Sri SAI RAM ENGINEERING COLLEGE
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Sairam INSTITUTIONS

SAIRAM DIGITAL RESOURCES

YEAR II | SEM III

MA8351

DISCRETE MATHEMATICS (COMMON TO CSE & IT)

Unit 1
LOGIC AND PROOFS

1.4 PREDICATES AND QUANTIFIERS

SCIENCE & HUMANITIES

PREDICATES AND QUANTIFIERS

PREDICATES:

Let us consider the two statements i) John is a bachelor

ii) smith is a bachelor

Denote the predicate " is a bachelor " by symbolically by the predicate letter

B john by j and smith by s the statements (i) and (ii) can be written as B(j) and B(s) respectively.

In general any state of the type p is Q where Q is the predicate and p is the subject can be denoted by Q (p)

Consider the statement “x >2”.Here there are two parts, One is x and the Second is greater than 2. 1st part represents subject and 2nd part represents the predicate of the statement.

We denote the statement x >2 by P(x).Where P denotes predicate “greater than 2”, and x is a variable.

The statement P(x) is also said to be the value of the propositional function P at x, once the value has been assigned to the variable x, the statement P(x),becomes proposition and has a truth value.

Example: 1

P(x): x is greater than 2

P(4): 4 is greater than 2

Hence Truth value of P(x) is true at x=4.

Consider P(1). We know that 1 is less than 2. Hence P(x) is false

Example: 2

Let R(x ,y ,z) denotes the statement that “x + y = z”, when the values are assigned to the variables x, y, z.

R(1,2,3) has the truth value as True.

R(0,0,1) has the truth value as False.

In general , statement involving variables $x_1, x_2, \ldots.. \; x_n$ can be denoted by P($x_1, x_2, \ldots.., \; x_n$) then P is called n place predicate or n- ary predicate.

QUANTIFIER:

When the variables in a propositional function are assigned values, the resulting statement becomes a proposition with a certain truth value. However, there is another important way, called quantification, to create a proposition from a propositional function. Quantification expresses the extent to which a predicate is true over a range of elements. In English, the words *all*, *some*, *many*, *none*, and *few* are used in quantifications.

Area of logic which deals with predicate and quantifiers is called predicate calculus or predicate logic.

There are two type of Quantifiers

1.Universal quantifiers.

2. Existential quantifiers.

Universal Quantifier:

Which tells us that predicate is true that every element is under consideration.

The universal quantification of P(x) is the statement of “P(x) for all values of x in the domain”. The statement $\forall x\, P(x)$ denotes the universal quantification of P(x) . We read $\forall x\, P(x)$ as "for all x P(x)” or “for every x P(x)”.

Many mathematical statements assert that a property is true for all values of a variable in a particular domain, called the domain of discourse (or the universe of discourse), often just referred to as the domain.

Existential quantifier:

Which tells us that is one at more element under consideration for which predicate is true. The existential qualifiers of P(x) is a proposition, “There exist element x in the domain such that P(x). $\exists x\, P(x)$ implies that the symbol is used to denote existential qualifiers.

1.What is the truth value of $\forall x\, P(x)$ where P (x) is the statement x^2 < 10" and the domain consists of positive integers not exceeding 4.

Solution:

Domain = {1,2,3,4},P(x): <10

P(1) is T

P(2) is T

P(3) is T

P(4) is F

$\forall x\, P(x)$ = P(1) ∧ P(2)∧ P(3)∧ P(4) = T∧ T∧ T ∧ F = F

Hence truth value of is false.

2.Let $Q(x)$ be denotes the statement "$x=x+1$" . What is the truth value of quantification. $\exists x\, P(x)$,where domain consists of all real numbers.

Solution:

Given that domain is set of all real numbers and we know that there is no real number satisfying the above statement.

Hence truth value of the quantification is false.

3)Find the truth value of $\exists x\, P(x)$,where P(x) is the statement and the domain consists of the positive integers not exceeding 4.

Solution:

Domain={1,2,3,4}

$$\exists x\, P(x) = P(1) \lor P(2) \lor P(3) \lor P(4)$$
$$= T \lor T \lor T \lor F$$
$$= T$$

4.Which of the following are statements?

i. $(x)(P(x) \vee Q(x)) \wedge R$.

Ii .$(x)(P(x) \wedge Q(x)) \wedge S(x)$

Solution:

i. is not a statement

ii. is a statement

5.Write the negation of the statement $\exists x\ \forall x\ P(x , y)$

Solution:

Given statement is $\exists x\ \forall x\ P(x , y)$

Its Negation is $(\forall x)\ (\exists x)\ \rceil P(x , y)$

5.Give the symbolic form of “Some men are giant”.

Solution:

M(x) : x is a man. G(x) : x is giant

$\exists x\,(M(x) \rightarrow G(x))$

7. Express the statement ,”Some people who trust others are rewarded” in symbolic form.

Solution:

P(x): x is a person T(x): x trust others R(x): x is rewarded

$\exists x P(x) T(x) R(x)$

6. Write in symbolic form “Some men are clever"

Solution:

The above statement can be restated as

“There is an x such that x is a man and x is clever”

We will translate it into symbolic form using Existential quantifier.

Let $M(x)$: x is a man

$C(x)$: x is clever

we write symbolic form us $(\exists x)\left(M(x) \wedge C(x)\right)$

7.Symbolize the expression "x is the father of the mother of y"

Solution:

P(x) : x is a person

F(x, y): x is a father of y

M(x, y): x is a mother of y

We symbolize this as $(\exists z)(P(z) \wedge F(x,z) \wedge M(z,y))$

8. Let G(x,y):x is taller than y Translate the following into formula:

Solution:

For any x and for any y if x is taller than y then it is not true that y is taller than x

Given G(x ,y) : x is taller than y

$\therefore$ G(y ,x) : y is taller than x

$\therefore$ the symbolic form of the given statement is

$$(\forall x)(\forall y)(G(x,y) \rightarrow \neg G(y,x))$$

9. Symbolize the following statement:
"All the world loves a lover"

Solution:

Let P(x) : x is a person
L(x) : x is a lover
R(x,y): x loves y

The given statement can be restated as "For all x if x is the person then for all y , if y is the person and y is a lover, then x loves y"

∴ Its symbolic form:

$$(\forall x)(P(x) \to (\forall y)(P(y) \wedge L(y) \to R(x,y)))$$

10.Let E= {-1,0,1,2} denote the universe of discourse. If p(x,y) : x+y = 1, find the truth value of $(\forall x)(\exists y)\, p(x,y)$.

Solution:

If $x = 1,\ y = 0,\quad p(x, y) = 1 + 0 = 1$

If $x = -1, y = 2,\ p(x, y) = -1 + 2 = 1$

If $x = 0, y = 1,\quad p(x, y) = 0 + 1 = 1$

If $x = 2, y = -1,\ p(x, y) = 2 + (-1) = 1$

∴The truth value of $(\forall x)(\exists y)\, p(x,y)$ is true.

11. Prove that $(\exists x)(P(x) \wedge Q(x)) \Rightarrow (\exists x) P(x) \wedge (\exists x) Q(x)$

Solution:

(1)	{1}	$(\ni x)(P(x) \wedge Q(x))$	*Rule P*
(2)	{1}	$P(y) \wedge Q(y)$	*Rule ES*
(3)	{1}	$P(y)$	*Rule T*
(4)	{1}	$Q(y)$	*Rule T*
(5)	{1}	$(\ni x) P(x)$	*Rule EG*
(6)	{1}	$(\ni x) Q(x)$	*Rule EG*
(7)	{1}	$(\ni x) P(x) \wedge (\ni x) Q(x)$	*Rule T*

12) By using indirect method, show that $(x)(P(x) \vee Q(x)) \Rightarrow (x)P(x) \vee (\exists x)Q(x)$

Solution:

(1)	$\{1\}$	$\neg\left((x)P(x) \vee (\ni x)Q(x)\right)$	$Rule\ P(assumed)$
(2)	$\{1\}$	$(\ni x)\neg P(x) \wedge (\forall x)\neg Q(x)$	$Rule\ T$
(3)	$\{1\}$	$(\ni x)\neg P(x)$	$Rule\ T$
(4)	$\{1\}$	$\neg P(y)$	$Rule\ ES$
(5)	$\{1\}$	$(x)\neg Q(x)$	$Rule\ T$
(6)	$\{1\}$	$\neg Q(y)$	$Rule\ US$
(7)	$\{1\}$	$\neg P(y) \wedge \neg Q(y)$	$Rule\ T$
(8)	$\{1\}$	$\neg\left(P(y) \vee Q(y)\right)$	$Rule\ T$

(9)	{9}	$(x)(P(x) \vee Q(x))$	*Rule P*
(10)	{10}	$P(y) \vee Q(y)$	*Rule US*
(11)	{1, 10}	$[P(y) \vee Q(y)] \wedge \neg [P(y) \vee Q(y)]$	*Rule T (Contradiction)*

13).Using Rule CP, obtain the following implication. $(\forall x)(P(x)\rightarrow Q(x)),(\forall x)(R(x)\rightarrow \neg Q(x))\Rightarrow(\forall x)(R(x)\rightarrow \neg P(x))$

Solution:

(1)	$\{1\}$	$(\forall x)(R(x)\rightarrow \neg Q(x))$	*Rule P*
(2)	$\{1\}$	$R(y)\rightarrow \neg Q(y)$	*Rule US*
(3)	$\{2\}$	$R(y)$	*Assumed*
(4)	$\{1,2\}$	$\neg Q(y)$	*Rule T*
(5)	$\{5\}$	$(\forall x)(P(x)\rightarrow Q(x))$	*Rule P*
(6)	$\{5\}$	$P(y)\rightarrow Q(y)$	*Rule US*
(7)	$\{1,2,5\}$	$\neg P(y)$	*Rule T*
(8)	$\{1,2,5\}$	$R(y)\rightarrow \neg P(y)$	*Rule CP*
(9)	$\{1,5\}$	$(\forall x)(R(x)\rightarrow \neg P(x))$	*Rule UG*

14.Show that $(\forall x)(P(x) \vee Q(x)) \Rightarrow (\forall x) P(x) \vee (\exists x) Q(x)$

Solution: **We Use Indirect Proof**

1	$\neg[(\forall x)P(x) \vee (\ni x)Q(x)]$	Assumed premise
2	$\neg(\forall x)P(x) \wedge \neg(\ni x)Q(x)$	Rule T (1)
3	$(\ni x)\neg P(x) \wedge (\forall x)\neg Q(x)$	Rule T (2)
4	$(\ni x)\neg P(x)$	Rule T (3) Simplification
5	$\neg P(y)$	Rule ES (4)
6	$(\forall x)\neg Q(x)$	Rule T (3) Simplification
7	$\neg Q(y)$	Rule US (6)
8	$(\forall x)[P(x) \vee Q(x)]$	Rule P

9	$P(y) \vee Q(y)$	Rule US (8)
10	$\neg P(y) \rightarrow Q(y)$	Rule T (9)
11	$P(y)$	Rule T (7, 10)
12	$P(y) \wedge \neg P(y)$	Rule T (11, 5)
13	F	Rule T (12)

15).Show that the premises" A student in this class has not read the book and "Every one in the class passed the first exam" imply the conclusion "some one who passed the first exam has not read the book.

Solution:

P(x) : x is in the class
Q(x) : x has read the book.
R(x) : Passed the first exam

$$(\ni x)\left(P(x)\wedge\neg Q(x)\right),(\forall x)\left(P(x)\rightarrow R(x)\right)\Rightarrow(\ni x)\left(R(x)\wedge\neg Q(x)\right)$$

(1)	$\{1\}$	$(\exists x) P(x) \wedge \neg Q(x)$	Rule P
(2)	$\{1\}$	$P(y) \wedge \neg Q(y)$	Rule ES
(3)	$\{1\}$	$P(y)$	Rule T
(4)	$\{4\}$	$(\forall x)(P(x) \rightarrow R(x))$	Rule P
(5)	$\{4\}$	$P(y) \rightarrow R(y)$	Rule US
(6)	$\{1,4\}$	$R(y)$	Rule T
(7)	$\{1\}$	$\neg Q(y)$	Rule T
(8)	$\{1,4\}$	$R(y) \wedge \neg Q(y)$	Rule T
(9)	$\{1,4\}$	$(\exists x)(R(x) \wedge \neg Q(x))$	Rule EG

16. Every living thing is a plant or an animal John's gold fish is alive and it is not a plant. All animals have hearts. Therefore John's gold fish has a heart.

Solution:

P(x) : is a plant
A(x) : is an animal
H(x) : has a heart
g : John's gold fish

$$(\forall x)\left(P(x)\vee A(x)\right),\neg P(g),(\forall x)\left(A(x)\rightarrow H(x)\right)\Rightarrow H(g)$$

(1)	$\{1\}$	$(\forall x)(P(x) \vee A(x))$	$Rule\ P$
(2)	$\{1\}$	$P(g) \vee A(g)$	$Rule\ US$
(3)	$\{3\}$	$\neg P(g)$	$Rule\ P$
(4)	$\{1,3\}$	$A(g)$	$Rule\ T$
(5)	$\{5\}$	$(\forall x)(A(x) \rightarrow H(x))$	$Rule\ P$
(6)	$\{5\}$	$A(g) \rightarrow H(g)$	$Rule\ US$
(7)	$\{1,3,5\}$	$H(g)$	$Rule\ T$

17. Using the rule CP or otherwise show the following implications

$(\exists x)P(x) \rightarrow (x)Q(x) \Rightarrow (x)(P(x) \rightarrow Q(x))$

Solution:

(1)	{1}	$(\ni x)P(x) \rightarrow (x)Q(x)$	*Rule P*
(2)	{1}	$\neg(\ni x)P(x) \vee (x)Q(x)$	*Rule T*
(3)	{1}	$(x)\neg P(x) \vee (x)Q(x)$	*Rule T*
(4)	{1}	$(x)(\neg P(x) \vee Q(x))$	*Rule T*
(5)	{1}	$(x)(P(x) \rightarrow Q(x))$	*Rule T*

18. All integers are rational numbers. Some integers are powers of 2. Therefore, some rational numbers are powers of 2.

Solution:

Let P(x) : is an integer
Q(x) : is a rational number
R(x) : is a power of 2.

$$(\forall x)\big(P(x)\rightarrow R(x)\big),(\ni x)\big(P(x)\wedge S(x)\big)\Rightarrow(\ni x)\big(R(x)\wedge S(x)\big)$$

(1)	$\{1\}$	$(\forall x)(P(x) \to R(x))$	Rule P
(2)	$\{1\}$	$P(x) \to R(x)$	Rule US
(3)	$\{3\}$	$(\ni x)(P(x) \wedge S(x))$	Rule P
(4)	$\{3\}$	$P(x) \wedge S(x)$	Rule ES
(5)	$\{3\}$	$P(x)$	Rule T
(6)	$\{3\}$	$S(x)$	Rule T
(7)	$\{1,3\}$	$R(x)$	Rule T
(8)	$\{1,3\}$	$R(x) \wedge S(x)$	Rule T
(9)	$\{1,3\}$	$(\ni x)(R(x) \wedge S(x))$	Rule EG

Free and Bound Variables:

When the quantifiers is used on the variable x, we say that the occurrence of the variable is bounded. An occurrence of the variable is not bounded by the quantifier or a set, particular value set to be free. The part of the logical expression for which the quantifiers applied is called the “scope ”of the quantifier.

Example:

1) $\exists x\,(x+y=1)$**. In this statement x is bounded variable ,y is free variable.**

Scope of $\exists x$ is $x+y=1$ **.**

2) **Statement:** $\exists x(P(x) \wedge Q(x)) \vee (\forall x\, R(x))$

x is bounded variable.

Scope of $\exists x$, $P(x) \wedge Q(x)$

Scope of $\forall x$, $R(x)$

NEGATING QUANTIFIED EXPRESSION

Example:

1)"Every Student in your class has taken a course in Calculus."

Find the negation.

Domain = The students in your class

P(x) , x is the student in the class studying Calculus.

Hence given statement is $\forall x \ P(x)$

Negation is given by

$\sim(\forall x) Px = \exists x \sim P(x)$

DE MORGAN'S LAW FOR QUANTIFIERS

$\sim(\exists x)\, P(x) = (\forall x) \sim P(x)$

$\sim(\forall x)\, P(x) = (\exists x) \sim P(x)$

Example:

What is the negation of the following statements.
1."There is an honest politician".

Domain = Honest politician
H(x) : x is a honest politician,
Hence given statement is $(\exists x) H(x)$

Negation:

$\sim(\exists x)\, H(x) = (\forall x) \sim H(x)$

2. “All Americans eats Cheese burgers”.

Domain= All Americans.

P(x): x is all Americans eat cheese burgers.

Given statement is $(\forall x)P(x)$

Negation :$\sim(\forall x)P(x)=(\exists x)\sim P(x)$

3.**Express the statement “Every student in the class has studied DPSD”.**

Domain= The student in the class.

P(x): x is a student studied DPSD.

Given statement is $(\forall x)P(x)$

Aliter:

By means of changing slightly the domain as set of peoples then

P(x): x is a student in the class.

Q(x): x has studied DPSD.

Now the statement is $(\forall x)\,(P(x)\rightarrow Q(x))$

4. Express the following statements.

(i) "Some student in the class has visited Mexico".

Domain = Set of people

P(x): x is a person in this class.

Q(x): x has visited Mexico

Given statement is $(\exists x)\,(P(x) \wedge Q(x))$

(ii) "Every student in the class has visited either Mexico or Canada".

Domain = Set of people.

P(x): x is a person in the class.

Q(x): x has visited Mexico.

R(x): x has visited Canada.

Given statement is $(\forall x)(P(x) \rightarrow (Q(x) \vee R(x).)$