Sri SAI RAM ENGINEERING COLLEGE
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Sairam INSTITUTIONS

SAIRAM DIGITAL RESOURCES

YEAR II

SEM III

MA8351

DISCRETE MATHEMATICS (COMMON TO CSE & IT)

Unit 1 LOGIC AND PROOFS

1.6 INTRODUCTION TO PROOFS

SCIENCE & HUMANITIES

4 QUALITY EDUCATION

INTRODUCTION TO PROOFS

INTRODUCTION :

A proof is a valid argument that establishes the truth of a mathematical statement. A proof can use the hypotheses of the theorem, if any, axioms assumed to be true, and previously proven theorems.

METHODS OF PROVING THEOREMS:

Proving mathematical theorems can be difficult. To construct proofs we need all available ammunition, including a powerful battery of different proof methods. These methods provide the overall approach and strategy of proofs .Understanding these methods is a key component of learning how to read and construct mathematical proofs.

To prove a theorem of the form $\forall x\,(P(x) \rightarrow Q(x))$, our goal is to show that $P(c) \rightarrow Q(c)$ is true, where *c* is an arbitrary element of the domain, and then apply universal generalization.In this proof, we need to show that a conditional statement is true. Because of this, we now focus on methods that show that conditional statements are true.

DIRECT PROOFS:

A direct proof of a conditional statement $p \rightarrow q$ is constructed when the first step is the assumption that p is true; subsequent steps are constructed using rules of inference, with the final step showing that q must also be true. A direct proof shows that a conditional statement $p \rightarrow q$ is true by showing that if p is true, then q must also be true, so that the combination p true and q false never occurs.

DEFINITION 1

The integer n is even if there exists an integer k such that $n = 2k$, and n is odd if there exists an integer k such that $n = 2k + 1$. Two integers have the same parity when both are even or both are odd; they have opposite parity when one is even and the other is odd.

DEFINITION 2

The real number r is rational if there exist integers p and q with $q \neq 0$ such that $r = p/q$. A real number that is not rational is called irrational.

PROBLEMS

1) Give a direct proof of the theorem "If n is an odd integer, then n^2 is odd."

Solution:

Assume that n is odd.

By the definition of an odd integer

$n = 2k + 1$, where k is some integer.

To prove n^2 is also odd.

squaring on both sides of the above equation we get,

$$n^2 = (2k + 1)^2$$

$$= 4k^2 + 4k + 1$$

$$= 2(2k^2 + 2k) + 1.$$

By the definition of an odd integer, n^2 is an odd integer.

2) Give a direct proof that if *m* and *n* are both perfect squares, then nm is also a perfect.

Solution:

Assume that m and n are both perfect squares.

By the definition of a perfect square,

there are integers s and t such that $m = s^2$ and $n = t^2$.

To show that mn must also be a perfect square.

Now $m\,n = s^2 t^2$.

$= (s\,s)\,(t\,t)$

$= (s\,t)(s\,t)$

$= (s\,t)^2$, (using commutativity and associativity)

By the definition of perfect square,

mn is also a perfect square.

3) Prove that the sum of two rational numbers is rational.

Solution:

By direct proof method,

Suppose that r and s are rational numbers.

From the definition of a rational number,

there are integers p and q, with $q \neq 0$, such that $r = p/q$,

and integers t and u, with $u \neq 0$, such that $s = t/u$.

$r + s = p/q + t/u$

$= (pu + qt) / qu$

Because $q \neq 0$ and $u \neq 0 \Rightarrow qu \neq 0$.

ie) we have expressed $r + s$ as the ratio of two integers, $pu + qt$ and qu, where

$qu \neq 0$.

$\Rightarrow$ $r + s$ is rational . ie) the sum of two rational numbers is rational.

PROOF BY CONTRAPOSITION:

Proofs, that do not start with the premises and end with the conclusion, are called indirect proofs**.** An extremely useful type of indirect proof is known as proof by contraposition.

Proofs by contraposition make use of the fact that the conditional statement $p \to q$ is equivalent to its contrapositive, $\neg q \to \neg p$. This means that the conditional statement $p \to q$ can be proved by showing that its contrapositive, $\neg q \to \neg p$, is true.

PROBLEMS

1) Prove that if n is an integer and $3n + 2$ is odd, then n is odd

Solution:

Assume that the conclusion of the conditional statement is false;
ie, assume that n is even.
Then, by the definition of an even integer, $n = 2k$ for some integer k.
Substituting $2k$ for n, we find that

$$3n + 2 = 3(2k) + 2$$
$$= 6k + 2$$
$$= 2(3k + 1).$$

ie) $3n + 2$ is even
ie) negation of the conclusion of the conditional statement implies that the hypothesis is false,
ie) original conditional statement is true
ie) "If $3n + 2$ is odd, then n is odd."

2) Prove that if $n = ab$, where a and b are positive integers, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

Solution:

Assume that the conclusion of the conditional statement is false.

ie) $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$" is false.

$\Rightarrow$ both $a \leq \sqrt{n}$ and $b \leq \sqrt{n}$ are false.

$\Rightarrow$ $a > \sqrt{n}$ and $b > \sqrt{n}$.

$\Rightarrow$ $ab > \sqrt{n} \cdot \sqrt{n} = n$.

$\Rightarrow$ $ab > n$

which contradicts the statement $n = ab$.

ie) the negation of the conclusion of the conditional statement implies that the hypothesis is false. ie) the original conditional statement is true.

ie) if $n = ab$, where a and b are positive integers,then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

3) Prove that if *n* is an integer and n^2 is odd, then *n* is odd.

Solution:

Assume that the conclusion of the conditional statement is false.

ie) n is even

ie) $n = 2k$

squaring both sides of this equation, we obtain

$n^2 = 4k^2$

$= 2(2k^2)$,

$\Rightarrow$ n^2 is even

We have proved that if n is an integer and n^2 is odd,then n is odd.

VACUOUS AND TRIVIAL PROOFS

A conditional statement $p \to q$ is true when we know that p is false, because $p \to q$ must be true when p is false. Consequently, if we can show that p is false, then we have a proof, called a **vacuous proof**, of the conditional statement $p \to q$. Vacuous proofs are often used to establish special cases of theorems that state that a conditional statement is true for all positive integers .

1) Show that the proposition *P(0)* is true, where *P(n)* is "If $n > 1$, then $n^2 > n$" and the domain consists of all integers.

Solution:

P(0) is If $0 > 1$, then $0^2 > 0$." We can show *P(0)* using a vacuous proof.

Indeed, the hypothesis $0 > 1$ is false. This tells us that *P(0)* is automatically true.

2) Let $P(n)$ be "If a and b are positive integers with $a \geq b$, then $a^n \geq b^n$," where the domain consists of all nonnegative integers. Show that $P(0)$ is true.

Solution:

The proposition $P(0)$ is "If $a \geq b$, then $a^0 \geq b^0$."

Because $a^0 = b^0 = 1$,

ie) the conclusion of the conditional statement "If $a \geq b$, then $a^0 \geq b^0$" is true.

Hence, this conditional statement, which is $P(0)$, is true.

PROOFS BY CONTRADICTION:

Suppose we want to prove that a statement p is true. Furthermore, suppose that we can find a contradiction q such that $\neg p \rightarrow q$ is true. Because q is false, but $\neg p \rightarrow q$ is true, we can conclude that $\neg p$ is false, which means that p is true.

1) Show that at least four of any 22 days must fall on the same day of the week.

Solution:

Let *p* be the proposition "At least four of 22 chosen days fall on the same day of the week."

Suppose that ┐ *p* is true.

This means that at most three of the 22 days fall on the same day of the week.

Because there are seven days of the week, this implies that at most 21 days could have been chosen, as for each of the days of the week, at most three of the chosen days could fall on that day.

This contradicts the premise that we have 22 days under consideration.

⇒ at least four of 22 chosen days fall on the same day of the week.

2) Prove that $\sqrt{2}$ is irrational by giving a proof by contradiction.

Solution:

Let p be the proposition " $\sqrt{2}$ is irrational."

suppose that $\neg p$ is true.

ie)$\neg p$ is $\sqrt{2}$ is rational

If $\sqrt{2}$ is rational, there exist integers a and b with

$\sqrt{2} = a/b$, where $b \neq 0$ and a and b have no common factors

Squaring on both sides we get

$2 = a^2 / b^2$.

$\Rightarrow 2b^2 = a^2$.

By the definition of an even integer, a^2 is even.

$\Rightarrow$ a is even

$\Rightarrow$ a = 2c for some integer c.

$\Rightarrow 2b^2 = 4c^2$.

$\Rightarrow b^2 = 2c^2$.

$\Rightarrow b^2$ is even.

$\Rightarrow$ b is even

ie) ┐p leads to the equation $\sqrt{2} = a/b$, where both a and b are even,

ie) 2 divides both a and b.

But a and b have no common factors.

This leads to the contradiction that 2 divides both a and b and 2 does not divide both a and b

$\Rightarrow$┐p must be false. ie) $\sqrt{2}$ is irrational.

3) Give a proof by contradiction of the theorem "If 3*n* + 2 is odd, then *n* is odd."

Solution:

Let p be "$3n + 2$ is odd" and q be "n is odd."

Assume that both p and $\neg q$ are true.

ie , assume that $3n + 2$ is odd and that n is not odd.

Because n is not odd, it is even.

Because n is even, there is an integer k such that $n = 2k$.

$\Rightarrow 3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1)$.

$\Rightarrow 3n + 2$ is even

$\Rightarrow$ both p and $\neg p$ are true

which is a contradiction. ie) if $3n + 2$ is odd, then n is odd.