Vector calculus

Gradient (vector)

For a scalar function φ = φ(x), $d\varphi = \underbrace{\left(\frac{d\varphi}{dx}\right)}_{\text{slope}} dx$

For φ = φ(x,y,z), $d\varphi = \left(\frac{\partial\varphi}{\partial x}\right) dx + \left(\frac{\partial\varphi}{\partial y}\right) dy + \left(\frac{\partial\varphi}{\partial z}\right) dz$

$$= \left(\frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k}\right) . \overbrace{(dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k})}^{\text{length element}}$$

$$= \underbrace{\left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)}_{\text{Vector operator DEL } \vec{\nabla}} \boldsymbol{\varphi} . (dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k})$$

$$= \underbrace{\vec{\nabla}\varphi}_{} . \vec{dl}$$

GRADIENT of scalar function φ which is a vector

$$dT = \vec{\nabla} T . \overrightarrow{dl}$$

$$dT = |\vec{\nabla} T| \, |\overrightarrow{dl}| \cos\theta$$ where θ is the angle between $\vec{\nabla} T$ and $\overrightarrow{dl}$

For a given *dl*, *dT* is maximum when θ = 0, ie, along the direction of ∇T

GRADIENT of a function points in the direction of max. increase of the function

$|\vec{\nabla} T|$ gives the slope along the maximum direction

Example: *r(x,y,z)* $\vec{\nabla} \mathbf{r}$? $\quad \vec{\nabla} \mathbf{r} = |\vec{\nabla} \mathbf{r}| \, |\overrightarrow{\mathbf{dl}}| \cos\theta$

Direction?

***r* increases fastest along the radial direction** $\Longrightarrow \hat{\mathbf{r}}$

dl = dr $\Longrightarrow |\vec{\nabla} \mathbf{r}| = 1$ $\qquad \vec{\nabla} r = \hat{r}$

Spherical coordinates
(r, θ, ϕ)

$$\vec{\nabla}T = \frac{\partial T}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\theta} + \frac{1}{r sin\theta}\frac{\partial T}{\partial \phi}\hat{\phi}$$

cylindrical coordinates
(ρ, ϕ, z)

$$\vec{\nabla}T = \frac{\partial T}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial T}{\partial \phi}\hat{\phi} + \frac{\partial T}{\partial z}\hat{z}$$

Divergence (scalar)

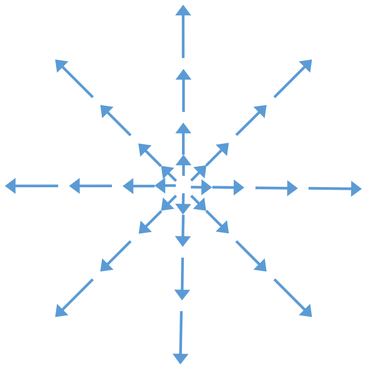
$\vec{\nabla}$ acts on a vector function through dot product

$$\vec{\nabla}.\vec{v} = \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right)\cdot\left(\hat{i}\,v_x + \hat{j}\,v_y + \hat{k}\,v_z\right)$$

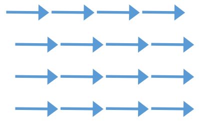
$$= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Physical meaning:

- **measures the spread out (divergence) of a vector at a given point**
- **net amount of flux through a given volume**



$$\vec{v} = \vec{r} = \left(\hat{i}\,x + \hat{j}\,y + \hat{k}\,z\right)$$



$$\vec{v} = v_0\hat{k}$$

$$\vec{\nabla}.\vec{v} = 0$$

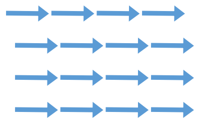
Spherical coordinates (r, θ, ϕ)	$$\vec{\nabla}T = \frac{\partial T}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\phi}$$ $$\vec{\nabla}\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\, v_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} v_\phi$$
cylindrical coordinates (ρ, ϕ, z)	$$\vec{\nabla}T = \frac{\partial T}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial T}{\partial \phi}\hat{\phi} + \frac{\partial T}{\partial z}\hat{z}$$ $$\vec{\nabla}\cdot\vec{v} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho v_\rho\right) + \frac{1}{\rho}\frac{\partial}{\partial \phi} v_\phi + \frac{\partial v_z}{\partial z}$$

Curl (vector)

$\vec{\nabla}$ acts on a vector function through cross product

$$\vec{\nabla} \times \vec{v} = \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right) \times \left(\hat{i}\,v_x + \hat{j}\,v_y + \hat{k}\,v_z\right)$$

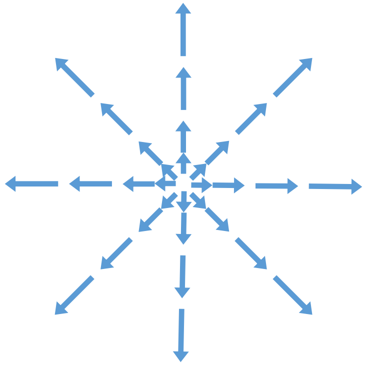
$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$$



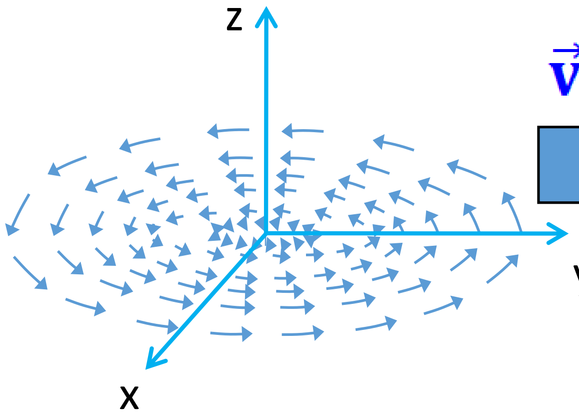
$$\vec{v} = v_0 \hat{k}$$
$$\vec{\nabla} \times \vec{v} = 0$$

Physical meaning:

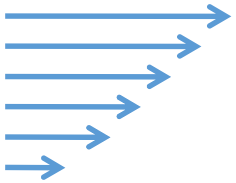
- **measures the circulation (curl) of a vector at a given point**



$$\vec{v} = \vec{r}$$
$$\vec{\nabla} \times \vec{v} = 0$$



$$\vec{v} = (-y\,\hat{i} + x\,\hat{j})$$



$$\vec{\nabla} \times \vec{v} \neq 0$$

Gradient Theorem

For a scalar function $\Theta(x,y,z)$ in the interval (r_a, r_b)

$$\int_{r_a}^{r_b} \vec{\nabla}\Theta \cdot \vec{\mathbf{dl}} = \Theta(\mathbf{r_b}) - \Theta(\mathbf{r_a}) \quad \Longleftarrow \quad \{\vec{\nabla}\varphi \, . \, \vec{\mathbf{dl}} = d\varphi\}$$

$\Rightarrow$ Depends only on the end points

$\Rightarrow$ Independent of path

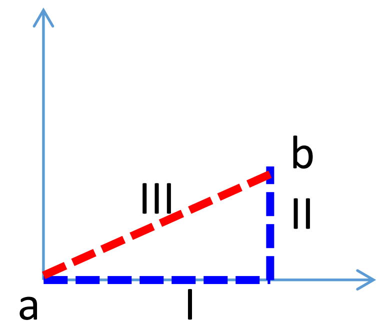
$\Rightarrow\Rightarrow$ $\oint \vec{\nabla}\Theta \cdot \vec{\mathbf{dl}} = 0$ over a closed path

Example :

$T = xy^2$, $\quad a(0,0,0)$ and $b(2,1,0)$

$$\vec{\nabla}T = \hat{\imath}y^2 + \hat{\jmath}2xy$$

$$\vec{\nabla}T \cdot \vec{dl} = (\hat{\imath}y^2 + \hat{\jmath}2xy) \cdot (dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k})$$
$$= y^2 dx + 2xy dy$$



Divergence theorem (Gauss' Theorem / Green's Theorem)

$$\iiint_{V} \underbrace{(\vec{\nabla}.\vec{v})}_{}\underbrace{d\tau}_{\text{volume element}} = \oiint_{S} \overbrace{\vec{v}\,.\overrightarrow{da}}^{\text{Flux of the vector function}}$$

Flux of the vector function

volume

volume element

area

area element

Divergence of a vector function in a given volume is equivalent

to the flux passing through any surface <u>bounding</u> the volume

Divergence theorem - contradictions

Let $\vec{v} = \frac{\hat{r}}{r^2}$ be a vector field. Then divergence theorem implies that

$$\iiint_V \left(\vec{\nabla} \cdot \frac{\hat{r}}{r^2}\right) dV = \oiint_S \frac{\hat{r}}{r^2} \cdot \overrightarrow{dS}$$

$$\vec{\nabla} \cdot \frac{\hat{r}}{r^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{r^2}\right) = 0$$ **Hence LHS = 0**

If the volume is taken as a sphere of radius *R*, then RHS

$$= \int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=2\pi} \frac{\hat{r}}{R^2} \cdot (\hat{r}\, R^2 \sin\theta \, d\theta \, d\phi) = 4\pi$$

This contradiction is due to the fact that *r* = 0 is a singularity and such cases should be dealt with Dirac Delta-function method of integration.

Curl theorem (Green's theorem)

$$\iint_S (\vec{\nabla} \times \vec{v}) . \overrightarrow{da} = \oint_L \vec{v} . \overrightarrow{dl}$$

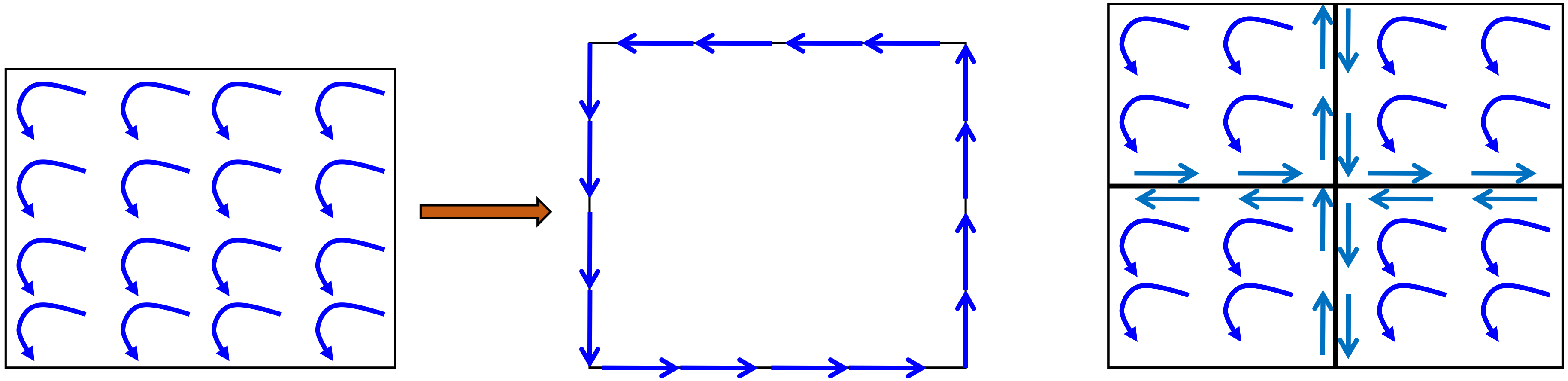
area

area element

length

length element

Curl of a vector function in a given surface is
equivalent to
value of function along the bounding line enclosing the surface



Electrostatics

Electric field, potential and conductors

ELECTROSTATICS

Assumption:

- all charges are stationary
- charges are given; i.e., neglect the internal structure of the charges or the energy needed to create them

AIM

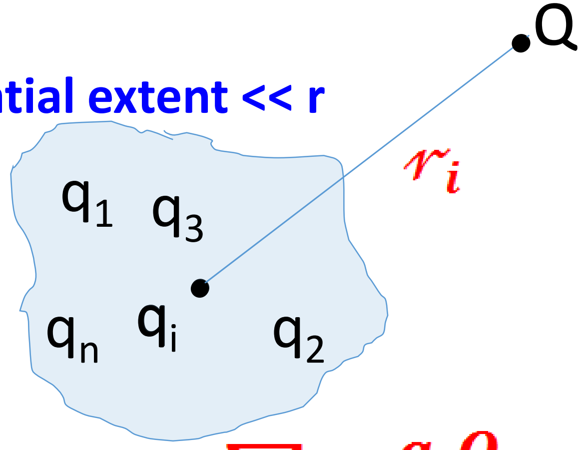
Force exerted on a charge Q by charges q_1, q_2, q_n? – Coloumb's Law (1st fundamental rule of electrostatics)

$$\vec{F} = \frac{qQ}{4\pi\varepsilon_0 r^2}\hat{r}$$

Attractive or repulsive depending on the charges

Valid only for point charges or

Charge distribution whose spatial extent << r



Superposition principle (2nd fundamental rule) :

•Interaction between 2 charges is completely unaffected by the presence of other charges. i.e., interaction between charges q_i can be neglected.

$$\vec{F} = \sum_i \frac{q_i Q}{4\pi\varepsilon_0 r_i^2}\hat{r}_i$$

ELECTRIC FIELD

How the charge distribution knows about the charge Q?

Action at a distance needs concept of Electric Field!

Electric field attaches itself as a local property to a charge system (compare with gravity).

Force exerted on a charge Q by charges $q_1, q_2, \ldots, q_n$

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \ldots + \vec{F}_n = \frac{1}{4\pi\varepsilon_0}\left\{\frac{q_1 Q}{r_1^2}\hat{r}_1 + \frac{q_2 Q}{r_2^2}\hat{r}_2 + \cdots \frac{q_n Q}{r_n^2}\hat{r}_n\right\}$$

E(P) refers only point P; completely removes reference of test charge

$$= Q\left\{\frac{q_1}{4\pi\varepsilon_0 r_1^2}\hat{r}_1 + \frac{q_2}{4\pi\varepsilon_0 r_2^2}\hat{r}_2 + \cdots + \frac{q_i}{4\pi\varepsilon_0 r_i^2}\hat{r}_i\right\}$$

$$\vec{F} = Q\vec{E} \quad \text{or} \quad \vec{E} = \lim_{Q\to 0}\frac{\vec{F}}{Q}$$

Test charge Q does not affect charge distribution $q_1, q_2, \ldots, q_n$

$$\vec{E}(P) = \sum \frac{q_i}{4\pi\varepsilon_0 r_i^2}\hat{r}_i$$

Electric Field at the point P where charge Q is located

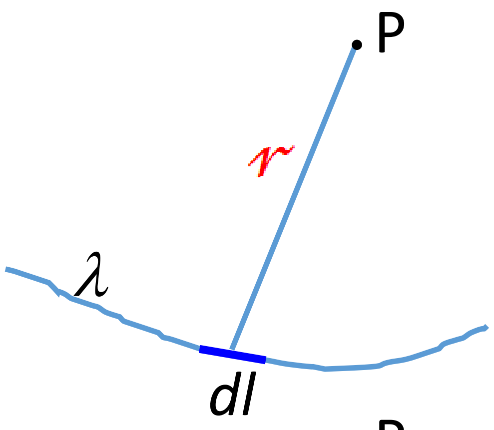
CONTINUOUS CHARGE DISTRIBUTIONS

LINE CHARGE

CHARGE DISTRIBUTED OVER A LENGTH L

LINEAR CHARGE DENSITY λ = CHARGE/LENGTH

$$\vec{E}(P) = \int_L \frac{\lambda dl}{4\pi\varepsilon_0 r^2}\hat{r}$$

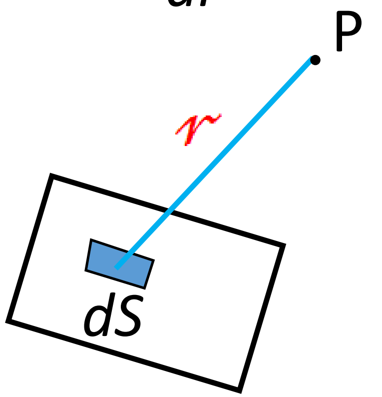


SURFACE CHARGE

CHARGE DISTRIBUTED OVER A SURFACE S

SURFACE CHARGE DENSITY σ = CHARGE/AREA

$$\vec{E}(P) = \iint_S \frac{\sigma ds}{4\pi\varepsilon_0 r^2}\hat{r}$$

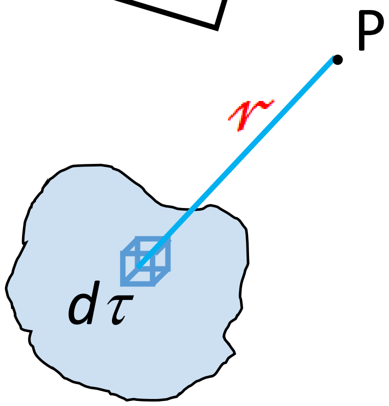


VOLUME CHARGE

CHARGE DISTRIBUTED OVER A VOLUME V

VOLUME CHARGE DENSITY ρ = CHARGE/VOLUME

$$\vec{E}(P) = \iiint_V \frac{\rho d\tau}{4\pi\varepsilon_0 r^2}\hat{r}$$



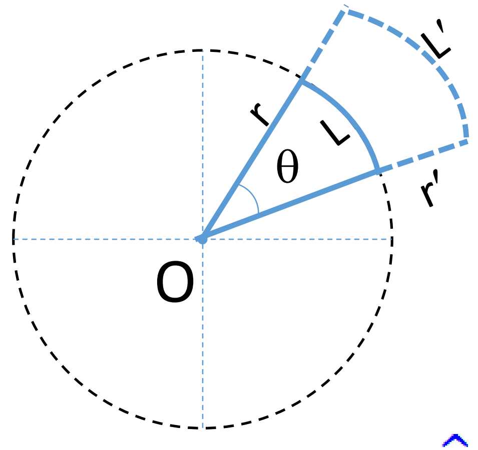
SOLID ANGLE

To define angle, circle of radius r is drawn with the apex as its centre.

Then $\theta = L/r$.

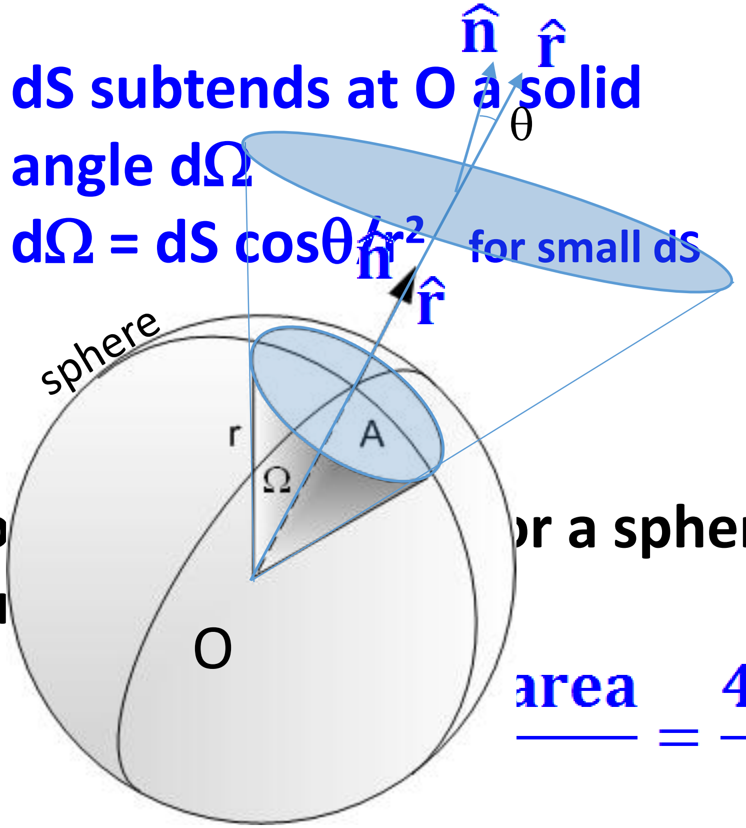
L is the length of arc subtending the angle

L/r is independent of the radius ($\theta = L'/r'$)



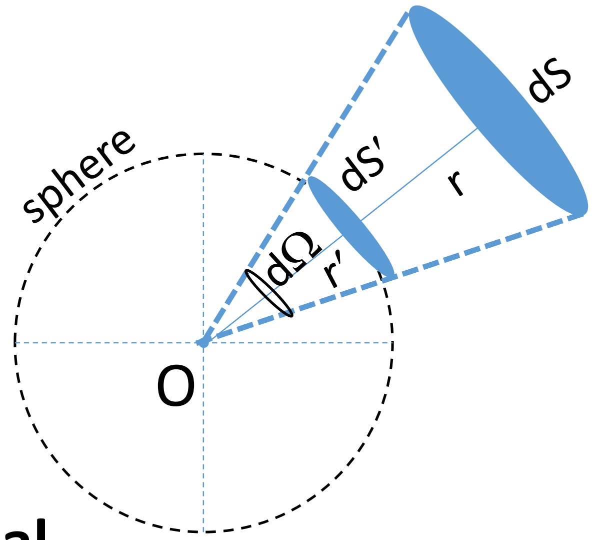
dS subtends at O a solid angle dΩ

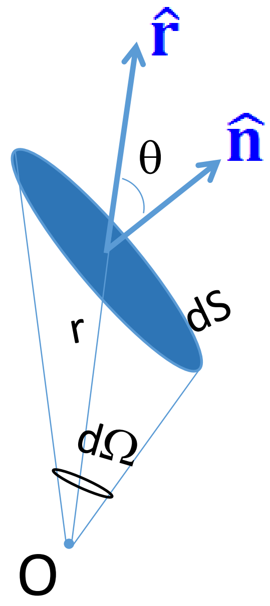
$d\Omega = dS \cos\theta / r^2$ for small dS



To ... or a spherical su...

$$\frac{\text{area}}{} = \frac{4\pi r^2}{r^2} = 4\pi$$



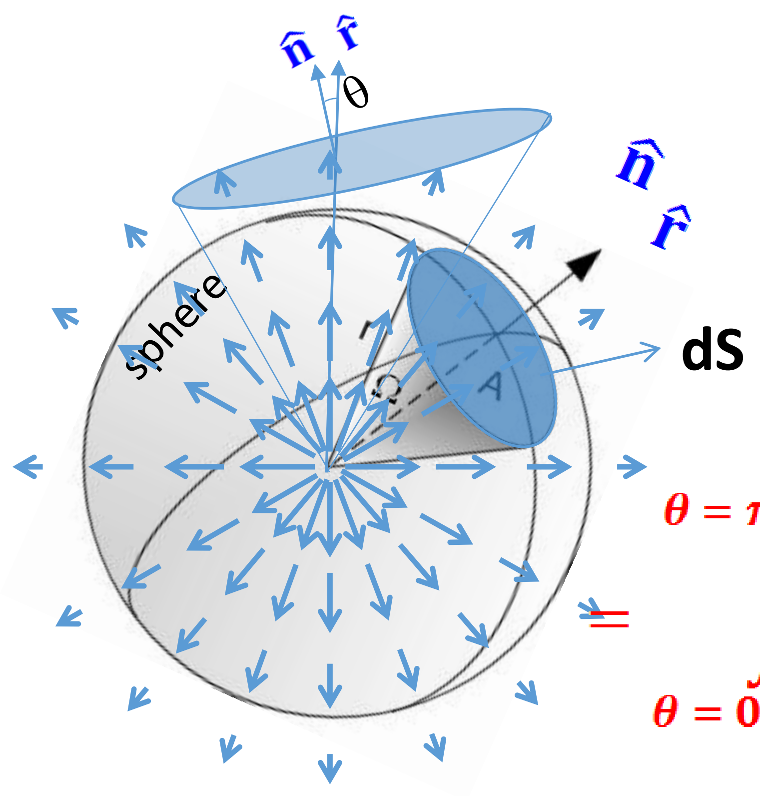


$$\Omega = \iint \frac{r^2 \sin\theta \, d\theta \, d\varphi}{r^2} = 4\pi$$

Flux of electric field and Gauss's Law

Point charge +q at the origin : E $\propto$ 1/r^2

Field lines



Flux of electric field $\propto$ field lines passing through a given surface

Flux through elemental area dS

$$d\Theta = \vec{E} \cdot \vec{dS}$$

Total flux $$\Theta = \int_S \vec{E} \cdot \vec{dS}$$

For the point charge +q at the origin of the sphere, total flux

$$= \iint_{\theta=0\ \phi=0}^{\theta=\pi\ \phi=2\pi} \frac{q}{4\pi\varepsilon_0 r^2}\,\hat{r} \cdot \left(r^2 \sin\theta\, d\theta\, d\phi\, \hat{r}\right)$$

$$= \frac{q}{\varepsilon_0}$$

$$\oiint \vec{E}.\vec{ds} = \frac{q}{\varepsilon_0}$$

constant, does not depend on *r*

surface area increases as r^2

e.f. decreases as $1/r^2$

source is enclosed by the surface

For a system of charges q1, q2, …., qn

$$\oiint \vec{E}.\vec{ds} = \frac{Q_{tot}}{\varepsilon_0}$$

Gauss's Law using SOLID ANGLE

$$d\Theta = \vec{E}\cdot\vec{dS} = \frac{q}{4\pi\varepsilon_0 r^2}\,\hat{r}\cdot\vec{dS} = \frac{q}{4\pi\varepsilon_0}\,\frac{\hat{r}\cdot\vec{dS}}{r^2}$$

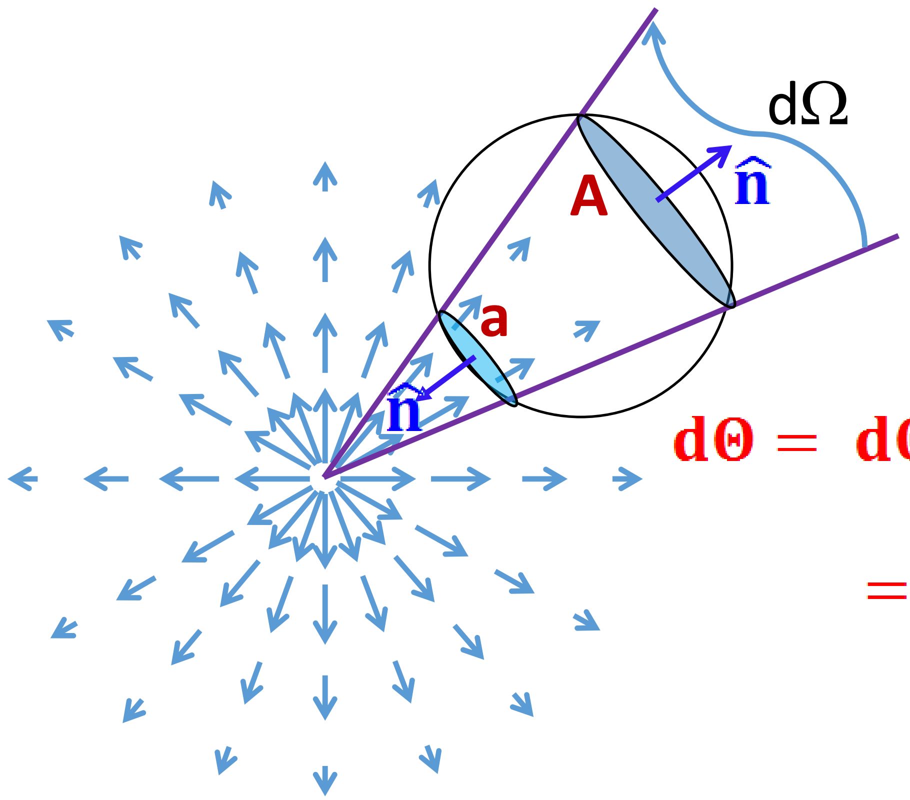
$$= \frac{q}{4\pi\varepsilon_0}\,d\Omega$$

$$\Theta = \int_S d\Theta = \frac{q}{\varepsilon_0}$$

Flux of E if charge is outside the volume

Net flux = incoming flux – outgoing flux

= 0



$$d\Theta = d\Theta_a + d\Theta_A$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{\hat{r}\cdot(-\overrightarrow{dS})}{r_a^2} + \frac{q}{4\pi\varepsilon_0}\frac{\hat{r}\cdot\overrightarrow{dS}}{r_A^2}$$

$$= \frac{q}{4\pi\varepsilon_0}\{-d\Omega_a + d\Omega_A\} = 0$$

Gauss's Law

Flux of electric field through a given surface,

$$\oint\!\!\oint \vec{E}.\overrightarrow{ds} = \frac{Q_{tot}}{\varepsilon_0}$$ if the surface encloses the charges

$$= 0$$ if the surface does not enclose the charges

$$\underbrace{\phantom{\oint\!\!\oint \vec{E}.\overrightarrow{ds} = \frac{Q_{tot}}{\varepsilon_0}}}_{\text{Integral form of Gauss's Law}}$$

Integral form of Gauss's Law

Differential form

Using divergence theorem,

$$\oint\!\!\oint \vec{E}.\overrightarrow{ds} = \iiint_V (\vec{\nabla}\cdot\vec{E})\,d\tau = \frac{Q_{tot}}{\varepsilon_0} = \frac{1}{\varepsilon_0}\iiint_V \rho\, d\tau$$

Since it is true for any volume,

$$\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$$ 1st Maxwell equation

Gauss's Law - Applications

$$\oiint \vec{E}.\vec{ds} = \frac{Q_{tot}}{\varepsilon_0}$$ if the surface encloses the charges

$= 0$ if the surface does not enclose the charges

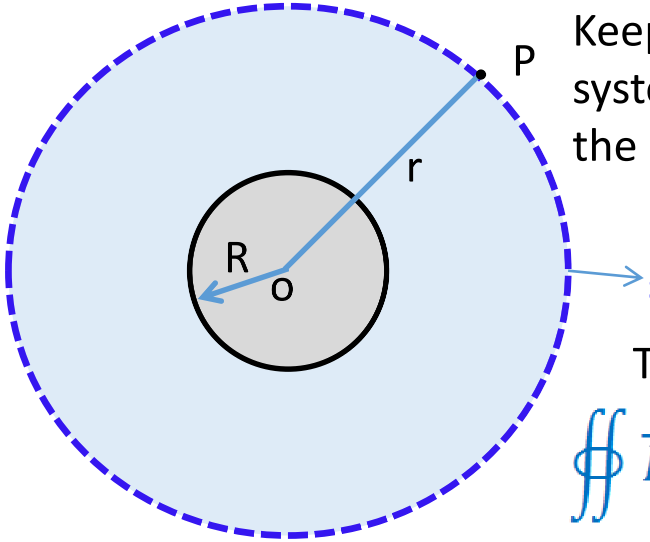
If a symmetry in the charge distribution exists, integral form of GL provides a far easier method to find $\vec{E}$

Possible symmetries

1. Spherical
2. Cylindrical
3. Planar

Gauss's Law - Applications

Spherical symmetry : Electric field outside and inside a uniformly charged sphere of radius R and volume charge density ρ



Keep the sphere with its centre at the origin of the coordinate system. Let P be the point outside at a distance r from O where the electric field is to be calculated.

Draw a spherical surface of radius *r* centred at O : Gaussian surface

Then for the flux through the spherical surface,

$$\oiint \vec{E}\cdot\vec{ds} = \frac{Q_{\text{tot,encl}}}{\varepsilon_0}$$

$$= \frac{1}{\varepsilon_0}\int_V \rho\, dV = \frac{\rho}{\varepsilon_0}\int_V dV = \frac{\rho}{\varepsilon_0}\left\{\frac{4}{3}\pi R^3\right\}$$

Symmetry arguments :

- **At the spherical surface, E and dS will be in $\hat{r}$ direction**
- **dot product goes away.**
- **E will be constant everywhere on the surface**
- **E comes out of the integral.**

This is the advantage of GL application in the case of symmetry

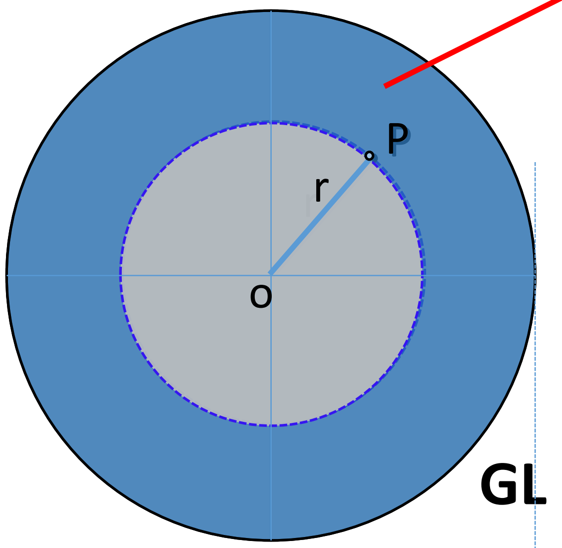
Gauss's Law - Applications

$$E\,4\pi r^2 = \frac{\rho}{\varepsilon_0}\left\{\frac{4}{3}\pi R^3\right\} \Longrightarrow E = \hat{r}\,\frac{\rho R^3}{3\varepsilon_0 r^2} = \hat{r}\,\frac{Q}{4\pi\varepsilon_0 r^2}$$

Total charge

Field outside is exactly the same as if the whole charge is concentrated at the centre!

Electric field inside

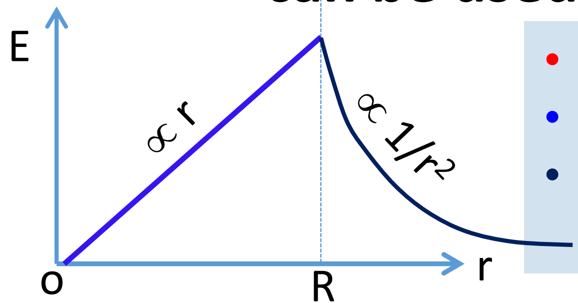


This volume does not contribute to flux since it lies outside the point, OR not enclosed by the surface

$$Q_{\text{encl}} = \frac{4}{3}\pi r^3 \rho$$

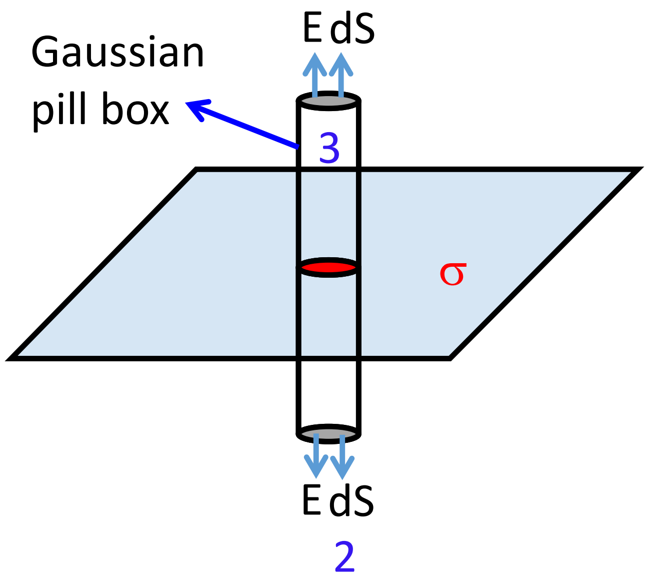
$$E.\,4\pi r^2 = \frac{1}{\varepsilon_0}\frac{4}{3}\pi r^3 \rho \Longrightarrow \vec{E} = \hat{r}\,\frac{\rho r}{3\varepsilon_0}$$

GL is valid for all charge distributions; integral form can be used for calculating E when symmetry allows



- GL is more general compared to Coloumb's law
- CL : applicable only in static cases
- Differential form of GL is the 1st Maxwell equation
  - Valid for em waves/moving charges

Gauss's Law – Applications : E due to infinite sheet, surface charge density σ



E is perpendicular to the infinite sheet of charge: By symmetry, it *cannot* point in any other direction

Due to symmetry, $E_1 = E_2 = E$
(equidistance surfaces from the plane)

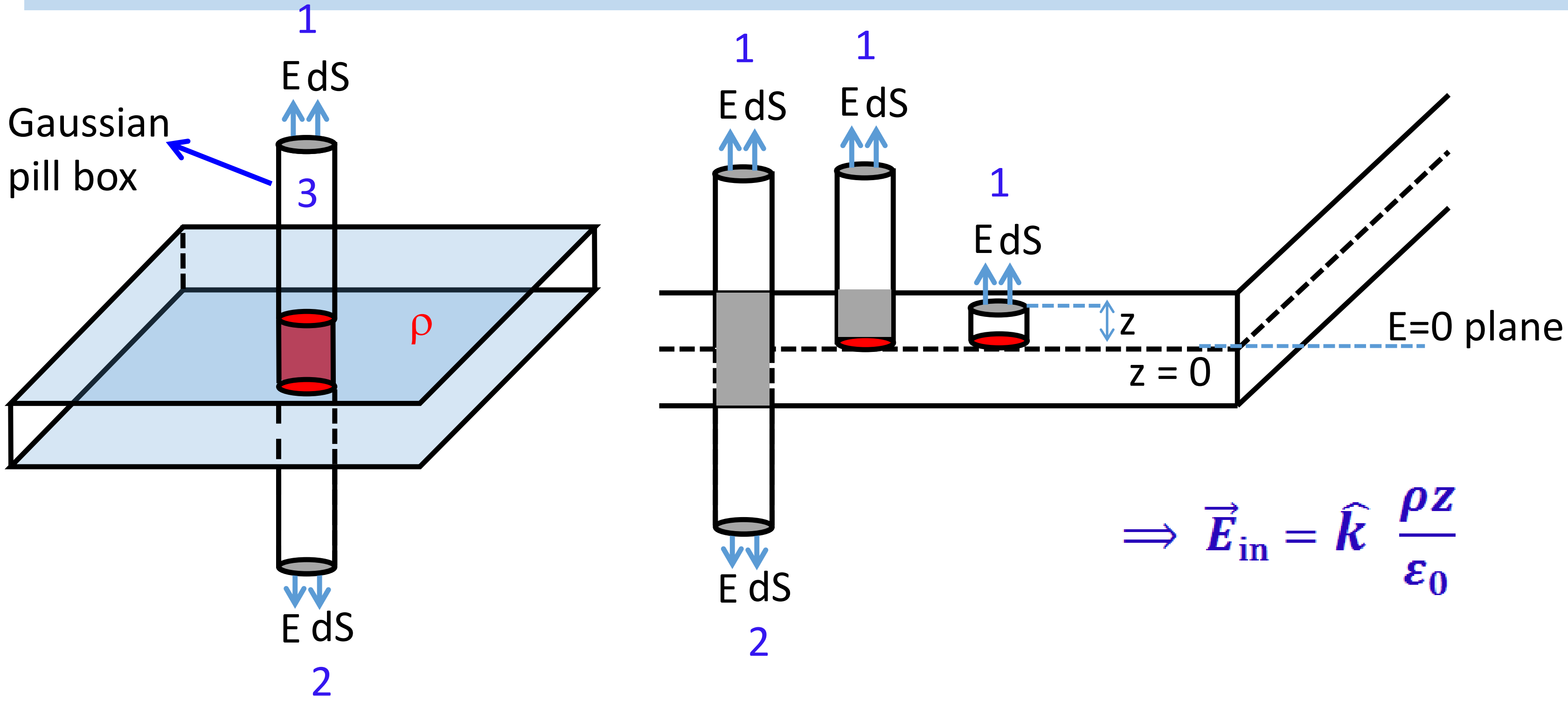
Since E is perpendicular to the plane , surface 3 (curved surface) contributes nothing (dot product goes to zero)

By Gauss's Law, Net Electric Flux $2\,E\,dS = \frac{\sigma}{\epsilon_0}\,dS$, i.e,

$$E = \frac{\sigma}{2\epsilon_0}$$

Constant! Does not depend on the distance

Gauss's Law – Applications : E due to infinite SLAB, thickness 2*d*, constant volume charge density ρ



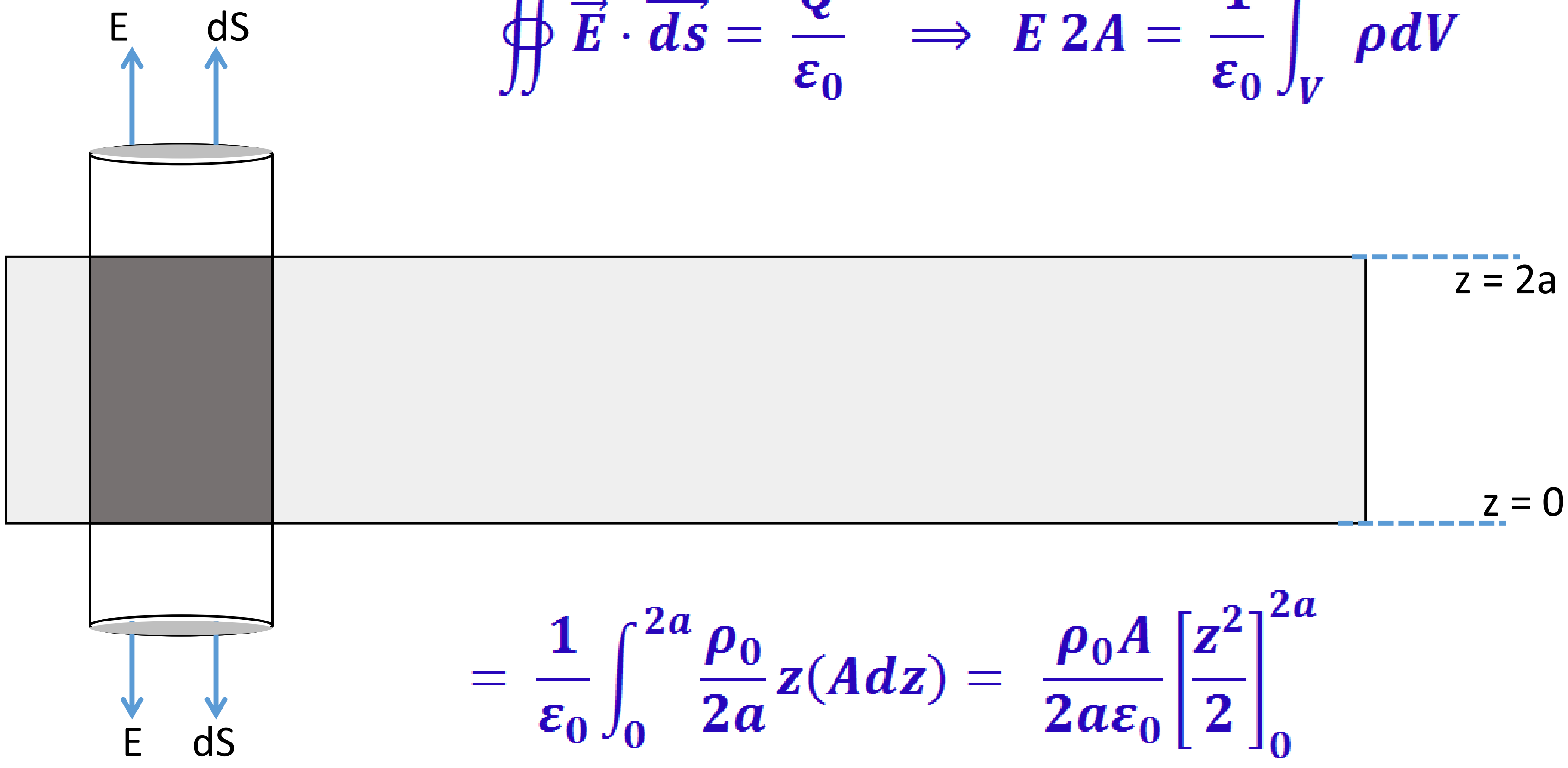
$$\Rightarrow \vec{E}_{in} = \hat{k}\,\frac{\rho z}{\varepsilon_0}$$

Consider the slab as infinite sheets in the z-direction. Then field outside the sheet is constant

$$\oiint \vec{E}\cdot\vec{ds} = \frac{Q}{\varepsilon_0} \Rightarrow E_{out}\,2A = \frac{\rho(2dA)}{\varepsilon_0} \Rightarrow \vec{E}_{out} = \pm\hat{k}\,\frac{\rho d}{\varepsilon_0} = \hat{k}\,\frac{\rho d}{\varepsilon_0}\,\frac{z}{|z|}$$

Infinite slab, thickness $2a$, volume charge density $\rho = \rho_0 z/2a$, kept in the x-y plane, Extends from z = 0 to z = $2a$ along the z-axis. Find the magnitude and direction of field both inside and outside

(a) Field outside



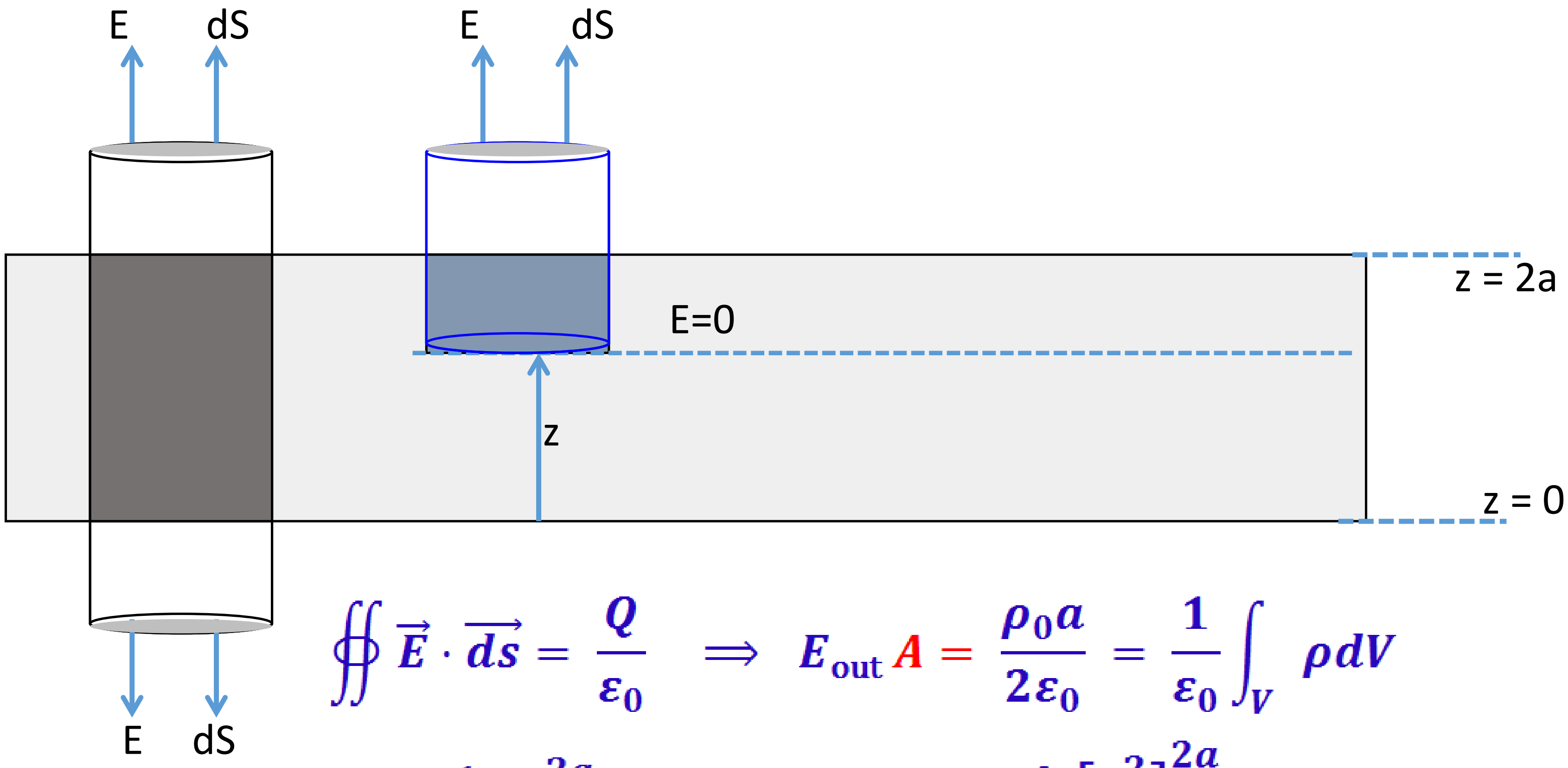
$$\oiint \vec{E} \cdot \vec{ds} = \frac{Q}{\varepsilon_0} \quad \Longrightarrow \quad E\,2A = \frac{1}{\varepsilon_0}\int_V \rho dV$$

$$= \frac{1}{\varepsilon_0}\int_0^{2a} \frac{\rho_0}{2a} z(Adz) = \frac{\rho_0 A}{2a\varepsilon_0}\left[\frac{z^2}{2}\right]_0^{2a}$$

$$= \frac{\rho_0 A a}{\varepsilon_0} \quad \Longrightarrow \quad \vec{E}_{\text{out}} = \pm\hat{k}\,\frac{\rho_0 a}{2\varepsilon_0}$$

$$= \text{constant}$$

(b) E = 0 plane : Let the E = 0 plane be at a distance z as shown



$$\oint\!\!\oint \vec{E}\cdot\vec{ds} = \frac{Q}{\varepsilon_0} \implies E_{out}A = \frac{\rho_0 a}{2\varepsilon_0} = \frac{1}{\varepsilon_0}\int_V \rho dV$$

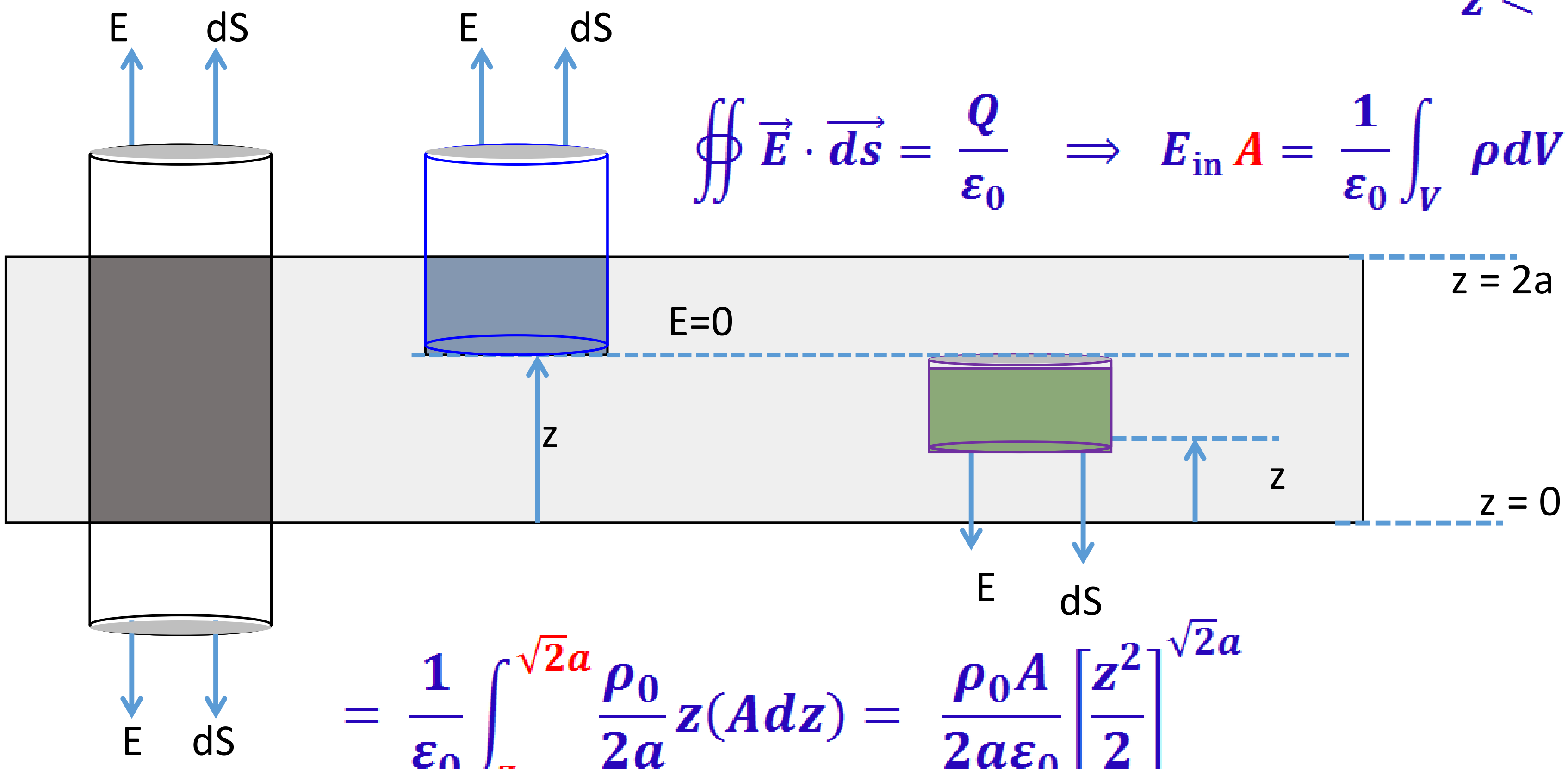
$$= \frac{1}{\varepsilon_0}\int_z^{2a} \frac{\rho_0}{2a} z(Adz) = \frac{\rho_0 A}{2a\varepsilon_0}\left[\frac{z^2}{2}\right]_z^{2a}$$

$$= \frac{\rho_0 A}{4a\varepsilon_0}\left(4a^2 - z^2\right) \implies z = \pm\sqrt{2}$$

(a) Field inside

Electric field is in $+\hat{k}$ direction if $z > \sqrt{2}a$ and in $-\hat{k}$ direction if $z < \sqrt{2}a$

$$\oiint \vec{E}\cdot\vec{ds} = \frac{Q}{\varepsilon_0} \quad\Longrightarrow\quad E_{\text{in}}\,A = \frac{1}{\varepsilon_0}\int_V \rho\, dV$$



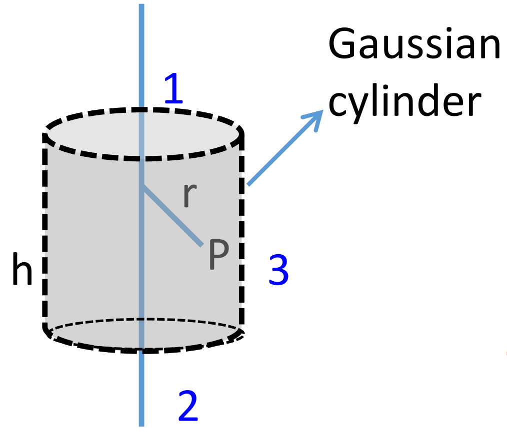
$$= \frac{1}{\varepsilon_0}\int_{z}^{\sqrt{2}a} \frac{\rho_0}{2a} z(A\,dz) = \frac{\rho_0 A}{2a\varepsilon_0}\left[\frac{z^2}{2}\right]_0^{\sqrt{2}a}$$

$$= \frac{\rho_0 A}{4a\varepsilon_0}\left(2a^2 - z^2\right)$$

$$\Longrightarrow\quad \vec{E}_{\text{in}} = -\hat{k}\,\frac{\rho_0 a}{4a\varepsilon_0}\left(2a^2 - z^2\right)$$

Gauss's Law – Applications : E due to infinite line charge, charge density λ

(A). Electric field at P at a distance *r* from the wire

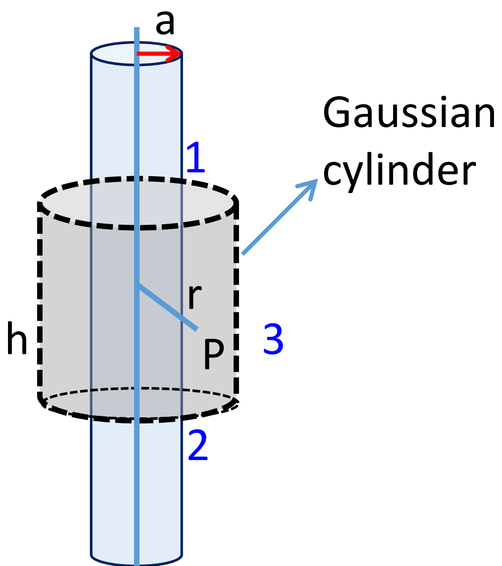


Draw Gaussian surface (cylinder of radius *r* and height *h*, enclosing the wire) such that the P is on the surface.

From symmetry, $\vec{E} = E\hat{r}$ $\Rightarrow$ **flux through flat surfaces (1 & 2) = 0, E $\perp$ dS**

$$\oiint \vec{E} \cdot \vec{ds} = E\,2\pi r h = \frac{\lambda h}{\varepsilon_0} \Longrightarrow \vec{E} = \pm\hat{r}\,\frac{\sigma}{2\pi\varepsilon_0 r}$$

(B). E due to infinite cylinder, volume charge density ρ



(i) E outside

$$\oiint \vec{E} \cdot \vec{ds} = E\,2\pi r h = \frac{\rho\pi a^2 h}{\varepsilon_0} \Longrightarrow \vec{E}_{\text{out}} = \hat{r}\,\frac{\rho a^2}{2\varepsilon_0 r}$$

(ii) E inside

$$\oiint \vec{E} \cdot \vec{ds} = E\,2\pi r h = \frac{\rho\pi r^2 h}{\varepsilon_0} \Longrightarrow \vec{E}_{\text{in}} = \hat{r}\,\frac{\rho r}{2\varepsilon_0}$$

If $\rho = \rho_0 r$, where ρ_0 is constant, then

$$Q_{\text{encl}} = \int \rho \, dV$$

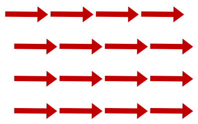
outside

$$= \int_{r=0}^{a} \int_{\phi=0}^{2\pi} \int_{z=0}^{h} \rho_0 r \, (r \, dr \, d\phi \, dz) = \rho_0 (2\pi) h \left(\frac{a^3}{3} \right)$$

inside

$$Q_{\text{encl}} = \int_{r=0}^{r} \int_{\phi=0}^{2\pi} \int_{z=0}^{h} \rho_0 r' \, (r' \, dr' \, d\phi \, dz) = \rho_0 (2\pi) h \left(\frac{r^3}{3} \right)$$

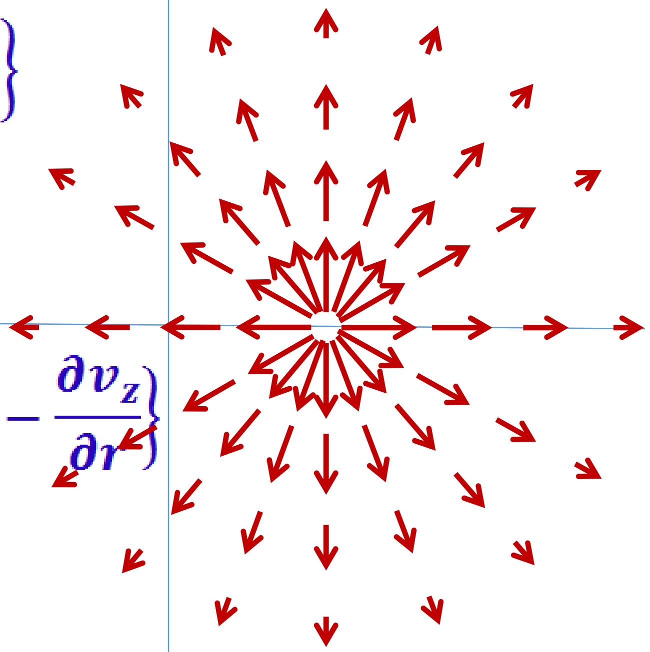
Curl of Electric field



$$\vec{v} = v_0 \hat{k}$$

Spherical coordinates
(r, θ, ϕ)

$$\vec{\nabla} \times \vec{v} = \hat{r}\frac{1}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right\} + \hat{\theta}\,\frac{1}{r}\left\{\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right\} + \hat{\phi}\,\frac{1}{r}\left\{\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right\}$$

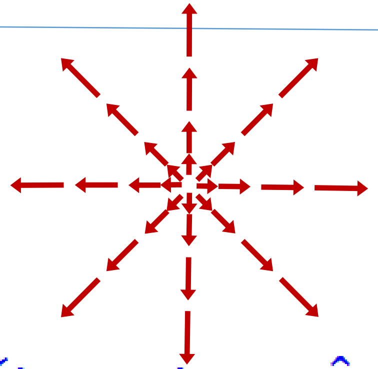


cylindrical coordinates
(r, ϕ, z)

$$\vec{\nabla} \times \vec{v} = \hat{r}\left\{\frac{1}{r}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right\} + \hat{\phi}\left\{\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right\} + \hat{z}\left\{\frac{\partial}{\partial r}(r v_\phi) - \frac{\partial v_r}{\partial\phi}\right\}$$

Cartesian coordinates
(x, y, z)

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$$

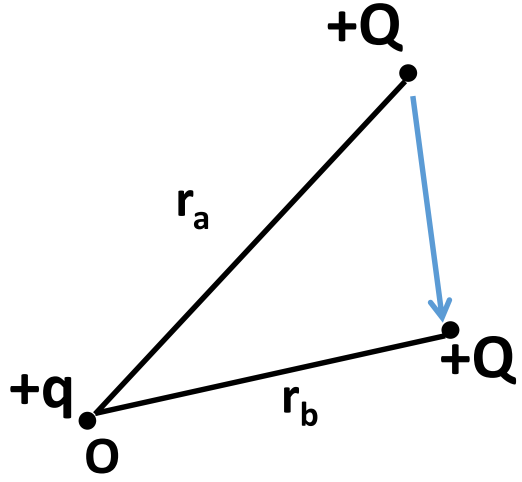


$$\vec{v} = \vec{r} = (\hat{i}\,x + \hat{j}\,y + \hat{k}\,z)$$

E = E(r) $\longrightarrow$ E is a central field $\longrightarrow$ $\vec{\nabla}\times\vec{E}=0 \longrightarrow \oint \vec{E}.\vec{dl}=0$

$\vec{\nabla}\cdot\vec{E}=\frac{\rho}{\varepsilon_0}$ Curl of Electric field $\vec{\nabla}\times\vec{E}=0 \quad \Longrightarrow \quad \vec{E}=\vec{\nabla}V$

Line integral of E



$$\vec{dl}=dr\hat{r}+rd\theta\hat{\theta}+r\sin\theta d\phi\hat{\phi}$$

$$\vec{E}=|E|\hat{r} \Longrightarrow \vec{E}\cdot\vec{dl}=Edr$$

$$\int_{r_a}^{r_b}\vec{E}\cdot\vec{dl}=\int_{r_a}^{r_b}Edr=\frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_a}-\frac{1}{r_b}\right)$$

Line integral is independent of path; depends only on the end positions $\oint \vec{E}.\vec{dl}=0$

Using Stoke's theorem

$$\oint \vec{E}.\vec{dl}=0 \Longrightarrow \iint_S (\vec{\nabla}\times\vec{E}).\vec{dS}=0 \Longrightarrow \vec{\nabla}\times\vec{E}=0$$

Line integral of E $\longrightarrow$ Work done per unit charge (Potential V)

$$\int_{\infty}^{r} F.dr = \int_{r}^{\infty} E.dr = \frac{q}{4\pi\varepsilon_0 r} \longrightarrow V(r) = -\int_{\infty}^{r} \vec{E}.\vec{dl} = -\int_{\vartheta}^{r} \vec{E}.\vec{dl}$$

any reference point

Usually Potential Difference is important

$$V(r_b) - V(r_a) = -\int_{r_a}^{r_b} \vec{E}.\vec{dl}$$

From Gradient theorem $$V(r_b) - V(r_a) = \int_{r_a}^{r_b} \vec{\nabla}V.\vec{dl} \longrightarrow \vec{E} = -\vec{\nabla}V$$

Potential V is a scalar function, –grad of which gives the electric field

Potential obeys superposition principle

$V = V_1 + V_2 + \ldots$.. A simple scalar sum