

ASSIGNMENT 5

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Download all python codes from

https://github.com/ArunSiddardha/EE3900/blob/main/Assignment_5/code/Assignment_5.py

and latex-tikz codes from

https://github.com/ArunSiddardha/EE3900/blob/main/Assignment_5/Assignment_5.tex

Calculating the eigen vectors corresponding to the $\lambda = 0, 3$ respectively,

$$\mathbf{V}\mathbf{x} = \lambda\mathbf{x} \quad (2.0.10)$$

$$\begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} = 0 \implies \mathbf{p}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (2.0.11)$$

$$\begin{pmatrix} 0 & 0 \\ 0 & -3 \end{pmatrix} \mathbf{x} = 3\mathbf{x} \implies \mathbf{p}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (2.0.12)$$

$$(2.0.13)$$

1 QUADRATIC FORMS/Q.2.21

Find the roots of the quadratic polynomials if they exist

a) $3x^2 - 5x + 2 = 0$

b) $x^2 + 4x + 5 = 0$

c) $2x^2 - 2\sqrt{2}x + 1 = 0$

Now by eigen decomposition on \mathbf{V} ,

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T \quad (2.0.14)$$

$$\text{where, } \mathbf{P} = (\mathbf{p}_1 \quad \mathbf{p}_2) \quad (2.0.15)$$

$$\mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (2.0.16)$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (2.0.17)$$

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix} \quad (2.0.18)$$

2 QUADRATIC FORMS/Q.2.21

Comparing the quadratic equations with standard equation,

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (2.0.1)$$

$$\therefore c = b = 0, e = -\frac{1}{2}, a = 3, d = \frac{-5}{2}, f = 2. \quad (2.0.2)$$

$$\therefore \mathbf{V} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \quad (2.0.3)$$

$$\therefore \mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} = \begin{pmatrix} \frac{-5}{2} \\ \frac{-1}{2} \end{pmatrix} \quad (2.0.4)$$

$$f = 2 \quad (2.0.5)$$

Hence equation becomes,

$$\mathbf{V} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (2.0.19)$$

$$\therefore \mathbf{V} = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \quad (2.0.20)$$

Finding the eigen values corresponding to \mathbf{V}

$$|\mathbf{V} - \lambda\mathbf{I}| = 0 \quad (2.0.6)$$

$$\begin{pmatrix} 3 - \lambda & 0 \\ 0 & -\lambda \end{pmatrix} = 0 \quad (2.0.7)$$

$$(3 - \lambda)(-\lambda) = 0 \quad (2.0.8)$$

$$\therefore \lambda = 0, 3 \quad (2.0.9)$$

To find the vertex of the parabola ,

$$\begin{pmatrix} \mathbf{u}^\top + \eta \mathbf{p}_1^\top \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \eta \mathbf{p}_1 - \mathbf{u} \end{pmatrix} \quad (2.0.21)$$

$$\text{where, } \eta = \mathbf{u}^\top \mathbf{p}_1 = \frac{-1}{2} \quad (2.0.22)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} -\frac{5}{2} & -1 \\ 3 & 0 \\ 0 & 0 \end{pmatrix} \quad (2.0.23)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} -2 \\ \frac{5}{2} \\ 0 \end{pmatrix} \quad (2.0.24)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} \frac{5}{6} \\ \frac{-1}{12} \end{pmatrix} \quad (2.0.25)$$

$$\therefore \mathbf{p}_1^\top \mathbf{c} = (0 \ 1) \begin{pmatrix} \frac{5}{6} \\ \frac{-1}{12} \end{pmatrix} \quad (2.0.26)$$

$$= \frac{-1}{12} \quad (2.0.27)$$

and,

$$\mathbf{p}_2^\top \mathbf{V} \mathbf{p}_2 = (1 \ 0) \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (2.0.28)$$

$$= 3 \quad (2.0.29)$$

$$\therefore (\mathbf{p}_1^\top \mathbf{c})(\mathbf{p}_2^\top \mathbf{V} \mathbf{p}_2) = \frac{-1}{4} < 0 \quad (2.0.30)$$

Hence, The given equation has real roots.

ii) Comparing the quadratic equations with standard

equation,

$$ax^2 + 2hxy + by^2 + 2dx + 2ey + f = 0 \quad (2.0.31)$$

$$\therefore c = b, e = \frac{-1}{2}, a = 1, d = 2, f = 5. \quad (2.0.32)$$

$$\therefore \mathbf{V} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (2.0.33)$$

$$\therefore \mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} = \begin{pmatrix} 2 \\ \frac{-1}{2} \end{pmatrix} \quad (2.0.34)$$

$$f = 5 \quad (2.0.35)$$

Finding the eigen values corresponding to \mathbf{V}

$$|\mathbf{V} - \lambda \mathbf{I}| = 0 \quad (2.0.36)$$

$$\begin{pmatrix} 1 - \lambda & 0 \\ 0 & -\lambda \end{pmatrix} = 0 \quad (2.0.37)$$

$$(1 - \lambda)(-\lambda) = 0 \quad (2.0.38)$$

$$\therefore \lambda = 0, 1 \quad (2.0.39)$$

Calculating the eigen vectors corresponding to the $\lambda = 0, 1$ respectively,

$$\mathbf{V} \mathbf{x} = \lambda \mathbf{x} \quad (2.0.40)$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} = 0 \Rightarrow \mathbf{p}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (2.0.41)$$

$$\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} = \mathbf{x} \Rightarrow \mathbf{p}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (2.0.42)$$

Now by eigen decomposition on \mathbf{V} ,

$$\mathbf{V} = \mathbf{P} \mathbf{D} \mathbf{P}^\top \quad (2.0.43)$$

$$\text{where, } \mathbf{P} = (\mathbf{p}_1 \ \mathbf{p}_2) \quad (2.0.44)$$

$$\mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (2.0.45)$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (2.0.46)$$

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (2.0.47)$$

Hence equation becomes,

$$\mathbf{V} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (2.0.48)$$

$$\therefore \mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (2.0.49)$$

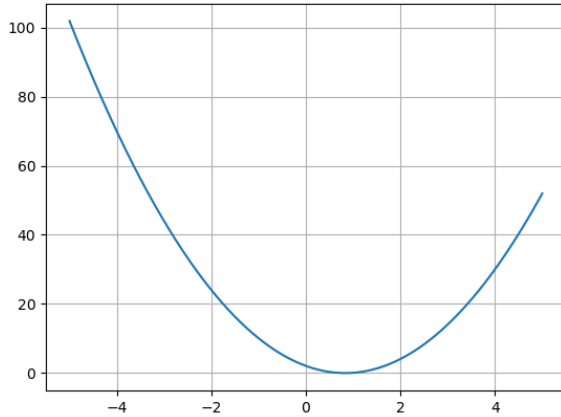


Fig. 0: $3x^2 - 5x + 2$

To find the vertex of the parabola ,

$$\begin{pmatrix} \mathbf{u}^\top + \eta \mathbf{p}_1^\top \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \eta \mathbf{p}_1 - \mathbf{u} \end{pmatrix} \quad (2.0.50)$$

$$\text{where, } \eta = \mathbf{u}^\top \mathbf{p}_1 \quad (2.0.51)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad (2.0.52)$$

$$\therefore \mathbf{p}_1^\top \mathbf{c} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad (2.0.53)$$

$$= 1 \quad (2.0.54)$$

and,

$$\mathbf{p}_2^\top \mathbf{V} \mathbf{p}_2 = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (2.0.55)$$

$$= 1 \quad (2.0.56)$$

$$\therefore (\mathbf{p}_1^\top \mathbf{c})(\mathbf{p}_2^\top \mathbf{V} \mathbf{p}_2) = 1 > 0 \quad (2.0.57)$$

Hence, The given equation does not have real roots.

iii) Comparing the quadratic equations with stan-

dard equation,

$$ax^2 + 2hxy + by^2 + 2dx + 2ey + f = 0 \quad (2.0.58)$$

$$\therefore c = b = 0, e = \frac{-1}{2}, a = 2, d = -\sqrt{2}, f = 1. \quad (2.0.59)$$

$$\therefore \mathbf{V} = \begin{pmatrix} 2 & b \\ b & c \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (2.0.60)$$

$$\therefore \mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} = \begin{pmatrix} -\sqrt{2} \\ -\frac{1}{2} \end{pmatrix} \quad (2.0.61)$$

$$f = 1 \quad (2.0.62)$$

Finding the eigen values corresponding to \mathbf{V}

$$|\mathbf{V} - \lambda \mathbf{I}| = 0 \quad (2.0.63)$$

$$\begin{pmatrix} 2 - \lambda & 0 \\ 0 & -\lambda \end{pmatrix} = 0 \quad (2.0.64)$$

$$(2 - \lambda)(-\lambda) = 0 \quad (2.0.65)$$

$$\therefore \lambda = 0, 2 \quad (2.0.66)$$

Calculating the eigen vectors corresponding to the $\lambda = 0, 1$ respectively,

$$\mathbf{V} \mathbf{x} = \lambda \mathbf{x} \quad (2.0.67)$$

$$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} = 0 \Rightarrow \mathbf{p}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (2.0.68)$$

$$\begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} \mathbf{x} = \mathbf{x} \Rightarrow \mathbf{p}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (2.0.69)$$

Now by eigen decomposition on \mathbf{V} ,

$$\mathbf{V} = \mathbf{P} \mathbf{D} \mathbf{P}^\top \quad (2.0.70)$$

$$\text{where, } \mathbf{P} = (\mathbf{p}_1 \quad \mathbf{p}_2) \quad (2.0.71)$$

$$\mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (2.0.72)$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (2.0.73)$$

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \quad (2.0.74)$$

Hence equation becomes,

$$\mathbf{V} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (2.0.75)$$

$$\therefore \mathbf{V} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \quad (2.0.76)$$

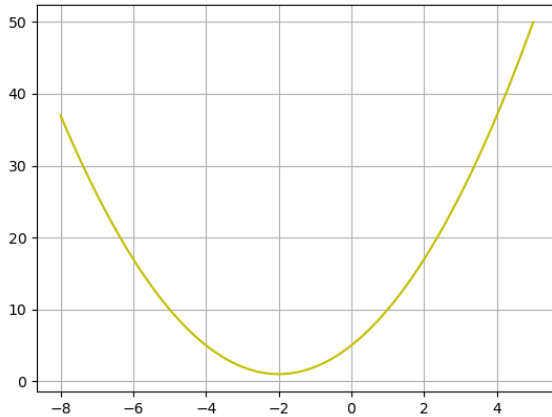


Fig. 0: $x^2 + 4x + 5 = 0$

To find the vertex of the parabola ,

$$\begin{pmatrix} \mathbf{u}^\top + \eta \mathbf{p}_1^\top \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \eta \mathbf{p}_1 - \mathbf{u} \end{pmatrix} \quad (2.0.77)$$

$$\text{where, } \eta = \mathbf{u}^\top \mathbf{p}_1 \quad (2.0.78)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \quad (2.0.79)$$

$$\because \mathbf{p}_1^\top \mathbf{c} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \quad (2.0.80)$$

$$= 0 \quad (2.0.81)$$

and,

$$\mathbf{p}_2^\top \mathbf{V} \mathbf{p}_2 = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (2.0.82)$$

$$= 1 \quad (2.0.83)$$

$$\because (\mathbf{p}_1^\top \mathbf{c})(\mathbf{p}_2^\top \mathbf{V} \mathbf{p}_2) = 0 = 0 \quad (2.0.84)$$

Hence, The given equation has equal real roots.

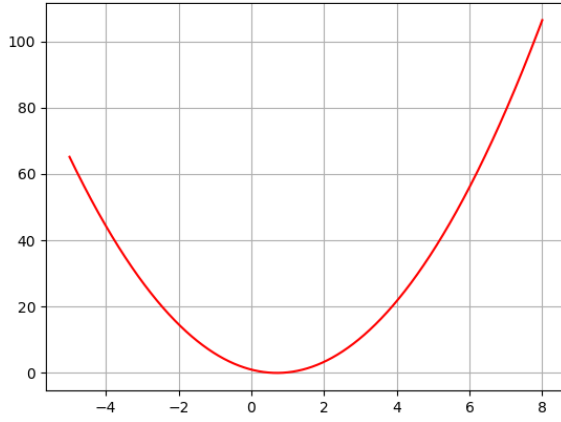


Fig. 0: $2x^2 - 2\sqrt{2}x + 1 = 0$

Our results:

Equation	V	D	P	η	c	p₁	$(\mathbf{p}_1^\top \mathbf{c})(\mathbf{p}_2^\top \mathbf{V} \mathbf{p}_2)$	result
$3x^2 - 5x + 2 = 0$	$\begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & -3 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\frac{-1}{2}$	$\begin{pmatrix} \frac{5}{6} \\ \frac{-1}{12} \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	< 0	Has real roots
$x^2 + 4x + 5 = 0$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\frac{-1}{2}$	$\begin{pmatrix} 2 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	> 0	Has no real roots
$2x^2 - 2\sqrt{2}x + 1 = 0$	$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\frac{-1}{2}$	$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$= 0$	Has equal roots