.1, \ u_5 = 1125.1$

When steady state condition prevail, the temperature distribution of the plate is represented by Laplace equation $u_{xx}+u_{yy}=0$.The temperature along the edges of the square plate of side 4 are given by along x=y=0,u=x^3 along y=4 and u=16y along x=4, divide the square plate into 16 square meshes of side h=1 ,compute the temperature a iteration process.

Solution of Poisson equation:-

An equation of the type $\nabla^2 u = f(x,y)$ i.e., is called $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f(x,y)$ poisson's equation where f(x,y) is a function of x and y.

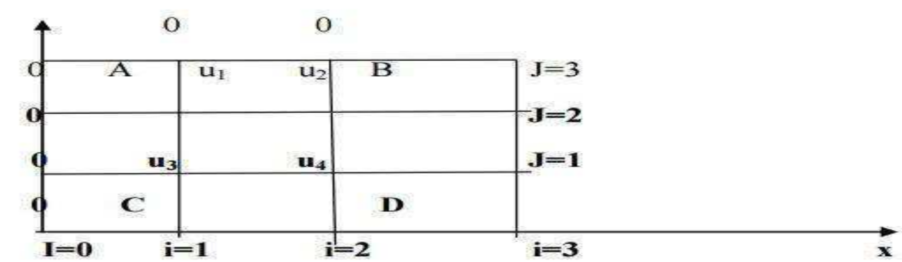
$$u_{i-1,j}+u_{i+1,j}+u_{i,j-1}-4u_{i,j}=h^2 f(ih,jh)$$

This expression is called the replacement formula.Applying this equation at each internal mesh point ,we get a system of linear equations in u_i,where u_i are the values of u at the internal mesh points.Solving the equations, the values u_i are known.

Problems

1.Solve the poisson equation $\nabla^2 u=-10(x^2+y^2+10)$ over the square mesh with sides x=0,y=0,x=3,y=3 and u=0 on the boundary .assume mesh length h=1 unit.

Sol:



Here the mesh length $\Delta x=h=1$

Replacement formula at the mesh point (i,j)

$$u_{i-1,j}+u_{i+1,j}+u_{i,j-1}+u_{i,j+1}-4u_{i,j}=-10(i^2+j^2+10) \qquad (1)$$

$$u_2+u_3-4u_1=-150$$
$$u_1+u_4-4u_3=-120$$
$$u_2+u_3-4u_4-150$$
$$u_1=u_4=75, u_2=82.5, u_3=67.5$$

2.Solve the poisson equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=-81xy, 0<x<1; 0<y<1$ and u(0,y)=u(x,0)=0,u(x,1)=u(1,y)=100 with the square meshes ,each of length h=1/3.

u(0,y)=u(x,0)=0,u(x,1)=u(1,y)=100 with the square meshes ,each of length h=1/3.

[II] Solution of One dimensional heat equation:-

In this session, we will discuss the finite difference solution of one dimensional heat flow equation by Explicit and implicit method

Explicit Method (Bender-Schmidt method)

Consider the one dimensional heat equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ .This equation is an example of parabolic equation.

$$u_{i,j+1} = \lambda u_{i+1,j} + 1 - 2\lambda u_{i,j} + \lambda u_{i-1,j} \qquad (1)$$

Where $\lambda = \frac{k}{ah^2}$

Expression (1) is called the explicit formula and it valid for $0 < \lambda \le \frac{1}{2}$

If λ=1/2 then (1) is reduced into

$$u_{i,j+1} = \frac{1}{2}[u_{i+1,j} + \lambda u_{i-1,j}] \qquad (2)$$

This formula is called Bender-Schmidt formula.

This formula is called Bender-Schmidt formula.

Implicit method (Crank-Nicholson method)

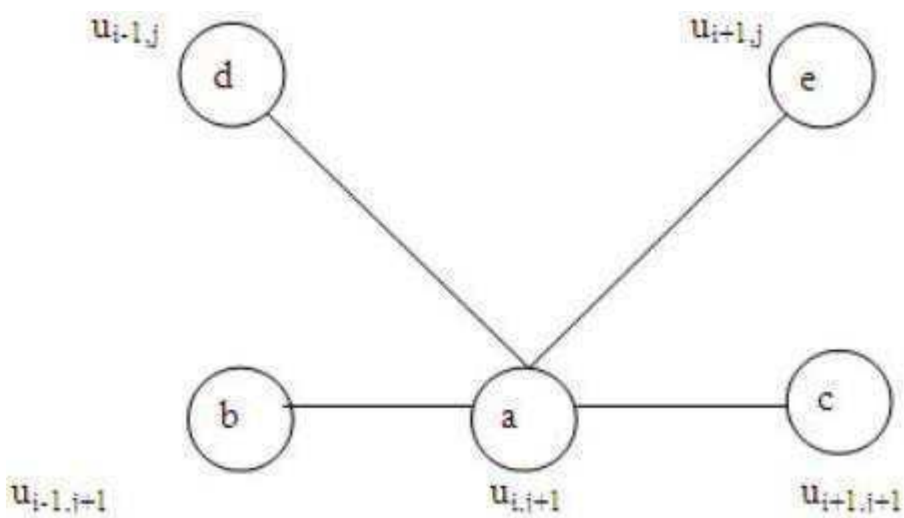
$$-\lambda u_{i-1,j+1} + 2(1+\lambda)u_{i,j+1} - \lambda u_{i+1,j+1} = \lambda u_{i-1,j} + 2(1-\lambda)u_{i,j} + \lambda u_{i+1,j}$$

This expression is called Crank-Nicholson’s implicit scheme. We note that Crank Nicholson’s scheme converges for all values of λ

When $\lambda=1$, i.e., $k=ah^2$ the simplest form of the formula is given by

$$\Rightarrow u_{i,j+1} = \frac{1}{4}[u_{i+1,j+1} + u_{i-1,j+1} + u_{i-1,j} + u_{i+1,j}]$$

The use of the above simplest scheme is given below.



The value of u at A=Average of the values of u at B, C, D, E

Note

In this scheme, the values of u at a time step are obtained by solving a system of linear equations in the unknown'su$_i$.

Example: when u(0,t)=0,u(4,t)=0 and with initial condition u(x,0)=x(4-x) upto t=sec

By initial conditions, u(x,0)=x(4-x) ,we have

1. **Solve** $u_{xx} = 2u_t$ **assuming** Δ**x=h=1**

Sol: By Bender-Schmidt recurrence relation,

$$u_{i,j+1} = \frac{1}{2}[u_{i+1,j} + \lambda u_{i-1,j}] \qquad (1)$$

$$k = \frac{ah^2}{2}$$

For applying eqn(1) ,we choose
Here a=2,h=1.Then k=1
By initial conditions, u(x,0)=x(4-x) ,we have

$$u_{i,0} = i(4-i) i = 1,2,3$$

$$, u_{1,0} = 3, u_{2,0} = 4, u_{3,0} = 3$$

By boundary conditions, u(0,t)=0,u_0=0,u(4,0)=0 $\Rightarrow u_{4,j} = 0 \forall j$

Values of u at t=1

$$u_{i,1} = \frac{1}{2}[u_{i-1,0} + u_{i+1,0}]$$

$$u_{1,1} = \frac{1}{2}[u_{0,0} + u_{2,0}] = 2$$

$$u_{2,1} = \frac{1}{2}[u_{1,0} + u_{3,0}] = 3$$

$$u_{3,1} = \frac{1}{2}[u_{2,0} + u_{4,0}] = 2$$

The values of u up to t=5 are tabulated below.

j\i	0	1	2	3	4
0	0	3	4	3	0
1	0	2	3	2	0
2	0	1.5	2	1.5	0
3	0	1	1.5	1	0
4	0	0.75	1	0.75	0
5	0	0	0.75	0.5	0

2.Solve the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ **subject to the conditions u(0,t)=u(5,t)=0 and** $u(x,0) = x^2(25 - x^2)$ **taking h=1 and k=1/2,tabulate the values of u upto t=4 sec.**

Sol:

Here a=1,h=1

For λ=1/2,we must choose $k=ah^2/2$

K=1/2

By boundary conditions

$$u(0,t)=0 \Rightarrow u_{0,j}=0 \forall j$$
$$u(5,t)=0 \Rightarrow u_{5,j}=0 \forall j$$
$$u(x,0)=x^2(25-x^2)$$
$$\Rightarrow u_{i,0}=i^2(25-i^2), i=0,1,2,3,4,5$$
$$u_{1,0}=24, u_{2,0}=84, u_{3,0}=144, u_{4,0}=144, u_{5,0}=0$$

By Bender-schmidt realtion,

$$u_{i,j+1}=\frac{1}{2}[u_{i+1,j}+u_{i-1,j}]$$

The values of u upto 4 sec are tabulated as follows

j\i	0	1	2	3	4	5
0	0	24	84	144	144	0
0.5	0	42	84	144	72	0
1	0	42	78	78	57	0
1.5	0	39	60	67.5	39	0
2	0	30	53.25	49.5	33.75	0
2.5	0	26.625	39.75	43.5	24.75	0
3	0	19.875	35.0625	32.25	21.75	0
3.5	0	17.5312	26.0625	28.4062	16.125	0
4	0	13.0312	22.9687	21.0938	14.2031	0

[III] Solution of One dimensional wave equation:-

Introduction

The one dimensional wave equation is of hyperbolic type. In this session, we discuss the finite difference solution of the one dimensional wave equation

$$u_{tt} = a^2 u_{xx}.$$

Explicit method to solve $u_{tt} = a^2 u_{xx}$

$$u_{i,j+1} = 2(1-\lambda^2 a^2)u_{i,j} + \lambda^2 a^2 u_{i-1,j} + u_{i+1,j} - u_{i,j-1} \qquad \textbf{(1)}$$

Where λ=k/h

Formula (1) is the explicit scheme for solving the wave equation.

Problems

1.Solve numerically , $4u_{xx} = u_{tt}$ with the boundary conditions u(0,t)=0,u(4,t)=0 and the initial conditions $u_t(x,0) = 0$ & $u(x,0) = x(4-x)$, taking h=1.Compute u upto t=3sec.

Sol:

Here a^2=4

A=2 and h=1

We choose k=h/a$\Rightarrow$k=1/2

The finite difference scheme is

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$$
$$u(0,t) = 0 \Rightarrow u_{0,j} \ \& \ u(4,t) = 0 \Rightarrow u_{4,j} = 0 \forall j$$
$$u(x,0) = x(4-x) \Rightarrow u_{i,0} = i(4-i), i = 0,1,2,3,4$$
$$u_{0,0} = 0, u_{1,0} = 3, u_{2,0} = 3, u_{4,0} = 0$$
$$u_{1,1} = 4 + 0/2 = 2$$
$$u_{2,1} = 3, u_{3,1} = 2$$

The values of u for steps t=1,1.5,2,2.5,3 are calculated using (1) and tabulated below.

j\i	**0**	**1**	**2**	**3**	**4**
0	0	3	4	3	0
1	0	2	3	2	0
2	0	0	0	0	0
3	0	-2	3	-2	0
4	0	-3	-4	3	0
5	0	-2	-3	-2	0
6	0	0	0	0	0

2.Solve $u_{xx} = u_{tt}$ given u(0,t)=0,u(4,t)=0,u(x,0)=$u(x,0)=\frac{x(4-x)}{2}$ & $u_t(x,0)=0$. Take h=1.Find the solution upto 5 steps in t-direction.

Sol:

Here $a^2=4$

A=2 and h=1

We choose k=h/a$\Rightarrow$k=1/2

The finite difference scheme is

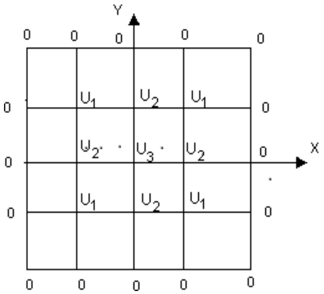
$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$$
$$u(0,t) = 0 \Rightarrow u_{0,j} \;\&\; u(4,t) = 0 \Rightarrow u_{4,j} = 0 \forall j$$
$$u(x,0) = x(4-x)/2 \Rightarrow u_{i,0} = i(4-i)/2, i = 0,1,2,3,4$$
$$u_{0,0} = 0, u_{1,0} = 1.5, u_{2,0} = 2, u_{3,0} = 1.5, u_{4,0} = 0$$
$$u_{1,1} = 1$$
$$u_{2,1} = 1.5, u_{3,1} = 1$$

The values of u upto t=5 are tabulated below.

j\i	0	1	2	3	4
0	0	1.5	2	1.5	0
1	0	1	1.5	1	0
2	0	0	0	0	0
3	0	-1	-1.5	-1	0
4	0	-1.5	-2	-1.5	0
5	0	-1	-1.5	-1	0

Practice Problems:

Example: Solve the equation $u_{xx} + u_{yy} = 8x^2 y^2$ over the square mesh of following figure with u x, y =0 on the boundary and mesh length=1.



Example: Solve the equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -10(x^2 + y^2 + 10)$ over the square with sides $x = y = 0$, to $x = y = 3$ with $u(x, y) = 0$ on the boundary and mesh length is one.

