Uniqueness of Tomography with Unknown View Angles

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***Abstract*—In the standard two-dimensional (2-D) parallel beam tomographic formulation, it is assumed that the angles at which the projections were acquired are known. In certain situations, however, these angles are known only approximately (as in the case of magnetic resonance imaging (MRI) of a moving patient), or are completely unknown. The latter occurs in a three-dimensional (3-D) version of the problem in the electron microscopy-based imaging of viral particles. We address the problem of determining the view angles directly from the projection data itself in the 2-D parallel beam case. We prove the surprising result that under some fairly mild conditions, the view angles are uniquely determined by the projection data. We present conditions for the unique recovery of these view angles based on the Helgasson–Ludwig consistency conditions for the Radon transform. We also show that when the projections are shifted by some random amount which must be jointly estimated with the view angles, unique recovery of both the shifts and view angles is possible.**

***Index Terms*—Motion, tomography, uniqueness, unknown view angle.**

I. Introduction

COMPUTERIZED tomography is a technique for estimating the cross section of an object from measurements that are essentially line integrals of that cross section at some set of orientations. The noninvasive nature of tomography has made it a very useful technique for a variety of applications, including medical imaging, nondestructive testing and evaluation, synthetic aperture radar (SAR), and electron-microscopy based tomography, among others [1]. Yet, the pervasiveness of the tomographic formulation in this variety of applications is coupled strongly to some key assumptions, which we will examine in this work.

Much of the current research in the field of computerized tomography has focused on the problem of estimating the object given a collection of its line integral projections at a finite set of angles. It is generally assumed in such developments that the angles at which the projections were acquired (i.e., the view angles) are known exactly. Yet in many practical applications, perfect knowledge of the object's orientation is unobtainable. In medical imaging, for example, involuntary motion of the patient can result in uncertainty as to the angles at which data was acquired. The problem can be further aggravated when the total scan time is large, requiring the patient to lie motionless for extended periods of time. A related problem arises in reconstructing a three-dimensional (3-D) model of a virus from a single projection of many identical units at random orientations in a substrate.

The problem of unknown view angles has been addressed in various limited contexts. In the case of magnetic resonance imaging (MRI), for example, a good deal of work has been done on studying and overcoming specific types of uncontrollable patient motion [2]–[5]. However, no underlying theory exists to suggest that the approaches described in these references produce unique and exact solutions. Furthermore, many of the approaches developed for MRI utilize the flexibility of the acquisition procedure, and are not applicable to other imaging modalities.

In 3-D, the problem of unknown view angles has been addressed in the viral imaging problem [6]. In this formulation of the problem, a single parallel beam projection of many virtually identical particles is available. The individual particles are at random orientations, which must be estimated from the data before a reconstruction can be attempted. Interestingly enough, the 3-D problem admits a unique solution, a fact that follows from the well-known 3-D Fourier Slice Theorem [7], [8]. The technique used to establish this fact, however, cannot be applied to the 2-D problem [6].

In this paper, we establish a theoretical foundation for the problem of determining the view angles from the 2-D parallel beam projection data. We prove the surprising result that under certain fairly general conditions, the angles can be recovered to within a rigid body transformation of the underlying object. The applications of this result are twofold. One direct application is in the tomographic imaging of 2-D objects undergoing (unknown) rigid body motion, where the per-projection scan time is small relative to the rate of motion, but the total scan time is not. A second application is in an entirely new means of tomographic imaging, one in which projections are acquired at random and unknown orientations in the plane of the object, and then the projections are postprocessed to extract the orientation information necessary to reconstruct the object.

Previous work has addressed the determination of projection angles using moments [9], [10]. Although primarily focused on the 3-D problem, both of these authors have addressed the 2-D problem which we examine. However, neither author provides a theoretical justification for their claims and conjectures of unique recovery of the projection angles. Indeed, based on

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the results derived in this paper, we can address nearly all of the conjectures made by both authors regarding unique recovery of the view angles. The relationship of our results to these works is dealt with in full detail in Section VII.

The paper is organized as follows. The problem is formulated, and some fundamental quantities defined, along with a brief review of the principles of tomography in Section II. This section also discusses equivalence of solutions to the problem due to inherent ambiguities. In Section III the main results of the work are stated: for almost all objects, under some mild conditions, the angle recovery problem can be uniquely solved. A similar result holds for the case in which the projections are shifted by an unknown amount as well. In Section IV, we derive general uniqueness conditions for the angle recovery problem. Specific conditions, along with geometric interpretations of these conditions, are detailed in Section V. The case in which the projections are shifted by an unknown amount is treated in Section VI. The relationship between our results and the claims made in [9] and [10] is examined in Section VII. Conclusions are discussed in Section VIII.

II. Problem Formulation

Background material on the tomographic problem can be found in many sources, including [1], [11] and the references therein. The image or object f we consider, is an element of the space $L_2(\mathbb{B}_2)$ of square-integrable real-valued functions supported on the closed unit ball $\mathbb{B}_2$ in the plane. Let us define the range of view angles to be $\Omega \triangleq [-\pi, \pi]$. We consider the reconstruction of f from a set of parallel-beam line integral projections, taken at P view angles $\theta_i, i \in I \triangleq \{1, \ldots, P\}$, the components of the vector $\boldsymbol{\theta} \in \Omega^P \triangleq [-\pi, \pi]^P$. Such a set of projections, known as a sinogram, is an element of $\{L_2(\mathbb{B})\}^P$, the P-wise Cartesian product of L_2 real-valued functions supported on the interval $\mathbb{B} = [-1, 1]$. The sinogram, in turn, is related to the original image by the discrete-angle Radon transform operator $R_\theta : L_2(\mathbb{B}_2) \to \{L_2(\mathbb{B})\}^P$ as $g = R_\theta f$, where each of the P projections, defined by

$$R_{\boldsymbol{\theta}} f(s, i) = \int_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}} f(s\cos\theta_i - t\sin\theta_i, s\sin\theta_i + t\cos\theta_i)\, dt \quad (1)$$

for $s \in [-1, 1]$ corresponds to the collection of line integrals through f in the direction perpendicular to θ_i.

As a simplification, we will assume throughout that the view angles are π-*distinct*. Two view angles are said to be π-distinct if they are not exactly π radians apart, i.e., $|\theta_1 - \theta_2| \neq \pi$. The necessity of making this definition comes about from the fact that projections at view angles that are *not* π-distinct are degenerately related. Specifically, suppose $\theta_i = \theta_j + \pi$ for some $i \neq j \in I$. Then we trivially have that $g(s, i) = g(-s, j)$ from (1). Hence, projections at angles that are not π-distinct do not constitute "new" information.

Our goal is to recover the view angles θ_i from the projection data $g(s, i)$. It is immediately clear, however, that unique recovery of the view angles is not possible. To see why, consider Fig. 1. The objects in this figure are related by orthogonal transformations, i.e., a rotation through some angle, plus a possible reflection about some axis through the origin in the plane of the object. Clearly, given a set of P projections from one of these objects, it is impossible to tell in which orientation the original object was. Hence, the set of projections could have come from an infinite number of objects, all related by orthogonal transformations. Furthermore, each of these objects will have an associated set of view angles, different from $\boldsymbol{\theta}$, which will yield exactly the same set of P projections. Thus, we cannot hope to recover the view angles from the projections alone. For practical purposes, however, this ambiguity is acceptable, unless we are directly interested in the orientation of the object.

Using the properties of the Radon transform, we can readily establish that orthogonal transformations of the object define an equivalence class on the view angles $\boldsymbol{\theta}$.

Definition 1 (Equivalence on Ω^P): Two vectors of view angles $\boldsymbol{\theta}, \hat{\boldsymbol{\theta}} \in \Omega^P$ are equivalent, denoted $\boldsymbol{\theta} \sim \hat{\boldsymbol{\theta}}$, if $\exists \sigma \in \{-1, 1\}, c \in \Omega$ such that for every $i \in I$, there exists an integer n_i for which

$$\theta_i = \sigma\hat{\theta}_i + c + 2\pi n_i.$$

A useful interpretation of this equivalence relation is provided by considering the reconstruction of f from the projection data. Suppose that f is estimated from g via the filtered backprojection (FBP) reconstruction algorithm. Then it follows from elementary properties of the Radon transform (and its adjoint, the backprojection) that if f_1 is an FBP reconstruction from projections g at view angles $\boldsymbol{\theta}$, and f_2 is an FBP reconstruction from projections g at view angles $\hat{\boldsymbol{\theta}} \sim \boldsymbol{\theta}$, then f_1 and f_2 will be related by an orthogonal transformation (rotation by $c \bmod 2\pi$, and possibly a reflection for $\sigma = -1$).

With this notion of equivalence in hand, we make the following definition.

Definition 2 [Angle Recovery Problem (ARP)]: Given a set of P projections, determine the angles θ_i at which they were acquired, or any equivalent set of angles.

In the scenarios in which an angle recovery problem might arise, there is generally also an unknown translation of the object associated with each projection. Recall that if $f'(x, y) = f(x - \delta_x, y - \delta_y)$ then $R_\theta f'(s, \theta) = R_\theta f(s - \delta_x \cos\theta - \delta_y \sin\theta, \theta)$ [11]. Thus, translations of the object being imaged result in shifts of the projections. Furthermore, using the same argument we used in association with Fig. 1, we argue that if the projections are each shifted by an unknown quantity δ_i, the vector of shifts $\boldsymbol{\delta}$ cannot be uniquely recovered from the projection data, because δ_i and $\delta_i - \delta_x \cos\theta_i - \delta_y \sin\theta_i$ for arbitrary δ_x and δ_y are both valid estimates of δ_i.

Hence, we define the following equivalence relation.

Definition 3 (Equivalence on $\mathbb{R}^P \times \Omega^P$): Two pairs of view angles and shift estimate vectors $(\boldsymbol{\delta}, \boldsymbol{\theta}), (\hat{\boldsymbol{\delta}}, \hat{\boldsymbol{\theta}}) \in \mathbb{R}^P \times \Omega^P$ are equivalent, denoted $(\boldsymbol{\delta}, \boldsymbol{\theta}) \sim (\hat{\boldsymbol{\delta}}, \hat{\boldsymbol{\theta}})$ iff $\exists \sigma \in \{-1, 1\}, c \in \Omega, \delta_x, \delta_y \in \mathbb{R}$ such that for every $i \in I$

$$\hat{\delta}_i = \delta_i + \delta_x \cos\theta_i + \delta_y \sin\theta_i \quad (2)$$

and there exists an integer n_i for which

$$\theta_i = \sigma\hat{\theta}_i + c + 2\pi n_i. \quad (3)$$

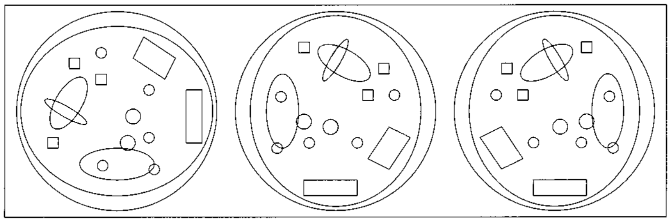

Fig. 1. Examples of the equivalence class transformations applied to a hypothetical geometrical object.

Our second goal is summarized in the following definition.

Definition 4 [Shift-Angle Recovery Problem (SHARP)]: Given a set of P projections, each shifted by some unknown amount, determine the shifts and view angles, or any equivalent set of shifts and view angles.

The ARP (respectively SHARP) will be said to admit a unique solution if all solutions to it are equivalent. The ARP formulation is less practical than the SHARP, because in most problems, unknown view angles are generally accompanied by unknown shifts in the projections as well. However, the ARP is considerably simpler to analyze, and the SHARP can be reduced to a related ARP through a simple transformation. We will formulate and present our main results for both problems simultaneously, but will treat the ARP in detail. Once uniqueness for the ARP, is established, the results for the SHARP quickly follow.

III. Consistency Conditions and Main Results

In this section, we will state the main results of our work, first laying the mathematical foundations for uniqueness in the form of consistency conditions. Physical interpretations of the conditions developed in this section, as well as detailed proofs, will be presented in subsequent sections.

The primary tool for studying the ARP will be the Helgasson–Ludwig (HL) consistency conditions, which characterize elements in the range space of R_θ. These conditions are generally phrased in terms of geometric moments of the object and its projections. Geometric moments of the object $f \in L_2(\mathbb{B}_2)$ being imaged are defined as

$$m_{i,k} = \iint_{\mathbb{B}_2} x^i y^k f(x,y)\, dx\, dy \quad i \geq 0, \quad k \geq 0. \tag{4}$$

Object moments of order d are those that satisfy $i + k = d$. We also need to define moments of elements[1] of $\{L_2(\mathbb{B})\}^P$

$$\mu_d(i) = \int_{\mathbb{B}} s^d g(s,i)\, ds \quad i \in I, \quad d \geq 0.$$

Throughout this section, we let D denote an arbitrary fixed set of moment orders. Then we denote by $\mathbf{m}$ the set of object moments of order $d \in D$, and $\eta(D) \triangleq |\mathbf{m}|$ (where $|A|$ denotes the cardinality of set A). We have suppressed the dependence of $\mathbf{m}$ on D to avoid complicating the notation.

[1]We do not include radial sampling in our formulation of the Radon transform. Because our results ultimately only use a finite set of moments of the projections, and not the continuous projections themselves, radial sampling can be incorporated into the problem formulation without changing the results.

Our focus will be on the following set of trigonometric polynomials, which are parameterized by the object moments

$$\check{\mathcal{Q}}_d(\theta; \mathbf{m}) = \sum_{r=0}^{d} \binom{d}{r} m_{r,d-r} \cos^r(\theta) \sin^{d-r}(\theta), \quad d \in D. \tag{5}$$

We will refer to the polynomials $\check{\mathcal{Q}}_d(\theta; \mathbf{m})$ as the HL polynomials. They form the basis of the HL conditions, which relate the moments of the object $m_{k,l}$ to the moments of the projections $\mu_i(k)$. The HL conditions can be found in [11]. We will use a variant of them that is more useful for our purposes.

Theorem 1 (HL Conditions): If $g = R_\theta f$ for some $\boldsymbol{\theta} \in \Omega^P$, and $f \in L_2(\mathbb{B}_2)$, then for $d \geq 0, i \in I$

$$\mu_d(i) = \check{\mathcal{Q}}_d(\theta_i; \mathbf{m}). \tag{6}$$

The HL conditions effectively state that the dth order moment of the projection at angle θ_i is a trigonometric polynomial of order d in θ_i with coefficients determined by the dth order object moments. Also, Theorem 1 incorporates no *a priori* assumptions about the object f, other than its square-integrability and support.

To use the HL conditions to solve the ARP, let us suppose that we fix a set D of moment orders. Then we seek to find $\hat{\boldsymbol{\theta}}, \hat{\mathbf{m}}$ that satisfy

$$\check{\mathcal{Q}}_d(\hat{\theta}_i; \hat{\mathbf{m}}) - \mu_d(i) = 0 \quad d \in D, \quad i \in I. \tag{7}$$

Thus, we propose to determine the view angles $\boldsymbol{\theta}$ by finding a set of object moments $\hat{\mathbf{m}}$ and view angles $\hat{\boldsymbol{\theta}}$ that produce projection moments that match $\mu_d(i)$ for $d \in D$. We will refer to (7) as "data matching", for obvious reasons. Note that (7) is a system of equations in $\hat{\boldsymbol{\theta}}$ and $\hat{\mathbf{m}}$, and that by Theorem 1, $(\hat{\boldsymbol{\theta}}, \hat{\mathbf{m}}) = (\boldsymbol{\theta}, \mathbf{m})$ is a solution to this system of equations. However, the problem remains to determine when this solution is unique. Unfortunately, no complete theory exists to answer such a question. Hence, we must examine these equations for particular choices of D to determine how many solutions exist, and how they relate to $(\boldsymbol{\theta}, \mathbf{m})$.

Before proceeding to examine this system, however, we must first deal with equivalent solutions as we discussed in Section II. We know that we can only determine a solution to the ARP to within an orthogonal transformation of the object, and that this defines an equivalence relation on Ω^P. We will also extend it to an equivalence relation on $\mathbb{R}^{\eta(D)}$.

Definition 5 (Equivalence of Moment Sets): $\mathbf{m} \sim \hat{\mathbf{m}}$ if there exist $s \in \{-1, 1\}$, and $c \in \Omega$ such that

$$\check{\mathcal{Q}}_d(s\theta + c; \mathbf{m}) - \check{\mathcal{Q}}_d(\theta; \hat{\mathbf{m}}) = 0 \quad \forall d \in D, \quad \theta \in \Omega. \tag{8}$$

It is trivial to show that this definition constitutes a legitimate equivalence relation. The physical interpretation is a bit trickier. If $\mathbf{m}$ and $\hat{\mathbf{m}}$ are moments for an object and an orthogonal transformation of that object respectively, then it follows by Theorem 1 that $\mathbf{m} \sim \hat{\mathbf{m}}$. However, in general, if $\mathbf{m} \sim \hat{\mathbf{m}}$, then it is not necessarily the case that $\mathbf{m}$ and $\hat{\mathbf{m}}$ are moments of the same object. Instead, we can think of $\mathbf{m}$ and $\hat{\mathbf{m}}$ as being sets of moments of two objects f and $\hat{f}$, respectively. If we replace f by its approximation f_D using only the moments of order D (with all other moments set to zero), and likewise for $\hat{f}$ to obtain $\hat{f}_D$, then we find that f_D and $\hat{f}_D$ are related by an orthogonal transformation. This interpretation also follows from Theorem 1.

In light of this equivalence relation on the moment sets, we can modify our uniqueness question. We now ask whether (7) has any solutions $(\hat{\boldsymbol{\theta}}, \hat{\mathbf{m}})$ where either $\hat{\boldsymbol{\theta}} \not\sim \boldsymbol{\theta}$ or $\hat{\mathbf{m}} \not\sim \mathbf{m}$. Such a situation will then be considered nonunique. Otherwise, if all solutions $(\hat{\boldsymbol{\theta}}, \hat{\mathbf{m}})$ to (7) satisfy $(\hat{\boldsymbol{\theta}}, \hat{\mathbf{m}}) \sim (\boldsymbol{\theta}, \mathbf{m})$, i.e., are equivalent to the correct value, we say that the solution is unique. This convention will simplify the analysis and proof of the main results. Although we are ultimately interested in establishing $\hat{\boldsymbol{\theta}} \sim \boldsymbol{\theta}$, considering the moments as well simplifies the solution. It turns out that the solution is almost always unique in this sense, even when D is very small. To make this notion of "almost always" precise (in a measure-theoretic sense), we first introduce the notion of a *D-generic set*. For two sets A, B, let $A \setminus B$ denote the set difference of A and B.

Definition 6 (D-Generic Set): A set S of objects is a D-generic set[2] if the set

$$B = \{D\text{-indexed moments of objects in } L_2(\mathbb{B}_2)\backslash S\}$$

is such that B is a nowhere dense set of measure zero in $\mathbb{R}^{\eta(D)}$.

Before continuing, let us briefly examine this concept of a D-generic set. First, note that if a property A holds for a D-generic set, then it holds with probability one for objects whose D-indexed moments are drawn at random from an absolutely continuous probability distribution. Secondly, if a set is D-generic, then we are guaranteed some measure of "robustness" with respect to membership in the set. Because the set B of Definition 6 is nowhere dense and measure zero, it follows that a D-generic set has a nonempty interior. Thus, given an object f that belongs to the interior of a D-generic set S, we can perturb that object and it will still belong to S, provided the perturbations of its D-indexed moments are sufficiently small. Finally, if a property A holds for a D-generic set, then it also holds for "almost all objects," i.e., the statement that A is true for a D-generic set is *stronger* than the statement that A is true for "almost all objects" (in the sense of Lebesgue measure on $L_2(\mathbb{B}_2)$ [12])[3]. The following lemma will also prove useful. It is a consequence of the fact that a finite union of nowhere dense sets of zero measure is a nowhere dense set of measure zero.

Lemma 1: Suppose that A and B are D-generic sets. Then $A \cap B$ is also D-generic.

With this notion of a D-generic set, we can state our uniqueness theorem for the angle recovery problem.

Theorem 2 (ARP Uniqueness Theorem): Suppose the projection data are acquired at P π-distinct view angles $\boldsymbol{\theta} = \{\theta_1, \theta_2, \ldots, \theta_P\}$. Let S be the set of objects for which the only view angles $\hat{\boldsymbol{\theta}}$ that produce the same projection moments of order $\{1, 2\}$ are equivalent to $\boldsymbol{\theta}$, i.e., $\boldsymbol{\theta} \sim \hat{\boldsymbol{\theta}}$. If $P > 8$, then S is $\{1, 2\}$-generic. Hence, the set of objects, for which the ARP has a unique solution for $P > 8$ is $\{1, 2\}$-generic.

In view of the previous discussion of D-generic, we obtain the following immediate (weaker) corollary.

Corollary 1: Suppose $\theta_1, \ldots, \theta_P$ are π-distinct. If $P > 8$, then the ARP has a unique solution for almost all objects.

The precise characterization of degenerate objects is discussed in Section V, along with the physical interpretation of those degeneracies. In general, however, the characterization of degenerate object moments as nowhere dense implies that degenerate objects are rare. These and some additional nuances are all addressed by the more technical Theorem 9 of Section IV.

A similar result also holds for the SHARP, which we state here for completeness.

Theorem 3 (SHARP Uniqueness Theorem): Suppose the projection data are acquired at P π-distinct view angles $\boldsymbol{\theta} = \{\theta_1, \theta_2, \ldots, \theta_P\}$ at shifts $\boldsymbol{\delta} = \{\delta_1, \delta_2, \ldots, \delta_P\}$. Let S be the set of objects for which the only view angles $\hat{\boldsymbol{\theta}}$ and shifts $\hat{\boldsymbol{\delta}}$ that produce the same projection moments of order $\{0, 1, 2, 3\}$ are equivalent to $\boldsymbol{\theta}$ and $\boldsymbol{\delta}$ respectively, i.e., $(\boldsymbol{\theta}, \boldsymbol{\delta}) \sim (\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\delta}})$. If $P > 24$, then S is $\{0, 1, 2, 3\}$-generic set. Hence, the set of objects, such the SHARP has a unique solution for $P > 24$, is $\{0, 1, 2, 3\}$-generic.

Corollary 2: Suppose $\theta_1, \ldots, \theta_P$ are π-distinct. If $P > 24$, then the SHARP has a unique solution for almost all objects.

IV. General Sufficient Conditions for the ARP

A. Arbitrary View Angles

We now turn to the problem of finding sufficient conditions for uniqueness of solutions to the ARP. Let us define the following bivariate trigonometric polynomials, parameterized by $\mathbf{m}$ and $\hat{\mathbf{m}}$

$$\check{\mathcal{P}}_d(\theta, \hat{\theta}; \mathbf{m}, \hat{\mathbf{m}}) = \check{\mathcal{Q}}_d(\theta; \mathbf{m}) - \check{\mathcal{Q}}_d(\hat{\theta}; \hat{\mathbf{m}}). \tag{9}$$

We can then rewrite (7) as

$$\check{\mathcal{P}}_d(\theta_i, \hat{\theta}_i; \mathbf{m}, \hat{\mathbf{m}}) = 0 \quad \forall i \in I, \quad d \in D. \tag{10}$$

We seek the set of $\boldsymbol{\theta}$ and $\mathbf{m}$ that lead to a unique solution of (10), i.e., the *nondegenerate set* $N^P_{(\theta, \mathbf{m})} \subset \Omega^P \times \mathbb{R}^{\eta(D)}$ defined by the following definition.

Definition 7 (Nondegenerate Angle and Moment Set): A pair of view angles and moments belongs to the nondegenerate set, denoted

$$(\boldsymbol{\theta}, \mathbf{m}) \in N^P_{(\boldsymbol{\theta}, \mathbf{m})}$$

if and only if the condition

$$\{\check{\mathcal{P}}_d(\theta_i, \hat{\theta}_i; \mathbf{m}, \hat{\mathbf{m}}) = 0 \, \forall i \in I, d \in D\}$$

implies that $\hat{\boldsymbol{\theta}} \sim \boldsymbol{\theta}$ and $\mathbf{m} \sim \hat{\mathbf{m}}$.

Hence, $N^P_{(\boldsymbol{\theta}, \mathbf{m})}$ is a collection of objects and view angles that lead to a unique solution of (10). Equivalently, view angles for projections from objects in $N^P_{(\boldsymbol{\theta}, \mathbf{m})}$ are uniquely determined by applying the data matching condition for this D. Our goal is to characterize the set $N^P_{(\boldsymbol{\theta}, \mathbf{m})}$, but we will first make a series of simplifications, whose only purpose is to cast the question of

[2]Although Definition 6 is stated for real objects, it (along with the results of this paper) can be extended to complex objects in a straightforward manner. For such objects, we require the set B of Definition 6 be a nowhere dense set of measure zero in $\mathbb{C}^{\eta(D)}$.

[3]In particular, using the sets defined in Definition 6, if B is a nowhere dense set of measure zero in $\mathbb{R}^{\eta(D)}$, then $L_2(\mathbb{B}_2)\backslash S$ must be a nowhere dense set of zero measure in $L_2(\mathbb{B}_2)$ (however, the converse is not necessarily true). Thus if a property A is true for a D-generic set S, then it can only fail on $\Gamma\backslash S$, which must have measure zero, and thus A is true for almost all objects.

uniqueness into a form that can be conveniently handled by existing theory. These simplifications take the form of a sequence of lemmas, each of which produce successively narrower sufficient conditions for membership in $N^P_{(\boldsymbol{\theta},\mathbf{m})}$. The proof of each lemma is extremely simple—it is the chain of reasoning that we wish to emphasize here.

Let us first examine the set $N^P_{\mathbf{m}}$ of object moments that lead to a unique solution of (10) with respect to $\mathbf{m}$, temporarily ignoring the possible nonuniqueness of $\boldsymbol{\theta}$. First, define the set $\bar{\Omega}^P = \{\boldsymbol{\theta} \in \Omega^P, \theta_i \neq \theta_j, i \neq j\}$ as the set of all vectors of P distinct (but not necessarily π-distinct) view angles.

Definition 8 (Nondegenerate Object Moments):

$$\begin{aligned} N^P_{\mathbf{m}} = \{\mathbf{m} : \forall \boldsymbol{\theta} \in \bar{\Omega}^P, \{\check{\mathcal{P}}_d(\theta_i, \hat{\theta}_i; \mathbf{m}, \hat{\mathbf{m}}) = 0,\\ \forall i \in I, d \in D \text{ for some } \hat{\boldsymbol{\theta}}(\boldsymbol{\theta}, \mathbf{m}), \hat{\mathbf{m}}(\boldsymbol{\theta}, \mathbf{m})\}\\ \Rightarrow \hat{\mathbf{m}} \sim \mathbf{m}\}. \end{aligned}$$

The set $N^P_{\mathbf{m}}$, is then the set for which the *moments* are uniquely determined by applying data matching conditions. The first lemma gives a sufficient condition for $\mathbf{m}$ to belong to $N^P_{\mathbf{m}}$.

Lemma 2: Let $\mathbf{m}$ be such that $\forall \hat{\mathbf{m}} \not\sim \mathbf{m}$

$$|\{(\theta, \hat{\theta}) \in \Omega^2 : \check{\mathcal{P}}_d(\theta, \hat{\theta}; \mathbf{m}, \hat{\mathbf{m}}) = 0, \forall d \in D\}| < P$$

i.e., if $\hat{\mathbf{m}}$ is not equivalent to $\mathbf{m}$, then (10) has less than P distinct solutions in Ω^2. Then $\mathbf{m} \in N^P_{\mathbf{m}}$.

Proof: Let $\boldsymbol{\theta} \in \bar{\Omega}^P, \hat{\boldsymbol{\theta}}$ and $\hat{\mathbf{m}}$ be such that $\check{\mathcal{P}}_d(\theta_i, \hat{\theta}_i; \mathbf{m}, \hat{\mathbf{m}}) = 0, \forall i \in I, d \in D$. Then we have that $|\{(\theta, \hat{\theta}) \in \Omega^2 : \check{\mathcal{P}}_d(\theta, \hat{\theta}; \mathbf{m}, \hat{\mathbf{m}}) = 0, \forall d \in D\}| \geq P$, which by assumption on $\mathbf{m}$, implies that $\mathbf{m} \sim \hat{\mathbf{m}}$. Thus, by Definition 8, $\mathbf{m} \in N^P_{\mathbf{m}}$. ■

This lemma is important because it defines nondegeneracy in terms of the set $\{(\theta, \hat{\theta}) \in \Omega^2 : \check{\mathcal{P}}_d(\theta, \hat{\theta}; \mathbf{m}, \hat{\mathbf{m}}) = 0, \forall d \in D\}$ for *fixed* $\mathbf{m}$ and $\hat{\mathbf{m}}$, which is the set of zeros to a system of trigonometric polynomials with fixed coefficients. Sets of this form can be handled using tools from algebraic geometry, which can provide bounds of the required form as a function of $\mathbf{m}$ and $\hat{\mathbf{m}}$. A few further transformations are necessary, however, before we can directly apply these tools.

Let us make the following extension from Ω to the complex plane:

$$\cos(\theta) \mapsto \frac{1}{2}(z + z^{-1}) \quad \sin(\theta) \mapsto \frac{1}{2j}(z - z^{-1})$$

where $z \in \mathbb{C}$. Note that this extension is an identity when $z = e^{j\theta}$. Substituting into (5) and regrouping coefficients and relabeling yields the following complex HL rational functions:

$$\mathcal{Q}_d(z; \mathbf{m}) = \sum_{\substack{n=1\\ n \text{ odd}}}^{d} (\alpha_{d,n}(\mathbf{m}) z^n + \alpha_{d,n}(\mathbf{m})^* z^{-n}) \tag{11}$$

for d odd, and

$$\begin{aligned} \mathcal{Q}_d(z; \mathbf{m}) = \alpha_{d,0}(\mathbf{m})\\ + \sum_{\substack{n=2\\ n \text{ even}}}^{d} (\alpha_{d,n}(\mathbf{m}) z^n + \alpha_{d,n}(\mathbf{m})^* z^{-n}) \end{aligned} \tag{12}$$

for d even. The HL polynomials and complex rational functions are related by $\check{\mathcal{Q}}_d(\theta; \mathbf{m}) = \mathcal{Q}_d(e^{j\theta}; \mathbf{m})$. For the sake of notational brevity, we will suppress the dependence of the individual coefficients on the set of moment parameters, and denote the coefficients of $\mathcal{Q}_d(z; \mathbf{m})$ and $\mathcal{Q}_d(z; \hat{\mathbf{m}})$ by $\alpha_{d,n}$ and $\hat{\alpha}_{d,n}$, respectively.

Similarly, the bivariate polynomials $\check{\mathcal{P}}_d(\theta, \hat{\theta}; \mathbf{m}, \hat{\mathbf{m}})$ of (9) can be extended (this time to $\mathbb{C}^2$) in the sense that

$$\mathcal{P}_d(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}}) = \mathcal{Q}_d(z_1; \mathbf{m}) - \mathcal{Q}_d(z_2; \hat{\mathbf{m}}) \tag{13}$$

so that

$$\mathcal{P}_d(e^{j\theta}, e^{j\hat{\theta}}; \mathbf{m}, \hat{\mathbf{m}}) = \check{\mathcal{P}}_d(\theta, \hat{\theta}; \mathbf{m}, \hat{\mathbf{m}}).$$

We refer to $\mathcal{Q}_d(z; \mathbf{m})$ (respectively $\mathcal{P}_d(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}})$) as *polynomial forms*, with the understanding that they can be readily converted into polynomials by multiplying by the appropriate monomial. Let us also define the unit torus

$$T = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| = |z_2| = 1\}$$

which is the set of points in $\mathbb{C}^2$ for which the rational functions "agree" with the HL polynomials.

In terms of these polynomial forms, we have the following lemma.

Lemma 3: Let $\mathbf{m}$ be such that $\forall \hat{\mathbf{m}} \not\sim \mathbf{m}$

$$|\{(z_1, z_2) \in \mathbb{C}^2 : \mathcal{P}_d(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}}) = 0, \ \forall d \in D\}| < P$$

i.e., if $\hat{\mathbf{m}}$ is not equivalent to $\mathbf{m}$, then the system of equations $\mathcal{P}_d(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}}) = 0, \forall d \in D$ has less than P solutions (total). Then $\mathbf{m} \in N^P_{\mathbf{m}}$.

Proof:

$$\begin{aligned} &|\{(\theta, \hat{\theta}) \in \Omega^2 : \check{\mathcal{P}}_d(\theta, \hat{\theta}; \mathbf{m}, \hat{\mathbf{m}}) = 0, \ \forall d \in D\}|\\ &= |\{(z_1, z_2) \in T : \mathcal{P}_d(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}}) = 0, \ \forall d \in D\}|\\ &\leq |\{(z_1, z_2) \in \mathbb{C}^2 : \mathcal{P}_d(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}}) = 0, \ \forall d \in D\}|. \end{aligned}$$

For $\hat{\mathbf{m}} \not\sim \mathbf{m}$, the rhs is less than P so that $\mathbf{m} \in N^P_{\mathbf{m}}$ by Lemma 2. ■

For the case that $D = \{k, l\}$, the conditions of Lemma 3 are addressed by the simplest form of Bezout's Theorem, which bounds the number of common zeros of a pair of polynomials in $\mathbb{C}^2$. The version of Bezout's Theorem cited below is a slight modification tailored to polynomials with rectangular coefficient support.

Theorem 4 (Zakhor and Izraelevitz [13]): If two bivariate polynomials of the form $p_A(x, y)$ and $p_B(x, y)$

$$\begin{aligned} p_A(x, y) = \sum_{q=0}^{A_x} \sum_{r=0}^{A_y} a(q, r) x^q y^r = 0\\ p_B(x, y) = \sum_{q=0}^{B_x} \sum_{r=0}^{B_y} b(q, r) x^q y^r = 0 \end{aligned} \tag{14}$$

have no common factors of degree greater than zero (i.e. they are relatively prime), then they have at most $A_x B_y + B_x A_y$ common zeros.

The following simple corollaries of Theorem 4 relate Lemma 3 to the structure of the two polynomials of interest, noting that the degrees of $z_1^k z_2^k \mathcal{P}_k(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}})$ and $z_1^l z_2^l \mathcal{P}_l(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}})$ are $2k$ and $2l$, respectively, when viewed as polynomials in either z_1 or z_2. Hence, we obtain for $p_A = z_1^k z_2^k \mathcal{P}_k(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}})$

and $p_B = z_1^l z_2^l \mathcal{P}_l(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}})$ the constants $A_x = A_y = 2k$ and $B_x = B_y = 2l$.

Corollary 3: If the polynomials $z_1^k z_2^k \mathcal{P}_k(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}})$ and $z_1^l z_2^l \mathcal{P}_l(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}})$ have no common factors of degree greater than zero, then they have at most $8kl$ common zeros.

Corollary 4: Suppose $D = \{k, l\}$, and let $\mathbf{m}$ be such that $\forall \hat{\mathbf{m}} \not\sim \mathbf{m}$, $z_1^k z_2^k \mathcal{P}_k(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}})$ and $z_1^l z_2^l \mathcal{P}_l(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}})$ are relatively prime. Then, $\mathbf{m} \in N_{\mathbf{m}}^P$ for $P > 8kl$.

The following theorem summarizes our characterization of $N_{\mathbf{m}}^P$ in terms of the polynomial forms $\mathcal{P}_k(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}})$ and $\mathcal{P}_l(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}})$.

Theorem 5: Let $\mathbf{m}$ be such that for any $\hat{\mathbf{m}}$ for which $\mathcal{P}_k(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}})$ and $\mathcal{P}_l(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}})$ have a common factor $\mathcal{F}(z_1, z_2)$ of nonzero degree, this is the only such factor, and has the form

$$\mathcal{F}(z_1, z_2) = \gamma \cdot \left(z_1 - e^{jc} z_2^{\sigma}\right) \tag{15}$$

for some $\sigma \in \{-1, 1\}$, $c \in \Omega$, $\gamma \in \mathbb{C}$. Then $\mathbf{m} \in N_{\mathbf{m}}^P$ for $P > 8kl$.

Proof: Let $\mathbf{m}$ satisfy the conditions of the theorem, and suppose $\hat{\mathbf{m}} \not\sim \mathbf{m}$. We claim that $\mathcal{P}_l(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}})$ and $\mathcal{P}_k(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}})$ are relatively prime. Suppose otherwise. Then by the assumptions on $\mathbf{m}$, the common factor is of the form (15). Now, $\mathcal{F}(e^{j(\sigma\theta + c)}, e^{j\theta}) = 0$ for all θ. But this implies that

$$\check{\mathcal{Q}}_d(s\theta + c; \mathbf{m}) = \check{\mathcal{Q}}_d(\theta; \hat{\mathbf{m}}), \quad \forall \theta \in \Omega$$

for all $d \in D = \{k, l\}$. Thus, by Definition 5, $\hat{\mathbf{m}} \sim \mathbf{m}$, which is a contradiction. Hence, $\mathcal{P}_l(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}})$ and $\mathcal{P}_k(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}})$ are relatively prime, and the result follows by Corollary 4. ■

In the next section, we will find specific conditions on $\mathbf{m}$ such that Theorem 5 applies. However, we cannot proceed without first dealing with the uniqueness of (10) with respect to $\boldsymbol{\theta}$. So far, we have simply demonstrated that under the right conditions, (10) has a unique solution for $\mathbf{m}$ (to within an equivalence class). However, we are far more interested in the acquisition angles. Fortunately, these results come fairly easily given the results we have derived so far.

Definition 9: The *identifiable angle set* (IAS) is a set valued function of $\mathbf{m}$ such that $\theta \in \mathrm{IAS}(\mathbf{m})$ if and only if $\{\check{\mathcal{P}}_d(\theta, \hat{\theta}; \mathbf{m}, \mathbf{m}) = 0 \, \forall d \in D\}$ implies that $\hat{\theta} = \theta$.

The terminology is meant to suggest that the θ that belong to $\mathrm{IAS}(\mathbf{m})$ are determined uniquely by $\mathbf{m}$, i.e., that for the given $\mathbf{m}$ (which in turn depends on D), there is no alternate view angle $\hat{\theta} \neq \theta$ that satisfies the HL conditions for all $d \in D$. We will also be interested in the complement of this set.

Definition 10: The *Unidentifiable Angle Set* (UAS) is defined by $\mathrm{UAS}(\mathbf{m}) = \Omega \backslash \mathrm{IAS}(\mathbf{m})$.

Theorem 6: If $\mathbf{m} \in N_{\mathbf{m}}^P, \boldsymbol{\theta} \in \overline{\Omega}^P$, and $\theta_i \in \mathrm{IAS}(\mathbf{m}), \forall i \in I$, then $(\boldsymbol{\theta}, \mathbf{m}) \in N_{(\boldsymbol{\theta}, \mathbf{m})}^P$. Thus, if $\mathbf{m}$ is a nondegenerate set of object moments, and all of the view angles $\boldsymbol{\theta}$ belong to the IAS for $\mathbf{m}$, then the pair $(\boldsymbol{\theta}, \mathbf{m})$ is a nondegenerate set of view angles and object moments.

Proof: Let $\mathbf{m}, \boldsymbol{\theta}$ satisfy the stated conditions. If for some $\hat{\mathbf{m}}$, $\hat{\boldsymbol{\theta}}$, we have

$$\check{\mathcal{P}}_d(\theta_i, \hat{\theta}_i; \mathbf{m}, \hat{\mathbf{m}}) = 0 \quad \forall i \in I, \quad d \in D$$

then by the definition of $N_{\mathbf{m}}^P$, we have $\hat{\mathbf{m}} \sim \mathbf{m}$. Furthermore, by the definitions of equivalence and $\check{\mathcal{P}}_d(\theta_i, \hat{\theta}_i; \mathbf{m}, \hat{\mathbf{m}})$, we then have

$$\begin{aligned} \check{\mathcal{Q}}_d(\theta_i; \mathbf{m}) &= \hat{\mathcal{Q}}_d(\hat{\theta}_i; \hat{\mathbf{m}}) \\ &= \check{\mathcal{Q}}_d(c\hat{\theta}_i + \sigma; \mathbf{m}) \quad \forall i \in I, \quad d \in D \end{aligned}$$

for some $\sigma \in \{-1, 1\}$ and $c \in \Omega$. Thus for $\theta_i' = c\hat{\theta}_i + \sigma$, $\check{\mathcal{P}}_d(\theta_i, \theta_i'; \mathbf{m}, \mathbf{m}) = 0$ for $i \in I, d \in D$. By Definition 9, we then have $\boldsymbol{\theta} = \boldsymbol{\theta}' \sim \hat{\boldsymbol{\theta}}$. ■

The following theorem demonstrates that the unidentified angle set is small for the objects that we have been discussing so far.

Theorem 7: Let $D = \{k, l\}$, and suppose $\mathbf{m}$ satisfies the conditions of Theorem 5. Then $|\mathrm{UAS}(\mathbf{m})| \leq 2(2k-1)(2l-1)$.

Proof: To prove this result, we need to determine the number of solutions $(\theta, \hat{\theta})$ to $\check{\mathcal{P}}_d(\theta, \hat{\theta}; \mathbf{m}, \mathbf{m}) = 0$, $\forall d \in D$ with $\theta \neq \hat{\theta}$. Equivalently, we can find the maximum number $\nu_{(k,\ell)}$ of solutions (z_1, z_2) with $z_1 \neq z_2$ on T to $\mathcal{P}_d(z_1, z_2; \mathbf{m}, \mathbf{m}) = 0$, $d \in D$. The symmetry of $\mathcal{P}_d(z_1, z_2; \mathbf{m}, \mathbf{m})$ implies that if $\mathcal{P}_d(e^{j\theta}, e^{j\hat{\theta}}; \mathbf{m}, \mathbf{m}) = 0$, then $\mathcal{P}_d(e^{j\hat{\theta}}, e^{j\theta}; \mathbf{m}, \mathbf{m}) = 0$. Hence each of the solution pairs $\{(e^{j\theta}, e^{j\hat{\theta}}), (e^{j\hat{\theta}}, e^{j\theta})\}$ contributes exactly two unique angles $\theta, \hat{\theta}$ to the UAS. Thus, we can bound $|\mathrm{UAS}(\mathbf{m})| \leq \nu_{(k,\ell)}$.

By (13), $\mathcal{P}_d(z_1, z_2; \mathbf{m}, \mathbf{m}) = 0$ for $z_1 = z_2$. Hence $z_1 - z_2$ is a factor of $\mathcal{P}_d(z_1, z_2; \mathbf{m}, \mathbf{m})$ for all $d \in D$, and we can write

$$z_1^d z_2^d \mathcal{P}_d(z_1, z_2; \mathbf{m}, \mathbf{m}) = (z_1 - z_2)\mathcal{G}_d(z_1, z_2; \mathbf{m}) \qquad d \in \{k, l\} \tag{16}$$

where $\mathcal{G}_k(z_1, z_2; \mathbf{m})$ and $\mathcal{G}_l(z_1, z_2; \mathbf{m})$ are polynomials of degree $2k - 1$ and $2l - 1$, respectively, and have no common factors by the conditions of Theorem 5 (note that $(z_1 - z_2)$ is of the form (15)). Furthermore, since $\nu_{(k,\ell)}$ counts solutions of the form (z_1, z_2) with $z_1 \neq z_2$, we can cancel the factor $(z_1 - z_2)$ common to both equations in (16), and apply Theorem 4 to the system

$$\begin{aligned} \mathcal{G}_k(z_1, z_2; \mathbf{m}) &= 0 \\ \mathcal{G}_l(z_1, z_2; \mathbf{m}) &= 0 \end{aligned} \tag{17}$$

to obtain a bound of $\nu_{(k,\ell)} \leq 2(2k - 1)(2l - 1)$. Hence the number of different angles in $\mathrm{UAS}(\mathbf{m})$ is bounded above by $2(2k - 1)(2l - 1)$. ■

B. Tightening the Conditions for π-Distinct Angles

The bound on the minimum P stated in Theorem 5 can be tightened by a factor of two if we know that the projection angles are π-distinct. Because many of the arguments are minor variations on the sequence of proofs used in the previous subsection, we summarize the argument in one step. We assume that $D = \{k, l\}$ as before.

Theorem 8: Let $\mathbf{m}$ be such that for any $\hat{\mathbf{m}}$ for which $\mathcal{P}_k(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}})$ and $\mathcal{P}_l(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}})$ have a common factor $\mathcal{F}(z_1, z_2)$ of nonzero degree, this is the only such factor, and has the form $\mathcal{F}(z_1, z_2) = \gamma \cdot (z_1 - e^{jc} z_2^{\sigma})$ for some $\sigma \in \{-1, 1\}, c \in \Omega, \gamma \in \mathbb{C}$. Suppose further that $\theta_i \in \mathrm{IAS}(\mathbf{m})$ for $i \in I$, where θ_i are π-distinct. Then $(\boldsymbol{\theta}, \mathbf{m}) \in N_{(\boldsymbol{\theta}, \mathbf{m})}^P$, for $P > 4k\ell$.

Proof: Suppose that for some $\hat{\mathbf{m}}$, $\hat{\boldsymbol{\theta}}$

$$\check{\mathcal{P}}_d(\theta_i, \hat{\theta}_i; \mathbf{m}, \hat{\mathbf{m}}) = 0 \quad i \in I, \quad d \in D.$$

Then the corresponding polynomial forms $\mathcal{P}_k(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}})$ and $\mathcal{P}_l(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}})$ have $>4kl$ zeros of the form $(e^{j\theta_i}, e^{j\hat{\theta}_i})$. Now, note from (11) and (12), that $\mathcal{Q}_d(-z; \mathbf{m}) = -\mathcal{Q}_d(z; \mathbf{m})$ for d odd, and $\mathcal{Q}_d(-z; \mathbf{m}) = \mathcal{Q}_d(z; \mathbf{m})$ for d even. Thus, if $\mathcal{P}_d(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}}) = 0$ then $\mathcal{P}_d(-z_1, -z_2; \mathbf{m}, \hat{\mathbf{m}}) = 0$ as well. Thus, $\mathcal{P}_k(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}})$ and $\mathcal{P}_l(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}})$ have $>4kl$ zeros of the form $(e^{j(\theta_i+\pi)}, e^{j(\hat{\theta}_i+\pi)})$, and by the assumption that the θ_i are π-distinct, $(e^{j(\theta_i+\pi)}, e^{j(\hat{\theta}_i+\pi)}) \neq (e^{j\theta_h}, e^{j\hat{\theta}_h})$ for $i \neq h$. Thus, $\mathcal{P}_k(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}})$ and $\mathcal{P}_l(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}})$ have $>8kl$ common zeros, which by Corollary 3 implies that they have a common factor of nonzero degree. We can then apply the exact same reasoning as employed in the proof of Theorem 5 to conclude $\mathbf{m} \sim \hat{\mathbf{m}}$ so that $\mathbf{m} \in N^P_{\mathbf{m}}$. Because $\theta_i \in \mathrm{IAS}(\mathbf{m})$ for $i \in I$, it follows by Theorem 6 that $\boldsymbol{\theta} \sim \hat{\boldsymbol{\theta}}$. ∎

V. ARP UNIQUENESS RESULTS FOR SPECIAL CASES

In this section, we focus on the characterization of $N^P_{(\boldsymbol{\theta},\mathbf{m})}$ with $D = \{k, l\}$ for specific k and l. In addition to yielding geometric interpretations of the degeneracies for specific choices of D, such characterizations pave the way for practical applications of the uniqueness results.

Our primary results provide sufficient conditions on a given $\mathbf{m}$ for membership in $N^P_{(\boldsymbol{\theta},\mathbf{m})}$ for $D = \{1, 2\}$ and $D = \{2, 3\}$. Note that Theorems 9 and 10 also tighten the bounds on the cardinality of the UAS by roughly a factor of two over Theorem 7. We begin by stating a condition on the moments of orders 1 and 2. The equivalent statement of this condition is derived directly from the definitions of the HL polynomials, and the proof omitted for brevity:

(i) $\nexists \beta, \gamma \in \mathbb{R}$ such that

$$\check{\mathcal{Q}}_2(\theta; \mathbf{m}) = \beta[\check{\mathcal{Q}}_1(\theta; \mathbf{m})]^2 + \gamma$$

for all $\theta \in \Omega$, or equivalently

$$\det \begin{bmatrix} m_{1,0}^2 & m_{2,0} & 1 \\ 2m_{1,0}m_{0,1} & m_{1,1} & 0 \\ m_{0,1}^2 & m_{0,2} & 1 \end{bmatrix} \neq 0.$$

Theorem 9: Let $D = \{1, 2\}$, and suppose $\mathbf{m}$ satisfies Condition (i). Then for $\boldsymbol{\theta} \in \Omega^P$ such that $\theta_i \in \mathrm{IAS}(\mathbf{m})$ for $i \in I$, θ_i π-distinct, we have $(\boldsymbol{\theta}, \mathbf{m}) \in N^9_{(\boldsymbol{\theta},\mathbf{m})}$. Furthermore,

$$\mathrm{UAS}(\mathbf{m}) = \left\{ \arg\left(\sqrt{\frac{-c_1^*}{c_1}} \right), \arg\left(-\sqrt{\frac{-c_1^*}{c_1}} \right) \right\} \tag{18}$$

where

$$c_1 = \frac{1}{2}(m_{1,0} - jm_{0,1}). \tag{19}$$

The following Corollary serves to interpret Theorem 9.

Corollary 5: Suppose the projection data are acquired at P π-distinct view angles $\boldsymbol{\theta}$, and $P > 8$. Furthermore, suppose that the set of object moments $\mathbf{m}$ satisfies Condition (i) of Theorem 9. If $\theta_i \notin \mathrm{UAS}(\mathbf{m})$ (defined by (18)), then the only other view angles $\hat{\boldsymbol{\theta}}$ that produce the same projection moments of order $D = \{1, 2\}$ are equivalent to $\boldsymbol{\theta}$, i.e., the ARP admits a unique solution.

The next theorem requires different conditions on the moments to be satisfied.

(ii) $\check{\mathcal{Q}}_2(\theta; \mathbf{m}) \not\equiv \beta$, or equivalently, $m_{1,1} \neq 0$ or $m_{2,0} \neq m_{0,2}$.

(iii) $\nexists \kappa \in \Omega$ such that $\check{\mathcal{Q}}_3(\kappa + \phi; \mathbf{m}) = \check{\mathcal{Q}}_3(\kappa - \phi; \mathbf{m})$ for all $\phi \in \Omega$, i.e., $\alpha_{3,1} \neq 0, \alpha_{3,3} \neq 0$ and $\arg \alpha_{3,3} \neq 3 \arg \alpha_{3,1}$, or equivalently,

$$3 \tan^{-1}\left(\frac{m_{1,2} + m_{3,0}}{m_{0,3} + m_{2,1}} \right) - \tan^{-1}\left(\frac{3m_{1,2} - m_{3,0}}{m_{0,3} - 3m_{2,1}} \right) \neq 0, \tag{20}$$

where $3m_{1,2} - m_{3,0}$ and $m_{0,3} - 3m_{2,1}$ are not both zero, and likewise $m_{1,2} + m_{3,0}$ and $m_{0,3} + m_{2,1}$ are not both zero.

Theorem 10: Let $D = \{2, 3\}$, and suppose $\mathbf{m}$ satisfies Conditions (ii) and (iii). Then for $\boldsymbol{\theta} \in \Omega^P$ such that $\theta_i \in \mathrm{IAS}(\mathbf{m})$ for $i \in I$, θ_i π-distinct, we have $(\boldsymbol{\theta}, \mathbf{m}) \in N^{25}_{(\boldsymbol{\theta},\mathbf{m})}$. Furthermore

$$\begin{aligned} \mathrm{UAS}(\mathbf{m}) = \{\theta : \check{\mathcal{Q}}_3(\theta; \mathbf{m}) = 0\} &\cup \{\theta : U(\theta, \rho) = 0\} \\ &\cup \{\theta : U(\theta, -\rho) = 0\} \end{aligned} \tag{21}$$

where

$$\begin{aligned} U(\theta, \rho) &= e^{j\theta}(\rho^3 c_1 - c_1^*) + \rho(\rho^3 c_1 + \rho^2 c_2 - \rho c_2^* - c_1^*)e^{j2\theta} \\ &\quad + \rho^5 c_1 - \rho^2 c_1^* \end{aligned} \tag{22}$$

$$\begin{aligned} c_1 &= \frac{1}{8}[(m_{3,0} - 3m_{1,2}) + j(m_{0,3} - 3m_{2,1})] \\ c_2 &= \frac{3}{8}[(m_{3,0} + m_{1,2}) + j(m_{0,3} + m_{2,1})] \end{aligned} \tag{23}$$

and

$$\rho = \sqrt{\frac{m_{2,0} - m_{0,2} + j2m_{1,1}}{m_{2,0} - m_{0,2} - j2m_{1,1}}}. \tag{24}$$

The following Corollary to Theorem 10 is analogous to Corollary 5.

Corollary 6: Suppose the projection data are acquired at P π-distinct view angles $\boldsymbol{\theta}$, and $P \geq 25$. Furthermore, suppose that the set of object moments $\mathbf{m}$ satisfies Conditions (ii) and (iii) of Theorem 10. If $\theta_i \notin \mathrm{UAS}(\mathbf{m})$ (defined by (21)), then the only other view angles $\hat{\boldsymbol{\theta}}$ that produce the same projection moments of order $D = \{2, 3\}$ are equivalent to $\boldsymbol{\theta}$, i.e., the ARP admits a unique solution.

While Theorem 9 will suffice to prove the ARP uniqueness result of Theorem 2, Theorem 10 will be necessary to establish the SHARP uniqueness results to be treated later. Because Theorems 9 and 10 are similar in nature, however, we present and prove them simultaneously.

Let us pause briefly to consider the geometric interpretation of Theorems 9 and 10. If we use only moments of order $D = \{1, 2\}$ to solve for the view angles, then nondegeneracy is defined in terms of Condition (i) of Theorem 9 on the first- and second-order object moments. Now, any given object can be replaced by an elliptical object of uniform density with identical first- and second-order moments, by an appropriate choice of minor and major axes, and proper rotation and translation to the object's center of mass. Condition (i) has the following implications.

- The first moment of the projections is not identically zero for all view angles.
- The vector through the origin and the object's center of mass is not colinear with either axis of the corresponding elliptical object. This guarantees that the second moment is not simply the square of the first.
- The object does not have spherical inertia, i.e., $\mu_2(\theta)$ varies as a function of θ.

For the $D = \{2, 3\}$ case, Condition (ii) implies that the object does not have spherical inertia. Condition (iii) has the following implications:

- not all third order object moments vanish;
- there is a lack of even symmetry on the third projection moment as a function of view angle.

We now turn our attention to proving Theorems 9, and 10; first establishing several auxiliary results on the factorization of certain polynomial forms. Proofs of the technical lemmas have been deferred to Appendix A.[4]

Lemma 4: The polynomial form $\mathcal{R}_1(x,y) = a_1 x + a_2 x^{-1} + a_3 y + a_4 y^{-1}$ is reducible if and only if $a_1 a_2 = a_3 a_4$, with the factorization

$$\mathcal{R}_1(x,y) = a_1 x^{-1}\left(x + \frac{a_2}{a_3} y^{-1}\right)\left(x + \frac{a_2}{a_4} y\right). \qquad (25)$$

Furthermore, if $|a_1| = |a_2|$ and $|a_3| = |a_4|$, then (25) can be written as

$$\mathcal{R}_1(x,y) = a_1 x^{-1}\left(x - e^{jc_1} y^{-1}\right)\left(x - e^{jc_2} y\right)$$

where $e^{jc_1} = -(a_2/a_3)$, and $e^{jc_2} = -(a_2/a_4)$.

Lemma 5: Suppose $2 \in D$, and let $\mathbf{m}$ and $\hat{\mathbf{m}}$ be such that $\nexists \beta, \beta' \in \mathbb{R}$ such that $\mathcal{Q}_2(z;\mathbf{m}) \equiv \beta$ or $\mathcal{Q}_2(z_2;\hat{\mathbf{m}}) \equiv \beta'$. Then $\mathcal{P}_2(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}})$ is reducible iff

$$|\alpha_{2,0} - \hat{\alpha}_{2,0}| = 2\left||\alpha_{2,2}|^2 \pm |\hat{\alpha}_{2,2}|^2\right|$$

with factorization (to within a constant)

$$\begin{aligned}\mathcal{P}_2(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}}) &= \left(b_1 z_1 + b_2 z_1^{-1} + b_3 z_2 + b_4 z_2^{-1}\right)\\ &\quad \cdot \left(b_1 z_1 + b_2 z_1^{-1} - b_3 z_2 - b_4 z_2^{-1}\right) \quad (26)\end{aligned}$$

where $b_1^2 = \alpha_{2,2}$, $b_2^2 = \alpha_{2,2}$, $b_3^2 = \hat{\alpha}_{2,2}$, and $b_4^2 = \hat{\alpha}_{2,2}$. $\mathcal{P}_2(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}})$ is further reducible iff $|\alpha_{2,2}|^2 = |\hat{\alpha}_{2,2}|^2$, with the factorization

$$\begin{aligned}\mathcal{P}_2(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}}) &= a_1 z_1^{-2}\left(z_1 + e^{jc_1} z_2^{-1}\right)\left(z_1 - e^{jc_1} z_2^{-1}\right)\\ &\quad \cdot \left(z_1 + e^{jc_2} z_2\right)\left(z_1 - e^{jc_2} z_2\right) \qquad (27)\end{aligned}$$

where $e^{jc_1} = b_2/b_3$, and $e^{jc_2} = b_2/b_4$.

Lemma 6: Suppose $\{1,2\} \subset D$, and let $\mathbf{m}$ and $\hat{\mathbf{m}}$ be such that $\nexists \beta, \beta' \in \mathbb{R}$ such that $\mathcal{Q}_2(z;\mathbf{m}) \equiv \beta$ or $\mathcal{Q}_2(z;\hat{\mathbf{m}}) \equiv \beta'$. If $\mathcal{P}_1(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}})$ is a factor of $\mathcal{P}_2(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}})$, then $\exists \beta, \gamma \in \mathbb{R}$ such that $\mathcal{Q}_2(z;\mathbf{m}) = \beta(\mathcal{Q}_1(z;\mathbf{m}))^2 + \gamma$.

Lemma 7: Let $k \in D$, and suppose $\mathbf{m}$ is such that $\nexists \beta \in \mathbb{R}$ for which $\mathcal{Q}_k(z;\mathbf{m}) \equiv \beta$, and suppose $\hat{\mathbf{m}}$ is such that

$$\mathcal{P}_k(z_1(i), z_2(i); \mathbf{m}, \hat{\mathbf{m}}) = 0 \quad i \in I \qquad (28)$$

where $\boldsymbol{\theta}, \hat{\boldsymbol{\theta}} \in \Omega^P$ for $P > 2k$. Then $\nexists \beta \in \mathbb{R}$ such that $\mathcal{Q}_k(z;\hat{\mathbf{m}}) \equiv \beta$.

[4]We have verified the technical lemmas (which are primarily algebraic in nature) with a computer algebra package. The details are available in [14]

With the aid of these results, we can now prove Theorem 9.

Proof: [Theorem 9]: Let $\hat{\mathbf{m}}$ be such that $\mathcal{P}_1(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}})$ and $\mathcal{P}_2(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}})$ have a common factor of nonzero degree. Lemma 7 together with Condition (i) imply that $\mathcal{Q}_1(z;\hat{\mathbf{m}}) \not\equiv 0$ and that $\nexists \beta \in \mathbb{R}$ for which $\mathcal{Q}_2(z;\hat{\mathbf{m}}) \equiv \beta$. We can now apply Lemma 6 and Condition (i) to conclude that $\mathcal{P}_1(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}})$ is not a factor of $\mathcal{P}_2(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}})$. Hence, $\mathcal{P}_1(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}})$ must be reducible. Applying Lemma 4 to $\mathcal{P}_1(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}})$ implies that the common factor must be of the form $(z_1 - e^{jc} z_2^s)$ where $s = \pm 1$. The result $(\boldsymbol{\theta}, \mathbf{m}) \in N^9_{(\boldsymbol{\theta}, \mathbf{m})}$ then follows by Theorem 8.

To complete the proof, we will explicitly derive the UAS for any $\mathbf{m}$ and $D = \{1,2\}$. Following the same arguments used in the proof of Theorem 7, we note that $\theta \in \mathrm{UAS}(\mathbf{m})$ if and only if $(e^{j\theta}, e^{j\phi})$ is a solution to the following system, (which is (17) specialized to the case $D = \{1,2\}$):

$$\begin{aligned}\mathcal{G}_1(z_1, z_2; \mathbf{m}) &= c_1\left(z_1 z_2 - \frac{c_1^*}{c_1}\right) = 0\\ \mathcal{G}_2(z_1, z_2; \mathbf{m}) &= b_1(z_1 + z_2)\left(z_1^2 z_2^2 - \frac{b_1^*}{b_1}\right) = 0 \qquad (29)\end{aligned}$$

with c_1 defined by (19), and b_1 given by

$$b_1 = \frac{1}{4}(m_{2,0} - m_{0,2}) - \frac{j}{2} m_{1,1}. \qquad (30)$$

It is simple to show by eliminating z_1, for example, that (29) has at exactly two solutions on T, which leads directly to (18). ∎

Proof: [Theorem 2]: It is straightforward to show that the set S_1 of all $\mathbf{m}$ that satisfy Condition (i) of Theorem 9 is the complement of an algebraic set (i.e., a set on which a system of polynomial equations vanish simultaneously), and is thus $\{1,2\}$-generic set. Similarly, for $\boldsymbol{\theta}$ fixed, the set of all $\mathbf{m}$ for which $\mathrm{UAS}(\mathbf{m}) \cap \boldsymbol{\theta} = \emptyset$ is also $\{1,2\}$-generic. By Corollary 5, for $\mathbf{m} \in S_1 \cap S_2$, the UAS admits a unique solution (to within an equivalence class). By Lemma 1, $S_1 \cap S_2$ is $\{1,2\}$-generic. Thus, the set S of objects for which the ARP has a unique solution must be $\{1,2\}$-generic, because it contains $S_1 \cap S_2$. ∎

The following are the results used to prove Theorem 10.

Lemma 8: Suppose $3 \in D$, and let $\mathbf{m}$ and $\hat{\mathbf{m}}$ be such that $\check{\mathcal{Q}}_3(\theta;\mathbf{m}) \not\equiv 0$, $\check{\mathcal{Q}}_3(\hat{\theta};\hat{\mathbf{m}}) \not\equiv 0$. Suppose further that $\mathcal{P}_3(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}})$ contains a factor of the form $b_1 z_1 + b_2 z_1^{-1} + b_3 z_2 + b_4 z_2^{-1}$ for $b_1, b_2, b_3, b_4 \neq 0$. Then $\exists \kappa \in \Omega$ such that $\check{\mathcal{Q}}_3(\kappa + \theta; \mathbf{m}) = \check{\mathcal{Q}}_3(\kappa - \theta; \mathbf{m})$ for all $\theta \in \Omega$.

Lemma 9: Suppose $\{2,3\} \subset D$, and let $\mathbf{m}$ and $\hat{\mathbf{m}}$ be such that $\mathcal{Q}_2(z;\mathbf{m}) \not\equiv \beta \in \mathbb{R}$, $\mathcal{Q}_3(z;\mathbf{m}) \not\equiv 0$, $\mathcal{Q}_2(z;\hat{\mathbf{m}}) \not\equiv \beta \in \mathbb{R}$ and $\mathcal{Q}_3(z;\hat{\mathbf{m}}) \not\equiv 0$. Then $\mathcal{P}_2(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}})$ cannot be a factor of $\mathcal{P}_3(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}})$.

Lemma 10: Suppose $3 \in D$, and let $\mathbf{m}$ be such that $\mathcal{Q}_3(z;\mathbf{m}) \not\equiv 0$. Then $\mathcal{P}_3(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}})$ contains no factors of the form $z_1^2 - e^{jc} z_2^{2\sigma}$ where $c \in \Omega$, and $\sigma \in \{1, -1\}$..

We can now prove Theorem 10.

Proof: [Theorem 10]: Let $\hat{\mathbf{m}}$ be such that $z_1^2 z_2^2 \mathcal{P}_2(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}})$, and $z_1^3 z_2^3 \mathcal{P}_3(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}})$ have a common factor of nonzero degree. Lemma 7 with Conditions (ii) and (iii) implies that $\mathcal{Q}_3(z;\hat{\mathbf{m}}) \not\equiv 0$, and that $\nexists \beta \in \mathbb{R}$ for which $\mathcal{Q}_2(z;\hat{\mathbf{m}}) \equiv \beta$. We can now apply Condition (iii) and Lemma 9 to conclude that $\mathcal{P}_2(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}})$ is not a factor of $\mathcal{P}_3(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}})$. Hence, $\mathcal{P}_2(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}})$ must be reducible, and by Lemma 5 must either factor in accordance with just (26) or both (26) and (27). Suppose now that $\mathcal{P}_2(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}})$ factors in accordance with (26) and not (27). Then the common factor must be of the form

$$b_1 z_1 + b_2 z_1^{-1} + b_3 z_2 + b_4 z_2^{-1}$$

for $b_1, b_2, b_3, b_4 \neq 0$. But this is a contradiction by Condition (iii) and Lemma 8. Thus by Lemma 5, $\mathcal{P}_2(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}})$ factors in accordance with (27). Factors of the form $[(z_1 + e^{jc_1} z_2)(z_1 - e^{jc_1} z_2)]$ and $[(z_1 + e^{jc_2} z_2^{-1})(z_1 - e^{jc_2} z_2^{-1})]$ cannot be common to $\mathcal{P}_2(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}})$ and $\mathcal{P}_3(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}})$ by Lemma 10. Factors of the form $[(z_1 \pm e^{jc_1} z_2)(z_1 \pm e^{jc_2} z_2^{-1})]$ can be written as $b_1 z_1 + b_2 z_1^{-1} + b_3 z_2 + b_4 z_2^{-1}$ for some $b_1, b_2, b_3, b_4 \neq 0$, and thus cannot be common to $\mathcal{P}_3(z_1, z_2; \mathbf{m}, \hat{\mathbf{m}})$ by Lemma 8. Hence, the common factor must be exactly one of the form $(z_1 - e^{jc} z_2)$ or $(z_1 - e^{jc} z_2^{-1})$ for some $c \in \Omega$. The result $(\hat{\boldsymbol{\theta}}, \mathbf{m}) \in N^{25}_{(\boldsymbol{\theta}, \mathbf{m})}$ then follows by Theorem 8.

To complete the proof, we will explicitly derive the UAS for any $\mathbf{m}$ and $D = \{2, 3\}$. As in Theorem 9, we are interested in solutions to (17) for $D = \{2, 3\}$. After some simple algebraic manipulations, (17) can be written in the following form:

$$\begin{aligned} \mathcal{G}_2(z_1, z_2; \mathbf{m}) &= b_1(z_1 + z_2)(z_1 z_2 - \rho)(z_1 z_2 + \rho) = 0 \\ \mathcal{G}_3(z_1, z_2; \mathbf{m}) &= z_1^3 z_2^3 \left(z_1^2 + z_1 z_2 + z_2^2\right) c_1 + z_1^3 z_2^3 c_2 \\ &\quad - z_1^2 z_2^2 c_2^* - \left(z_1^2 + z_1 z_2 + z_2^2\right) c_1^* = 0 \end{aligned} \quad (31)$$

where c_1, c_2 are defined by (23), ρ is defined by (24) and b_1 is defined by (30). Next, we note that if $\mathcal{G}_2(z_1, z_2; \mathbf{m}) = 0$, then either $z_2 = -z_1$, $z_2 = \rho z_1^{-1}$ or $z_2 = -\rho z_1^{-1}$. Substituting each of these possibilities into $\mathcal{G}_3(z_1, z_2; \mathbf{m})$ and simplifying then yields (21). ■

VI. Uniqueness Results for the SHARP

Having established uniqueness of the ARP under some conditions, we now turn to the SHARP. Recall that in solving the SHARP, we wish to recover the unknown shifts of each projection, as well as solve the associated ARP. In the absence of truncation effects (i.e., the translated projections remain supported on $\mathbb{B}$), we can model the measured sinogram as

$$\tilde{g}(s, i) = g(s - \delta_i, i)$$

where $\boldsymbol{\delta} \in \mathbb{R}^P$ is a vector of unknown shifts, and $g(s, i)$ is a sinogram in which the projections are untranslated.

Instead of attempting to prove Theorem 3 from scratch, we choose an indirect approach, by first using the first- and zero-order projection moments to estimate the δ_i's, and then solve an ARP. Let $\tilde{\mu}_k(i)$ be the geometric moments of $\tilde{g}$, and let $\tilde{\mu}_0 = \tilde{\mu}_0(i) = \mu_0(i)$, an immediate consequence of the HL conditions.

Lemma 11: Assume $\tilde{\mu}_0 \neq 0$, let

$$\hat{\delta}_i = \frac{-\tilde{\mu}_1(i)}{\tilde{\mu}_0} \quad (32)$$

and let $g'(s, i) = \tilde{g}(s - \hat{\delta}_i, i)$. Then $g' = R_{\boldsymbol{\theta}} f'$ where

$$f'(x, y) = f(x - m_{1,0}/m_{0,0}, y - m_{0,1}/m_{0,0}). \quad (33)$$

Proof: Follows directly from (1) and (6) for $d = 1$. ■

Lemma 11 suggests that we can convert the SHARP into an ARP for a shifted version of the underlying object *at the same acquisition angles*. The following lemma provides a sufficient condition for uniqueness of a solution to the SHARP based on this observation.

Lemma 12: Suppose the projection data are acquired at P π-distinct view angles $\boldsymbol{\theta}$, and $P > 25$. Furthermore, suppose that f is such that $m_{0,0} \neq 0$. Let $\mathbf{m}'$ denote the set of object moments corresponding to f', constructed via (33). If $\mathbf{m}'$ satisfies Conditions (ii) and Conditions (iii) of Theorem 10, and If $\theta_i \notin \mathrm{UAS}(\mathbf{m})$ [defined by (21)], then the SHARP admits a unique solution.

Proof: From Lemma 11, it follows that $\hat{\boldsymbol{\delta}} \sim \boldsymbol{\delta}$. By Corollary 6 it also follows that the ARP for f' has a unique solution, so that $\hat{\boldsymbol{\theta}} \sim \boldsymbol{\theta}$. From these two results, it follows that $(\boldsymbol{\delta}, \boldsymbol{\theta}) \sim (\hat{\boldsymbol{\delta}}, \hat{\boldsymbol{\theta}})$, and the SHARP has a unique solution. ■

Lemma 12 is sufficient to establish Theorem 3.

Proof: [Theorem 3]: For any object f with moments $\mathbf{m}$, let $\mathbf{m}'$ denote the moments of f', the translate of f with its center of mass at the origin. It is simple to show that $m'_{i,k}$ is a polynomial function of $m_{j,l}$ for $j + l \leq i + k$. Let S_1 denote the set of $\mathbf{m}$ such that $m_{0,0} \neq 0$. Then S_1 is trivially $\{0, 2, 3\}$-generic. Also, the set S_2 of $\mathbf{m}$ such that $\mathbf{m}'$ satisfies Condition (ii) of Theorem 10 is the complement of an algebraic set, and is thus $\{0, 2, 3\}$-generic. The set S_3 of $\mathbf{m}$ such that $\mathbf{m}'$ satisfies Condition (iii) of Theorem 10 is also the complement of an algebraic set, and is thus $\{0, 2, 3\}$-generic.[5] Finally, by (21), it follows that for $\boldsymbol{\theta}$ fixed, the set S_4 of $\mathbf{m}$ such that $\mathrm{UAS}(\mathbf{m}') \cap \boldsymbol{\theta} = \emptyset$ is $\{0, 2, 3\}$-generic. By Lemma 12, for objects such that $\mathbf{m} \in S_1 \cap S_2 \cap S_3 \cap S_4 \cap S_5$, the SHARP has a unique solution. But by Lemma 1, this set is $\{0, 2, 3\}$-generic. Thus, the set S of objects for which the SHARP has a unique solution must be $\{0, 2, 3\}$-generic, because it contains $\cap_{i=1}^5 S_i$. ■

To summarize, a solution equivalent to the correct one can be found for the SHARP by the following procedure:

1) shift all projections so that their center of mass is at the origin;
2) compute moments of order k and l;
3) solve the ARP for $D = \{k, l\}$.

VII. Relation to Existing Results

The SHARP has been partially addressed in the works of Salzman and Goncharov with varying degrees of success [9], [10]. Both Salzman and Goncharov use a moment based approach to solve the SHARP, but with different levels of justification. In [9], no explicit claim of uniqueness of the solution is either made or proven, the author relying on an *ad hoc* approach to justify the uniqueness of the solution. The approach in [10] is more rigorous, but the author makes a number of conjectures which run contrary to the results presented in this paper. We will address these issues, and simply note that the conjectures and claims made in [9] are a subset of those in [10].

The first conjecture made in [10] is that uniqueness of the solution to the SHARP requires only that the object being imaged not have vanishing second and third order moments, and that this is "generally" satisfied by asymmetric objects. While it is true that asymmetric objects will "generally" have nonzero

[5] It is possible, using tan addition formulae, to convert the left hand side of Condition (iii) into a rational function of the moments.

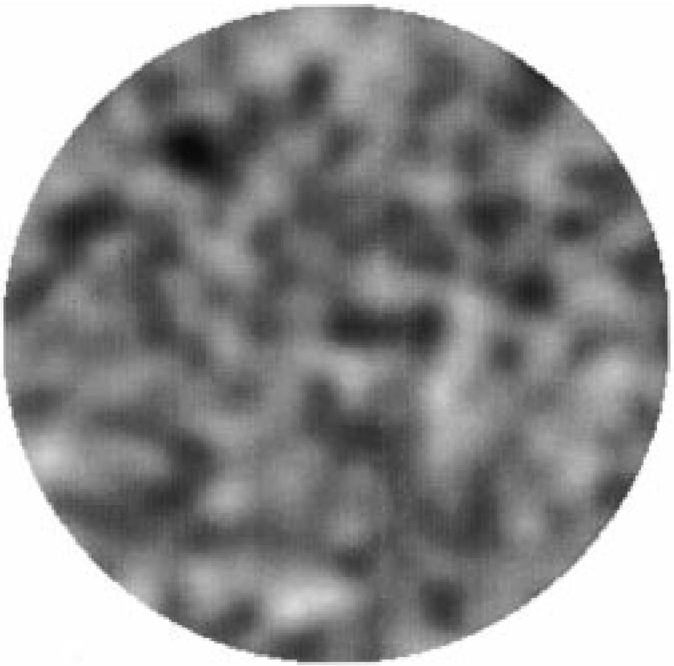

Fig. 2. Example of an asymmetric object with spherically symmetric inertia.

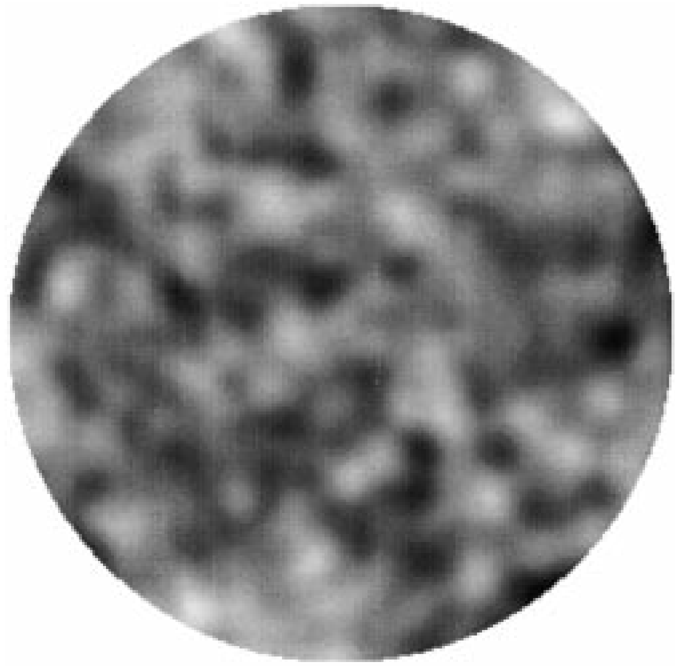

Fig. 4. Example of an asymmetric object whose third order moments violate the derived degeneracy conditions of Theorem 10.

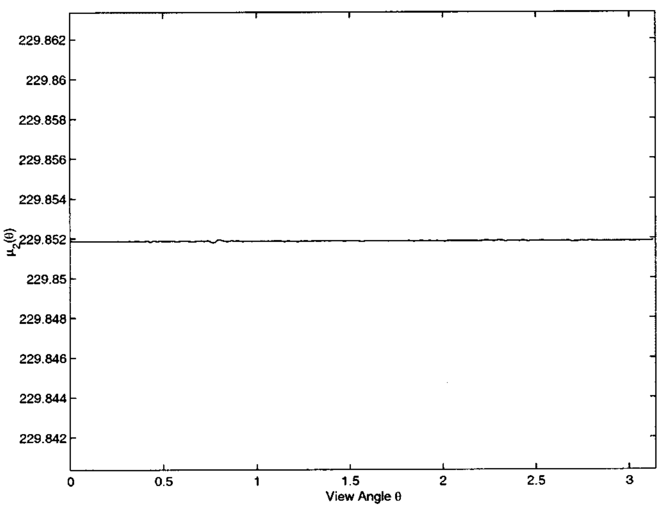

Fig. 3. Second moment of projections of Fig. 2 as a function of view angle.

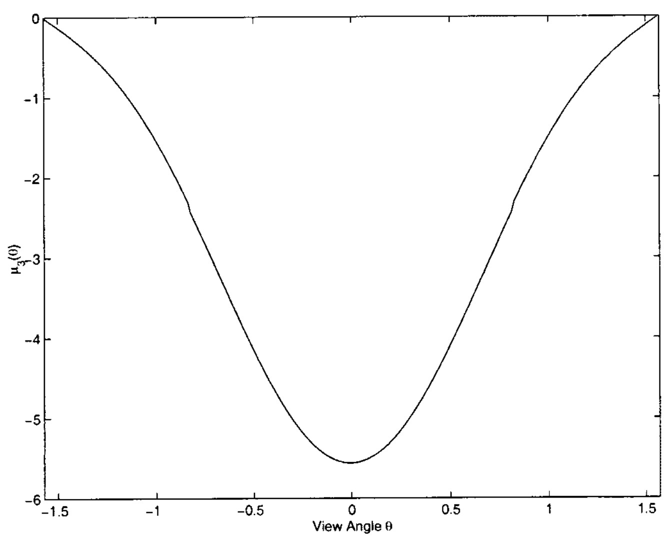

Fig. 5. Third moment of projections of Fig. 4 as a function of view angle. Note the even symmetry.

second and third order moments, this is not sufficient to guarantee uniqueness, as evidenced by the conditions of Theorem 10, and our geometrical interpretation thereafter. It is a simple matter to construct objects without vanishing second and third order moments, or any lines of symmetry, and yet a unique solution to the SHARP does not exist using only second and third order moments. Consider objects, for example, that have spherically symmetric inertia. One such object, is depicted in Fig. 2, along with Fig. 3 which shows the second moment of its projections as a function of view angle, and is constant to within the precision of the computations. This object was constructed by taking a smoothed noise field and modifying its moments so that it had spherical inertia. Clearly Goncharov's approach to solving for the view angles, which assumes that the second order moment is not a constant, would fail with this object. According to the conjectures made in [10], however, there is no reason to suspect that such an object would cause the proposed solution method to fail.

A second conjecture made in [10] is that if the system of equations

$$\check{\mathcal{P}}_d(\theta, \hat{\theta}; \mathbf{m}, \hat{\mathbf{m}}) = 0 \quad \forall d \in D \tag{34}$$

for $D = \{2, 3\}$ has more than 36 solutions, then $\theta = \hat{\theta}$. Casting aside the issue of equivalence of solutions, this conjecture is also false. Consider the counterexample depicted in Figs. 4 and 5. This object violates Condition (iii) of Theorem 10, which is our sufficient condition for unique recovery of the view angles from projection moments of order $\{2, 3\}$, and yet this object is highly asymmetric, and has no vanishing moments, so that it does not violate the assumptions of the conjectures in [10]. However, in this case, the third projection moment (but *not* the object) contains a line of symmetry with respect to view angle, such that *every* projection can be assigned to one of two view angles $\hat{\theta} = \phi_1$ or ϕ_2, both of which will satisfy $\check{\mathcal{P}}_d(\theta_i, \hat{\theta}_i; \mathbf{m}, \hat{\mathbf{m}}) = 0$ for $d \in \{2, 3\}$. Hence, for this object, it is impossible to uniquely determine the projection angles from the second and third order moment information alone.[6]

The final conjecture made in [10], is that $P > 7$ is a sufficient condition for uniqueness of the solution to the SHARP from projection moments of up to third order. Using the theory developed in this paper, it is fairly easy to construct counterexamples to this conjecture via two distinct approaches. The first approach is to note that there is no concept of the UAS in [10], and hence, for any object chosen at random, we can construct a set of $P = 8$ projection angles that are not uniquely determined, by picking one or more of these angles from the UAS, defined by (21)–(24). Note that because the UAS contains θ for which $\check{\mathcal{Q}}_3(\theta; \mathbf{m}) = 0$, it *must* be nonempty for every object, since it is trivial to show that $\check{\mathcal{Q}}_3(\theta; \mathbf{m}) = 0$ always has at least one solution.

The second sense in which this conjecture fails is more direct. We now choose a set of eight projection angles at random, and then search for objects that are not uniquely determine by these angles. However, we restrict our attention to synthetic object-angle pairs that are nondegenerate in the sense that $(\boldsymbol{\theta}, \mathbf{m}) \in N^{25}_{(\boldsymbol{\theta}, \mathbf{m})}$. Such a search can be performed numerically by minimizing a least squares cost function over $\hat{\boldsymbol{\theta}}, \mathbf{m}, \hat{\mathbf{m}}$ for N different starting points. We can then examine these numerical solutions to determine if they are in fact degenerate in the sense of Theorem 3, and if $\hat{\boldsymbol{\theta}} \sim \boldsymbol{\theta}$. If not, then they constitute a counterexample to the sufficiency of $P = 8$ projections. An example of such a solution is depicted in Fig. 6. This figure depicts the plots of the third versus the second order projection moments as a parametric function of view angle for two different objects. Each point at which the two curves intersect, corresponds to a pair of view angles $(\theta, \hat{\theta})$ at which the two (dissimilar) objects have identical second and third order projection moments. Furthermore, the location of these intersections was chosen *a priori* in a random manner. Hence, although with $P > 24$ projections it would be possible to uniquely determine the view angles, this is not possible with $P = 8$. Again, to demonstrate that this counterexample is not a freak occurrence, we repeated the construction for 100 randomly chosen $\boldsymbol{\theta}$. In all 100 cases, the construction succeeded (for $N = 100$), and $\boldsymbol{\theta} \not\sim \hat{\boldsymbol{\theta}}$. Thus, we have demonstrated experimentally that the conjecture that eight projections is sufficient is false in the following senses:

- for a randomly chosen object, there are sets of eight projections that are not uniquely determined;
- for a set of eight randomly chosen angles, there are nondegenerate objects for which these angles are nondegenerate, and yet the view angles are not uniquely determined.

Unlike the approaches taken in previous work, the framework that we have established to demonstrate uniqueness is sufficiently general to allow more powerful results to be developed using stronger algebraic results than those used here. For example, an open problem suggested by the approach we have taken is to derive general uniqueness conditions for arbitrary choices of D. Finding the tightest possible lower bounds on P for uniqueness is also an open problem, as are "generic" results that use generic properties of solutions to polynomial systems of equations.

[6]This example was generated in the same manner as Fig. 2, i.e., a smoothed noise field was perturbed so as to obtain the resulting symmetry.

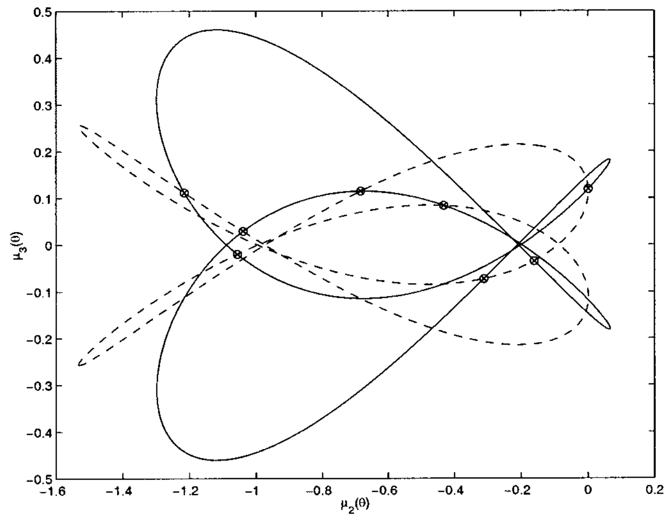

Fig. 6. Randomly generated example of insufficiency of $P = 8$ projections. Plotted is the second versus third projection moment for two objects as a function of angle. Crosses indicate eight randomly chosen θ_i at which the data matching condition for $D = \{2, 3\}$ is satisfied.

Finally, the generality of the approach taken in this work will hopefully set the stage for more powerful and general results. For example, the following lemmas illustrates two immediate generalizations of the results presented so far that can be useful in a practical context.

Lemma 13: If $D' \supset D$ then $\mathrm{IAS}(\mathbf{m}_{D'}) \supset \mathrm{IAS}(\mathbf{m}_D)$. Hence, adding additional projection moments can only decrease the $\mathrm{UAS}(\mathbf{m})$.

Proof: Follows immediately from the definition of $\mathrm{IAS}(\mathbf{m})$. ∎

Lemma 14: If $|D| \geq 3$, then the set of objects for which $\mathrm{UAS}(\mathbf{m}_D) = \emptyset$ is D-generic. Hence, with more than 3 projection moments, the $\mathrm{UAS}(\mathbf{m})$ is generically empty.

Proof: Follows from the fact that the set of coefficients for which an overdetermined algebraic system of equations has any solutions has Lebesgue measure zero, and is nowhere dense. For $|D| \geq 3$, the UAS is precisely the solution to such a system of equations. ∎

Furthermore, it may be possible to develop weaker results (in the sense of establishing conditions for degeneracy) than the ones derived here, but that are stated for general D. By relating uniqueness to the zeros of polynomial systems parameterized by $\mathbf{m}$ and $\hat{\mathbf{m}}$, such analysis may become feasible.

VIII. CONCLUSION

We have demonstrated that view angles can be uniquely recovered (to within an equivalence class) from a set of parallel beam projections. We have derived conditions for uniqueness of a general method, and then developed specific results for low-order methods. We also treated the case of unknown shifts, which together with an unknown view angle model can model unknown rigid body motion of the object between projections. We have suggested open problems related to establishing uniqueness, and have also presented a series of counterexamples that demonstrate the need for the care used in proving our results. Together, these results introduce a surprising and subtle new aspect to 2-D parallel beam tomography. The development of stability results, and the construction of estimators for the view angles from noisy data are addressed in a sequel to this paper [15]. Together with the uniqueness results of this paper, they show the potential for utilization of these results in practical applications.

APPENDIX
PROOFS

Proof: [Lemma 4]: We convert $\mathcal{R}_1(x,y)$ into a polynomial by multiplying by xy, to yield

$$xy\mathcal{R}_1(x,y) = a_1x^2y + a_2y + a_3xy^2 + a_4x.$$

The result then follows by applying Lemma 1 of [14]. ∎

Proof: [Lemma 5]: The general form of $\mathcal{P}_2$ is

$$\begin{aligned}\mathcal{P}_2(z_1,z_2;\mathbf{m},\hat{\mathbf{m}}) = {}& \alpha_{2,2}z_1^2 + \alpha_{2,0} + \alpha_{2,2}^*z_1^{-2}\\ &- \hat{\alpha}_{2,2}z_2^2 - \hat{\alpha}_{2,0} - \hat{\alpha}_{2,0}^*z_2^{-2}\end{aligned}$$

which (after multiplying both sides by $z_1^2z_2^2$) is of the form

$$a_1y^2x^4 - (a_2y^4 + ry^2 + a_3)x^2 + a_4y^2$$

with $x = z_1$, $y = z_2$, $a_1 = \alpha_{2,2}$, $a_4 = \alpha_{2,2}^*$, $a_2 = \hat{\alpha}_{2,2}$, $a_3 = \hat{\alpha}_{2,2}^*$, and $r = \alpha_{2,0} - \hat{\alpha}_{2,0}$. The assumptions that $\mathcal{Q}_2(z;\mathbf{m}) \not\equiv \beta$ and $\mathcal{Q}_2(z;\hat{\mathbf{m}}) \not\equiv \beta'$ imply that $a_1,a_2,a_3,a_4 \neq 0$. The result then follows from Lemma 2 of [14].

Proof: [Lemma 6]: If $\mathcal{P}_1$ (defined by (12) and (13)) is a factor of $\mathcal{P}_2$, then

$$\begin{aligned}&\mathcal{P}_2(z_1,z_2;\mathbf{m},\hat{\mathbf{m}})\\ &\quad= (\alpha_{1,1}z_1 + \alpha_{1,1}^*z_1^{-1} - \hat{\alpha}_{1,1}z_2 - \hat{\alpha}_{1,1}z_2^{-1}) \cdot \xi(z_1,z_2)\end{aligned}$$

for some polynomial form $\xi(z_1,z_2)$. By Lemma 5, $\xi(z_1,z_2)$ must be of the following form:

$$\xi(z_1,z_2) = r_1\left(\alpha_{1,1}z_1 + \alpha_{1,1}^*z_1^{-1} + \hat{\alpha}_{1,1}z_2 + \hat{\alpha}_{1,1}z_2^{-1}\right)$$

which leads to the following form for $\mathcal{P}_2$:

$$\begin{aligned}&\mathcal{P}_2(z_1,z_2;\mathbf{m},\hat{\mathbf{m}})\\ &\quad= r_1\left(\left(\alpha_{1,1}z_1 + \alpha_{1,1}^*z_1^{-1}\right)^2 - \left(\hat{\alpha}_{1,1}z_2 + \hat{\alpha}_{1,1}z_2^{-1}\right)^2\right).\end{aligned}$$

By comparison with (13), we note that there must exist $\beta,\gamma \in \mathbb{R}$ such that

$$\mathcal{Q}_2(z;\mathbf{m}) = \beta(\mathcal{Q}_1(z;\mathbf{m}))^2 + \gamma.$$

∎

Proof [Lemma 7] By Contradiction: Suppose that $\mathcal{Q}_k(z;\hat{\mathbf{m}}) \equiv \beta$. Then $\mathcal{P}_k(z_1,z_2;\mathbf{m},\hat{\mathbf{m}})$ is a polynomial of degree $2k$ in z_1. By the fundamental theorem of algebra, this polynomial can have at most $2k$ zeros (our assumption that $\mathcal{Q}_k(z;\mathbf{m}) \not\equiv \beta$ guarantees that $\mathcal{P}_k(z_1,z_2;\mathbf{m},\hat{\mathbf{m}})$ does not vanish everywhere). But this contradicts the assumption that (28) is satisfied for $P > 2k$ angles θ_i, which in turn implies that $e^{j\theta_i}$ are zeros of this polynomial. ∎

Proof: [Lemma 8]: The general form of $\mathcal{P}_3$ is

$$\begin{aligned}\mathcal{P}_3(z_1,z_2;\mathbf{m},\hat{\mathbf{m}}) = {}& \alpha_{3,3}z_1^3 + \alpha_{3,1}z_1 + \alpha_{3,1}^*z_1^{-1} + \alpha_{3,3}^*z_1^{-3}\\ &- \hat{\alpha}_{3,3}z_2^3 - \hat{\alpha}_{3,1}z_2 - \hat{\alpha}_{3,1}^*z_2^{-1} - \hat{\alpha}_{3,3}^*z_2^{-3}\end{aligned}$$

while we are considering factors of the form

$$\xi(z_1,z_2) = b_1z_1 + b_2z_1^{-1} + b_3z_2 + b_4z_2^{-1} \tag{A.1}$$

with $b_1,b_2,b_3,b_4 \neq 0$.

We first consider the cases in which either $\alpha_{3,3}$ or $\hat{\alpha}_{3,3}$ are zero. By the assumption that $\check{\mathcal{Q}}_3(\theta;\mathbf{m})$ is not identically zero, it follows that both $\alpha_{3,3}$ and $\alpha_{3,1}$ cannot be zero. Now suppose that $\alpha_{3,3} = 0$. Then $\mathcal{Q}_3(z;\mathbf{m})$ is given by

$$\mathcal{Q}_3(z;\mathbf{m}) = \alpha_{3,1}z + \alpha_{3,1}^*z^{-1}.$$

Letting $z = e^{j\theta}$, we have

$$\check{\mathcal{Q}}_3 = 2|\alpha_{3,1}|\cos(\phi + \theta)$$

where $\alpha_{3,1} = |\alpha_{3,1}|e^{j\phi}$. Then, if we choose $\kappa = -2\phi$, we have

$$\begin{aligned}\check{\mathcal{Q}}_3(\kappa - \theta;\mathbf{m}) &= 2|\alpha_{3,1}|\cos(\phi + \kappa - \theta)\\ &= 2|\alpha_{3,1}|\cos(-\theta - \phi) = \check{\mathcal{Q}}_3(\theta;\mathbf{m})\end{aligned}$$

and the proof is complete.

We can use the same arguments for the case $\alpha_{3,1} = 0$, and if we choose $\kappa = -2\phi/3$, where $\alpha_{3,3} = |\alpha_{3,3}|e^{j\theta}$, then we find

$$\check{\mathcal{Q}}_3(\kappa - \theta;\mathbf{m}) = \check{\mathcal{Q}}_3(\theta;\mathbf{m}).$$

In the final case to consider, $\alpha_{3,1} \neq 0$, and $\alpha_{3,3} \neq 0$, $\hat{\alpha}_{3,3} \neq 0$. We convert $\mathcal{P}_3$ into a polynomial by multiplying by $z_1^3z_2^3$. The result is

$$\begin{aligned}&a_1x^6y^3 + a_2x^4y^3 + a_3x^2y^3 + a_4y^3 + a_5x^3y^6\\ &\quad+ a_6x^3y^4 + a_7x^3y^2 + a_8x^3\end{aligned} \tag{A.2}$$

where $x = z_1$, $y = z_2$, $a_1 = \alpha_{3,3}$, $a_2 = \alpha_{3,1}$, $a_3 = \alpha_{3,1}^*$, $a_4 = \alpha_{3,3}^*$, $a_5 = \hat{\alpha}_{3,3}$, $a_6 = \hat{\alpha}_{3,1}$, $a_7 = \hat{\alpha}_{3,1}^*$, $a_8 = \hat{\alpha}_{3,3}^*$, and we assume that $a_i \neq 0$. By Lemma 3 of [14], it follows that (A.2) has a factor of the form

$$b_1x^2y + b_2y + b_3xy^2 + b_4x$$

with $b_1,b_2,b_3,b_4 \neq 0$ only if

$$\arg\alpha_{3,3} = 3\arg\alpha_{3,1}.$$

Writing $\alpha_{3,3} = |\alpha_{3,3}|e^{j3\phi}$ and $\alpha_{3,1} = |\alpha_{3,1}|e^{j\phi}$, we have

$$\begin{aligned}&\mathcal{Q}_3(z;\mathbf{m}) - \mathcal{Q}_3(e^{jc}z^{-1};\mathbf{m})\\ &\quad= z^3(\alpha_{3,3} - \alpha_{3,3}^*e^{-j3c}) + z(\alpha_{3,1} - \alpha_{3,1}^*e^{-jc})\\ &\qquad+ z^{-1}(\alpha_{3,1}^* - \alpha_{3,1}e^{jc}) + z^{-3}(\alpha_{3,3}^* - \alpha_{3,3}e^{j3c})\end{aligned}$$

which is identically zero for $c = 2\arg\alpha_{3,1}$. In terms of trigonometric quantities, we can thus state

$$\check{\mathcal{Q}}_3(\theta;\mathbf{m}) - \check{\mathcal{Q}}_3(c-\theta;\mathbf{m}) \equiv 0.$$

Choosing $\kappa = c/2$ then completes the proof. ∎

Proof: [Lemma 9]: We convert $\mathcal{P}_3$ to a polynomial as we did in the proof of Lemma 8, and $\mathcal{P}_2$ to a polynomial as we did in the proof of Lemma 5. Hence, we seek conditions under which a polynomial of the form

$$a_1x^6y^3 + a_2x^4y^3 + a_3x^2y^3 + a_4y^3 + a_5x^3y^6 + a_6x^3y^4 + a_7x^3y^2 + a_8x^3$$

has a factor of the form

$$b_1y^2x^4 - (b_2y^4 + ry^2 + b_3)x^2 + b_4y^2$$

with $b_1 = \alpha_{2,2}$, $b_4 = \alpha^*_{2,2}$, $b_2 = \hat{\alpha}_{2,2}$, $b_3 = \hat{\alpha}^*_{2,2}$, and $r = \alpha_{2,0} - \hat{\alpha}_{2,0}$. By the assumptions that $\mathcal{Q}_2(z;\mathbf{m}) \not\equiv \beta \in \mathbb{R}$, and $\mathcal{Q}_2(z;\hat{\mathbf{m}}) \not\equiv \beta \in \mathbb{R}$, we conclude that $b_1, b_2, b_3, b_4 \neq 0$. We can then apply Lemma 4 of [14] to complete the proof. ∎

Proof: [Lemma 10] (By Contradiction): Assume that $\mathcal{P}_3$ has a factor of the form $(z_1^2 - e^{jc}z_2^{-2\sigma})$ for $\sigma \in \{-1, 1\}$. Then, for all $u \in \mathbb{C}$, such that $u \neq 0$, let $v = \sqrt{e^{jc}u^{-2\sigma}}$. Both (v, u) and $(-v, u)$ are zeros of the proposed factor, and by assumption, zeros of $\mathcal{P}_3$. Thus

$$\mathcal{P}_3(v,u;\mathbf{m},\hat{\mathbf{m}}) = \mathcal{Q}_3(v;\mathbf{m}) - \mathcal{Q}_3(u;\hat{\mathbf{m}}) = 0$$
$$\mathcal{P}_3(-v,u;\mathbf{m},\hat{\mathbf{m}}) = \mathcal{Q}_3(-v;\mathbf{m}) - \mathcal{Q}_3(u;\hat{\mathbf{m}}) = 0$$

but because $\mathcal{Q}_3(z;\mathbf{m})$ contains only odd powers of z, $\mathcal{Q}_3(-v;\mathbf{m}) = -\mathcal{Q}_3(v;\mathbf{m})$. Thus,

$$\mathcal{Q}_3(v;\mathbf{m}) = 0$$

for all such u and v. This is possible iff $\mathcal{Q}_3(z;\mathbf{m}) \equiv 0$, hence a contradiction. ∎

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