Sri SAI RAM ENGINEERING COLLEGE
INSTITUTE OF TECHNOLOGY
West Tambaram, Chennai - 44

Sairam INSTITUTIONS

SAIRAM DIGITAL RESOURCES

Unit III GRAPHS

3.3 MATRIX REPRESENTATION AND ISOMORPHISM

YEAR II

SEM III

MA8351

DISCRETE MATHEMATICS
COMMON TO CSE & IT

SCIENCE & HUMANITIES

4 QUALITY EDUCATION

MATRIX REPRESENTATION OF GRAPHS

ADJACENCY MATRIX

When G is a simple graph with n vertices $v_1, v_2, \ldots\ldots, v_n$, the matrix A (or A_G) ≡ [a_{ij}], where

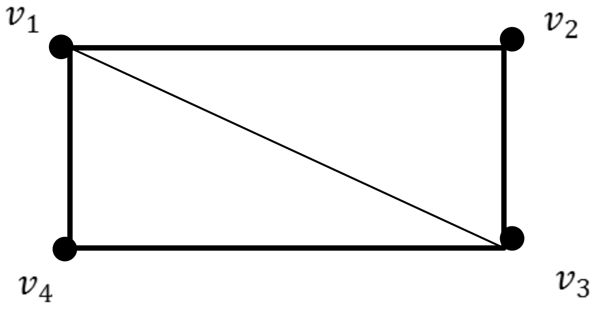
$$a_{ij} = \begin{cases} 1, & \text{if } v_i v_j \text{ is an edge of } G \\ 0, & \text{otherwise} \end{cases}$$

is called the adjacency matrix of G.

Example:

Let G be the given matrix . The adjacency matrix of G is given by

$$G = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$



Properties of an adjacency matrix:

- Since a simple graph has no loops, each diagonal entry of matrix G is zero i.e a_{ij}=0.
- The adjacency matrix of simple graph is symmetric i.e $a_{ij} = a_{ji}$
- Deg (v_i) is equal to the number of 1's in the i^{th} row or i^{th} column.
- Pseudographs, direct graphs and multi graphs can also be represented by adjacency matrix which may not be symmetric matrix.

INCIDENCE MATRIX

If G = (V,E) is an undirected graph with n vertices $v_1,v_2,\ldots\ldots,v_n$ and m edges $e_1,e_2,\ldots\ldots,e_m$, then the (n x m) matrix B =[b_{ij}], where

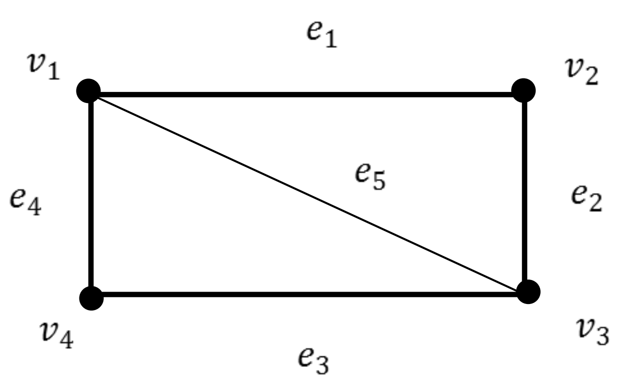
$$b_{ij} = \begin{cases} 1, & when\ edge\ \ e_j\ is\ incident\ \ on\ v_i \\ 0, & otherwise \end{cases}$$

is called the incidence matrix of G.

Example:

Let G be the given matrix . The incidence matrix of G is given by

$$\begin{array}{c c} & \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$



Properties of an incidence matrix:

- Each column of B contains exactly two unit entries.
- A row with all 0 entries corresponds to an isolated vertex.
- A row with a single unit entry corresponds to a pendant vertex.
- deg (v_i) is equal to the number of 1's in the i^{th} row.

ISOMORPHISM

Let $G_1 = (V_1 , E_1)$ and $G_2 = (V_2 , E_2)$ be two simple graphs. G_1 and G_2 are isomorphic if there is a one-to-one and onto function or map f from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only f(a) and f(b) are adjacent in G_2 $\forall$ a ,b $\in V_1$.

PERMUTATION MATRIX

A matrix whose rows are the rows of the unit matrix, but not necessarily in their natural order, is called a permutation.

THEOREM :1

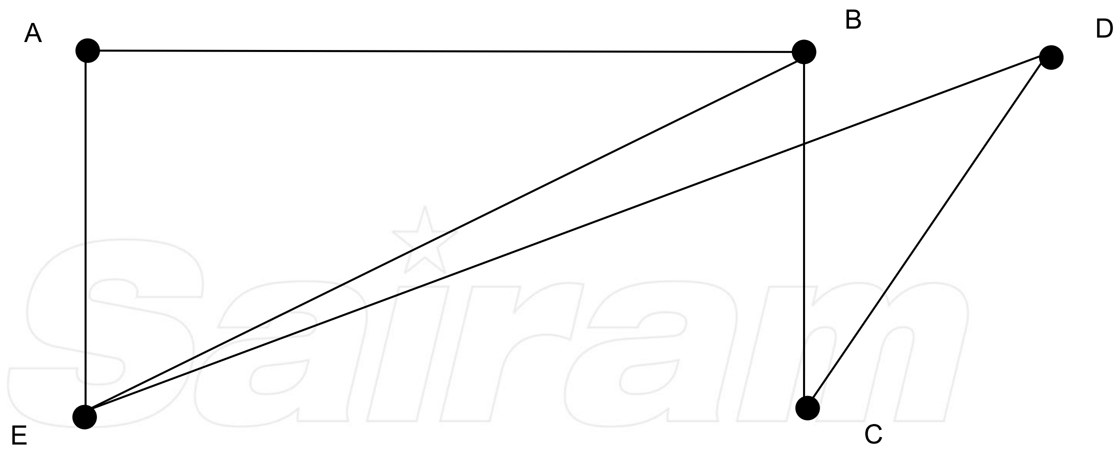
Two graphs are isomorphic, if and only if their vertices can be labeled in such a way that the corresponding adjacency matrices are equal.

THEOREM :2

Two labeled graphs G_1 and G_2 with adjacency matrices A_1 and A_2 respectively are isomorphic, if and only if, there exists a permutation matrix P such that $PA_1P^T = A_2$.

PROBLEMS

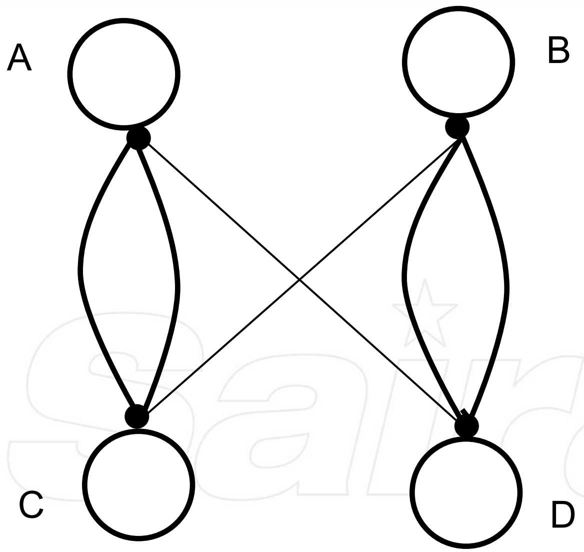
1) Represent the graph by adjacency matrices.



Solution:

$$
\begin{array}{c}
\\ A \\ B \\ C \\ D \\ E
\end{array}
\begin{array}{c}
\begin{array}{ccccc} A & B & C & D & E \end{array} \\
\begin{pmatrix}
0 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 0
\end{pmatrix}
\end{array}
$$

2) Draw the graph represented by the following adjacency matrix.



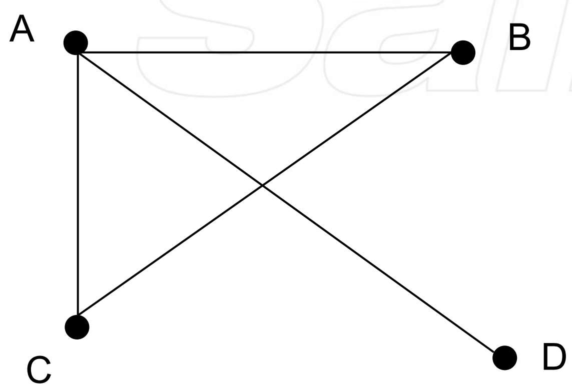
Solution:

$$\begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix}$$

3) Draw the graph represented by the following adjacency matrix.

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

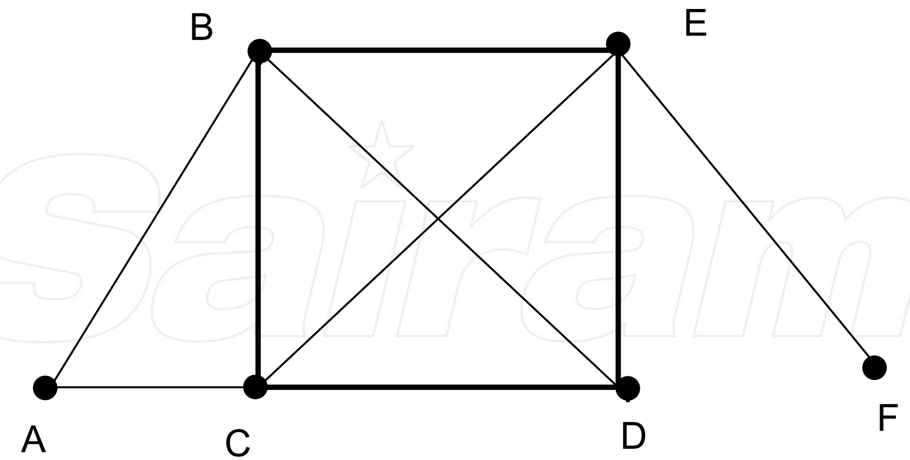
Solution:



4) Find the number of vertices, the number of edges and the degree of each vertex in the following undirected graphs. Verify also handshaking theorem in each case.

Solution:

i) Graph G_1



Number of vertices = 6

Number of edges =9

deg (A) =2, deg (B) =4, deg (C) =4, deg (D) =3, deg (E) =4, deg (F) =1,

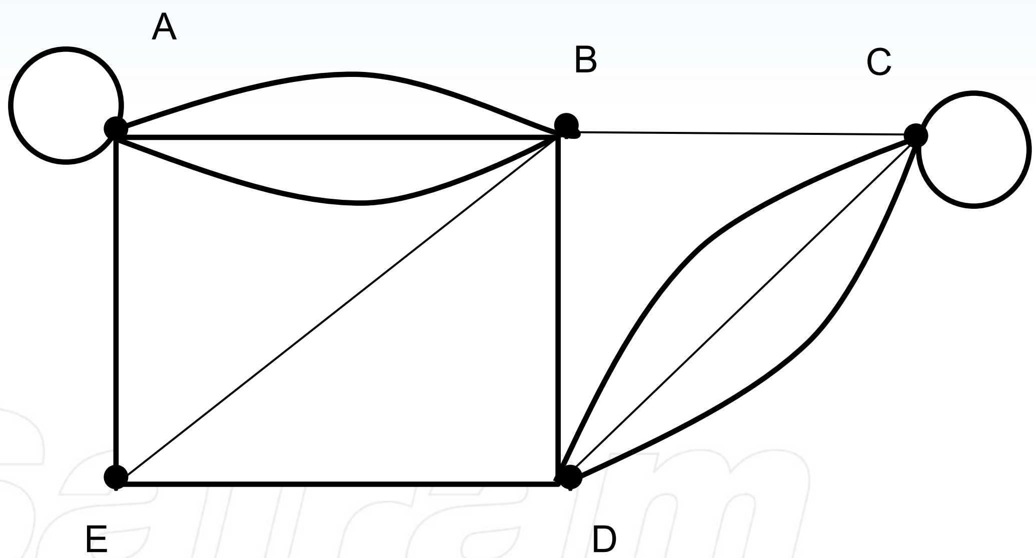
Handshaking theorem states that

Σdeg (vi) = 2e , (e is the number of edges)

$$\Sigma deg(v_i)=2+4+4+3+4+1=18=2x9$$

Hence the theorem is true.

ii) Graph G_2

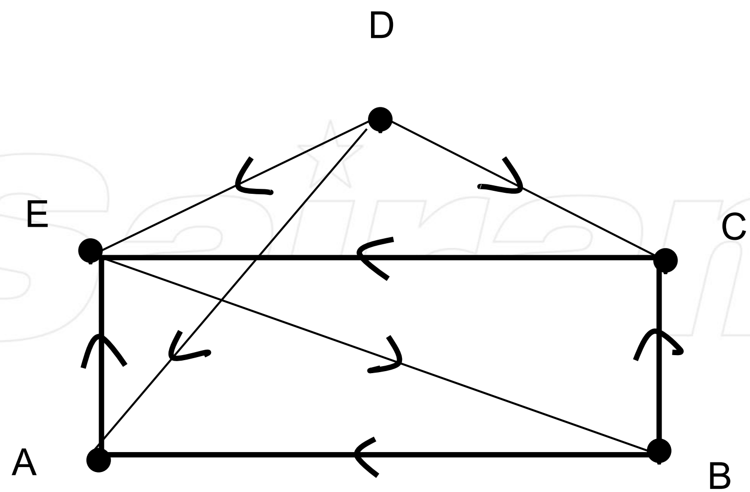


Number of vertices = 5
Number of edges = 13
deg (A) =6, deg(B) =6, deg(C) =6, deg(D) =5, deg(E) =3
Handshaking theorem states that
$\Sigma\deg(v_i)$ = 2e , (e is the number of edges)
Now,
$\Sigma\deg(v_i)$=6+6+6+5+3 =26=2x13. Hence the theorem is true.

5) Find the in-degree and out-degree of each vertex of each of the following directed graphs. Also verify that the sum of the in-degrees equals the number of edges.

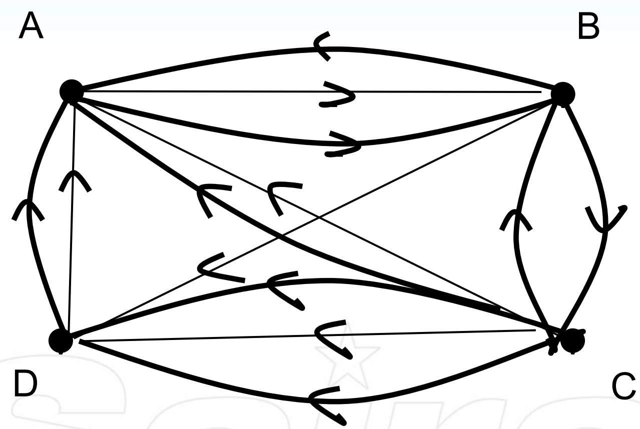
Solution:

i) Graph G_1



deg^-(A) = 2, deg^-(B) = 1, deg^-(C) = 2, deg^-(D) = 3 and deg^-(E) = 0,.
deg^+(A) = 1, deg^+(B) = 2, deg^+(C) = 1, deg^+(D) = 1 and deg^+(E) = 3.
We see that Σdeg^-(A) = Σdeg^+(A) = 8 = the number of edges of G_1.

ii) Graph G_2



$deg^{-}(\mathrm{A}) = 5$, $deg^{-}(\mathrm{B}) = 3$, $deg^{-}(\mathrm{C}) = 1$, $deg^{-}(\mathrm{D}) = 4$
$deg^{+}(\mathrm{A}) = 2$, $deg^{+}(\mathrm{B}) = 3$, $deg^{+}(\mathrm{C}) = 6$, $deg^{+}(\mathrm{D}) = 2$

We see that $\Sigma deg^{-}(\mathrm{A}) = \Sigma deg^{+}(\mathrm{A}) = 13$ = the number of edges of G_2.

6) For each of the following degree sequences, find if there exists graph. In each case, either draw a graph or explain why no graphs exists.

(i)4,4,4,3,2

Sum of the degrees of all the vertices = 17, which is an odd number. This is impossible. Hence no graphs exists with the given degree sequence.

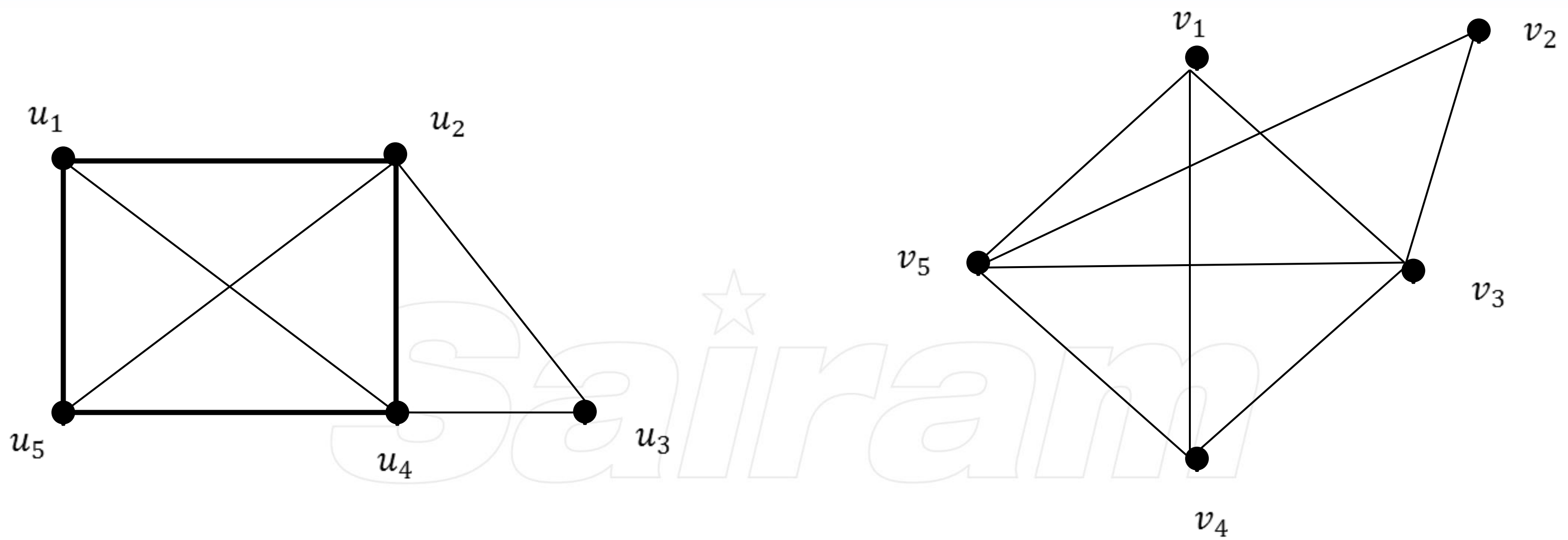
(ii)5,5,4,3,2,1

There are 6 vertices. Hence a vertex of degree 5 in the graph must be adjacent to all the vertices. As there are 2 vertices each of degree 5, all other vertices should be of degree at least 2. But the given degree sequence contains a 1. Hence, no graph is possible with the given sequence.

(iii)3,3,3,3,3,3

A simple graph with the given description is not possible only multigraph is possible.

7) Determine whether the following pairs of graphs are isomorphic.



Solution:

Each of the two graphs have 5 vertices and 8 edges . The vertices u_1 and u_5 are of degree 3 each, u_2 and u_4 are of degree 4 each and u_3 is of degree 2. Similarly v_1 and v_4 are of degree 3 each, v_3 and v_5 are of degree 4 each and v_2 is of degree 2.Thus the two graphs agree with respect to the 3 invariants.

To conclude that the two graphs are isomorphic , we need to prove that the adjacency matrices are same. For this purpose we assume arbitrarily that the vertex u_1 corresponds to v_1, $u_2$corresponds to v_5 and u_3 corresponds to v_2. The adjacency matrices are as follows.

$$
\begin{array}{c c}
 & \begin{array}{ccccc} u_1 & u_2 & u_3 & u_4 & u_5 \end{array} \\
\begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}
\end{array}
\qquad
\begin{array}{c c}
 & \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\
\begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}
\end{array}
$$

Since the two adjacency matrices are the same, the two graphs are isomorphic.

8) Test whether the graphs with the following adjacency matrices are isomorphic or not.

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

Solution: Let us interchange rows and columns

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \cong \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} C_2 \leftrightarrow C_3$$

$$\cong \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} R_2 \leftrightarrow R_3 \quad \cong \quad A_2$$

Therefore A_1 is similar to A_2 and hence $G_1 \cong G_2$.

9) Are the simple graphs with the following adjacency matrices isomorphic?

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

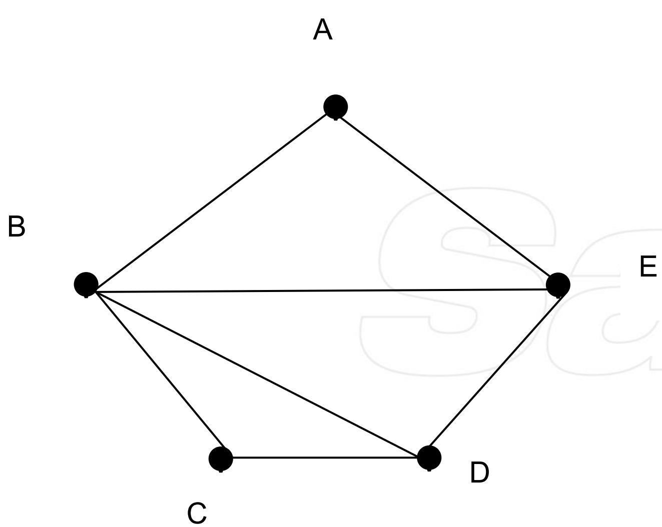
Solution: Let $A_1 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$

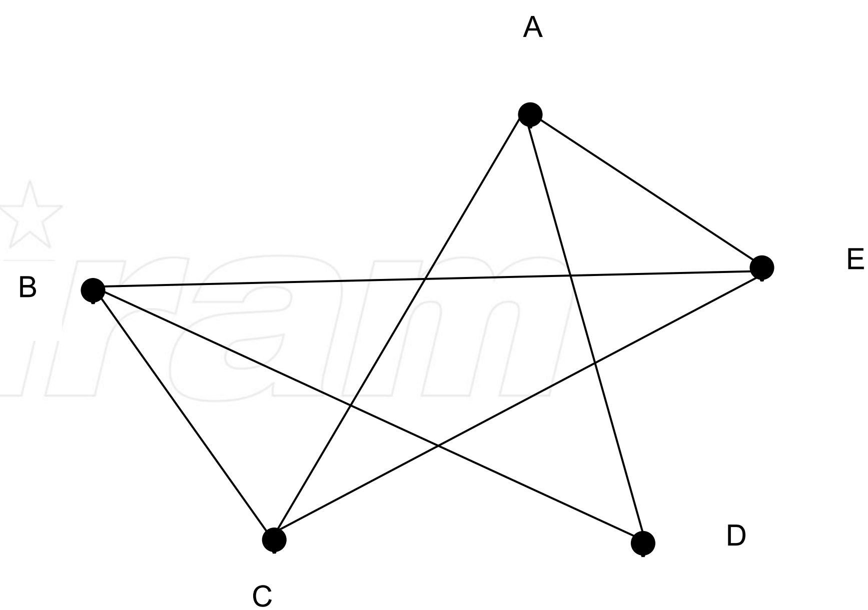
$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \cong \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} R_1 \leftrightarrow R_4$$

$$\cong \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} C_1 \leftrightarrow C_4$$

A_1 and A_2 cannot be similar. Hence the corresponding graphs are not isomorphic.

10) Examine whether the following pairs of graphs are isomorph. If not isomorphic, give the reasons.





Solution:

There are five vertices and seven edges in each of the two graphs. Here in the first graph deg(B)=4 and there is no corresponding vertex with degree four in the second graph. Hence the graphs are not isomorphic.

11) Draw the graph represented by the following incidence matrix.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

Solution: