PH 108: Electricity and Magnetism

Tutorial 2

1. Check the divergence theorem for the vector function $\boldsymbol{v} = r^2 \cos\theta\, \hat{r} + r^2 \cos\phi\, \hat{\theta} - r^2 \cos\theta \sin\phi\, \hat{\phi}$ using as the volume one octant of a sphere of radius *R*.
2. Compute the gradient and Laplacian of the function $T = r(\cos\theta + \sin\theta\cos\phi)$. Check the Laplacian by converting T to Cartesian co-ordinates. Check the gradient theorem for this function using the following path: along a semi-circular path in the XY plane from $(2,0,0)$ to $(0,2,0)$ and then a semi-circular path from $(0,2,0)$ to $(0,0,2)$ in the YZ plane.
3. Express the following derivatives in terms of the unit vectors $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$: $\frac{\partial \hat{\boldsymbol{r}}}{\partial \theta}, \frac{\partial \hat{\boldsymbol{r}}}{\partial \phi}, \frac{\partial \hat{\boldsymbol{\theta}}}{\partial \theta}, \frac{\partial \hat{\boldsymbol{\theta}}}{\partial \phi}, \frac{\partial \hat{\boldsymbol{\phi}}}{\partial \theta}$ and $\frac{\partial \hat{\boldsymbol{\phi}}}{\partial \phi}$
4. Compute the line integral of $\boldsymbol{v} = (r\cos^2\theta)\hat{r} - (r\cos\theta\sin\theta)\,\hat{\theta} + 3\,r\,\hat{\phi}$ along the flowing paths in sequence: (i) a semi-circular path in the XY plane from $(1,0,0)$ to $(0,1,0)$ (ii) straight line from $(0,1,0)$ to $(0,1,1)$ (iii) straight line joining $(0,1,1)$ to the origin (iv) straight line from origin to the starting point $(1,0,0)$. Do the calculation in spherical co-ordinates and check Stoke's theorem.
5. Imagine a sphere of radius R filled with negative charge of uniform density, the total charge being equivalent to that of two electrons. Imbed in this jelly two protons so that the total charge of the system is zero. Assume that the charge density is not altered because of the presence of the protons. Where must the protons be located so that the force on each on them is zero? This is a primitive model for the hydrogen molecule.
6. A sphere the size of a basketball is charged to a potential of -1000 V. About how many extra electrons are on it per square meter? ($Estimate$ the size of the basketball if you need it!)
7. Calculate the curl and divergence of the following vector functions. If the curl turns out to be zero, construct a scalar function ϕ of which the vector field is the gradient: (a) $F_x = x + y; F_y = -x + y; F_z = -2\,z$ (b) $G_x = 2\,y; G_y = 2x + 3z; G_z = 3y$ (c) $H_x = x^2 - z^2; H_y = 2; H_z = 2xz$

8. Describe the electric field and the charge distribution corresponding to the following potential: $V = x^2 + y^2 + z^2$ for $x^2 + y^2 + z^2 < a^2$ and $V = -a^2 + \frac{2a^3}{(x^2+y^2+z^2)^{\frac{1}{2}}}$ for $x^2 + y^2 + z^2 > a^2$.
9. Designate the corners of a square, 5 cm on a side, in clockwise order A, B, C, D. Put a charge 2 μC at A and -3 μC at B. Determine the value of the line integral of the electric field from point C to point D.
10. Show that the operator $\mathbf{L} = -i\, \mathbf{r} \times \boldsymbol{\nabla}$ satisfies $\mathbf{L} \times \mathbf{L}\, f = i\, \mathbf{L}\, f$ where f is an arbitrary test function. You can do this starting from the definition of the $\boldsymbol{\nabla}$ operator or by letting $\mathbf{E} = -\boldsymbol{\nabla} V$ and expanding $(\mathbf{r} \times \boldsymbol{\nabla}) \times (\mathbf{r} \times \mathbf{E})$ using the $\mathrm{BAC} - \mathrm{CAB}$ rule for vector triple products $\mathbf{A} \times \mathbf{B} \times \mathbf{C}$.
11. For the screened "Yukawa" potential $V(r) = \frac{qe^{-\mu r}}{r}$ find the electric field and the charge distribution that produces this potential.
12. Two long parallel wires, each with uniform linear charge density λ are a distance $\mathbf{a}$ apart, where $\mathbf{a}$ is a constant vector. Take the origin of co-ordinates as the mid-point between the two wires. Find the electric field in vector notation, using the two dimensional vectors $\mathbf{r}$ and $\mathbf{a}$. Write down the x and y components of $\mathbf{E}$, taking $\mathbf{a}$ in the x direction.
13. Two "semi-infinitely" long thin parallel wires, a distance $2b$ apart, are joined by a semi-circular wire piece of radius b placed at the top so as to form an (infinitely long) inverted "U" shape. The wire has uniform linear charge density λ. Find the electric field at the centre of the semi-circle.
14. Calculate the potential energy per ion for an infinite one dimensional ionic crystal, i.e, a row of equally spaced charges of magnitude e and alternating sign.
15. Compute the unit normal vector $\hat{\mathbf{n}}$ to the ellipsoidal surfaces defined by constant values of $\Phi(x, y, z) = V\left[\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right]$. What is $\hat{\mathbf{n}}$ when $a = b = c$?
16. The distributions of charge in different nuclei look very similar apart from a change of scale ,i.e, if $\rho_A(\mathbf{r})$ is the charge density of nucleus A and $\rho_B(\mathbf{r})$ the charge density of nucleus B, then the two are related by $\rho_B(\lambda\mathbf{r}) = \rho_A(\mathbf{r})$ where λ is a constant. Find the relation between the potentials $V_A(\mathbf{r})$ and $V_B(\mathbf{r})$ and between the electric fields $\mathbf{E}_A(\mathbf{r})$ and $\mathbf{E}_B(\mathbf{r})$.
17. Two point charges of strength 2 μC each, represented by black dots and two point charges of strength -1 μC, represented by white dots are symmetrically located in the xy plane as show in the figure, at $(0, \pm 2)$ and $(\pm 1, 0)$ respectively. Some of the equipotential curves are potted in the

figure. Find the value of the potential V on each of the curves A, B and C.