Lovasz Number and related convex relaxations

G Savarana Datta Reddy
Department of Artificial Intelligence
IIT Hyderabad
ai20btech11008@iith.ac.in

Adepu Adarsh Sai
Department of Artificial Intelligence
IIT Hyderabad
ai20btech11001@iith.ac.in

Rongala Arun Siddhardha
Department of Artificial Intelligence
IIT Hyderabad
ai20btech11019@iith.ac.in

***Abstract*—We worked on convex quadratic programming-based upper bound on the stability number $\alpha(G)$ of a graph G and several of its features were established. The goal of this study is to connect this bound (typically denoted by $v(G)$) mathematically using the well-known theta number of Lovasz$\vartheta(G)$. To begin, a new set of convex quadratic boundaries is introduced on $\alpha(G)$. A proposal is made to generalise and improve the bound $v(G)$. Then it is established that $\vartheta(G)$ is never worse than any other bound in this new collection of boundaries. The key outcome of this note asserts that one of these new boundaries is equal to $\vartheta(G)$, resulting in a new characterization of the theta number of Lovasz.**

I. Introduction

The Lovász number of a graph is a real number that is an upper bound on the graph's Shannon capacity. It is denoted by $\vartheta(G)$. László Lovász first mentioned this number in his 1979 paper On the Shannon Capacity of a Graph. Semidefinite programming and the ellipsoid approach can construct accurate numerical approximations to this quantity in polynomial time. It sits between any graph's chromatic number and clique number, and can be used to determine these values on graphs when they are equivalent, such as perfect graphs.

II. Definitions

A. Graph Strong Product

The typical product, often known as the AND product, is a strong product. Sabidussi was the first to introduce it in 1960. A graph that is the strong product $G \boxtimes H$ of graphs G and H is one that behaves such as

- The Cartesian product $V(G) \times V(H)$ is the vertex set of $G \boxtimes H$; and
- Distinct vertices (u,u') and (v,v') are adjacent in $G \boxtimes H$ if and only if:
  - u = v and u' is adjacent to v' , or
  - u' = v' and u is adjacent to v, or
  - u is adjacent to v and u' is adjacent to v'

B. Stability Number

A subset of nodes in V whose elements are pairwise nonadjacent is called a stable set (independent set) of G. The cardinality of a maximum stable set is defined as the stability number (or independence number) of G, which is commonly denoted by $\alpha(G)$: A stable set with $\alpha(G)$ nodes is the maximum stable set of $\alpha(G)$. The independence number of a graph is equal to the largest exponent in the graph's independence polynomial.

C. Shannon Capacity

A graph's Shannon capacity is a graph invariant defined by the number of independent sets of strong graph products. It is named after Claude Shannon, an American mathematician. It is upper bounded by the Lovász number and estimates the Shannon capacity of a communications channel defined from the graph. Mathematically it is defined as

$$\theta(G) = \sup_k \sqrt[k]{\alpha(G_k)} \tag{1}$$

where G_k is the k-fold strong product of G with itself.

D. Clique Number

A clique is a subset of vertices in an undirected graph where each pair of different vertices is adjacent. The cardinality of a maximum clique set is the clique number of G, which is commonly denoted by $\omega(G)$. Despite the fact that the goal of determining if a clique of a certain size exists in a graph (the clique problem) is NP-complete, various methods for detecting cliques have been developed.

E. Chromatic Number

A graph's chromatic number, $\chi(G)$, is the minimum number of colours utilised to represent the graph's vertices so that adjacent vertices are coloured differently. The task of calculating a graph's chromatic number is NP-complete. The chromatic number of a graph must be greater than or equal to its clique number.

F. Lovasz Number

Let $G = (V, E)$ be a graph on n vertices. An ordered set of n unit vectors $U = (u_i \mid i \in V) \subset \mathbb{R}^N$ is called an orthonormal representation of G in $\mathbb{R}^N$, if u_i and u_j are orthogonal whenever vertices i and j are not adjacent in G: The problem of finding Lovasz number can be expressed as a semi-definite programme, allowing it to be computed in polynomial time with arbitrary precision.

$$u_i^{\mathrm{T}} u_j = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i, j \notin E. \end{cases} \tag{2}$$

The Lovász number $\vartheta(G)$ of graph G is defined as follows:

$$\vartheta(G) = \min_{c,U} \max_{i \in V} \frac{1}{(c^{\mathrm{T}} u_i)^2}, \tag{3}$$

1) Alternative Definintion: Let G=(V,E) be an n-verticed graph. Let A be the set of all n times n symmetric matrices with $a_ij = 1$ whenever $i = j$ or when the vertices i and j are not adjacent, and λ_{max} be the largest eigenvalue of A. The following is an alternative method of calculating Lovász number of a graph G:

$$\vartheta(G) = \min_{A} \lambda_{\max}(A). \tag{4}$$

2) Properties of Lovasz Number:

$$\vartheta(G \boxtimes H) = \vartheta(G)\vartheta(H) \tag{5}$$

$$\vartheta(G)\vartheta(\overline{G}) \geq n, \tag{6}$$

where $\overline{G}$ is compliment graph of G.

3) Lovasz Sandwich Theorem: The Lovász "sandwich theorem" states that the Lovász number is always sandwiched between the clique number and the chromatic number which are two NP-complete numbers, i.e

$$\omega(G) \leq \vartheta(\bar{G}) \leq \chi(G) \tag{7}$$

where $\omega(G)$ is the clique number and $\chi(G)$ is the chromatic number of graph G.

4) Relation to Shannon capacity: The Lovász number is an upper bound on a graph's Shannon capacity.

$$\alpha(G) \leq \Theta(G) \leq \vartheta(G) \tag{8}$$

5) Lovasz Numbers of well known graphs:

Graph G	$\vartheta(G)$
cocktail party graph $K_{n\times 2}$	2
complete graph K_n	1
complete k-partite graph $K_(n_1, ..., n_k)$	$max_{(1<=i<=k)} n_i$
cycle graph C_n	$\begin{cases} n & \text{for } n = 1 \\ \frac{(n\cos(\pi/n))}{(1+cos(\pi/n))} & \text{for } n \geq 3, \text{‘ odd} \\ n/2 & \text{for n even} \end{cases}$
empty graph K_n	n
Kneser graph K(n,r)	$\binom{n-1}{r-1}$
Paley graph	$\sqrt{V(G)}$
cycle graph C_5	$\sqrt{5}$
Petersen graph P	4

III. Integer Programming Relaxation

Lovasz and Schrijver present a method for achieving ever stronger relaxations of 0-1 integer programming problems. They design multiple processes known as matrix-cut operators that yield a convex set that is an improved relaxation for the 0-1 (i.e. integral) vectors in P from a convex (e.g. linear programming) relaxation P $\subseteq [0, 1]^n$ of the issue. That is, the convex set that results is contained in P and contains all of P's 0-1 vectors. The matrix-cut operators take a lift-and-project strategy, in which they introduce additional variables and constraints to lift the convex relaxation P into a higher (quadratic) dimension, then project it back into the original space.

N and N_+ are the two primary matrix-cut operators of Lovasz and Schrijver.The distinction between the two operators is that the latter involves a positive semidefinite constraint in addition to the lifting. If P is a linear programming relaxation, then N(P) is a linear programming relaxation as well, but N_+(P) is a semidefinite programming relaxation.

The matrix-cut operators can be used iteratively, for example, r $\geq$ 0 times, and the iterated operators are designated N^r and N_+^r. The number of iterations of the N operator required to get the convex hull of the 0-1 vectors of P (i.e. a perfectly tight relaxation) is defined as the N-rank of a convex relaxation P. The $N+$-rank is defined in the same way. The N-rank of a relaxation, according to Lovasz and Schrijver, is always at most the dimension d (e.g., the number of variables in a linear programme).Because the N_+ operator is a stronger version of the N operator, the N_+-rank is always d. Goemans and Tuncel, as well as Cook and Dash, separately establish that relaxations with a N_+ rank that meets the upper bound d exist. Lovasz and Schrijver further demonstrate that the N and N_+ operators have the following crucial algorithmic characteristic. If linear objective functions can be efficiently optimised over a relaxation P, then the relaxation obtained by applying the operator on P can also be efficiently optimised. As a result, the iterated operators N^r and N_+^r satisfy this property for every fixed r $\geq$ 0.

IV. The Lovász-Schrijver matrix-cut operators

The matrix-cut operators consider a convex set P as a relaxation of the convex hull of its 0-1 vectors, and produce another relaxation that is tighter than P. In other words, these operators construct a convex set that lies between P and (the convex hull of) the 0-1 vectors in P in terms of containment. Furthermore, unless P is already tight, the resulting relaxation is strictly tighter than P.

V. Theta Function Relaxations

Lovasz number of a graph is also called the theta function of the graph. The theta function $\alpha(G)$ of Lovász is the most well-known semidefinite programming relaxation.

This was introduced as a relaxation of the maximum independent set issue, and it was used to demonstrate that the maximum independent set and minimal vertex colouring problems in perfect graphs may be solved in polynomial time. The theta function relaxation can also be used to approximate *maximum clique problem*, by formally referring to $\alpha(\overline{G})$.

The theta function looks to have little to offer in terms of approximation ratio. As established in [Fei97], the ratio between $\vartheta(G)$ and the independence number $\alpha(G)$ can be as great as $n^{1-o(1)}$. The Lovász theta function $\vartheta(G)$ and the independence number $\alpha(G)$ also have a substantial gap on an average. While the independence number of a random graph

$G_{n,1/2}$ is nearly certainly $2log_2n$, it is demonstrated that the value of the theta function is almost certainly $2log_2n$.

VI. Maximum independent set problem

Lovasz and Schrijver use their general technique of matrix-cut operators on a classical linear programming relaxation FRAC of the maximal independent set problem to get relaxations. Because the relaxation FRAC is a polynomial-sized linear algorithm, it can be easily optimised over N_+r for any fixed r $\geq$ 0. (FRAC). FRAC, on the other hand, has a dimension d (number of variables) equal to the number of vertices n in the graph, making optimization over N^n (FRAC) NP-hard.

The semidefinite programming relaxation N_+(FRAC) is at least as strong as the Lovasz theta function, according to Lovasz and Schrijver. It follows, for example, that the relaxation N_+^r(FRAC) for r $\geq$ 2 is stronger than the theta function for any graph on which the theta function is not tight.

The relaxation FRAC N-rank is used to determine a graph's N-rank. Similarly, the N_+-rank is defined. As a result, the maximal independent set issue can be solved in polynomial time for networks with bounded N_+-rank. Because the preceding link with the theta function implies that their N_+-rank is at most 1, this family comprises, for example, all perfect graphs.

The scenario where the n-vertex graph G is the line graph of a graph H on h vertices is studied by Stephen and Tuncel. They show that G's N_+-rank is at most h/2, and that this constraint is satisfied if H is a complete graph with an odd number of vertices, in which case n =$(\frac{h}{2})$, and G's N_+-rank is $\Omega(\sqrt{n})$. Because independent sets in G correspond to matchings in H and a maximum weight matching may be discovered quickly, there are graphs with unbounded (and fairly big) N_+-rank where the maximum (weighted) independent set issue can be solved in polynomial time.

Our results. The relaxations of Lovasz and Schrijver for the maximal independent set issue have asymptotic behaviour on the random graph $G_{n,1/2}$. We show that for $r = o(logn)$, the usual value of the semidefinite programming relaxation N_+^r(FRAC) on a random graph is around $\sqrt{n/2r}$.

Theorem 1.1. For every fixed $\delta > 0$ and $r = o(logn)$, the value of the relaxation N_+^r(FRAC) on a random graph $G_{n,1/2}$ is at least $\sqrt{n/(2+\delta)^(r+1)}$ and at most $4\sqrt{n/(2-\delta)^(r+1)}$ almost surely.

Remember that for $r = O(1)$, the most powerful relaxations of Lovasz and Schrijver whose value is known to be efficiently computed are N_+^r(FRAC). Theorem 1.1 states that on a random graph, the typical value of these relaxations is no more than a constant factor smaller than the theta function. In the hidden clique problem, a method of can improve the maximum clique size k that a heuristic can manage by an arbitrarily large constant factor, therefore it appears that the benefit afforded by these stronger relaxations can be accomplished by other approaches.

We utilise Theorem 1.1 to characterise the typical N_+-rank of a random graph $G_{n,1/2}$ up to a constant factor.

Theorem 1.2. The N_+-rank of a random graph $G_{n,1/2}$ is almost surely $\theta(\log n)$.

Our results for the N_+ operator are extended to a little stronger variation of Lovasz and Schrijver's matrix-cut operators. This operator, designated N_{FR+}, is specialised for the maximal independent set problem and retains the crucial algorithmic property of N_+, namely that a good optimization over P implies a good optimization over $N_{FR+}(P)$.

VII. Vertex cover problem

A vertex cover of an un-directed graph is a subset of its vertices in which either 'u' or 'v' is in the vertex cover for every graph edge (u, v). Despite its name, the Vertex Cover set spans all of the graph's edges. The vertex cover problem is to find the smallest vertex cover for an un-directed graph. We can formulate the problem of finding minimum vertex cover as the following intezer program

$$\begin{aligned} \text{Min } & \sum_i x_i \\ \text{such that } & x_i + x_j \geq 1 \quad (i,j) \in E \\ & x_i \in 0,1 \quad i \in V \end{aligned}$$

where $x_i = 1$ when i^{th} vertex is in the vertex cover, otherwise it is zero. Let us denote the optimal value of the problem as $vc(G)$. The problem of finding $vc(G)$ is NP-complete. So we apply *linear program relaxation* to approximate $vc(G)$ with in a factor of two.

A. Linear program relaxation

The Linear program relaxation can be formulated as

$$\begin{aligned} \text{Min } & \sum_i x_i \\ \text{such that } & x_i + x_j \geq 1 \quad (i,j) \in E \\ & 0 \leq x_i \leq 1 \quad i \in V \end{aligned}$$

Let the optimal value of the linear program relaxation be denoted by $lp(G)$. The above problem can run in polynomial time. It can be easily established that

$$2(lp(G)) \geq vc(G) \geq lp(G) \tag{9}$$

B. Theta function relaxation

The integer program for finding $vc(G)$ can be formulated as

$$\text{Min} \sum_{i \in V} (1 + y_0 y_i)/2 \tag{10}$$
$$\text{s.t.} \quad (y_0 - y_i)(y_0 - y_j) = 0 \quad (i,j) \in E \tag{11}$$
$$y_i \in -1, +1 \qquad i \in V \tag{12}$$
$$y_0 \in -1, +1 \tag{13}$$

where the vertex cover corresponds to the set of vertices i for which $y_i = y_0$. This formulation can be relaxed to a semidefinite program using the theta function as follows

$$\text{Min} \sum_{i \in V} (1 + y_0^\top y_i)/2 \tag{14}$$
$$\text{s.t.} \quad (y_0 - y_i)^\top (y_0 - y_j) = 0 \quad (i,j) \in E \tag{15}$$
$$||y_i||^2 \qquad i \in V \tag{16}$$
$$||y_0||^2 \tag{17}$$

where $y_i, y_0 \in \mathbb{R}^{n+1}$.
Let the optimal solution to the above relaxed problem be $sd(G)$. Then we can establish that $sd(G) = \theta(G) - n$. The exact approximation ratio achieved by $sd(G)$ is , $vc(G) \leq 2sd(G)$ but, for any $\epsilon > 0$, there exist instances for which $vc(G) > (2 - \epsilon)sd(G)$.It is worth noting, however, that on many natural examples, $sd(G)$ is a much tighter relaxation than $lp(G)$.

No polynomial time $2 - \epsilon$ approximation algorithm to solve the vertex cover problem is found for any $\epsilon > 0$. The best known approximation for a polynomial time algorithm is $2 - \frac{\log \log n}{2 \log n}$.

VIII. Codes

The code for finding the Lovasz Number of a graph can be found here.

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