⇒ Mechanics :-

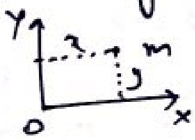
→ Moment of Inertia :

1) $I = \Sigma m_i r_i^2$ for discrete point masses.

$$= \int r^2 dm = \iint r^2 \sigma \, dA = \iiint r^2 \rho \, dV$$

Radius of gyration (k) st $I = Mk^2$

Product of Inertia = (wrt 2 mutually ⊥ axes) = mxy



2) Parallel Axis Theorem : $I_{\parallel} = I_{CM} + Md^2$

Perpendicular Axis Theorem : $I_{xx} = I_{yy} + I_{zz}$

$I_{yy} = I_{zz} + I_{xx}$

$I_{zz} = I_{xx} + I_{yy}$

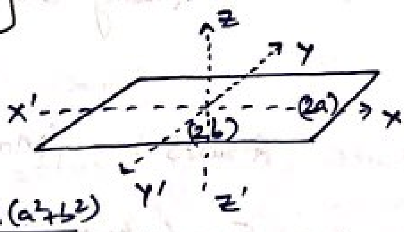
3) Standard Results :

(λ)

(a) Uniform Rod of length $(\underset{=2a}{l})$ & Mass M :

$I_{CM} = \frac{ML^2}{12} = \boxed{\frac{Ma^2}{3}}$ $\quad I_{endpt} = \frac{ML^2}{3} = \boxed{\frac{4Ma^2}{3}}$

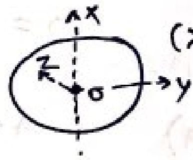
Don't confuse b/w length (l) or (2a)



(σ)

(b) Rectangular Lamina of mass M & sides (2a), (2b) :

$I_{xx'} = \frac{Ma^2}{3}$ $\quad I_{yy'} = \frac{Mb^2}{3}$ $\quad I_{zz} = \frac{M(a^2+b^2)}{3}$.

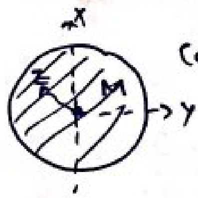


(λ)

(c) Circular Ring of radius 'a' & Mass M :-

About diameter = $\frac{Ma^2}{2}$ [using ↓]

About OZ = $\boxed{Ma^2}$ [directly]



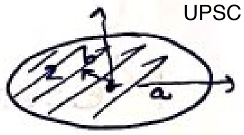
(σ)

(d) Circular Plate of radius 'a' & Mass M :-

About diameter = $\frac{Ma^2}{4}$

About ⊥ axis through centre : $\boxed{\frac{Ma^2}{2}}$

(e) Elliptic disc of axes $2a$, $2b$ & mass M

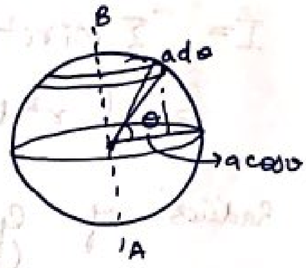


About axis $2a$: $\frac{Mb^2}{4}$

About axis $2b$: $\frac{Ma^2}{4}$

About axis $\perp$ to centre : $\frac{M}{4}(a^2+b^2)$

(f) Spherical shell of Mass 'M' :

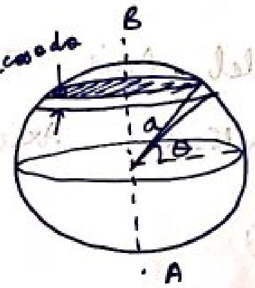


$$I_{AB} = \int_{-\pi/2}^{\pi/2} (\sigma \times a\,d\theta \times 2\pi a\cos\theta) \times (a\cos\theta)^2 \qquad \sigma = \frac{M}{4\pi a^2}$$

$$= 4\pi a^4 \sigma \int_0^{\pi/2} \cos^3\theta\, d\theta$$

$$= \boxed{\frac{2}{3} Ma^2}$$

(g) Solid sphere of Mass 'M' :



$$I = \int_{-\pi/2}^{\pi/2} \left[\rho \times a\cos\theta\, d\theta \times \pi(a\cos\theta)^2\right] \times \frac{(a\cos\theta)^2}{2} \qquad \rho = \frac{M}{\frac{4}{3}\pi r^3}$$

$$= \boxed{\frac{2}{5} Ma^2}$$

NOTE: Use $\int_0^{\pi/2} \sin^p x \cos^q x = \frac{1}{2} \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+1}{2}+\frac{q+1}{2}\right)}$ in exam.

very important to save time.

Don't forget

or $I = \int_0^a \left[\rho \times 4\pi r^2 dr\right] \times \frac{2r^2}{3}$ (using shells)

4) Some Results :- (Moment & Product of inertia wrt 3 mutually $\perp$ axes)

Let $A = \Sigma m(y^2+z^2)$ $\quad B = \Sigma m(z^2+x^2)$ $\quad C = \Sigma m(x^2+y^2)$

$D = \Sigma myz$ $\quad E = \Sigma mzx$ $\quad F = \Sigma mxy$

(a) $A+B > C$, $B+C > A$, $C+A > B$ [sum of any $2 >$ 3rd]

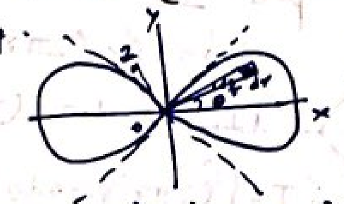
(b) $A+B+C =$ constant $= 2$ (MI about the point where the 3 axes meet)

$= 2\Sigma Mr^2$ (where $r^2 = x^2+y^2+z^2$)

Here it is origin

5) Example: Find MI of the area of lemniscate $r^2 = a^2\cos 2\theta$
(i) about the axis (ii) about a line through origin ⊥ to axis & in plane
(iii) — " — & ⊥ plane

Don't be oversmart. Approach crudely.



$$\text{Area} = 2\int_{\theta=-\pi/4}^{\pi/4}\int_0^{a\sqrt{\cos 2\theta}} r\,dr\,d\theta = \int_{-\pi/4}^{\pi/4} a^2\cos 2\theta\, d\theta = a^2$$

$\therefore \sigma = M/a^2$

$$I_{ox} = 2\int_{-\pi/4}^{\pi/4}\int_{r=0}^{a\sqrt{\cos 2\theta}} (\sigma \times dr \times r d\theta) \times r^2\sin^2\theta = 2\sigma\int \sin^2\theta \times \frac{a^4\cos^2 2\theta}{4}\, d\theta$$

$$= \frac{2\sigma a^4}{4}\times 2\int_0^{\pi/4}\frac{(1-\cos 2\theta)}{2}\cos^2 2\theta\, d\theta = \frac{\sigma a^4}{4}\int_0^{\pi/2}(1-\cos\phi)\cos^2\phi\, d\phi$$

$$= \frac{\sigma a^4}{4}\int_0^{\pi/2}(\cos^2\phi - \cos^3\phi)\, d\phi = \frac{\sigma a^4}{4}\left[\frac{1}{2}\frac{\Gamma\left(\frac{2+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{2+1}{2}+\frac{1}{2}\right)} - \frac{1}{2}\frac{\Gamma\left(\frac{3+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3+1}{2}+\frac{1}{2}\right)}\right]$$

$$= \frac{\sigma a^4}{4}\left[\frac{\frac{1}{2}\times\pi}{2} - \frac{1\times\sqrt{\pi}}{\cancel{2}\times\frac{3}{\cancel{2}}\times\frac{1}{2}\times\sqrt{\pi}}\right] = \frac{Ma^4}{4a^2}\left[\frac{\pi}{4} - \frac{2}{3}\right] = \frac{Ma^2}{16}\left(\pi - \frac{8}{3}\right)$$

Similarly $I_{oy} = \frac{Ma^2}{16}\left(\pi + \frac{8}{3}\right)$ $\therefore I_{oz} = I_{ox} + I_{oy} = (\pi M a^2)/8$

DIRICHILET THEOREM: $\iiint x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz = \dfrac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(l+m+n+1)}$

given $x + y + z \le 1$

6) MoI of ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ About axis OX:

$$I = \iiint (\rho\, dx\, dy\, dz)\,(y^2+z^2)$$

Let $x = u^{1/2}a$ $y = v^{1/2}b$ $z = w^{1/2}c$

$\therefore dx = \frac{a}{2}u^{-1/2}du$ $dy = \frac{b}{2}v^{-1/2}dv$ $dz = \frac{c}{2}w^{-1/2}dw$

Now $M = \iiint \rho\, dx\, dy\, dz$

$$\therefore M = 8\int_0^1\int_0^1\int_0^1 \rho \times \frac{abc}{8} u^{-1/2}v^{-1/2}w^{-1/2}\, du\, dv\, dw$$

$$= \rho\times abc\left[\frac{\Gamma(1/2)^3}{\Gamma\left(\frac{3}{2}+1\right)}\right] = \frac{\rho abc \times \pi\sqrt{\pi}}{\frac{3}{2}\times\frac{1}{2}\times\sqrt{\pi}} = \frac{4}{3}\pi abc\rho.$$

$$\therefore I = \iiint \rho\,(y^2+z^2)\, dx\, dy\, dz = 8\int_0^1\int_0^1\int_0^1 \frac{\rho abc}{8}\left(u^{-1/2}v^{-1/2}w^{-1/2}\right)(vb^2+wc^2)\, du\, dv\, dw$$

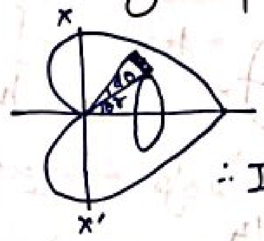
$$= \rho abc\iiint b^2 u^{-1/2}v^{1/2}w^{-1/2}\, du\, dv\, dw + \iiint c^2 u^{-1/2}v^{-1/2}w^{1/2}\, du\, dv\, dw$$

$$= \rho abc\left[(b^2+c^2)\left(\frac{\Gamma(1/2)\Gamma(1/2)\Gamma(3/2)}{\Gamma(5/2+1)}\right)\right] = \rho abc\,(b^2+c^2)\frac{\pi\,\Gamma(3/2)}{5/2\cdot\frac{3}{2}\Gamma(3/2)}$$

$$= \frac{M}{4/3\,\pi}\frac{(b^2+c^2)\pi\times 4}{5\times 3} = \frac{M}{5}(b^2+c^2).$$

Example: Solid body of density ρ is in the shape of the solid formed by the revolution of cardoid $r = a(1+\cos\theta)$ about initial line. Find its MI about straight line through pole $\perp$ initial line.

*Don't do the mistake of taking a disc. Go by ring instead



$dm = \rho \times (2\pi r \sin\theta) \times (r\,d\theta)(dr)$

$dI_{xx'} = dm\left[\frac{(r\sin\theta)^2}{2} + (r\cos\theta)^2\right]$

$\therefore I = \iint \rho \frac{2\pi r^2}{2} \sin\theta\, d\theta\, dr \times r^2 (1+\cos^2\theta)$

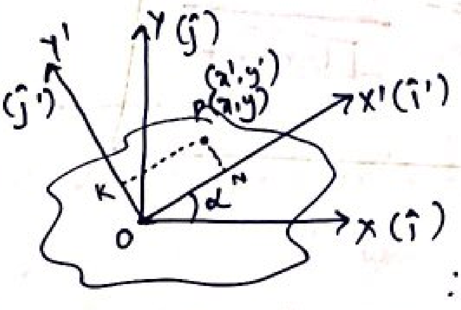
$$= I = \int_{\theta=0}^{\pi}\int_{r=0}^{a(1+\cos\theta)} \rho\pi r^4 (1+\cos^2\theta)\sin\theta\, dr\, d\theta = \frac{\rho\pi a^5}{5}\int_0^{\pi} (1+\cos^2\theta)(\sin\theta)(1+\cos\theta)^5\, d\theta$$

$$= \frac{\rho\pi a^5}{5}\int_2^0 [1+(t-1)^2]\, t^5\, (-dt) \qquad t = 1+\cos\theta$$

$$= \frac{\rho\pi a^5}{5}\int_0^2 [t^7 - 2t^6 + 2t^5]\,dt = \frac{\rho\pi a^5}{5}\left[32 - \frac{256}{7} + \frac{64}{3}\right]$$

$$= \frac{352\rho\pi a^5}{105}.$$

7) Moment & Product of Inertia of a plane lamina about a Line (l,m,n): $l = \cos\alpha$



M about OX & OY given as A & B

$\therefore$ Product is F

For $P(x,y)$: $A = \Sigma m x^2 \quad B = \Sigma m y^2 \quad F = \Sigma m xy$

Let OX', OY' be 2 $\perp$ axes through O st angle b/w OX & OX is α

$\therefore$ dc's of $OX' = [\cos\alpha, \sin\alpha, 0]$

dc's of $OY' = [-\sin\alpha, \cos\alpha, 0]$

$x' = PK = ON = \text{projection} = x\cos\alpha + y\sin\alpha$

$y' = PN = OK = -x\sin\alpha + y\cos\alpha$

$\therefore I_{OX'} = \Sigma m y'^2 = \Sigma m(x^2\sin^2\alpha + y^2\cos^2\alpha - 2xy\cos\alpha\sin\alpha) = B\sin^2\alpha + A\cos^2\alpha - F\sin 2\alpha$

$I_{x'y'} = \Sigma m x'y' = \Sigma m(-x^2\cos\alpha\sin\alpha + y^2\sin\alpha\cos\alpha + xy(\cos^2\alpha - \sin^2\alpha)) = F\cos 2\alpha + \frac{\sin 2\alpha}{2}[A-B]$

$\therefore\ I_{OX'} = A\cos^2\alpha + B\sin^2\alpha - F\sin 2\alpha$

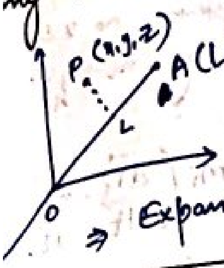
$I_{x'y'} = F\cos 2\alpha + \frac{\sin 2\alpha}{2}(A-B)$

for symmetric bodies $F = 0$,

If $x'y'$ are principal axes $\Rightarrow I_{x'y'} = 0$

$$\therefore \boxed{\tan 2\alpha = \frac{2F}{B-A}} \rightarrow \text{condition for principal axes}$$

> Given MI & P of Inertia about 3 mutually ⊥ axes, Find MI about any line (l, m, n) through origin:



$I_{OA} = \Sigma m \cdot PL^2$

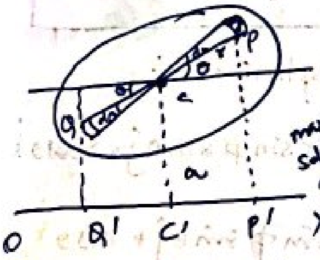
$PL^2 = (x^2+y^2+z^2) - (lx+my+nz)^2$

⇒ Expand & we get

$$\boxed{I_{OA} = Al^2 + Bm^2 + Cn^2 - 2Dmn - 2EnL - 2Flm}$$

9) Closed curve revolves around OX without intersection it. Show that MI of solid of revolution about OX = $M(a^2+3k^2)$ where a is the distance of centre C from OX & k is radius of gyration about line through C ∥ to OX.

$CC' = a$ area of closed surface



$M = \rho \times 2\pi a S$ mass of solid generated about OX [Pappus Theorem]

$S = 2\iint (r\,d\theta)(dr)$

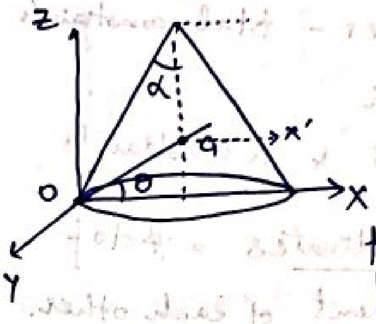
$\therefore I_{CA} = S\rho k^2 = 2\iint (r\sin\theta)^2 \rho\, r\,dr\,d\theta$
$= 2\rho \iint r^3 \sin^2\theta\, d\theta\, dr$

$I_{OX} = \iint [2\pi(a+r\sin\theta)\cdot(a+r\sin\theta)^2 + 2\pi(a-r\sin\theta)(a-r\sin\theta)^2]\rho r\,d\theta\,dr$
$= \iint 4\pi\rho(a^3 + 3ar^2\sin^2\theta) r\,d\theta\,dr$
$= 4\pi\rho a^2 \iint r\,d\theta\,dr + 12\pi\rho a \iint r^3 \sin^2\theta\,d\theta\,dr$
$= 2\pi\rho a^3 \cdot S + 6\pi\rho a \cdot S\rho k^2$
$= 2\pi\rho a S(a^2+3k^2) = M(a^2+3k^2)$

Solid Rubber tyre of mass M with radius 'b' & radius of cross section = 'a'

$\boxed{I = M\left(b^2 + \frac{3}{4}a^2\right)}$ For Hollow → $\boxed{I = M\left(\frac{2b^2+3a^2}{2}\right)}$

Example. Find the principal axes of a right circular cone at a point on the circumference of the base & show that one of them will pass through its CG if $2\alpha = 2\tan^{-1}(1/2)$



Let O be the pt on the circumference. Take OX as diameter of base, OY ⊥ OX in plane of base & OZ ⊥ the plane parallel to axis of cone

$I_{OX} = G_{x'} + \frac{Mh^2}{16} = \frac{M}{20}(3a^2+2h^2) = A$ [easy to calculate]

$I_{OY} = B = \frac{M}{20}(23a^2+2h^2)$

$I_{OZ} = C = \frac{3}{10}Ma^2 + Ma^2 = \frac{13}{10}Ma^2$

$I_{YZ} = D = 0$ $I_{ZX} = E = 0 + M(a)\left(\frac{h}{4}\right) = \frac{Mah}{4}$

$I_{XY} = F = 0$

$\because D = 0$ & $F = 0$ ⇒ OY is a principal axis.

Other 2 axes are in XZ plane.

Let one of it make angle θ with OX

then $\tan 2\theta = \frac{2E}{C-A} = \frac{\frac{Mah}{2}}{\frac{13}{10}Ma^2 - \frac{M}{20}(3a^2+2h^2)}$

$= \frac{10ah}{23a^2 - 2h^2}$

Other axis will be at $\left(\theta + \frac{\pi}{2}\right)$

Now if it passes through G, then

$\tan\theta = \frac{h}{4a}$ $\therefore \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{8ah}{16a^2-h^2}$

$\therefore \frac{8ah}{16a^2-h^2} = \frac{10ah}{23a^2-2h^2}$ ⇒ $h = 2a$

$\therefore \tan\alpha = \frac{a}{h} = \frac{1}{2}$

∴ vertical angle $2\alpha = 2\tan^{-1}(1/2)$

⇒ LAGRANGE'S EQUATIONS :

(ENERGY METHOD TO SOLVE EQ^n OF MOTION)

1) Degree of freedom = Min no of indept variables to describe the system.

n = total coordinates − total constraints

$= 3N - k$

for N particles & k constraints

2) Generalised coordinates = # dof

They are independent of each other.

$\vec{r}_i = \vec{r}_i(q_j)$ $\quad i = 1$ to N, $\quad j = 1$ to n

q_j → gen. coordinates.

3) Scleronomic Constraints = do not depend on time

Rheonomic = depend on time

Holonomic = can be expressed in equality form like $f(t, q_1, q_2 \dots) = 0$

4) $\vec{r}_i = \vec{r}_i(q_j)$

$$\delta \vec{r}_i = \sum \frac{\partial \vec{r}_i}{\partial q_j}\delta q_j + \frac{\partial \vec{r}_i}{\partial t}\delta t$$

$$\therefore \dot{\vec{r}}_i = \lim_{t \to 0}\frac{\delta \vec{r}_i}{\delta t} = \sum \frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j + \underbrace{\cancel{\frac{\partial \vec{r}_i}{\partial t}}}_{\text{0 for scleronomic constraints}}$$

$$\therefore \boxed{\dot{\vec{r}}_i = \sum \frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j}$$

5) $L = T - V$ $\quad T = (KE) = \frac{1}{2}\sum m_i \dot{r}_i^2$

$V = (PE) = -\int \vec{F}\cdot d\vec{r}$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \quad \text{— (1)}$$

LAGRANGIAN EQUATION

$$\equiv \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = \frac{\partial W}{\partial q_j} \quad \text{[for conservative]}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = -\frac{\partial V}{\partial q_j} + \frac{d}{dt}\underbrace{\left(\frac{\partial V}{\partial \dot{q}_j}\right)}_{0}$$

0 for conservative systems as V depends only on q_j

6) $T = \frac{1}{2}\sum m_i \dot{r}_i^2 = \frac{1}{2}\sum m_i\left(\frac{d\vec{r}_i}{dt}\cdot\frac{d\vec{r}_i}{dt}\right)$

$$= \frac{1}{2}\sum m_i\left(\frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \vec{r}_i}{\partial t}\right)\left(\frac{\partial \vec{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \vec{r}_i}{\partial t}\right)$$

$$= c_1 \dot{q}_j \dot{q}_k + \underbrace{c_2 \dot{q}_j + c_3}_{\text{only if non-scleronomic constraints.}}$$

7) REMEMBER : (polar coordinates)

$(x, y) \longrightarrow (r, \theta)$

$$\boxed{\begin{aligned} &\vec{r} = r\hat{r} \qquad \vec{V} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \\ &\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta} \end{aligned}}$$

Quick Derivation : (Spherical) :

$\vec{r} = r\cos\phi\sin\theta\,\hat{i} + r\sin\phi\sin\theta\,\hat{j} + r\cos\theta\,\hat{k}$

$\therefore \hat{r} = \cos\phi\sin\theta\,\hat{i} + \sin\phi\sin\theta\,\hat{j} + \cos\theta\,\hat{k}$

$\hat{\theta} = \cos\phi\cos\theta\,\hat{i} + \sin\phi\cos\theta\,\hat{j} - \sin\theta\,\hat{k}$

$\hat{\phi} = \hat{r} \times \hat{\theta}$

8) Velocity vector in diff^t systems :

$\vec{V} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}$

$= \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\dot{\phi}\hat{\phi}$

$= \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + \dot{z}\hat{z}$

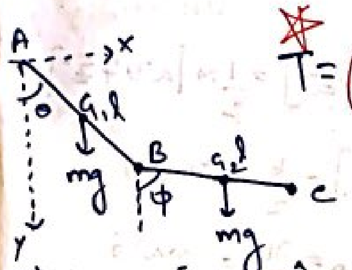
9) Steps to follow :

a) Express T & V using q_j

b) Get $L = T - V$

c) Put in (1) for different q_js to get no of eq^{n}s of motion.

10) Example: 2 equal rods AB & BC of length l each are joined at B & suspended vertically from A. ST the periods of small oscillations are $\frac{2\pi}{n}$ where $n^2 = \left(3 \pm \frac{6}{\sqrt{7}}\right)\frac{g}{l}$ Don't forget.



$$T = \left(\frac{1}{2} I_{G_1} \dot\theta^2 + \frac{1}{2} m V_{G_1}^2\right) + \frac{1}{2} I_{G_2} \dot\phi^2 + \frac{1}{2} m V_{G_2}^2$$

$\vec{r}_{G_1} = \frac{l}{2}\left[\sin\theta\, \hat{i} + \cos\theta\, \hat{j}\right]$

$\vec{V}_{G_1}^2 = \frac{1}{4} l^2 \dot\theta^2$ Sly $V_{G_2}^2 = l^2\left[\dot\theta^2 + \frac{\dot\phi^2}{4} + \dot\theta\dot\phi \cos(\theta-\phi)\right]$ 1 for small θ, ϕ

& $W = \frac{mgL}{2}\left[3\cos\theta + \cos\phi\right]$

$\therefore T = \frac{ML^2}{2}\left[\frac{4}{3}\dot\theta^2 + \frac{\dot\phi^2}{3} + \dot\theta\dot\phi\right]$

wrt $\theta \Rightarrow 8l\ddot\theta + 9g\theta + 3l\ddot\phi = 0$

wrt $\phi \Rightarrow 2l\ddot\phi + 3l\ddot\theta + 3g\phi = 0$

$\Rightarrow \left[(8LD^2 + 9g)(2LD^2 + 3g) - 3LD^2(3LD^2)\right]\theta = 0$

$\Rightarrow \left[7L^2D^4 + 42gLD^2 + 27g^2\right]\theta = 0$

If $t_p = \frac{2\pi}{n}$ $\quad \theta = A\cos(nt + B)$
$D^2\theta = -n^2\theta \quad D^4\theta = n^4\theta$

$\Rightarrow 7n^4L^2 - 42gLn^2 + 27g^2 = 0$

$\Rightarrow n^2 = \left[3 \pm \frac{6}{\sqrt{7}}\right]\frac{g}{l}$

$\Rightarrow$ HAMILTONIAN EQUATIONS (only for conservative systems)

1) We get 2n eq^{n}s for n gen. coordinates

2) $L = L(q_j, \dot q_j, t)$

$$\frac{dL}{dt} = \frac{\partial L}{\partial q_j}\dot q_j + \frac{\partial L}{\partial \dot q_j}\ddot q_j + \frac{\partial L}{\partial t} \quad \text{—(I)}$$

Also $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right) - \frac{\partial L}{\partial q_i} = 0$ —(II) Put (II) in (I)

$$\Rightarrow \frac{dL}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_j}\right)\dot q_j + \frac{\partial L}{\partial \dot q_j}\ddot q_j + \frac{\partial L}{\partial t}$$

$$= \frac{d}{dt}\left(\dot q_j \frac{\partial L}{\partial \dot q_j}\right) + \underbrace{\frac{\partial L}{\partial t}}_{0 \text{ if } L \text{ is indep of } t}$$

$$\Rightarrow \frac{d}{dt}\left(L - \dot q_j \frac{\partial L}{\partial \dot q_j}\right) = 0$$

$$\Rightarrow \boxed{H = \Sigma\left(\frac{\partial L}{\partial \dot q_j}\dot q_j\right) - L \ \ \& \ \frac{dH}{dt} = 0}$$

$\therefore$ If L is indep of t we get law of conserv^n of energy.

3) For conservative forces

$L = L(q_j, \dot q_j) = T - V$

$\frac{\partial L}{\partial \dot q_j} = \frac{\partial T}{\partial \dot q_j} - \cancel{\frac{\partial V}{\partial \dot q_j}} \to 0$ [V depends only on q_j]

$= \frac{\partial}{\partial \dot q_j}\left[\frac{1}{2} m_i \dot q_j \dot q_k\right] = m_i \dot q_k = p_j$

Generalised Momenta $= \frac{\partial L}{\partial \dot q_j} = p_j$

$\hookrightarrow$ TRICK: $L = \frac{1}{2}mv^2 + \phi$ (conservative force)
$\frac{\partial L}{\partial v} = mv = p$

$H = p_j \dot q_j - L$

Also $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_j}\right) = \frac{\partial L}{\partial q_j} = \dot p_j$

4) Hamiltonian function: $H = \Sigma p_j \dot q_j - L$

for scleronomic constraints: $T = \frac{\Sigma 1}{2} m \dot q_i \dot q_j$

$\therefore 2T = \Sigma p_j \dot q_j$

$\therefore H = 2T - (T - V) = T + V$
$\hookrightarrow$ useful in problems

5) Hamiltonian Equation:

$H = p_j \dot q_j - L \Rightarrow dH = dp_j(\dot q_j) + d\dot q_j\, p_j - dL$

$L = L(q_j, \dot q_j) \Rightarrow dL = \frac{\partial L}{\partial q_j}dq_j + \frac{\partial L}{\partial \dot q_j}d\dot q_j$

$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_j}\right)dq_j + \frac{\partial L}{\partial \dot q_j}d\dot q_j$

$= \dot p_j\, dq_j + p_j\, d\dot q_j$

$dH = \dot q_j\, dp_j + p_j\, d\dot q_j - \dot p_j\, dq_j - \cancel{p_j\, d\dot q_j}$

$dH = \dot q_j\, dp_j - \dot p_j\, dq_j$ ~~$H = f(p_j, q_j)$~~

Also $H = H(p_j, q_j)$ pos^n & momenta can define a system

$$\Rightarrow dH = \frac{\partial H}{\partial p_j} dp_j + \frac{\partial H}{\partial q_j} dq_j$$

$$= \dot{q}_j \, dp_j - \dot{p}_j \, dq_j$$

$$\Rightarrow \boxed{\frac{\partial H}{\partial p_j} = \dot{q}_j \quad \& \quad \frac{\partial H}{\partial q_j} = -\dot{p}_j}$$

can remember through Dimensional Analysis

See other notes to get the same result through Principle of Least Action

6) Steps to follow:

a) Express T & V as generalised coordinates

b) Get $L = T - V$ (generally independent of t)

V is also dependent only on q_j since we have considered conservative forces

c) Now find $p_j = \frac{\partial L}{\partial \dot{q}_j}$

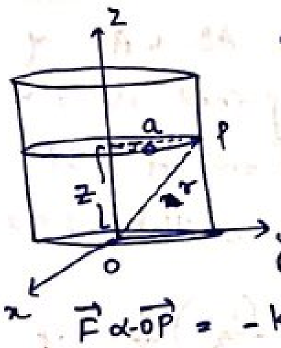
d) Now get $H = \Sigma p_j \dot{q}_j - L$

we generally get $H = T + V$ directly
they should have p_j terms

e) Now write the eq^ns $\frac{\partial H}{\partial p_j} = \dot{q}_j$ & $\frac{\partial H}{\partial q_j} = -\dot{p}_j$

[NOTHING FANCY]

Example: Particle of mass 'm' moves on ~~a~~ the surface of a cylinder. It is subjected to a force towards origin & proportional to distance of particle from origin. Construct H & equations of motions.



a = constant Therefore

$q_1 = z \quad q_2 = \theta$

$$\vec{r} = a\cos\theta\,\hat{i} + a\sin\theta\,\hat{j} + z\hat{k}$$

$$= a\hat{r} + z\hat{z}$$

$$\vec{F} \propto \vec{OP} = -kr \quad \therefore V = -\int F \cdot dr = \frac{kr^2}{2}$$

$$V = \frac{k(a^2 + z^2)}{2}$$

$$T = \frac{1}{2}m\left[\cancel{\dot{a}^2} + a^2\dot{\theta}^2 + \dot{z}^2\right] = \frac{1}{2}m\left[a^2\dot{\theta}^2 + \dot{z}^2\right]$$

$$\therefore L = \frac{1}{2}m\left[a^2\dot{\theta}^2 + \dot{z}^2\right] - \frac{k}{2}(a^2 + z^2)$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \qquad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ma^2\dot{\theta}$$

$$\Rightarrow \dot{z} = p_z/m \qquad \dot{\theta} = p_\theta / ma^2$$

$$\therefore H = p_\theta \dot{\theta} + p_z \dot{z} - L$$

$$= \frac{p_\theta p_\theta}{ma^2} + p_z \cdot \frac{p_z}{m} - \frac{1}{2}m\left[\frac{a^2 p_\theta^2}{m^2 a^4} + \frac{p_z^2}{m^2}\right] + \frac{k}{2}(a^2 + z^2)$$

$$= \frac{1}{2}\left[\frac{p_\theta^2}{ma^2} + \frac{p_z^2}{m}\right] + \frac{k}{2}(a^2 + z^2)$$

[essentially $H = T + V$]
but go like this

Now $\frac{\partial H}{\partial p_\theta} = +\dot{\theta}$ $\qquad \frac{\partial H}{\partial \theta} = -\dot{p}_\theta$

$\Rightarrow \frac{p_\theta}{ma^2} = \dot{\theta}$ & $\dot{p}_\theta = 0$

& $\frac{\partial H}{\partial p_z} = \dot{z}$ $\qquad \frac{\partial H}{\partial z} = -\dot{p}_z$

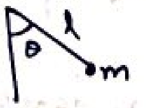
$\Rightarrow \frac{p_z}{m} = \dot{z}$ $\qquad -\dot{p}_z = kz$

These are the 4 H. equations

All questions to be done in this manner only.

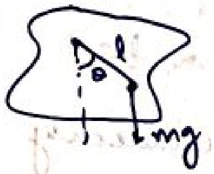
⇒ Different Pendulums :

a) SIMPLE



$T = \frac{1}{2} m (l\dot{\theta})^2 \quad V = -mgl\cos\theta$

b) COMPOUND



$T = \frac{1}{2} I \dot{\theta}^2 \quad I = mk^2$ ↳ radius of gyration

$V = -mgl\cos\theta$

c) SPHERICAL PENDULUM :

Object moves on a surface of sphere of radius 'a'.

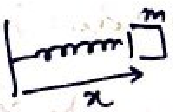
$x = a\sin\theta\cos\phi \quad y = a\sin\theta\sin\phi \quad z = a\cos\theta$

$\vec{v} = \underset{0}{\cancel{\dot{a}}}\hat{a} + a\dot{\theta}\,\hat{\theta} + a\sin\theta\,\dot{\phi}\,\hat{\phi}$

$\therefore T = \frac{1}{2} m \left[a^2\dot{\theta}^2 + a^2\sin^2\theta\,\dot{\phi}^2\right]$

$V = -mga\cos\theta$.

Example: Hamilton for SHM motion in 1D :



$T = \frac{1}{2} m\dot{x}^2 \quad V = -\int -kx\,dx = \frac{kx^2}{2}$

$\therefore L = \frac{1}{2} m\dot{x}^2 - \frac{kx^2}{2}$

$\therefore p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \Rightarrow \dot{x} = \frac{p_x}{m}$

$\therefore H = p_x\dot{x} - L = \frac{p_x^2}{2m} + \frac{kx^2}{2}$

$\Rightarrow \frac{\partial H}{\partial x} = -\dot{p}_x = kx$ & $\frac{\partial H}{\partial p_x} = \dot{x} = \frac{p_x}{m}$

$\therefore \ddot{x} = \frac{\dot{p}_x}{m} = -\frac{k}{m}x \rightarrow$ SHM !!

Using Lagrangian $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = \frac{d}{dt}(m\dot{x}) + kx = 0$

$\Rightarrow \ddot{x} = -\frac{kx}{m} \rightarrow$ SHM!! ∴

⇒ De Alemberts Principle :

1) Obtain all the necessary equations without writing eq^ns for all particles & without innerforces.

[∵ Newtons 3rd law - internal actions & reactions are in ⇌]

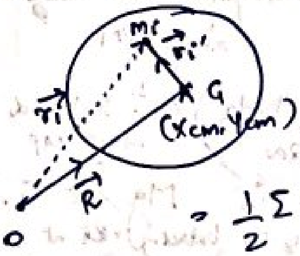
2) Impressed forces : External forces acting on a body

Effective force : product of mass × acc^n

3) Principle states that any point in the system is in ⇌ under the influence of external forces acting at that point & due to reverse forces ∴ $\vec{F}_{ext} + \left[-m\frac{d^2\vec{r}}{dt^2}\right] = 0$

4) Motion of Rigid Body =

Translation of COM + Rotation wrt com



$\vec{r}_i = \vec{r}_i' + \vec{R}$

$KE = \frac{1}{2}\Sigma m_i \dot{r}_i^2$

$= \frac{1}{2}\Sigma m_i (\dot{\vec{r}}_i' + \dot{\vec{R}})\cdot(\dot{\vec{r}}_i' + \dot{\vec{R}})$

$= \frac{1}{2}\Sigma m_i\left[\dot{r}_i'^2 + \dot{R}^2 + 2\dot{r}_i\dot{R}\right]$

$= \frac{1}{2} m\left[v'^2 + V_{cm}^2 + 2\vec{v}'\cdot\vec{V}_{cm}\right]$

$= KE$ wrt CM $+ \frac{1}{2}MV_{cm}^2 + \cancel{\Sigma m_i \vec{v}_i'}^{\,0}\cdot\vec{V}_{cm}$

net momentum abt CM → 0

$= KE_{wrt\,cm} + \frac{1}{2}MV_{cm}^2$

EQUATIONS : (i) $\boxed{M\frac{d^2X_{cm}}{dt^2} = \Sigma X \quad M\frac{d^2Y_{cm}}{dt^2} = \Sigma Y}$ where $F = \Sigma X i + \Sigma Y j$

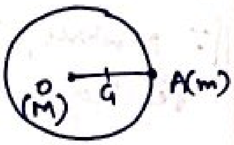
(ii) $\Sigma m_i \frac{d^2\vec{r}_i}{dt^2} = \Sigma F_i \Rightarrow \Sigma \vec{r}_i' \times m_i\frac{d^2\vec{r}_i}{dt^2} = \Sigma \vec{r}_i' \times \vec{F}_i$

Put $\vec{r}_i = \vec{R} + \vec{r}_i'$ & take moment abt G $\Rightarrow \boxed{Mk^2\ddot{\theta} = \Sigma(x'Y - Xy')}$

Examples:

a) Circular board is placed on a smooth horizontal surface & a boy runs around the edge at uniform rate, what is the motion of the board?

No external force ⇒ CM remains at rest

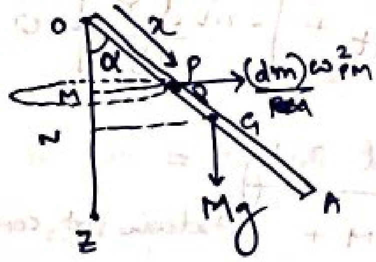


$\therefore OG = \dfrac{ma}{m+M}$

G doesn't move ⇒ O describe a circle of radius OG around G.

VVI - conceptual question

b) Uniform rod OA of length 2a, free to turn about O, revolves with uniform ω about vertical OZ. & is inclined at angle α to OZ. ST $\alpha = 0$ or $\cos^{-1}\left(\dfrac{3g}{4a\omega^2}\right)$



Let PQ = dx at x distance from O.

$\therefore dm = \dfrac{M}{2a} \cdot dx$

Only effective force only PQ is the centripetal force $\dfrac{dm\,\omega^2 r}{r}$ along PM

⇒ reverse eff. force $= \left(\dfrac{M}{2a} \cdot dx\right)\omega^2 . x\sin\alpha$ along MP

∴ Only external force is Mg at G. & its balancing rxn at O.

To avoid reaction at O, take moment about O:

$$\sum\left(\frac{M}{2a}dx\,\omega^2 x\sin\alpha\right)x\cos\alpha - Mg\,a\sin\alpha = 0$$

$$\left[\int_0^{2a} \frac{M\omega^2\cos\alpha}{2a}\,x^2 dx = Mga\right]\sin\alpha$$

$$\Rightarrow \frac{M\omega^2}{2a}\cos\alpha \cdot \frac{8a^3}{3} = Mg\,a\sin\alpha$$

$$\Rightarrow \alpha = \cos^{-1}\left(\frac{3g}{4a\omega^2}\right) \text{ or } \sin\alpha = 0 \Rightarrow \alpha = 0.$$

∴ STEPS:

ia) Find effective force for each particle

ii) Integrate & equate to $\vec{F}_{ext}$

Angular motion means taking moment about a point. We choose a pt so that unnecessary forces are avoided eg hinge points.

5> IMPULSIVE FORCES:

Infinitely large for acting for very small amt of time. Position doesn't change but momentum does

Equations become:

(i) $M(u'' - u') = \int_0^t X\,dt$

(ii) $M(v'' - v') = \int_0^t Y\,dt$

(iii) $Mk^2\ddot{\theta} = \int x'Y\,dt - \int y'X\,dt$ — see pt 4 on back page.

where $\vec{F} = X\hat{i} + Y\hat{j}$

& $\vec{r'} = x'\hat{i} + y'\hat{j}$ (wrt CoM)

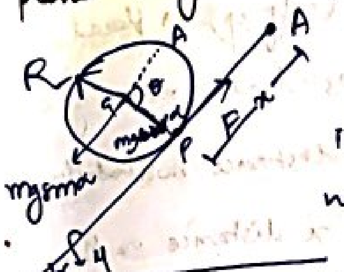
6> Impulse due to a force:

Force 'f' acting for time $t_1 \to t_2$

$$I = mV_2 - mV_1 = \int_{t_1}^{t_2} F \cdot dt$$

7> Angular momentum of Rigid Body = Angular momentum of CoM + Angular momentum wrt CoM

8> Motion of sphere down an inclined plane rough enough for no sliding :



Suppose it has rolled down a distance x. Initial contact pt is now forming angle θ.

$\therefore x = 2\pi a n + a\theta$

as no sliding $\therefore \dot{x} = a\dot{\theta}$ & $\ddot{x} = a\ddot{\theta}$

Also $M\ddot{x} = Mg\sin\alpha - F$ — (I) $F \to$ friction force.

& $M\ddot{y} = 0 = Mg\cos\alpha - R$ — (II)

Also motion about G : $Mk^2\ddot{\theta} = Fa$ — (III)

$\therefore Fa = \frac{Mk^2}{a} \cdot \frac{\ddot{x}}{a} = \frac{Mk^2}{a^2}\ddot{x}$

Put in (I) $\Rightarrow M\ddot{x} = Mg\sin\alpha - \frac{M\ddot{x}k^2}{a^2}$

$\Rightarrow \ddot{x} = \frac{a^2 g\sin\alpha}{a^2+k^2}$

$\Rightarrow \dot{x} = \frac{a^2 g\sin\alpha\, t}{a^2+k^2}$ $\left[\begin{matrix} c=0 \text{ as} \\ \text{at } t=0\ \dot{x}=0\end{matrix}\right]$

& $x = \frac{a^2 g\sin\alpha\, t^2}{2(a^2+k^2)}$ $\left[\begin{matrix} c=0 \text{ as} \\ \text{at } t=0\ x=0\end{matrix}\right]$

Solid sphere : $k^2 = \frac{2}{5}a^2 \Rightarrow \ddot{x} = \frac{5}{7}g\sin\alpha$

Hollow sphere $k^2 = \frac{2}{3}a^2 \Rightarrow \ddot{x} = \frac{3}{5}g\sin\alpha$

Circular Disc $k^2 = \frac{a^2}{2} \Rightarrow \ddot{x} = \frac{2}{3}g\sin\alpha$

Circular Ring $k^2 = a^2 \Rightarrow \ddot{x} = \frac{1}{2}g\sin\alpha$

For pure rolling :- $F = Mg\sin\alpha - M\ddot{x}$
$= \frac{2}{7}Mg\sin\alpha$ [for solid sphere]

$\Downarrow$

$\boxed{F < \mu R}$ → very important.

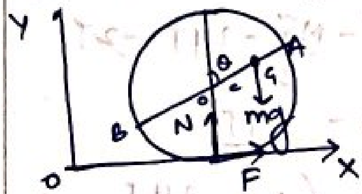
to prevent sliding

$\Rightarrow \frac{2}{7}Mg\sin\alpha < \mu Mg\cos\alpha$

$\Rightarrow \mu > \frac{2}{7}\tan\alpha$

NOTE: If it is given that it slides as well as rolls, then we take $F = \mu R$ directly [i.e. max friction]. o/w go for only F

Example: Circular cylinder whose COM is at distance c from axis rolls on a horizontal surface starting from the position of unstable equilibrium. Find N when COM is at lowest position. Given Radius of gyration = k.



$x_{cm} = a\theta + c\sin\theta$
$y_{cm} = a + c\cos\theta$

$\therefore F = M\frac{d^2}{dt^2}(a\theta + c\sin\theta) = \cancel{M}\frac{d}{dt}\dot{a\theta}$

$N - mg = \frac{M d^2}{dt^2}[a + c\cos\theta]$

Also $\boxed{\text{Work done} = \text{TE of system.}}$

$Mg(c - c\cos\theta) = \frac{1}{2}M[\dot{x}^2 + \dot{y}^2 + k^2\dot{\theta}^2]$

$\Rightarrow gc(1-\cos\theta) = \frac{1}{2}\left[\begin{matrix}(a\dot{\theta} + c\cos\theta\,\dot{\theta})^2 \\ + (-c\sin\theta\,\dot{\theta})^2 + k^2\dot{\theta}^2\end{matrix}\right]$

At $\theta = \pi$. Let $\dot{\theta}$ be ω

$\Rightarrow 2gc = \frac{1}{2}[(a-c)^2 + k^2]\omega^2$

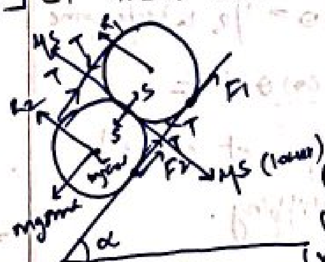
$\therefore \omega^2 = \frac{4gc}{k^2 + (a-c)^2}$

$N - mg = cM[-\cos\theta(\dot{\theta})^2 - \sin\theta\,\ddot{\theta}]$

$\Rightarrow N - Mg = -Mc[-\omega^2 - 0]$

$\Rightarrow N = M\left[g + \frac{4gc^2}{(a-c)^2 + k^2}\right]$

Example: 2 equal cylinder bound by string with tension T roll down an inclined plane. ST their acc^n = $\frac{2g}{3}\sin\alpha\left(1 - \frac{2\mu T}{Mg\sin\alpha}\right)$ μ coeff of friction b/w cylinders



(I) : $M\ddot{x} = Mg\sin\alpha + 2T - F_1 - S$

(II) : $M\ddot{y} = 0 = R_1 - Mg\cos\alpha + T S\mu$

(III) $Mk^2\ddot{\theta} = F_1 a - \mu S a$

(IV) $M\ddot{x} = Mg\sin\alpha - 2T - F_2 + S$

(V) $M\ddot{y} = 0 = R_2 - Mg\cos\alpha - \mu S$

(VI) $Mk^2\ddot{\theta} = F_2 a - \mu S a$

$\Rightarrow$ (iii) − (vi) $\Rightarrow$ $F_1 = F_2$

(i) − (iv) $\Rightarrow$ $S = 2T$

Put in (iii) $\Rightarrow$ $\dfrac{Ma^2\ddot{\theta}}{2} = F_1 a - 2\mu T a$

$\Rightarrow F_1 = \dfrac{Ma\ddot{\theta}}{2} + 2\mu T = \dfrac{M\ddot{x}}{2} + 2\mu T$

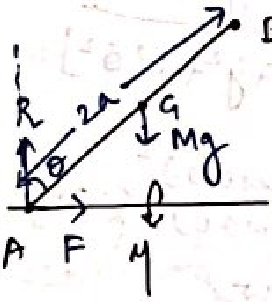
Put in ① $\Rightarrow$ $\left[\because x = a\theta,\ \ddot{x} = a\ddot{\theta}\right]$

$M\ddot{x} = Mg\sin\alpha + \cancel{2T} - \dfrac{M\ddot{x}}{2} - 2\mu T - \cancel{2T}$

$\dfrac{3M\ddot{x}}{2} = Mg\sin\alpha - 2\mu T$

$\therefore \ddot{x} = \dfrac{2}{3}g\sin\alpha\left(1 - \dfrac{2\mu T}{Mg\sin\alpha}\right)$

Example: Slipping of rod on a rough surface



$\bar{x} = a\sin\theta \quad \bar{y} = a\cos\theta$

$M\ddot{x} = F = Ma\left[\cos\theta\,\ddot{\theta} - \sin\theta\,(\dot{\theta})^2\right]$

$M\ddot{y} = R - Mg = -Ma\left[\sin\theta\,\ddot{\theta} + \cos\theta\,(\dot{\theta})^2\right]$

Moment about G $\Rightarrow$ $\dfrac{Ma^2\ddot{\theta}}{3} = Ra\sin\theta - Fa\cos\theta$

$\dfrac{Ma^2\ddot{\theta}}{3} = a\sin\theta\left[Mg - Ma(\sin\theta\,\ddot{\theta} + \cos\theta\,(\dot{\theta})^2)\right] - a\cos\theta\left[Ma(\cos\theta\,\ddot{\theta} - \sin\theta\,(\dot{\theta})^2)\right]$

$= Mga\sin\theta - Ma^2\left[\ddot{\theta} + 0\right]$

$\Rightarrow \dfrac{4}{3}Ma^2\ddot{\theta} = Mga\sin\theta \Rightarrow \ddot{\theta} = \dfrac{3g}{4a}\sin\theta$

$\Rightarrow \dot{\theta}^2 = \dfrac{3g}{2a}(1 - \cos\theta)$

$\therefore F = \dfrac{3}{4}Mg\sin\theta(3\cos\theta - 2) \quad R = \dfrac{1}{4}Mg(1 - 3\cos\theta)^2$

R vanishes when $\cos\theta = 1/3$ & keeps same sign

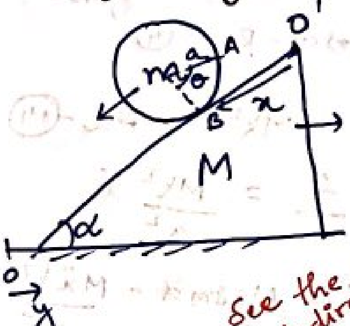
F changes sign when $\cos\theta < \dfrac{2}{3}$

$\mu = \dfrac{F}{R}$ needs to be ∞ o/w at $\cos\theta > \dfrac{1}{3}$ there will be slipping.

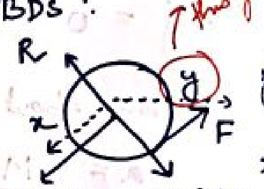
Example: Inclined plane of mass M can move on a smooth horizontal plane. Rough sphere (coeff μ) is placed on its face & it rolls down under gravity.

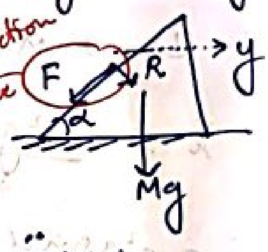
If planes moves y distance horizontally & sphere rolls down x distance on the face ST: $(M+m)y = mx\cos\alpha$

$\dfrac{7}{5}x - y\cos\alpha = \dfrac{1}{2}gt^2\sin\alpha$



FBDs:





On sphere:

(i) $mg\sin\alpha - F = m\ddot{x} - m\ddot{y}\cos\alpha$

(ii) $mg\cos\alpha - R = m\ddot{y}\sin\alpha$

(iii) $\dfrac{2}{5}ma^2\ddot{\theta} = Fa$ & $x = a\theta$

On plane: (iv) $M\ddot{y} = R\sin\alpha - F\cos\alpha$

$\therefore F = \dfrac{2}{5}m\ddot{x}$ Put in (i)

$\Rightarrow mg\sin\alpha = \dfrac{7}{5}m\ddot{x} - m\ddot{y}\cos\alpha$

$\int \Rightarrow g\sin\alpha\, t = \dfrac{7}{5}\dot{x} - \dot{y}\cos\alpha \quad \left[c = 0 \text{ as } \dot{x} = \dot{y} = 0 \text{ at } t = 0\right]$

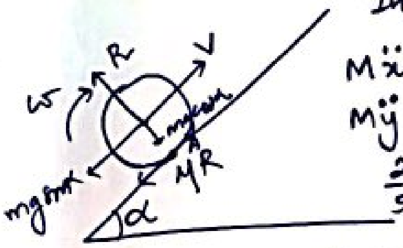
$\int \Rightarrow \dfrac{g\sin\alpha\, t^2}{2} = \dfrac{7}{5}x - y\cos\alpha \quad \left[c = 0 \text{ as } x, y = 0 \text{ at } t = 0\right]$

Using (i) (ii) & (iv) $\Rightarrow M\ddot{y} = (mg\cos\alpha - m\ddot{y}\sin\alpha)\sin\alpha - \left[mg\sin\alpha - m\ddot{x} + m\ddot{y}\cos\alpha\right]\cos\alpha$

$\Rightarrow M\ddot{y} = -m\ddot{y} + m\ddot{x}\cos\alpha \Rightarrow (M+m)\ddot{y} = m\ddot{x}\cos\alpha$

$\Rightarrow (M+m)y = mx\cos\alpha$ [C will be 0].

Example: sphere of Radius a is rolled up an inclined plane with velocity v & angular velocity ω to roll up further. $V > a\Omega$ & $\mu > \frac{2}{7}\tan\alpha$. Show that sphere will stop ascending at the end of time $(5V + 2a\Omega) / 5g\sin\alpha$



Initially, eq^n is

$$M\ddot{x} = -(Mg\sin\alpha + \mu R)$$

$$M\ddot{y} = 0 = R - Mg\cos\alpha$$

$$\frac{2}{5}Ma^2\ddot{\theta} = \mu R a$$

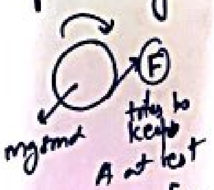
↳ [can't write $a\theta = x$ as there is sliding]

as at A, V & $a\Omega$ are in opposite direction. get the time t_1 when rolling starts (ie, $V_A = 0 \Rightarrow \dot{x} - a\dot{\theta}$)

After rolling

$$M\ddot{y} = -Mg\sin\alpha + F$$

$$\frac{2}{5}Ma^2\ddot{\theta} = -Fa$$



Now we can have $y = a\theta$

$$\therefore \ddot{y} = a\ddot{\theta}$$

[check here that $\mu R > F$ is satisfied !!]

Now sphere stops ascending when $\dot{y} = 0$

Let this time be t_2

$\therefore$ Total ascend time $= t_1 + t_2$.

NOTE: Always check $\mu R > F$ before commencing or considering pure rolling