Sri SAI RAM ENGINEERING COLLEGE
INSTITUTE OF TECHNOLOGY
West Tambaram, Chennai - 44

SAIRAM DIGITAL RESOURCES

YEAR II SEM IV

MA8391

PROBABILITY AND STATISTICS

DEPARTMENT OF INFORMATION TECHNOLOGY

UNIT I

PROBABILITY AND RANDOM VARIABLES

1.5 DISCRETE DISTRIBTION – BINOMIAL, POSSION, GEOMETRIC

SCIENCE & HUMANITIES

Binomial Distribution

A random variable X is said to follow binomial distribution if its probability mass function is $P[X = x] = nc_x p^x q^{n-x}, x = 0,1,2, \dots n \ where\ p + q = 1.$ It is denoted as $X \approx B(n, p)$ i.e., n, p are the parameters.

- It gives probability of x success in n trials.
- If the trial is repeated for N times, then the required probability is N p(x).
- If $X \approx B(n_1, p)$ and $Y \approx B(n_2, p)$ then $X + Y \approx B(n_1 + n_2, p)$.

Find the moment generating function of Binomial distribution and hence find the mean, variance.

Answer: Moment Generating Function of Binomial Distribution

The pmf of Binomial Distribution is $P[X=x] = nc_x p^x q^{n-x}, x = 0,1,2,\ldots\ldots$

$$M_X(t) = E[e^{tx}] = \sum_{x=0}^{n} e^{tx} p(x) = \sum_{x=0}^{n} e^{tx} nC_x p^x q^{n-x}$$

$$= \sum_{x=0}^{n} nC_x (e^t p)^x q^{n-x} = (e^t p + q)^n$$

Mean $E[X] = M_X'(0) = [n(pe^t + q)^{n-1} pe^t]_{t=0} = np$

$E[X^2] = M_X''(0)$

$= np[(pe^t + q)^{n-1} e^t + (n-1)(pe^t + q)^{n-2} pe^t]_{t=0}$

$= np[1 + (n-1)p] = np[1 + np - p]$

$= np + n^2p^2 - np^2$

$Var(X) = E[X^2] - (E[X])^2$

$= np + n^2p^2 - np^2 - n^2p^2 = np - np^2 = np(1-p)$

$= npq$

Q1. A random variable X follows Binomial distribution with mean 2, variance 4. Give your comment on this.

Answer Key: Since X follows Binomial distribution, mean = np and variance = npq.

Given Mean = 2 i.e. np = 2

Variance = 4, i.e. npq = 4

Therefore $\frac{npq}{np} = \frac{4}{2} thisimpliesq = 2$> 1. This is not possible. Hence given data are wrong.

Q2. For a Binomial distribution of mean 4 and variance 2, find the probability of getting i. at least 2 successes ii. utmost 2 successes iii. P(5<X<8)

Answer: Since X follows Binomial distribution, mean = np and variance = npq.

Given Mean = 4 i.e. np = 4 Variance = 2, i.e. npq = 2

$$\text{Therefore } \frac{npq}{np} = \frac{2}{4} = 0.5 \qquad \textit{this implies}$$

$$q = 0.5 \qquad \textit{and hence}$$

$$p = 1 - q = 0.5$$

$$\text{But } \; np = 2 \; gives \quad n(0.5) = 2$$

$$i.e. n = 4$$

The pmf of binomial distribution is $P[X=x] = nc_x p^x q^{n-x}, x = 0,1,2,\ldots\ldots$

$$P[X=x] = 4c_x (0.5)^x (0.5)^{4-x}, x = 0,1,2,\ldots\ldots$$

i. Probability of getting at least 2 success

$$P[X \geq 2] = P[X = 2,3,4,\ldots\ldots\ldots]$$
$$= 1 - P[X < 2] = 1 - P[X = 0,1]$$

$$= 1 - [4c_0(0.5)^0(0.5)^{4-0} + 4c_1(0.5)^1(0.5)^{4-1}]$$

$$= 1 - [(0.5)^4 + 4(0.5)(0.5)^3]$$

ii. Probability of getting utmost 2 successes

$$P[X \leq 2] = P[X = 0,1,2]$$
$$= [4c_0(0.5)^0(0.5)^{4-0} + 4c_1(0.5)^1(0.5)^{4-1}$$
$$+ +4c_2(0.5)^2(0.5)^{4-2}]$$
$$= [(0.5)^4 + 4(0.5)(0.5)^3 + 6(0.5)^2(0.5)^2]$$
$$= (0.5)^4[1 + 4 + 6]$$

iii. Probability of getting success lies between 5 to 8

$$P[5 < X < 8] = P[X = 6,7] = 0$$

Q3. **A Binomial variable X satisfies the relation $9P(X=4)=P(X=2)$ when $n=6$. Find the parameter p of the Binomial distribution.**

Answer:

The probability function for a binomial distribution is

$$P(X=x)=nC_x p^x q^{n-x}$$

$$P(X=4)=6C_4 p^4 q^2$$

$$P(X=2)=6C_2 p^2 q^4$$

$$9P(X=4)=P(X=2)$$

$$9P(X=4)=P(X=2)$$

$$9\,.6C_4p^4q^2=6C_2p^2q^4$$

$$135p^2=15q^2$$

$$9p^2-q^2=0$$

$$9p^2-(1-p)^2=0$$

$$9p^2-(1+p^2-2p)=0$$

$$9p^2-1-p^2+2p=0$$

$$8p^2+2p-1=0$$

$$p=\frac{-1}{2}\text{ or }\frac{1}{4}$$

$$p=\frac{1}{4}\ [\because p\text{ cannot be negative}]$$

Q4. **If, on an average, 9 ships out of 10 arrive safely to a port, Obtain the mean and standard deviation of the number of ships returning safely out of 150 ships.**

Answer: If p is the probability of safe arrival then

$$p = \frac{9}{10} \Rightarrow q = 1 - p = 1 - \frac{9}{10} = \frac{1}{10}$$

$$n = 150$$

$$\text{Mean} = np = 150 \times \frac{9}{10} = 135$$

$$\text{Variance} = npq = 150 \times \frac{9}{10} \times \frac{1}{10} = 13.5$$

$$\text{Standard deviation} = \sqrt{npq} = \sqrt{13.5} = 3.674$$

Q5. **There is rainfall in a certain place is 10 days in every thirty days. Find the probability that**

1. There is rainfall on at least 3 days of a given week.

2. The first four days of a given week will be wet and the remaining days dry.

Answer:

Given the probability of rainfall $p = \frac{10}{30} = \frac{1}{3} \Rightarrow$

$$q = \frac{2}{3} \quad n = 7$$

$$P(X = x) = 7C_x p^x q^{7-x}$$

1. $P(\text{there is rainfall on at least 3 days}) = P(X \geq 3)$

$$= 1 - P(X < 3)$$

$$= 1 - [P(X = 0) + P(X = 1) + P(X = 2)]$$

$$= 1 - \left[7C_0\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^{7-0} + 7C_1\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^{7-1} + 7C_2\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^{7-2}\right]$$

$$= 1 - \left[\left(\frac{2}{3}\right)^7 + 7\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^6 + 21\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^5\right]$$

$$= 1 - 0.5705 = 0.4295$$

2. $$P(w_1 w_2 w_3 w_4 d_1 d_2 d_3)$$

$$= P(w_1)P(w_2)P(w_3)P(w_4)P(d_1)P(d_2)P(d_3)$$

$$= [P(w)]^4 [P(d)]^3$$

$$= \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^3 = 0.0037$$

Q6. **6 dice are thrown 729 times. How many times do you expect at least three dice to show a five or a six?**

Answer:

Success=getting 5 or 6 with one die

p=Probability of getting 5 or 6 with one die=2/6=1/3

q=2/3

Let X be a random variable denoting the number of successes when 6 dice are thrown once

Given n=6

$$P(X=x)=6C_x\left(\frac{1}{3}\right)^x\left(\frac{2}{3}\right)^{6-x}$$

p(at least three dice to show a five or a six)= $P(X \geq 3)$

$$= P(X=3)+P(X=4)+P(X=5)+P(X=6)$$

$$= 6C_3\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^{6-3} + 6C_4\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^{6-4}$$

$$+6C_5\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right)^{6-5} + 6C_6\left(\frac{1}{3}\right)^6\left(\frac{2}{3}\right)^{6-6}$$

$$= \left(\frac{1}{3}\right)^6 [160 + 60 + 12 + 1] = \frac{233}{3^6}$$

6 dice are thrown 729 times. Hence, the expected number of times at least three dices to show a five or a six

$$= 729 \times \frac{233}{3^6} = 233 \text{ times}$$

Q7. A company produces screws. It is known that 0.01 of total production is defective. The company sells the screws in packages of 10 and the company announces that money will be given if at least 1 of the 10 screws is defective. What proportion of the packages sold must be replaced?

Answer:

If X be the number defective screws in the packages.

The probability of defective screws $p=0.01$

$q=0.99$

The probability that the package sold will have to be replaced.

$$P(\text{at least 1 of the 10 screws is defective}) = \mathrm{P}(\mathrm{X} > 1)$$

$$= 1 - \mathrm{P}(\mathrm{X} \leq 1)$$

$$= 1 - [P(X = 0) + P(X = 1)]$$

$$= 1 - [10C_0(p)^0(q)^{10-0} + 10C_1(p)^1(q)^{10-1}]$$

$$= 1 - [(0.99)^{10} + 10(0.01)^1(0.99)^9]$$

$$= 0.0052$$

Poisson Distribution

A random variable X is said to follow binomial distribution if its probability mass function is $P[X = x] = \frac{e^{-\lambda}\lambda^{x}}{x!}, x = 0,1,2,\ldots\ldots\ldots$ It is denoted as $X \approx P(\lambda)$ i.e., λ is the parameter.

- It gives probability of x success.
- * It is useful if n is large and p is small.

* If $X \approx P(\lambda_1)$ and $X \approx P(\lambda_2)$ then $X + Y \approx P(\lambda_1 + \lambda_2)$.

Find the moment generating function of Poisson distribution and hence find mean, variance.

Answer: The pmf of Poisson distribution is $P[X = x] = \frac{e^{-\lambda}\lambda^x}{x!}, x = 0,1,2,\ldots\ldots\ldots$

$$M_X(t) = E[e^{tx}]$$

$$= \sum_{x=0}^{\infty} e^{tx} p(x)$$

$$= \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!}$$

$$= e^{-\lambda} e^{\lambda e^t}$$

$$= e^{\lambda(e^t - 1)}$$

$$\text{Mean } E[X] = M_X'(0) = \left[e^{\lambda(e^1-1)}.\lambda e^t\right]_{t=0} = \lambda$$

$$E[X^2] = M_X''(0)$$

$$= \lambda\left[e^{\lambda(e^1-1)}.e^t + e^t e^{\lambda(e^1-1)}\lambda e^t\right]_{t=0}$$

$$= \lambda[1+\lambda]$$

$$= \lambda^2 + \lambda$$

$$Var(X) = E[X^2] - (E[X])^2$$

$$= \lambda^2 + \lambda - \lambda^2 = \lambda$$

Q1. Find the parameter λ of the Poisson distribution if P[X=1] = 2 P[X=2].

Answer Key:

Given P[X=1] = 2 P[X=2] and X follows Poisson distribution with

$$P[X = x] = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$\frac{e^{-\lambda}\lambda^{1}}{1!} = 2\,\frac{e^{-\lambda}\lambda^{2}}{2!}$$

Q2. If X be a random variable following Poisson distribution such that P(X=2) = 9 P(X=4) + 90 P(X=6). Find the mean, variance of X.

Answer : The pmf of Poisson distribution is

$$P[X = x] = \frac{e^{-\lambda}\lambda^{x}}{x!}, x = 0,1,2,3,\ldots\ldots$$

Given P(X=2) = 9 P(X=4) + 90 P(X=6)

Therefore $\frac{e^{-\lambda}\lambda^{2}}{2!} = 9\frac{e^{-\lambda}\lambda^{4}}{4!} + 90\frac{e^{-\lambda}\lambda^{6}}{6!}$

Cancelling $\frac{e^{-\lambda}\lambda^{2}}{2}$ on both sides, we get,

$$\frac{1}{1} = \frac{9}{12}\lambda^{2} + \frac{90}{360}\lambda^{4}$$

$$\frac{1}{1} = \frac{9}{12}\lambda^2 + \frac{90}{360}\lambda^4$$

$$\frac{1}{1} = \frac{3}{4}\lambda^2 + \frac{1}{4}\lambda^4,$$

$$whichgives$$

$$\lambda^4 + 3\lambda^2 - 4 = 0$$

$$(\lambda^2 - 1)(\lambda^2 + 4) = 0\lambda^2 = 1 = 0$$

$$\lambda = 1$$

For a Poisson distribution, mean λ = 1 and variance λ = 1

Q3. It is known that the probability of an item produced by a machine will be defective is 0.05 If the products are sold in packets of 20, find the number of packets containing at least, exactly and at most 2 defective items in a consignment of 1000 packets using Poisson distribution.

Answer : Let X represents the number of defective items produced and it follows Poisson distribution. Therefore

$$P[X = x] = \frac{e^{-\lambda}\lambda^{x}}{x!}, x = 0,1,2,\ldots\ldots$$

Probability for producing one defective item is
p = 0.05

Products are sold in packets of 20 and hence n = 20

Therefore $\lambda = np = 20 \times 0.05 = 1$

Probability of a packet containing exactly 2 defective is

$$P[X=2] = \frac{e^{-1}1^2}{2!} = \frac{1}{2e}$$

Therefore number of packets containing exactly 2 defectives in a consignment of 1000

$$packets\ is = 1000 \times \frac{1}{2e} = \frac{500}{e}.$$

Probability of a packet containing at most 2 defective

$$P[X \leq 2] = P[X = 0,1,2] = \frac{e^{-1}1^0}{0!} + \frac{e^{-1}1^1}{1!} + \frac{e^{-1}1^2}{2!}$$

$$= \frac{1}{e}\left[1 + 1 + \frac{1}{2}\right] = \frac{5}{2e}$$

Therefore number of packets containing at most 2 defectives in a consignment of 1000

$$packetsis = 1000 \times \frac{5}{2e} = \frac{2500}{e}.$$

Probability of a packet containing at least 2 defective

$$P[X \geq 2] = P[X = 2,3,4,\ldots\ldots] = 1 - P[X < 2]$$

$$= 1 - P[X = 0,1] = 1 - \left[\frac{e^{-1}1^0}{0!} + \frac{e^{-1}1^1}{1!}\right]$$

$$= 1 - \frac{1}{e}[1 + 1] = 1 - \frac{2}{e}$$

Therefore number of packets containing at most 2 defectives in a consignment of 1000

$$packets\ is = 1000 \times \left(1 - \frac{2}{e}\right).$$

Q4. A manufacturer of cotton pins knows that 5% of his product is defective. If he sells cotton pins in boxes of 100 and guarantees that not more than 10 pins will be defective. What is the probability that a box will fail to meet the guaranteed quality.

Answer : Let X represents the number of defective pins produced.

Probability for producing one defective item is $p = \frac{5}{100}$

Products are sold in packets of 100 and hence n = 100

Therefore $\lambda = np = 100 \times 0.05 = 5$

It follows Poisson distribution and hence

$$P[X = x] = \frac{e^{-\lambda}\lambda^{x}}{x!}, x = 0,1,2,\ldots\ldots$$

The manufacturer gives a guarantee that a packet may contain maximum 10defectives.

If a box contain more than 10 defective items, then the box will fail to meet the guarantee.

Probability of a packet containing more than 10 defective

$$P[X > 10] = P[X = 11,12,13,\ldots\ldots]$$
$$= 1 - P[X \leq 10] = 1 - P[X = 0,1,2,\ldots\ldots\ldots\ldots, 10]$$
$$= 1 - \left[\frac{e^{-5}5^{0}}{0!} + \frac{e^{-5}5^{1}}{1!} + \ldots\ldots\ldots\ldots + \frac{e^{-5}5^{10}}{10!}\right]$$
$$= 1 - e^{-5}\left[1 + \frac{5^{1}}{1!} + \frac{5^{2}}{2!} + \ldots\ldots\ldots\ldots + \frac{5^{10}}{10!}\right]$$

Geometric Distribution

A random variable X is said to follow geometric distribution if its probability mass function is $P[X = x] = pq^x, x = 0,1,2,\ldots\ldots\ldots (or)$

$$P[X = x] = pq^{x-1}, x = 1,2,3,\ldots\ldots\ldots$$

- It gives probability of first success after x failures.

Q1. Find the moment generating function and hence find the mean, variance of geometric distribution.

Answer Key: The pmf of geometric distribution is

$P[X = x] = q^{x-1}p, x = 1,2,3,\ldots\ldots\ldots$

$$M_X(t) = E[e^{tx}]$$

$$\sum_{x=0}^{n} e^{tx}p(x)$$

$$\sum_{x=1}^{\infty} e^{tx}q^{x-1}p$$

$$\frac{p}{q}\sum_{x=0}^{n}(qe^{t})^{x}$$

$$=\frac{p}{q}[1+(qe^{t})+(qe^{t})^{1}+(qe^{t})^{2}+\ldots\ldots\ldots]$$

$$=\frac{p}{q}[1-qe^{t}]^{-1}$$

$$=\frac{p}{q}\frac{1}{1-qe^{t}}$$

Mean $E[X]=M_X'(0)$

$$=\frac{p}{q}\frac{-(-qe^{t})}{(1-qe^{t})^{2}}=\left[\frac{pe^{t}}{(1-qe^{t})^{2}}\right]_{t=0}$$

$$=\frac{p}{p^{2}}=\frac{1}{p}$$

$$E[X^2] = M_X'(0)$$

$$= \left[\frac{(1-qe^t)^2pe^t - pe^t2(1-qe^t)(-qe^t)}{(1-qe^t)^4}\right]_{t=0}$$

$$= \frac{p^3 + 2p^2q}{p^4}$$

$$= \frac{p+2q}{p^2}$$

$$Var(X) = E[X^2] - (E[X])^2$$

$$= \frac{p+2q}{p^2} - \frac{1}{p^2}$$

$$= \frac{p+2q-1}{p^2} = \frac{q}{p^2}$$

State and prove memoryless property of Geometric Distribution

Answer : Statement: P [X > m + n / X > m] = P [X > n]

Here X follows Geometric Distribution and hence

$$P[X = x] = q^{x-1}p, x = 1,2,3,\ldots\ldots.$$

$$P[X > n] = \sum_{x=n+1}^{\infty} P(x) \sum_{x=n+1}^{\infty} q^{x-1}p$$

$$= p[q^n + q^{n+1} + q^{n+2} + \dots\dots\dots]$$
$$= pq^n[1 + q + q^2 + q^3 + \dots\dots\dots]$$
$$= pq^n[1 - q]^{-1}$$
$$= pq^n p^{-1}$$
$$= q^n$$

$$P[X > m + n/X > m] = \frac{P[X>m+n \cap X>m]}{P[X>m]}$$
$$= \frac{P[X > m + n]}{P[X > m]}$$
$$= \frac{q^{m+n}}{q^m}$$
$$= q^n$$
$$= P[X > n]$$

Q3. A die is cast until 6 appears. What is the probability that it must cast more than five times.

Answer: Let X represents the number of tosses required to get the first 6.

Probability of getting 6 is p = $\frac{1}{6}$

and hence q = 1 – p = $\frac{5}{6}$

Also $P[X = x] = q^{x-1}p = \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{x-1}, x = 1,2,\ldots\ldots..$

$$P[X>5]=1-P[X\le 5]$$

$$=1-\{P[X=1]+P[X=2]+P[X=3]+P[X=4]+P[X=5]\}$$

$$=1-\left\{\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^0+\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^1+\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^2+\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^3+\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^4\right\}$$

$$=1-\left(\frac{1}{6}\right)\left\{1+\left(\frac{5}{6}\right)^1+\left(\frac{5}{6}\right)^2+\left(\frac{5}{6}\right)^3+\left(\frac{5}{6}\right)^4\right\}=1-0.5981$$

$$=0.4019$$

Q4. A trainee soldier shoots a target in an independent fashion. If the probability that the target is shot on any one shot is 0.8. What is the probability that the target would be first hit at the 6th attempt?. What is the probability that it takes less than 5 shots?

Answer : Let X represents the number of shots required to hit the target first.

Probability of hitting the target is p = 0.8 and hence q = 1 – p = 0.2

Also $P[X = x] = q^{x-1}p = (0.8)(0.2)^{x-1}, x = 1,2,\ldots\ldots..$

(i) Probability of hitting the target on 6^{th} attempt

$$P[X=6] = (0.8)(0.2)^{6-1} = (0.8)(0.2)^5 = 0.00026$$

(ii) Probability of hitting the target in less than 5 attempt

$$P[X<5] = P[X=1] + P[X=2] + P[X=3] + P[X=4]$$

$$= (0.8)(0.2)^0 + (0.8)(0.2)^1 + (0.8)(0.2)^2 + (0.8)(0.2)^3$$
$$= (0.8)\{1 + (0.2)^1 + (0.2)^2 + (0.2)^3\} = 0.9984$$

Q5. A candidate is applying for driving license has the probability of 0.8 in passing the road test in a given trial. What is the probability that he will pass the test (i) on the fourth trial (ii) in less than four trials.

Answer : Let X represents the number of trails required to get the first success.

Probability of getting the license is p = 0.8 and hence q = 1 – p = 0.2

Also $P[X = x] = q^{x-1}p = (0.8)(0.2)^{x-1}, x = 1,2,\ldots\ldots$

(i) Probability of getting the license in the 4th trial

$$P[X=4]=(0.8)(0.2)^{4-1}=(0.8)(0.2)^3=0.0064$$

(ii) Probability of hitting the target in less than 4 attempt

$$P[X<4]=P[X=1]+P[X=2]+P[X=3]$$

$$=(0.8)(0.2)^0+(0.8)(0.2)^1+(0.8)(0.2)^2$$

$$=(0.8)\{1+(0.2)^1+(0.2)^2\}=0.992$$

(i) Probability of getting the license in the 4th trial

$$P[X = 4] = (0.8)(0.2)^{4-1} = (0.8)(0.2)^3 = 0.0064$$

(ii) Probability of hitting the target in less than 4 attempt

$$P[X < 4] = P[X = 1] + P[X = 2] + P[X = 3]$$

$$= (0.8)(0.2)^0 + (0.8)(0.2)^1 + (0.8)(0.2)^2$$

$$= (0.8)\{1 + (0.2)^1 + (0.2)^2\} = 0.992$$