Sri SAI RAM ENGINEERING COLLEGE
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Sairam INSTITUTIONS

SAIRAM DIGITAL RESOURCES

YEAR II

SEM III

MA 8351

DISCRETE MATHEMATICS
(COMMON TO CSE & IT)

UNIT 2 COMBINATORICS

2.5 SOLUTIONS OF RECURRENCE RELATIONS USING GENERATING FUNCTIONS

SCIENCE & HUMANITIES

GENERATING FUNCTIONS

Generating functions are used to solve many types of counting problems, such as the number of ways to select or distribute objects of different kinds, subject to a variety of constraints. Generating functions can be used to solve recurrence relations by translating a recurrence relation for the terms of a sequence into an equation involving a generating function. This equation can then be solved to find a closed form for the generating function. From this closed form, the coefficients of the power series for the generating function can be found, solving the original recurrence relation.

DEFINITION

The generating function for the sequence $a_0, a_1, \ldots, a_k, \ldots$ of real numbers is the infinite series

$$G(x) = a_0 + a_1 x + \cdots + a_k x^k + \cdots = \sum_{k=0}^{\infty} a_k x^k$$

EXAMPLE:

The generating function for the sequences $\{a_k\}$ with $a_k = 3$, $a_k = k+1$ and $a_k = 2^r$ are

$$\sum_{k=0}^{\infty} 3 x^k \qquad \sum_{k=0}^{\infty} (k+1) x^k \qquad \text{and} \qquad \sum_{k=0}^{\infty} 2^r x^k$$

respectively.

EXAMPLE:1

What is the generating function for the sequence 1, 1, 1, 1, 1, 1?

Solution:

The generating function of 1, 1, 1, 1, 1, 1 is

$$1 + x + x^2 + x^3 + x^4 + x^5.$$

But $(x^6 - 1)/(x - 1) = 1 + x + x^2 + x^3 + x^4 + x^5$ when $x \neq 1$

Consequently, $G(x) = (x^6 - 1)/(x - 1)$ is the generating function of the sequence 1, 1, 1, 1, 1, 1.

EXAMPLE:2

Let m be a positive integer. Let $a^k = C(m, k)$, for $k = 0, 1, 2, \ldots, m$. What is the generating function for the sequence $a^0, a^1, \ldots, a^m$?

Solution:

The generating function for this sequence is

$G(x) = C(m, 0) + C(m, 1)x + C(m, 2)x^2 + \cdots + C(m, m)x^m$.

EXAMPLE:3

The function $f(x) = 1/(1 - x)$ is the generating function of the sequence 1, 1, 1, 1, . . . , because

$$1/(1 - x) = 1 + x + x^2 + \cdots \quad \text{for } |x| < 1.$$

EXAMPLE:4

The function $f(x) = 1/(1 - ax)$ is the generating function of the sequence 1, a, a^2, a^3, . . . , because

$$1/(1 - ax) = 1 + ax + a^2x^2 + \cdots$$ when $|ax| < 1$, or equivalently, for $|x| < 1/|a|$ for $a \neq 0$.

USING GENERATING FUNCTIONS TO SOLVE RECURRENCE RELATIONS

1) **Solve the recurrence relation $a_k = 3a_{k-1}$ for k = 1, 2, 3, . . . and initial condition $a_0 = 2$.**

Solution:

Let G(x) be the generating function for the sequence $\{a_k\}$,

ie) $G(x) = \sum_{k=0}^{\infty} a_k x^k$

ie) $x\,G(x) = \sum_{k=0}^{\infty} a_k x^{k+1}$

$= \sum_{k=1}^{\infty} a_{k-1} x^k$

Using the recurrence relation, we see that

$$G(x) - 3xG(x) = \sum_{k=0}^{\infty} a_k x^k - 3 \sum_{k=1}^{\infty} a_{k-1} x^k$$

$$= a_0 + \sum_{k=1}^{\infty} (a_k - 3a_{k-1}) x^k$$

= 2,

because $a_0 = 2$ and $a_k = 3a_k - 1$.

Thus, $G(x) - 3xG(x) = (1 - 3x)G(x) = 2$.

ie) $G(x) = 2 / (1-3x)$

$$= 2 \sum_{k=0}^{\infty} 3^k x^k$$

$$a_k = 2 \cdot 3^k$$

2) Solve the recurrence relation $a_n = 8a_{n-1} + 10^{n-1}$ and the initial condition $a_0 = 1$ $a_1 = 9$.

Solution:

$$a_n = 8a_{n-1} + 10^{n-1}$$

Multiplying x^n on both sides , we get

$$a_n x^n = 8a_{n-1} x^n + 10^{n-1} x^n$$

Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$

be the generating function of the sequence $a_0, a_1, a_2, \ldots$. We sum both sides of the last equation starting with $n = 1$, to find that

$$G(x) - 1 = \sum_{n=1}^{\infty} a_n x^n$$

$$= \sum_{n=1}^{\infty}(8a_{n-1}x^n + 10^{n-1}x^n)$$

$$= 8\sum_{n=1}^{\infty}a_{n-1}x^n + \sum_{n=1}^{\infty}10^{n-1}x^n$$

$$= 8x\sum_{n=1}^{\infty}a_{n-1}x^{n-1} + x\sum_{n=1}^{\infty}10^{n-1}x^{n-1}$$

$$= 8x\sum_{n=0}^{\infty}a_n x^n + x\sum_{n=0}^{\infty}10^n x^n$$

= 8xG(x) + x/(1 − 10x),

ie) G(x) − 1 = 8xG(x) + x/(1 − 10x).

Solving for G(x) shows that

$$G(x) = \frac{1-9x}{(1-8x)(1-10x)}$$

Expanding the right-hand side of this equation into partial fractions gives

$$G(x) = \frac{1}{2}\left(\frac{1}{1-8x} + \frac{1}{1-10x}\right)$$

$$= \frac{1}{2}\left(\sum_{n=0}^{\infty} 8^n x^n + \sum_{n=0}^{\infty} 10^n x^n\right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{2}(8^n + 10^n)x^n$$

$$a_n = \frac{1}{2}(8^n + 10^n).$$

3) Use generating functions solve the recurrence relation $a_n - 7a_{n-1} + 6a_{n-2} = 0$, $n \geq 2$, $a_0 = 8$ $a_1 = 6$

Solution:

Let G(x) be the generating function for the sequence $\{a_k\}$,

ie) $G(x) = \sum_{n=0}^{\infty} a_n x^n$

Given equation is $a_n - 7a_{n-1} + 6a_{n-2} = 0$

Multi plying both sides by x^n and taking the summation on both sides, we get

$$a_n x^n - 7a_{n-1} x^n + 6a_{n-2} x^n = 0$$

$$\sum_{n=2}^{\infty} a_n x^n - 7x \sum_{n=2}^{\infty} a_{n-1} x^{n-1} + 6x^2 \sum_{n=2}^{\infty} a_{n-2} x^{n-2} = 0$$

$$\left(G(x) - a_0 - a_1 x\right) - 7x\left(G(x) - a_0\right) + 6x^2 G(x) = 0$$

$$G(x) - 8 - 6x - 7xG(x) + 7x(8) + 6x^2 G(x) = 0$$

$$\left(6x^2 - 7x + 1\right)G(x) = 6x + 8$$

$$G(x) = \frac{6x + 8 - 56x}{6x^2 - 7x + 1}$$

$$= \frac{8 - 50x}{6x^2 - 7x + 1}$$

Using partial fractions the above equation becomes

$$G(x)=\frac{8-50x}{(1-x)(1-6x)}=\frac{A}{1-x}+\frac{B}{1-6x}$$

$$A=\frac{42}{5} \quad B=\frac{-2}{5}$$

$$G(x)=\frac{8-50x}{(1-x)(1-6x)}=\frac{42/5}{1-x}+\frac{-2/5}{1-6x}$$

$$\sum_{n=0}^{\infty}a_n x^n=\frac{42}{5}\sum_{n=0}^{\infty}x^n-\frac{2}{6}(6x)^n$$

$$\boxed{a_n=\frac{42}{5}(1)^n-\frac{2}{5}(6)^n}$$

4) Use generating functions solve the recurrence relation $a_n = 3a_{n-1} + 2$, $n \geq 1$ with $a_0 = 1$

Solution:

Let G(x) be the generating function for the sequence $\{a_k\}$,

ie) $G(x) = \sum_{n=0}^{\infty} a_n x^n$

Given equation is $a_n = 3a_{n-1} + 2$

Multi plying both sides by x^n and taking the summation on both sides, we get

$$\sum_{n=1}^{\infty} a_n x^n = 3\sum_{n=1}^{\infty} a_{n-1} x^n + 2\sum_{n=1}^{\infty} x^n$$

$$= 3x\sum_{n=1}^{\infty} a_{n-1}x^{n-1} + 2x\sum_{n=1}^{\infty} x^{n-1}$$

$$G(x) - a_0 = 3x\sum_{n=1}^{\infty} a_{n-1}x^{n-1} + 2x\sum_{n=1}^{\infty} x^{n-1}$$

$$G(x) - 1 = 3x\,G(x) + 2x(1-x)^{-1}$$

$$G(x)[1-3x] = \frac{2x}{1-x} + 1$$

$$= \frac{1+x}{1-x}$$

$$G(x) = \frac{1+x}{(1-x)(1-3x)}$$

Using partial fractions, we get

$$G(x) = \frac{2}{(1-3x)} - \frac{1}{(1-x)}$$

$$G(x) = \frac{2}{(1-3x)} - \frac{1}{(1-x)}$$

$$\sum_{n=0}^{\infty} a_n x^n = 2\sum_{n=0}^{\infty}(3x^n) - \sum_{n=0}^{\infty} x^n$$

$$\Rightarrow a_n = 2(3^n) - 1$$

5) Use generating functions solve the recurrence relation $a_n = 4a_{n-1} - 4a_{n-2} + 4^n$, $n \geq 2$ given that $a_0 = 2$ and $a_1 = 8$

Solution:

Let G(x) be the generating function for the sequence $\{a_k\}$,

ie) $G(x) = \sum_{n=0}^{\infty} a_n x^n$

Given equation is $a_n = 4a_{n-1} - 4a_{n-2} + 4^n$

Multi plying both sides by x^n and taking the summation on both sides, we get

$$\sum_{n=2}^{\infty} a_n x^n = 4\sum_{n=2}^{\infty} a_{n-1} x^n - 4\sum_{n=2}^{\infty} a_{n-2} x^n + \sum_{n=2}^{\infty} 4^n x^n$$

$$\sum_{n=2}^{\infty} a_n x^n = 4x\sum_{n=2}^{\infty} a_{n-1} x^{n-1} - 4x^2\sum_{n=2}^{\infty} a_{n-2} x^{n-2} + \sum_{n=2}^{\infty} (4x)^n$$

$$\Rightarrow [G(x) - a_0 - a_1 x] - 4x[G(x) - a_0] + 4x^2 G(x) - (1-4x)^{-1} + 1 + 4x = 0$$

$$\Rightarrow [G(x) - 2 - 8x] - 4x[G(x) - 2] + 4x^2 G(x) - (1-4x)^{-1} + 1 + 4x = 0$$

$$\Rightarrow G(x)\left[1 - 4x + 4x^2\right] - 1 + 4x = \frac{1}{1-4x}$$

$$\Rightarrow G(x)\left[1 - 4x + 4x^2\right] = \frac{1}{1-4x} + 1 - 4x$$

$$= \frac{(1-4x)^2}{1-4x}$$

$$G(x) = \frac{(1-4x)^2}{(1-4x)(1-4x+4x^2)}$$

$$= \frac{(1-4x)^2}{(1-4x)(1-2x)^2}$$

Using partial fractions, we get,

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} a_{n-1} x^n + \sum_{n=1}^{\infty} 2(n-1) x^n$$

$$\sum_{n=1}^{\infty} a_n x^n = x\sum_{n=1}^{\infty} a_{n-1} x^{n-1} + 2\sum_{n=1}^{\infty} n x^n - 2\sum_{n=1}^{\infty} x^n$$

$$ie) \sum_{n=0}^{\infty} a_n x^n = 2(x+3x^2+6x^3+\ldots\ldots) - 2\sum_{n=0}^{\infty}(n+1)x^n + 5\sum_{n=0}^{\infty} x^n$$

$$ie) \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} n(n+1)x^n - 2\sum_{n=0}^{\infty}(n+1)x^n + 5\sum_{n=0}^{\infty} x^n$$

$$\Rightarrow \quad a_n = n^2 - n + 3$$

6)Use generating function to solve the recurrence relation $S(n+1) - 2S(n) = 4^n$

$S(0) = 1 \ , \ n \geq 0$

Solution.

Rewrite the given equation as $a_{n+1} - 2a_n = 4^n$ with $a_0=1$, $n\geq 0$
Let G(x) be the generating function for the sequence $\{a_k\}$,

ie) $G(x) = \sum_{n=0}^{\infty} a_n x^n$

Given equation is $a_{n+1} - 2a_n = 4^n$

Multi plying both sides by x^n and taking the summation on both sides, we get

$$a_{n+1} - 2a_n = 4^n$$

$$\sum_{n=0}^{\infty} a_{n+1}x^n - \sum_{n=0}^{\infty} 2a_n x^n = \sum_{n=0}^{\infty} 4^n x^n$$

$$\frac{1}{x}\sum_{n=0}^{\infty} a_{n+1}x^{n+1} - 2\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (4x)^n$$

$$\frac{1}{x}(G(x) - a_0) - 2G(x) = \frac{1}{1-4x}$$

$$G(x)[\frac{1}{x} - 2\ ] = \frac{1}{1-4x} + \frac{1}{x}$$

$$G(x)[\frac{1-2x}{x}\ ] = \frac{1-3x}{(1-4x)x}$$

$$G(x) = \frac{1-3x}{(1-4x)(1-2x)}$$

Using partial fractions the above equation becomes

$$\frac{1-3x}{(1-4x)(1-2x)} = \frac{A}{1-4x} + \frac{B}{1-2x}$$

$$solving\ this\ \ weget\ \ A = \frac{1}{2} \quad B = \frac{1}{2}$$

$$\Rightarrow G(x) = \frac{1}{2}\frac{1}{1-4x} + \frac{1}{2}\frac{1}{1-2x}$$

$$ie) \sum_{n=0}^{\infty} a_n x^n = \frac{1}{2}\sum_{n=0}^{\infty}(4x)^n + \frac{1}{2}\sum_{n=0}^{\infty}(2x)^n$$

$$\Rightarrow \ a_n = \frac{1}{2}4^n + \frac{1}{2}2^n$$

$$= 2.4^{n-1} + 2^{n-1}$$

7) Use generating functions solve the recurrence relation $a_n = 4a_{n-1} + 3n\,2^n$, $n \geq 1$ with $a_0 = 4$.

Solution:

Let G(x) be the generating function for the sequence $\{a_k\}$,

ie) $G(x) = \sum_{n=0}^{\infty} a_n x^n$

Given equation is $a_n = 4a_{n-1} + 3n\,2^n$
Multi plying both sides by x^n and taking the summation on both sides, we get

$$a_n = 4\, a_{n-1} + 3n\, 2^n$$

$$\sum_{n=1}^{\infty} a_n x^n = 4\sum_{n=1}^{\infty} a_{n-1} x^n + 3\sum_{n=1}^{\infty} n 2^n x^n$$

$$\sum_{n=1}^{\infty} a_n x^n = 4x\sum_{n=1}^{\infty} a_{n-1} x^{n-1} + 3\sum_{n=1}^{\infty} n(2x)^n$$

$$\Rightarrow G(x) - a_0 = 4xG(x) + 3.2x\sum_{n=1}^{\infty} n(2x)^{n-1}$$

$$\Rightarrow G(x)[1-4x] = 4 + 6x(1-2x)^{-2}$$

$$G(x) = \frac{4}{1-4x} + \frac{6x}{(1-4x)(1-2x)^2}$$

Using partial fractions the above equation becomes,

$$G(x) = \frac{10}{1-4x} - \frac{3}{1-2x} - \frac{3}{(1-2x)^2}$$

$$\sum_{n=0}^{\infty} a_n x^n = 10\sum_{n=0}^{\infty}(4x)^n - 3\sum_{n=0}^{\infty}(2x)^n - 3\sum_{n=0}^{\infty}(n+1)(2x)^n$$

$$\Rightarrow a_n = 10\,4^n - 3.2^n - 3(n+1)2^n$$

$$= 10\,4^n - (3n+6)2^n$$

8) Use generating functions solve the recurrence relation $a_{n+2} - 4a_n = 9n^2$, $n \geq 0$.

Solution:

Let G(x) be the generating function for the sequence $\{a_k\}$,

Ie) $G(x) = \sum_{n=0}^{\infty} a_n x^n$

Given equation is $a_{n+2} - 4a_n = 9n^2$

Multi plying both sides by x^n and taking the summation on both sides, we get

$$a_{n+2} - 4\,a_n = 9n^2$$

$$\sum_{n=0}^{\infty} a_{n+2}x^n - 4\sum_{n=0}^{\infty} a_n x^n = 9\sum_{n=0}^{\infty} n^2 x^n$$

$$\frac{1}{x^2}\sum_{n=0}^{\infty} a_{n+2}x^{n+2} - 4\sum_{n=0}^{\infty} a_n x^n = 9\sum_{n=0}^{\infty}[n(n+1)-n]x^n$$

$$\frac{1}{x^2}[G(x)-a_0-a_1x]-4G(x)=\frac{9.2x}{(1-x)^3}-\frac{9x}{(1-x)^2}$$

$$G(x)[\frac{1}{x^2}-4] = \frac{a_0+a_1x}{x^2}+\frac{18x}{(1-x)^3}-\frac{9x}{(1-x)^2}$$

$$= \frac{a_0+a_1x}{(1-2x)(1+2x)}+\frac{9x^3+9x^4}{(1-x)^3(1-2x)(1+2x)}$$

By splitting this according to the partial fraction technique, we get

$$=\frac{a_0+a_1x}{(1-2x)}+\frac{9x^3+9x^4}{(1-x)^3(1-2x)(1+2x)}$$

$$=\frac{A}{(1+2x)}+\frac{B}{(1-2x)}-\frac{17}{3(1-x)}+\frac{5}{(1-x)^2}-\frac{6}{(1-x)^3}-\frac{1}{12(1+2x)}+\frac{27}{4(1-2x)}$$

$$=\frac{k_1}{(1+2x)}+\frac{k_2}{(1-2x)}-\frac{17}{3(1-x)}+\frac{5}{(1-x)^2}-\frac{6}{(1-x)^3}$$

$$where\ k_1=A-\frac{1}{2}\ and\ k_2=B-\frac{1}{12}$$

$$\sum_{n=0}^{\infty}a_nx^n=k_1\sum_{n=0}^{\infty}(-2)^nx^n+k_2\sum_{n=0}^{\infty}2^nx^n-\frac{17}{3}\sum_{n=0}^{\infty}x^n+5\sum_{n=0}^{\infty}(n+1)x^n-6\sum_{n=0}^{\infty}(n+1)(n+2)x^n$$

$$a_n=k_1(-2)^n+k_2 2^n-\frac{17}{3}+5(n+1)-6(n+1)(n+2)$$