Linear Algebra

Santanu Dey

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Operations on matrices

- Set of all matrices of size $m \times n$ is denoted by $M_{m,n}$
- For $A = ((a_{ij})), B = ((b_{ij})) \in M_{m,n}$

$$A + B = ((a_{ij} + b_{ij})); \quad \alpha A = ((\alpha a_{ij}));$$
$$\mathbf{0} \text{ matrix with all entries } 0; \quad -A = ((-a_{ij}))$$

- $\mathbb{R}^n \longleftrightarrow M_{1,n}, \qquad$ similarly $\mathbb{R}^m \longleftrightarrow M_{m,1}$
- 'Transpose' operation introduced earlier can be extended to all matrices

$$A^T := ((b_{ij})) \in M_{n,m} \quad \text{where } b_{ij} := a_{ji}$$

- $(\alpha A + \beta B)^T = \alpha A^T + \beta B^T$
- Let $f : \mathbb{R}^n \to \mathbb{R}^m, g : \mathbb{R}^m \to \mathbb{R}^l$ be linear maps. If $A := \mathcal{M}_f$ and $B := \mathcal{M}_g$ then $\mathcal{M}_{g \circ f} = ?$

- We have

$$\begin{aligned}(g \circ f)(\mathbf{e}_j) &= g(\sum_i a_{ij}\mathbf{e}_i) = \sum_i a_{ij} g(\mathbf{e}_i) \\ &= \sum_i a_{ij}(\sum_k b_{ki}\mathbf{e}_k) = \sum_k (\sum_i b_{ki}a_{ij})\mathbf{e}_k\end{aligned}$$

- So $\mathcal{M}_{g \circ f} = C = ((c_{kj}))$, where $c_{kj} = \sum_{i=1}^{m} b_{ki}a_{ij}$
- Define

$$BA := C$$

- Properties:

1. Associativity: $A(BC) = (AB)C$ (if AB and BC are defined)
2. Right and Left Distributivity:
   $A(B + C) = AB + AC, \qquad (B + C)A = BA + CA$

3. Multiplicative identity: if $A \in M_{m,n}$, $B \in M_{n,k}$ then
$$AI_n = A \quad \text{and} \quad I_n B = B$$
4. $(AB)^T = B^T A^T$
5. Let $f : \mathbb{R}^n \to \mathbb{R}^m$, $g : \mathbb{R}^m \to \mathbb{R}^l$ be linear maps. Then $\mathcal{M}_{g \circ f} = \mathcal{M}_g \mathcal{M}_f$

Remark

$$M_{2,2} \longleftrightarrow \mathbb{R}^4$$

(preserves sum and scalar multiplication). Similarly

$$M_{m,n} \longleftrightarrow \mathbb{R}^{mn}$$

Definition

Any function $f : X \to Y$ is said to be invertible, if there exists $g : Y \to X$ such that

$$g \circ f = Id_X \quad \text{and} \quad f \circ g = Id_Y.$$

The inverse of a function if it exists is unique and is denoted by f^{-1}.

- A $n \times n$ matrix (i.e, square matrix) A is said to be *invertible* if there exists another $n \times n$ matrix B such that $AB = BA = I_n$. We call B an inverse of A.

  Remarks:

(i) An inverse of a matrix is unique. [if C is another matrix such that $CA = AC = I_n$ then

$$C = CI_n = C(AB) = (CA)B = I_nB = B.]$$

Denote it by A^{-1}.

(ii) If A_1, A_2 are invertible then so is $A_1 A_2$. What is its inverse?

(iii) Clearly I_n, and

$$diag(a_1, a_2, \ldots, a_n)$$

with $a_i \neq 0$ are invertible.

(iv) Let $B := A^{-1}$. If $f_A, f_B : \mathbb{R}^n \to \mathbb{R}^n$ are the linear maps associated with A, B resp., then it follows that

$$f_A \circ f_B = f_{AB} = Id.$$

Likewise $f_B \circ f_A = Id$. Even the converse holds. (viz, $\mathcal{M}_f \mathcal{M}_{f^{-1}} = I_n$).

(v) An invertible map is one-one and onto.

(vi) If $f : \mathbb{R}^n \to \mathbb{R}^n$ is an invertible linear map, then f^{-1} is linear.

Elementary and Permutation Matrix

- Consider a square matrix E_{ij} whose $(i,j)^{th}$ entry is 1 and all other entries are 0.
- If we multiply a matrix A by E_{ij} on the left, then what we get is a matrix whose i^{th} row is equal to the j^{th} row of A and all other rows are zero. In particular $E_{ij}E_{ij} = 0$ for $(i \neq j)$.
- It follows that for any α and $i \neq j$

$$(I + \alpha E_{ij})(I - \alpha E_{ij}) = I + \alpha E_{ij} - \alpha E_{ij} - \alpha^2 E_{ij}E_{ij} = I.$$

- $$(I + \alpha E_{ii})(I + \beta E_{ii}) = I + (\alpha + \beta + \alpha\beta)E_{ii}.$$

  So rhs to be equal to I then we must have $\alpha + \beta + \alpha\beta = 0$.
  Thus $I + \alpha E_{ii}$ is invertible if $\alpha \neq -1$.
  Alternatively, $I + \alpha E_{ii}$ is the diagonal matrix with all the diagonal entries equal to 1 except the $(i,i)^{th}$ one which is equal to $1 + \alpha$.

- Further consider $I + E_{ij} + E_{ji} - E_{ii} - E_{jj}$ which is similarly invertible. This matrix is nothing but the identity matrix after interchanging the i^{th} and j^{th} rows. These are called *transposition matrices.*
- The linear maps corresponding to them merely interchange the i^{th} and j^{th} coordinates. We shall denote them simply by T_{ij}. To sum up we have:

Theorem

The elementary matrices $I + \alpha E_{ij}\,(i \neq j),\;\; I + \alpha E_{ii}\,(\alpha \neq -1)$ *and* $T_{ij} = I + E_{ij} + E_{ji} - E_{ii} - E_{jj}$ *are all invertible with their respective inverses* $I - \alpha E_{ij},\;\; I - (\alpha/(1+\alpha))E_{ii}$ *and* T_{ij}.

- *Permutation matrices* are defined to be those square matrices which have all the entries in any given row (and column) equal to zero except one entry equal to 1.

- From a permutation matrix, for each $1 \leq i \leq n$ consider the i^{th}row. If $(i,j)^{th}$ entry is 1, then define $\sigma(i) = j$. Thus with $N := \{1, \ldots, n\}$ we can get a map $\sigma : N \to N$ which is one-to-one mapping.
- Conversely, given a permutation $\sigma : N \to N$, we define a matrix $P_\sigma = ((p_{ij}))$:

$$p_{ij} = \begin{cases} 0 & \text{if } j \neq \sigma(i) \\ 1 & \text{if } j = \sigma(i) \end{cases}$$

- A permutation matrix is obtained by merely shuffling the rows of the identity matrix (or by shuffling the columns)
- If A denotes a permutation matrix, then

$$AA^T = A^T A = I_n.$$

In particular, they are invertible.