Sri SAI RAM ENGINEERING COLLEGE
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Sairam INSTITUTIONS

SAIRAM DIGITAL RESOURCES

YEAR II

SEM III

MA8351

DISCRETE MATHEMATICS (COMMON TO CSE & IT)

UNIT-V

LATTICES AND BOOLEAN ALGEBRA

5.4 DIRECT PRODUCT, HOMOMORPHISM

SCIENCE & HUMANITIES

LATTICE HOMOMORPHISM

Let $(L_1,\wedge,\vee)$ and $(L_2,*,\oplus)$ be two given Lattices. A mapping $f: L_1 \to L_2$ is called Lattice Homomorphism if for every $a, b \in L_1$

(i) $f(a \wedge b) = f(a) * f(b)$

(ii) $f(a \vee b) = f(a) \oplus f(b)$

LATTICE ISOMORPHISM:

A homomorphism which is 1-1 and onto is called the lattice isomorphism.

DIRECT PRODUCT OF LATTICE:

Let $(L,*,\oplus)$ and $(S,\wedge,\vee)$ be two given Lattices. The algebraic system $(L \times S, \circ, +)$ in which the binary operations on $L \times S$ are such that for any (a_1, b_1) and (a_2, b_2) in $L \times S$

(i) $(a_1, b_1)\circ(a_2, b_2) = (a_1 * a_2, b_1 \wedge b_2)$

(ii) $(a_1, b_1) + (a_2, b_2) = (a_1 \oplus a_2, b_1 \vee b_2)$

Is called the Direct product of the lattice $(L,*,\oplus)$ and $(S,\wedge,\vee)$.

THEOREM: If $(L_1, \wedge, \vee)$ and $(L_2, *, \oplus)$ are two distributive lattices then $(L \times S, \circ, +)$ is also, a lattice.

PROOF: To prove $\circ$ *and* $+$ satisfies commutative, associative & absorption laws.

(i) $(x_1, y_1) + (x_2, y_2) = (x_1 \oplus x_2, y_1 \vee y_2) = (x_2 \oplus x_1, y_2 \vee y_1)$

$$= (x_2, y_2) + (x_1, y_1)$$

$$(x_1, y_1)^\circ(x_2, y_2) = (x_1 * x_2, y_1 \wedge y_2) = (x_2 * x_1, y_2 \wedge y_1)$$

$$= (x_2, y_2)^\circ(x_1, y_1)$$

$\circ$ *and* $+$ satisfies commutative laws.

(ii) $(x_1, y_1) + [(x_2, y_2) + (x_3, y_3)] = (x_1, y_1) + (x_2 \oplus x_3, y_2 \vee y_3)$

$$= [x_1 \oplus (x_2 \oplus x_3), y_1 \vee (y_2 \vee y_3)]$$

$$= [(x_1 \oplus x_2) \oplus x_3, (y_1 \vee y_2) \vee y_3]$$

$$= [(x_1, y_1) + (x_2, y_2)] + (x_3, y_3)$$

Similarly, $(x_1, y_1)^\circ[(x_2, y_2)^\circ(x_3, y_3)] = [(x_1, y_1)^\circ(x_2, y_2)]^\circ(x_3, y_3)$

$^\circ$ and $+$ satisfies associative laws.

(iii) $(x_1, y_1) + [(x_1, y_1)^\circ(x_2, y_2)] = (x_1, y_1) + [x_1 * x_2, y_1 \wedge y_2]$

$= (x_1 \oplus (x_1 * x_2), y_1 \vee (y_1 \wedge y_2)$

$= (x_1, y_1)$

Similarly, $(x_1, y_1)^\circ[(x_1, y_1) + (x_2, y_2)] = (x_1, y_1)$

$^\circ$ and $+$ satisfies absorption laws.

Hence $(L \times S, ^\circ, +)$ is a Lattice.

THEOREM: (DEMORGAN'S LAW OF LATTICES)

If $(\boldsymbol{L}, \wedge, \vee, \mathbf{0}, \mathbf{1})$ is a complemented lattice, the prove that

(i) $(a \wedge b)' = a' \vee b'$ (ii) $(a \vee b)' = a' \wedge b'$

PROOF:

(i) To prove $(a \wedge b) \wedge (a' \vee b') = 0$ and $(a \wedge b) \vee (a' \vee b') = 1$

$$(a \wedge b) \wedge (a' \vee b') = \left((a \wedge b) \wedge a'\right) \vee \left((a \wedge b) \wedge b'\right) = (0 \wedge b) \vee (a \wedge 0)$$

$$= 0 \vee 0 = 0$$

$$(a \wedge b) \vee (a' \vee b') = \left(a \vee (a' \vee b')\right) \wedge \left(b \vee (a' \vee b')\right) = (1 \vee b) \wedge (a' \vee 1)$$

$$= 1 \wedge 1 = 1$$

(ii) To prove $(a \vee b) \wedge (a' \wedge b') = 0$ and $(a \vee b) \vee (a' \wedge b') = 1$

$$(a \vee b) \vee (a' \wedge b') = \left((a \vee b) \vee a'\right) \wedge \left((a \vee b) \vee b'\right)$$

$$= (1 \vee b) \wedge (a \vee 1) = 1 \wedge 1 = 1$$

$$(a \vee b) \wedge (a' \wedge b') = \left(a \wedge (a' \wedge b')\right) \vee \left(b \wedge (a' \wedge b')\right)$$

$$= (0 \wedge b') \vee (0 \wedge a') = 0 \vee 0 = 0$$

THEOREM:

In a complemented distributive lattice, complement is unique.

PROOF: Let us assume x & y are two compliments of a.

x is complement of $a \Longrightarrow a \wedge x = 0, a \vee x = 0.$

y is complement of $a \Longrightarrow a \wedge y = 0, a \vee y = 1.$

Now $x = x \vee 0 = x \vee (a \wedge y) = (x \vee a) \wedge (x \vee y) = 1 \wedge (x \vee y) = x \vee y$.

$$y = y \vee 0 = y \vee (a \wedge x) = (y \vee a) \wedge (y \vee x) = 1 \wedge (y \vee x) = y \vee x.$$

Combining we have $x = x \vee y = y \vee x = y \Longrightarrow x = y$

Therefore, the complement is unique.

PROBLEM:

Show that in a distributive and complemented lattice,

$$a \leq b \Leftrightarrow a \wedge b' = 0 \Leftrightarrow a' \vee b = 1 \Leftrightarrow b' \leq a'$$

SOLUTION: (i) To prove $a \leq b \Longrightarrow a \wedge b' = 0$

$$a \leq b \Longrightarrow a \wedge b = a \,\&\, a \vee b = b$$

consider $a \wedge b' = \left((a \wedge b) \wedge b'\right) = (a \wedge b \wedge b') = a \wedge 0 = 0$

(ii) To prove $a \wedge b' = 0 \Longrightarrow a' \vee b = 1$

Let $a \wedge b' = 0$. Take complement on both sides, $(a \wedge b')' = (0)' \Longrightarrow a' \vee b = 1$

(iii) To prove $a' \vee b = 1 \Longrightarrow b' \leq a'$

Let $a' \vee b = 1 \Longrightarrow (a' \vee b) \wedge b' = 1 \wedge b' \Longrightarrow (a' \wedge b') \vee (b \wedge b') = b'$

$$\Longrightarrow (a' \wedge b') \vee 0 = b' \Longrightarrow a' \wedge b' = b'$$

$$\Longrightarrow b' \leq a'$$

(iv) To prove $b' \leq a' \Longrightarrow a \leq b$

Let $b' \leq a' \Longrightarrow a' \wedge b' = b'$. Taking complement on both sides

$$(a' \wedge b')' = (b')' \Longrightarrow a \vee b = b \Longrightarrow b \geq a \Longrightarrow a \leq b.$$

PROBLEM

If S_{42} is the set of all divisors of 42 and D is the relation "divisor of" on S_{42},

prove that $\{S_{42}, D\}$ is a complimented lattice.

SOLUTION: $S_{42} = \{1,2,3,6,7,14,21,24\}$

Here the least element is 1 and greatest element is 42.

LCM $\{1,42\} = 1 \vee 42 = 42$ and GCD $\{1,42\} = 1 \wedge 42 = 1$

Therefore, complement of 1 is 42.

since $2 \wedge 21 = 2$ *and* $2 \vee 21 = 21 \Rightarrow (2)' = 21.$

Similarly, complement of 3 and 6 are 14 and 7 respectively. Therefore, every element of S_{42} has complement. $\{S_{42}, D\}$ is a complimented lattice.