Sri SAI RAM ENGINEERING COLLEGE
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Sairam INSTITUTIONS

SAIRAM DIGITAL RESOURCES

Unit II COMBINATORICS

2.6 THE PRINCIPLE OF EXCLUSION AND INCLUSION

YEAR II SEM III

MA8351

DISCRETE MATHEMATICS
(COMMON TO CSE & IT)

SCIENCE & HUMANITIES

THE PRINCIPLE OF EXCLUSION AND INCLUSION

INTRODUCTION

To find the number of elements in the union of the two sets A and B is the sum of the numbers of elements in the sets minus the number of elements in their intersection. That is

$|\boldsymbol{A} \cup \boldsymbol{B}| = |\boldsymbol{A}| + |\boldsymbol{B}| - |\boldsymbol{A} \cap \boldsymbol{B}|$ or

$$\boldsymbol{n}(\boldsymbol{A} \cup \boldsymbol{B}) = \boldsymbol{n}(\boldsymbol{A}) + \boldsymbol{n}(\boldsymbol{B}) - \boldsymbol{n}(\boldsymbol{A} \cap \boldsymbol{B})$$

PROBLEMS

1. Find the number of integers between 1 and 250 that are not divisible by any of the integer 2, 3, 5 and 7.

Solution:

Let A,B,C,D are the set of integers between 1 and 250 that are divisible by 2,3,5,7 respectively.

$$|A| = \left[\frac{250}{2}\right] = 125\ ; |B| = \left[\frac{250}{3}\right] = 83;$$

$$|C| = \left[\frac{250}{5}\right] = 50; |D| = \left[\frac{250}{7}\right] = 35$$

$$|A \cap B| = \left[\frac{250}{2.3}\right] = \left[\frac{250}{5}\right] = 41$$

$$|A \cap C| = \left[\frac{250}{2.5}\right] = \left[\frac{250}{10}\right] = 25$$

$$|A \cap D| = \left\lfloor \frac{250}{2.7} \right\rfloor = \left\lfloor \frac{250}{14} \right\rfloor = 17$$

$$|A \cap B \cap C| = \left\lfloor \frac{250}{2.3.5} \right\rfloor = \left\lfloor \frac{250}{30} \right\rfloor = 8$$

$$|A \cap B \cap D| = \left\lfloor \frac{250}{2.3.7} \right\rfloor = \left\lfloor \frac{250}{42} \right\rfloor = 5$$

$$|A \cap C \cap D| = \left\lfloor \frac{250}{2.5.7} \right\rfloor = \left\lfloor \frac{250}{70} \right\rfloor = 3$$

$$|B \cap C \cap D| = \left\lfloor \frac{250}{3.5.7} \right\rfloor = \left\lfloor \frac{250}{105} \right\rfloor = 2$$

$$|A \cap B \cap C \cap D| = \left\lfloor \frac{250}{2.3.5.7} \right\rfloor = \left\lfloor \frac{250}{210} \right\rfloor = 1$$

$|A \cup B \cup C \cup D|$

$$= |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D|$$

$$- |C \cap D| + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D|$$

$$+|B \cap C \cap D| - |A \cap B \cap C \cap D| = 193$$

∴ The number of integers between 1 and 250 that are divisible by any of 2,3,5 and 7=193.

Hence the number of integers that are not divisible by 2, 3, 5 and 7 = 250 – 193 = 57.

2. Find the number of integers between 1 and 250 both inclusive that are not divisible by any of the integer 2 or 3 or 5 or 7.

Solution:

From the previous problem,

Number of integers not divisible by any of the integers 2, 3, 5 and 7

= Total number of integers-Number of integers divisible by any of the integer 2, 3, 5 and 7 = 250-193 = 57.

3. 100 cars are assembled in a factory. The options available are stereo, air conditioner, automatic doors. It is known that 45 of them have stereos, 27 of them have A.C, 20 of them have automatic doors. Further 10 of them have all the three options.
(i) Find the maximum number of cars having at least one option.
(ii) The minimum number of cars having no options.

Solution:

Let A, B, C be the sets of cars having stereo, A.C, automatic doors respectively.
Given $|A| = 45, |B| = 27, |C| = 20,$

$|A \cap B \cap C| = 10.$

(i)Required $|A \cup B \cup C|$

We cannot use Venn-diagram to solve this, because enough hypothesis is not given. So we can only find an upper bound.

We know

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

But $|A \cap B|, |B \cap C|, |C \cap A|$ contains $|A \cap B \cap C|$

$$\therefore |A \cap B| \geq |A \cap B \cap C| = 10$$

$$\Longrightarrow |A \cap B| \geq 10$$

Similarly, $|B \cap C| \geq 10, |C \cap A| \geq 10$

$\therefore -|A \cap B| \leq -10, -|B \cap C| \leq -10, -|C \cap A| \leq -10$

$$\therefore |A \cup B \cup C| \leq 45 = 27 = 20 - 10 - 10 - 10 + 10 \leq 72$$

$\therefore$ atmost 72 cars will have at least one option.

(ii) We know

$$(A \cup B \cup C) \cup (A \cup B \cup C)' = U, \text{the universal set.}$$

$$\Rightarrow (A \cup B \cup C) \cup \left(A' \cap B' \cap C'\right) = U \quad \text{[by DeMorgan's law]}$$

$$\therefore |A \cup B \cup C| + \left|A' \cap B' \cap C'\right| = |U| = 100$$

$$\Rightarrow \left|A' \cap B' \cap C'\right| = 100 - |A \cup B \cup C|$$

$$\therefore |A \cup B \cup C| \leq 72, -\therefore |A \cup B \cup C| \geq -72$$

$$\therefore \left|A' \cap B' \cap C'\right| \geq 100 - 72 = 28$$

$\therefore$ at least 28 cars will not have any of the options.

4. In a survey of 100 students, it was found that 40 studied maths, 64 studied physics, 35 studied chemistry 1 studied all the three subjects, 25 studied maths and physics, 3 studied maths and chemistry and 20 studied physics and chemistry. Find he number of students who studied chemistry only.

Solution:

$|A| = |Maths| = 40, |B| = |Physics| = 64, |C| = |Chemistry| = 35$
$|A \cap B \cap C| = 1, |A \cap B| = 25, |A \cap C| = 3, |B \cap C| = 20$

The number of students who studied any of the subject

$$|A \cup B \cup C| = 40 + 64 + 35 - 25 - 3 - 20 + 1 = 92$$

The number of students who studied none of the subjects = 100-92=8

Number of students who studied Physics and Chemistry alone is

$$|B \cap C| - |A \cap B \cap C| = 20 - 1 = 19$$

Number of students who studied Maths and Chemistry alone is

$$|A \cap C| - |A \cap B \cap C| = 3 - 1 = 2$$

Number of students who studied Chemistry alone $= 35 - 19 - 2 = 13$

5. Find the number of positive integers less than or equal to 3000 that are divisible by 3, 5 or 7.

Solution.

Let A, B, C denotes the set of positive integers ≤3000 which are divisible by 3,5,7 respectively. Now

$$|A \cup B \cup C|$$

$$= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$= \left\lfloor \frac{3000}{3} \right\rfloor + \left\lfloor \frac{3000}{5} \right\rfloor + \left\lfloor \frac{3000}{7} \right\rfloor - \left\lfloor \frac{3000}{3.5} \right\rfloor - \left\lfloor \frac{3000}{5.7} \right\rfloor - \left\lfloor \frac{3000}{7.3} \right\rfloor + \left\lfloor \frac{3000}{3.5.7} \right\rfloor$$

$$= \left\lfloor \frac{3000}{3} \right\rfloor + \left\lfloor \frac{3000}{5} \right\rfloor + \left\lfloor \frac{3000}{7} \right\rfloor - \left\lfloor \frac{3000}{15} \right\rfloor - \left\lfloor \frac{3000}{35} \right\rfloor - \left\lfloor \frac{3000}{21} \right\rfloor + \left\lfloor \frac{3000}{105} \right\rfloor$$

$$= 1000 + 600 + \lfloor 428.5 \rfloor - 200 - \lfloor 85.7 \rfloor - \lfloor 142.8 \rfloor + \lfloor 28.5 \rfloor$$

$$= 1000 + 600 + 428 - 200 - 142 - 85 + 28$$

=1629

6. There are 2500 students in a school, of these, 1700 have taken a course in C, 1000 have taken a course in pascal, and 550 have taken course in networking. Further, 750 have taken courses in both C and Pascal, 400 have taken course in both C and networking. If 200 of these students have taken course in C, pascal, and networking.

(i) How many of these 2500 students have taken a course in any of these three courses C, pascal, and networking?

(ii) How many of these 2500 students have not taken a course in any of these three courses C, pascal, and networking?

Solution:

Let E be the set of students in the school.

Let A be the set of students who have taken a course in C, B be the set of students who have taken a course in pascal, C be the set of students who have taken a course in networking.

$A \cap B$ be the set of students who have taken courses in both C and pascal.

Similarly, $A \cap C, B \cap C$ and $A \cap B \cap C$

$|E| = 2500$

$|A| = 1700, \ |B| = 1000, \ |C| = 550$

$|A \cap B| = 750, \ \ |A \cap C| = 400, |B \cap C| = 275, \ |A \cap B \cap C| = 200$

∴ By inclusion-exclusion principle we get,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$= 1700 + 1000 + 550 - 750 - 400 - 275 + 200$$

$$= 2025$$

∴2025 of 2500 students have taken a course in any of these three courses C,pascal and networking.

(ii) The total number of students who have not taken any of course = Total number of Students – The number of students who have taken one of these courses.

$= |E| - |A \cup B \cup C|$

$= 2500 - 2025$

$= 475$

∴ 475 of 2500 students have not taken a course in any of these three courses C, pascal, and networking.

7. How many solutions does the equation $x_1 + x_2 + x_3 = 11$ have, where x_1, x_2, x_3 are non-negative such that $x_1 \leq 3, x_2 \leq 4$ and $x_3 \leq 6$? Use the principle inclusion-exclusion.

Solution:

Let the total no. of solutions with no restriction be *N.*

Let P_1, P_2, P_3 denote respectively the properties $x_1 > 3, x_2 > 4$ and $x_3 > 6$.

Then the required no. of solutions is given by

$$N - \{|P_1| + |P_2| + |P_3| - |P_1 \cap P_2| - |P_2 \cap P_3| - |P_3 \cap P_1| + |P_1 \cap P_2 \cap P_3|\}$$

Now $N = C(3 + 11 - 1, 11) = 78$

$|P_1|$ = no. of solutions subject to P_1 (viz. $x_1 \geq 4$ or $x_1 = 4,5,6, \dots, 11$)

$= C(3 + 7 - 1, 7) = C(9,7) = 36$ $(\because x_2 \leq 7$ and $x_3 \leq 7)$

Similarly, $|P_2| = C(3 + 6 - 1, 6) = C(8,6) = 28$

$|P_3| = C(3+4-1,4) = C(6,4) = 15$

$|P_1 \cap P_2|$ = no. of solutions subject to $x_1 \geq 4$ and $x_2 \geq 5$

$= C(3+2-1,2) = C(4,2) = 6 \quad [\because x_3 \leq 2]$

Similarly, $|P_2 \cap P_3| = 0$ $(\therefore x_1 \leq -1)$ and $|P_3 \cap P_1| = C(3+0-1,0) = 1$

$|P_1 \cap P_2 \cap P_3|$ = no. of solutions subject to $x_1 \geq 4, x_2 \geq 5$ and $x_3 \geq 7$

$= 0$

$\therefore$ Required no. of solutions

$= 78 - \{(36+28+15) - (6+0+1) + 0\}$

$= 6.$