Sri SAI RAM ENGINEERING COLLEGE
INSTITUTE OF TECHNOLOGY
West Tambaram, Chennai - 44

Sairam INSTITUTIONS

SAIRAM DIGITAL RESOURCES

YEAR II

SEM III

MA8351

DISCRETE MATHEMATICS
(Common to CSE & IT)

UNIT V

LATTICE AND BOOLEAN ALGEBRA

5.6 Boolean Algebra

SCIENCE & HUMANITIES

4 QUALITY EDUCATION

UNIT V
5.6. BOOLEAN ALGEBRA

A complemented distributive Lattice is called Boolean algebra (or)
A Boolean algebra is a non empty set with two binary operations $\wedge$ and $\vee$ and is satisfied by the following conditions $\forall a, b \in L$.

Definition:

A non-empty set **B** together with two binary operations +, . on **B** (called addition and multiplication), a 'unary operation' (called complementation) and two distinct elements 0 and 1 is called a Boolean algebra if the following axioms are satisfied for all $a, b, c \in B$.

1. Commutative law

   $a + b = b + a$ and $a.b = b.a$

2. Associative laws

   $a + (b + c) = (a + b) + c$ and $a.(b.c) = (a.b).c$

3. Distributive laws

   $a + (b.c) = (a + b).(a + c)$ and $a.(b + c) = a.b + a.c$

4. Identity laws

   There exist $0,1 \in B$ such that $a + 0 = a$ and $a.1 = a$

5. Complement laws

   For each $a \in B$ there exists an element $a' \in B$ such that $a + a' = 1$ and $a.a' = 0$

   The Boolean algebra is usually denoted $(B, +, ., ', 0,1)$

Note:

1. The operations $+$, $.$ are not number addition and multiplication, but Boolean sum and Boolean product. They are binary operations on B. So B is closed under $+$ *and* $.$
2. a' is complement of a.
3. The elements 0 and 1of B are zero-element and the unit element respectively.
4. By the definition of Boolean algebra contains atleast two elements namely the zero element and the unit element.

Properties of Boolean algebra B

Theorem 1: Idempotent laws

$a + a = a$ and $a . a = a \; \forall a \in B$

Proof:

By axioms 4, $a = a + 0$

$= a + a . a'$ [by axiom, complement law $a . a' = 0$]

$= (a + a) . (a + a')$ [by distributive law]

$= (a + a) . 1$ [complement law $a + a' = 1$]

$= a + a$ [by identity law]

Thus $a + a = a \; \forall a \in B$

By identity axioms, $a = a . 1$

$= a . (a + a')$ [complement law 5]

$= a . a + a . a'$ [by distributive law]

$= a . a + 0$ [complement law]

$= a . a$ [identity law]

Thus $a . a = a \quad \forall \; a \in B$

Theorem 2:

The elements 0 and 1 of a Boolean algebra B are unique.

Proof:

(i) Assume 0_1 *and* 0_2 be two elements in B, then

$0_1 + 0_2 = 0_2$ taking 0_1 as zero element........(1)

and $0_1 + 0_2 = 0_1$ taking 0_2 as zero element........(2)

By commutativity $0_1 + 0_2 = 0_2 + 0_1$

$\Longrightarrow 0_1 = 0_2$ (using (1) and (2))

$\therefore$ *zero* element is unique.

(ii) Let I_1 *and* I_2 be two unit elements in B

Then $I_1.I_2 = I_1$ taking I_2 as unit element

and $I_2.I_1 = I_2$ taking I_1 as unit element

By Commutativity $I_1.I_2 = I_2.I_1$

$\Rightarrow I_1 = I_2$

Theorem 3:

In a Boolean algebra B, $0' = 1$ and $1' = 0$.

Proof:

We have $0' = 0 + 0'$ [by identity law]

$=1$ [by complement law]

and $1' = 1'.1$ [by identity law]

$=0$ [by complement law]

Theorem 4: Boundedness laws [or dominance laws]

$$(i)\ a + 1 = 1\ \ and\ (ii)\ a.0 = 0\ \ \forall a \in B$$

Proof:

$$\begin{aligned}(i)\ a + 1 &= a + (a + a') &&\text{[by complement law]}\\ &= (a + a) + a' &&\text{[associative law]}\\ &= a + a' &&\text{[idempotent law]}\\ &= 1 &&\text{[by complement law]}\end{aligned}$$

$$
\begin{aligned}
(ii)\ a.0 &= a.0 + 0 \quad \text{[by identity law]}\\
&= a.0 + a.a' \quad \text{[by complement law]}\\
&= a.(0 + a') \quad \text{[by distributive law]}\\
&= a.a' \quad \text{[by identity law]}\\
&= 0 \quad \text{[by complement law]}
\end{aligned}
$$

Theorem 5:

In a Boolean Algebra B, complement of every element is unique.

Proof:

Let $a \in B$ be any element.

If $a_1{}'$ and $a_2{}'$ be two complements of a in B.

Then $a + a_1{}' = 1$ and $a.a_1{}' = 0$

$a + a_2{}' = 1$ and $a.a_1{}' = 0$

Now $a_2' = a_2'.1$ [identity law]

$= a_2'.(a + a_1')$ [complement law]

$= a_2'.a + a_2'.a_1'$ [distributive law]

$= a.a_2' + a_1'.a_2'$ [commutative law]

$= 0 + a_1'.a_2'$ [complement law]

$= a.a_1' + a_1'.a_2'$ [complement law]

$= a_1'.a + a_1'.a_2'$ [commutative law]

$= a_1'.(a + a_2')$ [distributive law]

$= a_1'.1$ [complement law]

$a_2' = a_1'$ [identity law]

Theorem 6: Absorption laws

$(i)\ a.(a+b) = a$ and $(ii)\ a + (a.b) = a\ \forall\ a, b \in B.$

Proof:

$(i) a.(a+b) = (a+0).(a+b)$ [identity law]

$= a + 0.b$ [distributive law]

$= a + b.0$[commutative law and boundedness law]

$= a + 0 = a$ [identity law]

$(ii)\ \ a + a.b = a.1 + a.b$ [identity law]

$= a.(1+b)$ [distributive law]

$= a.(b+1)$ [commutative law]

$= a.1$ [boundedness law]

$= 0$ [identity law]

Theorem 7:Demorgan's law

$(i)\ (a+b)' = a'.b'$ and $(ii)\ (a.b)' = a'+b' \quad \forall\, a,b \in B.$

Proof:

Consider $(a+b) + a'.b' = [(a+b)+a'].[(a+b)+b']$

[by distributive law

a+(b.c)=(a+b).(a+c)]

$=[(b+a)+a'].[a+(b+b')]$ [by commutative and associativity law]

$=[b+(a+a')].[a+1]$

$=[b+1].[a+1]$ [complement law]

=1.1=1 [Boundedness and identity]

and $(a+b).(a'.b') = (a'.b').(a+b)$ [commutative law]

$= (a'.b').a + (a'.b').b$ [distributive law]

$= a.(a'.b') + a'.(b'.b')$ [by commutative and associativity law]

$= (a.a').b' + a'.(b'.b')$

$= (a.a').b' + a'.(b.b')$

$= 0.b' + a'.0$

$= b'.0 + a'.0$

$= 0 + 0$ [boundedness law]

$= 0$ [identity law]

$Thus\ (a+b) + (a'.b') = 1\ and\ (a+b).(a'.b') = 0$

Hence $a'.b'$ is the complement of $a+b$.

$$\Rightarrow (a+b)' = a'.b'$$

(ii) To prove the complement of $a.b$ is $a' + b'$, we have to prove

$(a.b) + (a' + b') = 1$ and $(a.b) + (a' + b') = 0.$

Now $(a.b) + (a' + b') = [a + (a' + b')].[b + (a' + b')].$

[by distributive law]

$= [(a + a') + b'].[(b + b') + a'].$

$= [1 + b'].[1 + a']$ [by boundedness]

$= [b' + 1].[a' + 1]$ [by commutative]

$= 1.1=1$ [boundedness and identity law]

and $(a.b).(a' + b') = (a.b).a' + (a.b).b'$ [by distributive law]

$= (b.a).a' + a.(b.b')$ [by associativity law]

$= b.(a.a') + a.(b.b')$

$= b.0 + a.0$ [complement law]

$= 0 + 0$ [by boundedness]

$= 0.$

Examples

1. In any Boolean algebra, prove that $(a+b).(a'+c) = ac + a'b + bc$

Solution:

Let B be a Boolean algebra and $a, b, c \in B$

Now $(a+b).(a'+c) = (a+b).a' + (a+b).c$ [Distributive law]

$= a.a' + b.a' + a.c + b.c$

$= 0 + a'.b + a.c + b.c$ [complement law]

$= ac + a'b + bc$

2. In any Boolean algebra, prove that the following statements are equivalent

(i) $a + b = b$, (2) $a.b = a$ (3) $a' + b = 1$, (4) $a.b' = 0$

Solution:

To prove (1) $\Longrightarrow$ (2)

Assume (1) ie. $a + b = b$

$Now\ \ a.b = a.(a + b)$

$= a$ [by absorption law]

$\therefore\ (1) \Rightarrow (2)$

To prove $(2) \Rightarrow (3)$

Assume (2) ie. $a.b = a$

Now $a' + b = (a.b)' + b$ [Assumption]

$= a' + b' + b$ [Demorgan's law]

$= a' + 1$ $[\therefore b' + b = 1]$

$= a' + 0'$ $[\therefore 0' = 1]$

$= (a.0)'$ [Demorgan's law]

$= 0' = 1$ [complement law]

$\therefore (2) \Rightarrow (3)$

To prove $(3) \Rightarrow (4)$

Assume (3) ie. $a' + b = 1$

$\therefore (a' + b)' = 1'$

$\Rightarrow (a')'.b' = 1'$ [Demorgan's law]

$\Rightarrow a.b' = 0$

$$\therefore \quad (3) \Rightarrow (4)$$

To prove $(4) \Rightarrow (1)$

Assume (4) ie. $a.b' = 0$

$\Rightarrow (a.b')' = 0'$

$\Rightarrow a' + (b')' = 0'$

$\Rightarrow a' + b = 1$ [complement law]

Now $a + b = (a + b).1$

$= (a + b).(a' + b)$

$= (b + a).(b + a')$ [by commutative]

$= b.(a.a')$ [Distributive law]

$= b + 0$ [complement law]

$= b$

$$\therefore \quad (4) \Rightarrow (1)$$

Thus

$(1) \Rightarrow (2), (2) \Rightarrow (3), (3) \Rightarrow (4), (4) \Rightarrow (1)$

Hence all the statements are equivalent.

Example 3:

If B is a Boolean algebra, then prove that for

$$a \in B, a + 1 = 1 \ and \ a.0 = 0$$

Solution:

Given B is a Boolean algebra, Let $a \in B$

$$\therefore a + 1 = (a + 1).1$$

$$= 1.(a+1)$$
$$= (a+a').(a+1)$$
$$= a + a' = 1$$
$$\therefore a+1 = 1 \ \ \forall a \in B$$

And $a.0 = a.1'$ $[\because 0 = 1']$
$= a.(a+1)'$ [using a+1=1]
$= a.(a'.1')$ [Demorgan's law]

$=(a.a').0 = 0.0 = 0$

Example 4: $a = b$

In a Boolean algebra show that that $ab' + a'b = 0$ if and only if $a = b$

Solution:

Let $(B, +, ., ', 0, 1)$ be a Boolean algebra.

Let $a, b \in B$ be any two elements

Let $a = b$, then $ab' + a'b = 0$

then $a + ab' + a'b = a$

$$\Rightarrow (a + ab') + a'b = a$$

$\Rightarrow a + a'b = a$ [absorption law]

$\Rightarrow (a + a').(a + b) = a$ [Distributive law]

$\Rightarrow 1.(a + b) = a$

$\Rightarrow 1.(a + b) = a$ ....(1)

Similarly, $ab' + a'b = 0$

$$\Rightarrow ab' + a'b + b = b$$

$\Rightarrow a'b + b = b$ [absorption law]

$\Rightarrow (a + b).(b' + b) = b$ [Distributive law]

$\Rightarrow (a + b).1 = b$

$\Rightarrow (a + b) = b$ .....(2)

From (1) and (2) we get $a = b$.

Example 5:

Prove that $a.b' = 0$ iff $a.b = a$

Proof:

Let $a.b' = 0$

$\Rightarrow (a.b')' = 0'$

$$\Rightarrow a' + (b')' = 1$$

$$\Rightarrow a' + b = 1$$

$$\therefore a.(a' + b) = a.1$$

$$\Rightarrow a.a' + a.b = a$$

$\Rightarrow 0 + a.b = a \qquad [\because a.a' = 0]$

$$\Rightarrow a.b = a$$