⇒ LINEAR ALGEBRA :

Vector spaces :

1) Let $(F, +, \cdot)$ be a field.
The elts of F are called scalars.
Let V be a non-empty set whose elts are called vectors.
V is called a vector space over F if
   a) Vector addition (ie, an <u>internal composition</u> in V is defined)
      ie, $\forall a, b \in V, \quad a+b \in V$.
      Scalar multiplication (ie, an <u>external composition</u> in V over F is defined)
      ie, $\forall a \in F, \alpha \in V \Rightarrow a \cdot \alpha \in V$.
   b) $(V, +)$ is an abelian group
   c) $\forall a, b \in F \; ; \; \alpha, \beta \in V$. $+^n$ & $\times^n$ satisfy :
      i) $a \cdot (\alpha+\beta) = a\alpha + a\beta$
      ii) $(a+b)\alpha = a\alpha + b\alpha$
      iii) $(ab)\alpha = a(b\alpha)$
      iv) $1\alpha = \alpha$; 1 is the unity elt of F

* If $V \subseteq F$, then $V(F)$ is not a vectorspace (except $V=\{0\}$)
   eg $I(Q)$
   ($\because$ Ext comp fails)
* If $F \subseteq V$, then $V(F)$ is a Vector space
   eg. $C(Q)$, $C(R)$, $R(Q)$

   * $F(F)$ is a vectorspace where F is a field
   * $K(F)$ is a vectorspace where K is a field & F a subfield of K

2) <u>Some notations</u> :
   a) $V_n(F)$ or $F^{(n)}$: VS of all ordered n-tuples over F
   b) $F[x] = \{ f(x) \mid f(x) = \sum_{i=0}^{\infty} a_i x^i = a_0 + a_1 x + a_2 x^2 + \ldots \quad a_i \in F \}$
      = set of all polynomials
   c) $\mathcal{C}[a,b]$ = set of all real valued fn defined & continuous in $[a,b]$

3> Properties : [$\vec{0}$ is zero vector of V]

a> $a\vec{0} = \vec{0}$ $\forall$ $a \in F$ b> $0\alpha = \vec{0}$ $\forall$ $\alpha \in V$

c> $a(-\alpha) = -(a\alpha)$ $\forall$ $a \in F, \alpha \in V$ d> $(-a)\alpha = -(a\alpha)$ $\forall a \in F, \alpha \in V$

e> $a(\alpha - \beta) = a\alpha - a\beta$ $\forall$ $a \in F$, $\alpha, \beta \in V$ f> $a\alpha = 0 \Rightarrow a = 0$ (or) $\alpha = \vec{0}$
using contradiction
using a^{-1}

step: $a\vec{0} = a(\vec{0}+\vec{0}) = a\vec{0} + a\vec{0}$

$a\vec{0} = \vec{0} + a\vec{0}$ [0 is identity for (V,+)]

$\therefore$ $a\vec{0} + a\vec{0} = \vec{0} + a\vec{0}$

$\Rightarrow$ $a\vec{0} = \vec{0}$ [$\because$ RCL in (V,+)]

4.> If $a, b \in F$ & $\alpha \in V$ $(\alpha \neq \vec{0})$, then $a\alpha = b\alpha \Rightarrow a = b$

If $\alpha, \beta \in V$ & $a \in F$ $(a \neq 0)$, then $a\alpha = a\beta \Rightarrow \alpha = \beta$

5.> To show V is not VS over F, it is enough to show that F is not a subfield of V.

eg $R^2(C)$ is not a VS [$\because$ C is not a subfield]
$Z_7(Z_5)$ is not a VS [Z_5 is not a subfield, $2+3=0$ in Z_5 but 5 in Z_7]
$C^2(Z)$ is not a VS [$\because$ Z is not a field]

6> Subspace : V(F) is a VS & $W \subseteq V$

W is a subspace of V(F) $\Longleftrightarrow$ internal & external compositions are defined in W, ie,

(We have 4 diff^t theorems)

i a> $\forall$ $\alpha, \beta \in W \Rightarrow \alpha + \beta \in W$
ii b> $\forall$ $a \in F, \alpha \in W \Rightarrow a\alpha \in W$

$\Longleftrightarrow$ $a, b \in F$, $\alpha, \beta \in W \Rightarrow a\alpha + b\beta \in W$

$\Longleftrightarrow$ i> $\forall$ $\alpha, \beta \in W \Rightarrow \alpha - \beta \in W$
ii> $a \in F$, $\alpha \in W \Rightarrow a\alpha \in W$

$\Longleftrightarrow$ $a \in F$, $\alpha, \beta \in W \Rightarrow a\alpha + \beta \in W$

7> **Algebra of subspaces**:

a> W_1, W_2 are subspaces of $V(F) \Rightarrow W_1 \cap W_2$ is also a subspace
($\text{So is } \bigcap_{i \in N} W_i$)

b> Union of 2 subspaces need not be a subspace.

eg $V_3(F) = \{(a_1, a_2, a_3) \mid a_1, a_2, a_3 \in F\}$
$W_1 = \{(0, a, b) \mid a, b \in F\} \subseteq V_3$
$W_2 = \{(x, 0, y) \mid x, y \in F\} \subseteq V_3$
$(0, 1, 2) \in W_1$ & $(1, 0, 2) \in W_2$
But $(0,1,2) + (1,0,2) = (1,1,4) \notin W_1 \cup W_2$

c> $W_1 \cup W_2$ is subspace iff $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$

SC: Suppose $W_1 \not\subseteq W_2$ or $W_2 \not\subseteq W_1$
If $W_1 \not\subseteq W_2$ let $\alpha \in W_1$, then $\alpha \notin W_2$
If $W_2 \not\subseteq W_1$ let $\beta \in W_2$ then $\beta \notin W_1$
Now $\alpha \in W_1, \beta \in W_2 \Rightarrow \alpha, \beta \in W_1 \cup W_2$
$\Rightarrow \alpha + \beta \in W_1 \cup W_2$ [$\because W_1 \cup W_2$ is SS]
$\Rightarrow \alpha + \beta \in W_1$ or $\alpha + \beta \in W_2$
If $\alpha \in W_1$ & $\alpha + \beta \in W_1 \Rightarrow \alpha + \beta - \alpha \in W_1 \Rightarrow \beta \in W_1 \Rightarrow \#$
Sly $\beta \in W_2$ & $\alpha + \beta \in W_2 \Rightarrow \alpha + \beta - \beta \in W_2 \Rightarrow \alpha \in W_2 \Rightarrow \#$ HP.

8> To show not a subspace, go with an example which fails.
eg. $W = \{(a,b,c) \mid a^2 + b^2 + c^2 \leq 1\}$ is ~~a~~ not a subspace of $\mathbb{R}^3$
$\alpha = (0,1,0)$ $\beta = (0,0,1)$ $\alpha + \beta = (0,1,1) \notin W$

9> Linear Combination: $\alpha = a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n$ where $a_i \in F$

Linear Span: $V(F)$ is a VS & $S = \{\alpha_1, \alpha_2, \dots, \alpha_n\} \subseteq V$
$L(S) = \{a_1\alpha_1 + \dots + a_n\alpha_n \mid \alpha_i \in S, \text{ \& } a_1, \dots, a_n \in F\}$.
↓
collection of all LCs of a finite no of elts of S.

10> Smallest subspace containing any subset of $V(F)$: If $S \subseteq V$ & U is a subspace of V st $S \subseteq U$. If X is any other subspace of V st $S \subseteq X$ $\Rightarrow U \subseteq X$, then U is smallest subspace containing S

The smallest subspace of V containing S is also called the subspace of V generated or spanned by S, i.e. $[\{S\} = V]$

11) ☆ L(S) is a subspace of V(F) generated by S
i.e, $L(S) = \{S\}$ [L(S) is smallest subspace of V containing S]

Proof. V(F) is a VS & $S = \{\alpha_1, \alpha_2, \dots, \alpha_n\} \subseteq V$

$$L(S) = \{ a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n \mid a_i \in F, \alpha_i \in S\}$$

Now let $\alpha, \beta \in L(S)$ $\quad \alpha = a_1\alpha_1 + \dots + a_n\alpha_n$
$\beta = b_1\alpha_1 + \dots b_n\alpha_n$

$$a\alpha + b\beta = (a a_1 + b b_1)\alpha_1 + \dots + (a a_n + b b_n)\alpha_n$$
$$= \text{LC of } \alpha_i\text{s}$$
$$\in L(S) \Rightarrow L(S) \text{ is subspace of } V$$

$\therefore$ ~~L(S) is a~~ ~~L(S) ⊆ S~~

Now for $\alpha_i \in S$, $\alpha_i \in L(S)$ for $a_j = 0$ $j \neq i$
$$\Rightarrow S \subseteq L(S)$$

Now suppose there is another subspace of W containing S.
if $\alpha \in L(S) \Rightarrow \alpha =$ LC of finite elts of S
$\in W$
$$\Rightarrow L(S) \subseteq W$$
$$\therefore S \subseteq L(S) \subseteq W \subseteq V$$
$\therefore$ L(S) is smallest subspace of V containing S.

NOTE: To prove $L(S) = V$, just prove that $V \subseteq L(S)$
i.e, $\forall e \in V$ if $e =$ LC's of finite elts of S
$\Rightarrow V \subseteq L(S)$.

eg PT. $W_1 + W_2 = L(W_1 \cup W_2)$ $\quad W_1 + W_2 = \{\alpha_i + \alpha_j \mid \alpha_i \in W_1, \alpha_j \in W_2\}$
easy to show that $W_1 + W_2$ is a subspace of V.
$\alpha_i \in W_1$, $0 \in W_2 \Rightarrow \alpha_i + 0 \in W_1 + W_2 \Rightarrow \alpha_i \in W_1 + W_2 \Rightarrow W_1 \subseteq W_1 + W_2$
Sly $W_2 \subseteq W_1 + W_2$
$\Downarrow$
$W_1 \cup W_2 \subseteq (W_1 + W_2)$

$\therefore L(W_1 \cup W_2) \subseteq W_1 + W_2$
(smallest subspace)

$\alpha \in W_1 + W_2 \Rightarrow \alpha = \alpha_i + \alpha_j = 1\cdot\alpha_i + 1\cdot\alpha_j =$ LC of elts of $W_1 \cup W_2$
$$\Rightarrow W_1 + W_2 \subseteq L(W_1 \cup W_2)$$

12> $S \subseteq T \Rightarrow L(S) \subseteq L(T)$

$L(S \cup T) = L(S) + L(T)$

$L(L(S)) = L(S)$ [$\because$ $L(L(S))$ is smallest subspace of V containing $L(S)$ but $L(S)$ itself is a subspace of V]

$\Rightarrow L(S) \subseteq L(L(S)) \subseteq L(S) \subseteq V$.

13> Linear Dependence of vectors : if $\exists$ atleast one non-zero $a_i \in F$ st $a_1\alpha_1 + \dots + a_n\alpha_n = 0$

Linear Independence of vectors : $a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n = 0 \Rightarrow a_i = 0 \ \forall i$ $1 \to n$

a> set containing $\vec{0}$ is always LD

b> subset of LI is LI

c> superset of LD is LD

d> eg. check if $(1,-2,1), (2,1,-1), (7,-4,1)$ are LD

They are LD if $x(1,-2,1) + y(2,1,-1) + z(7,-4,1) = 0$ has non-zero solution

$\Rightarrow x + 2y + 7z = 0$; $-2x + y - 4z = 0$; $x - y + z = 0$

$\Rightarrow A = \begin{vmatrix} 1 & 2 & 7 \\ -2 & 1 & -4 \\ 1 & -1 & 1 \end{vmatrix} = 0$ $\therefore$ non-zero sol^n $\Rightarrow$ LD

If $\rho(A) < 3 \Rightarrow$ LD.

eg Let $\alpha = (1,2,1)$ $\beta = (3,1,5)$ $\gamma = (3,-4,7)$

ST $L(S) = L(T)$ where $S = \{\alpha, \beta\}$ $T = \{\alpha, \beta, \gamma\}$

Sol^n: $S \subseteq T$

$\Rightarrow L(S) \subseteq L(T)$

Let $x \in L(T)$

then $x = a_1\alpha + a_2\beta + a_3\gamma$

Let $\gamma = b_1\alpha + b_2\beta$

$(3,-4,7) = b_1(1,2,1) + b_2(3,1,5) = (b_1 + 3b_2, 2b_1 + b_2, b_1 + 5b_2)$

$\Rightarrow b_1 = -3 ; b_2 = 2$

$\therefore x = a_1\alpha + a_2\beta + a_3[-3\alpha + 2\beta]$

$= (a_1 - 3a_3)\alpha + (a_2 + 2a_3)\beta$ = LC of $\alpha, \beta \in L(S) \Rightarrow L(T) \subseteq L(S)$ HP

⇒ Basis & Dimension :

1) $V(F)$ is a VS & $S = \{\alpha_1, \alpha_2, \dots, \alpha_n\} \subseteq V$.
S is called Basis of $V(F)$ if
 a) S is LI
 b) $L(S) = V$, ie, V is spanned by S.

2) Standard Basis :
 a) $\{(1,0,0\dots0), (0,1,0,\dots0), \dots, (0,0,\dots,0,1)\}$ for $V_n(F)$
 b) $\{1, i\}$ for $C(R)$
 c) $\{1, x, x^2, \dots, x^n\}$ for set of all polynomials of degree 'n'
 d) $\{(1,0), (0,1), (i,0), (0,i)\}$ for $C^2(\mathbb{R})$

3) FDVS : if there exists a finite subset S of V st $L(S) = V$

4) eg ST : $S = \{1, x, x^2, \dots, x^n, \dots\}$ is a basis of VS $F[x]$ of all polynomials over field F.
 a) To prove S is LI : use every finite subset is LI
 b) To prove $L(S) = F[x]$
  let $f(x) \in F(x)$
  ie $f(x) = b_0 + b_1x + \dots + b_mx^m$ for some m
  $= b_0(1) + b_1(x) + \dots + b_m(x^m) + 0 \cdot x^{m+1} + 0 \cdot \dots$
  $=$ LC of elts of S
  $\therefore f(x) \in L(S)$
  $\Rightarrow F[x] \subseteq L(S)$
  $\Rightarrow F[x] = L(S)$.

5) Every FDVS has a basis or If $L(S) = V$, then there exists a subset of S which spans V.

6) Any 2 bases of FDVS has same no of elts.
[use # to prove, $B_1 = \{\alpha_1, \dots \alpha_m\}$ $B_2 = \{\beta_1 \dots \beta_n\}$ & show that if $m \neq n$, either one of B_1 or B_2 is LD]

7> $\boxed{\text{Dimension of } V(F)}$ $\Rightarrow$ no of elts in any basis of $V(F)$

$\dim \mathbb{R}^2 = 2$

$\dim \mathbb{R}^3 = 3$

$\dim \mathbb{R}^n = n$

$\dim_{\mathbb{R}} C = 2$

$\dim_F F = 1 \Rightarrow \{1\}$ or $\{a, a \neq 0\}$ $a \in F$

dim V (set of all symm matrix) = $n(n+1)/2$

dim V (set of all anti-symm matrix) = $n(n-1)/2$

$\dim_{\mathbb{Q}} (\mathbb{Q}\sqrt{2}) = 2$ $[B\{1, \sqrt{2}\}]$

$\dim\{(0,0,0)\} = 0$

8> $\dim V(F) = n \iff n$ is the max. no of LI vectors in any subset of V

a> If $\dim V = n$, then any $(n+1)$ vectors are LD
[$\because$ sim to proof of pt. 6]

b> Every finite LI subset of FDVS can be extended to form a basis of V.

c> If $\dim V = n$ & $S = \{\alpha_1, \ldots, \alpha_n\}$ is a LI subset of V, then S is a basis of V.

d> If $\dim V = n$ & $S = \{\alpha_1, \ldots, \alpha_n\}$ & $L(S) = V$, then S is a basis of V.

Proof of the SC above: Let $S = (\alpha_1, \ldots, \alpha_n)$ be maximal LI subset
We need to prove $L(S) = V(F)$

For $\alpha \in V$, consider $S' = (\alpha_1, \ldots, \alpha_n, \alpha)$

S' can't be LI $\therefore \exists$ a_is not all zero st. $a_1\alpha_1 + a_2\alpha_2 + \ldots + a\alpha = 0$

If $a = 0$ we get all a_is $= 0$ $\Rightarrow$ not possible

$\therefore a \neq 0 \Rightarrow \alpha = -\left[\frac{a_1}{a}\alpha_1 + \frac{a_2}{a}\alpha_2 + \ldots + \frac{a_n}{a}\alpha_n\right]$

$= $ LC of α_is $\in S$

$\alpha \in L(S)$

$\Rightarrow V \subseteq L(S)$

$\Rightarrow L(S) = V$.

Remember this step in proofs.

eg ST $\mathbb{R}$ is a VS of infinite dimension over $\mathbb{Q}$.
we prove that $\{1, \pi, \pi^2, \dots, \pi^n\}$ is LI over $\mathbb{Q}$ for any $n \in \mathbb{Z}^+$

Let $\alpha_0 + \alpha_1\pi + \alpha_2\pi^2 + \dots + \alpha_n\pi^n = 0$ $(\alpha_i \in \mathbb{Q})$

then π is solⁿ to $\alpha_0 + \alpha_1 x + \dots + \alpha_n x^n = 0$

But π is transcendental $\Rightarrow$ # Hence $\mathbb{R}$ is of ∞ dimension

9) To extend any set to form basis, add any vector which is not in the span.

[using thm: if $S = \{\alpha_1, \dots \alpha_n\}$ is LI, $\alpha \in V(F)$ & $\alpha \notin L(S)$
then $S_1 = \{\alpha_0, \alpha_1, \dots \alpha_n\}$ is LI.]

10) $\boxed{\text{Dimension of a subspace}}$:

if W is ~~finite~~ subspace of a FDVS $\Rightarrow \dim W \le \dim V$

Also $\boxed{V = W \iff \dim V = \dim W}$

$V \subseteq W$
& $W \subseteq W$
$\Rightarrow \dim(V) \le \dim(W)$
& $\dim(W) \le \dim(V)$

Let S be basis of W
$\therefore S \subset W \subset V$
$L(S) = W$ & S is LI & $|S| = \dim V$
$\Rightarrow S$ is basis of V
$\Rightarrow L(S) = V$

11) $\boxed{\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)}$

Let $\dim(W_1 \cap W_2) = k$ & its basis be $(\gamma_1, \dots \gamma_k)$

Let $S_1 = \{\gamma_1, \dots, \gamma_k, \alpha_1, \dots \alpha_m)$ be basis of W_1
& $S_2 = \{\gamma_1, \dots, \gamma_k, \beta_1, \dots \beta_n)$ be basis of W_2

Now show that $S_3 = \{\gamma_1, \dots \gamma_k, \alpha_1, \dots \alpha_m, \beta_1, \dots \beta_n\}$ is basis of $W_1 + W_2$

12) Row space of A = Linear span of set containing row vectors.
Column sp(A) = Row sp(A^T)

13) a) Row equivalent matrices have same row space
b) Same row space iff A & B have same non-zero rows in row-reduced echelon form
c) Dim (row sp A) = max no of LI rows of A
= max no of LI rows of echelon matrix A
= no of non-zero rows of echelon matrix A.

** 14) To get basis & dimension of $W_1 + W_2$, consider the matrix of all vectors combined.

eg $V_1 = \{(1,1,0,-1), (1,2,3,0), (2,3,3,-1)\}$
$V_2 = \{(1,2,2,-2), (2,3,2,-3), (1,3,4,-3)\}$

To get dim V_1+V_2 Reduce matrix: $\begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 3 & -1 \\ 1 & 2 & 2 & -2 \\ 2 & 3 & 2 & -3 \\ 1 & 3 & 4 & -3 \end{bmatrix}$

eg $A = \{(x,y,0) \mid x,y \in \mathbb{R}\}$ $B = \{(0,y,z) \mid y,z \in \mathbb{R}\}$
Find dim $(A+B)$

Use dim $(A+B)$ = dim A + dim B - dim $(A \cap B)$
~~dem~~ $A \cap B = \{(0,y,0) \mid y \in \mathbb{R}\}$.

15) $\boxed{\text{Co-ordinates}}$ of α for basis $B = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$.
They are a_is st $\alpha = \sum a_i \alpha_i$ ~~[illegible]~~

⇒ Linear Transformation :

1> Let $U(F)$ & $V(F)$ be 2 VS.

Then $T: U \to V$ is a linear transformation of u into V

$$\boxed{\text{iff} \quad T(a\alpha + b\beta) = aT(\alpha) + bT(\beta) \quad \forall\, a, b \in F,\ \alpha, \beta \in U}$$

* $T: U \to U$ is called a linear operator on U
* $T: U \to F$ is called a linear function on U.
* $T: U \to V$ st $T(\alpha) = \hat{0}$ $\forall\, \alpha \in U$ where $\hat{0}$ is zero-vector of V is called zero transformation
* $T: U \to U$ st $T(\alpha) = \alpha$ $\forall\, \alpha \in U$ is called Identity operator

2> Properties : for $T: U \to V$

a> $T(\bar{0}) = \hat{0}$

b> $T(-\alpha) = -T(\alpha)$ $\forall\, \alpha \in U$

c> $T(\alpha - \beta) = T(\alpha) - T(\beta)$

3> **Determination of Linear Transformation**

Let $S = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be basis of $U(F)$ &

Let $\{\delta_1, \delta_2, \dots \delta_n\}$ be a set of vectors in V.

Then $\exists$ a unique T st $T(\alpha_i) = \delta_i$

⇒ define $T: U \to V$ st

$$T(\alpha) = a_1\delta_1 + a_2\delta_2 + \dots + a_n\delta_n$$

where $\alpha = a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n$

NOTE. S must be basis of $U(F)$

4> To write explicit Linear Transformation $T: U \to V$ where $T(\alpha_i) = \delta_i$ are given $[i = 1 \text{ to } n]$, do

a> Show that $S = \{\alpha_1, \dots \alpha_n\}$ is basis of V [using S is LI & $L(S) = U$]

b> for $\alpha \in U$, write $\alpha = a_1\alpha_1 + \dots + a_n\alpha_n$ & then do $T(\alpha) = a_1\delta_1 + \dots + a_n\delta_n$ by claiming existence of unique LT.

5) Sum of LTs T_1 & T_2 is also an LT $(T_1+T_2)(\alpha) = T_1(\alpha)+T_2(\alpha)$

Scalar x^n of LT is LT defd by $(aT)(\alpha) = a[T(\alpha)]$

6) $T: V \to W$ & $H: U \to V$ are 2 LTs, then

$(TH)(\alpha) = T[H(\alpha)]$ is a LT from $U \to W$

a) $T(H+H') = TH + TH'$
b) $(T+T')H = TH + T'H$
c) $a(TH) = (aT)H = T(aH)$

7) $\boxed{L(U,V)}$ is set of all LTs from $U \to V$, then

$L(U,V)$ is a vector space relative to vector +^n & scalar x^n op's

defd by a) $(T+H)(\alpha) = T(\alpha) + H(\alpha)$

b) $(aT)(\alpha) = a\,T(\alpha)$ $\forall$ $\alpha \in U$, $a \in F$ & $T, H \in L(U,V)$

To prove use the properties of $V(F)$ as VS directly.

8) $\boxed{\dim U = n \;,\; \dim V = m \;,\text{ then } \dim L(U,V) = mn}$

Let $B_U = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ $B_V = \{\beta_1, \beta_2, \ldots, \beta_m\}$

$\exists$ exists uniquely a LT T_{ij} from U to V st.

$T_{ij}(\alpha_i) = \beta_j$ & $T_{ij}(\alpha_k) = 0$ $k \neq i$

Now show that $\{T_{ij}\}$ is basis for $L(U,V)$

a) prove S is LI : b) $L(S) = L(U,V)$

eg. Find $\dim L(C^3, \mathbb{R}^2)$ ⇒ doesn't exists as VS's aren't compatible

Find $\dim L(V, \mathbb{R}^2)$ where V is $C^3(R)$ ⇒ $6 \times 2 = 12$

9) Range of $T: U \to V$ is a subspace of $V(F)$

Nullspace | Kernel of $T: U \to V$ > $N(T) = \{\alpha \in U / T(\alpha) = \hat{0} \in V\}$

$N(T)$ is a subspace of $U(F)$

10) $\rho(T) = \dim R(T)$ = Rank
$\nu(T) = \dim N(T)$ = Nullity
$\rho(T) + \nu(T) = \dim U$, i.e. $\dim R(T) + \dim N(T) = \dim U$

Proof: Let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ be basis of $N(T)$
S is extended to form $S' = \{\alpha_1, \ldots \alpha_k, \theta_1, \ldots \theta_m\}$ basis of $U(F)$
We have to show that $S'' = \{T(\theta_1), \ldots, T(\theta_m)\}$ is basis of $R(T)$

eg. Find $T: \mathbb{R}^3 \to \mathbb{R}^4$ whose range is spanned by $(1,2,0,-4)$ & $(2,0,-1,-3)$

Ans. Let's include vector $(0,0,0,0)$ in the set $\{(1,2,0,-4), (2,0,-1,-3)\}$ which doesn't affect the spanning property.

$S = \{(1,2,0,-4), (2,0,-1,-3), (0,0,0,0)\}$

Let $B = \{\alpha_1, \alpha_2, \alpha_3\}$ be the basis of $\mathbb{R}^3$

$\exists$ a LT st $T(\alpha_1) = (1,2,0,-4)$
$T(\alpha_2) = (2,0,-1,-3)$
$T(\alpha_3) = (0,0,0,0)$

If $\alpha \in \mathbb{R}^3 \Rightarrow \alpha = (a,b,c) = a\alpha_1 + b\alpha_2 + c\alpha_3$
$T(\alpha) = T(a,b,c) = a\,T(\alpha_1) + b\,T(\alpha_2) + c\,T(\alpha_3)$
$T(a,b,c) = (a+2b, 2a, -b, -4a-3b)$

eg Find range and rank of $T: \mathbb{R}^3 \to \mathbb{R}^2$ defd by $T(x_1, x_2, x_3) = (x_1 + x_2, 2x_3 - x_1)$

$R(T) = \{\beta \in \mathbb{R}^2 \mid T(\alpha) = \beta$ for all $\alpha \in \mathbb{R}^3\}$
It contains all the vectors of type $(x_1 + x_2, 2x_3 - x_1)$ $\forall (x_1, x_2, x_3) \in \mathbb{R}^3$
Let $\beta = (a,b) \in R(T)$
$\therefore \exists (x_1, x_2, x_3) \in \mathbb{R}^3$ st $T(x_1, x_2, x_3) = \beta = (a,b)$
$\Rightarrow a = x_1 + x_2$
$b = 2x_3 - x_1$
$(a,b) = x_1(1,-1) + x_2(1,0) + x_3(0,2)$
$S = \{(1,0),(0,1)\}$ is LI & $R(T) \subseteq L(S)$ & $L(S) \subseteq R(T)$
$\Rightarrow S$ is basis of $R(T)$
$\therefore \rho(T) = \dim S = 2$

11) * $\boxed{\text{Singular Transformation}}$: $N(T)$ contains atleast 1 non-zero vector.

ie, $\exists\ \alpha \in U$ st $T(\alpha) = \hat{0}$ for $\alpha \neq \bar{0}$

* $\boxed{\text{Non-singular transformation}}$: $N(T) = \{\bar{0}\}$ or $\alpha \neq \bar{0} \Rightarrow T(\alpha) \neq \hat{0}$

12) T is non-singular $\Leftrightarrow$ set of images of a linearly independent set is linearly independent

$\Leftrightarrow$ T is one-one

$\Leftrightarrow$ U and $R(T)$ have same dimension (given FDVS) $[\because \dim V = \rho(T) + \nu(T)]$

13) $\boxed{\text{Inverse function}}$: $T: U \to V$ is a one-one onto mapping

$T^{-1}: V \to U$ defd by $T^{-1}(\beta) = \alpha \Leftrightarrow T(\alpha) = \beta$
$\alpha \in U,\ \beta \in V$

is called inverse mapping of T.

14) $U(F)$ & $V(F)$ are 2 FDVS with $\dim U = \dim V$.

$T: U \to V$ is a LT. Following statements are equivalent

a) T is invertible
b) T is non-singular
c) $R(T) = V$
d) If $\{\alpha_1, \ldots, \alpha_n\}$ is Basis of U, $\{T(\alpha_1) \ldots T(\alpha_n)\}$ is Basis of V
e) There is some basis $\{\alpha_1 \ldots \alpha_n\}$ of U st $\{T(\alpha_1) - T(\alpha_n)\}$ is basis of V

15) $\boxed{\text{Matrix of Linear Transformation}}$: columns are co-ordinates wrt basis

$\alpha_i \in$ Basis of U & $T(\alpha_i) = a_{1i}\beta_1 + \ldots + a_{mi}\beta_m$ then

$\dim U = n$ — column 'i' of Matrix is $[a_{1i}\ a_{2i} \ldots a_{mi}]^T$

$\dim V = m$ — Matrix is of order $(m \times n)$

This matrix is constructed wrt particular bases B_1 & B_2.

eg $T(x, y, z) = (2x + 2y - 4z,\ x - 5y + 3z)$

$B_1 = \{(1,1,1), (1,1,0), (1,0,0)\}$ $B_2 = \{(1,3), (2,5)\}$

$(a, b) = (-5a + 2b)(1,3) + (3a - b)(2,5)$

$\therefore T(1,1,1) = -7(\ ) + 4(\ )$
$T(1,1,0) = -33(\ ) + 19(\ )$
$T(1,0,0) = -13(\ ) + 8(\ )$

$\therefore M = \begin{bmatrix} -7 & -33 & -13 \\ 4 & 19 & 8 \end{bmatrix}$

⇒ Examples :

Q. Let T be the linear operator on $\mathbb{R}^4$ which is represented in the standard ordered basis by the matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \end{bmatrix}$$

Under what conditions on a, b, c is T diagonalizable ?

Ans. $|A - \lambda I| = 0 \Rightarrow (0-\lambda)^4 = 0 \Rightarrow \lambda = 0, 0, 0, 0$

$\therefore$ 0 is the only char. root of A.

Let W_0 be the corresponding eigenspace. then

$W_0 = \{ v \in V \mid T(v) = 0 \} = \ker T$

T is diagonalizable $\Leftrightarrow \dim W_0 = 4$ [Same geometric & algebraic multiplicity]

$\Leftrightarrow \dim(\ker T) = 4$

$\Leftrightarrow$ nullity of $T = 4$

$\therefore$ Rank $T = 0$

ie, dimension of range space $= 0$

ie, Range space of $T = \{0\}$

$\Rightarrow T = 0$

$\Rightarrow a = b = c = 0$

⇒ Linear Equations :

1) $AX = B$ is consistent iff A & $[A|B]$ have same rank

Proof : $A = [c_1\ c_2 \cdots c_n]$ ⇒ $x_1c_1 + x_2c_2 + \cdots + x_nc_n = B$

If $\rho(A) = r$, then r columns are LI & rest a LC of them

WLOG, let those be the 1^{st} r columns.

NC : If there is a solⁿ ⇒ $B = x_1c_1 + x_2c_2 + \cdots + x_nc_n$
⇒ B is LC of $c_1 \cdots c_r$

∴ $[A|B]$ has only r LI column vectors
⇒ $\rho(A|B) = r = \rho(A)$

SC : If $\rho(A|B) = \rho(A) = r$ ⇒ B is LC of $c_1, \ldots, c_r$
∴ $\exists\ p_1, p_2 \cdots p_r$ st $\sum_{i=1}^{r} p_i c_i = B$

⇒ $x_i = p_i$ for $i = 1$ to r & $x_i = 0$ for $i > r$
is a solution

2) If $[A|B]$ & $[C|D]$ are row equivalent, $AX = B$ & $CX = D$ have exactly same solution.

3) If A is a non-singular matrix of order 'n' then $AX = B$ in n unknowns has a unique solution.

4) Solving $AX = B$; reduce $[A|B]$ to echelon form
a) $\rho(A) = \rho(A|B) =$ no of variable ⇒ unique solⁿ
b) $\rho(A) = \rho(A|B) <$ no of variable ⇒ infinite solⁿ
c) $\rho(A) \neq \rho(A|B)$ ⇒ no solution

5) **Cramer's Rule** of solving system of 'n' homogenous linear equations in 'n' unknowns :

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

Let $\Delta = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{n1} & \cdots & \cdots & a_{nn} \end{vmatrix}$

Works only for $\Delta \neq 0$

$x_i = \dfrac{\Delta_i}{\Delta}$ where Δ_i is obtd by replacing i^{th} column by $\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$

5) Homogenous Linear equations : $AX=0$ — ①

a) Trivial solution $\Rightarrow$ $x_1 = x_2 = \dots = x_n = 0$

b) If X_1, X_2 are 2 solⁿs of ①, then the solution set is a subspace of $V_n(F)$

c) No of LI solutions of $AX=0 \Rightarrow n - \rho(A)$

d) Steps : i) Reduce A to echelon form $o(A) = (m \times n)$
ii) $\rho(A) = r \Rightarrow n-r$ LI solⁿ

eg. Find a basis & dimension of the solⁿ space of

$$x + 2y - 2z + 2s - t = 0$$
$$x + 2y - z + 3s - 2t = 0$$
$$2x + 4y - 7z + s + t = 0$$

$$A = \begin{bmatrix} 1 & 2 & -2 & 2 & -1 \\ 1 & 2 & -1 & 3 & -2 \\ 2 & 4 & -7 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -2 & 2 & -1 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore \rho(A) = 2$
$\Rightarrow 5-2 = 3$ LI solⁿ

$$\Rightarrow \quad x + 2y - 2z + 2w - t = 0$$
$$z + w - t = 0$$

Let $t = k_1$, $w = k_2$ & $y = k_3$

$\Rightarrow z = k_1 - k_2$ & $x = 3k_1 - 4k_2 - 2k_3$

$\therefore$ Solⁿ are $\Rightarrow k_1[3\ 0\ 1\ 0\ 1]^T + k_2[-4\ 0\ -1\ 1\ 0]^T + k_3[-2\ 1\ 0\ 0\ 0]^T$

$\therefore$ Dimension of solution space = 3

& Basis $= \{(3,0,1,0,1)^T, (-4,0,-1,1,0)^T, (-2,1,0,0,0)^T\}$

e) Non-trivial solution iff $|A| = 0$

⇒ Eigen Values & Eigen Vectors :

1> Non-zero solution to $AX = \lambda X$ exists iff $|A - \lambda I| = 0$

$|A - \lambda I| = 0$ is characteristic equation of A and the roots are the char. values or Eigenvalues

Non-zero vector X st $AX = \lambda X$ is the eigenvector of A for eigenvalue λ.

2> Properties :

a> If X is e-vector for λ, then so is kX where k is non-zero scalar

b> If X is an e-vector, it can correspond to more than 1 e-value

c> <u>Eigen vectors corresponding to distinct Eigenvalues are LI.</u>

[∵ Let them be LD & $X_1, \ldots, X_r$ be LI.

so $k_1X_1 + k_2X_2 + \ldots + k_{r+1}X_{r+1} = 0$ — (1) ⇒ not all k_i are zero

⇒ $k_1(\lambda_1 X_1) + k_2(\lambda_2 X_2) + \ldots + k_{r+1}(\lambda_{r+1} X_{r+1}) = 0$ — (2)

(2) − λ_{r+1}(1) ⇒ $k_1(\lambda_1 - \lambda_{r+1})X_1 + \ldots + k_r(\lambda_r - \lambda_{r+1})\lambda_r = 0$

⇒ $k_1 = k_2 = \ldots = k_r = 0$

Put in (1) ⇒ $k_{r+1} = 0$ ⇒ #]

c> A & A^T have same E-Values

d> No e-value of an non-singular matrix is 0

Atleast 1 e-value of a singular matrix is 0.

e> Let λ is CR (char. root) of A with CV - 'X'

then $k \pm \lambda$ is CR of $A + kI$

λ^2 is CR of A^2 $[A^2X = A(AX) = A(\lambda X) = \lambda(AX) = \lambda^2 X]$

λ^{-1} is CR of A^{-1} $[AX = \lambda X \Rightarrow X = \lambda A^{-1}X \Rightarrow \lambda^{-1}X = A^{-1}X]$

$\lambda - k$ is CR of $A - kI$

f> CRs of A & $C^{-1}AC$ are same

To prove x is real st: $x = \bar{x}$

g> <u>CR of Hermitian Matrix are real</u>

$AX = \lambda X \Rightarrow (X^\theta AX)^\theta = (\lambda X^\theta X)^\theta \Rightarrow X^\theta A^\theta X = \bar{\lambda} X^\theta X$

$\Rightarrow X^\theta AX = \bar{\lambda} X^\theta X$

$\Rightarrow \lambda(X^\theta X) = \bar{\lambda} X^\theta X$

$\Rightarrow \lambda = \bar{\lambda}$

$\Rightarrow \lambda$ is real.

corollary: CR of Real symmetric matrix are all real

h) CR of a skew ~~sa~~ Hermitian matrix are either purely imaginary or zero.

$A^\theta = -A \qquad (iA)^\theta = \bar{i} A^\theta = (-i)(-A) = iA$

$\Rightarrow iA$ is Hermitian $\Rightarrow i\lambda$ is real

$\Rightarrow \lambda$ is purely img or 0.

Corr: CR of a real skew-symmetric matrix are either purely img or zero

i) CRs of a unitary matrix are of unit modulus.

$AX = \lambda X \Rightarrow (AX)^\theta = (\lambda X)^\theta \Rightarrow X^\theta A^\theta = \bar{\lambda} X^\theta$

$(X^\theta A^\theta)(AX) = \bar{\lambda} X^\theta \lambda X$

$\Rightarrow X^\theta A^\theta A X = \bar{\lambda}\lambda X^\theta X$

$\Rightarrow X^\theta X = |\lambda|^2 X^\theta X$

$\Rightarrow |\lambda|^2 = 1.$

Corr.: CRs of an orthogonal matrix are of unit modulus

~~Corr~~ [Real matrix is unitary $\Leftrightarrow$ it is orthogonal]

Corr: ± 1 can be the only CRs of orthogonal matrix.

j) CRs of A^θ are conjugate of CRs of A. & vice-versa

SUMMARY:

MATRIX		EIGENVALUES
Real symmetric Hermitian	$\}\longrightarrow$	Real
Skew symmetric Skew Hermitian	$\}\longrightarrow$	Purely img or 0
Unitary		$\lvert\lambda\rvert = 1$
Real orthogonal		$\lambda = \pm 1$

3) Construction of orthogonal matrices :

a) S is real skew-symmetric matrix

i) $I-S$ is non-singular [$\because$ property 2(h)]

ii) $A = (I+S)(I-S)^{-1}$ is orthogonal } [using $A^TA = I$ & the trick below]

iii) $A = (I-S)^{-1}(I+S)$

iv) If X is CV of S for CR 'λ', then X is also CV of A for CR $\frac{1+\lambda}{1-\lambda}$

TRICK | REMEMBER: $\boxed{(I-S)(I+S) = (I+S)(I-S)}$

for (iv) : $SX = \lambda X$

$X + SX = X + \lambda X \Rightarrow (I+S)X = (1+\lambda)X$

$(I-S)X = (1-\lambda)X$

$(I-S)^{-1}(I-S)X = (I-S)^{-1}(1-\lambda)X \Rightarrow X = (1-\lambda)(I-S)^{-1}X$

$\Rightarrow (1-\lambda)^{-1}X = (I-S)^{-1}X$ —— (1)

Also $(I-S)^{-1}(I+S)X = (1+\lambda)(I-S)^{-1}X = (1+\lambda)(1-\lambda)^{-1}X$

$AX = (1+\lambda)(1-\lambda)^{-1}X \Rightarrow \lambda$ for $A = \frac{1+\lambda}{1-\lambda}$.

Similar results for $(I+S)$ & $A = (I+S)^{-1}(I-S)$

b) If A is an orthogonal matrix with property that -1 is not a CR, then $A = (I+S)(I-S)^{-1}$ for some real skew-symm matrix 'S'.

$A = (I+S)(I-S)^{-1} \Rightarrow A(I-S) = (I+S) \Rightarrow A - I = AS + S$

$\Rightarrow A - I = (A+I)S \Rightarrow (A+I)^{-1}(A-I) = S$

Now prove $S^T = -S$ using manipulations ← Remember

we get $S^T = (A^T+I)^{-1}(A^T-I) = (A^T + A^TA)^{-1}(A^T - A^TA)$

$= (I+A)^{-1}(A^T)^{-1}A^T(I-A)$

$= -(A+I)^{-1}(A-I) = -S$.

4.) For generating the set of all eigenvectors, make sure that the paramaters are chosen properly.

eg. say we get $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x_1 + x_2 + x_3 = 0$

Let $x_2 = k_1$ & $x_3 = k_2$ but both not zero simultaneously

$\Rightarrow X = k_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = k_1 X_1 + k_2 X_2$ where k_i are params.

This is because X [eigenvectors] are non-zero.

5.) $\boxed{\text{Matrix Polynomial \& Cayley - Hamilton Theorem:}}$

Every square matrix satisfies its characteristic equation.

If $|A - \lambda I| = 0 \Rightarrow \lambda^n + \lambda^{n-1} a_1 + \lambda^{n-2} a_2 + \cdots + a_n = 0$

then $A^n + a_1 A^{n-1} + a_2 A^{n-2} + \cdots + a_n = 0$

Using it $\Rightarrow A^{-1} = \frac{(-1)}{a_n} \left[A^{n-1} + a_1 A^{n-2} + \cdots + a_{n-1} I \right]$

eg. $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ Find A^{-1}

$|A - \lambda I| = (1-\lambda)(4-\lambda) - 6 = \lambda^2 - 5\lambda - 2$

$\Rightarrow A^2 - 5A - 2I = 0$

$\Rightarrow A^{-1} = \frac{A - 5I}{2} = \frac{1}{2} \begin{bmatrix} -4 & 3 \\ 2 & -1 \end{bmatrix}$

eg. $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ st $A^n = A^{n-2} + A^2 - I$. Hence find A^{50}

Do it using induction & C-H Thm.

6.) $\boxed{\text{Minimum Polynomial of a matrix A:}}$

Lowest degree polynomial $f(x)$ st $f(A) = 0$

It is always a factor of characterestic polynomial but need not always be equal to it.

eg For I_n, char poly is $(x-1)^n = 0$ But minimum polynomial is $(x-1) = 0$

eg Find Minimal polynomial $m(t)$ of $M = \begin{bmatrix} 2 & 8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}$

$$M = \begin{bmatrix} A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & D \end{bmatrix} \quad \text{where } A = \begin{bmatrix} 2 & 8 \\ 0 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$$

$$D = [5]$$

$m(t)$ is the LCM of minimum polynomials of A, B, C, D.

min. poly of $A\ C\ D$ are $(t-2)^2,\ t^2,\ (t-5)$

min poly of B is $(t-2)(t-5)$

$\therefore\ m(t) = \text{LCM of } (t-2)^2,\ t^2, (t-5),\ (t-2)(t-5)$

$$= (t)^2 (t-2)^2 (t-5)$$

NOTE : * If eigenvalues are distinct, minimal & characteristic polynomial are same.

* If A is diagonalizable with eigenvalues λ_i with multiplicities t_i

then $m(x) = \prod_{i=1}^{r} (x - \lambda_i)$

But if A is not diagonalizable $m(x) = \prod_{i=1}^{r} (x-\lambda_i)^{t_i}$

i.e., for diagonalizable min. poly. is product of distinct linear factors

⇒ Matrices

1) $A_{mn} = m \times n$ m rows & n columns

* $m = n \longrightarrow$ square matrix   * $m \neq n$ rectangular matrix
* $m = 1 \Rightarrow$ row matrix   * $n = 1 \rightarrow$ column matrix
* $a_{ij} = 0 \ \forall i, j \Rightarrow$ zero matrix
* $a_{ij} = 0$ $i < j$ lower triangular
  $a_{ij} = 0$ $i > j$ upper triangular
  $a_{ij} = 0$ $i \neq j$ diagonal
  } only for square matrices

2) Idempotent Matrix → square matrix A with ($A^2 = A$)
Involuntary Matrix → $A^2 = I$
Nilpotent matrix ⇒ $\exists$ a $n \in Z^+$ st $A^n = 0$
lowest such n is index

3) Trace of matrix = $\operatorname{tr} A = \sum_{i=1}^{n} a_{ii}$

$\operatorname{tr}(\lambda A) = \lambda \operatorname{tr} A$
$\operatorname{tr}(A+B) = \operatorname{tr} A + \operatorname{tr} B$
$\operatorname{tr}(AB) = \operatorname{tr}(BA) \Rightarrow$ $\boxed{AB - BA = I \text{ is impossible}}$ (Remember)

4) Transpose of $A = A^T$ or $A' = [a_{ji}]_{n \times m}$ where $A = [a_{ij}]_{mn}$

$(A^T)^T = A$ $(-A)^T = -A^T$
$(A \pm B)^T = A^T \pm B^T$
$(kA)^T = kA^T$ (k is scalar)
$(AB)^T = B^T A^T$

Symmetric Matrix : $A^T = A$
Skew symmetric matrix : $A^T = -A$

Every matrix is sum of symm & skew symm matrices $\left[\frac{A+A^T}{2} \text{ \& } \frac{A-A^T}{2}\right]$ (unique)

5) Conjugate of a matrix $A = \bar{A} = [\bar{a}_{ij}]_{m\times n}$ where $A = [a_{ij}]_{m\times n}$

$\boxed{\text{Tranjugate of a matrix} = A^{\theta} = (\bar{A})^T}$

$(A^{\theta})^{\theta} = A$ $\quad (kA)^{\theta} = \bar{k} A^{\theta}$

$(A+B)^{\theta} = A^{\theta} + B^{\theta}$ $\quad (AB)^{\theta} = B^{\theta} A^{\theta}$

6) Hermitian Matrix - $A^{\theta} = A$

Skew - Hermitian - $A^{\theta} = -A$

* If A is Herm., iA is skew-Herm.

* Every square matrix is uniquely expressed as sum of a Herm. & skew - Herm. matrix $\left(\frac{1}{2}(A+A^{\theta}) \ \& \ \frac{1}{2}(A-A^{\theta})\right)$

* Every square matrix can be uniquely expressed as $P+iQ$ where P & Q are Herm.

$P = \frac{1}{2}(A+A^{\theta}) \qquad Q = \frac{1}{2i}(A-A^{\theta})$

7) Determinants

Cofactor $= (-1)^{i+j} M_{ij}$

a) $|A| = |A^T|$

b) Interchange (C_i, C_j) or $(R_i, R_j) \Rightarrow |A|$ is multiplied by (-1)

c) $\begin{vmatrix} ka_1 & b_1 \\ ka_2 & b_2 \end{vmatrix} = k \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$. Also $|kA| = k^n |A|$.

d) Sum of product of the elements of any row (col.) with the cofactors of corresponding elements of any other row (col.) $= 0$

e) If A is Herm, $|A|$ is real always

f) If A is skew - symm of odd order $|A| = 0$ $\quad [|A| = (-1)^n |A|]$

8) Adjoint Matrix = $\text{adj } A = [A_{ji}]_{n\times n}$ where A_{ji} = cofactor of a_{ji}
= transpose of cofactor matrix.

$$\boxed{A\cdot[\text{adj}(A)] = [\text{adj}(A)]\cdot A = |A|\, I_n} \quad \left[\because \sum_{k=1}^{n} A_{ki}\, a_{kj} = \begin{cases} |A| & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}\right.$$

$\therefore \quad A^{-1} = \frac{1}{|A|}\,\text{adj}(A)$ [No inverse for rectangular matrix $\because AB = BA = I$]

Invertible matrix has unique inverse. $\left[\begin{array}{l} B = BI = B(AC) \\ \quad = (BA)C \\ \quad = IC = C \end{array} \right. \begin{array}{l} AB = BA = I \\ AC = CA = I \end{array}$

$|A| \neq 0$ is necessary & sufficient for inverse to exist

$(AB)^{-1} = B^{-1}\cdot A^{-1}$; $(A^T)^{-1} = (A^{-1})^T$; $(A^{-1})^\theta = (A^\theta)^{-1}$; $\text{adj } A^T = (\text{adj } A)^T$

$|\text{adj } A| = |A|^{n-1}$ [use $A\cdot(\text{adj } A) = |A|\, I$, take $|\;|$ of both side & use $|kA| = k^n|A|$.]

$$\boxed{(\text{adj } AB) = (\text{adj } B)(\text{adj } A)}$$

9) Orthogonal Matrix : $A^TA = I = AA^T \Leftrightarrow A^T = A^{-1}$

10) $\boxed{\text{Unitary Matrix}}$: $A^\theta A = I = AA^\theta \Leftrightarrow A^\theta = A^{-1}$

REMEMBER: Real Matrix is unitary $\Leftrightarrow$ it is orthogonal

10) $\boxed{\text{Rank of a matrix}}$ $\rho(A) = r$ if

a) $\exists$ at least one minor of order r which $\neq 0$, &

b) Each minor of order $(r+1)$ vanishes

11) Elementary operations : R_{ij}, $R_i(k)$, $R_{ij}(k)$ $[R_i \to R_i + kR_j]$

12) Echelon Matrix : iff the # of 0's preceding the distinguished element in a row increases row by row and the elements of the last row or rows may be all zero

13) Row Reduced Echelon Form : (or $\boxed{\text{Row canonical form}}$) iff the distinguished elements are 1 & the only non-zero elements in the column. eg $\begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

14) $\rho(A)$ = no of non-zero rows in echelon form of A.

$\rho(A) = \rho(A^T)$

$A = [a_{ij}]_{m\times 1}$ $\quad B = [b_{ij}]_{1\times n}$ $\quad \rho(AB) = 1$

$A, B \neq 0$.

15) Matrix partition for multiplication :

$A = [a_{ij}]_{m\times n}$ $\quad B = [b_{ij}]_{n\times p}$ $\Rightarrow AB$ exists

Partitioning lines drawn parallel to the rows of B are in the same relative position as partitioning lines drawn parallel to columns of A.

eg
$$A = \left[\begin{array}{ccc|cc} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ \hline 11 & 12 & 13 & 14 & 15 \\ -1 & -2 & -3 & -4 & -5 \end{array}\right]_{4\times 5} \qquad B = \left[\begin{array}{c|cc|cc} 0 & -1 & -2 & -3 & -4 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right]_{5\times 5}$$

L_1 $\quad L_2$

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \qquad B = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \end{bmatrix}$$

L_1 & L_2 are corresponding partition lines.

REMEMBER: P, Q are non-singular

if $A = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}$ then $A^{-1} = \begin{bmatrix} P^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix}$

[$\because$ Let $B = \begin{bmatrix} M & N \\ R & S \end{bmatrix} = A^{-1}$ & solve $AB = I$]

* Rank of matrix doesn't alter on affixing any no of additional rows or columns of zeroes. [$\because M = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$ $\rho(M) \leq \rho(A)$]

16) $\boxed{\text{Elementary Matrix}}$: obtained by performing single elementary transformation on I

E_{ij}, $E_i(k)$, $E_{ij}(k)$, $E_{ij}'(k)$

17> Properties & Lemmas using E :-

a> Every elementary row transformation on $C = AB$ can be
Lemma affected by subjecting pre-factor A to the same row op^n

$$\sigma(AB) = (\sigma A)B \text{ where } \sigma \text{ is row transformation}$$

Similarly, $\sigma(AB) = A(\sigma B)$ where σ is column transformation

b> Theorem Every elementary row (column) transformation of a matrix can be obtd. by pre (post) - multiplication with the corresponding elementary matrix.

c> (i) $E_{ij}^{-1} = E_{ij}$ (ii) $[E_i(k)]^{-1} = E_i\left(\frac{1}{k}\right)$, $k \neq 0$

(iii) $[E_{ij}(k)]^{-1} = E_{ij}(-k)$, $k \neq 0$

d> Elementary operations do not change the rank.
[$\because$ solve for 3 diff op^ns by considering $|A_0| = r+1$ minor of A where $r = \rho(A)$]

e> Pre or Post multiplication by an elementary matrix or any such series do not change the rank of the matrix.

18> **Reduction to Normal or First Canonical Form :**

a> Not every matrix can be reduced using row (column) op^n only

Thm b> If A is $m \times n$ matrix of rank 'r', $\exists$ P, Q [$|P| \neq 0, |Q| \neq 0$]

st $PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

c> Every non-singular matrix is a product of elementary matrices.

d> $\rho(A)$ doesn't change by pre or post-multiplication with non-sing. matrix

eg Reduce $A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$ to canonical form & find rank.

Use both row & column op^n.

eg Find P & Q st $PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ where $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$

Note. P & Q are not unique

19) Express A as x^n of Elementary Matrices.
Apply row & colmn op^n on A to get I. Since op^n are equivalent to pre & post x^n with Elementary matrices, we get

$E_1E_2E_3E_4 A E_5E_6E_7E_8 = I$

$\Rightarrow A = (E_1E_2E_3E_4)^{-1}.(E_5E_6E_7E_8)^{-1}$ ...

20) $\boxed{\text{Equivalence of Matrices}}$

a) If B can be obtd from A by finite elementary op^{n}s then A is equivalent to B $[A \sim B]$

b) '$\sim$' is an equivalence relation

c) If $A \sim B$, $\rho(A) = \rho(B)$

d) If A & B have same order & $\rho(A) = \rho(B) \Rightarrow A \sim B$
$\left[\because A \sim \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, B \sim \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow A \sim B \text{ (transitivity)}\right]$

eg Is $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 2 & 1 & -3 \end{bmatrix} \sim I_3$? $|A| \neq 0$, so yes. $[\because \rho(A) = 3 = \rho(I_3)]$

21) If $A_{m\times n}$ & $\rho(A) = r$, $\exists$ P $[|P| \neq 0]$ st $PA = \begin{bmatrix} G \\ 0 \end{bmatrix}$ where $G_{r\times n}$ & $\rho(G) = r$ & O is $(m-r)\times n$.

Proof: $\exists$ P, Q st $PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow PA = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q^{-1}$

No Column op^n can change the last $(m-r)$ rows of $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$
$\Rightarrow \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q^{-1} = \begin{bmatrix} G \\ 0 \end{bmatrix}$ G is $r\times n$

Also as elem. op^n don't change rank $\rho(I_r) = \rho(G) = r$

Similarly, $\exists$ Q $[|Q| \neq 0]$ st $AQ = [H \ \ 0]$ H is $m\times r$ & O is $(m-r)\times(n-r)$

22) $\boxed{\rho(AB) \leq \rho(A) \ \ \& \ \ \rho(AB) \leq \rho(B)}$ $\Big[\because PAB = \begin{bmatrix} G \\ 0 \end{bmatrix} B$
$\rho(PAB) = \rho(AB) = r$
But $\rho\left(\begin{bmatrix} G \\ 0 \end{bmatrix} B\right) \leq r_1$
$\therefore r \leq r_1$
$\rho(AB) = \rho(AB)' = \rho(B'A')$
$\leq \rho(B')$
$\leq \rho(B)\Big]$

23) Finding $A^{-1} \Rightarrow A = I_n A$
Apply same E-row transformation to A & I_n

24) A^{-1} as multiplication of E_is.
$(E_1E_2 \cdots E_n)A = I \Rightarrow A = (E_1E_2 \cdots E_n)^{-1} \Rightarrow A^{-1} = (A)^{-1} = E_1E_2 \cdots E_n$.

⇒ Similarity of Matrices :

1) A & B are square matrices of order n.
B is similar to A iff $\exists$ a $n \times n$ invertible matrix C
st $AC = CB$ ie, $B = C^{-1}AC$ or $A = CBC^{-1}$

2) Similarity is an equivalence relation.

3) A is similar to $B \Rightarrow |A| = |B|$

4) Similar matrices have same char. polynomial & roots
But converse is not true.

5) A is similar to diagonal matrix D, then diag elements of D are the characteristic roots of A.

6) Diagonalizable Matrix :
Matrix A is diagonalizable if $\exists$ an invertible matrix P
st $P^{-1}AP = D$ where D is a diagonal matrix.

⇒ n-rowed ~~diago~~ square matrix is diagonalizable iff the matrix possesses n LI characteristic vectors.

Proof: Let A be diagonalizable & is similar to $D = \text{dia}(\lambda_1, \lambda_2 \cdot \lambda_n)$
$\exists$ $n \times n$ invertible matrix $P = [X_1, X_2, \ldots, X_n]$ st $P^{-1}AP = D$
↳ (inverse is prexy)

$$AP = PD$$
$$A[X_1 \ldots X_n] = [X_1 \ldots X_n][\text{diag}[\lambda_1, \ldots \lambda_n]]$$
$$[AX_1, AX_2, \ldots, AX_n] = [\lambda_1 X_1, \lambda_2 X_2, \ldots, \lambda_n X_n]$$

$\Rightarrow AX_1 = \lambda_1 X_1, \; AX_2 = \lambda_2 X_2 \ldots \; A_n X_n = \lambda_n X_n$

$\Rightarrow X_1, X_2, \ldots, X_n$ are char. vectors of A
since $|P| \neq 0 \Rightarrow X_i$'s are LI $\Rightarrow$ char vectors of A are LI

In converse, we can construct P as $[X_1, X_2, \ldots, X_n]$ & show that $AP = PD$

NOTE: If A is diagonalizable & P diagonalizes A, ~~then~~ $P^{-1}AP = D = [\text{diag}(\lambda_1, \ldots, \lambda_n)]$

iff j^{th} column of P is eigenvector of A corresponding to λ_j eigenvalue of A
⇒ Order of eigenvalues in D & column vectors in P are same

7> a) Every square matrix is not diagonalizable

b) If all eigenvalues are distinct ⇒ diagonalizable

c) 2 n×n matrices with <u>same set of n distinct eigenvalues</u> are similar. → Useful to prove similarity of 2 matrices directly

d) If some eigenvalues are repeated ⇒ may or may not be diagonalizable

8> It is best to stick to definition to prove some results instead of cramming tricks / methods.

eg A & B are non-singular, st AB & BA are similar

$A^{-1}(AB)A = (A^{-1}A)(BA) = BA$ ⇒ HP using def^n !!

9> <u>Algebraic multiplicity</u> (t) of char. root ⇒ order 't' of root λ in $|A-\lambda I|=0$

<u>Geometric multiplicity</u> (s) ⇒ No of LI char vectors for char root 'λ'

↳ No of LI sol^n to $(A-\lambda_1 I)X=0$ ⇒ s

and $\rho(A-\lambda_1 I) = n-s$

a) $s \le t$

b) <u>if for all λ_i, $s_{\lambda_i} = t_{\lambda_i}$</u> ⟺ A is diagonalizable

↳ use this to show non-diagonalizable.

10> <u>Orthogonal Vectors</u> :

a) Inner Product on <u>vectorspace</u> V(F) $f: V\times V \to F$ [$f(\alpha,\beta)=(\alpha,\beta)$ notation]

i) $(\alpha,\beta) = \overline{(\beta,\alpha)}$

ii) $(\alpha,\alpha) > 0$ for $\alpha \neq \bar{0}$ & $(\alpha,\alpha) = 0$ for $\alpha = \bar{0}$

iii) $(a\alpha + b\beta, \gamma) = a(\alpha,\gamma) + b(\beta,\gamma)$

b) vectorspace in which an IP is defd is Inner Product space

c) If V(F) is an IPS & F is field of real nos, V(F) is <u>euclidean space</u>

d) If V(F) is an IPS & F is field on C, V(F) is <u>unitary space</u>

e> If $\alpha = (a_1, a_2, \dots, a_n)$, $\beta = (b_1, b_2, \dots, b_n)$ are 2 vectors of VS $V_n(\mathbb{R})$, then $(\alpha, \beta) = \sum_{i=1}^{n} a_i b_i$ is called standard inner product

If $\alpha = (a_1, a_2 \dots a_n)$, $\beta = (b_1, b_2, \dots b_n)$ are elements of VS $V_n(C)$, then $(\alpha, \beta) = \sum_{i=1}^{n} a_i \bar{b}_i$ is SIP on $V_n(C)$

f> Inner Product of 2 vectors (column vectors)

$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ $Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ $(X, Y) = \bar{x}_1 y_1 + \bar{x}_2 y_2 + \dots + \bar{x}_n y_n = X^\theta Y$

for 2 row-vectors $(X, Y) = XY^\theta$

g> Norm of a vector (or length): $\Rightarrow \|X\| = \sqrt{X^\theta X} = \sqrt{|x_1|^2 + \dots + |x_n|^2}$

where $X = [x_1\ x_2 \dots x_n]^T$

Unit vector = $\|X\| = 1$ $\Rightarrow$ Normal vector

h> $\boxed{\text{Orthogonal vectors : } (X, Y) = X^\theta Y = 0}$

Orthogonal set : Any 2 distinct vectors in set are orthogonal

Orthonormal set : Orthogonal set where each vector is a unit vector

i> Orthogonally similar matrices : B is orthogonally similar to A if $\exists$ an orthogonal matrix P st $B = P^{-1} A P$

REMEMBER : Every real symmetric matrix is orthogonally similar to a diagonal matrix with real elements

A real symmetric matrix of order 'n' has n mutually orthogonal real eigen vectors.

THEOREM : Any 2 EVs corresponding to 2 distinct EValues of real symmetric matrix are orthogonal

$\lambda_1 X_2^T X_1 = X_2^T \lambda_1 X_1 = X_2^T A X_1 = X_2^T A^T X_1 = (AX_2)^T X_1 = \lambda_2 X_2^T X_1$

$\Rightarrow (\lambda_1 - \lambda_2) X_2^T X_1 = 0$

$\Rightarrow X_2^T X_1 = 0 \Rightarrow X_2^\theta X_1 = 0$ [real vectors] $\lambda_1 \neq \lambda_2$

i) For orthogonal reduction of a matrix, follow the same steps of getting λ & X for each λ. In this case normalise X vectors before forming P matrix.
then $D = P^{-1}AP$.

11) $\boxed{\text{Unitarily Similar Matrices}}$:

$\exists$ a unitary matrix P st $B = P^{-1}AP$

Every Hermitian matrix is unitarily similar to a diagonal matrix.

Any 2 EVectors corresponding to 2 distinct EValues of an Hermitian matrix are orthogonal.

12) To get the diagonal matrix & transformation matrix follow the same process & normalise the eigenvectors

$$\left\| \begin{bmatrix} 1+2i \\ 5 \end{bmatrix} \right\| = \sqrt{|1+2i|^2 + |5|^2} = \sqrt{x_1\bar{x}_1 + y_1\bar{y}_1} = \left\| \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \right\|$$