Sri SAI RAM ENGINEERING COLLEGE
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SAIRAM DIGITAL RESOURCES

YEAR II

SEM III

MA8351

DISCRETE MATHEMATICS

UNIT 2
COMBINATORICS

2.4 SOLVING LINEAR RECURRENCE RELATIONS

SCIENCE & HUMANITIES

4 QUALITY EDUCATION

RECURRENCE RELATIONS

Introduction

Permutations and Combinations are the basic techniques for solving counting problems. Some counting problems cannot be solved by these basic methods. Complex counting problems can be solved by advanced techniques such as recurrence relation and generating functions.

A recurrence relation of the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the earlier terms. A recurrence relation is also known as difference equation. Linear recurrence relation with constant coefficients can be solved by Iteration method, Characteristic roots method and Generating function method.

Characteristic method

Definition:

A recurrence relation of the form
$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots\ldots\ldots + c_k a_{n-k}$, where $c_1, c_2, \ldots\ldots, c_k$ are real numbers and $c_k \neq 0$ is called a linear homogeneous recurrence relation of degree k with constant co-efficients.

The recurrence relations in the above definition is linear because the right hand side is a sum of previous terms of the sequence each multiplied by a function of n. The recurrence relation is homogeneous because no terms occur that are not multiples of a_j 's . The co-efficients of the terms of the sequence are all constants, rather than function that depend on n.

The degree is k because a_n is expressed in terms of the previous k terms of the sequence.

Example:

The recurrence relation $f_n = f_{n-1} + f_{n-2}$ is a linear homogeneous recurrence relation of degree two.

The recurrence relation $a_n = a_{n-5}$ is a linear homogeneous recurrence relation of degree five.

The recurrence relation $a_n = a_{n-1} + a^2{}_{n-2}$ is not linear.

The recurrence relation $H_n = 2H_{n-1} + 1$ is not homogeneous.

The recurrence relation $B_n = nB_{n-1}$ does not have constant co-efficients.

Solving Linear Recurrence Relations

Let r_1 and r_2 be the roots of the characteristic equation of the recurrence relation of degree two.

Case(i)

If r_1 and r_2 are distinct roots. Then the solution $\{a_n\}$ of the recurrence relation is given by

$a_n = \alpha_1 {r_1}^n + \alpha_2 {r_2}^n$ for n=1,2,….. , where α_1 and α_2 are constants.

Case(ii)

If r_1 and r_2 are repeated roots. Then the solution $\{a_n\}$ of the recurrence relation is given by

$a_{n=} (\alpha_1 + n\alpha_2) r^n$, $r_1 = r_2$

Case(iii)

If r_1 and r_2 are complex conjugate. Then the solution $\{a_n\}$ of the recurrence relation is given by

$a_{n=}$ $(\alpha_1 cosn\theta+\alpha_2 sinn\theta)r^n$

Particular solutions

There is no general procedure for finding the particular solution of a recurrence solution. However for certain functions f(n) such as polynomials in n and powers of constants can be solved by the method of indetermined co-efficients. The following table gives certain forms of f(n) and the forms of the corresponding particular solution, on the assumption that f(n) is not a solution of the associated homogeneous relation.

Form of f(n)	
c	A
n	

Example:

The number of bacteria in a colony doubles every hour. If the colony begins with 5 bacteria, how many will be present in n^{th} hour?

Illustration:

We start with a_0=5,
a_1 =2(5) =2(a_0),
a_2 =2(10) =2(a_1),……..
The recurrence relation is a_n=2a_{n-1}.
Let a_n be the number of bacteria at the end of n hours. The explicit function i.e. without depending on the previous values.
a_0=5
a_1 =2(5)
a_2 =2(10)=2(2)(5)=2^2(5)……….a_n =2^n(5)

PROBLEMS

1) Find the solution of the recurrence relation $a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 2$ and $a_1 = 7$.

Solution:

The characteristic equation of the recurrence relation is

$$r^2 - r - 2 = 0$$
$$r^2 - 2r + r - 2 = 0$$
$$r(r-2) + 1(r-2) = 0$$
$$\boldsymbol{r} = 2, -1$$

The roots are $r_1 = 2$ & $r_2 = -1$. Hence the solution of the recurrence relation is

$$a_n = \alpha_1 r_1{}^n + \alpha_2 r_2{}^n$$

$$\boldsymbol{a_n} = \alpha_{\mathbf{1}} \mathbf{2}^{\boldsymbol{n}} + \alpha_{\mathbf{2}} (\mathbf{-1})^{\boldsymbol{n}} \ldots\ldots\ldots(1)$$

Given initial condition are $\boldsymbol{a_0 = 2}$ & $\boldsymbol{a_1 = 7}$

$$a_n = \alpha_1 2^n + \alpha_2 (-1)^n$$

$a_0=2$ $\quad a_0 = \alpha_1 2^0 + \alpha_2 (-1)^0 = 2$

$\alpha_1 + \alpha_2 = 2 \quad \ldots\ldots\ldots\ldots(2)$

$a_1=7$ $\quad a_1 = \alpha_1 2^1 + \alpha_2 (-1)^1 = 7$

$2\alpha_1 - \alpha_2 = 7 \quad \ldots\ldots\ldots\ldots(3)$

Solving (2) & (3) , we get

$\alpha_1 = 3, \alpha_2 = -1$

Therefore the solution of the recurrence relation is

$$\boldsymbol{a_n = 3.2^n - (-1)^n}$$

2) Find the explicit formula for the Fibonacci numbers.

Solution:

The recurrence relation of the Fibonacci numbers is given by $a_n = a_{n-1} + a_{n-2}$ satisfying the conditions $a_0 = 0$ & $a_1 = 1$.

The characteristic equation is given by

$$r^2 - r - 1 = 0$$

On factorizing, we get $r_1 = \frac{1+\sqrt{5}}{2}$, $r_2 = \frac{1-\sqrt{5}}{2}$

The solution of the recurrence relation is

$$a_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n \text{..........(1)}$$

Given $a_0 = 0$ & $a_1 = 1$.

$a_0 = 0 \quad a_0 = \alpha_1\left(\frac{1+\sqrt{5}}{2}\right)^0 + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right)^0$

$\alpha_1 + \alpha_2 = 0 \quad \ldots\ldots\ldots\ldots(2)$

$a_1 = 1 \quad a_1 = \alpha_1\left(\frac{1+\sqrt{5}}{2}\right)^1 + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right)^1$

$\alpha_1\left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right) = 1 \ldots\ldots\ldots(3)$

Solving (2) & (3), we get

$\alpha_1 = \frac{1}{\sqrt{5}}$ & $\alpha_2 = \frac{-1}{\sqrt{5}}$

$$a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

3) What is the solution of the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ with initial conditions $a_0 = 1$ and $a_1 = 6$?

Solution:

The characteristic equation of the recurrence relation is $r^2 - 6r + 9 = 0$. The roots are

$$r^2 - 6r + 9 = 0$$
$$r(r-3) - 3(r-3) = 0$$
$$r_1 = 3, \quad r_2 = 3$$

The roots are repeated. The solution if the recurrence relation is
$a_n = \alpha_1 3^n + \alpha_2 n 3^n$
Given $a_0 = 1$ & $a_1 = 6$

$a_0 = 1$

$a_0 = \alpha_1 + \alpha_2(0)=1$

$\alpha_1= 1$

$a_1 = 6 \quad 6 = \alpha_1 3+\alpha_2(1)3$

$6 = 3(1) + 3\alpha_2$

$\alpha_2=1$

$$\boldsymbol{a_n = 3^n + n3^n}$$

4) Find the solution to the recurrence relation $\mathbf{a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}}$ with the initial conditions $\mathbf{a_0=2}$, $\mathbf{a_1=5}$ and $\mathbf{a_2=15}$.

Solution:

The characteristic equation of the recurrence relation is $r^3 - 6r^2 + 11r - 6 = 0$.

$r_1=1$, $r_2=2$ & $r_3=3$

The solution of recurrence relation

$$a_n = \alpha_1(1)^n+\alpha_2(2)^n+\alpha_3(3)^n$$

Given $a_0 = 2, a_1 = 5 \ \& \ a_2 = 15$

$a_0 = 2$
$\alpha_1 + \alpha_2 + \alpha_3 \quad = 2 \dots\dots\dots\dots \quad (1)$
$a_1 = 5$
$\alpha_1 + 2\alpha_2 + 3\alpha_3 = 5 \dots\dots\dots\dots \quad (2)$
$a_2 = 15$
$\alpha_1 + 4\alpha_2 + 9\alpha_3 = 15 \dots\dots\dots. \quad (3)$

Solving (1) & (2),we get
$\alpha_2 + 2\alpha_3 = 3 \qquad \dots\dots\dots(4)$
Solving (2) & (3),we get
$\alpha_2 + 3\alpha_3 = 5 \qquad \dots\dots\dots(5)$
Solving (4) & (5),we get
$\alpha_3 = 2, \alpha_2 = -1, \alpha_1 = 1$

$\boldsymbol{a_n = (1)^n - 1.(2)^n + 2.(3)^n}$

5) A particle is moving in the horizontal direction. The distance it travels in each second is equal to two times the distance it travelled in the previous second. If a_r denotes the position of the particle in the r^{th} second, determine a_r, given that $a_0 = 3$ and $a_3=10$.

Solution:

Let a_r, a_{r+1}, a_{r+2} be the positions of the particle in the r^{th}, $(r+1)^{th}$, $(r+2)^{th}$ seconds. Then

$$a_{r+2} - a_{r+1} = 2(a_{r+1} - a_r)$$
$$a_{r+2} - 3a_{r+1} - 2a_r = 0$$

The characteristic equation is given by $m^2 - 3m + 2 = 0$

$$(m-1)(m-2) = 0$$
$$m = 1,2$$

$$a_r = c_1 1^r + c_2 2^r$$

To find the constants, we use the initial conditions

$$a_0 = 3 \qquad c_1 + c_2 = 3$$
$$a_3 = 10 \qquad c_1 + 8c_2 = 10$$

Solving the above equations, we get

$$c_1 = 2, c_2 = 1$$

The required solutions is

$\boldsymbol{a_r} = 2 + \mathbf{2^r}$

6) Solve the recurrence relation $\mathbf{a_n - 2a_{n-1} = 2^n, a_0 = 2}$.

Solution:

Given recurrence relation is $a_n - 2a_{n-1} = 2^n$

The homogeneous recurrence relation is $a_n - 2a_{n-1} = 0$

Since n-(n-1) = 1, it is first order equation.

The characteristic equation is $r-2=0$

$$r=2$$

The solution of homogeneous recurrence relation is $a_n^{(h)}=c.2^n$.

Given f(n) = 2^n , where 2 is a root of the characteristic equation.

a_n=An2^n is the particular solution.

$$An2^n-2.A(n-1)2^{(n-1)}=2^n$$

$$2^n[A\,(n-(n-1))]=2^n$$

$$[A(n-(n-1))]=1$$

$$A=1$$

Therefore $a_n^{(p)}=n.2^n$

Hence the general solution is $a_n=a_n^{(h)}+a_n^{(p)}$

$$a_n=c.\,2^n+n.\,2^n \ldots\ldots(1)$$

Substitute $a_0 = 2$ in (1)

$$a_0 = c.\ 2^0 + 0 = 2$$

$$2 = c$$

Therefore the general solution is

$$a_n = 2.\ 2^n + n.\ 2^n$$

$$\boldsymbol{a_n = (n+2)\ 2^n}$$

7) Solve the recurrence relation $a_n - 5a_{n-1} + 6a_{n-2} = 8n^2$, $a_0 = 4, a_1 = 7$.

Solution:

Given $a_n - 5a_{(n-1)} + 6a_{n-2} = 8n^2$

The homogeneous recurrence relation is

$a_n - 5a_{(n-1)} + 6a_{n-2} = 0$. Since n-(n-2) = 2, it is of order 2.

The characteristic equation is $r^2 - 5r + 6 = 0$

$$(r-2)(r-3) = 0$$

$$r = 2,3$$

Therefore the solution of the homogeneous recurrence solution is

$$a_n^{(h)} = A.2^n + B.3^n$$

Given f(n) = $8n^2$, which is a polynomial of degree 2 and 2 is the root of the characteristic equation.

Particular solution is $a_n = A_0 + A_1$n+A_2n^2, where A_0,A_1,A_2 are constants

$$A_0 + A_1\text{n}+A_2n^2 - 5(A_0 + A_1(\text{n-1})+A_2(n-1)^2+$$
$$6(A_0+A_1(\text{n-2})+A_2(n-2)^2) = 8n^2$$

$$n^2(A_2 - 5A_2 + 6A_2) + n(A_1 - 5A_1 + 10A_2 + 6A_1 - 24A_2)$$
$$+2A_0 - 7A_1 + 19A_2 = 8n^2$$

$$2A_2n^2 + (2A_1 - 14A_2)n + 2A_0 - 7A_1 + 19A_2 = 8n^2$$

Equating like co-efficients, we get

$$2A_2 = 8$$
$$A_2 = 4$$

$$2A_1 - 14A_2 = 0$$
$$A_1 = 7A_2$$
$$A_1 = 28$$
$$2A_0 - 7A_1 + 19A_2 = 0$$
$$A_0 = 60$$
$$a_n^{(p)} = 60 + 28n + 4n^2$$

Therefore the general solution is

$$a_n = a_n^{(h)} + a_n^{(p)}$$

$a_n = A.2^n + B.3^n + 60 + 28n + 4n^2 \ldots\ldots\ldots(1)$

To find A and B , we use $a_0 = 4, a_1 = 7$

$a_0 = 4$

$4 = A + B + 60$

$A + B = -56 \ldots\ldots\ldots\ldots (2)$

$a_1 = 7$

$2A + 3B = -85 \ldots\ldots\ldots\ldots (3)$

Solving (2) & (3), we get $A = -83, B = 27$

$$a_n = (-83).2^n + 27.3^n + 28n + 4n^2 + 60$$

8) Solve the recurrence relation $a_n = 4a_{n-1} - 4a_{n-2} + (n+1)2^n$.

Solution:

Given $a_n = 4a_{n-1} - 4a_{n-2} + (n+1).2^n$

$$a_n - 4a_{n-1} + 4a_{n-2} = (n+1).2^n$$

The homogeneous recurrence relation is

$$a_n - 4a_{n-1} + 4a_{n-2} = 0$$

Since n-(n-2)=2, it is of order 2.

Therefore the characteristic equation is $r^2 - 4r + 4 = 0$

$$(r-2)^2 = 0$$

$$r = 2,2$$

Therefore the solution of the homogeneous equation is

$$\boldsymbol{a_n^{(h)}} = (\boldsymbol{A} + \boldsymbol{Bn})2^{\boldsymbol{n}}.$$

$a_n = n^2(A_0 + A_1 n)\, 2^n$

Using this in Recurrence relation, we have

$$n^2(A_0+A_1 n).2^n - 4(n-1)^2\{A_0+A_1(n-1)\}2^{n-1}+4(n-2)^2\{A_0+A_1(n-2)\}2^{n-2} = (n+1)\,2^n$$

$$4n^2(A_0+A_1 n)-8(n-1)^2\{A_0+A_1(n-1)\}+4(n-2)^2\{A_0+A_1(n-2)\} = 4(n+1)$$

Equating co-efficients of n on both sides

$A_1 = \frac{1}{6}$

Equating constant terms on both sides

$A_0 = 1$

$$a_n^{(p)} = \left(n^2 + \frac{n^3}{6}\right) 2^n$$

Hence, the general solution of the recurrence relation is

$$a_n = a_n^{(h)} + a_n^{(p)}$$

$$a_n = \left((\boldsymbol{A} + \boldsymbol{Bn}) + n^2 + \frac{n^3}{6}\right) 2^n$$