Gate Assignment

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https://github.com/ArunSiddardha/EE900/tree/main/Gate_assignment/Gate_Assignment.tex

GATE-EC 2009 Q.41

Consider a system whose input x and input y are related by the equation,

$$y(t) = \int_{-\infty}^{\infty} x(t-\tau)\, h(2\tau)\, d\tau$$

where h(t) is shown in the graph. Which of the following properties are possesed by the system?
BIBO : Bounded input gives a bounded output
casual: the system is casual
LP: The system is low pass
LTI : The system is linear and time-variant

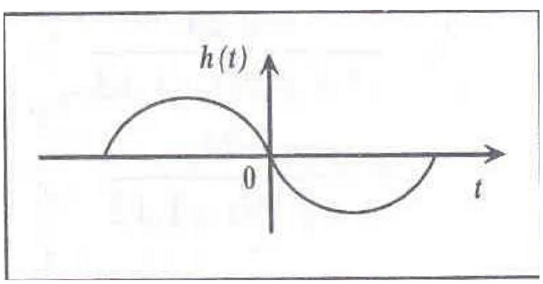


Fig. 1: $h(t)$

1) Casual,LP
2) BBIO, LTI
3) BIBO,casual,LTI
4) LP,LTI

SOLUTION

Answer: 2
Lets take **h(t)** as sin(t) in the interval of - π to π.
LTI:
We say that a system is **linear** if and only if it follows the Principle of Superposition, i.e Law of Additivity and Law of Homogeneity.
A system is said to be **time invariant** if the output signal does not depend on the absolute time, i.e a time delay on the input signal directly equates to the delay in the output signal.
The system relating the input signal $x(t)$ and output signal $y(t)$, given by

$$y(t) = \int_{-\infty}^{\infty} x(t-\tau)\, h(2\tau)\, d\tau \tag{0.0.1}$$

is linear and time invariant in nature.
From (0.0.1), we can say the system is linear if it follows both the laws of Additivity and Homogeneity.
Law of Additivity:
Let the two input signals be $x_1(t)$ and $x_2(t)$, and their corresponding output signals be $y_1(t)$ and $y_2(t)$, then:

$$y_1(t) = \int_{-\infty}^{\infty} x_1(t-\tau)\, h(2\tau)\, d\tau \tag{0.0.2}$$

$$y_2(t) = \int_{-\infty}^{\infty} x_2(t-\tau)\, h(2\tau)\, d\tau \tag{0.0.3}$$

$$y_1(t) + y_2(t) = \int_{t-T}^{t} [x_1(t-\tau)\, h(2\tau) + x_2(t-\tau)\, h(2\tau)]\, d\tau \tag{0.0.4}$$

Now, consider the input signal of $x_1(t) + x_2(t)$, then the corresponding output signal is given by $y'(t)$:

$$y'(t) = \int_{t-T}^{t} [x_1(t-\tau)\, h(2\tau) + x_2(t-\tau)\, h(2\tau)]\, d\tau \tag{0.0.5}$$

Clearly, from (0.0.4) and (0.0.5):

$$y'(t) = y_1(t) + y_2(t) \tag{0.0.6}$$

Thus, the Law of Additivity holds.
Law of Homogeneity:
Consider an input signal $kx(t)$, where k is any constant. Let the corresponding output be given by

$y'(t)$, then:

$$y'(t) = \int_{-\infty}^{\infty} kx(t-\tau)h(2\tau)\,d\tau \tag{0.0.7}$$

$$= k\int_{-\infty}^{\infty} x(t-\tau)h(2\tau)\,d\tau \tag{0.0.8}$$

$$= ky(t) \tag{0.0.9}$$

Clearly, from (0.0.9),

$$y'(t) = ky(t) \tag{0.0.10}$$

Thus, the Law of Homogeneity holds.
Since both the Laws hold, the system satisfies the Principle of Superposition, and is thus, a **linear system**.

BIBO:
We say that a system is BIBO stable if bounded input $x(t)$ gives bounded input $y(t)$. Here since $h(t)$ is bounded if we give a bounded input $x(t)$ and the integral is also is in betweeen two bounded inputs so we get a bounded value of $y(t)$. So the system is **BIBO stable**.

Therefore, The answer is **option 2**

LAPLACE

Taking the laplace transform form for the h(t)

$$H(s) = \int_{0}^{\infty} h(t)e^{-st}dt \tag{0.0.11}$$

$$H(s) = \int_{0}^{\infty} -sin(t)e^{-st}dt \tag{0.0.12}$$

$$H(s) = -\frac{-cos(t)+ssin(t)}{1+s^2}e^{-st}\Big|_{0}^{\infty} \tag{0.0.13}$$

$$H(s) = -\left(0 - \frac{1-0}{1+s^2}\right) \tag{0.0.14}$$

$$H(s) = \frac{1}{1+s^2} \tag{0.0.15}$$

The laplace transform of the above function also posses BBIO(because bounded input produces bounded output),LTI(trivial) properties.
So for the laplace transform also **option 2 is correct**

CONVOLUTION

$y(t)$ is defined as convolution of g(t) and x(t) $(taking\ \ h(2\tau)\ \ as\ \ g(\tau)\ \because\ h(t)\ is\ \ -\sin(t) \implies g(t)\ is\ -sin(2t))$

$$y(t) = g(t) * x(t) \tag{0.0.16}$$

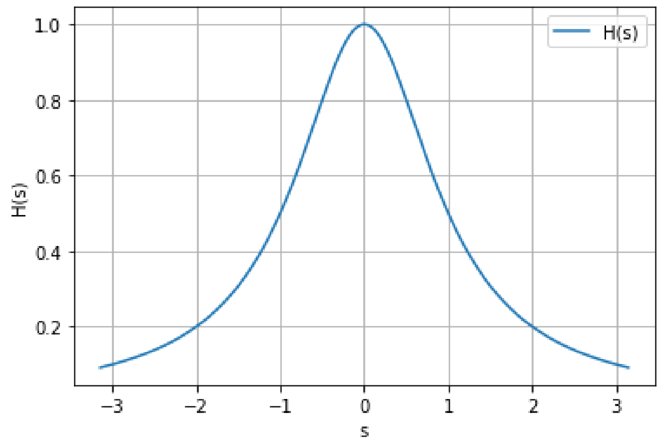


Fig. 2: Laplace transform

The convolution of the y(t) holds the following properties

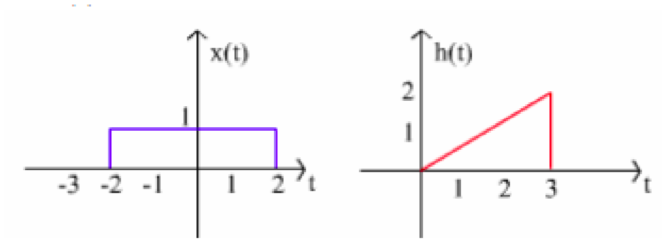
1) **Commutative property :** For the input function $x(t)$, and LTI system responses $g(t)$ the following expression is valid:
$x(t) * g(t) = g(t) * x(t) = \int_{-\infty}^{\infty} f(t–\tau)g(\tau)d\tau$
2) **Distributive property :** For the input function $x(t)$, and LTI system response $g_1(t), g_2(t)$ the following expression is valid:
$x(t) * (g_1(t) + g_2(t)) = x(t) * g_1(t) + x(t) * g_2(t)$
3) **Associative property:** For the input functions $x(t)$, and LTI system responses $g_1(t), g_2(t)$ the following expressions are valid:
$x(t) * (g_1(t) * g_2(t)) = (x(t) * g_1(t)) * g_2(t)$
4) **Inversion property:** If the LTI system is an inverting function, then $g_1(t) * g_2(t) = \delta(t)$, here $g_1(t), g_2(t)$ are responses of the input and output functions in the case of continuous-time functions.
5) **Stability property:** Let's assume that the input function is limited, so $x(t) < A$, then considering the convolution integral we can understand if the output function is stable. $y(t) < \int_{-\infty}^{\infty} Ag(\tau)d\tau$, so the output function $y(t)$ will be stable if the integral $\int_{-\infty}^{\infty} g(\tau)d\tau < \infty$. $\because$ Here $g(t)$ is an odd function the integral becomes 0.So the system is stable here.

Verification through example: Example of convolution functions

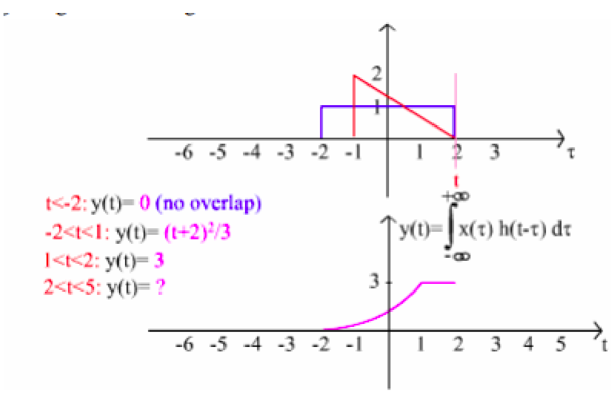
$$g(t) = \begin{cases} \frac{2}{3}t, & \text{for } 0 \leq n \leq 3 \\ 0, & otherwise \end{cases}$$

$$x(t) = \begin{cases} 1, & \text{for } -2 \leq n \leq 2 \\ 0, & otherwise \end{cases}$$

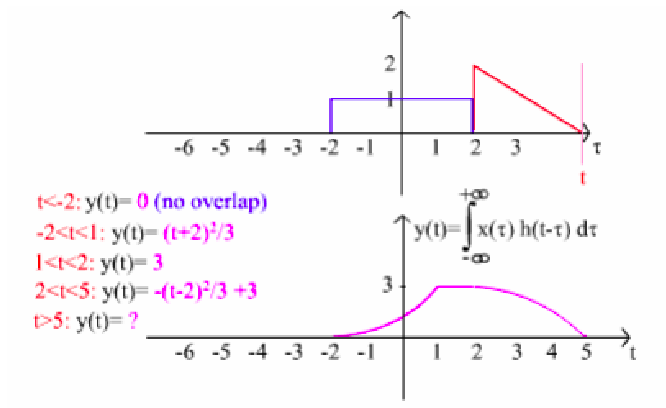
Here are our functions $x(t)$ and $g(t)$. The first step



is to reverse one since its **commutaive** of the functions. We choose to reverse h(t), which we now plot on the "dummy" time axis τ. We plot $x(\tau)$ on the same axis, and begin the process of shifting $g(-\tau)$ by t, and comparing it to $x(\tau)$. Since these are continuous (not discrete) functions, we take an integral (not the sum) when calculating the convolution. In the figure below, g is shifted by $t = -2$. For this value of shift, there is no overlap between $x(\tau)$ and $g(t-\tau)$, so $y(t) = 0$ As



we continue to shift $g(t-\tau)$ by changing t, there gets to be overlap between the two functions, as plotted below (the convolution, $y(t)$, is depicted in pink). The overlap is quantified by multiplying the two input functions on a point by point basis, and then integrating the resulting function. Once t reaches 5, there is no longer overlap between the two functions, and $y(t) = 0$ once again.



A useful thing to know about convolution is the Convolution Theorem, which states that convolvuting two functions in the time domain is the same as multiplying them in the frequency domain: If $y(t) = x(t) * h(t)$, then $Y(f) = X(f)H(f)$

Here the system is stable because for a limited input.The $\int_{-\infty}^{\infty} g(\tau)d\tau < \infty$ so the system is stable

Then the above remaining convolution properties will all hold for these two.