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The image moment method for the limited range CT image reconstruction and pattern recognition

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Abstract

Moment properties of the Radon transform have been discussed. A new concept of the image moment in the Radon transform has been introduced and described. A new approach to reconstruct the image from the projections within a limited range, called the *image moment method*, has been proposed. The new method has been validated through computer simulations. © 2001 Pattern Recognition Society. Published by Elsevier Science Ltd. All rights reserved.

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1. Introduction

In the past three decades, computerized tomography (CT) has attracted a great deal of attention in the digital imaging processing field. Along with the maturity of practical methods for image reconstruction from projections, its theoretical basis, Radon transformation received renewed interest. As a linear transform, Radon transform is based on the solutions of the multi-variable integral equation. Because of the explicit geometrical meaning of the Radon transform, it has shown many interesting properties and established varied forms of connections among the space, Frequency and Radon domains.

A number of studies have been carried out on this subject. For example, Lewitt has summarized a series of projection theorems and their corollaries [1]. Deans has collected many materials and provided a detailed description about the properties of Radon transform and its relationship to other transforms [2]. In this paper, a new property of the Radon transform, *moment property*, is presented. Three new moment theorems are introduced. An new method, the image moment method for image reconstruction, is proposed. The new method is applied to the problem of inverse Radon transform from limited range projections and evaluated by the simulation results.

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2. Definition of the Radon transform

Assume $f(x, y)$ is a two dimensional (2-D) real function in the x–y coordinate system, where $f(x, y)$ has a compact support. Assume a ray $L(s, \theta)$ is defined by two parameters, s and θ, where s is the distance from the origin to the ray, θ is the angle between the y-axis and the ray. Hence, the ray is described by the following equation (Fig. 1):

$$x\cos\theta + y\sin\theta = s. \tag{1}$$

Since s and θ can be any real values, the integral of $f(x, y)$ along the ray $L(s, \theta)$ defines a 2-D function, denoted as $P(s, \theta)$. We call $P(s, \theta)$ the *Radon transform* of $f(x, y)$.

$$P(s, \theta) = \int_{rayL(s,\,\theta)} f(x, y)\,\mathrm{d}s. \tag{2}$$

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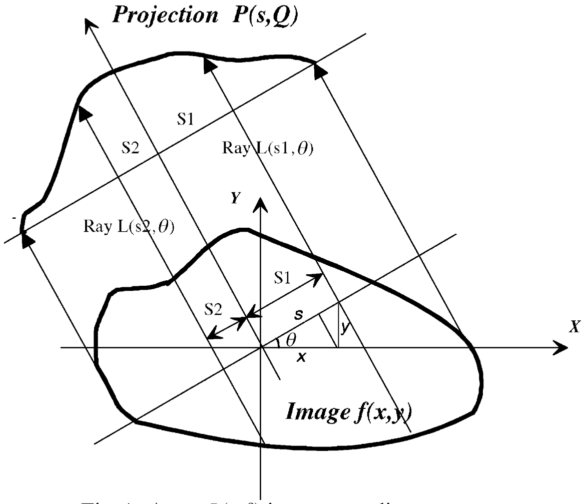


Fig. 1. A ray $L(s,\theta)$ in x–y coordinate system.

With the help of the 2-D δ distribution:

$$\delta(x,y)=\begin{cases}1 & \text{if } x=y=0,\\ 0 & \text{otherwise,}\end{cases} \tag{3}$$

this transform can be expressed as

$$P(s,\theta)=\iint_{L(s,\,\theta)}\delta(x,y)f(x,y)\,\mathrm{d}x\,\mathrm{d}y. \tag{4}$$

That is,

$$P(s,\theta)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)\delta(x\cos\theta+y\sin\theta-s)\,\mathrm{d}x\,\mathrm{d}y. \tag{5}$$

When θ is a constant, the 2-D function of $P(s,\theta)$ becomes a one-variable function of s, denoted by $P_\theta(s)$. Because $P_\theta(s)$ represents a collection of integrals along a set of parallel rays, $P_\theta(s)$ is also called parallel projections of $P(s,\theta)$ at view θ (see Fig. 2).

Therefore, the Radon transform also can be expressed as

$$P_\theta(s)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)\delta(x\cos\,\theta+y\sin\theta-s)\,\mathrm{d}x\,\mathrm{d}y. \tag{6}$$

The inverse Radon transform can be found by the *Convolution-backprojection* formula [3,4]

$$f(x,y)=\int_0^\pi Q_\theta(x\cos\theta+y\sin\theta)\,\mathrm{d}\theta, \tag{7}$$

where

$$Q_\theta(t)=P_\theta(t)\otimes h(t), \tag{8}$$

where $h(t)$ is a special convolving kernel.[1]

[1] The details of the derivation of $h(t)$ can be found from the works of both Herman and Kak. See Refs. [3,4].

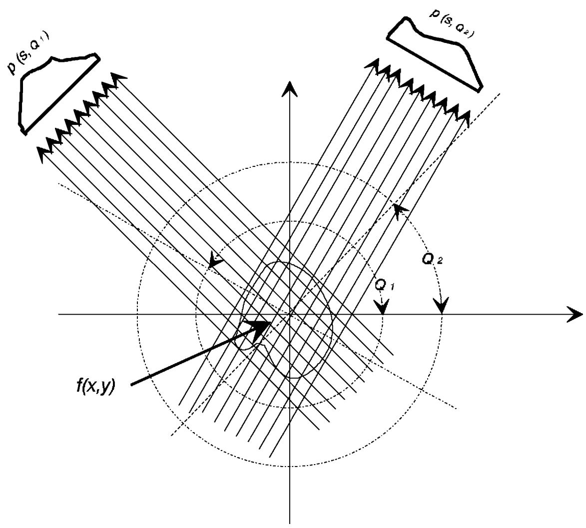


Fig. 2. Parallel projections of $P(s,\theta)$ at view θ_1 and view θ_2.

Although most today's CT scaners are using a fan-beam geometry, the nature of the Radon transformation remains the same as a parallel geometry. Many materials have described the adaptation process of Radon transformation between these two geometries, [5–7].

3. The projection moment and the projection moment theorem

The moments of $P_\theta(s)$ are called projection moments in the Radon domain [1]. The nth projection moment of $P_\theta(s)$ is defined as

$$M_\theta^{(n)}=\int_{-\infty}^{\infty}P_\theta(s)s^n\,\mathrm{d}s. \tag{9}$$

The widely known Uniqueness Theorem of the moments [8] assures that the moments of all orders, $M_\theta^{(n)}$, are uniquely determined by $P_\theta(s)$. Conversely, $M_\theta^{(n)}$ of all orders uniquely determine $P_\theta(s)$.

Substituting Eq. (6) into Eq. (9), we have

$$M_\theta^{(n)}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)\delta(x\cos\theta+y\sin\theta-s)s^n\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s. \tag{10}$$

Notice that the δ function has the following identity:

$$\int f(x)\,\delta(x-y)\,\mathrm{d}x=f(y). \tag{11}$$

Thus, the following equation holds.

$$M_\theta^{(n)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)(x\cos\theta + y\sin\theta)^n \,\mathrm{d}x\,\mathrm{d}y. \tag{12}$$

The above equation is called the *Projection Moment Theorem* [1]. It establishes a connection between projection moments and the space function, $f(x,y)$.

4. Image Moment and Radon Moment Theorems

Enlightened by the projection moment and the *Projection Moment Theorem*, the terminology of the *image moment* of the space domain is introduced into the Radon transformation [9]. Let p and q be any positive integers with $k = p + q$. It is known that the kth-order image moment of $f(x,y)$ is defined as [10]

$$M_{p,q} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)x^p y^q \,\mathrm{d}x\,\mathrm{d}y, \tag{13}$$

where $f(x,y)$ has a compact support.

The following table shows the image moment notations of the first three orders.

The order	The notation of $M_{p,q}$
0	$M_{0,0}$
1	$M_{1,0}, M_{0,1}$
2	$M_{2,0}, M_{1,1}, M_{0,2}$

The Uniqueness Theorem of the image moments [11] also assures that the image moment sequence, $M_{p,q}$, is uniquely determined by $f(x,y)$. Conversely, image moments of all orders uniquely determine $f(x,y)$.

By using the concepts of projection moments and the image moments, three moment theorems in Radon transform have been developed by the author [9] to describe the intrinsic properties of the Radon transform.

The Radon Moment Theorem I. *The projection moment at any view of θ, $M_\theta^{(n)}$, is determined by the image moments of the first n orders. The following equation holds*:

$$M_\theta^{(n)} = \sum_{l=0}^{n} A_{n,l}(\theta) M_{n-l,l}, \tag{14}$$

where

$$A_{n,l}(\theta) = C_n^{n-l}\cos^{n-l}\theta\sin^l\theta \tag{15}$$

C_n^{n-l} is a combinatorial coefficient.

Proof. In Eq. (12), expanding $(x\cos\theta + y\sin\theta)^n$, we have

$$(x\cos\theta + y\sin\theta)^n = \sum_{l=0}^{n} C_n^{n-l}(x\cos\theta)^{n-l}(y\sin\theta)^l. \tag{16}$$

Substitute Eq. (16) into Eq. (12):

$$M_\theta^{(n)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \times\left[f(x,y)\sum_{l=0}^{n} C_n^{n-l}(x\cos\theta)^{n-l}(y\sin\theta)^l\right]\mathrm{d}x\,\mathrm{d}y. \tag{17}$$

Exchange the integral sign and the summation sign:

$$M_\theta^{(n)} = \sum_{l=0}^{n} C_n^{n-l}(\cos\theta)^{n-l}(\sin\theta)^l \times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^{n-l}y^l f(x,y)\,\mathrm{d}x\,\mathrm{d}y, \tag{18}$$

$$M_\theta^{(n)} = \sum_{l=0}^{n} C_n^{n-l}(\cos\theta)^{n-l}(\sin\theta)^l M_{n-l,l}. \quad \square \tag{19}$$

Eq. (19) not only provides the solution for finding projection moments from image moments, but also the solution for finding image moments from projection moments.

In Eq. (19), for $n = 0$,

$$M_\theta^0 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = M_{0,0}. \tag{20}$$

In Eq. (19), for $n = 1$,

$$M_\theta^{(1)} = \cos\theta M_{1,0} + \sin\theta M_{0,1}. \tag{21}$$

Assume $M_\theta^{(1)}$ is known at θ_1 and θ_2, a set of equations is obtained as follows:

$$M_{\theta_1}^{(1)} = \cos\theta_1 M_{1,0} + \sin\theta_1 M_{0,1}, \tag{22}$$

$$M_{\theta_2}^{(1)} = \cos\theta_2 M_{1,0} + \sin\theta_2 M_{0,1}. \tag{23}$$

Thus, image moments of the first order can be found by solving the following equations:

$$\begin{pmatrix} M_{1,0} \\ M_{0,1} \end{pmatrix} = \begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ \cos\theta_2 & \sin\theta_2 \end{pmatrix}^{-1} \begin{pmatrix} M_{\theta_1}^{(1)} \\ M_{\theta_2}^{(1)} \end{pmatrix}. \tag{24}$$

In general, for $n = k$, assume $M_\theta^{(n)}$ is known at $k+1$ views: $\theta_1, \theta_2, \theta_3, \ldots, \theta_k, \theta_{k+1}$. The image moments of the

kth order can be determined as follows:

$$\begin{pmatrix} M_{k,0} \\ M_{k-1,0} \\ \vdots \\ M_{0,k} \end{pmatrix} = \begin{pmatrix} \cos^k\theta_1 & C_k^{k-1}\cos^{k-1}\theta_1\sin^1\theta_1 & \cdots & \sin^k\theta_1 \\ \cos^k\theta_2 & C_k^{k-1}\cos^{k-1}\theta_2\sin^1\theta_2 & \cdots & \sin^k\theta_2 \\ \vdots & \vdots & \vdots & \vdots \\ \cos^k\theta_{k+1} & C_k^{k-1}\cos^{k-1}\theta_{k+1}\sin^1\theta_{k+1} & \cdots & \sin^k\theta_{k+1} \end{pmatrix}^{-1} \cdot \begin{pmatrix} M_{\theta_1}^{(k)} \\ M_{\theta_2}^{(k)} \\ \vdots \\ M_{\theta_{k+1}}^{(k)} \end{pmatrix}. \quad (25)$$

Let $IM^{(k)}$ is the image moment vector of the kth order and $PM^{(k)}$ is the projection moment vector of the kth order from $k+1$ different views:

$$IM^{(k)} = \begin{pmatrix} M_{k,0} \\ M_{k-1,1} \\ \vdots \\ M_{1,k-1} \\ M_{0,k} \end{pmatrix}, \quad (26)$$

$$PM^{(k)} = \begin{pmatrix} M_{\theta_1}^{(k)} \\ M_{\theta_2}^{(k)} \\ \vdots \\ M_{\theta_k}^{(k)} \\ M_{\theta_{k+1}}^{(k)} \end{pmatrix}. \quad (27)$$

Let $\theta_1, \theta_2, \ldots, \theta_{k+1}$ represent $k+1$ different views. Denote A as a $(k+1)\times(k+1)$ system matrix which is determined by $\theta_1, \theta_2, \ldots, \theta_{k+1}$ as follows:

$$A = \begin{pmatrix} \cos^k\theta_1 & C_k^{k-1}\cos^{k-1}\theta_1\sin^1\theta_1 & \cdots & \sin^k\theta_1 \\ \cos^k\theta_2 & C_k^{k-1}\cos^{k-1}\theta_2\sin^1\theta_2 & \cdots & \sin^k\theta_2 \\ \vdots & \vdots & \vdots & \vdots \\ \cos^k\theta_k & C_k^{k-1}\cos^{k-1}\theta_k\sin^1\theta_k & \cdots & \sin^k\theta_k \\ \cos^k\theta_{k+1} & C_k^{k-1}\cos^{k-1}\theta_{k+1}\sin^1\theta_{k+1} & \cdots & \sin^k\theta_{k+1} \end{pmatrix}. \quad (28)$$

Thus, if A is invertible, Eq. (25) can be written in matrix form:

$$IM^{(k)} = A^{-1}PM^{(k)}. \quad (29)$$

We summarize Eq. (29) as a lemma:

Lemma 1. *The image moments of the kth order are determined by a set of projection moments of the kth order from* $k+1$ *given views.*[2]

In practical situations, projections from certain views are known while projections from other views are unknown. For the sake of convenience, the former projections are called *given projections* and those views are given views; the latter *unknown projections* and those views are unknown views.

Since the projection moments of the given views can always be found by the given projections, the second Radon moment theorem is introduced:

The Radon Moment Theorem II. *Image moments of the kth order can be found by a set of given projections,* $P_\theta(s)$, *from* $k+1$ *given views if A is an invertible matrix.*

Corollary. *Image moments of the first k orders can be found by a set of given projections from* $k+1$ *given views.*

Based upon two theorems developed above, the third Radon moment theorem may be formulated as follows:

The Radon Moment Theorem III. *The projection moment of the kth order at an unknown view* θ *is determined by a set of projections from* $k+1$ *given views if A is an invertible matrix.*

With the *Radon Moment Theorem III* in mind, not only $M_\theta^{(k)}$ is determined by a set of projections from $k+1$ given views, but also all lower-order projection moments, such as $M_\theta^{(k-1)}, M_\theta^{(k-2)}, M_\theta^{(1)}$ and $M_\theta^{(0)}$ at an unknown view are determined by the same set of projections. Thus, the following corollary is developed.

Corollary. *The projection moments,* $M_\theta^{(n)}$, *at an unknown view* θ, *where* $n = 1, 2, 3, \ldots, k$, *are determined by a set of given projections,* $P_{\theta_i}(s)$ *from* $k+1$ *given views* $(\theta_i, i = 1, 2, 3, \ldots, k, k+1)$.

The *Projection Moment Theorem* shows that the projection moments in the Radon domain can be directly

[2] The above results were first presented in author's Ph.D dessertation in 1986. The same results were again confirmed by Klebanov and Rachev in their paper, "The method of Moments in Computer Tomography" (1995) [12].

calculated from the space function of $f(x, y)$. The *Radon Moment Theorem I and II* provide the connection between the projection moments and the image moments of $f(x, y)$. The *Radon Moment Theorem III* provides the connection among the projection moments from different views.

5. Radon inverse formula with limited range projections

The Radon inverse formula gives a closed-form formula for finding $f(x, y)$ from $P(s, \theta)$. This formula needs projections, $P(s, \theta)$, for all s and all θ between 0 and π [13]. In many practical applications, it is desired to reconstruct $f(x, y)$ from $P(s, \theta)$ with θ in a limited range. It is known that "*A compactly supported function f is determined by the function $P(s, \theta)$ for any infinite set of θ*" [14]. This relation between the image and its limited range projections has been summarized by A.K. Louis and F. Natterer as the "*Uniqueness Theorem of The Projections*" [15] (1983): "*A function f is uniquely determined by any infinite number of projections*".

The Uniqueness Theorem of the Projections shows that Radon inverse transform can be determined by projections from a limited range of view instead of spreading over the entire range of view $(0, \pi)$. This theorem provides theoretical insight for seeking the solution for so-called image reconstruction from projections of the limited range.

As pointed out by Louis and Natterer in the same paper, "*Although* [*the above*] *theorem assures that f is uniquely determined, inversion* [*Radon transform*] *formulas in the* [*limited range*] *form do not exist*". Finding the closed-form formula for inverse Radon transform from limited range projections is not the intention of this paper. However, with the help of moment properties of the Radon transform, the complete projections can be estimated from limited range projections. Then, the closed-form formula of the inverse Radon transform for completed projections can be used for limited range projections.

6. The image moment method and the simulation verification

The moment properties of the Radon transform described in Section 4 can be used to estimate unknown projections from limited range projections. Two concepts, projection moment and image moment, will play a central role to establish connections between unknown projections and given projections from limited range.

Eq. (34) shows how to find image moments from given projection in a limited range. The following shows calculations of unknown projections from the image moments.

Assume that $P_\theta(s)$ is normalized in the unit circle as a two variable function, $P(s, \theta)$. $P(s, \theta)$ may be expanded by two kinds of orthogonal polynomials as follows [1,2]:

$$P(s,\theta) = \frac{2}{\pi}(1 - s^2)^{1/2} \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} C_{m,n} U_m(s) e^{in\theta}, \tag{30}$$

where $U_m(s)$'s are the Chebyshev polynomials of the second kind, $U_m(s)$ are orthogonal over $[-1, 1]$ with weight $(1 - s^2)^{1/2}$.

Rewrite the above equation, it follows

$$P(s,\theta) = \frac{2}{\pi}(1 - s^2)^{1/2} \sum_{m=0}^{\infty} \left(\sum_{n=-\infty}^{\infty} C_{m,n} e^{in\theta} \right) U_m(s). \tag{31}$$

Notice that $\sum_{n=-\infty}^{\infty} C_{m,n} e^{in\theta}$ is the coefficient of an orthogonal polynomials. Hence, we have

$$\sum_{n=-\infty}^{\infty} C_{m,n} e^{in\theta} = \int_{-1}^{1} P(s, \theta) U_m(s)\, ds. \tag{32}$$

Because $U_m(s)$ is a Chebychev polynomial of the mth order, it can be written as

$$U_m(s) = \sum_{k=0}^{m} A_{m,k} s^k, \tag{33}$$

where $A_{m,k}$'s are the coefficients of Chebychev polynomials.

Substituting the above equation into Eq. (32),

$$\begin{aligned} \sum_{n=-\infty}^{\infty} C_{m,n} e^{in\theta} &= \int_{-1}^{1} P(s,\theta) \sum_{k=0}^{m} A_{m,k} s^k\, ds \\ &= \sum_{k=0}^{m} A_{m,k} \int_{-1}^{1} P(s, \theta) s^k\, ds. \end{aligned} \tag{34}$$

That is,

$$\sum_{n=-\infty}^{\infty} C_{m,n} e^{in\theta} = \sum_{k=0}^{m} A_{m,k} M_\theta^{(k)}, \tag{35}$$

where $M_\theta^{(k)}$ is kth order projection moment of $P_\theta(s)$.

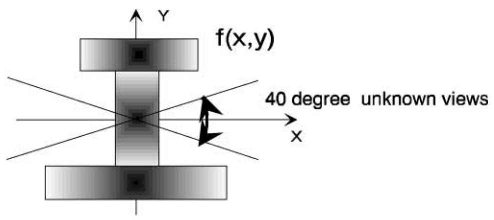
Substituting Eq. (35) into Eq. (31),

$$P(s,\theta) = \frac{2}{\pi}(1 - s^2)^{1/2} \sum_{m=0}^{\infty} \left(\sum_{k=0}^{m} A_{m,k} M_\theta^{(k)} \right) U_m(s). \tag{36}$$

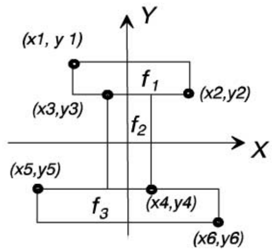
Using project moments of the first L orders to approximate $P(s, \theta)$ [1,2,9]

$$P(s,\theta) \approx \frac{2}{\pi}(1 - s^2)^{1/2} \sum_{m=0}^{L} \left(\sum_{k=0}^{m} A_{m,k} M_\theta^{(k)} \right) U_m(s). \tag{37}$$

This formula calculates $P(s, \theta)$ from its project moments.



(1) The Binary I-Beam image of f(x,y)



(2) The image of three rectangules

Fig. 3. A binary image of an I-beam, $f(x, y)$.

Using Radon Moment Theorem I and substituting Eq. (19) into Eq. (37),

$$P(s,\theta) \approx \frac{2}{\pi}(1-s^2)^{1/2} \sum_{m=0}^{L} U_m(s) \sum_{k=0}^{m} A_{m,k} \times \sum_{j=0}^{k} C_k^{k-j} \cos^{k-j}\theta \sin^j \theta M_{k-j,j}. \quad (38)$$

This formula establishes the connections between projections from any specific views and image moments.

To sum up, the projections of unknown views may be estimated from given projections through two steps as follows:

1. Obtain the Image moments of all orders from given projections in a limited range.
2. Find the unknown projection from the image moments of all orders.

Since projections of unknown views are estimated through the use of image moments, this procedure is called the *Image moment method*.

The Image moment method has been verified by computer simulations as follows.

A binary image of an I-Beam shown at Fig. 3 has been adopted as a 2-D image, $f(x, y)$. In order to verify the image moment method, without loss of generality, $f(x, y)$ consists of three rectangles. The vertices of these rectangles are also specified so that the image moment of $f(x, y)$ can be easily calculated by the definition of the image moment. Thus, these calculated image moments can be used as a reference for the verification purpose.

For the given image, $f(x, y)$, a projection simulation program is used to generate all the projection data from all the given views. In the following example, the projections from 15 given views are generated so that the first 15 image moments for $f(x, y)$ can be calculated from the Eq. (34).

Table 1 lists a few arbitrarily selected image moments, $M_{p,q}$, for $p = 0, 2, 5, 7$ and $q = 0, 2, 6, 8$. In the table, the figures of the image moments without parentheses are calculated from projections. The figures with parentheses are calculated from the image itself. This table shows that the errors between calculated image moments and the real image moments are relatively small.

With calculated image moments of first L orders and a specified θ, $P_\theta(s)$ can be estimated by Eq. (38).

Fig. 4 show the estimation of $P_\theta(s)$ from the image moments of the first 15 orders. Estimated $P_\theta(x)$ are plotted in 6 subplots at views of 0, 30, 60, 90, 120, 150°, respectively. For comparison, the real projections (calculated by a projection simulation program) from these views are also drawn on the same subplots.

Table 1

$M_{p,q}$ q / p	0	2	5	7
0	0.7584E + 00	0.2062E + 00	− 0.5205E − 01	− 0.2908E − 01
	(0.7636E + 00)	(0.2060E + 00)	(− 0.5202E − 01)	(− 0.2909E − 01)
2	0.4572E − 01	0.1751E − 01	− 0.5353E − 02	− 0.2975E − 02
	(0.4673E − 01)	(0.1751E − 01)	(− 0.5352E − 02)	(− 0.2974E − 02)
6	0.1414E − 02	0.6331E − 03	− 0.2208E − 03	− 0.1212E − 03
	(0.1413E − 02)	(0.6324E − 03)	(− 0.2200E − 03)	(− 0.1204E − 03)
8	0.3164E − 03	0.1453E − 03	− 0.5169E − 04	− 0.2776E − 04
	(0.3187E − 03)	(0.1472E − 03)	(− 0.5255E − 04)	(− 0.2866E − 04)

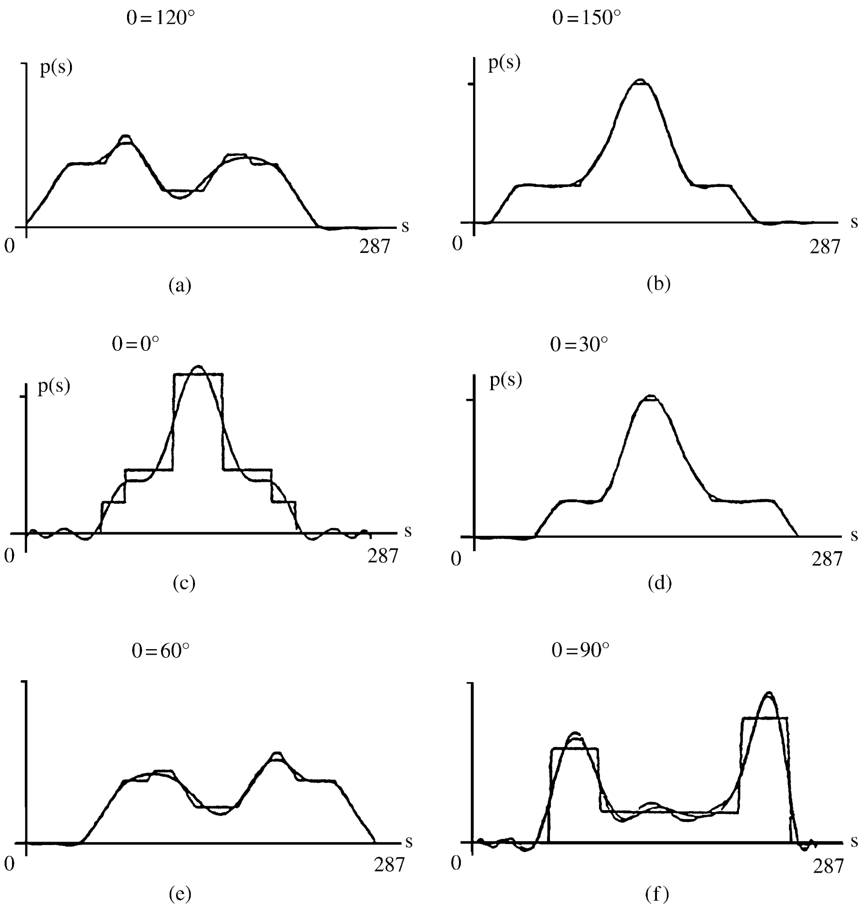


Fig. 4. The estimations of $P_\theta(s)$ from the image moments of the first 15 orders.

7. The image moment method and the limited range projection problem

The convolution-backprojection method is the most widely used algorithm for CT image reconstruction from projections. However, this algorithm needs the projection, $P(s,\theta)$, for all s and all θ between 0 and π [13]. In order to use the convolution-backprojection method for the problem of image reconstruction from limited range projections, we use the image moment method to estimate the unknown projections based upon the known projection from the limited range so that the convolution-backprojection algorithm can be ultimately adopted.

In the following example, the reconstruction of a steel I-beam image from limited range projections is shown with the assistance of the image moment method. The 3-D plot of a binary I-beam image is shown in Fig. 5. The given projections, $P(s,\theta)$, are calculated through a projection simulation program from the known image of $f(x,y)$.

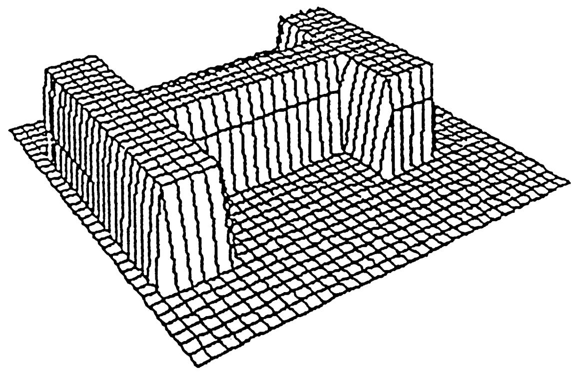


Fig. 5. The 3-D plot of the binary I-beam image.

The computer simulation is based upon the following configuration:

(1) Total views are 180 views between 0 and 179°. (2) View sampling rate is 1° per view. (3) Ray sampling rate is 287 rays per view.

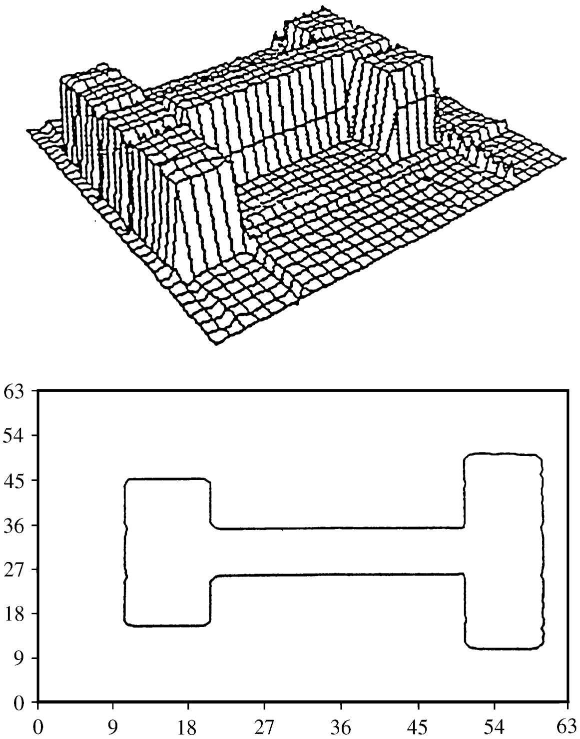


Fig. 6. The reconstruction of the I-beam image and its contour plot from the complete projections (0–179°).

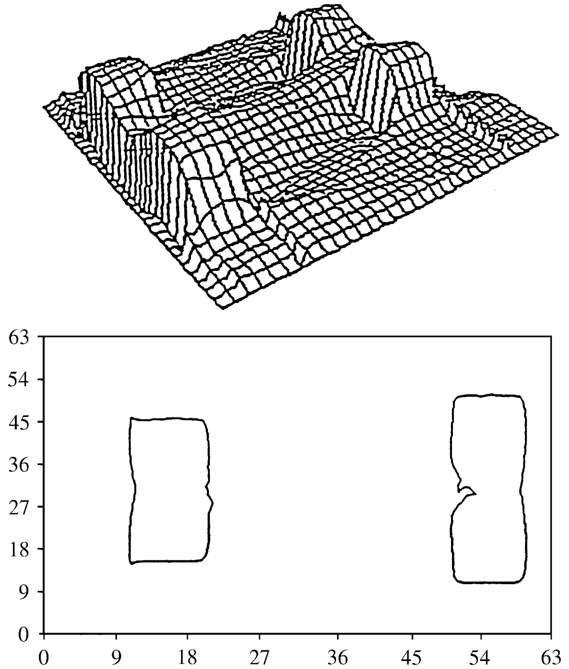


Fig. 7. The reconstruction of the I-beam image and its contour plot from incomplete projections (20–160°).

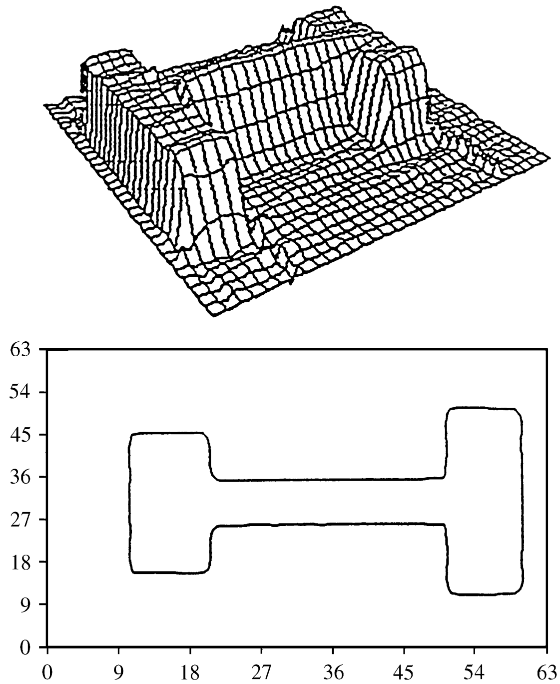


Fig. 8. The reconstruction of the I-beam image and its contour plot from both given and estimated projections (0–179°).

As a reference, Fig. 6 shows the reconstruction of the I-beam image and its contour plot from complete projection (0–179°) by using the convolution-backprojection method.

As a limited range projection problem, assume projections, $P_\theta(s)$, to be only available in the range of $20^\circ < \theta_i < 160^\circ$. Projections in the range of $0^\circ < \theta_i < 20^\circ$ and $160^\circ < \theta_i < 179^\circ$ are unknown. Fig. 7 shows the reconstruction of the I-beam image and its contour plot from incomplete projections (20–160°) by using the convolution-backprojection method.

By using the image moment method mentioned in Section 6, the first 24 orders image moments are calculated from given projections. The unknown projections are, then, estimated from these image moments. With both given projections from 140 views (20–160°) and estimated projections from 40 views (0–20, 160–179), the convolution-backprojection method is then applied to reconstruct the image. Fig. 8 shows the reconstruction of the I-beam image and its contour plot.

8. Conclusion

Three Radon moment theorems introduced in this paper set up two bridges through the projection moments and the image moments: one between Radon transform, $P(s,\theta)$, and the image, $f(x,y)$; another between the projections from different views. The use of the moment concepts achieves a better comprehension of the Radon transform and gives a great deal of flexibility when dealing with the problem of inverse Radon transform from limited range projections.

The concept of the image moment and the image moment method developed in this paper have been successfully applied to a practical CT image reconstruction problem, i.e., reconstruction from limited range projections. The simulation results show great improvement of a reconstructed image. The improvement of the reconstructed image comes from the fact that the image moment method can estimate all the unknown projections from limited range projection so that the existing closed form inverse Radon transform formula can still be used to reconstruct the image. The I-beam example used in the simulation has also shown the potential applications of the image moment method in the field of pattern recognition and the nondestructive measurement.

9. Summary

Radon transformation has been receiving increased attention along with the maturity of the computerized tomography (CT). Because of its explicit geometrical meaning, Radon transformation has many interesting properties and varied forms of connections among the space, frequency and Radon domains. This paper investigates the moment property of the Radon transformation.

There are two types of moments in the Radon transform: projection moments in Radon domain, and image moments in space domain. Three Radon moment theorems are introduced. The Radon Moment Theorem I and II establish the connection between the projections and the image moments. The Radon moment Theorem III establishes the connection among the projections from different views. The use of the moment concepts achieves a better comprehension of the Radon transform.

By using the theorems described above, a new method, the image moment method, is proposed to estimate the unknown projections and solve the problem of inverse Radon transform from limited range projections. The new method is evaluated through a practical reconstruction problem. The simulation results are presented.

The I-Beam example in the paper shows that the image moment method can also be applied to the problem of pattern recognition, and the nondestructive measurement.

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References

[1] R.M. Lewitt, Reconstruction algorithm: transform methods, Proc. IEEE 71 (3) (1983).
[2] S.R. Deans, The Radon transform and its applications, Academic Press, Inc. Ltd., New York, 1980.
[3] G.T. Herman, Image reconstruction from projections, Academic Press, Inc. Ltd., New York, 1980.
[4] A.C. Kak, Principles of computerized tomographic imaging, IEEE Press, New York, 1988.
[5] D.L. Parker, Optimal short scan convolution reconstruction for fan-bean CT, Med. Phys. 9 (2) (1982).
[6] A. Naparstek, Short-scan fan-beam algorithm for CT, IEEE Trans. Nucl. Sci. NS-27 (3) (1980).
[7] T.M. Peters, R.M. Lewitt, Computed tomography with fan-beam Geometry, Comp. Assis. Tomography 1 (4) (1977).
[8] A. Papoulis, Signal analysis, McGraw-Hill, New York, 1977.
[9] T.T. Wang, Image reconstruction from incomplete projections and some applications for industrial CT scanners, Ph.D. Dissertation, University of Pittsburgh, 1986.
[10] A. Rosenfeld, A.C. Kak, Digital Image Processing and Recognition, Academic Press, Inc. Ltd, New York, 1980.
[11] A. Papoulis, System and transforms with applications to optics, McGraw-Hill, New York, 1969.
[12] L. Klebanov, S. Rachev, The method of moments in computer tomography, Math. Sci. 20 (1) (1995).
[13] D. Ludwig, The Radon transform on Euclidean spaces, communications on pure and applied mathematics, Vol. 19, Wiley, New York.
[14] S. Helgason, The Radon transform, Birkhauser, Berlin, 1983.
[15] A.K. Louis, F. Natterer, Mathematical problem of computerized tomography, Proc. IEEE 71 (3) (1983).

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