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SAIRAM
DIGITAL RESOURCES

YEAR II | SEM IV

MA8391

PROBABILITY AND STATISTICS

(DEPARTMENT OF INFORMATION TECHNOLOGY)

UNIT I

PROBABILITY AND RANDOM VARIABLES

1.3 DISCRETE AND CONTINUOUS RANDOM VARIABLE

SCIENCE & HUMANITIES

RANDOM VARIABLES

Introduction

Consider an experiment of throwing a coin twice. The outcomes {HH, HT, TH, TT} constitute the sample space. Each of these outcomes can be associated with a number by specifying a rule of association with a number by specifying a rule of association (e.g., The number of heads). Such a rule of association is called a random variable. We denote a random variable by the capital letter (X, Y, etc.) and any particular value of the random variable by x and y.

Thus, a random variable X can be considered as a function that maps all elements in the sample space S into points on the real line. The notation X(S)=x means that x is the value associated with the outcomes S by the Random variable X.

SAMPLE SPACE

Consider an experiment of throwing a coin twice. The outcomes S = {HH, HT, TH, TT} constitute the sample space.

RANDOM VARIABLE

In this sample space each of these outcomes can be associated with a number by specifying a rule of association. Such a rule of association is called a random variable. In other words, A random variable is a rule that assigns a numerical value to each possible outcome of an experiment.

Definition: Let S be the sample space of an experiment. A random variable X is a real valued function defined on S. i.e., for each $s \in S$ there is a real number $X(s) = p$.

Example: Number of heads

We denote random variable by the letter (X, Y, etc.,) and any particular value of the random variable by x or y.

S = {HH, HT, TH, TT}

X(S) = {2, 1, 1, 0}

Thus, a random X can be the considered as a fun. That maps all elements in the sample space S into points on the real line. The notation X(S) = x means that x is the value associated with outcome s by the R.V.X.

Example 1: In the experiment of throwing a coin twice the sample space S is S = {HH, HT, TH, TT}. Let X be a random variable chosen such that X(S) = x (the number of heads).

Probability distribution function:

If X is a random variable, then the function F(x), defined by F(x) = P{X ≤ x} is called the distribution function of X.

Discrete Random Variable

Definition:

A random variable X is said to be ‘Discrete’ if it takes a finite number of values or countably infinite number of values. The below example is discrete.

Example: Suppose a coin is tossed twice. The sample space is S = {HH, TT, TH, HT}. Let X denote the ‘number of heads’ appeared. Then X is a random variable with values X(HH)=2, X(TH)=X(HT)=1, X(TT)=0. Therefore, the values of X are 0, 1, 2.

Probability Function

Probability Mass Function or Probability Function: Let X be a discrete random variable which takes the values $x_1, x_2, x_3, \ldots\ldots x_n$. Let $P[X = x_1] = p_1$ be the probability of x_1. Then the function p is called the probability mass function if $p(x_i) \geq 0$ for all *i* and $\sum_{i=1}^{n} p(x_i) = 1$.

Probability Distribution: The values assumed by the random variable X presented with corresponding probabilities is known as the probability distribution of X. The probability distribution (i.e., the values of X and its probability) is usually displayed in the form of a table.

X	x_1	$x_2 x_2$	$x_3 x_3$	.....	x_n
P(X=x)	$P(X = x_1)$ $p(x_1)$ p_1	$P(X = x_2)$ $p(x_2)$ p_2	$P(X = x_2)$ $p(x_3)$ p_3	.....	$P(X = x_n)$ $p(x_n)$ $p_n p_n$

Note: $P[X \leq x_3] = x_1 + x_2 + x_3$, $P[X < x_3] = x_1 + x_2$.

$P[X \geq x_3] = x_3 + x_4 + x_5 + \ldots\ldots + x_n = 1 - P[X < x_3]$, $P[X > x_3] = x_4 + x_5 + \ldots\ldots + x_n = 1 - P[X \leq x_3]$

Problems:

1. **Let X be the random variable which denotes the number of heads in three tosses of a fair coin. Determine the probability mass function of X.**

Solution: Sample space when tossing coins three times is

(H,H,H); (H,H,T); (H,T,H); (H,T,T); (T,H,H); (T,H,T); (T,T,H); (T,T,T)

Let x denotes the random, variable of getting heads

X	0	1	2	3
P(X)	1/8	3/8	3/8	1/8

2. **Let the random variable X denotes the sum obtained 'm' when rolling a pair of fair dice. Determine the probability mass function of X.**

Solution: Let the random variable X represent the sum of numbers on them when 2 dice are thrown

Possible (x,y)	Sum X= x+y	P(X=x)
(1,1)	2	1/36
(1,2), (2,1)	3	2/36
(1,3), (2,2), (3,1)	4	3/36
(1,4), (2,3), (3,2), (4,1)	5	4/36
(1,5), (2,4), (3,3), (4,2), (5,1)	6	5/36
(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)	7	6/36
(2,6), (3,5), (4,4), (5,3), (6,2)	8	5/36
(3,6), (4,5), (5,4), (6,3)	9	4/36
(4,6), (5,5), (6,4)	10	3/36
(5,6), (6,5)	11	2/36
(6,6)	12	1/36

3. If a random variable X takes the values 1,2,3,4, such that 2P(X=1)=3P(X=2)=P(X=3) =5P(X=4). Find the probability distribution of X.

Solution: Let P(X = 3) = k.

Then we get, P(X = 1) = k/2 , P(X = 2) = k/3 , P(X = 4) = k/5.

We Know that P(X = 1)+ P(X = 2)+ P(X = 3)+ P(X = 4)=1

$$\frac{k}{2}+\frac{k}{3}+k+\frac{k}{5}=1$$

$$\frac{15k+10k+30k+6k}{30}=1$$

$$\frac{61k}{30}=1.$$

Therefore, $k=\frac{30}{61}$

Required probability distribution is

$X=x$	1	2	3	4
$P(X=x)$	$\frac{15}{61}$	$\frac{10}{61}$	$\frac{30}{61}$	$\frac{6}{61}$

Cumulative distribution or distribution function of X:

The cumulative distribution function F(x) of a discrete random variable X with probability distribution P(x) is given by,

$F(x) = P(X \le x) = \sum_{t \le x} p(t)$, $x = -\infty, \ldots, -2, -1, 0, 1, 2, \ldots, \infty$.

Properties of distribution functions:

1. $F(-\infty) = 0$
2. $F(\infty) = 1$
3. $0 \le F(x) \le 1$
4. $F(x_1) \le F(x_2)$ if $x_1 \le x_2$
5. $P(x_1 < X \le x_2) = F(x_1) - F(x_2)$
6. $P(x_1 \le X \le x_2) = F(x_1) - F(x_2) + P[X = x_1]$
7. $P(x_1 < X < x_2) = F(x_1) - F(x_2) - P[X = x_2]$
8. $P(x_1 \le X < x_2) = F(x_1) - F(x_2) - P[X = x_2] + P[X = x_1]$

Results:

1. $P[X \leq \infty] = 1$
2. $P[X \leq -\infty] = 0$
3. If $x_1 \leq x_2$ then $P(X = x_1) \leq P(X = x_2)$
4. $P(X > x) = 1 - P[X \leq x]$
5. $P(X \leq x) = 1 - P[X > x]$

Expected value of a discrete random variable X:

Let X be a discrete random variable assuming values $x_1, x_2, x_3, \ldots\ldots x_n$ with corresponding probabilities $P_1, P_2, \ldots, P_n$. Then,

$$E[X] = \sum_i x_i p(x_i)$$

is called the Expected value of X.

The variance of a random variable X:

It is defined by $Var(X) = E[X - E(X)]^2$

The variance, which is equal to the expected value of the square of the difference between X and its expected value. It is a measure of the spread of the possible values of X.

A useful identity is that $Var(X) = E[X^2] - [E(X)]^2$. The quantity $\sqrt{Var(X)}$ is called the **standard deviation** of X.

Problems:

1. Evaluate the Mean of a random variable X if its probability distribution is as follows:

X	–2	–1	0	1	2
P(x)	a	a	2a	a	a

Solution: Mean $E(X) = \sum xp(x) = -2a - a + 0(2a) + a + 2a = 0$

2. Let X be a random variable such that $P(X=-2)=P(X=-1)=P(X=1)=P(X=2)$ and $P(X<0)=P(X=0)=P(X>0)$. Determine the probability mass function of X and distribution function of X.

Solution: Let $P(X=-2)=P(X=-1)=P(X=1)=P(X=2)=\mathrm{a}$

Then $\mathrm{P}(\mathrm{X}<0)=\mathrm{P}(\mathrm{X}=0)=\mathrm{P}(\mathrm{X}>0)=2\mathrm{a}$

The probability distribution is

X:	-2	-1	0	1	2
P (X = x):	a	a	2a	a	a

Therefore, the probability mass function of X and distribution function of X are given by

X:	-2	-1	0	1	2
P (X = x):	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$
F(x) = P (X$\leq x$):	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1

If the range of X is the set {0,1,2,3,4} and P {X = x} = 0.2, determine the mean and variance of the random variable.

Solution: We tabulate the values of X and its probabilities.

X	0	1	2	3	4
P(x)	0.2	0.2	0.2	0.2	0.2
xP(x)	0	0.2	0.4	0.6	0.8
x^2P(x)	0	0.2	0.8	1.8	3.2

We know that Mean = $\sum_0^4 xP(x)$ = 0+ 0.2 + 0.4 +0.6 +0.8 = 2

$E(x^2)$= $\sum_0^4 x^2P(x)$= 0 + 0.2 + 0.8 +1.8 + 3.2 = 6

Var (X) =$E(x^2) - [E(x)]^2 = 6 - (2)^2 = 6 - 4 = 2$

4. A random variable X has the following probability function:

X = x	0	1	2	3	4	5	6	7
P(x)	0	k	2k	2k	3k	k^2	$2k^2$	$7k^2$+k

(i) Find k

(ii) Evaluate P (X < 6), P (X ≥ 6), P (0 < X < 5)

(iii) Find the distribution function of X

(iv) Find the least value of ‘a’ such that P (X ≤ a) > 0.5

(v) Evaluate $P(1.5 < X < 4.5/X > 2)$.

Solution: (i) We know that $\sum P(x) = 1$

$$10k^2 + 9k = 1$$

$$10k^2 + 9k - 1 = 0$$

$$k = -1 \; or \; k = \frac{1}{10}$$

But P(x) cannot be negative. Hence $k = -1$ is neglected. Hence $k = \frac{1}{10}$

(ii) $P(X < 6) = P(0) + P(1) + \cdots \ldots . + P(5) = 8k + k^2 = \frac{8}{10} + \frac{1}{100} = \frac{81}{100}$

$$P(X \geq 6) = P(6) + P(7) = 9k^2 + k = \frac{9}{100} + \frac{1}{10} = \frac{19}{100}$$

or $\quad P(X \geq 6) = 1 - P(X < 6) = 1 - \frac{81}{100} = \frac{19}{100}$

$$P(0 < X < 5) = P(1) + \ldots\ldots + P(4) = 8k = \frac{8}{10}$$

(iii) $F(0) = P(X \leq 0) = P(0) = 0$

$$F(1) = P(X \leq 1) = P(0) + P(1) = k = \frac{1}{10}$$

$$F(2) = P(X \leq 2) = P(0) + P(1) + P(2) = 3k = \frac{3}{10}$$

$$F(3) = P(X \leq 3) = P(0) + \ldots\ldots + P(3) = 5k = \frac{5}{10}$$

$$F(4) = P(X \leq 4) = P(0) + \ldots\ldots + P(4) = 8k = \frac{8}{10}$$

$$F(5) = P(X \leq 5) = P(0) + \ldots\ldots + P(5) = 8k + k^2 = \frac{8}{10} + \frac{1}{100} = \frac{81}{100}$$

$$F(6) = P(X \leq 6) = P(0) + \ldots\ldots + P(6) = 8k + 3k^2 = \frac{8}{10} + \frac{3}{100} = \frac{83}{100}$$

$$F(7) = P(X \leq 7) = P(0) + \ldots\ldots + P(7) = 9k + 10k^2 = \frac{9}{10} + \frac{10}{100} = \frac{100}{100} = 1$$

(iv) $P(X \leq 3) = P(0) + \ldots\ldots + P(3) = 5k = \frac{5}{10} = \frac{1}{2}$

$P(X \leq 4) = P(0) + \ldots\ldots + P(4) = 8k = \frac{8}{10} > \frac{1}{2}$ and hence $a = 4$

(v) $P(1.5 < X < 4.5\ /\ X > 2) = \frac{P[(1.5<X<4.5)\cap(X>2)]}{P(X>2)}$

$$= \frac{P[2 < X < 4.5]}{1 - P[X \leq 2]} = \frac{P(3) + P(4)}{1 - [P(0) + P(1) + P(2)]}$$

$$= \frac{\frac{2}{10} + \frac{3}{10}}{1 - [0 + \frac{1}{10} + \frac{2}{10}]} = \frac{\frac{5}{10}}{1 - \frac{3}{10}} = \frac{5}{7}$$

5. A discrete random variable X has the following probability distribution.

X:	0	1	2	3	4	5	6	7	8
P (X = x):	a	3a	5a	7a	9a	11a	13a	15a	17a

(i) Find the value of “a”.

(ii) Find $P(X < 3), P(0 < X < 3), P(X \geq 3)$

(iii) Find the distribution function of X.

Solution: (i) We now know that,

$$\sum_i P(x_i) = 1 \Rightarrow \sum_{i=1}^{8} P(x_i) = 1$$

$$a + 3a + 5a + 7a + 9a + 11a + 13a + 15a + 17a = 1$$

$$81a = 1 \quad \Rightarrow a = \frac{1}{81}$$

(ii)
$$P(X < 3) = P(X = 0) + P(X = 1) + P(X = 2)$$

$$= a + 3a + 5a = 9a = 9 \times \frac{1}{81} = \frac{1}{9}$$

$$P(0 < X < 3) = P(X = 1) + P(X = 2) = 3a + 5a = a = 8 \times \frac{1}{81} = \frac{8}{81}$$

$$P(X \geq 3) = 1 - P(X < 3) = 1 - \frac{1}{9} = \frac{8}{9}$$

(iii) The distribution function F(x) of X is :

x_i	$p(x_i)$	$F(x_i)$
0	$p(0)=\frac{1}{81}$	$F(0)=p(0)=\frac{1}{81}$
1	$p(1)=\frac{3}{81}$	$F(1)=F(0)+p(1)=\frac{1}{81}+\frac{3}{81}=\frac{4}{81}$
2	$p(2)=\frac{5}{81}$	$F(2)=F(1)+p(2)=\frac{4}{81}+\frac{5}{81}=\frac{9}{81}$
3	$p(3)=\frac{7}{81}$	$F(3)=F(2)+p(3)=\frac{9}{81}+\frac{7}{81}=\frac{16}{81}$
4	$p(4)=\frac{9}{81}$	$F(4)=F(3)+p(4)=\frac{16}{81}+\frac{9}{81}=\frac{25}{81}$
5	$p(5)=\frac{11}{81}$	$F(5)=F(4)+p(5)=\frac{25}{81}+\frac{11}{81}=\frac{36}{81}$
6	$p(6)=\frac{13}{81}$	$F(6)=F(5)+p(6)=\frac{36}{81}+\frac{13}{81}=\frac{49}{81}$
7	$p(7)=\frac{15}{81}$	$F(7)=F(6)+p(7)=\frac{49}{81}+\frac{15}{81}=\frac{64}{81}$
8	$p(8)=\frac{17}{81}$	$F(8)=F(7)+p(8)=\frac{64}{81}+\frac{17}{81}=\frac{81}{81}=1$

1. The probability function of an infinite discrete distribution is given by $P[\mathrm{X}=\mathrm{j}]=\frac{1}{2^{\mathrm{j}}}$, $j=1,2,\ldots\ldots,\infty$. Find the mean and variance of the distribution. Also find $P[X\ is\ even]$, $P[X\geq 5]$ and $P[X\ is\ divisible\ by\ 3]$.

Solution: Given: $\mathbf{P}[\mathbf{X}=\mathbf{j}]=\frac{1}{2^{\mathrm{j}}}=\left(\frac{1}{2}\right)^{j}$

Mean = E[X] = $\sum_{j=1}^{\infty} x_j p(x_j) = (1)\left(\frac{1}{2}\right)+(2)\left(\frac{1}{2}\right)^2+(3)\left(\frac{1}{2}\right)^3+\cdots$

$$=\frac{1}{2}\left[1+2\left(\frac{1}{2}\right)+3\left(\frac{1}{2}\right)^2+\cdots\right]$$

$=\frac{1}{2}[1-x]^{-2}$ Here $x=\frac{1}{2}$

$$=\frac{1}{2}\left[1-\frac{1}{2}\right]^{-2}=\frac{1}{2}\left(\frac{1}{2}\right)^{-2}=\frac{1}{2}(4)=2$$

$$E(X^2)=\sum_{j=1}^{\infty} x_j^2 p(x_j)=\sum_{j=1}^{\infty}(x_j)(x_j+1)p(x_j)-\sum_{j=1}^{\infty} x_j p(x_j)$$

$= \left[(1)(2)\frac{1}{2} + (2)(3)\left(\frac{1}{2}\right)^2 + (3)(4)\left(\frac{1}{2}\right)^3 + \cdots\right]$-2

$$= \frac{1}{2}\left[1.2 + 2.3\left(\frac{1}{2}\right) + 3.4\left(\frac{1}{2}\right)^2 + \cdots\right] - 2$$

$$= \frac{1}{2}[2(1-x)^{-3}] - 2 \qquad \text{where } x = \frac{1}{2}$$

$$= \frac{1}{2}2\left(1 - \frac{1}{2}\right)^{-3} - 2 = \frac{1}{2}2\left(\frac{1}{2}\right)^{-3} - 2 = 8 - 2 = 6$$

[Formula: $(1-x)^{-2} = 1 + 2x + 3x^2 + \cdots$ and $(1-x)^{-3} = \frac{1}{2}[1.2 + 2.3x + 3.4x^2 + \cdots]$]

Variance of X = Var(X) = $E[X^2] - [E(X)]^2 = 6 - (2)^2 = 2$

(i) P[X is even] = $P[X = 2] + P[X = 4] + \cdots = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^6 + \cdots$

$$= \left(1 - \frac{1}{4}\right)^{-3} - 1 = \frac{4}{3} - 1 = \frac{1}{3}$$

(i) $P[X \geq 5] = P[X = 5] + P[X = 6] + \cdots = \left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^6 + \cdots$

$$= \left(\frac{1}{2}\right)^5 \left[1 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \cdots\right]$$

$$= \left(\frac{1}{2}\right)^5 \left[1 - \frac{1}{2}\right]^{-1} = \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^{-1} = \frac{1}{2^4} = \frac{1}{16}$$

(i) P [X is divisible by 3] = $P[X = 3] + P[X = 6] + \cdots$

$$= \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^6 + \left(\frac{1}{2}\right)^9 + \cdots$$

$$= \frac{1}{8} + \left(\frac{1}{8}\right)^2 + \left(\frac{1}{8}\right)^3 +.. = \left(1 - \frac{1}{8}\right)^{-1} - 1 = \left(\frac{7}{8}\right)^{-1} - 1 = \frac{1}{7}$$

Continuous Random Variable

Definition:

A random variable X is said to be continuous if it takes all possible values between certain limits say from real number ‘a’ to real number ‘b’.

Example: The length of time during which a vacuum tube installed in a circuit functions is a continuous random variable.

Note: If X is a continuous random variable for any x_1 and x_2, $P(x_1 < X \le x_2) = P(x_1 \le X \le x_2) = P(x_1 < X < x_2) = P(x_1 \le X < x_2)$

Probability Density Function or Probability Function:

Let X be a random variable which takes all values in an interval $(a \le X \le b)$. Then the function f(x) is called the probability density function of X if

1. $f(x) \geq 0 \; \forall \; x$

2. $\int_{-\infty}^{\infty} f(x)dx = 1$

3. P(a ≤ X ≤ b) = $\int_{a}^{b} f(x)dx$ = area under f(x) from a to b for any a and b

Cumulative Distribution Function:

The cumulative distribution function of a continuous random variable X is,

F(x) = P(X ≤ x) = $\int_{-\infty}^{x} f(x)dx$ for -∞ ≤x ≤ ∞.

The mean or expected value of a continuous random variable X:

Suppose X is a continuous random variable with probability density function f(x). The mean or expected value of X, denoted as μ or E(X) is,

$$\mu = E[X] = \int_{-\infty}^{\infty} xf(x)\,dx$$

The variance of a continuous random variable X:

The variance of X, denoted as V(X) or σ^2, is

$\sigma^2 = V(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx = \int_{-\infty}^{\infty} x^2 f(x)\, dx - \mu^2 = E[X^2] - [E(X)]^2$

Note: $V(aX+b) = a^2 V(X)$. The standard deviation of X is $\sigma = \sqrt{\mathrm{Var(X)}}$

Problems:

1. **A continuous random variable X has the pdf $f(x) = k(1+x), 2 \le x \le 5$. Find $P(X < 4)$**

Solution: Since f(x) is a pdf,

$\int_{-\infty}^{\infty} f(x)dx = 1$ $\int_2^5 k\,(1+x)dx = 1$ $k\left[\frac{(1+x)^2}{2}\right]_2^5 = 1$ $\frac{k}{2}[36-9] = 1$ $k = \frac{2}{27}$	$P(X<4) =$ $P(2<X<4) =$ $\int_2^4 f(x)dx =$ $\int_2^4 k\,(1+x)dx =$ $k\left[\frac{(1+x)^2}{2}\right]_2^4 =$ $\frac{k}{2}[25-9] = \frac{16}{27}$

2. If a random variable X has the pdf $f(x) = \begin{cases} 3x^2, 0 < x < 1 \\ 0, otherwise \end{cases}$, find 'k' such that P[X > k] = 0.05

Solution: Given that, P[X > k] = 0.05

P[k < X < 1] = 0.05

$$\int_k^1 f(x)dx = 0.05 \int_k^1 3x^2 dx = 0.05$$

$$3\left[\frac{x^3}{3}\right]_k^1 = 0.05$$

$$[1 - k^3] = \frac{5}{100}$$

$$k^3 = 1 - \frac{5}{100} = \frac{95}{100}$$

$$k = \sqrt[3]{\frac{95}{100}} = 0.983$$

3. If $f(x) = \begin{cases} Kxe^{-x}, & x > 0 \\ 0, & otherwise \end{cases}$ is the probability density function of a random variable X. Find K.

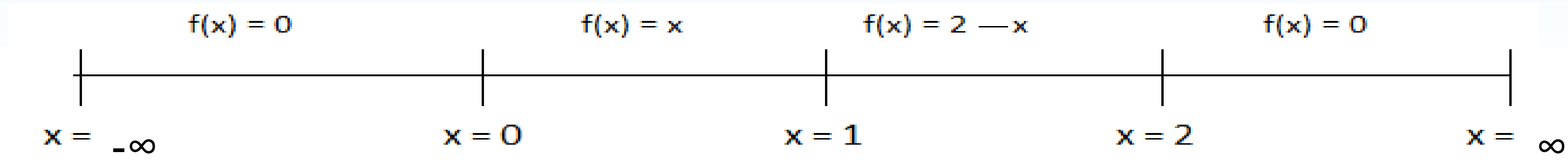
Solution: For a probability density function, $\int_{-\infty}^{\infty} f(x)dx = 1$

Here, $\int_0^{\infty} Kxe^{-x}dx = 1 \Rightarrow K\left[x\left(\frac{e^{-x}}{-1}\right) - (1)\left(\frac{e^{-x}}{(-1)^2}\right)\right]_0^{\infty} = 1$

$K[[-xe^{-x} - e^{-x}]_0^{\infty}] = 1 \implies K[(0-(0-1)] = 1 \Rightarrow \boldsymbol{K = 1}$

4. Find the CDF of the RV with pdf f(x) = x if 0 < x <1
= 2 – x if 1 < x < 2
= 0 if 2 < x < ∞

Also find the value of P (0.5 < x < 1.5), P (1 ≤ x ≤ 2), P (2 < x ≤ 2)



Solution: CDF in the interval $(-\infty, 0)$

$$F(x) = P(X < x) = P(-\infty < X < x) = \int_{-\infty}^{x} f(x)dx = \int_{-\infty}^{x} 0dx = 0$$

CDF in the interval (0, 1)

$$F(x) = P(X < x) = P(-\infty < X < x) = \int_{-\infty}^{x} f(x)dx = \int_{-\infty}^{0} 0dx + \int_{0}^{x} xdx = \left[\frac{x^2}{2}\right]_0^x = \frac{x^2}{2}$$

CDF in the interval (1, 2)

$$F(x) = P(-\infty < X < x) = \int_{-\infty}^{x} f(x)dx = \int_{-\infty}^{0} 0dx + \int_{0}^{1} xdx + \int_{1}^{x} (2-x)dx$$

$$= \left[\frac{x^2}{2}\right]_0^1 + \left[\frac{(2-x)^2}{-2}\right]_1^x = \frac{1}{2} + \frac{(2-x)^2}{-2} + \frac{1}{2} = 1 - \frac{(2-x)^2}{2}$$

CDF in the interval $(2, \infty)$

$$F(x) = P(-\infty < X < x) = \int_{-\infty}^{x} f(x)dx = \int_{-\infty}^{0} 0dx + \int_{0}^{1} xdx + \int_{1}^{2}(2-x)dx + \int_{2}^{\infty} 0dx = \left[\frac{x^2}{2}\right]_0^1 + \left[\frac{(2-x)^2}{-2}\right]_1^2 = \frac{1}{2} + \frac{1}{2} = 1$$

Therefore $F(x) = 0$, in $-\infty \leq x < 0$

$F(x) = \frac{x^2}{2}$, in $0 \leq x < 1$

$F(x) = 1 - \frac{(2-x)^2}{2}$, in $1 \leq x < 2$

$F(x) = 1$, in $2 \leq x < \infty$

$P(0.5 < x < 1.5) = F(1.5) - F(0.5) = 1 - \frac{(2-1.5)^2}{2} - \frac{(0.5)^2}{2}$

$P(1 \leq x \leq 2) = F(2) - F(1) = 1 - \left(1 - \frac{1}{2}\right)$ and $P(2 < x \leq 3) = F(3) - F(2) = 1 - 1 = 0$

5. The mileage X (in thousands of miles) which car owners get with a certain kind of tyre is a random variable having a p.d.f $f(x) = \frac{1}{20} e^{-\frac{x}{20}}, x > 0$. Find the probabilities that one of these tyres will last i) at most 10,000 miles (ii) at least 30,000 miles (iii) anywhere from 16,000 to 24,000 miles.

Solution: Given $f(x) = \frac{1}{20} e^{-\frac{x}{20}}, 0 < x\infty$

(i) Probability that the tyre will last at most 10,000 miles

$$P[X \leq 10] = P[0 \leq X \leq 10] = \int_0^{10} \frac{1}{20} e^{-\frac{1}{20}x} dx = \frac{1}{20}\left[\frac{e^{-\frac{1}{20}x}}{-\frac{1}{20}}\right]_0^{10} = -\left[e^{-\frac{1}{2}} - 1\right] = 1 - e^{-\frac{1}{2}}$$

(ii) Probability that the tyre will last at least 30,000 miles is $P[X \geq 30] =$

$$P[30 \leq X \leq \infty] = \int_{30}^{\infty} \frac{1}{20} e^{-\frac{1}{20}x} dx = \frac{1}{20}\left[\frac{e^{-\frac{1}{20}x}}{-\frac{1}{20}}\right]_{30}^{\infty} = -\left[0 - e^{-\frac{3}{2}}\right] = e^{-\frac{3}{2}}$$

(i) Probability that the tyre will last between 16,000 to 24,000 miles is

$$P[16 \leq X \leq 24] = \int_{16}^{24} \frac{1}{20} e^{-\frac{1}{20}x} dx = \frac{1}{20}\left[\frac{e^{-\frac{1}{20}x}}{-\frac{1}{20}}\right]_{16}^{24} = -\left[e^{-\frac{24}{20}} - e^{-\frac{16}{20}}\right] = e^{-\frac{16}{20}} - e^{-\frac{24}{20}}$$

6. Find the pdf of the random variable X if its CDF is F(x) = 1 – (1 + X) $e^{-x}, x \geq 0$.

Solution**:** The pdf is $f(x) = \frac{d}{dx} F(x) = \frac{d}{dx}[1 - e^{-x} - xe^{-x}] = e^{-x} - (-xe^{-x} + e^{-x}) = xe^{-x}$

7. Evaluate $E[X^2]$ if the probability density function of a random variable is $f(x) = xe^{-x}, x \geq 0$.

Solution: $E[X^2] = \int_0^\infty x^2 f(x) dx = \int_0^\infty x^3 e^{-x} dx = \Gamma 4 = 3! = 6$

8. Find the constant K if $f(x) = Kx^2, 0 < x < 1$ is the pdf of a continuous random variable X.

Solution: We know that,

$$\int_{-\infty}^{\infty} f(x) = 1 \int_0^1 K\, x^2 dx = 1$$

$$K\left[\frac{x^3}{3}\right]_0^1 = 1$$

$$K\left[\frac{1}{3}\right] = 1$$

$$\therefore K = 3$$

9. Write any two properties of CDF F(x).

Solution:

(i) If X is a distribution function of the random variable X and if a < b, then, $P(a < X \leq b) = F(b) - F(a)$.

(i) If F is the distribution function of a one-dimensional random variable X, then (i) $0 \leq F(x) \leq 1$, (ii) $F(x) \leq F(y)$, if x < y.

10. If $f(x) = \begin{cases} Ke^{-x}, & x > 0 \\ 0, & otherwise \end{cases}$ is the pdf of a random variable X, then find the value of K.

Solution: Since $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\int_0^{\infty} K\, e^{-x}\, dx = 1$$

$$K\left[\frac{e^{-x}}{-1}\right]_0^{\infty} = 1$$

$$K[0-(-1)] = 1$$

$$\therefore K = 1$$

11. Assume that X is a continuous random variable with the probability density function $f(x) = \begin{cases} \frac{3}{4}(2x - x^2), 0 < x < 2 \\ 0, \quad otherwise \end{cases}$**. Find** $P(X > 1)$

Solution: Given: $f(x) = \begin{cases} \frac{3}{4}(2x - x^2), 0 < x < 2 \\ 0, \quad otherwise \end{cases}$

$$P(X > 1) = \int_1^2 \frac{3}{4}(2x - x^2)dx$$

$$= \frac{3}{4}\left[2\frac{x^2}{2} - \frac{x^3}{3}\right]_1^2 = \frac{3}{4}\left[\left(4 - \frac{8}{3}\right) - \left(1 - \frac{1}{3}\right)\right] = \frac{3}{4} \times \frac{2}{3} = \frac{1}{2}$$

12. The pdf of a random variable X is given by $f(x) = \begin{cases} 2x, 0 \le x \le b \\ 0, \quad otherwise \end{cases}$ **for what value of b is** $f(x)$ **a valid pdf?. Also find the cdf of the random variable X with the above pdf.**

Solution: Since $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\int_0^b 2xdx = 1 \Rightarrow \left[\frac{2x^2}{2}\right]_0^b = 1$$

$$b^2 - 0 = 1, \qquad b = \pm 1$$

$$take b = 1 \quad since\ a < b$$

The cdf of X is $F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)dx$

$(i)\ \ If x < 0\,, \qquad F(x) = \displaystyle\int_{-\infty}^{x} f(x)dx = 0$

$(ii)\ If\ 0 < x < 1\,, \qquad F(x) = \displaystyle\int_{-\infty}^{0} f(x)dx + \int_0^x f(x)dx$

$$= 0 + \int_0^x 2xdx = \left[\frac{2x^2}{2}\right]_0^x = x^2$$

$(iii)\ If\ x > 1$

$$F(x) = \int_{-\infty}^{0} f(x)dx + \int_{0}^{1} f(x)dx + \int_{1}^{x} f(x)dx$$

$$= 0 + \left[\frac{2x^2}{2}\right]_0^1 + 0$$

$$= 1$$

The cdf of X is $F(x) = \begin{cases} 0, & x < 0 \\ x^2, & 0 < x < 1 \\ 1, & x > 1 \end{cases}$

13. Let X be a continuous random variable with probability density function

$$\boldsymbol{f(x) = \begin{cases} ax & : 0 \le x \le 1 \\ a : & 1 \le x \le 2 \\ -ax + 3a & : 2 \le x < 3 \\ 0 : & otherwise \end{cases}}$$

(i) Determine the constant "a"

(ii) The cumulative distribution function of X.

(iii)Compute $P(X > 1.5)$

Solution: (i) Since, f(x) is a probability density function, then

$$\int_{-\infty}^{\infty} \mathrm{f(x)dx} = 1 \Longrightarrow \int_0^3 f(x)dx = 1$$

$$\int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^3 (3a - ax)dx = 1 \quad \text{.....(A)}$$

$$\frac{a}{2} + a + \frac{a}{2} = 1 \;\; \Rightarrow 2a = 1 \;\; \Longrightarrow a = \frac{1}{2}$$

$$\int_0^x ax\,dx = \frac{1}{2}\left[\frac{x^2}{2}\right]_0^x = \frac{1}{2}\left[\frac{x^2}{2} - 0\right] = \frac{x^2}{4} \quad \text{...... (1)}$$

$$\int_1^x a\,dx = \frac{1}{2}[x]_1^x = \frac{1}{2}[x - 1] = \frac{x}{2} - \frac{1}{2} \quad \text{.......(2)}$$

$$\int_2^x (3a - ax)dx = \frac{1}{2}\left[3x - \frac{x^2}{2}\right]_2^x = \frac{1}{2}\left[\left(3x - \frac{x^2}{2}\right) - (6 - 2)\right] = \frac{3}{2}x - \frac{x^2}{4} - 2 \quad \text{.....(3)}$$

(ii) (a) If x < 0, then F[x] = 0

(b) If $0 \leq x \leq 1$, then $F[x] = \int_0^x ax\,dx = \frac{x^2}{4}$ { by (1)}1

(c) If $1 \leq x \leq 2$, then $F[x] = \int_0^1 ax\,dx + \int_1^x a\,dx = \frac{1}{4} + \frac{x}{2} - \frac{1}{2} = \frac{x}{2} - \frac{1}{4}$

(d) If $2 \leq x \leq 3$, then $F[x] = \int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^x (3a - ax)dx$

$$= \frac{1}{4} + \frac{1}{2} + \left[\frac{3}{2}x - \frac{x^2}{4} - 2\right] = -\frac{x^2}{4} + \frac{3}{2}x - \frac{5}{4}$$

(b) If $x > 3$, then $F[x] = \int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^3 (3a - ax)dx + \int_3^x f(x)dx$

$$= \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1$$

$$(iii) \quad P(X > 1.5) = \int_{1.5}^{3} f(x)dx = \int_{1.5}^{2} adx + \int_{2}^{3} (3a - ax)dx$$

$$= \frac{1}{2}[x]_{1.5}^{2} + \frac{1}{4} = \frac{1}{2}[2 - 1.5] + \frac{1}{4} = \frac{1}{2}$$