Tomographic reconstructions under unknown angles and shifts

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Problem Statement

- A standard problem in tomography is to reconstruct the object from a sample of its projections.
- Tomographic reconstruction algorithms such as the filtered back projection algorithm assume that the angles at which the projection are taken are known and reconstruct the image successfully.
- In some cases the projection angles may be unknown, for example when reconstructing certain biological proteins or moving objects.

Moment based approach of estimating angles

- There are n+1 n^{th} order moments for the image, for every pair of integers p and q such that p + q = n. The moments are given by

$$v_{p,q} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x,y) x^p y^q \, dx \, dy$$

- For any angle θ, there exists a single n^{th} order moment of the projection $g(\rho,\theta)$ which is given by

$$m_{\theta,n} = \int_{-\infty}^{+\infty} g(\rho,\theta) \rho^n d\rho$$

- The Helgasson Ludwig Consistency Conditions (HLCC) gives a relationship between the geometric moments of the underlying image and its projections.

Moment based approach of estimating angles

- $m_\theta^n = \sum_{j=0}^{n} \binom{n}{j} \cos(\theta)^{n-j} \sin(\theta)^j \, v_{n-j,j}$
- These relations can be used to find out the unknown angles by iteratively solving for the unknown angles.
- Using coordinate descent strategy we achieved a first estimate for the angles of the projection.
- Unfortunately as moments are highly susceptible to noise, we need to further fine tune the estimates given by this algorithm.

Fine tuning the estimates

- To fine tune the estimates we decided to use alternate minimization to estimate the reconstructed image as well the unknown angles.
- We use the estimates given to us by the moment based approach as initial estimates for the angles. Then we shall perform two steps:
  - Using the theta estimates we can reconstruct the new image.
  - Using the new image, we can estimate the new thetas
- We repeat the above two steps until the algorithm has reached convergence.
- We will next describe each of the above steps in detail.

Fine tuning of estimates – Reconstruction of Image

- For estimating the image, we decided to formulate it as a optimization problem in the compressed sensing framework. The loss function is given by –
- $L(\{\theta_i\}, \mathrm{B}) = \sum_{i=1}^{Q} \left\| y_i - R_{\theta_i}(U\mathrm{B}) \right\|^2 + \lambda \|\mathrm{B}\|_1$
- θ_i *are the Q unknown angles,*
- y_i *are the actually measured projections,*
- R_{θ_i} *represents the Radon Transform,*
- *U denotes the Inverse DCT basis,*
- B is the vector of DCT coefficients to be reconstructed

Fine tuning of estimates – Estimate of angles

- Taking the reconstructed image, fine tune on an angle estimate using the following algorithm:
  - We assume the refined angle lies within a fixed delta of the original estimate.
  - Within $\theta_i - \delta$ to $\theta_i + \delta$, we look for the optimum angle which gives the projection closest to the measured projection. We calculate the optimum angle using brute search.
- We follow the above procedure for each of the angles, and obtain a new set of optimum angles.

Fine tuning the estimates

- We define the loss function as

$$L(\{\theta_i\}, \mathrm{B}) = \sum_{i=1}^{Q} \left\| y_i - R_{\theta_i}(U\mathrm{B}) \right\|^2 + \lambda \|\mathrm{B}\|_1$$

- Algorithm:
  - Given θ_i, we reconstruct B, using L1-LS package which solves the problem by formulating it as regularized least squares programs.
  - Given reconstructed image B, we can optimize θ_i using the following procedure:
    - For each θ_i:
      - $\theta_i^{t+1} = \min_{\theta} \|y_i - R_\theta(U\mathrm{B})\|^2$
      - We solve the above equation using a brute force search on θ, from $\theta - \delta$ to $\theta + \delta$ in step sizes of q.

Fine tuning on the estimates

- We follow the described procedure by first reconstructed image, followed by obtaining the angles estimates and we repeat this until the algorithm has converged and the difference in error is very small.
- While optimizing on the angles, we define a fixed step size which defines with what precision we want to look into the delta space.
- This parameter is decreased, to obtain greater precision in the theta estimates if we find the error hasn’t decreased in an iteration.

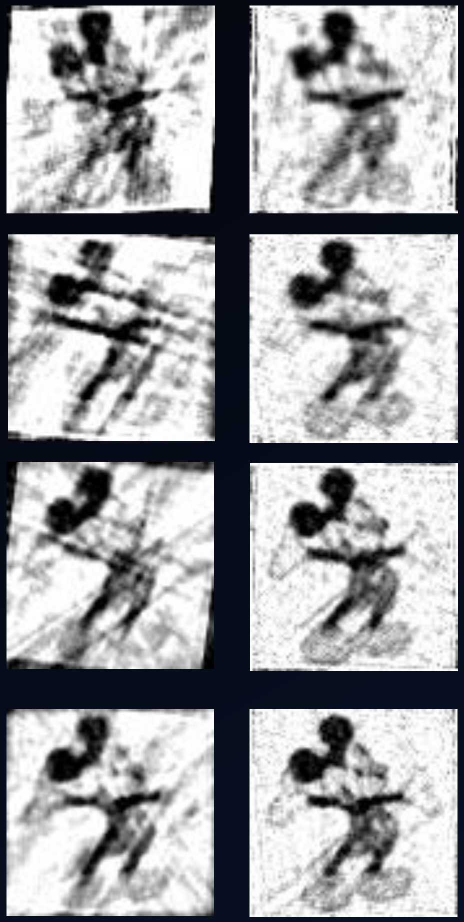
Results

- Environment conditions –
  - Original image size – 80*80
  - Noise in the projections – 5% Gaussian noise
  - Angles are completely unknown
  - In the next slide, we show the reconstruction results for a number of projection angles.
- Original Image –

Results

Moment based estimated image – It is the image that is constructed using the estimates of angles given by moment based approach.

Final image – The image reconstructed using the estimates fine tuned by the algorithm described above.



Number of Projection Angles	Relative Error of moment based estimated image	Relative Error of final image
20	15.47%	10.54%
30	17.03%	5.48%
50	2.99%	0.22%
80	1.57%	0.101%

Adding 50% noise to projections

- We took the problem statement one step further, and decided to add 50% noise to the projections.
- Our previous algorithm fails to reconstruct the image from such noisy projections.
- So we had to modify the algorithm majorly focusing on how we can reduce the noise and make our algorithm robust to noise.

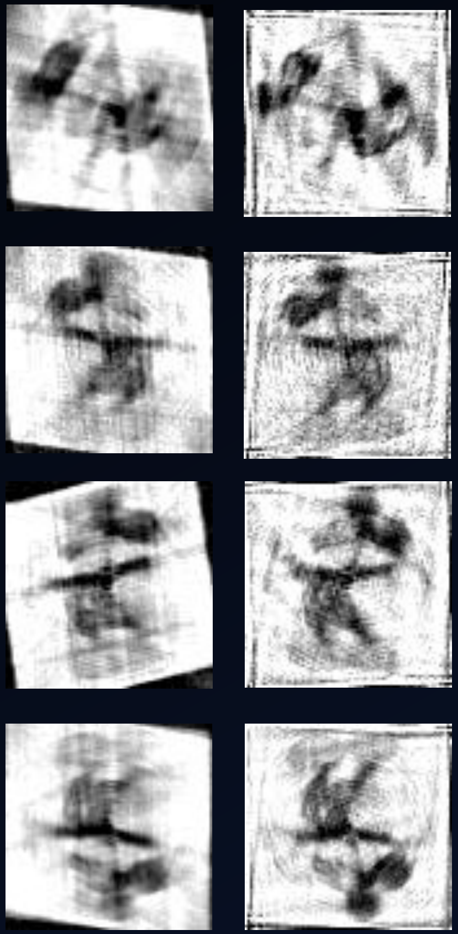
Denoising of projections

- The first step, involves using a patch-based PCA denoising method to reduce the noise in the projections.
- However just this step is not enough to denoise in the projections.
- We decided cluster the projections to a small set of projections using K-means clustering and take the cluster centres as our actual projections.
- In the process of taking the cluster centres, the averaging over many projections would filter the noise out.
- Taking the cluster centers our algorithm proceeded in the same way as before.

Results

Moment based estimated image – It is the image that is constructed using the estimates of angles given by moment based approach.

Final image – The image reconstructed using the estimates fine tuned by the algorithm described above.



Number of Projection Angles	Number of Clusters	Relative Error of moment based estimated image	Relative Error of moment based final image
10000	50	28.26%	28.92%
20000	50	17.43%	14.14%
40000	50	14.56%	13.1%
40000	90	12.35%	9.54%

Some more results

Original Image | Estimated Image | Final Image

Unknown shifts in Projections

- Along with unknown angles, we now allow projections to have unknown shifts in them as well.
- Very often in tomography, the projections might be shifted by a small amount due to uncontrolled motion of the subject.
- Having shifted projections seriously hamper our reconstruction results and give poor results.
- We over here have developed algorithms to estimate these shifts, correct the projections and then reconstruct the image.

Initial Estimate of Shifts

- After a thorough survey of the literature, we learnt that shifts are mostly corrected by assuming that the center of masses of all projections should lie in the center, which makes sense for most objects as they would have the more mass near the center rather than the edges.
- So we decided to shift the projections such that they have their center of masses at the origin.
- This corrected some of the shifts but the results were not accurate enough to give good reconstruction results.

Refining the shift estimates

- We decided to incorporate the shift estimates in the Helgasson Ludwig Consistency Conditions (HLCC) and modified the algorithm to estimate shifts as well.
- The energy function defined by HLCC condition is given by

$$E(\theta, v, s_i) = \sum_{n=0}^{k} \sum_{i=1}^{p} \left(m_{\theta_i, s_i}^{n} - \sum_{j=0}^{n} A_{i,j}^{n} v_{n-j,j} \right)^2$$

- For each s_i from $-\delta$ to $+\delta$ we evaluate the value of the energy function and choose the estimate which returns the lowest value. This gave a reasonable shift estimate for all the shifts although we found that it required a lot of multi-starts to give us the best estimate.
- We further pass these estimates to the alternate minimization scheme described above and try to correct for any erroneous estimate.

Fine tuning the estimates

- We define the loss function as

$$L(\{\theta_i\}, \mathrm{B}, \{s_i\}) = \sum_{i=1}^{Q} \left\| y_{s_i,i} - R_{\theta_i}(U\mathrm{B}) \right\|^2 + \lambda \|\mathrm{B}\|_1$$

- $\theta_i\ are\ the\ Q\ unknown\ angles,$
- $y_i\ are\ the\ actually\ measured\ projections,$
- $R_{\theta_i}\ represents\ the\ Radon\ Transform,$
- $U\ denotes\ the\ Inverse\ DCT\ basis,$
- B is the vector of DCT coefficients to be reconstructed,
- $y_{s_i,i}$ denotes y_i shifted by s_i

Fine tuning the estimates

- Algorithm:
  - Given θ_i and s_i, we reconstruct $\mathrm{B}^{\mathrm{t+1}}$, using L1-LS package which solves the problem by formulating it as regularized least squares programs.
  - Given reconstructed image $\mathrm{B}^{\mathrm{t+1}}$ and the s_i, we can optimize θ_i using the following procedure:
    - For each θ_i:
      - $\theta_i^{t+1} = \min_{\theta} \left\| y_{i,s_i} - R_\theta(U\mathrm{B}) \right\|^2$
      - We solve the above equation using a brute force search on θ, from $\theta - \delta$ to $\theta + \delta$ in step sizes of q.
  - Given θ_i^{t+1} and reconstructed image $\mathrm{B}^{\mathrm{t+1}}$, we construct the new estimate by the following procedure:
    - For each s_i:
      - $s_i^{t+1} = \min_{s_i} \left\| y_{i,s_i} - R_\theta(U\mathrm{B}) \right\|^2$
      - We solve the above equation using a brute force search on s_i, from $-\delta$ to $+\delta$.

Results

- Experiment conditions
  - Size of original image – 80*80
  - Number of projections – 50
  - Noise in the projections – 5% Gaussian noise
  - Angles – Unknown
  - Maximum shift in the projections - ± 2

Relative error of estimated image	Relative error of final image
15.14%	9.54%

Conclusion

- In this project we have devised and implemented algorithms which substantially improve the reconstruction offered by the moment based approach.
- We have presented a modification of our algorithm which can work with projections having 50% gaussian noise and shown that with a large number of projections we can accurately reconstruct back the image.
- We have also accounted for shifts in the projections and modified our algorithms to estimate the shifts and provide accurate reconstructions.

Conclusion

- By solving these problems, we have solved some of the severe bottle necks which occur in real life tomographic reconstructions.
- In reconstructing the 3D structure of biological proteins, the projections often have high amount of noise.
- Also it might be the case that the projections are slightly shifted from the center.
- We have shown algorithms to solve each of the problems mentioned above and proved their validity using experiments.

Future Work

- We have demonstrated our algorithms in the 2D domain. To show the validity of our algorithms in the real world we need to extend it to the 3D domain.
- Using these algorithms, we can improve upon the 3D reconstructions of biological proteins which is an active area of research.
- We can improve on the reconstruction algorithm by accounting for the symmetry of algorithms along a particular axis.

Thank you

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