Sri SAI RAM ENGINEERING COLLEGE
INSTITUTE OF TECHNOLOGY
West Tambaram, Chennai - 44

Sairam INSTITUTIONS

SAIRAM DIGITAL RESOURCES

YEAR II

SEM III

MA8351

DISCRETE MATHEMATICS
(COMMON TO CSE & IT)

UNIT III

GRAPHS

3.4 CONNECTIVITY

SCIENCE & HUMANITIES

4 QUALITY EDUCATION

CONNECTIVITY

Definition:

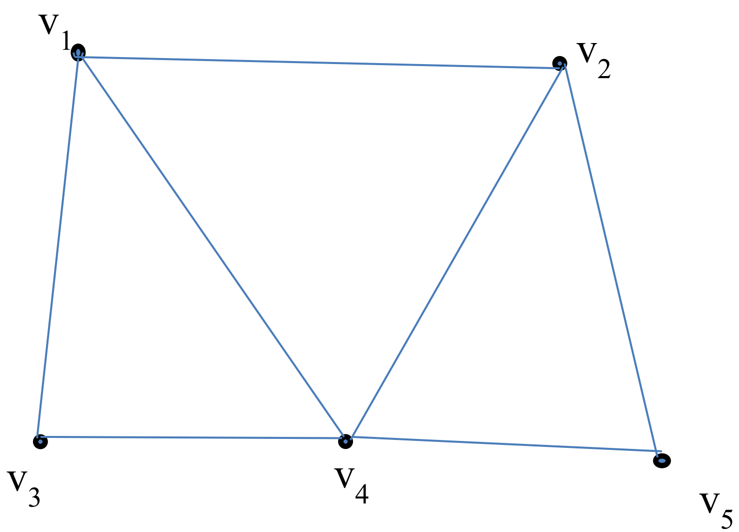
A graph is said to be connected if there is a path between every pair of distinct vertices of the graph.

A graph that is not connected is called disconnected.

ie) a graph G is connected if given any vertices u and v , it is possible to travel from u to v along a sequence of adjacent edges of the graph.

Example: 1

Connected graph



A disconnected graph is the union of two or more connected components .

A disconnected graph is the union of two or more connected sub graphs, each pair of which has no vertex in common .These disjoint connected sub graphs are called the connected components of the graph.

Example:

Theorem :1

If a graph G(connected or disconnected) has exactly two vertices of odd degree then there must be a path joining these two vertices.

Proof:

Case (i) : Let G be connected.

Let v_1 and v_2 be the only vertices of G which are of odd degree .But we have already proved that the number of odd degree vertices in a graph is always even. That is v_1 and v_2 belongs to the same component. Since G is connected clearly there is a path connecting v_1 and v_2 .

Case (i) : Let G be disconnected.

Then the components of G are connected. Hence v_1 and v_2 should belongs to the same component of G. Hence there is a path between v_1 and v_2 .

Theorem :2

The maximum number of edges in a simple disconnected graph G with n vertices and k components is $\dfrac{(n-k)(n-k-1)}{2}$

Proof:

Let the number of vertices in the i^{th} component of G be n_i $(n_i \geq 1)$

$$Then \quad n_1 + n_2 + n_3 + \ldots\ldots + n_k = n$$

$$ie) \sum_{i=1}^{k} n_i = n$$

$$\sum_{i=1}^{k} (n_i - 1) = n - k$$

$$\left\{ \sum_{i=1}^{k} (n_i - 1) \right\}^2 = n^2 - 2nk + k^2$$

$$ie) \quad \sum_{i=1}^{k} (n_i - 1)^2 + 2\sum_{i \neq j} (n_i - 1)(n_j - 1) = n^2 - 2nk + k^2$$

$$ie) \quad \sum_{i=1}^{k} (n_i - 1)^2 \leq n^2 - 2nk + k^2$$

$$ie) \sum_{i=1}^{k} (n_i^{\,2} - 2n_i + 1) \leq n^2 - 2nk + k^2$$

$$ie) \sum_{i=1}^{k} n_i^{\,2} \leq n^2 - 2nk + k^2 + 2n + k$$

Now the maximum number of edges in the i[th] component of $G = \frac{1}{2} n_i (ni - 1)$

Hence the maximum number of edges of G

$$= \frac{1}{2} \sum_{i=1}^{k} n_i (n_i - 1)$$

$$= \frac{1}{2} \sum_{i=1}^{k} n_i^{2} - \frac{1}{2} \sum_{i=1}^{k} n_i$$

$$\leq \frac{1}{2} \left(n^2 - 2nk + k^2 + 2n - k \right) - \frac{1}{2} n$$

$$= \frac{1}{2} [(n-k)^2 + (n-k)]$$

$$= \frac{1}{2} [(n-k) \ + (n-k+1)]$$

ie) maximum number of edges in a simple disconnected graph G with n vertices and k components is $\frac{(n-k)(n-k-1)}{2}$

Theorem :3

A graph G is said to be connected iff its vertex set V can be partitioned in two subsets V_1 and V_2 such that there exists and edge in G whose one vertex is in subset of V_1 and the other is in subset of V_2

Proof : Suppose G is connected . Let V be partitioned in to two subsets V_1 and V_2
Let $u \; \varepsilon \; V_1$ and $v \; \varepsilon \; V_2$. Since G is connected, there exists a uv path in G, say

$$u = v_0, v_1, v_2, \ldots\ldots\ldots v_n = v$$

Let i be the least positive integer such that $v_i \; \varepsilon \; V_2$ such an i exists since $v_n = v \; \varepsilon \; V_2$

Then $v_{i-1} \; \varepsilon \; V_1$ and $v_{i-1} \; v_i$ are adjacent. Thus there is an edge between $v_{i-1} \; \varepsilon \; V_1$ and $v_i \; \varepsilon \; V_2$

Conversely, assume that V can be partitioned in to two subsets V_1 and V_2 such that there exists an edge in G whose one vertex in V_1 and the other in V_2 .

To prove G is connected.
Suppose G is not connected. Then G contains at least two components.
Let V_1 denote the set of all vertices of one component and V_2 denote the set of all remaining vertices of G.Clearly $V = V_1 \cup V_2$ is a partition of V and there is no line joining any vertices of V_1 to any other vertices of V_2, which is a contradiction to our assumption.
Hence G is connected.

Theorem:4

If G is not connected then $\overline{G}$ is connected

Proof:

Let G be a disconnected graph. Then G has more than one component. Let u,v be any two points in G.It is enough to prove that there is a path u-v in $\overline{G}$ If u,v belongs to different components in G,then they are not adjacent in G and hence they are adjacent in $\overline{G}$.If u,v lie in the same of G,choose w in a different component.

Then u,v,w represent a u-v path in $\overline{G}$. Hence $\overline{G}$is connected.

CONNECTEDNESS IN DIRECTED GRAPHS

DEFINITIONS:

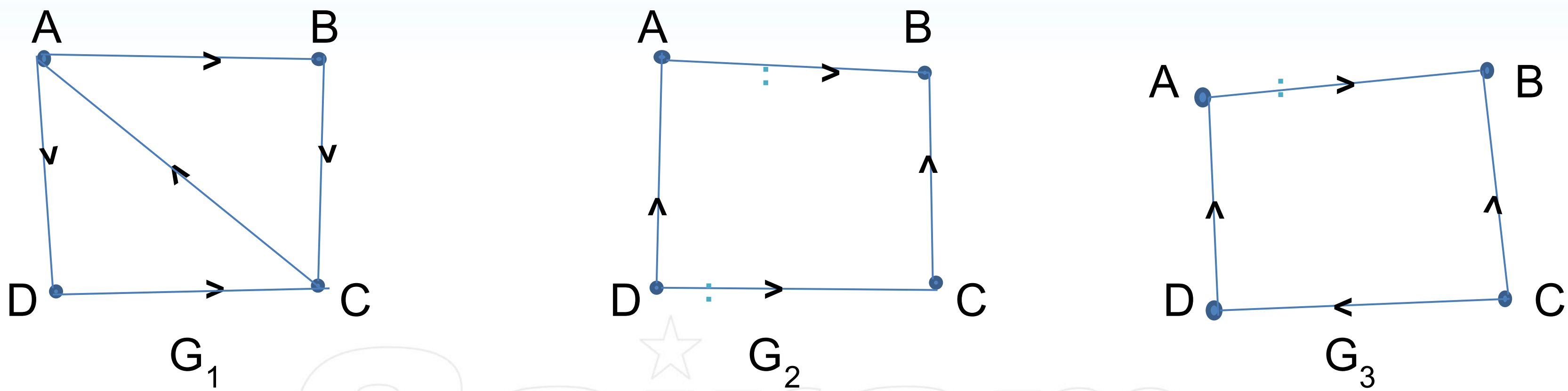
A directed graph is said to be strongly connected , if there is a path from V_i to V_j and from V_j to V_i where V_i and V_j are any pair of vertices of the graph.

For a directed graph to be strongly connected, there must be a sequence of directed edges from any vertex in the graph to any other vertex.

A directed graph is said to be weakly connected , if there is a path between every two vertices in the underlying undirected graph. In other words, a directed graph is weakly connected if and only if there is always a path between every two vertices when the direction of the edged are disregarded. Clearly any strongly connected directed graph is also a weakly connected.

A simple directed graphs is said to be unilaterally connected, if for any pair of vertices of the graph, at least one of the vertices of the pair is reachable from the other vertex. Note that unilaterally connected digraph is weakly conneced,but a weakly connected digraph is not necessarily connected. A strongly connected digraph is both unilaterally and weakly connected.

Example:



G_1 is a strongly connected graph, as the possible pairs of vertices in G_1 are (A,B),(A,C),(A,D),(B,C),(B,D) AND (C,D) and there is a path from the first vertex to the second and from second vertex to the first in all the pairs.

For example, let us take the pair (A,B). Clearly the path from A to B is A – B and the path from B to A is B – C – A .

Similarly if we take the pair (B,D),the path from B to D is B – C – A – D and the path from D to B is D – C – A – B .

Clearly G_2 is only a weakly connected graph.

G_3 is unilaterally connected, since there is a path from A to B , but there is no path from B to A. Similarly ,there is a path from D to B , but there is a path from B to D .