ASSIGNMENT 3

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Download all python codes from

https://github.com/ArunSiddardha/EE3900/blob/main/Assignment_2/code/Assignment_3.py

and latex-tikz codes from

https://github.com/ArunSiddardha/EE3900/blob/main/Assignment_2/Assignment_3.tex

1 Ramsey/tangent and normals/ Q.19

Prove that the circle $\mathbf{x}^\top\mathbf{x} - \begin{pmatrix} 6 & 4 \end{pmatrix}\mathbf{x} + 9$ subtends an angle $\tan^{-1}\frac{12}{5}$ at the origin

2 Solution

The general equation of a second dergree can be expressed as :

$$\mathbf{x}^\top\mathbf{V}\mathbf{x} + 2\mathbf{u}^\top\mathbf{x} + f = 0 \tag{2.0.1}$$

let the equationn of the tangent be

$$\begin{pmatrix} m & -1 \end{pmatrix}\mathbf{x} = c \tag{2.0.2}$$

since, The tangent passes through origin c=0.
We know that, for a circle,

$$\mathbf{V} = \mathbf{I} \tag{2.0.3}$$
$$\mathbf{c} = -\mathbf{u} \tag{2.0.4}$$

$$\mathbf{u} = \begin{pmatrix} -3 \\ -2 \end{pmatrix}, f = 9 \tag{2.0.5}$$

$$\mathbf{c} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \tag{2.0.6}$$

$$\mathbf{n} = \begin{pmatrix} m & -1 \end{pmatrix}^\top \textit{ and } \mathbf{u} = \begin{pmatrix} -3 & -2 \end{pmatrix}^\top \tag{2.0.7}$$

The points of contact $\mathbf{q}$, of a line with a normal vector $\mathbf{n}$ to the conics are given by:

$$\mathbf{q} = \mathbf{V}^{-1}\left(\kappa\mathbf{n} - \mathbf{u}\right) \tag{2.0.8}$$

$$\kappa = \pm\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n}}} \tag{2.0.9}$$

We know that, for a circle,

$$\mathbf{V} = \mathbf{I} \tag{2.0.10}$$

If r is radius and $\mathbf{c}$ is the centre of the circle we have:

$$f = \mathbf{u}^T\mathbf{u} - r^2 \tag{2.0.11}$$
$$\mathbf{c} = -\mathbf{u} \tag{2.0.12}$$

and from the properties of an Identity matrix,

$$\mathbf{I}^{-1} = \mathbf{I} \tag{2.0.13}$$
$$\mathbf{IX} = \mathbf{X} \tag{2.0.14}$$

$$\kappa = \pm\sqrt{\frac{r^2}{\begin{pmatrix} m & -1 \end{pmatrix}\begin{pmatrix} m \\ -1 \end{pmatrix}}} \tag{2.0.15}$$
$$= \pm\sqrt{\frac{4}{m^2+1}} \tag{2.0.16}$$
$$= \pm\frac{2}{\sqrt{m^2+1}} \tag{2.0.17}$$

So,

$$\mathbf{q} = \begin{pmatrix} \pm\frac{2m}{\sqrt{m^2+1}} \\ \pm\frac{-2}{\sqrt{m^2+1}} \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \end{pmatrix} \tag{2.0.18}$$
$$\mathbf{q} = \begin{pmatrix} \pm\frac{2m}{\sqrt{m^2+1}} + 3 \\ \pm\frac{-2}{\sqrt{m^2+1}} + 2 \end{pmatrix} \tag{2.0.19}$$

Now $\mathbf{q}$ lies on the line therefore,

$$\begin{pmatrix} m & -1 \end{pmatrix}\begin{pmatrix} \pm\frac{2m}{\sqrt{m^2+1}} + 3 \\ \pm\frac{-2}{\sqrt{m^2+1}} + 2 \end{pmatrix} = 0 \tag{2.0.20}$$

$$\pm\frac{2m^2}{\sqrt{m^2+1}} + 3m \pm \frac{2}{\sqrt{m^2+1}} - 2 = 0 \tag{2.0.21}$$
$$\pm\left(\frac{2m^2+2}{\sqrt{m^2+1}}\right) = 2 - 3m \tag{2.0.22}$$
$$\pm\left(2\sqrt{m^2+1}\right) = 2 - 3m \tag{2.0.23}$$

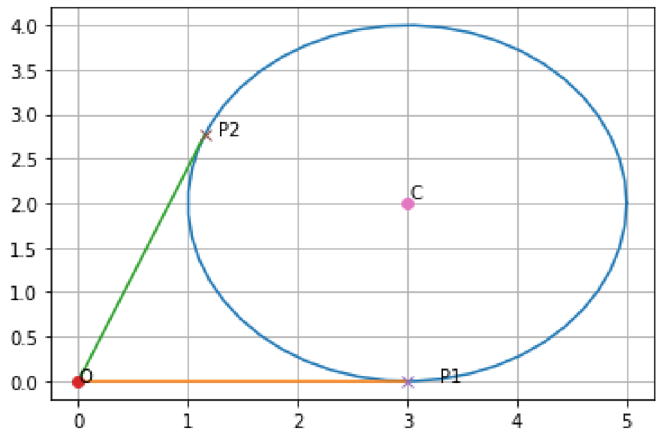

Fig. 0: graph

S.O.B.S

$$4\left(m^2+1\right) = 4 + 9m^2 - 12m \tag{2.0.24}$$

$$5m^2 - 12m = 0 \tag{2.0.25}$$

$$m(5m-12) = 0 \tag{2.0.26}$$

$$m = 0 \ or \ m = \frac{12}{5} \tag{2.0.27}$$

[htp] Therefore slopes of the tangents from origin to the circle is given by $m_1 = \frac{12}{5}$ and $m_2 = 0$.
Angle between lines is given by

$$tan^{-1}\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right) \tag{2.0.28}$$

$$tan^{-1}\left(\frac{\frac{12}{5} - 0}{1 + \frac{12}{5} \times 0}\right) \tag{2.0.29}$$

$$tan^{-1}\left(\frac{12}{5}\right) \tag{2.0.30}$$

Therefore angle between the lines is $tan^{-1}\left(\frac{12}{5}\right)$
Hence proved.