⇒ 3D GEOMETRY :

⇒ Coordinate system :

1> 3 axes , 8 octants

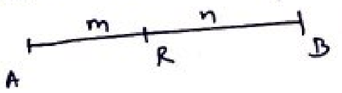
2> Rectangle ⟶ opp. sides are equal & diagonals are equal
Rhombus ⟶ all sides are equal but unequal diagonals
Diagonals bisect each other in ||gm , rectangle , rhombus , square
($\perp$ for rhombus & square)

3> Collinear Points (A,B,C): AB + BC = AC or AC + CB = AB

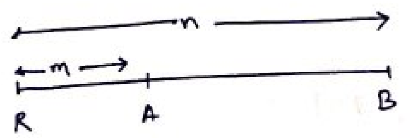
Necessary & Sufficient Condition : $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

4> go for length of sides & diagonals to find nature of Δ or $\square$.

5> Internal section formula $R\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}, \frac{mz_2+nz_1}{m+n}\right)$



External section formula $R\left(\frac{mx_2-nx_1}{m-n}, \frac{my_2-ny_1}{m-n}, \frac{mz_2-nz_1}{m-n}\right)$



6> If $P(x,y,z)$ lies on $A(x_1,y_1,z_1)$, $B(x_2,y_2,z_2) \Rightarrow \frac{x_1-x}{x_1-x_2} = \frac{y_1-y}{y_1-y_2} = \frac{z_1-z}{z_1-z_2}$

7> Δ Centroid divides median in ratio 1:2 (1 near side) $C\left(\frac{\Sigma x_1}{3}, \frac{\Sigma y_1}{3}, \frac{\Sigma z_1}{3}\right)$

Tetrahedron centroid divides in ratio 3:1 (1 near side) $C = \left(\frac{\Sigma x_1}{4}, \frac{\Sigma y_1}{4}, \frac{\Sigma z_1}{4}\right)$

8> All edges are equal in regular tetrahedron

9> $P = ax+by+cz+d = 0$ divides $(x_1,y_1,z_1),(x_2,y_2,z_2)$ in ratio $-\frac{(ax_1+by_1+cz_1+d)}{(ax_2+by_2+cz_2+d)}$

10> ||gm 3 vertex given , find 4th using midpt of diagonal

11> Locus : set of all points satisfying a given condition

⇒ Direction Cosines & Direction Ratios :

1) - dc's are $\cos\alpha$, $\cos\beta$, $\cos\gamma$ where α, β, γ are angles the line makes with x, y & z-axis respectively.

$l = \cos\alpha \quad m = \cos\beta \quad n = \cos\gamma$

$\Sigma l^2 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

If $\vec{P}(x,y,z)$ & $OP = r$ then $P = (lr, mr, nr)$

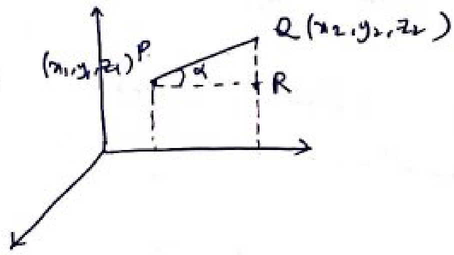
$\downarrow$ dc's (l, m, n)

2) Direction Ratio : 3 nos a, b, c proportional to dc's.

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$$

If dr's are a, b, c $\quad dc = \pm\frac{a}{\sqrt{\Sigma a^2}}, \pm\frac{b}{\Sigma a^2}, \pm\frac{c}{\Sigma a^2}$ [all +ve or all −ve]

3) DR of line joining 2 points :



$x_2 - x_1 = PQ\cos\alpha$

$y_2 - y_1 = PQ\cos\beta$

$z_2 - z_1 = PQ\cos\gamma$

$$\therefore \frac{x_2 - x_1}{\cos\alpha} = \frac{y_2 - y_1}{\cos\beta} = \frac{z_2 - z_1}{\cos\gamma} = PQ$$

$\therefore$ dr's are $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$

4) Lagrange Identity : $(l_1^2 + m_1^2 + n_1^2)(l_2^2 + m_2^2 + n_2^2) - (l_1 l_2 + m_1 m_2 + n_1 n_2)^2$

$$= (l_1 m_2 - m_1 l_2)^2 + (m_1 n_2 - m_2 n_1)^2 + (n_1 l_2 - n_2 l_1)^2$$

$$(\Sigma l_1^2)(\Sigma l_2^2) - (\Sigma l_1 l_2)^2 = \Sigma (l_1 m_2 - m_1 l_2)^2$$

5) Angle b/w 2 lines : $\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$ $\quad l_i, m_i, n_i$ are dc's

Remember : In a ΔABC ; $AB^2 + AC^2 - BC^2 = 2\,AB \cdot AC \cdot \cos\theta$

$\sin\theta = \pm\sqrt{\Sigma (l_1 m_2 - m_1 l_2)^2}$

−ve $\cos\theta$ gives obtuse angle & +ve $\cos\theta$ gives acute angle
↳ more interested in this

Angle if dr's are given $\cos\theta = \dfrac{\Sigma a_1 a_2}{\Sigma a_1^2 \, \Sigma a_2^2}$

6> conditions : $\perp$ lines $\Rightarrow$ $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

$a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$

$\parallel$ lines $\Rightarrow$ $l_1 = l_2 \quad m_1 = m_2 \quad n_1 = n_2$

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \frac{\sqrt{a_1^2+b_1^2+c_1^2}}{\sqrt{a_2^2+b_2^2+c_2^2}}$$

eg Relation b/w dc's of 2 lines given & find constraints to satisfy conditions like

dc's given $ul + vm + wn = 0$ & $al^2 + bm^2 + cn^2 = 0$

$\perp$ if $u^2(b+c) + v^2(a+c) + w^2(a+b) = 0$

$\parallel$ if $\dfrac{u^2}{a} + \dfrac{v^2}{b} + \dfrac{w^2}{c} = 0$

To solve, eliminate either one of l, m or n & get quadratic eq^n

like $(aw^2 + cu^2)\left(\frac{l}{m}\right)^2 + 2cuv\left(\frac{l}{m}\right) + (bw^2 + cv^2) = 0$ —(1)

It's roots are $\dfrac{l_1}{m_1}$ & $\dfrac{l_2}{m_2}$

Product of roots = $\dfrac{l_1}{m_1} \cdot \dfrac{l_2}{m_2} = \dfrac{bw^2 + cv^2}{aw^2 + cu^2}$

we get $\dfrac{l_1 l_2}{bw^2 + cv^2} = \dfrac{m_1 m_2}{aw^2 + cu^2} = \dfrac{n_1 n_2}{bu^2 + av^2}$

for $\perp$, $\Sigma l_1 l_2 = 0$; for $\parallel$ roots of eq^n (1) are equal so put $D = b^2 - 4ac = 0$

7> 3 concurrent lines are co-planar if $\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0$

Also $\vec{a} \cdot [\vec{b} \times \vec{c}] = 0$ [using common normal condition]

REMEMBER: $a_1 x + b_1 y + c_1 z = 0$

$a_2 x + b_2 y + c_2 z = 0$

$$\begin{matrix} a_1 & & b_1 & & c_1 & & a_1 \\ & \times & & \times & & \times & \\ a_2 & & b_2 & & c_2 & & a_2 \end{matrix}$$

for z, for x, for y.

By Cramer's Rule: $\dfrac{x}{b_1 c_2 - b_2 c_1} = \dfrac{-y}{a_1 c_2 - a_2 c_1} = \dfrac{z}{a_1 b_2 - a_2 b_1}$

8) Projection of a point & segment P on a line

$P'Q' =$ Projection of $\vec{PQ}$ on $\vec{AB} = |PQ| \cos\theta$ where θ is angle b/w $\vec{PQ}$ & $\vec{AB}$

$P = (x_1, y_1, z_1)$ $Q = (x_2, y_2, z_2)$ $\vec{AB}$ has dc's (l, m, n)

$\therefore$ $P'Q' = l(x_1 - x_2) + m(y_1 - y_2) + n(z_1 - z_2)$

⇒ The Plane

* Line joining any 2 pts on the surface lies on the surface

1) General form : $ax + by + cz + d = 0$ $[\because a^2+b^2+c^2 \neq 0]$

One-point form : $a(x-x_1) + b(y-y_1) + c(z-z_1) = 0$

Intercept form : $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ — $A(a,0,0)$, $B(0,b,0)$, $C(0,0,c)$ are intercepts on coordinate axes.

Normal form : $lx + my + nz = p$ — (l,m,n) are dc's of ⊥ from O to plane & 'p' is the length of the ⊥

↳ $(\text{coeff } x^2) + (\text{coeff } y^2) + (\text{coeff } z^2) = 1$

& RHS = constant > 0

Reduction from General to Normal form :

$Ax + By + Cz + D = 0 \Rightarrow \frac{A}{\Sigma A^2}x + \frac{B}{\Sigma A^2}y + \frac{C}{\Sigma A^2}z + \frac{D}{\sqrt{\Sigma A^2}} = 0$

so dr's of normal to plane are (coeff of x, coeff y, coeff z)

2) Angle b/w planes : $a_1x + b_1y + c_1z + d_1 = 0$
$a_2x + b_2y + c_2z + d_2 = 0$

$$\cos\theta = \Sigma a_1a_2 / (\sqrt{\Sigma a_1^2}) \cdot (\sqrt{\Sigma a_2^2})$$

$$\sin\theta = \pm \frac{\sqrt{\Sigma (a_1b_2 - b_1a_2)^2}}{\sqrt{\Sigma a_1^2}\sqrt{\Sigma a_2^2}}$$

⊥ planes : $a_1a_2 + b_1b_2 + c_1c_2 = 0$

∥ planes : $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

3) Plane through 3 given pts :

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

For 4 co-planar pts, above determinant is 0.

4) Particular Planes

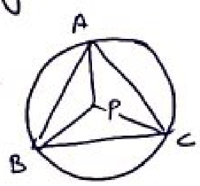
a) YZ-plane : $x=0$ XZ-plane : $y=0$ XY-plane : $z=0$

b) Plane $\perp$ to YZ-plane $By+Cz+D=0$
$\perp$ to XZ-plane $Ax+Cz+D=0$
$\perp$ to XY-plane $Ax+By+D=0$

c) Plane $\perp$ to $x=0$ $\Rightarrow$ $\parallel$ to YZ-plane = $x=a$

d) Plane $\perp$ to YZ-plane $\Rightarrow$, XZ & XY by putting $a=0, b=0, c=0$ in $ax+by+cz+d=0$ respectively

TRICK : Proving circumcentre :



1) $PA = PB = PC$
2) P, A, B, C are coplanar

5) Plane through intersection of 2 given planes : $P_1 + \lambda P_2 = 0$

6) 2 sides of a plane $P = ax+by+cz+d$
$A(x_1, y_1, z_1)$ $B = (x_2, y_2, z_2)$

a) $(ax_1+by_1+cz_1+d)$ & $(ax_2+by_2+cz_2+d)$ of ~~same~~ same sign then ~~opposite~~ same side

b) If opposite sign the ~~same~~ opposite side

ratio $k = -\dfrac{(ax_1+by_1+cz_1+d)}{(ax_2+by_2+cy_2+d)}$

7) Length of $\perp$ from pt to a plane : $dis = \dfrac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$

Distance b/w $\parallel$ planes : $d = \dfrac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}$

TRICK : change of coordinate axes with same origin, use the property that $\perp$ distance of plane from origin remains same.

REMEMBER: Equation of plane through $P(a,b,c)$ st OP is $\perp$ to plane

$$a(a-x)+b(b-y)+c(c-z)=0 \Rightarrow ax+by+cz=a^2+b^2+c^2$$

eg Plane $lx+my=0$ rotated about its line of intersection with $z=0$ by α ~~degrees~~.

New plane: $lx+my \pm z\sqrt{l^2+m^2}\tan\alpha=0$

Sol:- Let the new plane be $(lx+my)+\lambda(z)=0$

$\cos\alpha = \Sigma l_1 l_2$ $\quad \sin\alpha = \Sigma\sqrt{(l_1m_2-m_2l_1)^2}$, get λ in form of $\tan\alpha$.

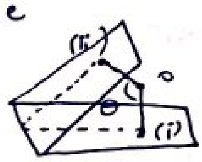
8> $\boxed{\text{Position of origin wrt angle b/w 2 planes}}$

Put planes in form $a_1x+b_1y+c_1z+d_1=0$
$a_2x+b_2y+c_2z+d_2=0$ $\quad$ d_1, d_2 both +ve

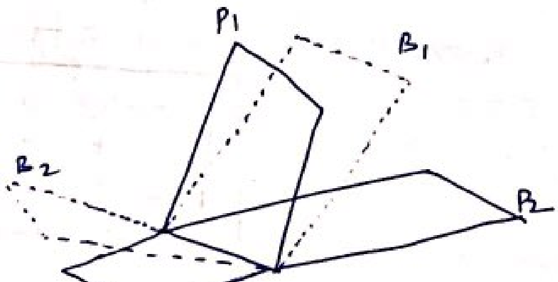
$a_1a_2+b_1b_2+c_1c_2<0 \Rightarrow$ origin in acute angle

$a_1a_2+b_1b_2+c_1c_2>0 \Rightarrow$ origin in obtuse angle

[$\because$ using the angle b/w normals from origin to the planes



9> $\boxed{\text{Angle Bisectors of 2 planes}}$



$P_1 = a_1x+b_1y+c_1z+d_1=0$
$P_2 = a_2x+b_2y+c_2z+d_2=0$

B_1 & B_2 are given by $\dfrac{P_1}{\Sigma a_1^2} = \pm\dfrac{P_2}{\Sigma a_2^2}$

* Find $\tan\theta$ b/w P_1 & B_1

i> $\tan\theta<1 \Rightarrow \theta<45^\circ \Rightarrow B_1$ is acute angle bisector

ii> $\tan\theta>1 \Rightarrow \theta>45^\circ \Rightarrow B_1$ is obtuse angle bisector

* If $d_1, d_2>0$, then $\dfrac{P_1}{\Sigma a_1^2} = \dfrac{P_2}{\Sigma a_2^2}$ ~~contains~~ bisects the angle containing origin

10> Projection on a plane: $ax+by+cz+d=0$

i> Foot of $\perp$ from $P(x_1,y_1,z_1)$ on plane: (x_1+at, y_1+bt, z_1+ct)
Find t by putting in eq^n of plane

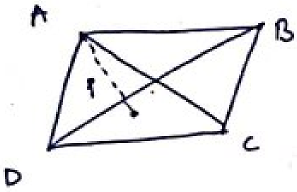
ii> Length of projection = $PQ\cos\theta$

iii> Area of projection = $S\cos\theta$

11) Area of Δ with vertices $(x_1,y_1)\ (x_2,y_2)\ (x_3,y_3) = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

Area of $\Delta = \sqrt{\Delta_x^2 + \Delta_y^2 + \Delta_z^2}$ where $\Delta_x, \Delta_y, \Delta_z$ are projections of Δ on yz, xz & xy planes respectively

12) Volume of tetrahedron (x_i, y_i, z_i) $i = 1$ to 4 $= \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$



$V = \frac{1}{3} P \times \Delta_{BCD}$

& use eq^n of BCD plane as $|\ | = 0$ form

13) $\boxed{\text{Equation for pair of planes :}}$

$$(a_1x + b_1y + c_1z + d_1)(a_2x + b_2y + c_2z + d_2) = 0$$

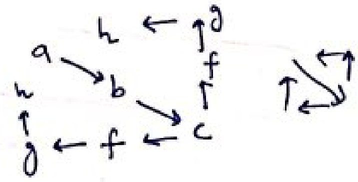
CONDITION : general homogenous equation of deg. 2 representing a pair of planes

EQ^n ⇒ $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$

CONDITION ⇒ $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

or $\boxed{\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0}$

TRICK :



* If θ is the angle, then $\cos\theta = \dfrac{a+b+c}{\sqrt{(a+b+c)^2 + 4(f^2+g^2+h^2-ab-bc-ca)}}$

$\tan\theta = \dfrac{2\sqrt{f^2+g^2+h^2-ab-bc-ca}}{a+b+c}$

$\perp$ iff $a+b+c = 0$

coincident iff $f^2 = bc$ $g^2 = ca$
$h^2 = ab$

TRICKS : i) If origin, length & angle are given assume $l\ m\ n$ & proceed
eg length ~~from~~ ~~orig~~ of sides of tetrahedron & angle b/w them is given
Let the vertex be $(0,0,0), (l_1a, m_1a, n_1a), (l_2b, m_2b, n_2b), (l_3c, m_3c, n_3c)$

If $D = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$ then $D^2 = \begin{vmatrix} \Sigma l_1^2 & \Sigma l_1 l_2 & \Sigma l_1 l_3 \\ \Sigma l_2 l_1 & \Sigma l_2^2 & \Sigma l_2 l_3 \\ \Sigma l_3 l_1 & \Sigma l_3 l_2 & \Sigma l_3^2 \end{vmatrix}$

⇒ The Straight Line :

* General equations of 1st degree taken together represent a line

1) General form : $\Pi_1 = 0 \;,\; \Pi_2 = 0$

Symmetrical form : $\dfrac{x-x_1}{l} = \dfrac{y-y_1}{m} = \dfrac{z-z_1}{n}$ → passing through (x_1, y_1, z_1) with dc's (l, m, n)

Parametric form : $x = x_1 + lr \quad y = y_1 + mr \quad z = z_1 + nr$; r is parameter

Two point form : $\dfrac{x-x_1}{x_2-x_1} = \dfrac{y-y_1}{y_2-y_1} = \dfrac{z-z_1}{z_2-z_1}$

2) Image of a pt in a plane : Find foot of the ⊥ & use midpt formula

3) Unsymmetrical to symmetrical form :

a) Find dr's using Cramer's rule

b) Put either one of x, y, z as 0 & solve eqn to get other 2 to find one pt on the line

$ax + by + cz + d = 0$ & $a_1x + b_1y + c_1z + d_1 = 0$

line ⇒ $\dfrac{x - \left(\dfrac{bd_1 - b_1d}{ab_1 - a_1b}\right)}{bc_1 - b_1c} = \dfrac{y - \left(\dfrac{a_1d - ad_1}{ab_1 - a_1b}\right)}{ca_1 - a_1c} = \dfrac{z - 0}{ab_1 - a_1b}$

4) Angle b/w line & plane = complementary of angle b/w line & normal

$\sin\theta = \dfrac{al + bm + cn}{\Sigma l^2 \; \Sigma a^2}$

line : $\dfrac{x-x_1}{l} = \dfrac{y-y_1}{m} = \dfrac{z-z_1}{n}$

plane : $ax + by + cz + d = 0$

a) line is ∥ ⇒ $al + bm + cn = 0$

b) line is ⊥ ⇒ $\dfrac{l}{a} = \dfrac{m}{b} = \dfrac{n}{c}$

c) line is ∥ & doesn't lie on the plane ⇒ $al + bm + cn = 0$, $ax_1 + by_1 + cz_1 + d \neq 0$

line lies on the plane ⇒ $al + bm + cn = 0$, $ax_1 + by_1 + cz_1 + d = 0$

eg. 1 Plane passing through (x_1, y_1, z_1) & $\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}$

$A(x-x_1)+B(y-y_1)+C(z-z_1)=0$
$A(a-x_1)+B(b-y_1)+C(c-z_1)=0$
$Al + Bm + Cn = 0$

$\Rightarrow \begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ a-x_1 & b-y_1 & c-z_1 \\ l & m & n \end{vmatrix} = 0$

eg. 2 Find Equation to the line through (f, g, h) which is parallel to $lx+my+nz=0$ & intersects line $ax+by+cz+d=0=a'x+b'y+c'z+d'$

Plane parallel to $lx+my+nz=0$ & passing through (f, g, h)

$\Rightarrow \quad l(x-f)+m(y-g)+n(z-h)=0 \quad -(1)$

Also any plane through 2nd line is $(ax+by+cz+d)+\lambda(a'x+b'y+c'z+d')=0$

If it goes through $(f, g, h) \Rightarrow \lambda = -\frac{\Sigma af}{\Sigma a'f}$

$\therefore$ plane is $\frac{\Sigma ax}{\Sigma af} = \frac{\Sigma a'x}{\Sigma a'f} \quad -(2)$

$\therefore$ line is (1) and (2)

6) Coplanar Lines | Intersecting Lines :

a) Lines : $\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$

$\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$

Coplanar if $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$

Eq^n of plane : $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$

$\Rightarrow$ Coplanar $\Leftrightarrow$ (intersect or parallel)

Working : (i) Write coordinates in parametric form
(ii) Use any 2 coordinates to get r_1 & r_2
(iii) Show r_1 & r_2 satisfy 3rd coordinate also

b) Lines : $\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1} \quad -(1)$ & $ax+by+cz+d=0=a'x+b'y+c'z+d' \quad -(2)$

Eq^n of plane containing line (2) $\Rightarrow \Sigma ax + \lambda\, \Sigma a'x = 0 \quad -(3)$

If (1) lies on (3) $\Rightarrow l_1(a+\lambda a')+m_1(b+\lambda b')+n_1(c+\lambda c')=0$
and $\Sigma ax_1 + \lambda \Sigma a'x_1 = 0$

$\Rightarrow \frac{\Sigma ax_1}{\Sigma a'x_1} = \frac{\Sigma la}{\Sigma la'} \Rightarrow \frac{\Sigma ax_1}{\Sigma al_1} = \frac{\Sigma a'x_1}{\Sigma a'l_1}$

c) Lines: $L_1 = \pi_1 = 0 = \pi_2$, $L_2 = \pi_3 = 0 = \pi_4$ Condition = $\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = 0$

c) Symmetrical to General form.

$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$. $\Rightarrow$ pick any 2 & expand

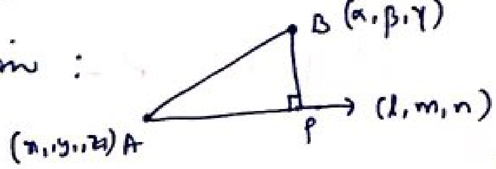
$\therefore$ $m(x-x_1) = l(y-y_1)$

$n(y-y_1) = m(z-z_1)$

$\Rightarrow$ $x = ay + b$ $y = cz + d$ $\Rightarrow$ 4 arbitrary constants

7) $\boxed{\perp \text{ distance of pt } (\alpha, \beta, \gamma) \text{ to line } \frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}}$

a) By projection:



$BP = \sqrt{AB^2 - AP^2}$ $AB^2 = (x_1-\alpha)^2 + (y_1-\beta)^2 + (z_1-\gamma)^2$

$AP = l(x_1-\alpha) + m(y_1-\beta) + n(z_1-\gamma)$

b) Let $P = (x_1 + lr, y_1 + mr, z_1 + nr)$

Now $BP \perp AP \Rightarrow$ $l.(x_1 + lr - \alpha) + m(y_1 + mr - \beta) + n(z_1 + nr - \gamma) = 0$

Find 'r' & use it to find P, BP & also eqn of BP.

8) $\boxed{\text{Any line intersecting 2 given lines}}$

$u_1 = 0, v_1 = 0$ & $u_2 = 0, v_2 = 0$

Remember x-axis is $y = 0 = z$

Reqd line is $\boxed{u_1 + k_1 v_1 = 0 = u_2 + k_2 v_2}$ k_1, k_2 are constt depending upon some constraints

If lines are in symmetrical form, get the coordinates in parametric form & satisfy the constraints to get the params

$(x_1 + l_1 r_1, y_1 + m_1 r_1, z_1 + n_1 r_1)$ & $(x_2 + l_2 r_2, y_2 + m_2 r_2, z_2 + n_2 r_2)$

9) $\boxed{\text{Shortest Distance b/w 2 skew lines}}$

a) Projection Method: (for symmetrical form)

$L_1: \frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$ & $L_2 = \frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$

a i) Find the dc's of $\perp^r$ using cramer's

ii) Take projection of (x_1, y_1, z_1) — (x_2, y_2, z_2) on the $\perp$

iii) For SD equation, get the eq^n of 2 planes

$$\Pi_1 = \begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_p & m_p & n_p \end{vmatrix} = 0. \quad \Pi_2 = \begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_2 & m_2 & n_2 \\ l_p & m_p & n_p \end{vmatrix} = 0$$

where l_p, m_p, n_p are dc's of $\perp^r$(SD)

iv) For points, get them in parametric form & ~~go~~ equate dr's with that of SD($\perp$)

b) One line in general form, other in symmetrical (Parallel Plane Method)

$L_1 = \Pi_1 = 0 = \Pi_2$

Find $\Pi_3 = \Pi_1 + \lambda \Pi_2$ s.t. Π_3 is $\parallel$ L_2

SD = $\perp$ from a point on L_2 to Π_3

Eq^n: – plane containing L_1 & $\perp$ to Π_3
– plane containing L_2 & $\perp$ to Π_3

c) Both in general form

$\Pi_1 = 0 = \Pi_2$ & $\Pi_3 = 0 = \Pi_4$

Find λ_1, λ_2 st $\Pi_1 + \lambda_1 \Pi_2$ & $\Pi_3 + \lambda_2 \Pi_4$ are parallel

SD = distance b/w $\parallel$ planes

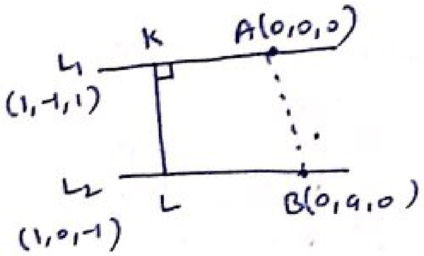
Eq^n: $\Pi_1 + \lambda_3 \Pi_2$ which is $\perp$ to $\Pi_1 + \lambda_1 \Pi_2$

& $\Pi_3 + \lambda_4 \Pi_4$ which is $\perp$ to $\Pi_3 + \lambda_2 \Pi_4$

eg. 1. Show that the SD b/w any 2 opposite edges of the tetrahedron formed by planes $y+z=0$, $x+z=0$, $x+y=0$, $x+y+z=a$ is $\frac{2a}{\sqrt{6}}$ and the 3 SD lines intersect at $x=y=z=-a$

Ans) $L_1 = x+y=0=y+z$ $\quad L_2 = x+z=0, x+y+z=a$

$L_1 = \frac{x}{1} = \frac{y}{-1} = \frac{z}{1}$ $\quad L_2 = \frac{x}{1} = \frac{y-a}{0} = \frac{z}{-1}$



Let dc's of KL be (l, m, n)

then $l(1)+m(-1)+n(1)=0$

$l+n=m$

& $l(1)+m(0)+n(-1)=0$

$l=n$

$\therefore$ KL dc's are $\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}$

$\therefore$ SD = projection of AB on KL $\Rightarrow \frac{0\times1}{\sqrt6} + \frac{a\times2}{\sqrt6} + \frac{0\times1}{\sqrt6} = \frac{2a}{\sqrt6}$

SD eq^n. is given by $\Pi_{AKL} = 0 = \Pi_{BLK}$

$$\Pi_{AKL} = \begin{vmatrix} x & y & z \\ 1 & -1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = x(-1-2)-y(1-1)+z(2+1)=0$$

$$x-z=0$$

$$\Pi_{BLK} = \begin{vmatrix} x & y-a & z \\ 1 & 0 & -1 \\ 1 & 2 & 1 \end{vmatrix} = x(2)-(y-a)(2)+z(2)=0$$

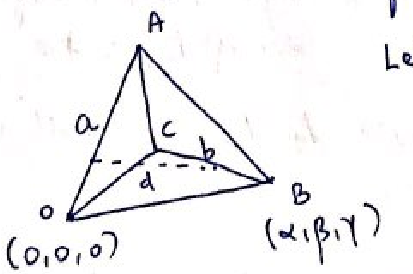
$$x+a-y+z=0$$

$$x-y+z+a=0$$

$\therefore$ SD is $x-z=0=x-y+z+a$ which is satisfied by $x=y=z=-a$

eg. 2. Lengths of opposite edges of tetrahedron are 'a' & 'b' & angle b/w them is θ ; SD is 'd'. Prove that

Vol of tetrahedron $= \frac{1}{6} abd \sin\theta$



Let dc of OA be $l_1, m_1, n_1 \Rightarrow A(l_1a, m_1a, n_1a)$

Let dc of BC be l_2, m_2, n_2 & B be (α, β, γ)

then C is $(\alpha + l_2 b, \beta + m_2 b, \gamma + n_2 b)$

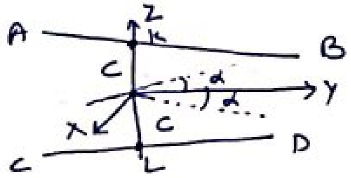
$\sin\theta = \sqrt{\Sigma(l_1m_2 - m_1l_2)^2}$

d = projection of OA on SD $\Rightarrow$ we get $d = \frac{1}{\sin\theta}\begin{vmatrix} \alpha & \beta & \gamma \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}$

$$\text{Vol} = \frac{1}{6}\begin{vmatrix} \text{Coord O} \\ \text{Coord A} \\ \text{Coord B} \\ \text{Coord C} \end{vmatrix} \Rightarrow \text{Volume} = \frac{1}{6} abd \sin\theta$$ [upon putting it gets simplified]

REMEMBER: By proper choice of axes, equations of any skew lines can be put in the form $y = x\tan\alpha$ & $z = c$
$y = -x\tan\alpha$ & $z = -c$

Take origin as midpt of SD, SD as z-axis. Draw line || to original lines & take X, Y axes as angle bisectors



dc's of AB $\Rightarrow (\sin\alpha, \cos\alpha, 0)$
CD $\Rightarrow$ - - - & so on.

10) $\boxed{\text{Locus of line intersecting 3 lines}}$: $u_1 = 0 = v_1$; $u_2 = 0 = v_2$; $u_3 = 0 = v_3$

Line should intersect $u_1 + k_1 v_1 = 0 = u_2 + k_2 v_2$; put it in 3rd line to get relation b/w k_1 & k_2 & then put $k_1 = \frac{-u_1}{v_1}$ & $k_2 = \frac{-u_2}{v_2}$

eg. $y = mx,\ z = c$; $y = -mx,\ z = -c$; $y = z,\ mx = -c$

we get $y - mx + k_1(z-c) = 0 = y + mx + k_2(z+c)$

Now put $y = z$ & $mx = -c$ $\Rightarrow$ we get $k_1 k_2 = +1$

$\therefore \frac{mx - y}{z - c} \cdot \frac{-(mx+y)}{(z+c)} = 1 \Rightarrow (m^2x^2 - y^2) + z^2 - c^2 = 0$

NOTE: Here instead of L_3 any other curve can be given also

11) $\boxed{\text{Intersection of 3 planes}}$: $\pi_i = a_ix + b_iy + c_iz + d_i = 0$ $i = 1, 2, 3$

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix}$$

get line of intersection of 2 planes (L_1) & see its behavior with 3rd plane

a) Single intersection pt : L_1 is not || to π_3 $\Rightarrow \Delta_4 \neq 0$

b) Prism : $L_1 \parallel \pi_3$, pt on L_1 doesn't lie on $\pi_3 \Rightarrow \Delta_4 = 0$ & any one of $\Delta_{1,2,3} \neq 0$

c) Intersection in line : L_1 lies in $\pi_3 \Rightarrow \Delta_1, \Delta_2, \Delta_3, \Delta_4$ all 0

TRICK: If $a^2 + b^2 + c^2 + 2abc = 1$
then $(ac+b)^2 = (1-a^2)(1-c^2)$
$(ab+c)^2 = (1-a^2)(1-b^2)$
$(a+bc)^2 = (1-b^2)(1-c^2)$.

⇒ The Sphere

* locus of a pt whose distance from a fixed pt is constant

1) standard form : $x^2+y^2+z^2=a^2$ $\quad c(0,0,0)$ $r=a$

central form : $(x-a)^2+(y-b)^2+(z-c)^2=r^2$ $\quad c(a,b,c)$ $r=r$

General form : $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ $\quad c(-u,-v,-w)$
Remember: (coeff are 1) $\quad r=\sqrt{u^2+v^2+w^2-d}$

conditions :
a) 2nd degree in x,y,z, &
b) coeff of x^2, y^2, z^2 are equal, &
c) No xy, yz, xz term

2) Four point form :

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ x_1^2+y_1^2+z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2+y_2^2+z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2+y_3^2+z_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2+y_4^2+z_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

3) Diameter form : $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$
[∵ using · product]

4) Eq^n of OABC, $O(0,0,0)$, $A(a,0,0)$, $B(0,b,0)$, $C(0,0,c)$ ⇒ $\boxed{x^2+y^2+z^2=ax+by+cz}$
⇓
Eq^n of sphere circumscribing the tetrahedron OABC

REMEMBER: Sphere inscribed in tetrahedron OABC ⇒ choose appropriate sign for distance of centre from plane ABC as inside the tetrahedron so pick the side with origin [make 'd' +ve in eq^n of plane & open | | with + sign - Pt#6 in 'Plane' notes]

TRICK: OA, OB, OC with dc's l_i, m_i, n_i ($i=1$ to 3) are mutually ⊥
OA = a OB = b OC = c Find eq^n of sphere OABC.

as it has origin : so $\quad x^2+y^2+z^2+ux+vy+wz=0$

$A(al_1, am_1, an_1)$
$B(bl_2, bm_2, bn_2)$
$C(cl_3, cm_3, cn_3)$

we get
$a+l_1u+m_1v+n_1w=0$ — (1)
$b+l_2u+m_2v+n_2w=0$ — (2)
$c+l_3u+m_3v+n_3w=0$ — (3)

do : (1)$\times l_1$ + (2)$\times l_2$ + (3)$\times l_3$ ⇒ $(al_1+bl_2+cl_3)+u=0$
& so on

Since they are mutually ⊥ ← Trick is $\quad l_1^2+l_2^2+l_3^2=1$ $\quad m_1^2+m_2^2+m_3^2=1$ $\quad n_1^2+n_2^2+n_3^2=1$
$l_1m_1+l_2m_2+l_3m_3 = l_1n_1+l_2n_2+l_3n_3 = n_1m_1+n_2m_2+n_3m_3=0$

6) Plane Section of a sphere is a circle

Eq^n of a circle : $S=0$ & $P=0$ taken together

$\rightarrow$ $x^2+y^2+z^2+2ux+2vy+2wz+d=0$

& $lx+my+nz=p$

Centre : foot of $\perp$ from sphere centre to the plane

Radius : $\sqrt{r^2-p^2}$ $r=$ radius of sphere

$p=$ $\perp$ distance of centre from plane

If centre of sphere lies on the plane then the circle is great circle

eg. Circle circumscribing $A(a,0,0)$, $B(0,b,0)$ & $C(0,0,c)$ is

$$\boxed{x^2+y^2+z^2-ax-by-cz=0 \quad \& \quad \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1}$$

REMEMBER: 4 points are concylic if circle through any 3 points passes through the fourth point.

eg. To show concylic points, get eq the plane ABC & the sphere OABC [using $x^2+y^2+z^2+2ux+2vy+2wz=0$]

If D lies on ABC & OABC, then it lies on the same circle.

Here for easy calculation, we choose $(0,0,0)$ to construct sphere

7) Locus Point of intersection of 2 spheres is a circle $\rightarrow S_1-S_2=0$

Set of spheres through the circle $S=0$ & $P=0$ is $S+\lambda P=0$ {λ is a param}

Set of spheres through the circle $S=0$ & $S'=0$ Intersection is $S+kS'=0$ {k is a param}

$\Rightarrow S+k(S-S')=0$

REMEMBER: For general eq^n of sphere through circle $x^2+y^2+2gx+2fy+c=0,\ z=0$

is $x^2+y^2+z^2+2gx+2fy+2\lambda z+c=0$

↳ don't forget this z^2, as S_1 represents cylinder alone

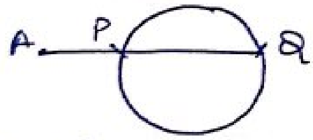
8> $\boxed{\text{Tangent plane}}$: $\perp$ distance from centre to the plane is equal to radius.

At pt (x_1, y_1, z_1) on sphere:

$T = xx_1 + yy_1 + zz_1 = a^2$

& $T = xx_1 + yy_1 + zz_1 + u(x+x_1) + v(y+y_1) + w(z+z_1) + d = 0$

9> $\boxed{\text{Power of a point wrt a sphere}}$



If from a pt A, lines are drawn in any direction to intersect the sphere at P & Q.

$AP \cdot AQ$ = constant = Power of pt $A(x_1, y_1, z_1)$

Power of (x_1, y_1, z_1) is $x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d = 0$
$\Rightarrow S_1 = 0$

10> REMEMBER: For problems involving chords & its midpts, the best way to proceed is take line as $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} = r$ where (x_1, y_1, z_1) is the mid pt. If the chord intersects the sphere for $r = r_1$ & r_2 we have $r_1 + r_2 = 0$

11> TRICKS: a> To prove tangent plane, show that $\perp^r$ distance = r

b> To find TPs || to a given plane, ($\pi_1 = lx + my + nz = p_1$)
choose π_2 as $lx + my + nz = p_2$ & satisfy $\perp$ dist = r

c> Sphere touching another sphere at a pt. A :
Find plane at A touching S_1 as P_1
Now $S_2 = S_1 + \lambda P_1$ Find λ as per the constraints.

eg $S_1 = x^2 + y^2 + z^2 - x + 3y + 2z - 3 = 0$ Find S_2 touching S_1 at $(1,1,-1)$ & passing through $(0,0,0)$

TP is $P_1 = x + 5y - 6 = 0$

Let $S_2 = S_1 + \lambda P_1 = x^2 + y^2 + z^2 - x + 3y + 2z - 3 + \lambda(x + 5y - 6) = 0$

We get $\lambda = -1/2$ ..

d> To find pt of contact of TP to S, find foot of $\perp$ from centre

e> Eq^n of tangent plane passing through line $L = \pi_1 = 0 = \pi_2$
Let plane be $\pi_1 + k\pi_2 = 0$ & put $\perp^r = r$
If L is in symmetrical form, convert to general form & continue as above

eg Find the locus of the centre of a sphere of constant radius which passes through a given pt & touches a given line.

Let the pt be $(0,0,c)$, line be X-axis $(y=0=z)$

& sphere be $x^2+y^2+z^2+2ux+2vy+2wz+d=0$

$$u^2+v^2+w^2-d=\lambda^2 \quad \text{— (I)}$$

Now putting $(0,0,c) \Rightarrow \quad c^2+2wc+d=0 \quad$ — (II)

Putting $y=0=z \Rightarrow \quad x^2+2ux+d=0 \quad$ & $(2u)^2-4(d)(1)=0$

$$\Rightarrow d=u^2 \quad \text{— (III)}$$

$\therefore \quad v^2+w^2=\lambda^2$

& $\quad u^2+c^2+2wc=0$

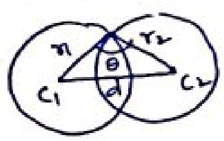
$\therefore$ Locus is the curve $\quad C_1 = y^2+z^2=\lambda^2 \quad$ & $C_2 = x^2+c^2-2cz=0$

12) Touching sphere : $\quad C_1C_2 = r_1+r_2$ (externally) $\quad$ or $\quad C_1C_2 = |r_1-r_2|$ (internally)

13) **Angle of intersection of 2 spheres** :

= Angle b/w the TPs at common point

= Angle b/w line joining common point to the centres.



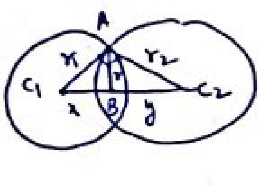
$$\boxed{d^2 = r_1^2+r_2^2-2r_1r_2\cos\theta}$$

$$\therefore \theta = \cos^{-1}\left(\frac{r_1^2+r_2^2-d^2}{2r_1r_2}\right)$$

Orthogonal spheres : $\quad \theta = \pi/2 \Rightarrow \boxed{r_1^2+r_2^2=d^2}$

Also $\Rightarrow \quad \boxed{2u_1u_2+2v_1v_2+2w_1w_2=d_1+d_2}$

eg. 2 spheres of radii r_1 and r_2 cut orthogonally. Show that radius of common circle is $\frac{r_1r_2}{\sqrt{r_1^2+r_2^2}}$



$\cos(\angle AC_1B) = \frac{x}{r_1} = \frac{r_1}{C_1C_2} \quad \cos(\angle AC_2B) = \frac{y}{r_2} = \frac{r_2}{C_1C_2}$

$\therefore \frac{x}{r_1^2} = \frac{y}{r_2^2} = \frac{1}{C_1C_2} \quad C_1C_2 = x+y \Rightarrow C_1C_2 = \sqrt{r_1^2+r_2^2}$

$\tan(\angle AC_1B) = \frac{r}{x} = \cot(\angle AC_2B) = \frac{y}{r} \Rightarrow r^2 = xy = \frac{r_1^2r_2^2}{C_1C_2^2} \Rightarrow r = \frac{r_1r_2}{C_1C_2} = \frac{r_1r_2}{\sqrt{r_1^2+r_2^2}}$

14.) **Length of tangent from a pt (x_1, y_1, z_1)** to the sphere $x_1^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 = S$

$\boxed{\text{Length} = \sqrt{\text{power of the pt}}} = \sqrt{S_1}$

$= \sqrt{x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d}$

use pythagoras $\boxed{d^2 = l^2 + r^2}$

NOTE: Coeff of sq. terms should be 1.

15.) **Radical plane of two spheres**

Locus of a pt whose powers wrt 2 spheres are equal

$\Rightarrow S_1 - S_2 = 0$

a) for 2 spheres $\Rightarrow$ $2(u_1 - u_2)x + 2(v_1 - v_2)y + 2(w_1 - w_2)z + (d_1 - d_2) = 0$

It is $\perp$ to line joining the centres

For intersecting spheres it is the plane of common circle.

b) for 3 spheres : $S_1 - S_2 = 0$ and $S_2 - S_3 = 0$

for 4 spheres : $S_1 - S_2 = 0$ and $S_2 - S_3 = 0$ and $S_3 - S_1 = 0$ and $S_2 - S_4 = 0$

16.) **Co-axial spheres**: System of spheres where any two spheres have the same radical plane

If 2 spheres are given, the system is $S_1 + \lambda S_2 = 0$ or $S_1 + \lambda(S_1 - S_2) = 0$

eg. Show that eq^n of a co-axial system of spheres can be put in the form $x^2 + y^2 + z^2 + \lambda x + d = 0$ λ is a param

TRICK Let S_1 & S_2 be the 2 spheres. Take line joining their centres as x-axis and the radical plane as yz-plane & proceed

17.) $\boxed{\text{Limiting points of a co-axial system}}$

$\Rightarrow$ centre of 2 spheres of the system with zero radius

For $x^2+y^2+z^2+2ux+d=0$

the limiting points are $(\sqrt{d},0,0)$, $(-\sqrt{d},0,0)$

18.) Every sphere which passes through the limiting points of a co-axial system cuts every other sphere of the system orthogonally.

$\Rightarrow$ Let the system be $x^2+y^2+z^2+2\lambda x+d=0$

& the sphere through limiting pts be $x^2+y^2+z^2+2ux+2vy+2wz+c=0$

We get $c=-d$ & $u=0$

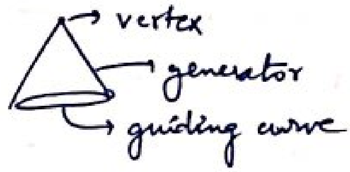
$\therefore$ sphere is $x^2+y^2+z^2+2vy+2wz-d=0$

Now use $2u_1u_2+2v_1v_2+2w_1w_2=d_1+d_2$

for orthogonality

⇒ Cone :

* surface generated by a ~~fixed point~~ straight line passing through a fixed pt & satisfying 1 or more conditions



1) Cone with second degree equation is called QUADRIC CONE

2) Eq^n of a cone with vertex $(0,0,0)$ is homogenous in x, y, z

Conversely, any homogenous eq^n in x, y, z represents a cone with vertex as origin

∴ $\boxed{\text{General Form: } ax^2+by^2+cz^2+2fyz+2gxz+2hxy=0}$

3) Method to make non-homogenous equation homogenous

a) Introduce appropriate powers of 't' to make eq^{n}s homogenous where t stands for 1 & then eliminate 't'

eg. $x^2+y^2+z^2-x-1=0=x^2+y^2+z^2+y-2$

$\Rightarrow x^2+y^2+z^2-tx-t^2=0$ (1) $\quad x^2+y^2+z^2+ty-2t^2=0$ (2)

(2) − (1) $\Rightarrow t(x+y)-t^2=0 \Rightarrow t=x+y$ as $t\neq 0$

$\therefore x^2+y^2+z^2-(x+y)x-(x+y)^2=0.$

$\Rightarrow x^2+3xy-z^2=0$

b) Direct manipulation ⇒ eg. $ax^2+by^2+cz^2=1$ & $lx+my+nz=p$

$\frac{lx+my+nz}{p}=1$ so $ax^2+by^2+cz^2=\frac{(lx+my+nz)^2}{p^2}$

4) $\boxed{\text{Equation of a cone with given vertex and guiding curve.}}$

$V=(\alpha,\beta,\gamma) \quad f(x,y)=ax^2+by^2+2hxy+2fy+2gx+c=0,\ z=0$

Let generator be $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ (1) $\Rightarrow$ for $z=0 \Rightarrow x_1=\alpha-\frac{l\gamma}{n}$, $y_1=\beta-\frac{m\gamma}{n}$

$\Rightarrow (x_1, y_1, 0)$ lies on the base conic

$\Rightarrow a\left[\alpha-\frac{l\gamma}{n}\right]^2+b\left[\beta-\frac{m\gamma}{n}\right]^2+2h\left[\alpha-\frac{l\gamma}{n}\right]\left[\beta-\frac{m\gamma}{n}\right]+2g\left[\alpha-\frac{l\gamma}{n}\right]+c=0$ (11)

Eliminate l, m, n in (11) using (1) $\left[\frac{l}{n}=\frac{x-\alpha}{z-\gamma},\ \frac{m}{n}=\frac{y-\beta}{z-\gamma}\right]$

REMEMBER : Rectangular Hyperbola in yz-plane has coeff of y^2 + coeff of $z^2 = 0$

5.) $\boxed{\text{Enveloping cone of a sphere}}$: Locus of the tangent from a given point to the sphere $\Rightarrow SS_1 = T^2$

STEP : $P(x_1, y_1, z_1)$ Let $Q(x, y, z)$ be a pt on the tangent & pt of contact divides PQ in $k:1$ ratio.

Put $b^2 - 4ac = 0$ in subsequent equation

6.) $\boxed{\text{Cone of second degree passing through axes}}$: $fyz + gzx + hxy = 0$

A cone can be found so as to contain any 2 given sets of 3 mutually perpendicular lines as generators.

$\Rightarrow$ pick one set as coordinate axes ; other as (l_i, m_i, n_i) $i = 1$ to 3

7.) CONDITION : General second degree equation represents a cone

EQ^n : $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$

$$\Rightarrow \boxed{\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} = 0}$$

STEPS : a) Make the equation homogenous by introducing 't' $= F(x, y, z, t) = 0$

b) Get 4 eq^n: F_x, F_y, F_z, F_t where $t = 1$

c) Solve any 3 of the 4 to get x, y, z

d) If x, y, z satisfies the remaining eq^n, then it's a cone with vertex (x, y, z)

8.) Angle b/w the 2 lines in which a plane through the vertex cuts a cone

If plane passes through origin & vertex of cone is origin

Let the lines be $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ Put x, y, z as l, m, n in both eq^n & find the 2 values of l, m, n.

If plane $\Rightarrow ux + vy + wz = 0$
& cone $= f(x,y,z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$

then $\tan\theta = \dfrac{2P\sqrt{u^2+v^2+w^2}}{(a+b+c)(u^2+v^2+w^2) - F(u,v,w)}$

where $P = \begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & 0 \end{vmatrix}$ & $F(u,v,w) = au^2 + bv^2 + cw^2 + 2fvw + 2gwu + 2huv$

9.) $\boxed{\text{Mutually perpendicular generators of a cone}}$

NECESSARY & SUFFICIENT CONDITION: $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 + 2ux + 2vy + 2wz + d$

$\Leftrightarrow \quad a+b+c = 0$

eg. PT $ax + by + cz = 0$ cuts the cone $yz + zx + xy = 0$ in $\perp$ lines if $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$

REMEMBER: ~~If~~ plane cuts the cone in $\perp$ lines, if, normal to the plane through the vertex lies on the cone

dr's of normal $\Rightarrow (a, b, c)$
$\Rightarrow ab + bc + ca = 0 \Rightarrow 1/a + 1/b + 1/c = 0$

10.) Tangent plane at the point (x_1, y_1, z_1) to the cone
$C = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$

The generator line $\dfrac{x-x_1}{l} = \dfrac{y-y_1}{m} = \dfrac{z-z_1}{n} = r$ lies on the cone

Put $(x_1 + lr, y_1 + mr, z_1 + nr)$ in $C \Rightarrow$ we get $r[r(\;) + 2(\;)] = 0$ ↳ this has to be 0 as $r \neq 0$

we get ~~$x\,fax$~~ $axx_1 + byy_1 + czz_1 + f(yz_1 + y_1z) + g(zx_1 + z_1x) + h(xy_1 + x_1y) = 0$

TRICK: To get tangent plane at (x_1, y_1, z_1) for any given surface do these changes:

$x^2 \rightarrow xx_1 \qquad y^2 \rightarrow yy_1 \qquad z^2 \rightarrow zz_1$

$xy \rightarrow \frac{1}{2}(x_1y + xy_1) \qquad yz \rightarrow \frac{1}{2}(y_1z + yz_1) \qquad zx \rightarrow \frac{1}{2}(z_1x + zx_1)$

$x \rightarrow \frac{1}{2}(x + x_1) \qquad y \rightarrow \frac{1}{2}(y + y_1) \qquad z \rightarrow \frac{1}{2}(z + z_1)$

* Tangent plane passes through the vertex & touches the cone along the generator at (x_1, y_1, z_1)

11) **Condition of tangency of a plane & cone**

$P \Rightarrow lx + my + nz = 0 \qquad C \Rightarrow ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$

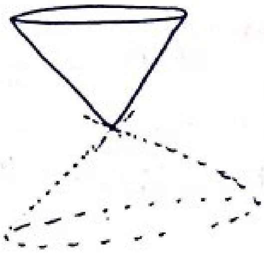
is $Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm = 0$

where A, B, C, F, G, H are cofactors of a, b, c, f, g, h

in $D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

12) **Reciprocal Cone**

The locus of normals to the tangent planes through vertex of the cone is another cone called Reciprocal cone



$C: \; ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$

$RC: \; Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=0$

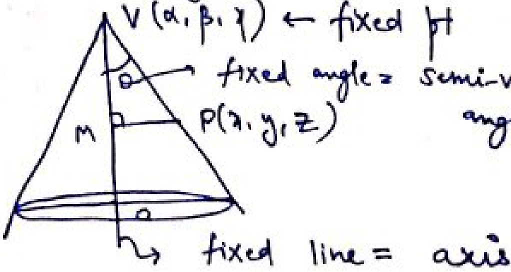
REMEMBER: for cone C to have 3 mutually $\perp$ tangent planes its RC should have 3 $\perp$ generators.

$\therefore \; A+B+C=0 \Rightarrow f^2+g^2+h^2 = ab+bc+ca$

eg. Eq^n for a cone touching 3 coordinate plane
$= RC$ of a cone through 3 axis
$= RC$ of $(fyz + gzx + hxy = 0)$
find A, B, C, F, G, H $\Rightarrow$ RC is $\sqrt{fx} \pm \sqrt{gy} \pm \sqrt{hz} = 0$

13) **Right Circular Cone**

surface generated by a straight line which passes through a fixed pt & makes a constant angle with a fixed line through the fixed point.



a) Standard form : $x^2+y^2 = z^2 \tan\alpha$

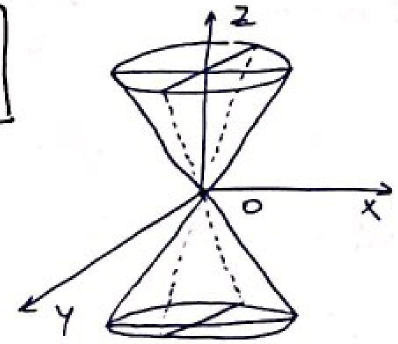
general form : $V(\alpha,\beta,\gamma)$, angle $= \theta$, dc's of axis = l, m, n

$$\Rightarrow \left[l(x-\alpha)+m(y-\beta)+n(z-\gamma)\right]^2 = \left[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2\right]\cos^2\theta$$

[$\because$ using trigonometry in ΔVPM]

* θ for a right circular cone with 3 $\perp$ generators is $\tan^{-1}\sqrt{2}$

14.) Simple form for cone $\Rightarrow$ $\boxed{\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0}$



Vertex is at origin

Generated by variable ellipse

$z=k,\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{k^2}{c^2}$

* Cone is a central conicoid with centre as vertex

Read general conicoid first then come here

a) Std form : $ax^2+by^2+cz^2=0$

i) TP at (x_1,y_1,z_1) = $axx_1+byy_1+czz_1=0$

ii) Polar plane of (x_1,y_1,z_1) = $axx_1+byy_1+czz_1=0$

iii) $lx+my+nz$ is TP if $\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=0$

iv) Plane which cuts the cone in a conic with centre (x_1,y_1,z_1) is $T=S_1$

v) Normal plane of the cone through generator (OP) $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ is the plane through OP & $\perp$ to TP at any pt of OP

$$\Rightarrow \left(\frac{b-c}{l}\right)x+\left(\frac{c-a}{m}\right)y+\left(\frac{a-b}{n}\right)z=0$$

eg. Lines are drawn $\perp$ from origin to NPs of $ax^2+by^2+cz^2=0$. Find their locus.

Let OP $x/l = y/m = z/n$ be a generator, then $al^2+bm^2+cn^2=0$

Also NP for OP $\rightarrow$ is from above. Normal becomes: $\frac{x}{\frac{b-c}{l}}=\frac{y}{\frac{c-a}{m}}=\frac{z}{\frac{a-b}{n}}$.

Put l, m, n as $\frac{b-c}{x}$, $\frac{c-a}{y}$ & $\frac{a-b}{z}$. HP.

REMEMBER: 2 lines given by $ul+vm+wn=0$ & $al^2+bm^2+cn^2=0$ are $\perp$ if $u^2(b+c)+v^2(c+a)+w^2(a+b)=0$

vi> locus of asymptotes drawn from origin to $ax^2+by^2+cz^2=1$ is the asymptotic cone $ax^2+by^2+cz^2=0$

Let it be $\frac{x}{L}=\frac{y}{M}=\frac{z}{n}=r \Rightarrow (\Sigma aL^2)=\frac{1}{r^2}$

But r is ∞ since asymptote meets at ∞ $\therefore \Sigma al^2=0$

$\therefore ax^2+by^2+cz^2=0$ is the locus

REMEMBER. eg. Any plane whose normal lies on the cone $(b+c)x^2+(c+a)y^2+(a+b)z^2=0$ cuts the surface $ax^2+by^2+cz^2=1$ in a rectangular hyperbola.

Sol. = Let the plane be $ux+vy+wz=0$ — (I)

& it cuts the surface in Rect HB.

Let asymptote of the hyperbola be $\frac{x}{L}=\frac{y}{m}=\frac{z}{n}$ — (II)

Since (II) lies on (I) $\therefore$ $ul+vm+wn=0$ — (III)

Also any pt on (II) is (lr, mr, nr). It lies on surface if $(\Sigma al^2)=1/r^2$ But $r\to\infty$ $\therefore \Sigma al^2=0$ — (IV)

From (III) & (IV) : $al^2+bm^2+cn^2=0$

$ul+vm+wn=0$

Since asymptotes of RHB are $\perp$

$\Rightarrow u^2(b+c)+v^2(c+a)+w^2(a+b)=0$

$\therefore$ Normal $\frac{x}{u}=\frac{y}{v}=\frac{z}{w}$ lies on

$(b+c)x^2+(c+a)y^2+(a+b)z^2=0$

⇒ Cylinder :

Surface generated by a variable line which is always parallel to a fixed line & intersects a given curve (or touches a given surface).

1) Generator ⇒ $\parallel$ to → $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ Base conic : $ax^2+by^2+2hxy+2gx+2fy+c=0$, $z=0$

Put $x = x - \frac{l}{n}z$ $y = y - \frac{mz}{n}$

If generator is $\parallel$ to z-axis cylinder ⇒ $ax^2+by^2+2hxy+2gx+2fy+c=0$ (free from z)

TRICK | STEP : To find eqⁿ of the cylinder whose generators are parallel to z-axis, then eliminate 'z' from the eqⁿs of the conic.

eg. Curve : $ax^2+by^2 = 2z$ & $lx+my+nz = p$ & generator $\parallel$ to z-axis

cylinder ⇒ $ax^2+by^2 = 2\frac{(p-lx-my)}{n}$.

2) **Enveloping cylinder of a sphere**

generators are parallel to $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ & touch $x^2+y^2+z^2=a^2$

Put (x_1+lr, y_1+mr, z_1+nr) in sphere with $b^2-4ac=0$

⇒ $(lx+my+nz)^2 = (l^2+m^2+n^2)(x^2+y^2+z^2-a^2)$

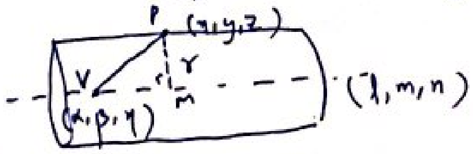
3) **Right circular cylinder**

guiding curve is a circle and generator is ⊥ to the plane of the circle.

a) standard form : $x^2+y^2=a^2$

b) general form : axis : $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ & radius = r

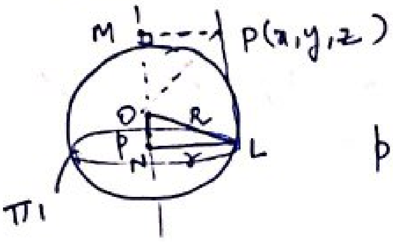
$$\Rightarrow (x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2 = \frac{[l(x-\alpha)+m(y-\beta)+n(z-\gamma)]^2}{l^2+m^2+n^2} + r^2$$



[∵ using pythagoras theorem in ΔVPM]

eg. Find the right circular cylinder whose guiding curve is the circle through $A(a,0,0)$, $B(0,b,0)$, $C(0,0,c)$. Also find the axis

Circle $C = x^2+y^2+z^2-ax-by-cz=0$ & $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \to \pi_1$



$$R^2 = \frac{a^2+b^2+c^2}{4}$$

$p = \perp$ distance of $C\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$ from π_1

$$= \left| \frac{\frac{3}{2}-1}{\sqrt{\Sigma a^{-2}}} \right| = \frac{1}{2\sqrt{\Sigma a^{-2}}}$$

$$\therefore r^2 = R^2 - p^2 = \frac{a^2+b^2+c^2}{4} - \frac{1}{4\Sigma a^{-2}}$$

In ΔOPM: $OP^2 = OM^2 + PM^2$

$$OP^2 = \left(x+\frac{a}{2}\right)^2 + \left(y+\frac{b}{2}\right)^2 + \left(z+\frac{c}{2}\right)^2$$

$$PM^2 = r^2 = \frac{a^2+b^2+c^2}{4} - \frac{1}{4\Sigma a^{-2}}$$

dr's of axis $= \left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$

$$\therefore OM = \frac{\frac{(x+a/2)}{a} + \frac{(y+b/2)}{b} + \frac{(z+c/2)}{c}}{\sqrt{\Sigma a^{-2}}}$$

$$\Rightarrow \frac{1}{\Sigma a^{-2}}\left[\left(\frac{x}{a}+\frac{1}{2}\right)^2 + \left(\frac{y}{b}+\frac{1}{2}\right)^2 + \left(\frac{z}{c}+\frac{1}{2}\right)^2\right] + \frac{a^2+b^2+c^2}{4} - \frac{1}{4\Sigma a^{-2}}$$
$$= \left(x+\frac{a}{2}\right)^2 + \left(y+\frac{b}{2}\right)^2 + \left(z+\frac{c}{2}\right)^2$$

Axis $= \frac{x-a/2}{1/a} = \frac{y-b/2}{1/b} = \frac{z-c/2}{1/c}$.

$$\to (x^2+ax)\left(\frac{1}{b^2}+\frac{1}{c^2}\right) + (y^2+by)\left(\frac{1}{c^2}+\frac{1}{a^2}\right) + (z^2+cz)\left(\frac{1}{a^2}+\frac{1}{b^2}\right) = \frac{1}{2}$$

⇒ The Conicoid :

* surface whose equation is of the second degree in x, y, z

⇒ $$\boxed{ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0}$$

1) There are 9 independent constants (divide by a), so 9 conditions can help in determining the conicoid

2) **Standard Forms** (can be achieved by suitable axes' transformation)

a) $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ Ellipsoid

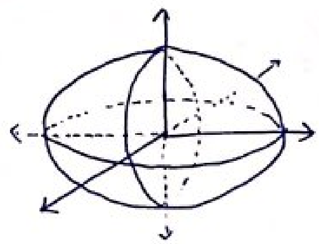
b) $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ Hyperboloid of 1 sheet

c) $\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ Hyperboloid of 2 sheet

d) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2z}{c}$ Elliptic paraboloid

e) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{2z}{c}$ Hyperbolic paraboloid

3) $$\boxed{\text{Ellipsoid : } \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1}$$



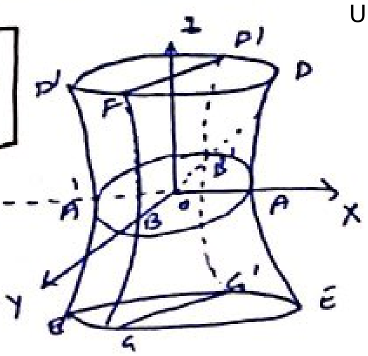
a) Origin is centre since it bisects any chord passing through it

b) Coordinate planes are principal planes as surface is symmetric about them.
Their intersection, ie, the coordinate axes are principal axes.
Coordinate planes bisect the chords ⊥ to them.

c) It is generated by ellipses of different sizes as planes ∥ to coordinate planes cut the surface in ellipses.

d) It is a closed surface as $|x|\le a$, $|y|\le b$ & $|z|\le c$

4.> $\boxed{\text{Hyperboloid of 1 sheet : } \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1}$



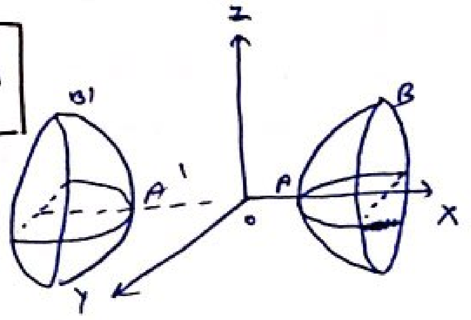
Center : origin

Principal plane : coordinate planes

Principal axes : coordinate axes

Made up of ellipses of varying sizes with no limits on $z=k$

5.> $\boxed{\text{Hyperboloid of 2 sheet : } \frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1}$



Center : origin

Principal plane : coordinate planes

Principal axes : coordinate axes

No portion b/w $|x|<a$

$x=k$ gives ellipses of diff^t sizes but real only when $|k|>a$

6.> Central Conicoid

A conicoid whose all chords through origin are bisected at origin $\Rightarrow$ $ax^2+by^2+cz^2=1$

a) Diameter : chord of a central conicoid passing through the centre.

b) Pt of intersection of line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ with $ax^2+by^2+cz^2=1$

Put $(x_1+lr,\ y_1+mr,\ z_1+nr)$ in the eq^n of conicoid.

If (l,m,n) are dc's, r_1 & r_2 are measures of distance from (x_1,y_1,z_1) to the pts of intersection

For diameter, $x_1=y_1=z_1=0$ & $r_1=r_2$

c) $\boxed{\text{Tangent plane}}$ at (x_1,y_1,z_1) put $(x_1+lr_1,\ y_1+mr_1,\ z_1+nr_1)$ in conicoid & do $\Sigma ax_1^2+1=0$ $\Rightarrow$ $r[r()+2()]=0$ $\Rightarrow$ coeff of r is 0

$\therefore$ Tangent Plane at (x_1, y_1, z_1) is $axx_1 + byy_1 + czz_1 = 1$

CONDITION FOR TANGENCY : $lx + my + nz = p$ is tangent

if $\boxed{\dfrac{l^2}{a} + \dfrac{m^2}{b} + \dfrac{n^2}{c} = p^2}$ & pt is $\left(\dfrac{l}{ap}, \dfrac{m}{bp}, \dfrac{n}{cp}\right)$

✱ REMEMBER : Always put conicoid in $ax^2+by^2+cz^2=1$ form & then only proceed.

Any tangent plane is of type : $lx + my + nz = \sqrt{\dfrac{l^2}{a} + \dfrac{m^2}{b} + \dfrac{n^2}{c}}$

d) Director sphere : Locus of the pt of intersection of 3 mutually $\perp$ tangent planes to $ax^2 + by^2 + cz^2 = 1$

$$\Rightarrow \boxed{x^2 + y^2 + z^2 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}}$$ [$\because$ square & add the 3 plane eq^{n}s]

eg. Find eq^n of the 2 tangent planes which contain the line $7x + 10y - 30 = 0 = 5y - 3z$ & touch

$7x^2 + 5y^2 + 3z^2 = 60$

Let the plane be $7x + 5(k+2)y - 3kz = 30$

Put the above condition for conicoid $\dfrac{7x^2}{60} + \dfrac{5}{60}y^2 + \dfrac{3}{60}z^2 = 1$

$$\therefore \frac{7^2}{7/60} + \frac{[5(k+2)]^2}{5/60} + \frac{(-3k)^2}{3/60} = (30)^2$$

$$\Rightarrow k = -1, -3/2$$

eg. If the line of intersection of 2 $\perp$ tangent planes to the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ passes through $(0,0,k)$, then st it lies on the cone $x^2(b^2+c^2-k^2) + y^2(c^2+a^2-k^2) + (z-k)^2(a^2+b^2) = 0$

Sol. Let the tangent planes be $l_1x + m_1y + n_1z = p_1$ & $l_2x + m_2y + n_2z = p_2$

Let the line be $\dfrac{x}{l_3} = \dfrac{y}{m_3} = \dfrac{z-k}{n_3}$. If line passes through $(0,0,k)$ then so do the planes $\therefore$ $p_1 = n_1k$ $p_2 = n_2k$

Also, $a^2l_1^2 + b^2m_1^2 + c^2n_1^2 = p_1^2$ & $a^2l_2^2 + b^2m_2^2 + c^2n_2^2 = p_2^2$

Adding $\Rightarrow$ $a^2(l_1^2 + l_2^2) + b^2(m_1^2 + m_2^2) = (k^2 - c^2)(n_1^2 + n_2^2)$ [$\because$ put $p_1 = n_1k$ & $p_2 = n_2k$]

Since l_1, l_2, l_3 are $\perp$ $\Rightarrow$ $l_1^2 + l_2^2 + l_3^2 = l_3^2 + m_3^2 + n_3^2$

We get $\Rightarrow$ $a^2(m_3^2 + n_3^2) + b^2(l_3^2 + n_3^2) + c^2(l_3^2 + m_3^2) - k^2(l_3^2 + m_3^2) = 0$

$\Rightarrow$ $l_3^2(b^2 + c^2 - k^2) + m_3^2(c^2 + a^2 - k^2) + n_3^2(a^2 + b^2) = 0$

Now put $l_3 = x$ $m_3 = y$ & $n_3 = (z-k)$ H.P.

REMEMBER: $lx + my = p$ ~~...~~ is tangent to ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

if $a^2 l^2 + b^2 m^2 = p^2$

e) $\boxed{\text{Normal}}$: line $\perp$ to the tangent plane through the pt of contact.

i) $\frac{x - x_1}{a x_1} = \frac{y - y_1}{b y_1} = \frac{z - z_1}{c z_1}$ (for $ax^2 + by^2 + cz^2 = 1$)

$\Rightarrow \frac{x - x_1}{a x_1 p} = \frac{y - y_1}{b y_1 p} = \frac{z - z_1}{c z_1 p}$ (in dc's form)

$p =$ $\perp$ distance from the centre to the TP.

(ii) From (α, β, γ), 6 normals can be drawn to ~~the~~ an ellipsoid

Foot of normal : $\boxed{\left(\frac{a^2 \alpha}{a^2 + \lambda}, \frac{b^2 \beta}{b^2 + \lambda}, \frac{c^2 \gamma}{c^2 + \lambda} \right)}$ ellipsoid is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$\left[\because \frac{\alpha - x_1}{x_1/a^2} = \frac{\beta - y_1}{y_1/b^2} = \frac{\gamma - z_1}{z_1/c^2} = \lambda \right]$ Put in, we get a λ^6 equation

(iii) Feet of the normal from (α, β, γ) are pt of intersection of ellipsoid with a cubic curve

For proof, consider an arbitrary plane & satisfy the above pt we get a cubic in λ eq^n.

(iv) The six normals lie on a cone of second degree

We have $x_1 = \frac{a^2 \alpha}{a^2 + \lambda}$ $y_1 = \frac{b^2 \beta}{b^2 + \lambda}$ $z_1 = \frac{c^2 \gamma}{c^2 + \lambda}$

Normal at (x_1, y_1, z_1) is $\frac{x - x_1}{p x_1 / a^2} = \frac{y - y_1}{p y_1 / b^2} = \frac{z - z_1}{p z_1 / c^2}$

which is also $\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n}$

$\therefore l = \frac{p x_1}{a^2} = \frac{p \alpha}{a^2 + \lambda}$ $m = \frac{p \beta}{b^2 + \lambda}$ $n = \frac{p \gamma}{c^2 + \lambda}$

$$(b^2-c^2)(a^2+\lambda) + (c^2-a^2)(b^2+\lambda) + (a^2-b^2)(c^2+\lambda)$$

$$= (b^2-c^2)\frac{\alpha p}{\lambda} + (c^2-a^2)\frac{\beta p}{m} + (a^2-b^2)\frac{\gamma p}{n}$$

$$0 = \frac{\alpha(b^2-c^2)}{\lambda} + \frac{\beta(c^2-a^2)}{m} + \frac{\gamma(a^2-b^2)}{n}$$

Put $\lambda = x-\alpha$ $m = y-\beta$ & $n = z-\gamma$.

(V) Feet of six normals lie on the cone

$$\frac{a^2(b^2-c^2)\alpha}{x} + \frac{b^2(c^2-a^2)\beta}{y} + \frac{c^2(a^2-b^2)\gamma}{z} = 0$$

where pt from which normal is drawn is (α, β, γ) &

ellipsoid is $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$

In above proof, continue in same way w/o replacing x_1, y_1, z_1.

REMEMBER: In the eq^n of cone through feet of six normals coeff of $x^2 = 0$, coeff of $y^2 = 0$ & coeff of $z^2 = 0$ & constant $= 0$

eg. If 3 feet pass through $lx + my + nz = p$, then other 3 pass through $x/a^2l + y/b^2m + z/c^2n + 1/p = 0$

Sol^n: Let plane be $l'x + m'y + n'z = p'$

Eq^n of conicoid through ellipsoid & pair of planes is

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right) + k(lx + my + nz - p)(l'x + m'y + n'z - p') = 0$$

Now put coeff of $x^2 = y^2 = z^2 = 0$ & constt of 0 [above cond]

$\Rightarrow$ $l' = -1/kla^2$ $m' = -1/kmb^2$ $n' = -1/knc^2$ $p' = 1/kp$

$$\Rightarrow P_2 = \frac{x}{a^2l} + \frac{y}{b^2m} + \frac{z}{c^2n} + \frac{1}{p} = 0.$$

7> $\boxed{\text{Polar plane of a point}}$

a) Harmonic Division → P, Q cut A & B in the same ratio internally & externally. They are Harmonic conjugates of AB.

b) Locus of the harmonic conjugate of a pt 'P' which cuts the ellipsoid in A & B is called polar plane of P.

→ locus of Q in PP

$\Rightarrow \quad axx_1 + byy_1 + czz_1 = 1$

$\Rightarrow \quad \underline{\underline{T = 0}}$ [$\underline{\text{same as tangent}}$]

Remember: A, B divides PQ in the same but opp ratio
∴ get quad in k whose Sum = 0

c) Pole of a given plane $lx + my + nz = p$ wrt $ax^2 + by^2 + cz^2 = 1$ is $\left(\frac{l}{ap}, \frac{m}{bp}, \frac{n}{cp}\right)$

d) $\boxed{\text{Conjugate points}}$: their polar planes pass through each other

Such planes are conjugate planes.

$\Rightarrow \quad ax_1x_2 + by_1y_2 + cz_1z_2 = 1$

e) $\boxed{\text{Polar Lines}}$: 2 lines st. polar plane of any point on one line passes through the other line

Polar of line: $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ wrt $ax^2 + by^2 + cz^2 = 1$

$$\left.\begin{aligned} \Rightarrow \quad & axx_1 + byy_1 + czz_1 - 1 = 0 \\ \& \quad & alx + bmy + cnz = 0 \end{aligned}\right] \begin{aligned} & - T \text{ with } (x_1, y_1, z_1) \\ & - T \text{ with } (l, m, n) \text{ except constant} \end{aligned}$$

eg. Find the locus of straight lines drawn through a fixed pt (α, β, γ) at 90° to their polars wrt $ax^2+by^2+cz^2=1$

Let line be $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = L_1$

Polar of L_1 wrt conic is $\rightarrow$ $a\alpha x + b\beta y + c\gamma z = 1$
$alx + bmy + cnz = 0$

dr's of polar line will be $\frac{l'}{bc(n\beta - m\gamma)} = \frac{m'}{ca(l\gamma - n\alpha)} = \frac{n'}{ab(m\alpha - l\beta)}$

since L_1 is $\perp$ to polar

$lbc(n\beta - m\gamma) + mca(l\gamma - n\alpha) + nab(m\alpha - l\beta) = 0$

$\Rightarrow \alpha mna(b-c) + \beta nlb(c-a) + \gamma lmc(a-b) = 0$

$\Rightarrow \Sigma \frac{\alpha}{l}\left(\frac{1}{c} - \frac{1}{b}\right) = 0$

$\Rightarrow \Sigma \frac{\alpha}{(x-\alpha)}\left(\frac{1}{c} - \frac{1}{b}\right) = 0$.

REMEMBER: Polar of PQ is line of intersection of polar planes of P & Q

$\Rightarrow axx_1 + byy_1 + czz_1 = 1$ & $axx_2 + byy_2 + czz_2 = 1$

8) $\boxed{\text{Enveloping Cone}}$ from (x_1, y_1, z_1) to $ax^2+by^2+cz^2=1$

$\Rightarrow \boxed{SS_1 = T^2}$

$\Rightarrow (ax^2+by^2+cz^2-1)(ax_1^2+by_1^2+cz_1^2-1) = \left[(axx_1+byy_1+czz_1-1)^2\right]$

[$\because$ using (x_1,y_1,z_1) P — k — m 1 — Q (x,y,z) $M = \frac{P+kQ}{K+1}$ & $b^2-4ac=0$ in quadratic eqn of k^2]

9) $\boxed{\text{Enveloping cylinder}}$ of $ax^2+by^2+cz^2=1$ $\parallel$ to $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$

STEP: Treat (l, m, n) as a pt & get 't' & 's' for it

$t = alx + bmy + cnz$ [TP at (l,m,n) ignoring constant]

$S_1 = al^2 + bm^2 + cn^2$ [put (l,m,n) in std. conic eqn]

$S = ax^2+by^2+cz^2-1$

$\therefore$ cylinder $= \boxed{SS_1 = t^2}$

[$\because$ can also do it by taking x_1, y_1, z_1 as pt on generator & satisfy x_1+lr, y_1+mr, z_1+nr in conicoid for 2 equal roots of r.]

10) REMEMBER: In 2-D for $ax^2 + by^2 + 2hxy + ux + vy + d = 0$

a) for circle: coeff x^2 = coeff y^2 ⇒ $a = b$

and coeff of xy ⇒ $h = 0$

b) for parabola: $ab = h^2$

11) $\boxed{\text{Sections with a given centre}}$:

a) Locus of chords of the conicoid which are bisected at the given point (ie, their centre is given) [Centre ⇒ chords are bisected]

$$\underline{T = S_1}$$

$$\Pi_1 \Rightarrow \quad axx_1 + byy_1 + czz_1 = ax_1^2 + by_1^2 + cz_1^2$$

REMEMBER: Π_1 meets the given conicoid in a conic with centre (x_1, y_1, z_1)

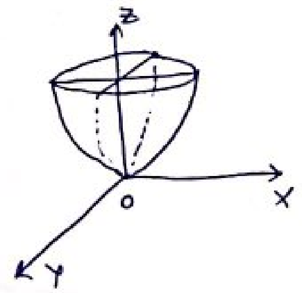
b) Locus of middle pts of a system of chords of the conicoid parallel to $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$

$$\Rightarrow \quad \underline{alx + bmy + cnz = 0}$$

⇒ plane through centre of the conicoid

12) $\boxed{\text{PARABOLOID}}$.

a) Elliptic Paraboloid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{2z}{c}$



Generated by ellipse

$z = k \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{2k}{c}$

b) Hyperbolic Paraboloid $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{2z}{c}$

section by coordinate planes ⇒ upward & downward parabolas

generated by hyperbolas

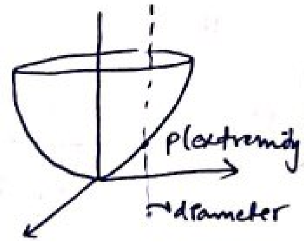
$z = k, \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{2k}{c}$

c) General Eqⁿ of paraboloid : $ax^2+by^2=2z$

i) a, b same sign ⇒ elliptic
opp sign ⇒ hyperbolic

ii) Every line meets it in 2 points, i.e, every plane section of a paraboloid is a conic

d) Line ∥ to z-axis is called Diameter which cuts paraboloid at a finite pt (called extremity) & at ∞.



Diameter ⊥ to the TP at its ∞ extremity is called the axis & that extremity is the vertex

e) i) TP at (x_1, y_1, z_1) : $axx_1 + byy_1 = z+z_1$

ii) Condition for $lx+my+nz=p$ to be a TP

$$\frac{l^2}{a} + \frac{m^2}{b} = -2np.$$

& pt of contact $\left(\frac{-l}{an}, \frac{-m}{bn}, \frac{-p}{n}\right)$

iii) Polar plane for (x_1, y_1, z_1) : $axx_1 + byy_1 = z+z_1$

iv) Enveloping Cone : $SS_1 = T^2$

v) Plane section for given centre : $S_1 = T_1$

TRICK eg : Finding locus of points from which 3 mutually ⊥ tangents can be drawn to a surface S.

⇒ Find the enveloping cone $SS_1 = T^2$ & put the condition that coeff of $x^2+y^2+z^2=0$

vi) From given pt (α, β, γ), 5 normals can be drawn with feet $x = \frac{a^2\alpha}{a^2+\lambda}$ $y = \frac{b^2\beta}{b^2+\lambda}$ $z = \gamma+\lambda$

13.) $\boxed{\text{CONJUGATE DIAMETERS}}$:

a) Locus of mid pts of chords to $ax^2+by^2+cz^2=1$ with dr's l, m, n is $alx + bmy + cnz = 0$

This plane is diametral plane conjugate to l, m, n

For $Ax+By+Cz=0$ dr's are $\frac{al}{A} = \frac{bm}{B} = \frac{cn}{C}$ ⇒ Every central plane is diametral to certain direcⁿ

Following is meant for only ellipsoids

b) Diametral plane of OP: If P is any pt on ellipsoid, then plane joining midpts of chords ∥ to OP is dplane of OP.

c) Conjugate semi-diameters: Any 3 semi-diameter are conjugate if the plane containing any 2 of them is diametral to 3rd

d) Conjugate planes: Any 3 diametral planes if intersection of any 2 of them gives a line whose diametral plane is the 3rd.

e) Relations b/w extremities of 3 conjugate semi-diameters for ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ & extremities (x_i, y_i, z_i) $i=1,2,3$

TRICK: $\left(\frac{x_i}{a}, \frac{y_i}{b}, \frac{z_i}{c}\right)$ act like dc's of 3 mutually ⊥ lines.

i) $\Sigma \frac{x_i^2}{a^2}=1$ ii) $\frac{x_1^2}{a^2}+\frac{x_2^2}{a^2}+\frac{x_3^2}{a^2}=1 \Rightarrow x_1^2+x_2^2+x_3^2=a^2$

Also $y_1^2+y_2^2+y_3^2=b^2$

$z_1^2+z_2^2+z_3^2=c^2$

$\frac{x_i^2}{a^2}+\frac{y_i^2}{b^2}+\frac{z_i^2}{c^2}=1$

iii) $\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}+\frac{z_1z_2}{c^2}=0$ iv) $\frac{x_1y_1}{ab}+\frac{x_2y_2}{ab}+\frac{x_3y_3}{ab}=0 \Rightarrow x_1y_1+x_2y_2+x_3y_3=0$

Also $\Sigma y_iz_i = \Sigma x_iz_i = 0$

* We do not know if the 3 semidiameters are ⊥ though

REMEMBER: Volume of parallelopiped of OP, OQ, QR = 6 × volume of tetrahedron OP, QR

eg. (i) sum of squares of 3 conjugate SD's of an ellipsoid $= a^2+b^2+c^2=$ constant

(ii) ——"—— projections of ——"—— on any line or plane is constant

(iii) Equation of plane through extremities

$lx+my+nz=p$ put (x_i, y_i, z_i), multiply by $x_i|y_i|z_i$ & add

$$\Rightarrow \frac{x}{a^2}(x_1+x_2+x_3)+\frac{y}{b^2}(y_1+y_2+y_3)+\frac{z}{c^2}(z_1+z_2+z_3)=1$$

14.) $\boxed{\text{GENERATING LINES OF CONICOIDS}}$ lie completely on the surface

a) Hyperboloid of 1 sheet : $\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

2 system of lines $\rightarrow$ $\frac{x}{a} - \frac{z}{c} = \lambda\left(1 - \frac{y}{b}\right)$ & $\frac{x}{a} + \frac{z}{c} = \frac{1}{\lambda}\left(1 + \frac{y}{b}\right)$

$\hookrightarrow$ $\frac{x}{a} + \frac{z}{c} = \mu\left(1 - \frac{y}{b}\right)$ & $\frac{x}{a} - \frac{z}{c} = \frac{1}{\mu}\left(1 + \frac{y}{b}\right)$

b) Developable Surface : consecutive generators intersect. eg cone - vertex, cylinder - whole length)

Skew surface : consecutive generators do not intersect eg. Hyperbolic paraboloid

c) Properties of generators above (for hyperboloid) :

i) one generator of each system passes through every point of the hyperboloid

ii) No 2 generators of the same system intersect.

iii) Any 2 generators of different systems intersect

Parametric Eq^n : $\boxed{x = \frac{a(1+\lambda\mu)}{\lambda+\mu} \quad y = \frac{b(\lambda-\mu)}{\lambda+\mu} \quad z = \frac{c(1-\lambda\mu)}{\lambda+\mu}}$

d) If 3 pts of $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ lies on conicoid $F(x,y,z)=0$ then whole line lies on the conicoid. [$\because$ 3 roots to quadratic in $r^2 \Rightarrow$ identity eq^n]

e) CONDITION : $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ is generator of $ax^2+by^2+cz^2=1$

i) $al^2 + bm^2 + cn^2 = 0$, &
ii) $al\alpha + bm\beta + cn\gamma = 0$, &
iii) $a\alpha^2 + b\beta^2 + c\gamma^2 - 1 = 0$

Put $(\alpha+lr, \beta+mr, \gamma+nr)$ & make coeff of r^2, r & constt as 0.

eg. CP, CQ are any 2 conjugate SD's of ellipse $x^2/a^2 + y^2/b^2 = 1, z=c$
CP', CQ' are the conjugate SD's of $x^2/a^2 + y^2/b^2 = 1, z=-c$ in direction of CP & CQ. Prove that $\frac{2x^2}{a^2} + \frac{2y^2}{b^2} - \frac{z^2}{c^2} = 1$ is generated by PQ' or P'Q

P, Q, P', Q' are $P(a\cos\theta, b\sin\theta, c)$, $Q(-a\sin\theta, b\cos\theta, c)$
$P'(a\cos\theta, b\sin\theta, -c)$, $Q'(-a\sin\theta, b\cos\theta, -c)$

PQ' $\Rightarrow$ $\frac{x-a\cos\theta}{-a(\sin\theta+\cos\theta)} = \frac{y-b\sin\theta}{b(\cos\theta-\sin\theta)} = \frac{z-c}{-2c} = r$ $\rightarrow$ Eliminate 'r', we get the above hyperboloid.

Point on elliptic section through generator

$$\frac{x-a\cos\alpha}{a\sin\alpha} = \frac{y-b\sin\alpha}{-b\cos\alpha} = \frac{z-0}{c}$$

$\downarrow \alpha = \alpha+\pi$

$P(a\cos\alpha, b\sin\alpha, c)$
$Q(-a\cos\alpha, -b\sin\alpha, c)$

$$\frac{x+a\cos\alpha}{-a\sin\alpha} = \frac{y+b\sin\alpha}{b\cos\alpha} = \frac{z}{c}$$

$\Rightarrow$ Reduction of conic

$D = \begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \Rightarrow \lambda_1, \lambda_2, \lambda_3$

$\boxed{\text{Principal directions}}$: $\rightarrow$ put λ_i in D. Rows give the ratio for $\perp$ lines eg $\begin{vmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \\ g_1 & h_1 & i_1 \end{vmatrix} \rightarrow$ solve $a_1 l + b_1 m + c_1 n = 0$, $d_1 l + e_1 m + f_1 n = 0$ to get l, m, n

In case of all rows being equal $\Rightarrow$ choose 1 l, m, n arbitrary satisfying $al + bm + cn$ and then take l_2, m_2, n_2 $\perp$ to (l_1, m_1, n_1) & satisfying

Principal plane : $\boxed{\lambda(lx + my + nz) + (ul + vm + wn) = 0}$ — ①

Transformed equation : $\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + 2x(\Sigma u l_1) + 2y(\Sigma u l_2) + 2z(\Sigma u l_3) + d = 0$

Centre : $F_x = 0 \quad F_y = 0 \quad F_z = 0 \Rightarrow [\alpha, \beta, \gamma]$

↳ Direct reduction : $d' = \alpha u + \beta v + \gamma w + d$

$\Downarrow$

$\boxed{\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + d' = 0}$ — ②

If $ul + vm + wn \neq 0 \rightarrow$ use ①

If $ul + vm + wn = 0 \Rightarrow$ Find centre & use — ②

If there are equal non-zero roots $\Rightarrow$ Quadric of Revolution

$\Downarrow$

Find l, m, n using the principal value of λ which is different.