Sri SAI RAM ENGINEERING COLLEGE
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Sairam INSTITUTIONS

SAIRAM DIGITAL RESOURCES

YEAR II

SEM III

MA8351

# DISCRETE MATHEMATICS

(Common to CSE & IT)

## UNIT V

## LATTICES AND BOOLEAN ALGEBRA

### 5.3 SUB-LATTICES

SCIENCE & HUMANITIES

## Sub- Lattice:

Let $(L,*,\oplus)$ be a lattice. A non-empty subset M of L is called a sub-lattice of L if and only if M is closed under te same operations $*$ and $\oplus$ of L.

i.e, $a*b\in M$ & $a\oplus b\in M$ .

## Problem 1 :

If $(L,*,\oplus)$ is a lattice and let $a,b\in L$ such that $a\leq b$, then the closed interval $[a,b]$ is defined as te set of all $x\in L$ such that $a\leq x\leq b$. Prove that $[a,b]$ is a lattice of L.

## Solution:

Clearly $[a,b]$ is a non-empty subset of L.

Let $x,y\in[a,b]$ then $x,y\in L\Rightarrow x*y,\ x\oplus y\in L,$ by the closure of $*$ and $\oplus$ in L as a Lattice.

Since $a \le x$ , $a \le y$ , a is a lover bound for $\{x, y\}$ and $x * y$ is the GLB $\{x, y\}$ .

$\therefore a \le x * y$

Since $x \le b;\ x * y \le b * y$ [By Isotonic Property]

Since $y \le b;\ y * b = y$ $[\because a \le b \text{ iff } a * b = a]$

$\Rightarrow b * y = y \le b$

$\therefore x * y \le b,$

Hence $a \le x * y \le b \Rightarrow x * y \in [a, b]$

Similarly $a \le x \oplus y \le b \Rightarrow x \oplus y \in [a, b]$

So, the subset $[a, b]$ is closed under $*$ and $\oplus$ .

Hence $[a, b]$ is a lattice of L.

## Problem 2 :

$N$ *or* $Z^+$ with partial order ≤ given by divisibility, i.e, $a \le b$ if $a \mid b$ is a lattice with $a * b = \gcd(a,b)$ and $a \oplus b = lcm(a,b)$. $S_n$ is the set of divisors of n is a sub lattice-verify.

## Solution:

We know $N = \{1,2,3,...\}, (N,*,\oplus)$ is a lattice. $S_n$ is the set of divisors of a fixed number n. Clearly $S_n \subset N$.

Let $a,b \in S_n$ then $\gcd(a,b)$ and $lcm(a,b)$ are also divisors of n and so $S_n$ is closed under the operations $*$ and $\oplus$.

Hence $S_n$ is a sub lattice.

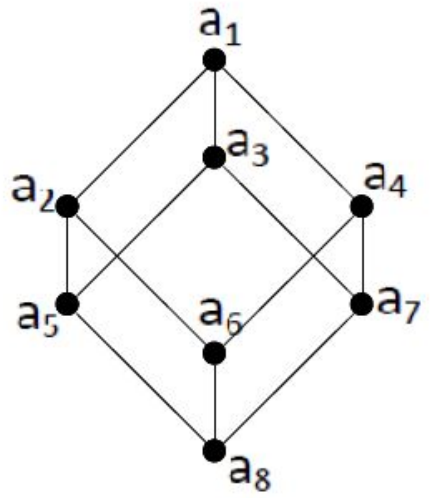
## Problem 3 :

Let $L=\{a_1,a_2,a_3,\ldots,a_8\}$ and let $(L,\leq)$ be a lattice. Let $S_1,S_2,S_3$ be subsets of L given by $S_1=\{a_1,a_2,a_3,a_6\}$, $S_2=\{a_3,a_5,a_7,a_8\}$ and $S_3=\{a_1,a_2,a_4,a_8\}$ which of these are sub lattices of L.

## Solution:

$L=\{a_1,a_2,a_3,\ldots,a_8\}$ and $(L,\leq)$ be a lattice.

The Hasse diagram of the lattice is shown here.



$S_1 = \{a_1, a_2, a_4, a_6\}$

$(S_1, \leq)$ is a sub lattice as the elements are consecutive vertices.

$S_2 = \{a_3, a_5, a_7, a_6\}$

$(S_2, \leq)$ is a sub lattice as the elements are consecutive vertices.

$S_3 = \{a_1, a_2, a_4, a_8\}$

$(S_3, \leq)$ is not a sub lattice as the elements are the vertices are not consecutive.

In other words $a_2, a_4 \in S_3$, but $a_2 * a_4 = a_6 \notin S_3$. So $S_3$ is not closed under $*$.