Sri SAI RAM ENGINEERING COLLEGE
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SAIRAM DIGITAL RESOURCES

YEAR II

SEM IV

MA8391

PROBABILITY AND STATISTICS

UNIT II

TWO DIMENSIONAL RANDOM VARIABLES

2.5 TRANSFORM ATION OF RANDOM VARIABLES

SCIENCE & HUMANITIES

Transformation of Random Variables

Let (X,Y) be a continuous random variable with joint probability density function $f(x,y)$. Let $U \text{ and } V$ be transformation such that $U = u(x,y), V = v(x,y)$. The joint probability density function of (U,V) is

$g(u,v) = f(x,y)|J|$, where J is the Jacobian of the transformation.

i.e. $J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$ and J is the Jacobian of the transformation.

Note $f(x,y)$ must be expressed in terms of $u \text{ and } v$.

Problems

1. The joint p.d.f. of a two dimensional random variable (X, Y) is given by

$f(x,y) = \begin{cases} 4xye^{-(x^2+y^2)}; & x \geq 0, y \geq 0 \\ 0; & otherwise \end{cases}$ Find the density function of $U = \sqrt{X^2+Y^2}$

Solution:

Given $u = \sqrt{x^2+y^2}$, take $v = y$. Then

$$u^2 = x^2 + y^2, v = y$$

$$x^2 = u^2 - y^2, v = y$$

$$x = \sqrt{u^2 - v^2}, v = y$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{u}{\sqrt{u^2-v^2}}, & \frac{-v}{\sqrt{u^2-v^2}}, \\ 0 & 1 \end{vmatrix}$$

$$= \frac{u}{\sqrt{u^2-v^2}}$$

Now the joint p.d.f of (U,V) is $g(u,v) = f(x,y)|J|$,

$$= 4xye^{-(x^2+y^2)}\frac{u}{\sqrt{u^2-v^2}}$$

$$= 4v\sqrt{u^2-v^2}e^{-u^2}\frac{u}{\sqrt{u^2-v^2}}$$

$$= 4uve^{-u^2}$$

Range space

$x \geq 0 \Rightarrow x^2 > 0 \Rightarrow u^2 - v^2 \geq 0$

$\Rightarrow u \geq v$

$y \geq 0 \Rightarrow v \geq 0$

$$g(u,v) = \begin{cases} 4uv\, e^{-u^2} & ,u \geq 0, 0 \leq v \leq u \\ 0, & otherwise \end{cases}$$

To find the density function of U

$$\underset{U}{g}(u) = \int_0^u f(u,v)dv$$

$$= \int_0^u 4uve^{-u^2}\,dv$$

$$= 4ue^{-u^2}\int_0^u vdv$$

$$= 4ue^{-u^2}\left[\frac{v^2}{2}\right]_0^u$$

$$\underset{U}{g}(u) = \begin{cases} 2u^3e^{-u^2}; u \geq 0 \\ 0; otherwise \end{cases}$$

Problem 2:

Let (X,Y) be a two dimensional random variable whose jont p.d.f is given by $f(x,y) = e^{-(x+y)}$; $x > 0$, $y > 0$. Find the p.d.f of $U = \frac{X+Y}{2}$.

Solution :

Given, $= \frac{X+Y}{2}$. Let $V = Y \implies u = \frac{x+y}{2}$, $v = y$

i.e $x = 2u - v, v = y$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} = 2$$

The joint p.d.f. of $U \text{ and } V$ is

$g(u,v) = f(x,y)|J|$

$= e^{-(x+y)}|2|$

$= 2e^{-(2u-v+v)}$

$= 2e^{-2u}$

Range Space :

Given $x > 0 , y > 0$

i.e.$2u - v > 0, v > 0$

$2u > v, v > 0$

i.e.$0 < v < 2u , u > 0$.

The density function of u is

$g_U(u) = \int_0^{2u} 2\, e^{-2u}\, dv$

$= 2e^{-2u}[v]_0^{2u}$

$g_U(u) = 4ue^{-2u} \; ; u > 0$

Problem 3.

Let $X \; and \; Y$ are normally distributed independent random variable with mean 0 and variance σ^2. Find the p.d.f.s of $R = \sqrt{x^2 + y^2}$ and $= tan^{-1}\left(\frac{y}{x}\right)$.

Solution :

Given $X \; and \; Y$ are normally distributed with mean 0 and variance σ^2

$\therefore$ Their p.d.f.s are

$$f_X(x) \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} \;\; ; -\infty < x < \infty$$

$$f_Y(y) \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{y^2}{2\sigma^2}} \;\; ; -\infty < x < \infty$$

The jontp.d.f is $f(x, y) = \frac{1}{2\pi\sigma^2} \; e^{-(\frac{x^2+y^2)}{2\sigma^2}} - \infty < x < \infty$

[$\because$ they are independent]

Given $R = \sqrt{x^2 + y^2}$ and $\theta = tan^{-1}\left(\frac{y}{x}\right)$

$$\Longrightarrow x = R\; cos\theta \; , y = R\; sin\theta$$

Range Space:

$-\infty < x < \infty, -\infty < y < \infty \implies 0 \le R < \infty, 0 \le \theta \le 2\pi$

$$J = \begin{vmatrix} \frac{\partial x}{\partial R} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial R} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} cos\theta & -Rsin\theta \\ sin\theta & Rcos\theta \end{vmatrix} = R$$

Now the joint p.d.f of R and θ is

$$g(R,\theta) = f(x,y)|J|$$

$$= \frac{1}{2\pi\sigma^2} e^{-(\frac{x^2+y^2)}{2\sigma^2}} |R|$$

$$= \frac{R}{2\pi\sigma^2} e^{-(\frac{R^2}{2\sigma^2})} ; R \ge 0, 0 \le \theta \le 2\pi$$

To find the marginal density function of R

$$g(R) = \int_0^{2\pi} \frac{R}{2\pi\sigma^2} e^{-(\frac{R^2}{2\sigma^2})} d\theta$$

$$= \frac{R}{2\pi\sigma^2} e^{-(\frac{R^2)}{2\sigma^2}} [\theta]_0^{2\pi}$$

$$= \frac{R}{\sigma^2} e^{-(\frac{R^2)}{2\sigma^2}}, R \ge 0$$

Which is the p.d.f of Rayleigh distribution.

The marginal p.d.f of θ

$$g(\theta) = \int_0^{\infty} \frac{R}{2\pi\sigma^2} e^{-(\frac{R^2}{2\sigma^2})} dR$$

$$put \frac{R^2}{2\sigma^2} = t, when\ R = 0, t = 0$$

$$when\ R = \infty, t = \infty$$

$$\frac{2R}{2\sigma^2} dR = dt \implies RdR = \sigma^2 dt$$

Substituting $g(\theta) = \frac{1}{2\pi} \int_0^{\infty} e^{-t}\, dt$

$$= \frac{1}{2\pi} \left[\frac{e^{-t}}{-1}\right]_0^{\infty}$$

$$= -\frac{1}{2\pi}[0 - 1] = \frac{1}{2\pi}, 0 \le \theta \le 2\pi$$

Which is the p.d.f of uniform distribution in $(0,2\pi)$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ -v & 1-u \end{vmatrix} = v(1-u) + uv = v$$

The joint p.d.f of (U, V) is given by

$g(u, v) = f(x, y)|J|$

$= e^{-(x+y)}|v|$

$= \; ve^{-(uv+v(1-u))}$

$= ve^{-v}$

Range Space

Given $x \geq 0 \; and \; y \geq 0$

$\Longrightarrow uv \geq 0 \; and \; v - uv \geq 0$

$\Longrightarrow uv \geq 0 \; and \; v > uv \Longrightarrow v \geq 0$

$v(1-u) \geq 0 \Longrightarrow 1 - u \geq 0 \Longrightarrow u \leq 1$

$uv \geq 0, v \geq 0 \; \Longrightarrow u \geq 0$

$\therefore 0 \leq \mathrm{u} \leq 1 \text{ and } \mathrm{v} \geq 0$

$g(u, v) = \; ve^{-v}, 0 \leq \mathrm{u} \leq 1 \text{ and } \mathrm{v} \geq 0$

The p.d.f. U is $\underset{U}{g}(u) = \int_0^{\infty} g(u,v)\,dv$

$$= \int_0^{\infty} ve^{-v} dv$$

$$= \left[\frac{ve^{-v}}{-1} - \frac{e^{-v}}{-1^2}\right]_0^{\infty}$$

$$= \left[ve^{-v} - e^{-v}\right]_0^{\infty}$$

$$= -(0-1) = 1$$

$$g(u) = 1, 0 \le \mathrm{u} \le 1$$

The p.d.f. of V is $\underset{V}{g}(v) = \int_0^{1} g(u,v)\,du$

$$= \int_0^{1} ve^{-v} du$$

$$ve^{-v}[u]_0^1$$

$$g(v) = ve^{-v}, \mathrm{v} \ge 0$$

$$g(u,v) = g(u).\,g(v).\, Hence\ U\ and\ V\ are\ independent$$

Problem 5

Solution: If the joint p.d.f of the R.v's X and Y are given by

$f(x,y) = \begin{cases} 2; & 0 < x < y < 1 \\ 0; & otherwise \end{cases}$ Find the p.d.f of the R.V

$U = \frac{X}{Y}$

Solution:

Given f$(x,y) = \begin{cases} 2; & 0 < x < y < 1 \\ 0; & otherwise \end{cases}$

$and\ U = \frac{X}{Y}\ assume\ V = Y$

$X = UY = UV$

$y = v, x = uv$

$$\therefore J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial x}{\partial v} & \frac{\partial x}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = 1$$

Hence the joint p.d.f of (U, V) is given by

$$g(u,v) = f(x,y)\,|J|$$
$$= f(uv, v)v = 2v$$

$\therefore$ The joint pdf of u and v is

$$g(u,v) = \begin{cases} 2v\,;\ 0 < u < 1, 0 < v < 1 \\ 0;\ otherwise \end{cases}$$

$\therefore$ *the pdf of* u *is*

$$g(u) = \int_0^1 g(u,v)dv$$

$$g(u) = \int_0^1 g(u,v)dv$$

$$= \int_0^1 2v\,dv$$

$$= 2\left[\frac{v^2}{2}\right]_0^1 = 1$$

The pdf is

$$g(u) = \begin{cases} 1, 0 < u < 1 \\ 0\ otherwise \end{cases}$$