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- In particular, if $i \neq j$, then $E_{ij}E_{ij} = 0$ and so for any α,

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- On the other hand, if $i = j$, then

$$(I + \alpha E_{ii}) = \mathrm{diag}(1, \ldots, 1, 1 + \alpha, 1, \ldots, 1)$$

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and this is invertible if $\alpha \neq -1$.
- Matrices of the type $I + \alpha E_{ij}$, with $\alpha \in \mathbb{R}$ and $i \neq j$ or of the type $I + \alpha E_{ii}$ with $\alpha \neq -1$ provide simple examples of invertible matrices whose inverse is of a similar type. These are two among 3 possible types of elementary matrices.

The third remaining type of elementary matrix and its basic propery is described in the following easy exercise.

Exercise: Given any $i \neq j$, show that the square matrix

$$T_{ij} := I + E_{ij} + E_{ji} - E_{ii} - E_{jj}$$

is precisely the matrix obtained from the identity matrix by interchanging the i^{th} and j^{th} rows. Also show that $T_{ij}T_{ij} = I$ and deduce that T_{ij} is invertible and $T_{ij}^{-1} = T_{ij}$.

A square matrix is said to be elementary if it of the type

$$T_{ij}\ (i \neq j) \quad \text{or} \quad I + \alpha E_{ii}\ (\alpha \neq -1) \quad \text{or} \quad I + \alpha E_{ij}\ (i \neq j).$$

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Thanks to the discussion in the last slide and the exercise above, we have the following result.

Theorem

Every elementary matrix is invertible and its inverse is an elementary matrix of the same type.

Permutation Matrices

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$$p_{ij} = \begin{cases} 0 & \text{if } i \neq \sigma(j) \\ 1 & \text{if } i = \sigma(j) \end{cases}$$

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- A permutation matrix is obtained by shuffling the rows of the identity matrix (or by shuffling the columns).
- If A is a permutation matrix, then

$$AA^T = A^T A = I_n.$$

  In particular, permutation matrices are invertible.

Gaussian Elimination

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$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots &+ a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots &+ a_{2n}x_n = b_2 \\ \dots &\dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots &+ a_{mn}x_n = b_m \end{aligned}$$

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- Basic observation: Operations of three types on these equations do not alter the solutions:

1. Interchanging two equations.
2. Multiplying all the terms of an equation by a nonzero scalar.
3. Adding to one equation a multiple of another equation.

- The above system of linear equations can be written in matrix form as follows.

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \qquad (*)$$

or in short as $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}.$$

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- The $m \times n$ matrix $A = (a_{ij})$ is called the *coefficient matrix* of the system. By a *solution* of $(*)$ we mean any choice of $x_1, x_2, \dots, x_n$ which satisfies all the equations in the system.

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- If each $b_i = 0$, then the system is said to be *homogeneous.* Otherwise it is called an *inhomogeneous system.*

- All the known data in the system $(*)$ is captured in the $m \times (n+1)$ matrix

$$(A|\mathbf{b}) := \left(\begin{array}{ccc|c} a_{11} & \dots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} & b_m \end{array}\right).$$

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- Now the above three operations on the equations in the linear system correspond to the following operations on the rows of the augmented matrix:
  (i) interchanging two rows,
  (ii) multiply a row by a nonzero scalar,
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- Gaussian Elimination Method consists of reducing the augmented matrix to a simpler matrix from which solutions can be easily found. This reduction is by means of elementary row operations.

- Example 1 (A system with a unique solution):

$$\begin{aligned} x - 2y + z &= 5 \\ 2x - 5y + 4z &= -3 \\ x - 4y + 6z &= 10. \end{aligned}$$

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The augmented matrix for this system is the 3×4 matrix

$$
\left(\begin{array}{ccc|c}
1 & -2 & 1 & 5 \\
2 & -5 & 4 & -3 \\
1 & -4 & 6 & 10
\end{array}\right)
$$

The elementary row operations mentioned above will be performed on the rows of this augmented matrix,

First we add -2 times the first row to the second row. Then we subtract the first row from the third row.

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$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 5 \\ 0 & -1 & 2 & -13 \\ 0 & -2 & 5 & 5 \end{array}\right)$$

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The circled entry is the first nonzero entry in the first row and all the entries below this are 0. Such a circled entry is called a pivot. This next step is called 'sweeping' a column. Here we repeat the process for the smaller matrix:

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$$\left(\begin{array}{cc|c} -1 & 2 & -13 \\ -2 & 5 & 5 \end{array}\right)$$

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Put back the rows and columns that has been cut out earlier:

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 5 \\ 0 & -1 & 2 & -13 \\ 0 & 0 & 1 & 31 \end{array}\right) \qquad (*)$$

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The matrix represents the linear system:

$$
\begin{aligned}
x - 2y + z &= 5 \\
-y + 2z &= -13 \\
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These can be solved successively by backward substitution

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$$\begin{aligned} &z = 31; \\ &y = 13 + 2z = 75; \\ &x = 5 + 2y - z = 124. \end{aligned}$$

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This gives

$$\Rightarrow \qquad \left(\begin{array}{ccc|c} 1 & 0 & 0 & 124 \\ 0 & 1 & 0 & 75 \\ 0 & 0 & 1 & 31 \end{array}\right)$$

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- It is useful to have a shorthand notation for the three types of elementary row operations.
  Notation: Let R_i denote the i^{th} row of a given matrix.

Operation	**Notation**
Interchange R_i and R_j	$R_i \leftrightarrow R_j$
Multiply R_i by a (nonzero) scalar c	cR_i
Multiply R_j by a scalar c and add to R_i	$R_i + cR_j$

- Example 2 (A system with infinitely many solutions):

$$\begin{aligned} x - 2y + z - u + v &= 5 \\ 2x - 5y + 4z + u - v &= -3 \\ x - 4y + 6z + 2u - v &= 10 \end{aligned}$$

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$$\begin{array}{c} R_2-2R_1\\ R_3-R_1\\ \longrightarrow \end{array} \quad \left(\begin{array}{ccccc|c} 1 & -2 & 1 & -1 & 1 & 5\\ 0 & -1 & 2 & 3 & -3 & -13\\ 0 & -2 & 5 & 3 & -2 & 5 \end{array}\right)$$

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$$\xrightarrow{R_3 - 2R_2} \left(\begin{array}{ccccc|c} 1 & -2 & 1 & -1 & 1 & 5 \\ 0 & -1 & 2 & 3 & -3 & -13 \\ 0 & 0 & 1 & -3 & 4 & 31 \end{array}\right)$$

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The system of linear equations corresponding to the last augmented matrix is:

$$\begin{aligned} x &= 124 + 16u - 19v \\ y &= 75 + 9u - 11v \\ z &= 31 + 3u - 4v. \end{aligned}$$

- We say that u and v are independent (or free) variables and x, y, z are dependent (or basic) variables. The general solution is given by

$$(x, y, z, u, v)^T$$
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Theorem

Suppose $A\mathbf{x} = \mathbf{b}$ *is a system of linear equations where* $A = ((a_{ij}))$ *is a* $m \times n$ *matrix and* $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$*,* $\mathbf{b} = (b_1, b_2, \ldots, b_m)^T$.

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Hence $\mathbf{c} + \mathbf{v}$ is a solution to $A\mathbf{x} = \mathbf{b}$. $\square$

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$\rightarrow$ Hence the system has no solutions.

The basic idea is to reduce the augmented matrix $[A|\mathbf{b}]$ by elementary row operations to $[A'|\mathbf{c}]$, where A' is simple, or more precisely, in REF. This can always be achieved by the Gaussian Elimination Algorithm, which consists of the following steps.

1. Search the first column of $[A|\mathbf{b}]$ from the top to the bottom for the first non-zero entry, and then if necessary, the second column (the case where all the coefficients corresponding to the first variable are zero), and then the third column, and so on. The entry thus found is called the **current pivot**.

2. Interchange, if necessary, the row containing the current pivot with the first row.

3. Keeping the row containing the pivot (that is, the first row) untouched, subtract appropriate multiples of the first row from all the other rows to obtain all zeroes below the current pivot in its column.

4. Repeat the preceding steps on the **submatrix** consisting of all those elements which are **below** and **to the right** of the current pivot.

5. Stop when no further pivot can be found.

REF

The $m \times n$ coefficient matrix A of the linear system $A\mathbf{x} = \mathbf{b}$ is thus reduced to an $(m \times n)$ matrix A' in row echelon form and so the augmented matrix $[A|\mathbf{b}]$ becomes $[A'|\mathbf{c}]$, which looks like

$$\left[\begin{array}{cccccccccccc|c}
0 & \dots & p_1 & * & * & * & * & * & * & * & \dots & * & c_1 \\
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\vdots & & \vdots & \vdots & \vdots & 0 & 0 & \dots & \vdots & \vdots & & \vdots & \vdots \\
0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & p_r & * & \dots & * & c_r \\
0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & c_{r+1} \\
\vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
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In fact, there are $n - r$ free variables, where n is the number of columns (unknowns) of A (and hence of U).

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Remark: The general solution of a consistent system of m equations in n unknowns will involve $n - r$ free variables or free parameters (often denoted by $t_1, t_2, \ldots$ or $s_1, s_2, \ldots$), where r is the number of pivots in a REF of the coefficient matrix. The numbers r and $n - r$ associated with the matrix A are important quantities and deserve a name. Thus we define

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row-rank(A) = number of non-zero rows in REF of A

nullity(A) = number of free variables in the solution of $AX = 0$.

Since the homogeneous system $A\mathbf{x} = \mathbf{0}$ is always consistent, we see that nullity(A) $= n -$row-rank(A).

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If a matrix in echelon form satisfies the following additional conditions, then it is in row reduced echelon form:
(d) The leading entry in each nonzero row is 1.
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Examples

1. $$\begin{bmatrix} 2 & 3 & -2 & 4 \\ 0 & -9 & 7 & -8 \\ 0 & 0 & 0 & 20 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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4. Different sequences of row operations might involve a different set of pivots.

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Determine whether the following statements are true or false.

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More Examples - REF, RREF

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1. $$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}$$

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Note that the left side of a matrix in RREF need not be identity matrix.

Solution given REF

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Consider the REF form of the augmented matrix given by

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Note that $\boxed{x_3 \text{ is a free variable}}$

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Consider the REF form of the augmented matrix given by

$$\begin{bmatrix} 1 & 1 & 2 & 3 & 2 \\ 0 & 1 & 5 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

In this case $r = 3$ (number of pivots) and $n = 4$ (number of variables).

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We get infinitely many solutions (since x_3 is a free parameter).

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Example : Consider I_3. $R_3 \to 7R_3$ gives $\mathcal{E}_3(7) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix}$.

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Proposition 1

Let A be an $m \times n$ matrix. If $\widetilde{A}$ is obtained from A by an elementary row operation, and $\mathcal{E}$ is the corresponding $m \times m$ elementary matrix, then $\mathcal{E}A = \widetilde{A}$.

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Elementary matrices are invertible and the inverses are also elementary matrices.

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To see this, let E be an elementary matrix. Then E is obtained from I by an elementary row operation. There is clearly an "inverse elementary row operation", which will transform E back to I. Now if Suppose $\widetilde{E}$ is the elementary matrix obtained by making this "inverse elementary row operation" on I. Then by the previous proposition, we see that

$$\widetilde{E}E = I \quad \text{and} \quad E\widetilde{E} = I.$$

Thus E is invertible and E^{-1} is the elementary matrix $\widetilde{E}$.

Exercise : With notation as in the previous slides, show that
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Exercise : With notation as in the previous slides, show that (i) $\mathcal{E}_{i,j}^{-1} = \mathcal{E}_{i,j}$, (ii) $\mathcal{E}_i(k)^{-1} = \mathcal{E}_i(1/k)$ and (iii) $\mathcal{E}_{i,j}(k)^{-1} = \mathcal{E}_{i,j}(-k)$.

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Inverse of A : an example

$$[A|I] = \left[\begin{array}{ccc:ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ & & & & & \\ & & & & & \end{array}\right]$$

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Example continued...

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Example continued...

$$[B|L] \xrightarrow{R_2 \to R_2 + (2)R_3} \xrightarrow{R_1 \to R_1 + (-1)R_3} \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 2 & -1 & -1 \\ 0 & -8 & 0 & -4 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 + (1/8)R_2} \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & \frac{12}{8} & -\frac{5}{8} & -\frac{6}{8} \\ 0 & -8 & 0 & -4 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

Example continued...

$$[B|L] \xrightarrow{R_2 \to R_2 + (2)R_3} \xrightarrow{R_1 \to R_1 + (-1)R_3} \left[\begin{array}{ccc:ccc} 2 & 1 & 0 & 2 & -1 & -1 \\ 0 & -8 & 0 & -4 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 + (1/8)R_2} \left[\begin{array}{ccc:ccc} 2 & 0 & 0 & \frac{12}{8} & -\frac{5}{8} & -\frac{6}{8} \\ 0 & -8 & 0 & -4 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1/2} \xrightarrow{R_2 \to R_2/-8} \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & \frac{12}{16} & -\frac{5}{16} & -\frac{6}{16} \\ 0 & 1 & 0 & \frac{4}{8} & -\frac{3}{8} & -\frac{2}{8} \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] = [I|A^{-1}].$$

Example continued...

$$[B|L] \xrightarrow{R_2 \to R_2 + (2)R_3} \xrightarrow{R_1 \to R_1 + (-1)R_3} \begin{bmatrix} 2 & 1 & 0 & \vdots & 2 & -1 & -1 \\ 0 & -8 & 0 & \vdots & -4 & 3 & 2 \\ 0 & 0 & 1 & \vdots & -1 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 + (1/8)R_2} \begin{bmatrix} 2 & 0 & 0 & \vdots & \frac{12}{8} & -\frac{5}{8} & -\frac{6}{8} \\ 0 & -8 & 0 & \vdots & -4 & 3 & 2 \\ 0 & 0 & 1 & \vdots & -1 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1/2} \xrightarrow{R_2 \to R_2/-8} \begin{bmatrix} 1 & 0 & 0 & \vdots & \frac{12}{16} & -\frac{5}{16} & -\frac{6}{16} \\ 0 & 1 & 0 & \vdots & \frac{4}{8} & -\frac{3}{8} & -\frac{2}{8} \\ 0 & 0 & 1 & \vdots & -1 & 1 & 1 \end{bmatrix} = [I|A^{-1}].$$