ASSIGNMENT 3

RONGALA ARUN SIDDARDHA
AI20BTECH110019

Download all python codes from

https://github.com/ArunSiddardha/EE3900/blob/main/Assignment_2/code/Assignment_3.py

and latex-tikz codes from

https://github.com/ArunSiddardha/EE3900/blob/main/Assignment_2/Assignment_3.tex

1 Ramsey/tangent and normals/ Q.19

Prove that the circle $\mathbf{x}^\top\mathbf{x} - \begin{pmatrix}6 & 4\end{pmatrix}\mathbf{x} + 9$ subtends an angle $\tan^{-1}\frac{12}{5}$ at the origin

2 Solution

The general equation of a second dergree can be expressed as :

$$\mathbf{x}^\top\mathbf{V}\mathbf{x} + 2\mathbf{u}^\top\mathbf{x} + f = 0 \tag{2.0.1}$$

We know that, for a circle,

$$\mathbf{V} = \mathbf{I} \tag{2.0.2}$$

$$\mathbf{c} = -\mathbf{u} \tag{2.0.3}$$

$$\mathbf{u} = \begin{pmatrix}-3\\-2\end{pmatrix}, f = 9 \tag{2.0.4}$$

$$\mathbf{c} = \begin{pmatrix}3\\2\end{pmatrix} \tag{2.0.5}$$

$$\mathbf{u} = \begin{pmatrix}-3 & -2\end{pmatrix}^\top \tag{2.0.6}$$

The points of contact $\mathbf{q}$, of a line with a normal vector $\mathbf{n}$ to the conics are given by:

$$\mathbf{q} = \mathbf{V}^{-1}(\kappa\mathbf{n} - \mathbf{u}) \tag{2.0.7}$$

$$\kappa = \pm\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n}}} \tag{2.0.8}$$

We know that, for a circle,

$$\mathbf{V} = \mathbf{I} \tag{2.0.9}$$

If r is radius and $\mathbf{c}$ is the centre of the circle we have:

$$f = \mathbf{u}^T\mathbf{u} - r^2 \tag{2.0.10}$$

$$\mathbf{c} = -\mathbf{u} \tag{2.0.11}$$

and from the properties of an Identity matrix,

$$\mathbf{I}^{-1} = \mathbf{I} \tag{2.0.12}$$

$$\mathbf{I}\mathbf{X} = \mathbf{X} \tag{2.0.13}$$

$$\kappa = \pm\sqrt{\frac{r^2}{\mathbf{n}^\top\mathbf{n}}} \tag{2.0.14}$$

$$= \pm\sqrt{\frac{4}{\mathbf{n}^\top\mathbf{n}}} \tag{2.0.15}$$

$$= \pm\frac{2}{\sqrt{\mathbf{n}^\top\mathbf{n}}} \tag{2.0.16}$$

So,

$$\mathbf{q} = \pm\frac{2\mathbf{n}}{\sqrt{\mathbf{n}^\top\mathbf{n}}} + \mathbf{u} \tag{2.0.17}$$

$$\tag{2.0.18}$$

Now $\mathbf{q}$ lies on the line therefore,

$$\mathbf{n}^\top\left(\pm\frac{2\mathbf{n}}{\sqrt{\mathbf{n}^\top\mathbf{n}}} + \mathbf{u}\right) = 0 \tag{2.0.19}$$

$$\pm 2\sqrt{\mathbf{n}^\top\mathbf{n}} + \mathbf{n}^\top\mathbf{u} = 0 \tag{2.0.20}$$

$$Since,\ \mathbf{n} = \begin{pmatrix}m & -1\end{pmatrix}^\top\ for\ the\ tangent \tag{2.0.21}$$

$$\implies \pm\left(2\sqrt{m^2+1}\right) = 2 - 3m \tag{2.0.22}$$

S.O.B.S

$$4\left(m^2+1\right) = 4 + 9m^2 - 12m \tag{2.0.23}$$

$$5m^2 - 12m = 0 \tag{2.0.24}$$

$$m(5m-12) = 0 \tag{2.0.25}$$

$$m = 0\ or\ m = \frac{12}{5} \tag{2.0.26}$$

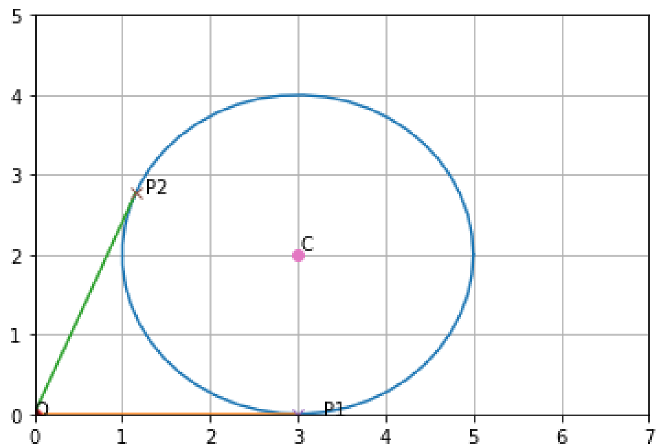

Fig. 0: graph

So,

$$\mathbf{n_1} = \begin{pmatrix} \frac{12}{5} & 1 \end{pmatrix}^\top \quad \mathbf{n_2} = \begin{pmatrix} 0 & 1 \end{pmatrix}^\top \tag{2.0.27}$$

$$cos\theta = \frac{\mathbf{n_1}^\top \mathbf{n_2}}{\|\mathbf{n_1}\|\|\mathbf{n_2}\|} \tag{2.0.28}$$

$$= \frac{\begin{pmatrix} \frac{12}{5} & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}}{\frac{13}{5} \times 1} \tag{2.0.29}$$

$$cos\theta = \frac{5}{13} \tag{2.0.30}$$

Therefore angle between the lines is $tan^{-1}\left(\frac{12}{5}\right)$
Hence proved.