Sri SAI RAM ENGINEERING COLLEGE
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SAIRAM DIGITAL RESOURCES

YEAR II | SEM III

MA8351

DISCRETE MATHEMATICS
(Common to CSE & IT)

UNIT V
LATTICES AND BOOLEAN ALGEBRA

5.2 PROPERTIES OF LATTICES

SCIENCE & HUMANITIES

Properties of Lattices:

Theorem 1: Let $(L, \leq)$ be a lattice with binary operations meet and join, denoted by $*$ and $\oplus$. Then for any $a, b, c \in L$ the following are true.

1. **Idempotent laws**

   $a * a = a$ and $a \oplus a = a$

2. **Commutative laws**

   $a * b = b * a$ and $a \oplus b = b \oplus a$

3. **Associative laws:**

   $a * (b * c) = (a * b) * c$ and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$

4. **Absorption laws:**

   $a * (a \oplus b) = a$ and $a \oplus (a * b) = a$

Proof:

Given $(L, \leq)$ is a lattice and so for any $a, b \in L$, $a * b$ and $a \oplus b$ exist uniquely. $a * b = GLB\{a, b\}$, $a \oplus b = LUB\{a, b\}$

1. Let $a \in L$

$$a * a = GLB\{a, a\} = GLB\{a\} = a$$

$$a \oplus a = LUB\{a, a\} = LUB\{a\} = a$$

[Since $a \leq a, a$ is an upper bound and lower bound for a]

2. Since $\{a, b\} = \{b, a\}$ as sets,

$$a * b = GLB\ \{a\ , b\} = GLB\ \{b\ , a\} = b * a$$

and $a \oplus b = LUB\ \{a, b\} = LUB\ \{b\ , a\} = b \oplus a.$

3. We shall prove that $a * (b * c) = (a * b) * c$

For simplicity, let $x = a * (b * c)$ and $y = (a * b) * c$.

$$x = a * (b * c) = GLB\{a\ , b * c\} \leq a\ , b * c.$$

$\therefore$ $x \leq a$ and $x \leq b * c$.

But $b * c = GLB\{b, c\} \leq b, c$.

So, by transitivity $x \leq b,\ x \leq c$.

Thus $\quad x \leq a,\ \ x \leq b,\ \ x \leq c$

$\Rightarrow x \leq a * b,\ \ x \leq c$

$\Rightarrow x \leq (a * b) * c \Rightarrow \ x \leq y$

Now $y = (a * b) * c \Rightarrow y \leq a * b,\ y \leq c$ (Proceeding as above)

$\Rightarrow y \leq a,\ \ y \leq b,\ \ y \leq c$

$\Rightarrow y \leq a,\ \ y \leq b * c$

$\Rightarrow y \leq a * (b * c) \Rightarrow \ y \leq x$

Thus $x \leq y$ and $y \leq x$.

As $\leq$ is a partial order it is anti-symmetric.

$\therefore\ x \leq y$ and $y \leq x \Rightarrow x = y$

$\Rightarrow a * (b * c) = (a * b) * c$

Now we shall prove $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

Again, for simplicity, let $x = a \oplus (b \oplus c)$ and $y = (a \oplus b) \oplus c$.

$x = a \oplus (b \oplus c) = LUB\{a, b \oplus c\} \Rightarrow a \le x, b \oplus c \le x$.

As, $b \le b \oplus c$ and $c \le b \oplus c$.

So, by transitivity $b \le x$ and $c \le x$

Thus $\quad a \le x, \quad b \le x, \quad c \le x$

$$\Rightarrow \quad a \oplus b \le x, \; c \le x$$

$$\Rightarrow \quad (a \oplus b) \oplus c \le x \quad \Rightarrow y \le x$$

Now $y = (a \oplus b) \oplus c$

Proceeding as above we get $a \oplus b \le y, \; c \le y$

$$\Rightarrow a \le y, \quad b \le y, \quad c \le y$$

$$\Rightarrow a \le y, \quad b \oplus c \le y$$

$$\Longrightarrow a \oplus (b \oplus c) \leq y \qquad \Longrightarrow x \leq y$$

Thus, we have $y \leq x$ and $x \leq y$.

Since $\leq$ is antisymmetric, we get

$x = y \Longrightarrow a \oplus (b \oplus c)$ and $y = (a \oplus b) \oplus c$.

4. We shall prove $a * \{a \oplus b\} = a$

For any $a \in (L,)$ we have $a \leq a$

Since $a \oplus b = LUB\ \{a, b\}$, as an upper bound of a, b

We have $a \leq a \oplus b$

$\therefore\ a \leq a,\ a \leq a \oplus b \Rightarrow a \leq GLB\ \{a, a \oplus b\}$

$$\Longrightarrow a \leq a * (a \oplus b)$$

Now $a * (a \oplus b) = GLB\ \{a, a \oplus b\}$.

As a lower bound, we have $a * (a \oplus b) \leq a$

Thus $a \leq a * (a \oplus b)$ and $a * (a \oplus b) \leq a$.

By antisymmetric of $\leq$, we get $a \oplus (a * b) = a$

Applying duality principle, we get $a \oplus (a * b) = a$

Theorem 2:

If $(L, \leq)$ is a lattice in which meet and join are denoted by $*$ and $\oplus$, then prove that for any $a, b \in L$, $a \leq b \Longrightarrow a * b = a \Longrightarrow a \oplus b = b$

Proof: First we shall prove that $a \leq b \Longrightarrow a * b = a$

Assume that $a \leq b$. We know that $a \leq a$.

$\therefore\ a \leq GLB\ \{a, b\} \Longrightarrow a \leq a * b$

Now $a * b = GLB\ \{a, b\}$

As a lower bound of a and b.

We have $a * b \leq a$

Thus $a \leq a * b$ and $a * b \leq a$.

So, by antisymmetric of $\leq$, we have $a * b = a$.

Hence $a \leq b \Rightarrow a * b = a$ ... (1)

To prove the reverse implications, assume $a * b = a$

$\Longrightarrow a = GLB\,\{a\,,b\} \leq b$ [as a lower bound of b]

$\therefore\ a * b = a \Longrightarrow a \leq b$ ... (2)

From (1) and (2) we get $a \leq b \Leftrightarrow a * b = a$

We shall now prove that $a \leq b \Leftrightarrow a \oplus b = b$

Assume that $a \leq b$. We know that $b \leq b$.

$\therefore\ LUB\,\{a\,,b\} \leq b \Longrightarrow a \oplus b \leq b$

Since $a \oplus b$ is an upper bound of a and b, we have

$$b \leq a \oplus b$$

Thus $a \oplus b \leq b$ and $b \leq a \oplus b$.

So, by antisymmetric of $\leq$ we have $a \oplus b = b$.

Thus $a \leq b \implies a \oplus b = b$ … (3)

Now assume $a \oplus b = b$.

Since $a \oplus b$ is an upper bound of a and b,

$$a \leq a \oplus b \quad \implies a \leq b$$

Hence $a \oplus b = b \implies a \leq b$ … (4)

From (3) and (4) we get $a \leq b \Leftrightarrow a \oplus b = b$

Note:

1. This theorem gives the following results.

   (i) $a * b = a$ if and only if $a \leq b$.

   (ii) $a \oplus b = b$ if and only if $a \leq b$.

   (iii) $a * b = a$ if and only if $a \oplus b = b$.

2. This theorem gives a connection between the partial order $\leq$ and the binary operations $*$ and $\oplus$ in lattice, which will enable us to define lattice as an algebraic system.

Theorem 3: (Isotonic property)

Let $(L, \leq)$ be a lattice. For any $a, b, c \in L$ the following properties called isotonicity hold.

If $b \leq c$ then (i) $a * b \leq a * c$ (ii) $a \oplus b \leq a \oplus c$

Proof: Given $(L, \leq)$ is a lattice with the operations meet $*$ and join $\oplus$ defined.

By **theorem (2)** we know that $a \leq b$ is equivalent to $a * b = a$.

So to prove $a * b \leq a * c$, it is enough we prove that

$(a * b) * (a * c) = a * b$.

Now $(a * b) * (a * c) = a * (b * a) * c$ [associativity]

$= a * (a * b) * c$ [commutativity]

$= (a * a) * (b * c)$ [associativity]

$= a * (b * c)$ [idempotent law]

Given $b \leq c$, then **by theorem (2)**, $b * c = b$

$\therefore (a * b) * (a * c) = a * b.$

Hence by **theorem (2)** $a * b \leq a * c$

Thus, $b \leq c \Rightarrow a * b \leq a * c.$

In the same way we shall prove the 2nd part.

Given $b \leq c$ then $b \oplus c = c$ by theorem 2

To prove $a \oplus b \leq a \oplus c$, by theorem (2), it is enough we prove that

$(a \oplus b) \oplus (a \oplus c) = a \oplus c.$

Now $(a \oplus b) \oplus (a \oplus c) = a \oplus (b \oplus a) \oplus c$ [associativity]

$= a \oplus (a \oplus b) \oplus c$ [commutativity]

$= (a \oplus a) \oplus (b \oplus c)$ [associativity]

$= a \oplus (b \oplus c)$ [idempotent law]

$= a \oplus c$ [by hypothesis]

$\therefore$ by theorem (2) $a \oplus b \leq a \oplus c$

Thus, $b \leq c \Rightarrow a \oplus b \leq a \oplus c$.

Theorem 4: (Distributive inequalities)

Let $(L, \leq)$ be a lattice. For any $a,\ b, c \in L$ the following inequalities known as distributive inequalities hold.

(i) $a \oplus (b * c) \leq (a \oplus b) * (a \oplus c)$

(ii) $a * (b \oplus c) \geq (a * b) \oplus (a * c)$

Proof : Given $(L, \leq)$ is a lattice and $a,\ b,\ c\ \in L$.

We know $a \oplus b = LUB\ \{a, b\}$

$\therefore\ a \oplus b$ is an upper bound of a

$\therefore a \leq\ a \oplus b$

Similarly, $a \leq\ a \oplus c$

$\therefore a$ is a lower bound for $a \oplus b\ , a \oplus c$

$\Longrightarrow a \leq\ GLB\ \{a \oplus b\ ,\ a \oplus c\}$

$\Longrightarrow a \leq\ (a \oplus b\ ) *\ (a \oplus c)$ ... (1)

Now $b * c\ = GLB\ \{b,\ c\ \} \leq b$

But $b \leq\ LUB\ \{a, b\ \} =\ a \oplus b$

$\therefore b * c\ \leq\ a \oplus b,$ by transitivity of $\leq$

Similarly, $b * c\ = GLB\ \{b,\ c\ \} \leq c$ and $c \leq\ LUB\ \{a,\ c\ \} =\ a \oplus c$

$\therefore\ b * c \le a \oplus c$

Thus $b * c \le a \oplus b$ and $a \oplus c$

$$\Rightarrow b * c \le GLB\{a \oplus b, a \oplus c\}$$

$$\Rightarrow b * c \le (a \oplus b) * (a \oplus c) \qquad \dots (2)$$

From (1) and (2) we get $(a \oplus b) * (a \oplus c)$ is an upper bound for a and $b * c$.

Hence $LUB \{a, b * c\} \le (a \oplus b) * (a \oplus c)$

$$\Rightarrow a \oplus (b * c) \le (a \oplus b) * (a \oplus c).$$

(ii) By applying the principle of duality to (i) we get

$a * (b \oplus c) \ge (a * b) \oplus (a * c)$, which is (ii).

Theorem 5: (Modular inequality)

Let $(L, \leq)$ be a lattice. Then for any $a,\ b,\ c\ \in L$ the following inequality known as modular inequality holds. $a \leq c \Leftrightarrow a \oplus (b * c) \leq (a \oplus b) * c$

Proof:

Given $(L, \leq)$ is a lattice and $a,\ b,\ c\ \in L$.

Assume $a \leq c$ then $a \oplus c = c$

Now by distributive property, we have

$$a \oplus (b * c) \leq (a \oplus b) * (a \oplus c)$$

$$\Longrightarrow a \oplus (b * c) \leq (a \oplus b) * c$$

Thus $a \leq c \Rightarrow a \oplus (b * c) \leq (a \oplus b) * c$ ... (1)

For the reverse implication, assume $a \oplus (b * c) \leq (a \oplus b) * c$

Now $a \leq a \oplus (b * c)$ and $(a \oplus b) * c \leq c$, by definition of LUB and GLB

Since $a \oplus (b * c) \leq (a \oplus b) * c$, by transitivity, we get $a \leq c$

Hence $a \oplus (b * c) \leq (a \oplus b) * c \Rightarrow a \leq c$ ... (2)

From (1) and (2), $a \leq c \Leftrightarrow a \oplus (b * c) \leq (a \oplus b) * c$

The modular inequality can be expressed in different forms as below.

(i) $(a * b) \oplus (a * c) \leq a * [b \oplus (a * c)]$

(ii) $(a \oplus b) * (a \oplus c) \geq a \oplus [b * (a \oplus c)]$

Proof: $(a * b) \oplus (a * c) = (a * c) \oplus (a * b)$ [commutative law]

$\leq [(a * c) \oplus a] * [(a * c)\ b]$ [distributive law]

$\leq [a \oplus (a * c)] * [b \oplus (a * c)]$ [commutative law]

But $a \oplus (a * c) = a,$

$\therefore (a * b) \oplus (a * c) \leq a * [b \oplus (a * c)]$

By applying principle of duality, we get

$(a \oplus b) * (a \oplus c) \geq a \oplus [b * (a \oplus c)]$