MA 108 ODE: Linear DE-s of Higher Order

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Recall

Recall: constant coefficients, Cauchy-Euler, undetermined coefficients.

Method of Variation of Parameters

Though the method of undetermined coefficients was sort of easy, it was applicable only when $r(x)$ was of a certain form. Also we had to guess the form of a solution in advance.

Another method to find a particular solution of a non-homogeneous ODE is the method of variation of parameters. Here, we vary the constants c_1, c_2 coming in the general solution

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

of the associated homogeneous equation

$$y'' + p(x)y' + q(x)y = 0.$$

Method of Variation of Parameters

We replace the constants c_1, c_2 by functions $v_1(x), v_2(x)$, so that

$$y(x) = v_1(x)y_1(x) + v_2(x)y_2(x)$$

is a solution of

$$y'' + p(x)y' + q(x)y = r(x).$$

Now,

$$y' = v_1 y_1' + v_2 y_2' + v_1' y_1 + v_2' y_2.$$

Let's also demand

$$v_1' y_1 + v_2' y_2 = 0.$$

Thus,

$$y'' = v_1 y_1'' + v_1' y_1' + v_2 y_2'' + v_2' y_2'.$$

Method of Variation of Parameters

Substituting y, y', y'' in the given non-homogeneous ODE, and rearranging terms, we get:

$$v_1(y_1'' + py_1' + qy_1) + v_2(y_2'' + py_2' + qy_2) + v_1'y_1' + v_2'y_2' = r(x).$$

Thus,

$$v_1'y_1' + v_2'y_2' = r(x).$$

Recall that we also have

$$v_1'y_1 + v_2'y_2 = 0.$$

Thus, we have:

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 \\ r(x) \end{bmatrix}.$$

Therefore,

$$v_1' = \frac{\begin{vmatrix} 0 & y_2 \\ r(x) & y_2' \end{vmatrix}}{W(y_1, y_2)}, \quad v_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & r(x) \end{vmatrix}}{W(y_1, y_2)}.$$

Thus,

$$v_1 = -\int \frac{y_2 r(x)}{W(y_1, y_2)} dx, \quad v_2 = \int \frac{y_1 r(x)}{W(y_1, y_2)} dx.$$

Hence,

$$\begin{aligned} y &= v_1 y_1 + v_2 y_2 \\ &= y_2 \int \frac{y_1 r(x)}{W(y_1, y_2)} dx - y_1 \int \frac{y_2 r(x)}{W(y_1, y_2)} dx. \end{aligned}$$

Example: Find a particular solution of

$$y'' + y = \csc x.$$

Step I: Find a basis of solutions for the associated homogeneous equation

$$y'' + y = 0.$$

The general solution of this is

$$y(x) = c_1 \sin x + c_2 \cos x.$$

Step II: Calculate the Wronskian $W(y_1, y_2)$:

$$W(y_1, y_2) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -1.$$

Now,

$$v_1 = -\int \frac{y_2 r(x)}{W(y_1, y_2)} dx = -\int \frac{\cos x \csc x}{-1} dx = \ln|\sin x|,$$

and

$$v_2 = \int \frac{y_1 r(x)}{W(y_1, y_2)} dx = \int \frac{\sin x \csc x}{-1} dx = -x.$$

Hence, a particular solution is given by

$$y(x) = \sin x \ln|\sin x| - x \cos x.$$

Method of Variation of Parameters

Example: Find a particular solution of

$$y'' + 4y = 3\cos 2t.$$

Recall that via the method of undetermined coefficients, you had to modify the proposed initial solution with multiplication by t, and you got the answer as $\frac{3}{4}t\sin 2t$.
Now in variation of parameters,

$$y_1 = \cos 2t, \;\; y_2 = \sin 2t,$$

and

$$v_1 = -\int \frac{\sin 2t \cdot 3\cos 2t}{2}dt = -\frac{3}{16}\cos 4t,$$

$$v_2 = \int \frac{\cos 2t \cdot 3\cos 2t}{2}dt = \frac{3}{16}\sin 4t + \frac{3}{4}t.$$

Thus, a particular solution is

$$v_1y_1 + v_2y_2 = -\frac{3}{16}\cos 2t + \frac{3}{4}t\sin 2t.$$

Consider an n-th order linear DE:

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_n(x)y = g(x).$$

Assume that the functions $a_0(x), a_1(x), \ldots, a_n(x), g(x)$ are continuous on an interval I. Also assume that $a_0(x) \neq 0$ for every $x \in I$. Such an equation is called non-singular. Such an equation can be put into standard form:

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = r(x).$$

Note that $p_1(x), \ldots, p_n(x), r(x)$ are continuous on I. When will you call such an equation homogeneous?

Linear DE's of Higher Order

Our systematic study of second order linear differential equations will guide us in our study of n-th order linear equations. Quick recall:

- Convert the problem to MA 106 via $L : C^2(I) \to C(I)$
- Dimension theorem about the dimension of Ker L
- Existence and uniqueness for IVP's (assume continuity of coefficients, get existence and uniqueness on the same I)
- Linear dependence and independence in terms of Wronskian
- Could write down the general solution in the constant coefficient case (and for Cauchy-Euler)
- Non-homogeneous case handled by undetermined coefficients and variation of parameters.

Define the vector space

$$C^n(I) = \{f : I \to \mathbb{R} \mid f, f^1, \ldots, f^{(n)} \text{ are continuous}\}.$$

This is a vector space under usual addition and scalar multiplication for functions. Define

$$L : C^n(I) \to C(I)$$

by

$$L(f) = f^{(n)} + p_1(x)f^{(n-1)} + \ldots + p_n(x)f.$$

Then L is a linear transformation. Sometimes we write

$$L = D^n + p_1 D^{n-1} + \ldots + p_{n-1}D + p_n,$$

where $D_k = \frac{d^k}{dx^k}$, and call L, a linear differential operator.

Ker L is a subspace of $C^n(I)$ and consists precisely of solutions of the homogeneous DE

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = 0.$$

Theorem (Dimension Theorem)

$$\dim \mathrm{Ker}\ L = n.$$

An IVP in this context will be of the form

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = 0$$
$$y(x_0) = k_0, y^1(x_0) = k_1, \ldots, y^{n-1}(x_0) = k_{n-1}$$

with $x_0 \in I$.

Theorem (Existence and Uniqueness)

If $p_i(x)$ are continuous throughout an interval I containing x_0, then the IVP has a unique solution on I.

Note that both existence and uniqueness are guaranteed on the same I where continuity of the coefficients is given.

Can we give an explicit basis of Ker L? Recall the theorem we had: if f, g are two solutions of a homogeneous second order linear ODE (with continuous coefficients) and if $(f(x_0), f'(x_0))$ and $(g(x_0), g'(x_0))$ are linearly independent vectors in $\mathbb{R}^2$, for some x_0, then, the solution space is the linear span of f and g. If h is a solution, can solve for c, d such that

$$\begin{aligned} h(x_0) &= cf(x_0) + dg(x_0) \\ h'(x_0) &= cf'(x_0) + dg'(x_0). \end{aligned}$$

since $W(f, g)(x_0) \neq 0$ from the given condition, and apply uniqueness theorem to $y_1 = h - cf - dg$ and $y_2 \equiv 0$.

Wronskian:

$$W(y_1, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}.$$

If $y_1, y_2, \ldots, y_n$ are solutions of an n-th order linear homogeneous ODE, and if

$$W(y_1, \ldots, y_n; x_0) \neq 0,$$

for some x_0, then any solution can be written as a linear combination of $y_1, y_2, \ldots, y_n$.

Proof? Let y be any solution. Solve for the unique $c_1, \ldots, c_n$ such that

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) & \ldots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \ldots & y_n'(x_0) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \ldots & y_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ \cdot \\ c_n \end{bmatrix} = \begin{bmatrix} y(x_0) \\ y'(x_0) \\ \cdot \\ \cdot \\ \cdot \\ y^{(n-1)}(x_0) \end{bmatrix}.$$

Now apply uniqueness theorem to

$$f = y - c_1 y_1 - \ldots - c_n y_n, \; g \equiv 0.$$

Wronskian and Linear Independence

We have seen that the Wronskian of any two solutions $f(x), g(x)$ of $y'' + p(x)y' + q(x)y = 0$ is given by

$$W(f, g; x) = W(f, g; a)e^{-\int_a^x p(t)dt},$$

for any $a \in I$. We did notice that the answer depends only on $p(x)$ and not on $q(x)$. What should be the n-th order analogue?

Theorem

If $y_1, y_2, \ldots, y_n$ are solutions of

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = 0,$$

then,

$$W(y_1, \ldots, y_n; x) = W(y_1, \ldots, y_n; a)e^{-\int_a^x p_1(t)dt}.$$

Once again, our earlier proof will go through. If you remember, in the earlier proof, we just had to claim:

$$W' = -p_1(x)W.$$

We make the same claim now too. Notice that the derivative of

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

is

$$\begin{vmatrix} y_1' & y_2' \\ y_1' & y_2' \end{vmatrix} + \begin{vmatrix} y_1 & y_2 \\ y_1'' & y_2'' \end{vmatrix},$$

which is

$$\begin{vmatrix} y_1 & y_2 \\ y_1'' & y_2'' \end{vmatrix}.$$

In this, we had substituted $y_i'' = -p_1(x)y_i' - p_2(x)y_i$ to conclude the claim.

We do the same thing. The derivative of $W(y_1, y_2, \ldots, y_n)$ is the sum of n determinants with derivative being taken in the first row in the first one, in the second row in the second one, etc. Except the last one, all vanish because two rows are identical. In the last one, make the substitution

$$y_i^{(n)} = -p_1(x)y_i^{(n-1)} - p_2(x)y_i^{(n-2)} - \ldots - p_n(x)y_i.$$

Expand this into sum of n determinants. Once again, all but one vanish. The non-vanishing one gives

$$-p_1(x) \cdot W(y_1, y_2, \ldots, y_n).$$

Thus, for solutions of linear homogeneous DE's in standard form, the Wronskian is either never zero or identically zero on I!

Wronskian and Linear Independence

If $y_1, \ldots, y_n$ (not necessarily solutions of an ODE) are linearly dependent, then there exist $(c_1, \ldots, c_n) \neq (0, \ldots, 0)$ such that

$$c_1 y_1 + \ldots + c_n y_n = 0$$

on I. Differentiating, we get

$$c_1 y_1^{(i)} + \ldots + c_n y_n^{(i)} = 0,$$

for $i = 1, \ldots, n-1$. This implies

$$W(y_1, \ldots, y_n) = 0.$$

Wronskian and Linear Independence

On the other hand, suppose $y_1, \ldots, y_n$ are solutions of an ODE in standard form with continuous coefficients in an open interval I, and that

$$W(y_1, \ldots, y_n; x_0) = 0,$$

for some $x_0 \in I$. We get $(c_1, \ldots, c_n) \neq (0, \ldots, 0)$ such that

$$c_1 y_1^{(i)}(x_0) + \ldots + c_n y_n^{(i)}(x_0) = 0,$$

for $i = 0, 1, \ldots, n-1$. Now consider

$$f = c_1 y_1 + \ldots + c_n y_n,$$

and

$$g \equiv 0$$

and apply uniqueness theorem to conclude that $\{y_1, \ldots, y_n\}$ is linearly dependent. (What's the IVP here?)