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Contents

Numerical Differentiation

In numerical analysis, numerical differentiation describes algorithms for estimating the derivative of a mathematical function or function subroutine using values of the function and perhaps other knowledge about the function. i.e. Numerical different ion is the process of calculating the derivative of a function at some particular value of the independent variable by means of a set of given values of that function.

[A] **Derivative at any point** :- The general method of numerical differentiation of a function consists in obtaining an explicit analytical relation $y = f(x)$ with the help of an interpolation formula ,and then differentiating y with respect to x as many times as required.

Let the function $y = f(x)$ be obtained by the Newton Gregory's forward interpolation formula

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Where $u = \frac{x - x_0}{h}$,

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$$\frac{dy}{du} = \Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2-6+2}{3!}\Delta^3 y_0 + \frac{4u^3-18u^2+22u-6}{4!}\Delta^4 y_0 + \cdots ....$$

and $\frac{du}{dx} = \frac{1}{h}$

Therefore, $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$$= \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2-6+2}{3!}\Delta^3 y_0 + \frac{4u^3-18u^2+22u-6}{4!}\Delta^4 y_0 + \cdots \ldots\right] \ldots.(2)$$

The second derivative $\frac{d^2y}{dx^2}$ may be determined by differentiating (2) with respect to x

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + u - 1\,\Delta^3 y_0 + \frac{6u^2-18u+11}{12}\Delta^4 y_0 + \frac{2u^3-12u^2+21u-10}{12}\Delta^5 y_0 + \cdots \ldots\right]$$

Similarly, derivatives of other higher orders may be obtained.

Example: A rod is rotating in a plane about one of it's ends .If the following table gives the angle θ radians throughh ,which the rod has turned for different values of time t seconds, find it's angular velocity and angular acceleration ,when $t = 0.7$ second.

t seconds	0.0	0.2	0.4	0.6	0.8	1.0
θ radians	0.0	0.12	0.48	1.10	2.0	3.20

Solution: First , we will construct the difference table :

t seconds	**θ radians**	$\Delta\theta$	$\Delta^2\theta$	$\Delta^3\theta$	$\Delta^4\theta$	$\Delta^5\theta$
0.0	0.0	0.12				
0.2	0.12	0.36	0.24	0.02	0.00	0.00

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0.4	0.48	0.62	0.26	0.02	0.00	
0.6	1.10	0.90	0.28	0.02		
0.8	2.0	1.20	0.30			
1.0	3.20					

Then the angular velocity $\frac{d\theta}{dt}$ is

$$=\frac{1}{h}\left[\Delta\theta_0+\frac{2u-1}{2!}\Delta^2\theta_0+\frac{3u^2-6u+2}{3!}\Delta^3\theta_0+\frac{4u^3-18u^2+22u-6}{4!}\Delta^4\theta_0+\cdots\ldots\right]\ldots\ldots\ldots(1)$$

Where $u=\frac{t-t_0}{h},=\frac{t-0.0}{0.2}=5t$

At $t=0.7$,then $u=3.5$

Therefore, the angular velocity $\frac{d\theta}{dt}=4.496$ radian per second.

The angular acceleration $\frac{d^2\theta}{dt^2}$ is obtained with the help of equation (1),

$$\frac{d^2\theta}{dt^2}=\frac{1}{h}\left[0+\frac{2}{2!}\Delta^2\theta_0+\frac{6u-6}{3!}\Delta^3\theta_0\right]\frac{du}{dt}=\frac{1}{h^2}[\Delta^2\theta_0+(u-1)\Delta^3\theta_0]$$

At $t=0.7$, then $u=3.5$

Therefore, the angular acceleration

$$\frac{d^2\theta}{dt^2}=\frac{1}{(0.2)^2}[0.24+(3.5-1)0.02]=7.25 \text{ radian per } second^2.$$

[B] **Derivative at tabulated points**: -In particular case ,When the derivatives are required at one of the tabulated points (x_0, y_0), (x_1, y_1),..........................(x_n, y_n), and not at the intermediate points, the following formula may be used with advantage.

If D denotes the differential operator $\frac{d}{dx}$ and Δ is the difference operator defined by

$\Delta f(x)=f(x+h)-f(x)$,then there holds the operational relation

$$hD=\log(1+\Delta)$$

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i.e.$D \equiv \frac{1}{h}[\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \cdots \ldots \ldots .]$

Where, h is the interval between any two successive values of x ,at which the values of y are prescribed.

Similarly,.$D^2 \equiv \frac{1}{h^2}[\Delta^2 - \Delta^3 + \frac{11}{12}\Delta^4 - \frac{5}{6}\Delta^5 + \cdots \ldots \ldots .]$

Example :A slider in a machine moves along a fixed straight rod. It's distance x cm along the road are given in the following table for various values of the time t seconds

t seconds	0.0	0.1	0.2	0.3	0.4	0.5
$\boldsymbol{x}$ **cm**	30.1	31.6	32.9	33.6	40.0	33.8

Find the velocity and acceleration, when $t = 0.3$ second

Solution : First we will construct the difference table :

t seconds	$\boldsymbol{x}$ **cm**	$\boldsymbol{\Delta x}$	$\boldsymbol{\Delta^2 x}$	$\boldsymbol{\Delta^3 x}$	$\boldsymbol{\Delta^4 x}$	$\boldsymbol{\Delta^5 x}$
0.0	30.1	1.5				
0.1	31.6	1.3	-0.2	-0.4	6.7	-31.3
0.2	32.9	0.7	-0.6	6.3	-24.6	
0.3	33.6	6.4	5.7	-18.3		
0.4	40.0	-6.2	-12.6			
0.5	33.8					

Then the velocity $\frac{\mathrm{dx}}{\mathrm{dt}}$ is

.$D \equiv \frac{1}{h}[\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \cdots \ldots \ldots .]$

Where $\mathrm{h} = 0.1$

Therefore, the velocity $\frac{\mathrm{dx}}{\mathrm{dt}} = \mathrm{Dx}$ at $t = 0.3$ is given by

$Dx_3 = \frac{1}{0.1}\Delta x_3 - \frac{1}{2}\Delta^2 x_3 + \frac{1}{3}\Delta^3 x_3 - \cdots \ldots \ldots . = 10\ \Delta x_3 - \frac{1}{2}\Delta^2 x_3$ =127 cm per second

Since other differences are not available in the table.

Similarly, acceleration, when$t = 0.3$ second

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$D^2 x_3 = \frac{1}{h^2}[\Delta^2 x_3 - \Delta^3 x_3 + \frac{11}{12}\Delta x_3^4 - \frac{5}{6}\Delta^5 x_3 + \cdots \ldots \ldots .]$= 100[$\Delta^2 x_3 - 0$]= -1260 cmper$second^2$.

Numerical integration

The process of computing the value of a definite integral $\int_a^b f(x)\,dx$ from a set of numerical values of the integrand is called Numerical integration. When applied to the integration of a function of one variable, the process is known as quadrature.

A General Quadrature Formula for Equidistant Ordinates:-

.We shall, now establish a general formula for the numerical integration, from which other special formulas will be deduced. For this, we assume that the values of $\boldsymbol{x}$ are at equal intervals .i.e$a = x_0$, $x_1 = x_0 + \mathrm{h}$, $x_2 = x_0 + 2\mathrm{h} \ldots \ldots \ldots \ldots \ldots .\ x_n = x_0 + n\mathrm{h} = \mathrm{b}$

.Let the function $y = f(x)$ be given by Newton-Gregory's forward interpolation formula's as

$$y = f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (1)$$

Where$u = \frac{x - x_0}{\mathrm{h},} => x = x_0 + u\mathrm{h}$

Integrate (1) on the both sides w.r.t.x over the limits e$a = x_0$ to $x_n = x_0 + n\mathrm{h} = \mathrm{b}$

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_0+n\mathrm{h}} f(x)\,dx$$

But $= x_0 + u\mathrm{h}$, $\therefore \mathrm{dx} = \mathrm{h\ du}$, When $x = x_0$, $then u = 0\ and x = x_0 + n\mathrm{h}$, then $\mathrm{u} = \mathrm{n}$

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$$\int_{x_0}^{x_0+nh} f(x)\,dx$$

$$= h\int_0^n \left[y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \dots\dots\dots\dots\dots \right] du$$

$$\Rightarrow \int_{x_0}^{x_0+nh} f(x)\,dx = h\left[n y_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2!}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\Delta^2 y_0 + \frac{1}{3!}\left(\frac{n^4}{4} - n^3 + n^2\right)\Delta^3 y_0 + \dots \right] \dots\dots\dots(2)$$ Where $h = \frac{b-a}{n}$

This is required general quadrature formula.

[I] Trapezoidal rule :- If ,we put $n = 1$ in (2) ,Then there are only two ordinates y_0, y_1, and only one finite difference $\Delta y_0 = y_1 - y_0$, exists, all other higher order finite differences become zero.

$$\int_{x_0}^{x_1} f(x)\,dx = h\left(y_0 + \frac{1}{2}\Delta y_0\right) = h\left[y_0 + \frac{1}{2}(y_1 - y_0)\right] = \frac{h}{2}[y_0 + y_1]$$

The geometrical meaning of this result is that the area between the curve $y = f(x)$,the x-axis and the ordinates y_0 and y_1 ,is approximated by the area of the trapezium who parallel sides are y_0 and y_1, and whose breadth is $h = x_1 - x_0$,

Similarly $\int_{x_1}^{x_2} f(x)\,dx = \frac{h}{2}[y_1 + y_2]$,

$$\int_{x_2}^{x_3} f(x)\,dx = \frac{h}{2}[y_2 + y_3]$$

And so on

$$\int_{n-1}^{x_n} f(x)\,dx = \frac{h}{2}[y_{n-1} + y_n]$$

Therefore, adding all the above results, We get

$$\int_a^b f(x)\,dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \cdots\dots\dots\dots + y_{n-1})]$$ Where $h = \frac{b-a}{n}$

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This is required **Trapezoidal rule.**

[II] Simpson's 1/3[rd] rule : If ,we put $n = 2$ in (2) ,Then there are only three ordinates y_0, y_1, y_2, and therefore only two finite differences Δy_0 and $\Delta^2 y_0$ are exist and all other differences become zero.

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}[y_0 + 4y_1 + y_2]$$

The geometrical meaning of this result is that the curve $y = f(x)$ between the lines $x = x_0$ and $x = x_2$ is approximated by the area of the parabola whose equation is

$y = y_0 + \frac{x-x_0}{h}\Delta y_0 + \frac{(x-x_0)(x-x_1)}{2!\,h^2}\Delta^2 y_0$ and which passes through the points $(x_0, y_0), (x_1, y_1), (x_2, y_2)$.

Proceeding in a similar way, we will get

$$\int_{x_2}^{x_4} f(x)\,dx = \frac{h}{3}[y_2 + 4y_3 + y_4],$$

$$\int_{x_4}^{x_6} f(x)\,dx = \frac{h}{3}[y_4 + 4y_5 + y_6],$$

And so on ……….

$$\int_{x_{n-2}}^{x_n} f(x)\,dx = \frac{h}{3}[y_{n-2} + 4y_{n-1} + y_n]$$

Therefore, adding all the above results, We get

$$\int_a^b f(x)\,dx = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2})]$$

This is required **Simpson's 1/3[rd] rule.** Where $h = \frac{b-a}{n}$

[III] Simpson's 3/8rule :- If ,we put $n = 3$ in (2) ,Then there are only four ordinates y_0, y_1, y_2, y_3, and therefore only three finite differences Δy_0, $\Delta^2 y_0$, and $\Delta^3 y_0$ are exist and all other differences become zero

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$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}[y_0 + 3(y_1 + y_2) + y_3]$$

The geometrical meaning of this result is that the curve $y = f(x)$ between the lines $x = x_0$ and $x = x_3$ is approximated by the area of the cubical parabola whose equation is

$$y = y_0 + \frac{x-x_0}{h}\Delta y_0 + \frac{(x-x_0)(x-x_1)}{2!\,h^2}\Delta^2 y_0 + \frac{(x-x_0)(x-x_1)(x-x_2)}{3!\,h^3}\Delta^3 y_0$$ and which

passes through the points $(x_0, y_0), (x_1, y_1), (x_2, y_2)$, and (x_3, y_3).

Proceeding in a similar way, we will get

$$\int_{x_3}^{x_6} f(x)\,dx = \frac{3h}{8}[y_3 + 3(y_4 + y_5) + y_6]$$

And so on ……….

$$\int_{x_{n-3}}^{x_n} f(x)\,dx = \frac{3h}{8}[y_{n-3} + 3(y_{n-2} + y_{n-1}) + y_n]$$

Therefore, adding all the above results, We get

$$\int_a^b f(x)\,dx = \frac{3h}{8}[(y_0 + y_n) + 2(y_3 + y_6 + \cdots .. + y_{n-3}) + 3(y_1 + y_2 + y_4 + \cdots + y_{n-1})]$$

This is required **Simpson's 3/8th rule.** Where $h = \frac{b-a}{n}$

Example : Evaluate the integral $\int_0^1 \frac{1}{1+x^2}dx$ by taking no. of subinterval 4 through, Trapezoidal rule, Simpson's 1/3rd rule and Simpson's 3/8th rule**.**

Solution: We know that
Where $h = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4}$

x	$y = \frac{1}{1+x^2}$
$x_0 = 0$	$y_0 = 1$
$x_1 = x_0 + h = \frac{1}{4}$	$y_1 = \frac{16}{17}$
$x_2 = x_0 + 2h = \frac{1}{2}$	$y_2 = \frac{4}{5}$

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$x_3 = x_0 + 3 ? = \frac{3}{4}$	$y_3 = \frac{16}{25}$
$x_4 = x_0 + 4 ? = 1$	$y_4 = \frac{1}{2}$

We know that by Trapezoidal rule

$$\int_a^b f(x)\,dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \cdots \ldots \ldots .. + y_{n-1})]$$

Hence from above table the formula becomes

$$\int_0^1 \frac{1}{1+x^2}dx = \frac{h}{2}[(y_0 + y_4) + 2(y_1 + y_2 + y_3)] = 0.782775$$

Now by Simpson's 1/3rd rule

$$\int_a^b f(x)\,dx = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + \cdots . + y_{n-1})$$
$$+ 2(y_2 + y_4 + \cdots y_{n-2})]$$

Hence from the above table the formula becomes

$$\int_0^1 \frac{1}{1+x^2}dx = \frac{h}{3}[(y_0 + y_4) + 4(y_1 + y_3) + 2(y_2)] = 0.7854.$$

Now by Simpson's 3/8th rule

$$\int_a^b f(x)\,dx = \frac{3h}{8}[(y_0 + y_n) + 2(y_3 + y_6 + \cdots .. + y_{n-3})$$
$$+ 3(y_1 + y_2 + y_4 + \cdots + y_{n-1})]$$

Hence from above table the formula becomes

$$\int_a^b f(x)\,dx = \frac{3h}{8}[(y_0 + y_4) + 2(y_3) + 3(y_1 + y_2)] = 0.7503$$

Numerical Solution of Simultaneous Linear Algebraic Equations

Simultaneous Linear Algebraic Equations are very common in various fields of Engineering and Science. We used matrix inversion method or Cramer's rule to solve these equations in general. But these methods prove to be tedious, when

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the system of equations contain a large number of unknowns. To solve such equations there are other numerical methods, which are particularly suited for computer operations. These are of two types:-

[I] **Direct Method**:

(1) Gauss's Elimination Method.

(2) Gauss's Jordan Method.

(3) Crout's methods .

[II] Iterative Method

(1) Gauss's Jacobi's Method
(2) Gauss-Seidal Method
(3) Relaxation method

[I] Direct Method

(1) Gauss's Elimination Method: In this method, the unknowns are eliminated successively and the system is reduced to an upper triangular system from which, the unknowns are found by back substitutions. The method is quite general and is well adapted for computer operations.

Let us consider a system of m **equations and in** n unknowns

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \text{--} [1]
\end{aligned}
$$

In this method for solving the above equations, we proceed stepwise asfollows :

Step-1:- Elimination of x_1 from the second, third, …..nt ⍰ equation. We assume here that the order of the equation and the order of unknowns in each equations are such that $a_{11} \neq 0$. The variable x_1 can then be eliminated from the second equation by subtracting $\left(\frac{a_{21}}{a_{11}}\right)$ times the first equation from the

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second equation $(\frac{a_{31}}{a_{11}})$ times the first equation from the third equation, e.t.c.

This gives new system say as follows:

$$a_{11}x_1 + a_{12}x_2 + \dots\dots\dots + a_{1n}x_n = b_1$$
$$a'_{22}x_2 + \dots\dots\dots. + a'_{2n}x_n = b'_2$$
$$\dots\dots\dots \quad \dots\dots\dots \quad \dots\dots\dots \qquad \text{-------------------------------- [2]}$$
$$a'_{n2}x_2 + \dots\dots\dots. + a'_{nn}x_n = b'_n$$

Here the first equation is called the pivotal equation and a_{11} is called the first pivot.

Step-2. Now, Elimination of x_2 from the third, ….. nt equation in (2).

If the coefficient $a'_{22}, \dots\dots.., a'_{nn}$ in (2) are not all zero,we may assume that the order of equation and the unknowns is such that $a'_{22} \neq 0$ .Then ,we may eliminate x_2 from the third, ….. nt equation of (2) by subtracting

$(\frac{a'_{32}}{a'_{22}})$times the second equation from the third equation,

$(\frac{a'_{42}}{a'_{22}})$times the second equation from the fourth equation e.t.c.

The further steps are now obvious. In the third step, we eliminate x_3 and in the fourth step, we eliminate x_4 e.t.c.

By successive elimination, we arrive at a single equation in the unknown x_n ,which can be solved and substituting this in the preceding equation, we obtain the value of x_{n-1}. In this,manner ,we find x_n when the elimination is completed. Also when the elimination is complete the system takes the form.

$$c_{11}x_1 + c_{12}x_2 + \dots\dots\dots + c_{1n}x_n = d_1$$
$$c_{22}x_2 + \dots\dots\dots. + c_{2n}x_n = d_2$$
$$\dots\dots\dots \quad \dots\dots\dots \quad \dots\dots\dots$$
$$c_{nn}x_n = d_n$$

In this case there exists a unique solution.The new coefficient matrix is an upper triangular matrix ; the diagonal element c_{ii} are usually equal to 1.

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Example: - Apply Gauss's Elimination Method to solve the equations

$$\boldsymbol{x + 4y - z = -5}$$

$$\boldsymbol{x + y - 6z = -12}$$

$$\boldsymbol{3x - y - z = 4}$$

Solution: Given system of equations can be written in matrix form

$$\approx \begin{matrix} 1 & 4 & -1 \\ 1 & 1 & 6 \\ 3 & -1 & -1 \end{matrix} \begin{matrix} x \\ y \\ z \end{matrix} = \begin{matrix} -5 \\ -12 \\ 4 \end{matrix}$$

$\rightarrow R_2 - R_1 , \rightarrow R_3 - 3R_1$

$$\approx \begin{matrix} 1 & 4 & -1 \\ 0 & -3 & -5 \\ 0 & -13 & 2 \end{matrix} \begin{matrix} x \\ y \\ z \end{matrix} = \begin{matrix} -5 \\ -7 \\ 19 \end{matrix}$$

$$\rightarrow R_3 - \frac{13}{3} R_2$$

$$\approx \begin{matrix} 1 & 4 & -1 \\ 0 & -3 & -5 \\ 0 & 0 & 71/3 \end{matrix} \begin{matrix} x \\ y \\ z \end{matrix} = \begin{matrix} -5 \\ -7 \\ 148/3 \end{matrix}$$

This is an upper triangular matrix of coefficient matrix A

Therefore, the algebraic form of above Matrix form is

$$\boldsymbol{x + 4y - z = -5} \dots \dots . . (\boldsymbol{1})$$

$$\boldsymbol{-3y - 5z = -7} \dots \dots . . (\boldsymbol{2})$$

$$\frac{71}{3}\boldsymbol{z} = 148/3 \dots \dots . . (\boldsymbol{2})$$

Hence from above by back substitution, we get the desiredapproximate solution

$$z = 2.0845 \, , y = -1.1408 \, , x = 1.6479$$

(2) **Gauss's Jordan Method: -** This is a modification ofGauss's Elimination Method. In this method, elimination of unknowns is performed not in the equations below but in the equations above also. Ultimatelyreducingthesystem

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to a diagonal matrix. i.e. each equation involving only one unknown. Thus in this method, thelabor of back substitution for finding the unknowns is saved at the cost of additional calculations. This method is well explained by the following example.

Example: - Apply Gauss's Jordan Method to solve the equations

$$x + y + z = 9$$

$$2x - 3y + 4z = 13$$

$$3x + 4y + 5z = 40$$

Solution: Given system of equations is

$$x + y + z = 9 \dots \dots \dots . (1)$$

$$2x - 3y + 4z = 13 \dots \dots \dots (2)$$

$$3x + 4y + 5z = 40 \dots \dots \dots . (3)$$

Step-1:- Operate (2)-2(1) and (3)-3(1) to eliminate x from (2) and (3**).**

$x + y + z = 9$ ....(4)

$$-5y + 2z = -5 \dots \dots \dots (5)$$

$y + 2z = 13$ ....(6)

Step-2:- operate (4) +1/5 (5) and (6) +1/5 (5) to eliminate y from (4) and (6**).**

$x + 7/5z = 8$ ....(7)

$$-5y + 2z = -5 \dots \dots \dots (8)$$

$\frac{12}{5}z = 12$ ....(9)

Step-3:- operate (7) -7/12 (9) and (8) -5/6 (9) to eliminate z from (7) and (8**).**

$$x = 1$$
$$y = 3$$
$$z = 5$$

(4) Crout's method:- Computation scheme : Let us consider

$$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = b_1$$

$$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 = b_2$$

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$$a_{31}\,x_1 + a_{32}\,x_2 + a_{33}\,x_3 = b_3$$

The Augmented Matrix of given system of equations is

$$A : B \;=\; \begin{matrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{matrix}$$

Now from the above matrix, the Matrix of 12 unknowns,so called derived matrix or Auxiliary Matrix is $D : M = \begin{matrix} l_{11} & u_{12} & u_{13} & y_1 \\ l_{21} & l_{22} & u_{23} & y_2 \\ l_{31} & l_{32} & l_{33} & y_3 \end{matrix}$ and is to be calculated as follows:

Step-1:- The first column of the derived matrix is identical with the first column of $A : B$.

Step-2: The first row to the right of the first column of the $D : M$ is obtained by dividing the corresponding element in $A : B$ by the leading diagonal element of that row.

Step-3: Remaining second column of $D : M$ is calculated as follows:

$$l_{22} = a_{22} - l_{21}\,u_{12}\,;\, l_{32} = a_{32} - l_{31}\,u_{12}$$

Step-4: Remaining elements of second row of $D : M$ is calculated as follows ;

$$u_{23} = \frac{a_{23} - l_{21}\,u_{13}}{l_{22}} \;;\, y_2 = \frac{b_2 - l_{21}\,y_1}{l_{22}}$$

Step-5: Remaining element of the third column of $D : M$ is calculated as follows:

$$l_{33} = a_{33} - (\,l_{31}\,u_{13} + l_{32}\,u_{23}\,)$$

Step-6: Remaining element of third row of $D : M$ is calculated as follows ;

$$y_3 = \frac{b_3 - (\;l_{31}\;y_1\;+\;l_{32}\;y_2)}{l_{33}}$$

Hence, from above $D : M\ \ Matrix\,,\,t\,?\,e\ required\ solution\ is$ obtained as follows:

$x_3 = y_3;$

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$x_2 = y_2 - u_{23}\, x_3$;

$$x_1 = y_1 - [\, x_2\, u_{12} + x_3\, u_{13}]$$

Example: Apply **Crout's** Method to solve the equations

$$2\, x_1 - 6\, x_2 + 8\, x_3 \;= 24$$

$$5\, x_1 + 4\, x_2 - 3\, x_3 \;= 2$$

$$3\, x_1 + x_2 \;+ 2\, x_3 = 16$$

Solution: The Augmented Matrix of given system of equations is

$$A : B \;= \begin{matrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{matrix}$$

Now from the above matrix, the Matrix of 12 unknowns,so called derived matrix or Auxiliary Matrix is $D : M = \begin{matrix} l_{11} & u_{12} & u_{13} & y_1 \\ l_{21} & l_{22} & u_{23} & y_2 \\ l_{31} & l_{32} & l_{33} & y_3 \end{matrix}$ and is to be calculated as follows:

Step-1:- The first column of the derived matrix $D : M$ is identical with the first column of $A : B$.

i.e $l_{11} = a_{11} = 2$; $l_{21} = a_{21} = 5$; $l_{31} = a_{31} = 3$

Step-2: The first row to the right of the first column of the $D : M$ is obtained by dividing the corresponding element in $A : B$ by the leading diagonal element of that row.i.e.

$$u_{12} = \frac{a_{12}}{a_{11}} = \frac{-6}{2} = -3 \;;\; u_{13} = \quad \frac{a_{13}}{a_{11}} = \frac{8}{2} = 4 \;;\; \text{y} = \frac{b_1}{a_{11}} = \frac{24}{2} = 12$$

Step-3: Remaining second column of $D : M$ is calculated as follows:

$$l_{22} = a_{22} - l_{21}\, u_{12} = 4 - 5\; -3 = 19 \;;\; l_{32} = a_{32} - l_{31}\, u_{12}$$
$$= 1 - 3\; -3 = 10$$

Step-4: Remaining elements of second row of $D : M$ is calculated as follows ;

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$$u_{23} = \frac{a_{23} - l_{21}u_{13}}{l_{22}} = \frac{-3 - 5(4)}{19} = -\frac{23}{19}\,;\; y_2 = \frac{b_2 - l_{21}y_1}{l_{22}} = \frac{2 - 5(12)}{19} = -\frac{58}{19}$$

Step-5: Remaining elements of the third column of $D : M$ is calculated as follows:

$$l_{33} = a_{33} - l_{31}u_{13} + l_{32}u_{23} = 2 - [3\;4 + 10 - \frac{23}{19} = \frac{40}{19}$$

Step-6: Remaining element of third row of $D : M$ is calculated as follows ;

$$y_3 = \frac{b_3 - (l_{31}y_1 + l_{32}y_2)}{l_{33}} = \frac{16 - [3\;12 + 10(-\frac{58}{19})}{\frac{40}{19}} = 5$$

Hence, from above $D : M\;\; Matrix\,, t\boxed{?}e\ required\ solution\ is$ obtained as follows:

$x_3 = y_{3=5}$;

$x_2 = y_2 - u_{23}\,x_3 = -\frac{58}{19}\,5 - \frac{23}{19} = 2$;

$x_1 = y_1 - [\,x_2\,u_{12} + x_3\,u_{13}]$=12- [3(-3)+5(4)] =1

[II] **Iterative Methods:** In these methods, we start from an initial approximation to the true solution and obtain better and better approximations from a computation cycle repeated as often as may be necessary for achieving a desired accuracy. For large systems, iterative methods may be faster than the direct methods. Even the round -off errors in iterative methods are smaller. In fact, iteration is a self-correcting process and any error made at any stage of computation gets automatically corrected in the subsequent steps.

(1) **Gauss's Jacobi's Method :** Let us consider the system of equations

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2 \ldots\ldots\ldots\ldots\ldots(1)$$

$$a_3x + b_3y + c_3z = d_3$$

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If a_1, b_2, c_3 are large as compared to other coefficients, solve the above system for x, y, z respectively. Then system of equations can be written as

$$x = \frac{1}{a_1}(d_1 - b_1 y - c_1 z)$$

$$y = \frac{1}{b_2}(d_2 - a_2 x - c_2 z) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$z = \frac{1}{c_3}(d_3 - a_3 x - b_3 y)$$

Let us start with the initial approximation x_0, y_0, z_0 for the values of x, y, z respectively.Substituting these on the right sides of above, the first approximations are given by

$$x_1 = \frac{1}{a_1}(d_1 - b_1 y_0 - c_1 z_0)$$

$$y_1 = \frac{1}{b_2}(d_2 - a_2 x_0 - c_2 z_0)$$

$$z_1 = \frac{1}{c_3}(d_3 - a_3 x_0 - b_3 y_0)$$

Now for second approximations, Substituting thex_1,y_1, z_1 on the right hand sides of (2).

$$x_2 = \frac{1}{a_1}(d_1 - b_1 y_1 - c_1 z_1)$$

$$y_2 = \frac{1}{b_2}(d_2 - a_2 x_1 - c_2 z_1)$$

$$z_2 = \frac{1}{c_3}(d_3 - a_3 x_1 - b_3 y_1)$$

This process is repeated till the diffence between two consecutive approximations is negligible.

Note :In the absence of any better estimates for x_0, y_0, z_0,these may each be taken as zero.

Example : Solve by **Gauss's Jacobi's Method**

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$$20x + y - 2z = 17$$

$$3x + 20y - z = -18 \ldots\ldots\ldots\ldots\ldots(1)$$

$$2x - 3y + 20z = 25$$

Solution : Then system of equations can be written as

$$x = \frac{1}{20}(17 - y + 2z)$$

$$y = \frac{1}{20}(-18 - 3x + z) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$z = \frac{1}{20}(25 - 2x + 3y)$$

For first approximations, substituting $x_0 = 0,\ y_0 = 0,\ z_0 = 0$ for the values of x, y, z respectively in above,

$$x_1 = \frac{1}{20} 17 - 0 + 2.0 = 0.85;$$

$$y_1 = \frac{1}{20}\ -18 - 3.0 + 0\ = -0.9;$$

$$z_1 = \frac{1}{20}\ 25 - 2.0 + 3.0\ = 1.25.$$

Now for second approximations, Substituting the x_1, y_1, z_1 on the right hand sides of (2).

$$x_2 = \frac{1}{20}\ 17 - y_1 + 2z_1\ = 1.02;$$

$$y_2 = \frac{1}{20} -18 - 3x_1 + z_1 = -0.965;$$

$$z_2 = \frac{1}{20} 25 - 2x_1 + 3y_1 = 1.03.$$

Now for third approximations, Substituting the x_2, y_2, z_2 on the right hand sides of (2).

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$$x_3 = \frac{1}{20}17 - y_2 + 2\, z_2 = 1.00125;$$

$$y_3 = \frac{1}{20}-18 - 3\, x_2 + z_2 = -1.0015;$$

$$z_3 = \frac{1}{20}25 - 2x_2 + 3y_2 = 1.00325.$$

Now for fourth approximations, substituting these values on the right hand sides of (2).

$$x_4 = \frac{1}{20}\, 17 - y_3 + 2\, z_3 = 1.0004;$$

$$y_4 = \frac{1}{20}\, -18 - 3\, x_3 + z_3 = -1.000025;$$

$$z_4 = \frac{1}{20}\, 25 - 2x_3 + 3y_3 = 0.9965.$$

Now for fifth approximations, substituting these values on the right hand sides of (2).

$$x_5 = \frac{1}{20}17 - y_4 + 2\, z_4 = 0.999966;$$

$$y_5 = \frac{1}{20}-18 - 3\, x_4 + z_4 = -1.000078;$$

$$z_5 = \frac{1}{20}25 - 2x_4 + 3y_4 = 0.999956.$$

Now for sixth approximations, substituting these values on the right hand sides of (2).

$$x_6 = \frac{1}{20}\, 17 - y_5 + 2\, z_5 = 1.0000;$$

$$y_6 = \frac{1}{20}\, -18 - 3\, x_5 + z_5 = -0.999997;$$

$$z_6 = \frac{1}{20}\, 25 - 2x_5 + 3y_5 = 0.999992.$$

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As the values in the 5th and 6th approximations, i.e. Iterations being practically the same.We can stop now, Hence the solution is

$$x = 1; \quad y = -1; \quad z = 1$$

(2) **Gauss-Seidal Method** : This is a modification of Jacobi's method . Let us consider the system of equations

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \quad \ldots\ldots\ldots\ldots\ldots(1) \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

If a_1, b_2, c_3 are large as compared to other coefficients, solve the above system for x, y, z respectively. Then system of equations can be written as

$$x = \frac{1}{a_1}(d_1 - b_1y - c_1z)$$

$$y = \frac{1}{b_2}(d_2 - a_2x - c_2z) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$z = \frac{1}{c_3}(d_3 - a_3x - b_3y)$$

Let us start with the initial approximation x_0, y_0, z_0 for the values of x, y, z respectively. Which may each be taken as zero .Now, Substituting $y = y_0, z = z_0$on the right sides of above first equation, the first approximations are given by

$$x_1 = \frac{1}{a_1}(d_1 - b_1y_0 - c_1z_0)$$

Then putting $x = x_1, z = z_0$ in the second of the equation of (2),we obtain

$$y_1 = \frac{1}{b_2}(d_2 - a_2x_1 - c_2z_0)$$

Next putting $x = x_1, y = y_1$ in the third of the equation of (2),we obtain

$$z_1 = \frac{1}{c_3}(d_3 - a_3x_1 - b_3y_1)$$

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And so on i.e. as soon as a new approximation for an unknown is found, it is immediately used in the next step.

This process of iteration is repeated till the values of x, y, z are obtained to desired degree of accuracy.

Note :- (1) Since the most recent approximations of the unknowns are used, while proceeding to the next step,the convergence in the **Gauss-Seidal Method is twice as fast as in Gauss's Jacobi's Method.**

(3) Gauss's Jacobi's Method and Gauss-Seidal Methods converge for any choice of the initial approximations if in each equation of the system, the absolute value of the largest coefficient is almost equal to or in at least one equation greater than the sum of the absolute values of all the remaining coefficients**.**

Example: Apply Gauss-Seidal Method

$$20x + y - 2z = 17$$

$$3x + 20y - z = -18 \quad \text{.................(1)}$$

$$2x - 3y + 20z = 25$$

Solution: Then system of equations can be written as

$x = \frac{1}{20}(17 - y + 2z)$ ....(i)

$y = \frac{1}{20}(-18 - 3x + z)$ ....(ii)

$$z = \frac{1}{20}(25 - 2x + 3y) \quad \text{..............(iii)}$$

For First approximations / Iteration,

Substituting $y = y_0 = 0, z = z_0 = 0$ for the values of y, z respectively in (i),

$$x_1 = \frac{1}{20}\, 17 - 0 + 2.0 = 0.8500;$$

Substituting $x = x_1, z = z_0$ for the values of x , z respectively in (ii)

$$y_1 = \frac{1}{20}\, -18 - 3x_1 + z_0 = -1.0275;$$

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Substituting $x = x_1, y = y_1$ for the values of x , y respectively in (iii)

$$z_1 = \frac{1}{20}\left(25 - 2x_1 + 3y_1\right) = 1.0109$$

For Second approximations / Iteration

Substituting $y = y_1, z = z_1$ for the values of y, z respectively in (i),

$$x_2 = \frac{1}{20}\left(17 - y_1 + 2z_1\right) = 1.0025;$$

Substituting $x = x_2, z = z_1$ for the values of x , z respectively in (ii)

$$y_2 = \frac{1}{20}\left(-18 - 3x_2 + z_1\right) = -0.9998;$$

Substituting $x = x_2, y = y_2$ for the values of x , y respectively in (iii)

$$z_2 = \frac{1}{20}\left(25 - 2x_2 + 3y_2\right) = 0.9998$$

For Third approximations / Iteration

Substituting $y = y_2, z = z_2$ for the values of y, z respectively in (i),

$$x_3 = \frac{1}{20}\left(17 - y_2 + 2z_2\right) = 1.0000;$$

Substituting $x = x_3, z = z_2$ for the values of x , z respectively in (ii)

$$y_3 = \frac{1}{20}\left(-18 - 3x_3 + z_2\right) = -1.0000;$$

Substituting $x = x_3, y = y_3$ for the values of x , y respectively in (iii)

$$z_3 = \frac{1}{20}\left(25 - 2x_3 + 3y_3\right) = 1.0000$$

Hence,the values in 2nd and 3rd Approximations being practically the same,
Now ,we can stop,

The solution is $x = 1$; $y = -1$; $z = 1$.

Therefore,we seen that the convergence is quite fast in **Gauss-Seidal Method as compared to Gauss's Jacobi's Method.**

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(3) Relaxation method: This method was original developed by R.V. South well in 1935,for application to Structural engg. Problems.

Let us consider the system of equations

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2 \ldots\ldots\ldots\ldots\ldots(1)$$

$$a_3x + b_3y + c_3z = d_3$$

We define the residuals R_x , R_y R_z , by the relations

$$R_x = d_1 - a_1x - b_1y - c_1z$$

$$R_y = d_2 - a_2x - b_2y - c_2z \ldots\ldots\ldots\ldots\ldots (2)$$

$$R_z = d_3 - a_3x - b_3y - c_3z$$

To start with we assume $x = 0, y = 0, z = 0$ and calculate the initial residuals. Then the residuals are reduced step by step, by giving increments to the variables. For this purpose, we construct the following operation table:

	δR_x	δR_y	δR_z
$\delta x = 1$	$-a_1$	$-a_2$	$-a_3$
$\delta y = 1$	$-b_1$	$-b_2$	$-b_3$
$\delta z = 1$	$-c_1$	$-c_2$	$-c_3$

We note from the equations (2) that if x is increased by 1 (keepingy, z constant), R_x, R_y and R_z decrease by a_1, a_2, a_3 respectively.This is shown in the above table along with the effects on the residuals when $y, and\ z$ are given unit increments (The table is the transpose of the coefficient matrix)

At each step, the numerically largest residual is reduced to almost zero,To reduce a particular residual, the value of the corresponding variable is changed.

When all the residuals have been reduced to almost zero, the increments in $x, y, and\ z$ are added separately to give the desired solution.

Example: Apply Relaxation method

$$10x - 2y - 3z = 205$$

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$-2x + 10y - 2z = 154$.................(1)

$$-2x - y + 10z = 120$$

Solution: We define the residuals R_x , R_y R_z , by the relations

$$R_x = 205 - 10x + 2y + 3z$$

$R_y = 154 + 2x - 10y + 2z$ .... (2)

$$R_z = 120 + 2x + y - 10z$$

The operation table is

	$\delta\ R_x$	$\delta\ R_y$	$\delta\ R_z$
$\delta x = 1$	$-\ a_1$=-10	$-a_2 = 2$	$-\ a_3 = 2$
$\delta y = 1$	$-\ b_1 = 2$	$-\ b_2 = 10$	$-\ b_3$=1
$\delta z = 1$	$-\ c_1 = 3$	$-\ c_2 = 2$	$-\ c_3$=-10

The **Relaxation table is**

	R_x	R_y	R_z
$x = 0, y = 0, z = 0$	205	154	120
$\delta x = 20$	5	194	160
$\delta y = 19$	43	4	179
$\delta z = 18$	97	40	-1
$\delta x = 10$	-3	60	19
$\delta y = 6$	9	0	25
$\delta z = 2$	15	4	5
$\delta x = 2$	-5	8	9
$\delta y = 1$	-2	10	-1
$\delta z = 1$	0	0	0

$$R_x\ = 32, \qquad R_y = 26, \qquad R_z = 21$$

Hence The solution is $x\ = 32$; $y = 26$; $z = 21$.

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