Sri SAI RAM ENGINEERING COLLEGE
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SAIRAM DIGITAL RESOURCES

YEAR II SEM IV

MA8391

PROBABILITY AND STATISTICS

(DEPARTMENT OF INFORMATION TECHNOLOGY)

UNIT II

TW0-DIMENSIONAL RANDOM VARIABLES

2.1. JOINT, MARGINAL AND CONDITIONAL DISTRIBUTIONS

SCIENCE & HUMANITIES

JOINT PROBABILITY DISTRIBUTION:

If X and Y are two random variables, the probability distribution for their simultaneous occurrences can be represented by a function $f(x,y)$, for any pair of values (x,y) within the range of the random variables X and Y. This function is known as Joint Probability distribution of X and Y.

$$f(x,y) = P(X = x, Y = y)$$

JOINT PROBABILITY MASS FUNCTION OF (X, Y):

The function $P(x,y)$ is the joint probability mass function of the discrete random variable (X,Y) if

(i) $P(X = x_i, Y = y_j) \geq 0$

(ii) $\sum_i \sum_j P\left(X = x_i, Y = y_j\right) = 1$, where $i = 1,2, \dots \dots, n$ and $j = 1,2, \dots \dots, m$

JOINT PROBABILITY DENSITY FUNCTION:

If (X, Y) is a two-dimensional continuous random variable such that

$$P\left\{x - \frac{dx}{2} \le X \le x + \frac{dx}{2} \quad and \quad y - \frac{dy}{2} \le Y \le y + \frac{dy}{2}\right\} = f(x, y)dx\, dy,$$

Then $f(x, y)$ is called the joint probability density function of (X, Y), provided $f(x, y)$ satisfies the following conditions:

$(i)\ f(x, y) \ge 0,\ \forall\ (x, y) \in R$
$(ii) \iint f(x, y)dx\, dy = 1$

JOINT CUMULATIVE DISTRIBUTION FUNCTION:

For the random variable (X, Y) the cumulative distribution function is

$$F(x, y) = P[X \le x, Y \le y]$$

(i) Discrete case:

$$F(x, y) = \sum_{\substack{j \\ y_j \le y}} \sum_{\substack{i \\ x_i \le x}} p_{ij}$$

(i) Continuous case:

$$F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(x,y)dy\,dx$$

MARGINAL DISTRIBUTIONS:

Let (X,Y) be a two-dimensional random variable

Discrete Case:

The marginal distribution for X alone is given by

$$P[X = x_i] = \sum_{j} P[X = x_i, Y = y_j] = p_i$$

The marginal distribution for Y alone is given by

$$P[Y = y_j] = \sum_{i} P[X = x_i, Y = y_j] = p_j$$

Continuous Case:

The marginal distribution for X alone is given by

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

The marginal distribution for Y alone is given by

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

CONDITIONAL PROBABILITY DISTRIBUTION:

Discrete Case:

Let $P[X = x_i, Y = y_j]$ be the joint probability function of a two dimensional random variable (X,Y). Then the conditional probability function of X given $Y = y_j$ is

$$P[X = x_i / Y = y_j] = \frac{P[X = x_i \cap Y = y_j]}{P[Y = y_j]} = \frac{p_{ij}}{p_j}$$

Similarly, the conditional probability function of Y given $X = x_i$ is defined by

$$P[Y = y_j/X = x_i] = \frac{P[X = x_i \cap Y = y_j]}{P[X = x_i]} = \frac{p_{ij}}{p_i}$$

Continuous Case:

Let (X, Y) be the two-dimensional continuous random variables with joint probability density function $f(x, y)$. Then the conditional probability density function of X given Y is

$$f(x/y) = \frac{f(x, y)}{f_Y(y)}$$

Where $f_Y(y)$ is the marginal probability density function of Y. Similarly, the conditional probability density function of Y given X is

$$f(y/x) = \frac{f(x, y)}{f_X(x)}$$

Where $f_X(x)$ is the marginal probability density function of X.

INDEPENDENCE OF TWO RANDOM VARIABLES:

Discrete Case:

If (X,Y) is a two-dimensional discrete random variable such that $P\left[X = x_i/Y = y_j\right] = P[X = x_i]$

i.e., $p_{ij} = p_i \times p_j$ for all i, j the X and Y are said to be independent random variables.

Continuous Case:

If (X,Y) is a two- dimensional continuous random variable with joint probability density function $f(x,y)$ such that $f(x,y) = f_X(x).f_Y(y)$, then X and Y are said to be independent random variable.

PROBLEMS:

1. From the following table for bivariate distribution of (X,Y). Find

(i) $P(X \leq 1)$

(ii) $P(Y \leq 3)$

$(iii)\ P(X \leq 1, Y \leq 3)$
(iv) $P(X \leq 1/Y \leq 3)$
(v) $P(Y \leq 3/X \leq 1)$
$(vi) P(X + Y \leq 4)$
(vii) The marginal distribution of X and Y
(viii)The conditional distribution of X given Y = 2
(ix) Examine X and Y are independent.
(x) $E[Y - 2X]$

X \ Y	1	2	3	4	5	6
0	0	0	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{2}{32}$	$\frac{3}{32}$
1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
2	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{64}$	0	$\frac{2}{64}$

Solution:

X \ Y	1	2	3	4	5	6	$P_X(x) = f(x)$
0	0	0	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{2}{32}$	$\frac{3}{32}$	$P(X=0) = \frac{8}{32}$
1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$P(X=1) = \frac{20}{32}$
2	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{64}$	0	$\frac{2}{64}$	$P(X=2) = \frac{4}{32}$
$P_Y(y) = f(y)$	$P(Y=1) = \frac{3}{32}$	$P(Y=2) = \frac{3}{32}$	$P(Y=3) = \frac{11}{64}$	$P(Y=4) = \frac{13}{64}$	$P(Y=5) = \frac{6}{32}$	$P(Y=6) = \frac{16}{64}$	1

$$(i)\ P(X \le 1) = P(X = 0) + P(X = 1) = \frac{8}{32} + \frac{20}{32} = \frac{7}{8}$$

$$(i)\ P(Y \le 3) = P(Y = 1) + P(Y = 2) + P(Y = 3)$$
$$= \frac{3}{32} + \frac{3}{32} + \frac{11}{64} = \frac{23}{64}$$

(iii) $P(X \le 1, Y \le 3) = P(0,1) + P(0,2) + P(0,3) + P(1,1) + P(1,2) + P(1,3)$
$$= 0 + 0 + \frac{1}{32} + \frac{1}{16} + \frac{1}{16} + \frac{1}{8} = \frac{9}{32}$$

(iv) $P(X \le 1/Y \le 3) = \frac{P(X\le 1, Y\le 3)}{P(Y\le 3)} = \frac{\frac{9}{32}}{\frac{23}{64}} = \frac{18}{23}$

$$(v)\ P\left(Y \le \frac{3}{X} \le 1\right) = \frac{P(X \le 1, Y \le 3)}{P(X \le 1)} = \frac{\frac{9}{32}}{\frac{7}{8}} = \frac{9}{28}$$

(vi) $P(X + Y \le 4) = P(0,1) + P(0,2) + P(0,3) + P(0,4) + P(1,1) + P(1,2) + P(1,3) +$
$P(2,1) + P(2,2) = 0 + 0 + \frac{1}{32} + \frac{2}{32} + \frac{1}{16} + \frac{1}{16} + \frac{1}{8} + \frac{1}{32} + \frac{1}{32} = \frac{13}{32}$

(vii) The marginal distribution of X is :

$$P(X=0)=\frac{8}{32},\qquad P(X=1)=\frac{20}{32},\qquad P(X=2)=\frac{4}{32}$$

The marginal distribution of Y is :

$$P(Y=1)=\frac{3}{32},\qquad P(Y=2)=\frac{3}{32},\qquad P(Y=3)=\frac{11}{64}$$

$$P(Y=4)=\frac{13}{64},\qquad P(Y=5)=\frac{6}{32},\qquad P(Y=6)=\frac{16}{64}$$

(viii) The conditional distribution of X given Y = 2 is

$$P\left(X=\frac{x_i}{Y}=2\right), x_i \to 0,1,2$$

$$P[X=0/Y=2]=\frac{P[X=0,Y=2]}{P[Y=2]}=\frac{P[0,2]}{P[Y=2]}=\frac{0}{\left(\frac{3}{32}\right)}=0$$

$$P[X=1/Y=2]=\frac{P[X=1,Y=2]}{P[Y=2]}=\frac{P[1,2]}{P[Y=2]}=\frac{\frac{1}{16}}{\left(\frac{3}{32}\right)}=\frac{2}{3}$$

$$P[X=2/Y=2]=\frac{P[X=2,Y=2]}{P[Y=2]}=\frac{P[2,2]}{P[Y=2]}=\frac{\frac{1}{32}}{\left(\frac{3}{32}\right)}=\frac{1}{3}$$

(ix) Formula for X and Y are independent.

$$[P(X=i)]\times P[Y=j]=P[i,j],\qquad \forall\, i \text{ and } j$$

Here, X and Y are not independent.

Since, $P[X=0]\times P[Y=1]\neq P[0,1]$

i.e., $\left(\frac{8}{32}\right)\times\left(\frac{3}{32}\right)\neq 0$

(xi) $E[X]=\sum x_i p(x_i)=x_0p(x_0)+x_1p(x_1)+x_2p(x_2)$

$$=(0)\left(\frac{8}{32}\right)+(1)\left(\frac{20}{32}\right)+(2)\left(\frac{4}{32}\right)=\frac{20}{32}+\frac{8}{32}=\frac{28}{32}$$

(x) $E[Y]=\sum y_j p(y_j)=y_1p(y_1)+y_2p(y_2)+y_3p(y_3)+y_4p(y_4)+y_5p(y_5)+y_6p(y_6)$

$$=(1)\left(\frac{3}{32}\right)+(2)\left(\frac{3}{32}\right)+(3)\left(\frac{11}{64}\right)+(4)\left(\frac{13}{64}\right)+(5)\left(\frac{6}{32}\right)+(6)\left(\frac{16}{64}\right)$$

$$=\frac{3}{32}+\frac{6}{32}+\frac{33}{64}+\frac{52}{64}+\frac{30}{32}+\frac{96}{64}=\frac{259}{64}$$

$$E[Y-2X]=E[Y]-2E[X]=\frac{259}{64}-2\left(\frac{28}{32}\right)=\frac{147}{64}=2.297$$

2. The joint probability function (X, Y) is given by $P(x, y) = k(2x + 3y)$;

$$x = 0,1,2 \;;\quad y = 1,2,3$$

(i) Find the marginal distributions.

(ii) Find the probability distribution of $(X + Y)$.

(iii) Find all conditional probability distributions.

Solution:

Y / x	1	2	3	
0	3k	6k	9k	18k
1	5k	8k	11k	24k
2	7k	10k	13k	30k
	15k	24k	33k	72k

We know that, $\sum p_{ij} = 1 \implies 72k = 1$

$$\therefore \boldsymbol{k = \frac{1}{72}}$$

Hence the joint probability function is given by

Y / x	1	2	3	Total $p_X(x)$
0	$\frac{3}{72}$	$\frac{6}{72}$	$\frac{9}{72}$	$\frac{18}{72}$
1	$\frac{5}{72}$	$\frac{8}{72}$	$\frac{11}{72}$	$\frac{24}{72}$
2	$\frac{7}{72}$	$\frac{10}{72}$	$\frac{13}{72}$	$\frac{30}{72}$
Total $p_Y(y)$	$\frac{15}{72}$	$\frac{24}{72}$	$\frac{33}{72}$	1

(i) The marginal distribution of X and Y is:

$X = x$	0	1	2
$P(X = x)$	$\frac{18}{72}$	$\frac{24}{72}$	$\frac{30}{72}$

$Y = y$	0	1	2
$P(Y = y)$	$\frac{15}{72}$	$\frac{24}{72}$	$\frac{33}{72}$

(ii) To find the probability distribution of $X + Y$ is:

$X + Y$	p
1 (0,1)	$\frac{3}{72}$
2 (0,2) , (1,1)	$\frac{5}{72} + \frac{6}{72} = \frac{11}{72}$
3 (0,3),(1,2),(2,1)	$\frac{7}{72} + \frac{8}{72} + \frac{9}{72} = \frac{24}{72}$
4 (1,3),(2,2)	$\frac{11}{72} + \frac{10}{72} = \frac{21}{72}$
5 (2,3)	$\frac{13}{72}$

(iii) The conditional distribution of X given Y is

$$P(X = 0/Y = 1) = \frac{P(X = 0, Y = 1)}{P(Y = 1)} = \frac{\frac{3}{72}}{\frac{15}{72}} = \frac{1}{5}$$

$$P(X = 1/Y = 1) = \frac{P(X = 1, Y = 1)}{P(Y = 1)} = \frac{\frac{5}{72}}{\frac{15}{72}} = \frac{1}{3}$$

$$P(X = 2/Y = 1) = \frac{P(X = 2, Y = 1)}{P(Y = 1)} = \frac{\frac{7}{72}}{\frac{15}{72}} = \frac{7}{15}$$

$$P(X = 0/Y = 2) = \frac{P(X = 0, Y = 2)}{P(Y = 2)} = \frac{\frac{6}{72}}{\frac{24}{72}} = \frac{1}{4}$$

$$P(X = 1/Y = 2) = \frac{P(X = 1, Y = 2)}{P(Y = 2)} = \frac{\frac{8}{72}}{\frac{24}{72}} = \frac{1}{3}$$

$$P(X=2/Y=2)=\frac{P(X=2,Y=2)}{P(Y=2)}=\frac{\frac{10}{72}}{\frac{24}{72}}=\frac{5}{12}$$

$$P(X=0/Y=3)=\frac{P(X=0,Y=3)}{P(Y=3)}=\frac{\frac{9}{72}}{\frac{33}{72}}=\frac{9}{33}$$

$$P(X=1/Y=3)=\frac{P(X=1,Y=3)}{P(Y=3)}=\frac{\frac{11}{72}}{\frac{33}{72}}=\frac{1}{3}$$

$$P(X=2/Y=3)=\frac{P(X=2,Y=3)}{P(Y=3)}=\frac{\frac{13}{72}}{\frac{33}{72}}=\frac{13}{33}$$

The conditional distribution of Y given X is:

$$P(Y=1/X=0)=\frac{P(Y=1,X=0)}{P(X=0)}=\frac{\frac{3}{72}}{\frac{18}{72}}=\frac{1}{6}$$

$$P(Y=2/X=0)=\frac{P(Y=2,X=0)}{P(X=0)}=\frac{\frac{6}{72}}{\frac{18}{72}}=\frac{1}{3}$$

$$P(Y=3/X=0)=\frac{P(Y=3,X=0)}{P(X=0)}=\frac{\frac{9}{72}}{\frac{18}{72}}=\frac{1}{2}$$

$$P(Y=1/X=1)=\frac{P(Y=1,X=1)}{P(X=1)}=\frac{\frac{5}{72}}{\frac{24}{72}}=\frac{5}{24}$$

$$P(Y=2/X=1)=\frac{P(Y=2,X=1)}{P(X=1)}=\frac{\frac{8}{72}}{\frac{24}{72}}=\frac{1}{3}$$

$$P(Y=3/X=1)=\frac{P(Y=3,X=1)}{P(X=1)}=\frac{\frac{11}{72}}{\frac{24}{72}}=\frac{11}{24}$$

$$P(Y=1/X=2)=\frac{P(Y=1,X=2)}{P(X=2)}=\frac{\frac{7}{72}}{\frac{30}{72}}=\frac{7}{30}$$

$$P(Y=2/X=2)=\frac{P(Y=2,X=2)}{P(X=2)}=\frac{\frac{10}{72}}{\frac{30}{72}}=\frac{1}{3}$$

$$P(Y=3/X=2)=\frac{P(Y=3,X=2)}{P(X=2)}=\frac{\frac{13}{72}}{\frac{30}{72}}=\frac{13}{30}$$

3. The joint distribution of X and Y is given by $f(x,y)=\frac{x+y}{21}$, $x=1,2,3; y=1,2$. Find the marginal distribution. Also find $E[XY]$.

Solution: Given $f(x,y)=\frac{x+y}{21}$, $x=1,2,3; y=1,2$

Y / X	1	2	3	$P_Y(y)$
1	$\frac{2}{21}$	$\frac{3}{21}$	$\frac{4}{21}$	$P(Y=1)=\frac{9}{21}$
2	$\frac{3}{21}$	$\frac{4}{21}$	$\frac{5}{21}$	$P(Y=2)=\frac{12}{21}$
$P_X(x)$	$P(X=1)=\frac{5}{21}$	$P(X=2)=\frac{7}{21}$	$P(X=3)=\frac{9}{21}$	1

The marginal distribution of X:

$$P(X=1)=\frac{5}{21}\ ;\ P(X=2)=\frac{7}{21}\ ;\ P(X=3)=\frac{9}{21}$$

The marginal distribution of Y:

$$P(Y=1)=\frac{9}{21}\ ;\ P(Y=2)=\frac{12}{21}$$

$$E[XY] = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} x_i y_j P_X(x_i) P_Y(y_j)$$

$$= x_1y_1P(x_1)P(y_1) + x_1y_2P(x_1)P(y_2) + x_2y_1P(x_2)P(y_1) + x_2y_2P(x_2)P(y_2) + x_3y_1P(x_3)P(y_1) + x_3y_2P(x_3)P(y_2)$$

$$= (1)(1)\left(\frac{5}{21}\right)\left(\frac{9}{21}\right) + (1)(2)\left(\frac{5}{21}\right)\left(\frac{12}{21}\right) + (2)(1)\left(\frac{7}{21}\right)\left(\frac{9}{21}\right) + (2)(2)\left(\frac{7}{21}\right)\left(\frac{12}{21}\right) + (3)(1)\left(\frac{9}{21}\right)\left(\frac{9}{21}\right) + (3)(2)\left(\frac{9}{21}\right)\left(\frac{12}{21}\right)$$

$$= \frac{5}{49} + \frac{40}{147} + \frac{2}{7} + \frac{16}{21} + \frac{27}{49} + \frac{72}{49} = \frac{506}{147}$$