ASSIGNMENT 5

RONGALA ARUN SIDDARDHA
AI20BTECH110019

Download all python codes from

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https://github.com/ArunSiddardha/EE3900/blob/main/Assignment_5/code/Assignment_5.py
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and latex-tikz codes from

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https://github.com/ArunSiddardha/EE3900/blob/main/Assignment_5/Assignment_5.tex
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1 Quadratic Forms/Q.2.21

Find the roots of the quadratic polynomials if they exist

a)
$$3x^2 - 5x + 2 = 0$$

b)
$$x^2 + 4x + 5 = 0$$

c)
$$2x^2 - 2\sqrt{2}x + 1 = 0$$

2 Quadratic Forms/Q.2.21

Comparing the quadratic equations with standard equation,

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{2.0.1}$$

$$\therefore c = b = 0, e = -\frac{1}{2}, a = 3, d = \frac{-5}{2}, f = 2. \tag{2.0.2}$$

$$\therefore \mathbf{V} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.0.3}$$

$$\therefore \mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} = \begin{pmatrix} \frac{-5}{2} \\ \frac{-1}{2} \end{pmatrix} \tag{2.0.4}$$

$$f = 2 \tag{2.0.5}$$

Finding the eigen values corresponding to $\mathbf{V}$

$$\mid \mathbf{V} - \lambda\mathbf{I} \mid = 0 \tag{2.0.6}$$

$$\begin{pmatrix} 3-\lambda & 0 \\ 0 & -\lambda \end{pmatrix} = 0 \tag{2.0.7}$$

$$(3-\lambda)(-\lambda) = 0 \tag{2.0.8}$$

$$\therefore \lambda = 0, 3 \tag{2.0.9}$$

Calculating the eigen vectors corresponding to the $\lambda = 0$,3 respectively,

$$\mathbf{Vx} = \lambda\mathbf{x} \tag{2.0.10}$$

$$\begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}\mathbf{x} = 0 \implies \mathbf{p_1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.0.11}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & -3 \end{pmatrix}\mathbf{x} = 3\mathbf{x} \implies \mathbf{p_2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.0.12}$$

$$\tag{2.0.13}$$

Now by eigen decomposition on $\mathbf{V}$,

$$\mathbf{V} = \mathbf{PDP}^\top \tag{2.0.14}$$

$$where, \mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} \tag{2.0.15}$$

$$\mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.0.16}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{2.0.17}$$

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix} \tag{2.0.18}$$

Hence equation becomes,

$$\mathbf{V} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.0.19}$$

$$\therefore \mathbf{V} = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.0.20}$$

To find the vertex of the parabola ,

$$\begin{pmatrix} \mathbf{u}^\top + \eta \mathbf{p_1}^\top \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \eta \mathbf{p_1} - \mathbf{u} \end{pmatrix} \tag{2.0.21}$$

$$where, \eta = \mathbf{u}^\top \mathbf{p_1} = \frac{-1}{2} \tag{2.0.22}$$

$$\implies \mathbf{c} = \begin{pmatrix} -\frac{5}{2} & -1 \\ 3 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.0.23}$$

$$\implies \mathbf{c} = \begin{pmatrix} -2 \\ \frac{5}{2} \\ 0 \end{pmatrix} \tag{2.0.24}$$

$$\implies \mathbf{c} = \begin{pmatrix} \frac{5}{6} \\ \frac{-1}{12} \end{pmatrix} \tag{2.0.25}$$

$$\because \mathbf{p_1}^\top \mathbf{c} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{5}{6} \\ \frac{-1}{12} \end{pmatrix} \tag{2.0.26}$$

$$= \frac{-1}{12} \tag{2.0.27}$$

and,

$$\mathbf{p_2}^\top \mathbf{V} \mathbf{p_2} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.0.28}$$

$$= 3 \tag{2.0.29}$$

$$\because (\mathbf{p_1}^\top \mathbf{c})(\mathbf{p_2}^\top \mathbf{V} \mathbf{p_2}) = \frac{-1}{4} < 0 \tag{2.0.30}$$

Hence, The given equation has real roots.

ii) Comparing the quadratic equations with standard equation,

$$ax^2 + 2hxy + by^2 + 2dx + 2ey + f = 0 \tag{2.0.31}$$

$$\therefore c = b, e = \frac{-1}{2}, a = 1, d = 2, f = 5. \tag{2.0.32}$$

$$\therefore \mathbf{V} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.0.33}$$

$$\therefore \mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} = \begin{pmatrix} 2 \\ \frac{-1}{2} \end{pmatrix} \tag{2.0.34}$$

$$f = 5 \tag{2.0.35}$$

Finding the eigen values corresponding to $\mathbf{V}$

$$| \mathbf{V} - \lambda \mathbf{I} | = 0 \tag{2.0.36}$$

$$\begin{pmatrix} 1 - \lambda & 0 \\ 0 & -\lambda \end{pmatrix} = 0 \tag{2.0.37}$$

$$(1 - \lambda)(-\lambda) = 0 \tag{2.0.38}$$

$$\therefore \lambda = 0, 1 \tag{2.0.39}$$

Calculating the eigen vectors corresponding to the $\lambda = 0$,1 respectively,

$$\mathbf{V}\mathbf{x} = \lambda \mathbf{x} \tag{2.0.40}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} = 0 \implies \mathbf{p_1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.0.41}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} = \mathbf{x} \implies \mathbf{p_2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.0.42}$$

Now by eigen decomposition on $\mathbf{V}$,

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^\top \tag{2.0.43}$$

$$where, \mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} \tag{2.0.44}$$

$$\mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.0.45}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{2.0.46}$$

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.0.47}$$

Hence equation becomes,

$$\mathbf{V} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.0.48}$$

$$\therefore \mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.0.49}$$

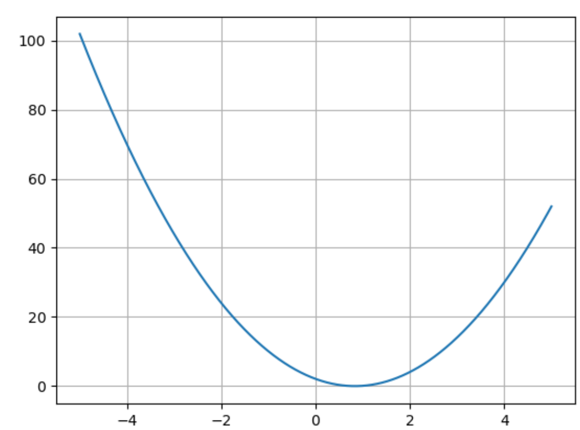

Fig. 0: $3x^2 - 5x + 2$

To find the vertex of the parabola ,

$$\begin{pmatrix} \mathbf{u}^\top + \eta \mathbf{p_1}^\top \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \eta \mathbf{p_1} - \mathbf{u} \end{pmatrix} \tag{2.0.50}$$

$$where, \eta = \mathbf{u}^\top \mathbf{p_1} \tag{2.0.51}$$

$$\implies \mathbf{c} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{2.0.52}$$

$$\because \mathbf{p_1}^\top \mathbf{c} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{2.0.53}$$

$$= 1 \tag{2.0.54}$$

and,

$$\mathbf{p_2}^\top \mathbf{V} \mathbf{p_2} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.0.55}$$

$$= 1 \tag{2.0.56}$$

$$\because (\mathbf{p_1}^\top \mathbf{c})(\mathbf{p_2}^\top \mathbf{V} \mathbf{p_2}) = 1 > 0 \tag{2.0.57}$$

Hence, The given equation does not have real roots.

iii) Comparing the quadratic equations with standard equation,

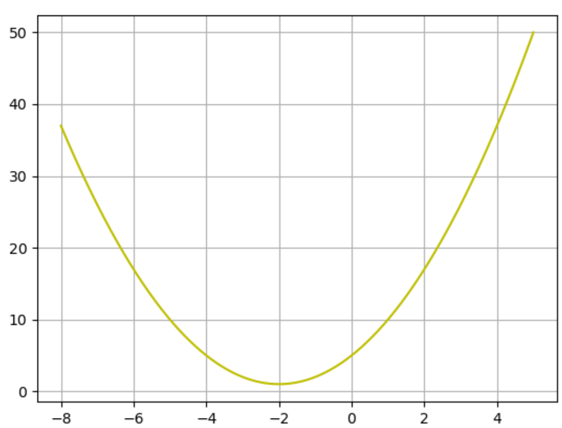

Fig. 0: $x^2 + 4x + 5 = 0$

$$ax^2 + 2hxy + by^2 + 2dx + 2ey + f = 0 \tag{2.0.58}$$

$$\therefore c = b = 0, e = \frac{-1}{2}, a = 2, d = -\sqrt{2}, f = 1. \tag{2.0.59}$$

$$\therefore \mathbf{V} = \begin{pmatrix} 2 & b \\ b & c \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.0.60}$$

$$\therefore \mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} = \begin{pmatrix} -\sqrt{2} \\ -\frac{1}{2} \end{pmatrix} \tag{2.0.61}$$

$$f = 1 \tag{2.0.62}$$

Finding the eigen values corresponding to $\mathbf{V}$

$$| \mathbf{V} - \lambda \mathbf{I} | = 0 \tag{2.0.63}$$

$$\begin{pmatrix} 2-\lambda & 0 \\ 0 & -\lambda \end{pmatrix} = 0 \tag{2.0.64}$$

$$(2-\lambda)(-\lambda) = 0 \tag{2.0.65}$$

$$\therefore \lambda = 0, 2 \tag{2.0.66}$$

Calculating the eigen vectors corresponding to the $\lambda = 0$,1 respectively,

$$\mathbf{V}\mathbf{x} = \lambda \mathbf{x} \tag{2.0.67}$$

$$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} = 0 \implies \mathbf{p_1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.0.68}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} \mathbf{x} = \mathbf{x} \implies \mathbf{p_2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.0.69}$$

Now by eigen decomposition on $\mathbf{V}$,

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^\top \tag{2.0.70}$$

$$where, \mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} \tag{2.0.71}$$

$$\mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.0.72}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{2.0.73}$$

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \tag{2.0.74}$$

Hence equation becomes,

$$\mathbf{V} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.0.75}$$

$$\therefore \mathbf{V} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.0.76}$$

To find the vertex of the parabola ,

$$\begin{pmatrix} \mathbf{u}^\top + \eta \mathbf{p_1}^\top \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \eta \mathbf{p_1} - \mathbf{u} \end{pmatrix} \tag{2.0.77}$$

$$where, \eta = \mathbf{u}^\top \mathbf{p_1} \tag{2.0.78}$$

$$\implies \mathbf{c} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \tag{2.0.79}$$

$$\because \mathbf{p_1}^\top \mathbf{c} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \tag{2.0.80}$$

$$= 0 \tag{2.0.81}$$

and,

$$\mathbf{p_2}^\top \mathbf{V} \mathbf{p_2} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.0.82}$$

$$= 1 \tag{2.0.83}$$

$$\because (\mathbf{p_1}^\top \mathbf{c})(\mathbf{p_2}^\top \mathbf{V} \mathbf{p_2}) = 0 = 0 \tag{2.0.84}$$

Hence, The given equation has equal real roots.

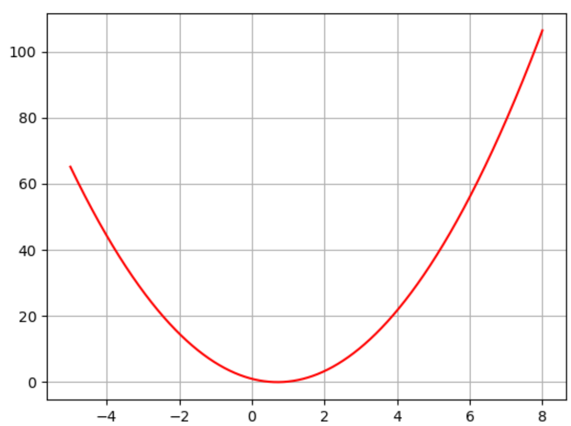

Fig. 0: $2x^2 - 2\sqrt{2}x + 1 = 0$

Our results:

$3x^2 - 5x + 2 = 0$	$(\mathbf{p_1}^\top \mathbf{c})(\mathbf{p_2}^\top \mathbf{V} \mathbf{p_2}) < 0$	It have real roots
$x^2 + 4x + 5 = 0$	$(\mathbf{p_1}^\top \mathbf{c})(\mathbf{p_2}^\top \mathbf{V} \mathbf{p_2}) > 0$	It doesnt have real roots
$2x^2 - 2\sqrt{2}x + 1 = 0$	$(\mathbf{p_1}^\top \mathbf{c})(\mathbf{p_2}^\top \mathbf{V} \mathbf{p_2}) = 0$	It has equal roots