⇒ FLUID DYNAMICS :

Fluid → isotropic material - property doesn't change with direction

Ideal fluid → (Perfect or Non-viscous fluid)
↳ No shear force. Pressure is always normal to surface

Real fluid → viscous fluid
↳ has shear force

Lagrangian Approach → focus on a single fluid particle

Euler's Approach → consider a volume element in a region

$$\delta f = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial z}\delta z + \frac{\partial f}{\partial t}\delta t$$

where f is any property $f(x,y,z,t)$

$$\therefore \lim_{\delta t \to 0} \frac{\delta f}{\delta t} = \frac{\partial f}{\partial x}\underbrace{\frac{\delta x}{\delta t}}_{u} + \frac{\partial f}{\partial y}\underbrace{\frac{\delta y}{\delta t}}_{v} + \frac{\partial f}{\partial z}\underbrace{\frac{\delta z}{\delta t}}_{w} + \frac{\partial f}{\partial t}$$

$$\Rightarrow \frac{Df}{Dt} = \frac{\partial f}{\partial t} + (q\cdot\vec{\nabla}) f$$

$$\Rightarrow \boxed{\frac{D}{Dt} = \frac{\partial}{\partial t} + q\cdot\vec{\nabla}}$$

3) CONDITION FOR CONTINUITY :

conservation of Mass ⇒ in a region inflow = outflow

$$\therefore \frac{\partial}{\partial t}\int_V \rho \, d\tau = -\int_S \rho \vec{q}\cdot \vec{ds}$$

$$\Rightarrow \int_V \frac{\partial \rho}{\partial t} d\tau = -\int_V (\vec{\nabla}\cdot\rho\vec{q})\, d\tau$$

$$\Rightarrow \boxed{\vec{\nabla}\cdot\rho\vec{q} + \frac{\partial \rho}{\partial t} = 0}$$

Incompressible fluid ⇒ $\frac{\partial \rho}{\partial t} = 0$

∴ Condition is $\boxed{\vec{\nabla}\cdot\rho\vec{q} = 0} \Rightarrow \vec{\nabla}\cdot\vec{q} = 0$

REMEMBER : $\vec{\nabla}$ forms :

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \quad \text{cartesian}$$

$$= \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\hat{\phi} \quad \text{polar}$$

$$= \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\theta} + \frac{\partial}{\partial z}\hat{z} \quad \text{cylindrical}$$

$$\therefore \text{If } \vec{A} = A_r\hat{r} + A_\theta\hat{\theta} + A_\phi\hat{\phi}$$

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}A_\phi$$

Eg Spherical

or Spherical

$$= \frac{1}{r}\frac{\partial}{\partial r}(r A_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$$

4) VELOCITY POTENTIAL (ϕ) : use to check if ϕ exists or not

$$\vec{q} = -\vec{\nabla}\phi \Rightarrow \nabla\times\vec{q} = 0$$

velocity ⇒ $u = -\frac{\partial \phi}{\partial x}$ $v = -\frac{\partial \phi}{\partial y}$ $w = -\frac{\partial \phi}{\partial z}$

5) STREAMLINE → flow of all particles at a particular instant of time

PATHLINE → path of a particle over time

STEADY FLOW ≡ (Streamline = Pathline)

∵ velocity is tangential to surface :

$$\boxed{\vec{q}\times d\vec{r} = 0} \Rightarrow \boxed{\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}}$$

⇓ gives streamlines

for pathlines : $\frac{dx}{dt} = u$, $\frac{dy}{dt} = v$, $\frac{dz}{dt} = w$

Surface ⊥ to streamlines = EQUIPOTENTIAL LINE

$$\boxed{u\,dx + v\,dy + w\,dz = 0}$$

In polarform $$\boxed{\frac{dr}{-\frac{\partial \Phi}{\partial r}} = \frac{r\,d\theta}{-\frac{1}{r}\frac{\partial \Phi}{\partial \theta}} = \frac{r\sin\theta\, d\phi}{-\frac{1}{r\sin\theta}\frac{\partial \Phi}{\partial \theta}}}$$ where $\Phi(r,\theta,\phi)$ is velocity potential

6) VORTICITY (ω) = measure of rotation of the motion

$$\boxed{\vec{\Omega} = \vec{\nabla} \times \vec{q}} = m\hat{i} + n\hat{j} + L\hat{k}$$

$\therefore$ Irrotational Motion $\Rightarrow \vec{\nabla} \times \vec{q} = 0$

$\Rightarrow \vec{q}$ can be written as $-\nabla\phi$

Vortex lines: $\vec{\omega} \times d\vec{r} = 0$

$\Rightarrow \frac{dx}{m} = \frac{dy}{n} = \frac{dz}{L}$

7) MOTION IN 2-DIMENSION ($\omega = 0$)

(a) $\therefore$ By continuity condition $= \frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$

Also streamline $\Rightarrow \frac{dx}{u} = \frac{dy}{v} \Rightarrow v\,dx - u\,dy = 0$

$\therefore v\,dx - u\,dy = d\psi = 0$ [exact differential]

$\Rightarrow \boxed{\psi = \text{constant}}$ = STREAM FUNCTION = eqn of streamline

always exist in 2D irrespective of rotation.

$$d\psi = \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy = v\,dx - u\,dy$$

$$\Rightarrow \boxed{\frac{\partial\psi}{\partial x} = v \quad \& \quad \frac{\partial\psi}{\partial y} = -u}$$

(b) Now, if motion is irrotational $q = -\nabla\phi$

$\boxed{\psi \;\&\; \phi}$ are harmonic conjugates

& satisfy laplace condition ($\nabla^2\phi = \nabla^2\psi = 0$)

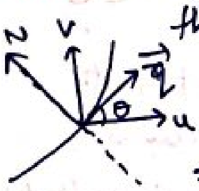
and Cauchy-Reiman $\Rightarrow$ $\boxed{\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} = -u}$ & $\boxed{\frac{\partial\phi}{\partial y} = \frac{-\partial\psi}{\partial x} = -v}$

In polar form:

$$\boxed{\begin{aligned} q_r &= -\frac{\partial\phi}{\partial r} = -\frac{1}{r}\frac{\partial\psi}{\partial\theta} \\ q_\theta &= -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{\partial\psi}{\partial r} \end{aligned}}$$

$q_r = \dot{r}$, $q_\theta = r\dot{\theta}$

GUIDE: $d\psi$ represents flow $\perp$ to the motion



$$\perp \text{ flow} = \int (v\cos\theta - u\sin\theta)\,ds$$
$$= \int \left(\frac{\partial\psi}{\partial x}\cdot\frac{dx}{ds} + \frac{\partial\psi}{\partial y}\frac{dy}{ds}\right)ds$$
$$= \int \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy = \int d\psi$$

(c) Let $w = w(z) = \phi + i\psi$

$$z = x + iy = re^{i\theta} = \sqrt{x^2+y^2}\,e^{i\tan^{-1}\left(\frac{y}{x}\right)}$$

$$\ln z = \underbrace{\frac{1}{2}\ln(x^2+y^2)}_{\phi} + i\underbrace{\tan^{-1}\left(\frac{y}{x}\right)}_{\psi}$$

$\phi + i\psi$

$$\frac{dw}{dz} = \frac{\partial\phi}{\partial x}\frac{dx}{dz} + i\frac{\partial\psi}{\partial x}\frac{dx}{dz}$$
$$= \frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x} = -u + iv$$

$$\therefore \boxed{q = |\vec{q}| = \left|\frac{dw}{dz}\right|}$$

where W = COMPLEX POTENTIAL

Example: $w = -\frac{1}{2}n(x+iy)^2 e^{2int}$. Find lines of flow in 2D. Also prove the paths of the particles of the fluid may be obtained by eliminating t from

$r(\cos(nt+\theta)) - x_0 = r\sin(nt+\theta) - y = nt(x_0 - y_0)$

$$w = -\frac{nr^2}{2}e^{2i(nt+\theta)}$$

$\therefore$ lines of flow $= \psi = $ constt $= -\frac{nr^2}{2}\sin 2(nt+\theta) = $ constt

$$\phi = -\frac{nr^2}{2}\cos[2(nt+\theta)]$$

$$\dot{r} = \frac{-\partial\phi}{\partial r} = nr\cos[2(nt+\theta)]$$

$$r\dot{\theta} = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = -nr\sin[2(nt+\theta)]$$

low: $\frac{d}{dt}\left[r\cos(nt+\theta)\right] = \dot{r}\cos(nt+\theta) - r\sin(nt+\theta)(n+\dot{\theta})$

$= nr\cos 2(nt+\theta)\cos(nt+\theta) - nr\sin(nt+\theta) + nr\sin 2(nt+\theta)\sin(nt+\theta)$

$= nr\left[\cos(nt+\theta) - \sin(nt+\theta)\right]$ — ①

y: $\frac{d}{dt}\left[r\sin(nt+\theta)\right] = \dot{r}\sin(nt+\theta) + r\cos(nt+\theta)(n+\dot{\theta})$

$= nr\cos\left[2(nt+\theta)\right]\sin\left[nt+\theta\right] + nr\cos(nt+\theta) - nr\sin\left[2(nt+\theta)\right]\cos(nt+\theta)$

$= nr\left[\cos(nt+\theta) - \sin(nt+\theta)\right]$

$\frac{d}{dt}\left[r\cos(nt+\theta) - r\sin(nt+\theta)\right] = 0$

$r\cos(nt+\theta) - r\sin(nt+\theta) = r_0\cos\theta_0 - r_0\sin\theta_0$

$r\cos(nt+\theta) - x_0 = r\sin(nt+\theta) - y_0$

B) EQUATION OF MOTION OF IDEAL FLUID: [Incompressible, Streamline, & non-viscous]

$\vec{F}_{ext} = F(r)$ = per unit mass external conservative force acting on particles

Pressure gradient force $= \int_S (-p)\, ds\, \hat{n} = -\int \vec{\nabla} p\, d\tau$

$\vec{F}_{net}$ = mass × acc^n $= \int (\rho\, d\tau) \times \frac{d\vec{q}}{dt} = \int \frac{d\vec{q}}{dt}\rho\, d\tau$

we $\vec{F}_{net} = \frac{d(\vec{M})}{dt}$ (momentum) $= \frac{d}{dt}\int(\rho\, d\tau)\vec{q}$

$= \frac{d}{dt}\left[\int \vec{q}\rho\, d\tau\right] = \int \frac{d\vec{q}}{dt}\rho\, d\tau + \int \vec{q}\,\cancel{\frac{d}{dt}(\rho\, d\tau)}^{\,0}$ (constant mass)

$= \frac{d}{dt}\int \rho\, d\tau$

∴ Equation of motion → $\vec{F}_{net} = \vec{F}_{ext} + \vec{P}_{GF}$

$\frac{d\vec{M}}{dt} = \Sigma F_{ext}\rho\, d\tau - \int \vec{\nabla} p\, d\tau$
is per unit mass

$\Rightarrow \int \frac{d\vec{q}}{dt}\rho\, d\tau = \int(\rho\vec{F}_{ext} - \vec{\nabla} p)\, d\tau$

$\Rightarrow \rho\frac{d\vec{q}}{dt} - \rho\vec{F}_{ext} + \vec{\nabla} p = 0$

$\Rightarrow \boxed{\frac{d\vec{q}}{dt} = \vec{F}_{ext} - \frac{\vec{\nabla} p}{\rho}}$ EULER'S EQ^N OF MOTION

$\Downarrow$

$\boxed{\frac{\partial\vec{q}}{\partial t} + (\nabla\cdot\vec{q})\vec{q} = \vec{F}_{ext} - \frac{1}{\rho}\vec{\nabla} p}$ — ①

In cartesian coordinates ⇒ 3 equations

In polar, we are only interested when $\vec{q} = v(r)\hat{r}$

$\Rightarrow \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} = F(r) - \frac{1}{\rho}\frac{\partial p}{\partial r}$

otherwise

$\frac{\partial q_r}{\partial t} + q_r\frac{\partial q_r}{\partial r} + q_\theta\frac{1}{r}\frac{\partial q_r}{\partial\theta} + q_\phi\frac{1}{r\sin\theta}\frac{\partial q_r}{\partial\phi} = F(r) - \frac{1}{\rho}\frac{\partial p}{\partial r}$

condition for integration of ①

a) Forces are conservative $\vec{F} = -\vec{\nabla} V$

b) Flow is irrotational, ie, $\vec{\nabla}\times\vec{q} = 0$ & $\vec{q} = -\nabla\phi$

When we expand into x, y, z terms, multiply by dx, dy, dz & add we get

$-\left[\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial t}\right)dx + \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial t}\right)dy + \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial t}\right)dz\right] +$

$+\, u\,\Sigma\frac{\partial u}{\partial x}dx + v\,\Sigma\frac{\partial v}{\partial x}dx + w\,\Sigma\frac{\partial w}{\partial x}dx$

$= -\Sigma\frac{\partial V}{\partial x}dx - \frac{1}{\rho}\Sigma\frac{\partial p}{\partial x}dx$

$\Rightarrow -d\left[\frac{\partial\phi}{\partial t}\right] + \frac{1}{2}d\left[u^2+v^2+w^2\right] + dV + \frac{1}{\rho}dp = 0$

$\Rightarrow \frac{\partial\phi}{\partial t} + \frac{q^2}{2} + V + \int\frac{dp}{\rho} = F(t)$

9) BERNOULLI'S EQUATION:

Using previous integration where

$q = -\vec{\nabla}\phi$ & $F = -\vec{\nabla}V$

$$\Rightarrow \boxed{V + \int \frac{dp}{\rho} + \frac{q^2}{2} = \frac{\partial \phi}{\partial t} + \text{constt}}$$

10) BOYLE'S LAW: Ideal gases (pt particle & no interaction)

$PV = \text{constant}$

$\frac{PM}{\rho} = k \Rightarrow p = k\rho$ [per unit mass]

FOR STEADY FLOW: $\frac{\partial}{\partial t}(\,) = 0$ $\therefore \vec{\nabla}\cdot\rho\vec{q} = 0$ is cont. eq^n

11) WORKING RULE:

(a) Write Euler's equation

(b) Write continuity condition

$$\boxed{\begin{array}{ll} \frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\rho\vec{q} = 0 & \text{(general)} \\ r\,V(r) = \text{constant} = f(t) & \text{(cyl.)} \\ r^2\,V(r) = \text{constant} = f(t) & \text{(spherical)} \end{array}}$$

(c) Write boundary conditions

(i) at $r = \infty$, $v = 0$, $p = 0$

(ii) at $r = r'$ (radius of cavity) $p = 0$ $v = \dot{r}' = v(r')$

(iii) at $r = c$ (initial stage), $v = 0$

(d) Get relation b/w r, v & t through manipulations & find result.

Example: $F(r) = -\mu r^{-3/2}$ per unit mass. Fluid at rest initially with a cavity of radius $r = c$, find time taken to fill up the cavity.

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} = -\frac{\mu}{r^{3/2}} - \frac{1}{\rho}\frac{\partial p}{\partial r}$$

& $r^2 V = F(t) \Rightarrow V = F(t)/r^2$

$\therefore \frac{\partial v}{\partial t} = \frac{F'(t)}{r^2}$ $\quad v\frac{\partial v}{\partial r} = \frac{V \times F(t) \times -2}{r^3}$

$$\Rightarrow \frac{F'(t)}{r^2} - \frac{2F^2(t)}{r^5} = -\frac{\mu}{r^{3/2}} - \frac{1}{\rho}\frac{\partial p}{\partial r}$$

Integrate wrt r:

$$-\frac{F'(t)}{r} + \frac{V^2}{2} = \frac{2\mu}{\sqrt{r}} - \frac{p}{\rho} + C_1$$

Now at $r = \infty$, $v = 0$, $p = 0 \Rightarrow C_1 = 0$

$$\Rightarrow -\frac{F'(t)}{r} + \frac{V^2}{2} = \frac{2\mu}{\sqrt{r}} - \frac{p}{\rho} \quad \text{—(i)}$$

At $r = r'$, $p = 0$, $v = \dot{r}'$

$$\Rightarrow -\frac{F'(t)}{r'} + \frac{\dot{r}'^2}{2} = \frac{2\mu}{\sqrt{r'}} \quad \text{—(ii)}$$

$\cancel{r^2(\dot{r}) = F(t)}$

$\Rightarrow \cancel{r^2\,dr = F(t)\,dt}$ ~~multiply wrt~~

$\cancel{-\frac{2F(t)F'(t)dt}{r'} + \frac{F^2(t)}{r^4}r^2\,dr = \frac{2\mu}{\sqrt{r}}r^2\,dr}$

$\Rightarrow \cancel{F(t) = r^2 V}$

$\cancel{F'(t) = 2rV^2 + r^2 v\frac{dv}{dr}}$ (replaced $\frac{\partial v}{\partial t} = v\frac{dv}{dr}$)

~~Put into (1)~~

$\cancel{-2V^2 - rv\frac{dv}{dr} + \frac{V^2}{2} = \frac{2\mu}{\sqrt{r}} - \frac{p}{\rho}}$

In question, better to go with x in place of r & r in place of r'

$r'^2(\dot{r}') = F(t)$

$\Rightarrow r'^2\,dr' = F(t)\,dt$

Put $\dot{r}' = \frac{F(t)}{r'^2}$ & multiply by $F(t)dt$ on LHS & r'^2dr' on RHS

$$\Rightarrow -\frac{2F(t)F'(t)}{r'} + \frac{(F(t))^2}{r'^2} = 4\mu(r')^{3/2}dr'$$

$$\Rightarrow -d\left[\frac{(F(t))^2}{r'}\right] = 4\mu(r')^{3/2}dr'$$

$$\Rightarrow -\frac{F^2(t)}{r'} = 4\mu\frac{2}{5}r'^{5/2} + C_1$$

At $r' = c$, $v = 0 \Rightarrow F(t) = 0$

$$\Rightarrow C_1 = -\frac{8\mu}{5}c^{5/2}$$

$$\Rightarrow \frac{V^2(r'^4)}{r'} = \frac{8\mu}{5}\left[c^{5/2} - r'^{5/2}\right]$$

$$\Rightarrow v^2 = \frac{8\mu}{5r^3}\left(c^{5/2} - r^{5/2}\right)^{1/2}$$

$$\Rightarrow \frac{dr'}{dt} = -\sqrt{\frac{8\mu}{5r'^3}\left(c^{5/2} - r'^{5/2}\right)}$$ (-ve denotes that $\frac{dr'}{dt}$ is in direction of $\downarrow r'$)

$$\Rightarrow \int_c^0 \frac{dr'\sqrt{r'^3}}{\sqrt{c^{5/2} - r'^{5/2}}} = \int_0^T -\sqrt{\frac{8\mu}{5}}\,dt$$

Put $r'^{5/2} = c^{5/2}\sin^2\theta$ $\quad \frac{5}{2}r^{3/2}dr' = 2c\sin\theta\cos\theta\,d\theta$

$$\Rightarrow \int_{\pi/2}^{0} \frac{2}{5}2c\sin\theta\cos\theta\,d\theta\left(\frac{\cancel{\;}}{c^{5/2}\cos^2\theta}\right)^{1/2} = -\int_0^T\sqrt{\frac{8\mu}{5}}dt$$

$$\frac{4}{5}c^{5/4}\int_0^{\pi/2}\sin\theta\,d\theta = \sqrt{\frac{8\mu}{5}}\,T$$

$$\Rightarrow T = \left(\frac{2}{5\mu}\right)^{1/2} c^{5/4}.$$

MISCELLANEOUS EXAMPLES:

ST: $\frac{x^2}{a^2}f(t) + \frac{y^2}{b^2}\phi(t) = 1$ where $f(t)\phi(t) =$ constt is a possible boundary surface.

$$\boxed{F \text{ is a boundary surface if } \frac{DF}{Dt} = 0}$$

$$\Rightarrow F = \frac{x^2}{a^2}f(t) + \frac{y^2}{b^2}\phi(t) - 1 = 0$$

$$\frac{DF}{Dt} = u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} + \frac{\partial F}{\partial t} = 0$$

$$\Rightarrow u\left[\frac{2x}{a^2}f\right] + v\left[\frac{2y}{b^2}f\right] + 0 + \frac{x^2}{a^2}f' + \frac{y^2}{b^2}\phi' = 0$$

$$\Rightarrow \frac{2x}{a^2}f\left(u + \frac{x}{2}\frac{f'}{f}\right) + \frac{2y}{b^2}\phi\left(v + \frac{y\phi'}{2\phi}\right) = 0$$

So we take $u + \frac{x}{2}\frac{f'}{f} = 0$ & $v + \frac{y\phi'}{2\phi} = 0$

This is valid if continuity is satisfied

$$\Rightarrow -\frac{1}{2}\frac{f'}{f} + \frac{-1}{2}\frac{\phi'}{\phi} = 0 \Rightarrow \phi f = \text{constt}$$

which is given.

An infinite fluid in which a spherical hollow shell of radius a is initially at rest under the action of no forces. If Π is constant pressure at ∞, find time to fill up cavity $= \pi^2 a\left(\frac{\rho}{\Pi}\right)^{1/2} 2^{5/6}\,\Gamma\left(\frac{1}{3}\right)^{-3}$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \frac{-\partial P}{\rho\,\partial x} \quad \& \quad x^2 v = f(t)$$

$$\Rightarrow -\frac{f'(t)}{x} + \frac{v^2}{2} = \frac{-P}{\rho} + c$$

(i) at $x = \infty$ $\quad P = \Pi$ & $v = 0 \Rightarrow c = \frac{\Pi}{\rho}$

(ii) at $x = r$ (radius of cavity) $\quad P = 0$

$$-\frac{f'(t)}{r} + \frac{v^2}{2} = \frac{\Pi}{\rho}$$

$$r^2\frac{dr}{dt} = f(t) \Rightarrow r^2 dr = f(t)dt$$

$$-\frac{2f(t)f'(t)dt}{r} + \frac{f^2(t)\times 2r^2dr}{2r^4} = \frac{2\Pi}{\rho}r^2dr$$

$$\Rightarrow -\frac{2f(t)f'(t)dt}{r} + \frac{f^2(t)\,dr}{r^2} = \frac{2\Pi}{\rho}r^2dr$$

$$\Rightarrow -d\left[\frac{f^2(t)}{r}\right] = \frac{2\Pi r^2 dr}{\rho}$$

$$\Rightarrow \frac{f^2(t)}{r} = -\frac{2\Pi r^3}{3\rho} + c_2$$

(iii) At $r = a$, $v = 0 \Rightarrow f(t) = 0 \Rightarrow c_2 = \frac{2\Pi a^3}{3\rho}$

$$\therefore \frac{f^2(t)}{r} = \frac{2\Pi}{3\rho}(a^3 - r^3) = v^2 r^3$$

$$\Rightarrow v^2 = \frac{2\Pi}{3\rho}\left(\frac{a^3 - r^3}{r^3}\right)$$

$$\frac{dr}{dt} = -\left[\frac{2\Pi}{3\rho}\left(\frac{a^3 - r^3}{r^3}\right)\right]^{1/2}$$

$$\Rightarrow \int_a^0\left(\frac{r^3}{a^3 - r^3}\right)^{1/2}dr = -\int_0^t\left(\frac{2\Pi}{3\rho}\right)^{1/2}dt$$

$r^3 = a^3\sin^2\theta \Rightarrow r = a\sin^{2/3}\theta$

$dr = \frac{2}{3}a\sin^{-1/3}\theta\cos\theta\,d\theta$

$$\Rightarrow \int_0^{\pi/2} \cancel{a} \frac{\sin\theta}{\cos\theta}\times\frac{2}{3}\times a\sin^{-1/3}\theta\cos\theta\,d\theta = \sqrt{\frac{2\Pi}{3\rho}}\,t$$

$$\Rightarrow \sqrt{\frac{2\rho}{3\Pi}}\,(a)\int_0^{\pi/2}\sin^{2/3}\theta\,d\theta = t$$

upon manipulating we get the result

⇒ SOURCES & SINKS:

(Flow must be irrotational)

1) Source: fluid is emitted in all directions radially & symmetrically

Sink: fluid is absorbed.

Strength (m): Total volume of flow across any small circle surrounding the source is $(2\pi m)\left[=(q_r)(2\pi r)\right]$

2) $-\frac{\partial\phi}{\partial r}\,r = m \Rightarrow \phi = -m\ln r$

$\frac{\partial\phi}{\partial r} = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = -\frac{m}{r} \Rightarrow \psi = -m\theta$

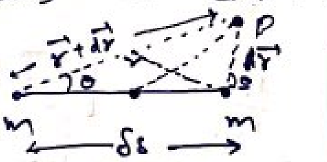
∴ Complex Potential $= W = \phi + i\psi$
(due to a source at origin) $= -m\ln r - mi\theta$
$= -m\ln[re^{i\theta}]$
$= -m\ln z$

$$\boxed{W = -m\ln z}$$

3) If source at $z=a \Rightarrow W = -m\ln(z-a)$

sources kept at $a_1, a_2, a_3 - \Rightarrow W = -\sum m_i\ln(z-a_i)$

4) DOUBLET:



strength of doublet $= m\,\delta s = \mu$

$\phi = -m\ln r + m\ln(r+dr)$
$= m\ln\left(1+\frac{dr}{r}\right) \approx \frac{m\,dr}{r}$
$= \frac{m\,\delta s\cos\theta}{r} = \frac{\mu\cos\theta}{r}$

$\frac{\partial\phi}{\partial r} = \frac{1}{r}\frac{\partial\psi}{\partial\theta} \Rightarrow \psi = -\frac{\mu\sin\theta}{r}$

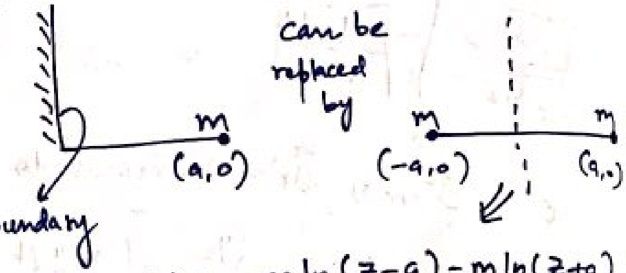
$\therefore W = \frac{\mu}{r}(\cos\theta - i\sin\theta) = \frac{\mu}{re^{i\theta}}$

$\boxed{W \text{ (due to doublet)} = \frac{\mu}{z}}$ if shifted $\Rightarrow \boxed{\frac{\mu}{z-a}}$

5) BOUNDARIES, SOURCE, SINK, DOUBLET

there is no flow across the boundary ⇒ represents a streamline ⇒ $\psi = 0$

(a) eg



$W = -m\ln(z-a) - m\ln(z+a)$
$= -m\ln(z^2-a^2)$
$= -m\ln(x^2-y^2-a^2+2ixy)$

$W = -m\left[\frac{1}{2}\ln\sqrt{x^2-y^2-a^2} + i\tan^{-1}\left(\frac{2ixy}{x^2-y^2-a^2}\right)\right]$

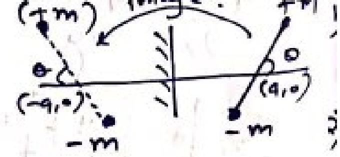
$\therefore \psi = -m\tan^{-1}\left(\frac{2ixy}{x^2-y^2-a^2}\right)$

at x-axis ⇒ $x=0$ ⇒ $\psi = 0$ ⇒ boundary!

6) If doublet forms angle α with x-axis $\boxed{W = \frac{\mu e^{i\alpha}}{z-a}}$

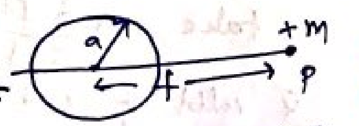
7) Image of sources, sinks:

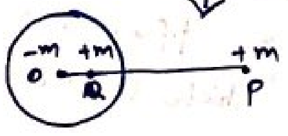
(a) in a line, just take image & place source, sink doublet of same strength.



∴ curve or boundary across which image is taken is a streamline with no across flow

(b) in a circle:





(i) sink (-m) at O
(ii) source (+m) at Q st $(OP)(OQ) = a^2$

Example: 2 sources each of strength m are placed at $(-a,0)$ & $(a,0)$ & a sink of $2m$ at $(0,0)$. Show that stream lines are $(x^2+y^2)^2 = a^2(x^2-y^2+\lambda xy)$

$$W = -m\ln(z+a) - m\ln(z-a) + 2m\ln z$$

$$\therefore \psi = m\left(-\tan^{-1}\left(\frac{y}{x+a}\right) - \tan^{-1}\left(\frac{y}{x-a}\right) + 2\tan^{-1}\left(\frac{y}{x}\right)\right)$$

$$= m\left(-\tan^{-1}\left[\frac{2xy}{x^2-y^2-a^2}\right] + 2\tan^{-1}\left(\frac{y}{x}\right)\right)$$

$$= -m\left[\tan^{-1}\left(\frac{2xy}{x^2-y^2-a^2}\right) - \tan^{-1}\left(\frac{2yx}{x^2-y^2}\right)\right]$$

$$= -m\left[\tan^{-1}\left[\frac{2xy\,a^2}{(x^2-y^2-a^2)(x^2-y^2)+4x^2y^2}\right]\right]$$

$\psi =$ constant $= -m\tan^{-1}\left(\frac{2}{\lambda}\right)$ say

$$\Rightarrow \tan^{-1}\left[\frac{2a^2xy}{(x^2-y^2)^2 - a^2(x^2-y^2)+4x^2y^2}\right] = \tan^{-1}\left(\frac{2}{\lambda}\right)$$

$$\Rightarrow \frac{2a^2xy}{(x^2+y^2)^2 - a^2(x^2-y^2)} = \frac{2}{\lambda}$$

$$\Rightarrow (x^2+y^2)^2 = a^2\left[x^2-y^2+\lambda xy\right].$$

⇒ AXISYMMETRIC MOTION:

1) Motion is symmetric to axis - sphere & cylinder

2) Irrotational motion in 2D

$\Rightarrow \vec{\nabla}\times\vec{q} = 0 \Rightarrow \vec{q} = -\nabla\phi$

Continuity condition. $\vec{\nabla}\cdot\vec{q} = 0 \Rightarrow \nabla^2\phi = 0$ & $\nabla^2\psi = 0$.

$\phi = \phi(r,\theta)$

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) = 0$$

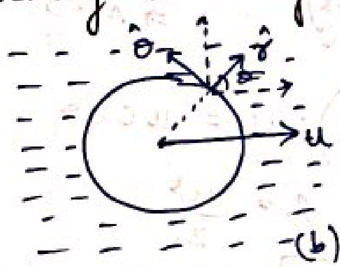
$q_r = u = -\partial\phi/\partial r \qquad q_\theta = v = -\frac{1}{r}\partial r/\partial\theta$

3) We work with 2 boundary conditions

(a) Fluid is at rest at $r=\infty \Rightarrow \left(\frac{\partial\phi}{\partial r}\right)_\infty = 0$

(b) Normal component of velocity at boundary of fluid = Normal component of velocity of sphere

4) Motion of sphere moving with velocity u along x-axis in ∞ fluid:



$q_r = u\cos\theta$

$q_\theta = -u\sin\theta$

(a) at $r=a \Rightarrow -\frac{\partial\phi}{\partial r} = u\cos\theta$

(b) at $r=\infty$ $\left(\frac{\partial\phi}{\partial r}\right) = 0$

Now we solve $\nabla^2\phi = 0$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) = 0$$

$$\Rightarrow \frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) = 0$$

Lets take $\phi = R(r)\,\Theta(\theta) = R(r)\cos\theta$

$$\therefore \frac{\partial\phi}{\partial r} = \Theta\frac{dR}{dr} \qquad \frac{\partial\phi}{\partial\theta} = R\frac{\partial\Theta}{\partial\theta}$$

$$\Rightarrow \Theta\frac{\partial}{\partial r}\left(r^2\frac{dR}{dr}\right) + \frac{R}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = 0$$

$$\Rightarrow \frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \frac{-1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = n(n+1) \text{ (say)}$$

$$\Rightarrow \frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - n(n+1)R = 0$$

$$\Rightarrow 2r\frac{dR}{dr} + r^2\frac{d^2R}{dr^2} - n(n+1)R = 0$$

take $R = r^\alpha \Rightarrow \alpha(\alpha-1) + 2\alpha - n(n+1) = 0$

$(\alpha-n)(\alpha+(n+1)) = 0$

$\alpha = n, -(n+1)$.

$$\therefore R = \sum c_1 r^n + c_2 r^{-(n+1)}$$

since $\left(\frac{\partial R}{\partial r}\right)_{r=\infty} = 0 \Rightarrow n = 1$

$$\Rightarrow R = Ar + \frac{B}{r^2}$$

Also: $\frac{-1}{\Theta \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) = n(n+1)$

$\Rightarrow \frac{\cos\theta}{\sin\theta}\frac{d\Theta}{d\theta} + \frac{d^2\Theta}{d\theta^2} + n(n+1)\Theta = 0$

$\Theta = \cos\theta$ satisfies

$\therefore \boxed{\phi = \left(Ar + \frac{B}{r^2}\right)\cos\theta}$

$\left(\frac{\partial\phi}{\partial r}\right)_{r\to\infty} = \left(A - \frac{2B}{r^3}\right)\cos\theta = A\cos\theta = 0 \Rightarrow A = 0$

$-\left(\frac{\partial\phi}{\partial r}\right)_{r=a} = -\left(\frac{-2B}{a^3}\right)\cos\theta = u\cos\theta$

$\Rightarrow B = \frac{ua^3}{2}$

$\therefore \boxed{\phi = \frac{ua^3}{2r^2}\cos\theta}$

$\Rightarrow \frac{dr}{-\frac{\partial\phi}{\partial r}} = \frac{r\,d\theta}{-\frac{1}{r}\frac{\partial\phi}{\partial\theta}} \Rightarrow \frac{dr}{\frac{ua^3\cos\theta}{r^3}} = \frac{r\,d\theta}{\frac{ua^3\sin\theta}{2r^3}}$

$\Rightarrow \frac{dr}{r} = \frac{2\cos\theta\,d\theta}{\sin\theta}$

$\Rightarrow \boxed{r = c\sin^2\theta} \Rightarrow$ streamline

5) Sphere at rest & fluid moving →

$\left(\frac{\partial\phi}{\partial r}\right)_{r=a} = 0; \; -\left(\frac{\partial\phi}{\partial r}\right)_{r\to\infty} = u\cos\theta$

$\phi = \left(Ar + \frac{B}{r^2}\right)\cos\theta \Rightarrow \frac{\partial\phi}{\partial r} = \left(A - \frac{2B}{r^3}\right)\cos\theta$

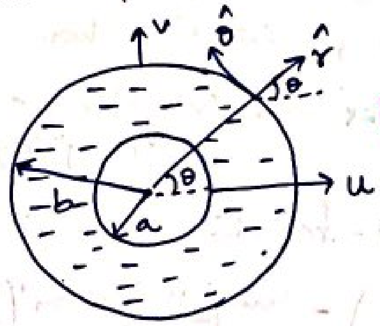
$\therefore A\cos\theta = -u\cos\theta \Rightarrow A = -u$

$-u - \frac{2B}{a^3} = 0 \Rightarrow B = -\frac{ua^3}{2}$

$\therefore \phi = -\left[ur + \frac{ua^3}{2r^2}\right]\cos\theta$

$= -ur\left[1 + \frac{a^3}{2r^3}\right]\cos\theta$

6) Motion of concentric spheres with fluid in between :-



$\nabla^2\phi = 0$

$\phi(r,\theta) = \left(Ar+\frac{B}{r^2}\right)\cos\theta + \left(Cr + \frac{D}{r^2}\right)\sin\theta$

$-\left(\frac{\partial\phi}{\partial r}\right)_{r=a} = u\cos\theta \qquad -\left(\frac{\partial\phi}{\partial r}\right)_{r=b} = V\sin\theta$

$-\frac{\partial\phi}{\partial r} = -A\cos\theta + \frac{2B}{r^3}\cos\theta - C\sin\theta + \frac{2D}{r^3}\sin\theta$

$\therefore -A\cos\theta + \frac{2B}{a^3}\cos\theta - C\sin\theta + \frac{2D}{a^3}\sin\theta = u\cos\theta$

$-A\cos\theta + \frac{2B}{b^3}\cos\theta - C\sin\theta + \frac{2D}{b^3}\sin\theta = V\sin\theta$

$-A + \frac{2B}{a^3} = u \qquad -C + \frac{2D}{a^3} = 0$

$-A + \frac{2B}{b^3} = 0 \qquad -C + \frac{2D}{b^3} = V$

$\Rightarrow u = 2B\left(\frac{b^3-a^3}{a^3b^3}\right) \qquad V = -2D\left(\frac{b^3-a^3}{a^3b^3}\right)$

$B = \frac{ua^3b^3}{2(b^3-a^3)} \qquad D = \frac{-Va^3b^3}{2(b^3-a^3)}$

$A = \frac{ua^3}{(b^3-a^3)} \qquad C = \frac{-Vb^3}{(b^3-a^3)}$

$\therefore \phi = \frac{ua^3 r}{b^3-a^3}\left[1 + \frac{b^3}{2r^3}\right]\cos\theta - \frac{Vb^3 r}{b^3-a^3}\left[1+\frac{a^3}{2r^3}\right]\cos\theta$

Example: sphere of radius 'a' moving with uniform velocity 'u' in a perfectly incompressible fluid, the acceleration of particle of fluid at (r,θ) is

$3u^2\left(\frac{a^3}{r^4} - \frac{a^6}{r^7}\right)$

→ Basically its all about the Initial conditions for $\phi = \left(Ar + \frac{B}{r^2}\right)\cos\theta$

Also remember whether sphere is moving or the fluid & whose acc^n we are to calculate.

Here we need for fluid so we take the motion relative to sphere.

$\therefore \left(\frac{\partial \phi}{\partial r}\right)_{r=0} = 0 \qquad \left(\frac{\partial \phi}{\partial r}\right)_{\infty} = -u\cos\theta$

$$\Rightarrow \phi = ur\left(1 + \frac{a^3}{2r^3}\right)\cos\theta$$

$$\dot{r} = \frac{-\partial\phi}{\partial r} = -\left[u\left(1+\frac{a^3}{2r^3}\right)\cos\theta - \frac{3ua^3\cos\theta}{2r^3}\right]$$

$$= -\left[u\left(1-\frac{a^3}{2r^3}\right)\cos\theta\right]$$

$$r\dot{\theta} = \frac{-1}{r}\frac{\partial\phi}{\partial\theta} = \frac{-1}{r}\times ur\left(1+\frac{a^3}{2r^3}\right)(-\sin\theta)$$

$$\therefore \dot{\theta} = \frac{u}{r}\left(1+\frac{a^3}{2r^3}\right)\sin\theta$$

Now $\boxed{acc^n = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}}$

$$\ddot{r} = -u\left[\frac{3a^3}{r^4}\dot{r}\cos\theta + \left(1-\frac{a^3}{r^3}\right)-\sin\theta\,\dot{\theta}\right]$$

$$\ddot{\theta} = u\left[\left(\frac{-1}{r^2} - \frac{2a^3}{r^5}\right)\sin\theta\,\dot{r} + \left(\frac{1}{r}+\frac{a^3}{2r^4}\right)\cos\theta\,\dot{\theta}\right]$$

at $(r,0)$: $\dot{r} = -u\left(1-\frac{a^3}{r^3}\right)$

$\dot{\theta} = 0 \quad \ddot{r} = -\frac{3ua^3}{r^4}\dot{r} \quad \ddot{\theta} = 0$

$$\therefore acc^n = \ddot{r}\,\hat{r} = \frac{3u^2a^3}{r^4}\left(1-\frac{a^3}{r^3}\right)$$

$$= 3u^2\left(\frac{a^3}{r^4} - \frac{a^6}{r^7}\right).$$

7) Motion of cylinder in 2D:

$$\nabla^2\phi = 0 \Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial z}\right) = 0$$

$$\Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0$$

↳ Starting pt

$$\phi = \phi(r,\theta) = R(r)\Theta(\theta)$$

$$\Rightarrow \frac{1}{r}\frac{d}{dr}\left[r\frac{dR}{dr}\Theta\right] + \frac{1}{r^2}R\frac{d^2\Theta}{d\theta^2} = 0$$

$$\Rightarrow \frac{r}{R}\frac{d}{dr}\left[r\frac{dR}{dr}\right] + \frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = 0$$

$$\Rightarrow \frac{r}{R}\frac{d}{dr}\left[r\frac{dR}{dr}\right] = \frac{-1}{\Theta}\frac{d^2\Theta}{d\theta^2} = m$$

$$\Rightarrow \boxed{\phi = \left(Ar + \frac{B}{r}\right)\cos\theta}$$

(a) Cylinder moving with u:

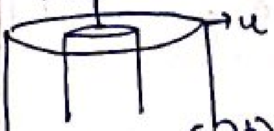
$-\left(\frac{\partial\phi}{\partial r}\right)_{r=\infty} = 0 \qquad -\left(\frac{\partial\phi}{\partial r}\right)_{r=a} = u\cos\theta$

$$\Rightarrow \phi = \frac{ua^2\cos\theta}{r} \qquad \psi = -\frac{ua^2\sin\theta}{r}$$

$$w = \phi + i\psi = \frac{ua^2}{re^{i\theta}} = \frac{ua^2}{z}$$

Streamline $= \frac{ua^2\sin\theta}{r} = $ constant

(b) Co-axial cylinders:



$$\phi = \left(Ar + \frac{B}{r}\right)\cos\theta + \left(Cr + \frac{D}{r}\right)\sin\theta$$

$-\left(\frac{\partial\phi}{\partial r}\right)_{r=a} = u\cos\theta \quad -\left(\frac{\partial\phi}{\partial r}\right)_{r=b} = v\sin\theta$

$$\Rightarrow \phi = \frac{ua^2r}{b^2-a^2}\left(1+\frac{b^2}{r^2}\right)\cos\theta - \frac{vb^2r}{b^2-a^2}\left(1+\frac{a^2}{r^2}\right)\sin\theta$$

⇒ NAVIER - STOKES EQUATIONS :

* viscous flow of newtonian fluids

1) $dm = \rho\, dx\, dy\, dz$

$(\rho\, dx\, dy\, dz) \dfrac{D\vec{q}}{Dt} = \vec{F}_{net}$

2 types of forces :

a) Body forces : uniformly spread throughout $\vec{B}$ = force per unit mass

b) surface forces :
→ stress → normal to surface
→ shear → tangential to surface

force per unit area :

$$\text{Stress} = \lim_{\delta S \to 0} \frac{F_{ns}}{\delta S} = \sigma_{xx}\, \sigma_{yy}\, \sigma_{zz}$$

$$\text{Shear} = \sigma_{xy}\ \sigma_{yz}\ \sigma_{xz}$$

$$F_x = \sigma_{xx}\hat{i} + \sigma_{xy}\hat{j} + \sigma_{xz}\hat{k}$$

$$F_y = \sigma_{yx}\hat{i} + \sigma_{yy}\hat{j} + \sigma_{yz}\hat{k}$$

$$F_z = \sigma_{zx}\hat{i} + \sigma_{zy}\hat{j} + \sigma_{zz}\hat{k}$$

$$\overleftrightarrow{\sigma} \text{ (tensor)} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

2) $\vec{q} = u\hat{i} + v\hat{j} + w\hat{k}$

$\mu = \dfrac{\eta}{\rho}$ η = coefficient of viscosity
μ = kinematic viscosity

STRESSES :

3) $$\sigma_{xx} = 2\mu \frac{\partial u}{\partial x} - \frac{2\mu}{3} \vec{\nabla}\cdot\vec{q} - p$$

$$\sigma_{yy} = 2\mu \frac{\partial v}{\partial y} - \frac{2\mu}{3} \vec{\nabla}\cdot\vec{q} - p$$

$$\sigma_{zz} = 2\mu \frac{\partial w}{\partial z} - \frac{2\mu}{3} \vec{\nabla}\cdot\vec{q} - p$$

SHEAR STRAINS :

$$\sigma_{xy} = \sigma_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\sigma_{yz} = \sigma_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

$$\sigma_{xz} = \sigma_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

4) Equation of motion of an infinitisimle small mass element of fluid :

$$(\rho\, \delta x \delta y \delta z) \frac{Du}{Dt} = (B_x)\rho\, \delta x \delta y \delta z + \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z}\right)\delta x \delta y \delta z$$

$$\Rightarrow \frac{Du}{Dt} = B_x + \frac{1}{\rho}\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z}\right)$$

$$\frac{Dv}{Dt} = B_y + \frac{1}{\rho}\left(\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z}\right)$$

$$\frac{Dw}{Dt} = B_z + \frac{1}{\rho}\left(\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}\right)$$

$$\Rightarrow \boxed{\frac{D\vec{q}}{Dt} = \vec{B} + \frac{\mu}{\rho}\nabla^2\vec{q} + \frac{\mu}{3\rho}\vec{\nabla}(\vec{\nabla}\cdot\vec{q}) - \frac{\vec{\nabla}p}{\rho}}$$

3) For incompressible fluid $\vec{\nabla}\cdot\rho\vec{q}=0$

$$\Rightarrow \boxed{\frac{D\vec{q}}{Dt} = \vec{B} + \nu\nabla^2\vec{q} - \frac{\vec{\nabla}p}{\rho}}$$

4) N-S eq^n for a steady laminar flow of a viscous incompressible fluid b/w 2 infinite parallel plates. CORVETTE FLOW

steady $\Rightarrow \frac{\partial}{\partial t}(\,)=0$ laminar = No turbulence

incompressible = $\nabla\cdot\vec{q}=0$

$\vec{q} = u\hat{i}+v\hat{j}+w\hat{k}$ $\left.\begin{matrix} v=w=0 \\ \& \; u=u(x,y)\end{matrix}\right\}$ b/w plates

$$\therefore u\frac{\partial u}{\partial x} = 0 + \nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

For continuity $\frac{\partial u}{\partial x}=0 \Rightarrow u = u(y)$

$$\therefore 0 = \nu\frac{\partial^2 u}{\partial y^2} - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\Rightarrow \frac{\partial p}{\partial x} = \rho\nu\left(\frac{\partial^2 u}{\partial y^2}\right) = \mu\frac{\partial^2 u}{\partial y^2} \quad \left[\nu = \frac{\mu}{\rho}\right]$$

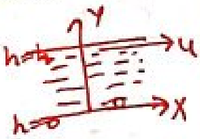
Also $\frac{\partial p}{\partial y} = 0 = \frac{\partial p}{\partial z}$ $\therefore p = f(x)$

$$\frac{\partial^2 u}{\partial y^2} = \frac{+1}{\mu}\frac{\partial p}{\partial x} \Rightarrow \frac{d}{dx}\left(\frac{\partial^2 u}{\partial y^2}\right) = \frac{1}{\mu}\frac{d}{dx}\left(\frac{dp}{dx}\right)$$

$$\Rightarrow \frac{dp}{dx} = \text{constt} = (\text{say})\, P \Rightarrow \frac{d^2u}{dy^2} = \frac{P}{\mu}$$

$\therefore u = \frac{Py^2}{2\mu} + Ay + B$ at $y=0$ $u=0$, at $y=h$ $u=U$

$$\Rightarrow u = \frac{Py^2}{2\mu} + \frac{Uy}{h} - \frac{Phy}{2\mu}$$



use this to get $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$

$\Rightarrow$ VORTEX MOTION :

$\vec{\Omega}$ = vorticity = $\vec{\nabla}\times\vec{q}$

1) Rectilinear vertices $\rightarrow w=0$

Streamlines & vortex lines are $\perp$.

2) Eq^n of vortex line = $\frac{dx}{\Omega_x} = \frac{dy}{\Omega_y} = \frac{dz}{\Omega_z}$

streamline : $\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$

$\perp$ iff $u\Omega_x + v\Omega_y + w\Omega_z = 0$

$$u\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)+v\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)+z\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)=0$$

when $w=0$, $\frac{\partial}{\partial z}(\;)=0$

$\Omega_x = 0$ $\Omega_y = 0$ $\Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$

$\therefore v\,dx - u\,dy = 0 = d\psi$

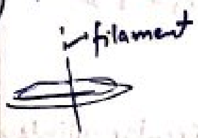
$v = \frac{d\psi}{dx}$ $u = -\frac{\partial\psi}{\partial y}$

$$\Omega_z = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = \nabla^2\psi$$

$\therefore$ $\boxed{\text{On} \;|\; \text{vortex filament} \quad \nabla^2\psi = \Omega_z}$

outside it $\nabla^2\psi = 0 \Rightarrow$ irrotational & ϕ exists

i.e, vorticity exists on the filament



3) Find effect of filament at some distance 'r'

$\nabla^2(\psi) = 0$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) = 0$$

$$\Rightarrow r\frac{\partial\psi}{\partial r} = \text{constt}$$

$\Rightarrow \frac{d\psi}{dr} = \frac{c}{r} \Rightarrow \psi = c \ln r$

$\boxed{c = \frac{k}{2\pi} \Rightarrow \text{strength of the vortex} = k}$

$\psi = \frac{k}{2\pi} \ln r$

$k = \text{circulation} = \oint \vec{q} \cdot \vec{dl}$
$= \int (\vec{\nabla} \times \vec{q}) \cdot \vec{ds}$
$k = \int \vec{\Omega} \cdot \vec{ds}$

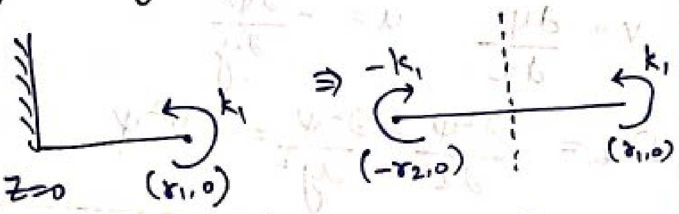
ϕ exists outside filament

$\therefore \phi = \frac{-k\theta}{2\pi}$ $\quad \therefore w = \frac{-k\theta}{2\pi} + \frac{ik \ln r}{2\pi}$

$= \frac{ki}{2\pi} \ln z$

$\therefore$ $\boxed{\begin{array}{ll} \text{Motion of Rectilinear Vortex} = & \\ w = i\frac{k}{2\pi} \ln z & \text{vortex at } (0,0) \\ = i\frac{k}{2\pi} \ln(z - z_0) & \text{vortex at } z_0 \end{array}}$

4) Image of vortex :

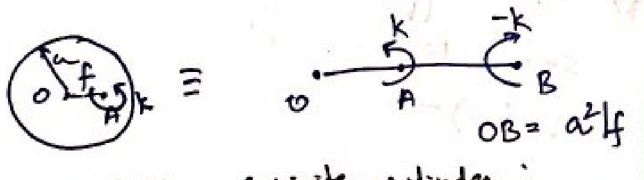


$w = \frac{ik_1}{2\pi} \ln (z - z_1) - \frac{ik_1}{2\pi} \ln(z - z_2)$

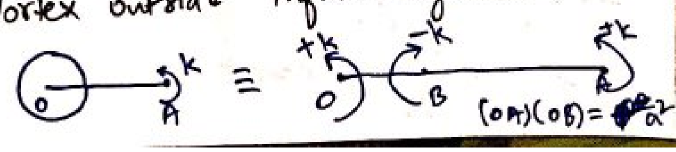
$\phi + i\psi = \frac{ik_1}{2\pi} \left[\ln r_1 + i\theta_1 - \ln r_2 - i\theta_2 \right]$

$\psi = \frac{k_1}{2\pi} (\ln r_1 - \ln r_2) \Rightarrow r_1 = r_2$

5) Vortex inside infinite circular cylinder



Vortex outside infinite cylinder :



Remember formula :

$\sin\theta = \theta\left(1 - \frac{\theta^2}{\pi^2}\right)\left(1 - \frac{\theta^2}{2^2\pi^2}\right)\left(1 - \frac{\theta^2}{3^2\pi^2}\right)$

6) Necessary & Sufficient condition for streamlines & vortex lines to be $\perp$

vortex $: \frac{dx}{\Omega_x} = \frac{dy}{\Omega_y} = \frac{dz}{\Omega_z} = \mu$

$\perp$ if $u\Omega_x + v\Omega_y + w\Omega_z = 0$ — (I)

if they are $\perp$, above eqⁿ holds & we get an exact differential

$\frac{u}{\mu} dx + \frac{v}{\mu} dy + \frac{w}{\mu} dz = d\phi$

$\frac{\partial\phi}{\partial x} dx + \frac{\partial\phi}{\partial y} dy + \frac{\partial\phi}{\partial z} dz = d\phi$

$\therefore u = \mu \frac{\partial\phi}{\partial x} \quad v = \mu \frac{\partial\phi}{\partial y} \quad w = \mu \frac{\partial\phi}{\partial z}$ — (II)

Now if (II) is given, we can easily get (I) $\Rightarrow$ they are $\perp$.

$\therefore$ NC & SC proved.

7) In incompressible fluid, vorticity at every point is constt in magnitude & direction ; PT u, v, w satisfy Laplace

$\Omega^2 = \Omega_x^2 + \Omega_y^2 + \Omega_z^2$

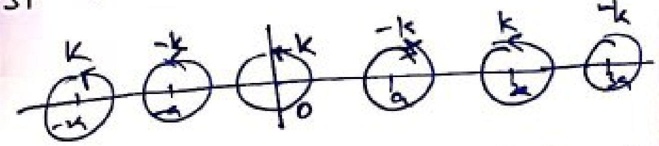
dc's $= \frac{\Omega_x}{\Omega}, \frac{\Omega_y}{\Omega}, \frac{\Omega_z}{\Omega}$ All constant

$\Rightarrow \frac{\partial\Omega_z}{\partial y} - \frac{\partial\Omega_y}{\partial z} = 0$

$\Rightarrow \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 w}{\partial x \partial z} = 0$

$\Rightarrow \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \frac{\partial}{\partial x}\left(\underbrace{\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}}_{-\frac{\partial u}{\partial x} \text{ by cont}}\right) = 0$ Hence proved

Example: Infinite row of equidistant vortices of alternating strengths are kept. Find the complex function, ϕ & ST vortices are at rest.



$$W = \frac{ik}{2\pi}\left[\log z + \log(z-2a) + \log(z+2a) + \dots\right]$$

$$-\frac{ik}{2\pi}\left[\log(z+a) + \log(z-a) + \dots\right]$$

$$= \frac{ik}{2\pi}\log\left[\frac{z(z^2-2^2a^2)(z^2-4^2a^2)\dots}{(z^2-a^2)(z^2-3^2a^2)(z^2-5^2a^2)\dots}\right]$$

$$= \frac{ik}{2\pi}\log\left[\frac{\frac{z}{2a}\left[1-\left(\frac{z}{2a}\right)^2\right]\left[1-\left(\frac{z}{4a}\right)^2\right]\dots}{\left(1-\left(\frac{z}{a}\right)^2\right)\left(1-\left(\frac{z}{3a}\right)^2\right)\dots}\right] + \text{constt}$$

$$= \frac{ik}{2\pi}\log\left(\frac{\sin(\pi z/2a)}{\cos(\pi z/2a)}\right) = \frac{ik}{2\pi}\log\tan\frac{\pi z}{2a}$$

$$\phi + i\psi = \frac{ik}{2\pi}\log\tan\frac{\pi z}{2a} \quad \text{— (I)}$$

$$\phi - i\psi = \frac{-ik}{2\pi}\log\tan\frac{\pi\bar{z}}{2a} \quad \text{— (II)}$$

$$\text{(I)+(II)} \Rightarrow \psi = \frac{k}{4\pi}\log\left(\tan\frac{\pi z}{2a}\right)\left(\tan\frac{\pi\bar{z}}{2a}\right)$$

$$\psi = \frac{k}{4\pi}\left\{\log\left(\frac{\cosh\frac{\pi y}{a} - \cos\frac{\pi x}{a}}{\cosh\frac{\pi y}{a} + \cos\frac{\pi x}{a}}\right)\right\}$$

$\psi =$ constt gives $\Rightarrow \cosh\frac{\pi y}{a} = b\cos\frac{\pi x}{a}$

$$\text{(I)+(II)} \Rightarrow \phi = \frac{ik}{4\pi}\log\frac{\tan(\pi z/2a)}{\tan(\pi\bar{z}/2a)}$$

$$= \frac{ik}{4\pi}\log\frac{\sin(\pi x/a) + i\sinh(\pi y/a)}{\sin(\pi x/a) - i\sinh(\pi y/a)}$$

$$\phi = \frac{-k}{4\pi}\left[\tan^{-1}\frac{\sinh(\pi y/a)}{\sin(\pi x/a)} + \tan^{-1}\frac{\sinh(\pi y/a)}{\sin(\pi x/a)}\right]$$

Vortex at origin at $z=0$

$$W_1 = W - \frac{ik}{2\pi}\log z$$

Let its velocity be u_0, v_0

$$-u_0 + iv_0 = \left(\frac{dW_1}{dz}\right)_{z=0}$$

$$= \frac{ik}{2\pi}\left[\frac{\sec^2(\pi z/2a)}{\tan(\pi z/2a)}\cdot\frac{\pi}{2a} - \frac{1}{z}\right]_{z=0} = 0$$

$\therefore$ vortex is at rest.