Sri SAI RAM ENGINEERING COLLEGE

INSTITUTE OF TECHNOLOGY

West Tambaram, Chennai - 44

Sairam INSTITUTIONS

SAIRAM DIGITAL RESOURCES

YEAR II | SEM IV

MA8391

PROBABILITY AND STATISTICS.

UNIT NO 1

PROBABILITY AND RANDOM VARIABLES

- CONDITIONAL PROBABILITY
- BAYE'S THEOREM

SCIENCE & HUMANITIES

CONDITIONAL PROBABLITY:

The probability of an event B occurring when it is known that some event A has occurred is called a conditional probability and is denoted by $P(B|A)$. The symbol $P(B|A)$ is usually read "the probability that B occurs given that A occurs" or simply "the probability of B, given A."

$$P(B/A) = P(A \cap B)/P(A)$$
$$P(A \cap B) = P(A)\, P(B/A)$$

The conditional probability of A given B is

$$P(A/B) = P(A \cap B)/P(B)$$
$$P(A \cap B) = P(B)\, P(A/B)$$

1. The probability that a regularly scheduled flight departs on time is P(D) =0.83; the probability that it arrives on time is P(A) = 0 .82; and the probability that it departs and arrives on time is P(D∩A) =0.78; Find the probability that a plane

(i) Arrives on time, given that it departed on time and

(ii) Departed on time, given that it has arrived on time

Solution: Given P(D) =0.83; P(A) =0.82; P(D∩A) =0.78

(i)P(A/D) = P(D∩A) /P(D) = (0.78)/(0.83)=.0.93

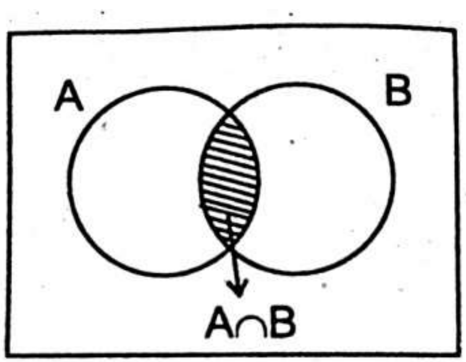
(ii)P(D/A)= P(D∩A) /P(A) = (0.78)/(0.82) =0.95

2. If A and B are two independent events, then show that**,**

i) $\overline{A}$ and $\overline{B}$ are also independent.

ii) $\overline{A}$ and B are also independent.

iii) A and $\overline{B}$ are also independent.



i) $\overline{A}$ and $\overline{B}$ are also independent

Proof:

We know that

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

$$P\left(\overline{A} \cap \overline{B}\right) = P\left(\overline{A \cup B}\right)$$

$$= 1 - P(A \cup B)$$

$$= 1 - \left[P(A) + P(B) - P(A \cap B)\right] - - - - (1)$$

Now A and B are independent events, hence $P(A \cap B) = P(A).P(B)$

$\therefore$ By (1), $P\left(\overline{A} \cap \overline{B}\right) = 1 - P(A) - P(B) + P(A).P(B)$

$$= \left(1 - P(A)\right)\left(1 - P(B)\right)$$

$$P\left(\overline{A} \cap \overline{B}\right) = P\left(\overline{A}\right).P\left(\overline{B}\right)$$

$\overline{A}$ and $\overline{B}$ are also independent

ii) Proof:

The events $A \cap B$ and $\overline{A} \cap B$ are mutually exclusive

$$\therefore \quad (A \cap B) \cup (\overline{A} \cap B) = B$$

$$\therefore \quad P\left[(A \cap B) \cup (\overline{A} \cap B)\right] = P(B)$$

$$P(A \cap B) + P(\overline{A} \cap B) = P(B)$$

$$P(\overline{A} \cap B) = P(B) - P(A \cap B)$$

$$= P(B) - P(A).P(B)$$

$\because$ A and B are independent $\quad P(A \cap B) = P(A).P(B)$

$$P(\overline{A} \cap B) = P(B).\left[1 - P(A)\right]$$

$$P(\overline{A} \cap B) = P(\overline{A}).P(B)$$

$\overline{A}$ and B are independent

iii) **Proof:**

$A \cap B$ and $A \cap \overline{B}$ are mutually exclusive

$$A = (A \cap B) \cup (A \cap \overline{B})$$

$$P(A) = P(A \cap B) + P(A \cap \overline{B})$$

$$\therefore P(A \cap \overline{B}) = P(A) - P(A \cap B)$$

$$= P(A) - P(A).P(B)$$

$$= P(A).[1 - P(B)]$$

$$P(A \cap \overline{B}) = P(A).P(\overline{B})$$

$\therefore$ A and $\overline{B}$ are independent

3. In a shooting test, the probability of hitting the target are $\frac{1}{2}$ for A, $\frac{2}{3}$ for B, $\frac{3}{4}$ for C. If all of them fire at the target, find the probability that

i) None of them hits the target
ii) At least one of them hits the target
iii) Exactly one of them hits the target.

Solution:

The three events **A,B, C** are independent

i)
$$P(\overline{A}\cap\overline{B}\cap\overline{C}) = P(\overline{A}).P(\overline{B}).P(\overline{C})$$
$$=\left(1-\frac{1}{2}\right).\left(1-\frac{2}{3}\right).\left(1-\frac{3}{4}\right)$$
$$=\frac{1}{2}\times\frac{1}{3}\times\frac{1}{4}=\frac{1}{24}$$

ii)
$$P(A\cup B\cup C)=1-P\left(\overline{A\cup B\cup C}\right)$$

$$=1-P\left(\overline{A}\cap\overline{B}\cap\overline{C}\right)$$

$$=1-\frac{1}{24}=\frac{23}{24}$$

iii) $P\left(A\cap\overline{B}\cap\overline{C}\right)+P\left(\overline{A}\cap B\cap\overline{C}\right)+P\left(\overline{A}\cap\overline{B}\cap C\right)$

$$=\left(\frac{1}{2}\times\frac{1}{3}\times\frac{1}{4}\right)+\left(\frac{1}{2}\times\frac{2}{3}\times\frac{1}{4}\right)+\left(\frac{1}{2}\times\frac{1}{3}\times\frac{3}{4}\right)$$

$$=\frac{6}{24}=\frac{1}{4}$$

Theorem of Total Probability

If $B_1, B_{2,}, \dots, B_n$ be a set of exhaustive and mutually exclusive events, and A is another event associated with B_i, then

$$P(A) = \sum_{i=1}^{n} P(B_i) P\left({}^{A}/_{B_i}\right)$$

Proof: The inner circle represents the event A, A can occur along with $B_1, B_{2,}, \dots, B_n$ be a set of exhaustive and mutually exclusive.

$AB_1, AB_{2,} \dots., AB_n$ are also mutually exclusive.

$A = AB_1 + AB_{2+} \dots. + AB_n$ (By addition theorem)

$$P(A) = P\sum (A\,B_i)$$

$=\sum P(AB_i)$ (Since $AB_1, AB_{2,} \dots., AB_n$ are mutually exclusive)

$$P(A) = \sum_{i=1}^{n} P(B_i) P\left({}^{A}/_{B_i}\right)$$

BAYE'S THEOREM :

If $B_1, B_{2,}, \ldots, B_n$ be a set of exhaustive and mutually exclusive events, associated with a random experiment and A is another event associated with B_i, then

$$P(B_i/A) = \frac{P(B_{i)}P(A/B_i)}{\sum_{i=1}^{n} P(B_i)P\left(A/_{B_i}\right)}, \text{ i=1,2,3..n}$$

Proof: $P(B_i \cap A) = P(B_i)P(A/B_i) = P(A)P(B_i/A)$

$$\therefore\ P(B_i/A) = \frac{P(B_{i)}P(A/B_i)}{P(A)}$$

$$P(B_i/A) = \frac{P(B_{i)}P(A/B_i)}{\sum_{i=1}^{n} P(B_i)P\left(A/_{B_i}\right)}, \quad \text{i=1,2,3..n}$$

1. The contents of urns I, II and III are as follows: 1 white, 2 black and 3 red balls. 2 white, 1 black and 1 red ball, and 4 white, 5 black and 3 red balls respectively. One urn is chosen at random and two balls are drawn from it. They happen to be white and red. What is the probability that they come from urns I, II or III.

Solution:

Let A_1, A_2, A_3 denote the events that the urn I, II, III is selected respectively and let B be the event that the two balls taken from the selected urn are white and red.

Now,

$$P(A_1)=\frac{1}{3}\,;\,P(A_2)=\frac{1}{3}\,;\,P(A_3)=\frac{1}{3}$$

$$P(B/A_1)=\frac{1C_1\times 3C_1}{6C_2}=\frac{1}{5}$$

$$P(B/A_2)=\frac{2C_1\times 1C_1}{4C_2}=\frac{1}{3}$$

$$P(B/A_3)=\frac{4C_1\times 3C_1}{12C_2}=\frac{2}{11}$$

$$\sum_{i=1}^{3}P(A_i)P(B/A_i)=P(A_1).P(B/A_1)+P(A_2).P(B/A_2)+P(A_3).P(B/A_3)$$

$$=\frac{1}{3}\times\frac{1}{5}+\frac{1}{3}\times\frac{1}{3}+\frac{1}{3}\times\frac{2}{11}=0.2384$$

$$P(A_1/B)=\frac{P(A_1)\,.\,P(B/A_1)}{\sum_{i=1}^{3}P(A_i)P(B/A_i)}=\frac{\frac{1}{3}\times\frac{1}{5}}{0.2384}=0.2796$$

Similarly

,

$$P(A_2/B)=\frac{P(A_2)\,.\,P(B/A_2)}{\sum_{i=1}^{3}P(A_i)P(B/A_i)}=\frac{\frac{1}{3}\times\frac{1}{3}}{0.2384}=0.4661$$

$$P(A_3/B) = \frac{P(A_3)\,.\,P(B/A_3)}{\sum_{i=1}^{3} P(A_i)P(B/A_i)} = \frac{\frac{1}{3}\times\frac{2}{11}}{0.2384} = 0.2542$$

2. Suppose that 37% of a community are at least 45 years old. If 80% of the time a person who is 45 or older tells the truth, and 65% of the time, a person below 45 tells the truth. What is the probability that a randomly selected person answers a question truthfully?

Solution:

Let A be the event that a person is at least 45 years old. B be the event that a person speaks the truth.

$$P(A) = 0.37,\ P(\overline{A}) = 0.63$$

$$P(B/A) = 0.80,\ P(B/\overline{A}) = 0.65$$

By total probability theorem, $P(B) = P(A).P(B/A) + P(\overline{A}).P(B/\overline{A})$

$$= (0.37)(0.80) + (0.63)(0.65) = 0.7055$$

3. For a certain binary communication channel the probability that a transmitted '0' is received as a '0' is 0.95, and the probability that a transmitted 1 is received as a 1 is 0.90. If the probability that a '0' is transmitted is 0.4, find the probability that

i) a '1' is received .

ii) a '1' was transmitted given that a '1' was received.

Solution:

Let A = event of transmitting 1. B = event of receiving a '1'.

$\therefore$ $\overline{A}$ = event of transmitting 0 $\therefore$ $\overline{B}$ = event of receiving 0

$$\therefore P\left(\overline{A}\right)=0.4; \quad P(B/A)=0.9; \quad P\left(\overline{B}/\overline{A}\right)=0.95$$

$$P(A)=0.6, \quad P\left(B/\overline{A}\right)=0.05$$

i) By theorem of total probability,

$$P(B)=P(A).P(B/A)+P\left(\overline{A}\right).P\left(B/\overline{A}\right)$$

$$= 0.6 \times 0.9 + 0.4 \times 0.05 = 0.56$$

By Baye's theorem

$$P(A/B) = \frac{P(A).P(B/A)}{P(B)} = \frac{0.6 \times 0.9}{0.56} = 0.9643$$