Sri SAI RAM ENGINEERING COLLEGE
INSTITUTE OF TECHNOLOGY
West Tambaram, Chennai - 44

Sairam INSTITUTIONS

SAIRAM DIGITAL RESOURCES

YEAR II

SEM IV

MA8391

PROBABILITY AND STATISTICS (IT)

UNIT I

PROBABILITY AND RANDOM VARIABLES

1.6 CONTINUOUS DISTRIBUTION – UNIFORM , EXPONENTIAL , NORMAL

SCIENCE & HUMANITIES

Uniform Distribution

Let X be a uniform distribution defined in the interval (a, b) then its probability density function is of the form $f(x) = \frac{1}{b-a}, a < x < b$.

Moment generating function and mean, variance of uniform distribution where $X \approx U(a,b)$.

Here X follows uniform distribution in (a , b) and hence $f(x) = \frac{1}{b-a}, a < x < b$

The moment generating function is given by

$$M_X(t) = E[e^{tx}] \quad = \int_a^b e^{tx} f(x)dx$$

$$= \frac{1}{b-a}\int_a^b e^{tx}dx$$

$$= \frac{1}{b-a}\left[\frac{e^{tx}}{t}\right]_a^b$$

$$= \frac{1}{b-a}\left[\frac{e^{tb}}{t} - \frac{e^{ta}}{t}\right]$$

$$E[X] = \int_0^1 xf(x)dx \qquad\qquad E[X^2] = \int_0^1 xf(x)dx$$

$$= \frac{1}{b-a}\int_a^b xdx \qquad\qquad = \frac{1}{b-a}\int_a^b x^2dx$$

$$= \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b \qquad\qquad = \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b$$

$$= \frac{1}{b-a}\left[\frac{b^2-a^2}{2}\right] \qquad\qquad = \frac{1}{b-a}\left[\frac{b^3-a^3}{2}\right]$$

$$= \frac{a+b}{2} \qquad\qquad = \frac{1}{b-a}\left[\frac{(b-a)\left(b^2+ab+a^2\right)}{3}\right]$$

$$Var(X) = E[X^2] - [E(X)]^2 = \frac{a^2+b^2+ab}{3} - \frac{(a+b)^2}{4} = \frac{(a-b)^2}{12}$$

1.Find the value of ‘a’ if X follows uniform distribution in the interval (a , 9) and $P[3 < X < 5] = \frac{2}{7}$.

Solution : Given X follows uniform distribution in the interval (a, 9) and hence

$$f(x) = \frac{1}{9-a}, a < x < 9$$

Also given that

$$P[3 < X < 5] = \frac{2}{7}$$

$$\int_3^5 f(x)dx = \frac{2}{7}$$

$$\int_3^5 \frac{1}{9-a} dx = \frac{2}{7}$$

$$\frac{1}{9-a}[x]_3^5 = \frac{2}{7}$$

$$\frac{2}{9-a} = \frac{2}{7} \quad \Rightarrow \quad 14 = 18 - 2a \quad \Rightarrow \quad 2a = 4 \therefore a = 2$$

2. Find the m.g.f of uniform distribution in the interval (0,1). Also find the mean, variance of it.

Solution: Here X follows uniform distribution in (0 , 1) and hence

$$f(x) = \frac{1}{1-0}, 0 < x < 1$$

The moment generating function is given by

$$M_X(t) = E[e^{tx}]$$

$$= \int_0^1 e^{tx} f(x) dx$$

$$= \frac{1}{1} \int_0^1 e^{tx} dx$$

$$= \left[\frac{e^{tx}}{t}\right]_0^1$$

$$= \left[\frac{e^t}{t} - \frac{1}{t}\right]$$

$$E[X] = \int_0^1 xf(x)dx$$

$$= \int_0^1 xdx$$

$$= \left[\frac{x^2}{2}\right]_0^1$$

$$= \frac{1}{2}$$

$$E[X^2] = \int_0^1 xf(x)dx$$

$$= \int_0^1 x^2dx$$

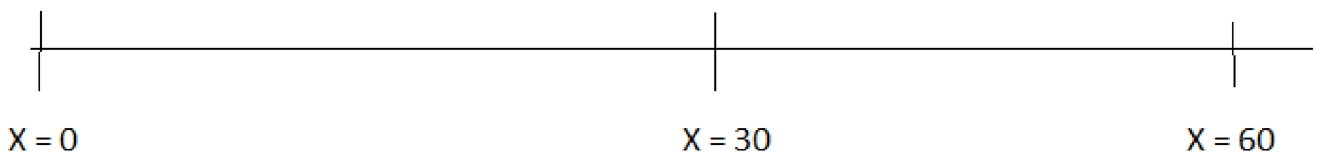
$$= \left[\frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{3}$$

$$Var(X) = E[X^2] - [E(X)]^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

3. Electric trains in a route run every half an hour. Find the probability that passenger entering the station will have to wait (i) at least 20 minutes (ii) less than 10 minutes.

Solution : Electric trains in a route run every half an hour. Therefore, it follows uniform distribution with pdf $f(x) = \frac{1}{30}, 0 < x < 30$



Probability for a passenger to wait at least 20 minutes. It happens if he arrives in between 0 to 10 minutes.

$$P[0 < X < 10] = \int_0^{10} f(x)dx \quad = \frac{1}{30}\int_0^{10} dx \quad = \frac{1}{30}[x]_0^{10} \quad = \frac{10}{30} \quad = \frac{1}{3}$$

Probability for a passenger to wait less than 10 minutes. It happens if he arrives in between 20 to 30 minutes.

$$P[20 < X < 30] = \int_{20}^{30} f(x)dx \quad = \frac{1}{30}\int_{20}^{30} dx \quad = \frac{1}{30}[x]_{20}^{30} \quad = \frac{10}{30} \quad = \frac{1}{3}.$$

4. A random variable X has an uniform over the interval (– 3 , 3). Compute

$i)P[X < 2]$ $ii)P[|X| < 2]$ $iii)$ Find k such that $P[X > k] = \frac{1}{3}$

Solution: Here X follows uniform in the interval (– 3, 3) and hence

$$f(x) = \frac{1}{3+3}, -3 < x < 3$$

$$P[X < 2] = P[-3 < X < 2] = \int_{-3}^{2} f(x)dx \quad = \frac{1}{6}\int_{-3}^{2} dx \quad = \frac{1}{6}[x]_{-3}^{2} \quad = \frac{5}{6}$$

$$P[|X| < 2] = P[-2 < X < 2] = \int_{-2}^{2} f(x)dx \quad = \frac{1}{6}\int_{-2}^{2} dx \quad = \frac{1}{6}[x]_{-2}^{2} \quad = \frac{4}{6} \quad = \frac{2}{3}$$

Given $P[X > k] = \frac{1}{3}$

$$P[k < x < 3] = \frac{1}{3}$$

$$\int_k^3 f(x)dx = \frac{1}{3}$$

$$\frac{1}{6}[x]_k^3 = \frac{1}{3}$$

$$\frac{3-k}{6} = \frac{1}{3}$$

$$k = 1$$

5. Let X be a uniformly distributed random variable with mean 1 and variance $\frac{4}{3}$. Find $P[X < 0]$.

Solution: We know that, if X follows uniform distribution in (a, b), then

$$Mean = \frac{a+b}{2}, \quad Var = \frac{(b-a)^2}{12}$$

Given mean = 1 and Variance = 4/3

Therefore $1 = \frac{a+b}{2}$ and $\frac{4}{3} = \frac{(b-a)^2}{12}$

i.e. $a + b = 2 \ldots\ldots (i)$ and $(b-a)^2 = 16 i.e. b - a = 4 \ldots\ldots\ldots\ldots (ii)$

Solving (i) and (ii), we get $a = -1$ and $b = 3$

Therefore $f(x) = \frac{1}{b-a} = \frac{1}{4}, -1 < x < 3$

$P[X < 0] = P[-1 < X < 0] \quad = \int_{-1}^{0} f(x)dx \quad = \frac{1}{4}\int_{-1}^{0} dx \quad = \frac{1}{4}[x]_{-1}^{0} \quad = \frac{1}{4}$

Exponential Distribution

A continuous random variable X is said to have exponential distribution with parameter $\lambda > 0$ if its probability density function is of the form $f(x) = \lambda e^{-\lambda x}, 0 < x < \infty$

Moment generating function, mean, variance of exponential distribution.

The moment generating function of exponential distribution is given by

$$M_X(t) = E[e^{tx}]$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x) dx$$

$$= \lambda \int_0^{\infty} e^{tx} e^{-\lambda x} dx$$

$$= \lambda \int_0^{\infty} e^{-(\lambda - t)x} dx$$

$$= \lambda \left[\frac{e^{-(\lambda-t)x}}{-(\lambda-t)}\right]_0^{\infty}$$

$$= \lambda \left[0 - \frac{1}{-(\lambda-t)}\right] \quad = \frac{\lambda}{(\lambda-t)}$$

$$M'_X(t) = \frac{(\lambda-t)(0)-\lambda(-1)}{(\lambda-t)^2} = \frac{\lambda}{(\lambda-t)^2}$$

$$M''_X(t) = \frac{(\lambda-t)^2(0)-\lambda 2(\lambda-t)(-1)}{(\lambda-t)^4} = \frac{2\lambda}{(\lambda-t)^3}$$

$$Mean E(X) = M_X'(0) = \frac{1}{\lambda}$$

$$E(X^2) = M_X''(0) = \frac{2}{\lambda^2}$$

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

1. State and prove memoryless property of exponential distribution.

proof: If X is exponentially distributed with parameter λ, then for any two positive integer m and n, $P[X > m + n/X > m] = P[X > n]$

$$P[X > n] = P[n < X < \infty]$$

$$= \int_n^{\infty} f(x)dx$$

$$= \lambda \int_n^{\infty} e^{-\lambda x} dx$$

$$= \lambda \left[\frac{e^{-\lambda x}}{-\lambda}\right]_n^{\infty}$$

$$= e^{-n\lambda} ------(1)$$

$$P[X > m + n/X > m] = \frac{P[X>m+n \cap X>m]}{P[X>m]}$$

$$= \frac{P[X>m+n]}{P[X>m]}$$

$$= \frac{e^{-(m+n)}}{e^{-m}}, \quad \text{using } (1)$$

$$= e^{-n} \quad = P[X > n]$$

2. Find $P[X > 10]$ if the probability density function of X is $f(x) = e^{-x}, x > 0$.

Solution: Given $f(x) = e^{-x}, x > 0$.

$$P[X > 10] = P[10 < X < \infty]$$

$$= \int_{10}^{\infty} f(x)dx$$

$$= \int_{10}^{\infty} e^{-x} dx$$

$$= \left[\frac{e^{-x}}{-1}\right]_{10}^{\infty} \quad = e^{-10}$$

3. Suppose that during the rainy season in an island, the length of the shower has an exponential distribution with average 2 minutes. Find the probability that the shower will be there for more than 3 minutes. If the shower has already lasted for 2 minutes, what is the probability that it will last for al least 1 more minute.

Solution : Let X represents the length of the shower in minutes and it follows exponential distribution with mean 2. But exponential distribution has mean $\frac{1}{\lambda}$.

Therefore $\frac{1}{\lambda} = 2 \Rightarrow \lambda = \frac{1}{2}$.

The pdf of exponential distribution is $f(x) = \lambda e^{-\lambda x} = \frac{1}{2} e^{-\frac{1}{2}x}, 0 < x < \infty$

(i) Probability that the shower lasts more than 3 minutes

$$P[X > 3] = P[3 < X < \infty]$$

$$= \int_3^\infty f(x)dx$$

$$= \frac{1}{2}\int_3^\infty e^{-\frac{1}{2}x} dx$$

$$= \frac{1}{2}\left[\frac{e^{-\frac{1}{2}x}}{-\frac{1}{2}}\right]_3^\infty = e^{-\frac{3}{2}}$$

(ii) Probability that the shower will last at least one minute, given that it had lasted 2 minutes

$$P[X > 3/X > 2] = P[X > 1]$$
$$= P[1 < X < \infty]$$
$$= \int_1^{\infty} f(x)dx$$
$$= \frac{1}{2}\int_1^{\infty} e^{-\frac{1}{2}x}dx$$
$$= \frac{1}{2}\left[\frac{e^{-\frac{1}{2}x}}{-\frac{1}{2}}\right]_1^{\infty} = e^{-\frac{1}{2}}$$

4. The life of a lamp (in 1000's of hours) is exponentially distributed with parameter $\lambda = \frac{1}{3}$. Find (i) the probability that the lamp will last longer than its mean life of 3000 hours. (ii) the probability that the lamp will last for another 1000 hours given that it is operating after 2500 hours.

Solution: Let X represents the life of the bulb (in 1000 hours) and it follows exponential distribution with parameter $\lambda = \frac{1}{3}$. Therefore, its average life is 3000 hours.

The pdf of exponential distribution is $f(x) = \lambda e^{-\lambda x} = \frac{1}{3} e^{-\frac{1}{3}x}, 0 < x < \infty$

(i) Probability that the life of the bulb lasts more than its average life3000 hours.

$$P[X > 3] = P[3 < X < \infty]$$
$$= \int_3^{\infty} f(x)dx$$
$$= \frac{1}{3}\int_3^{\infty} e^{-\frac{1}{3}x}dx$$
$$= \frac{1}{3}\left[\frac{e^{-\frac{1}{3}x}}{-\frac{1}{3}}\right]_3^{\infty} = e^{-1}$$

(ii) Probability that the shower will last more than 1000 hours, given that it had already lasted 2500 hours

$$P[X > 3.5/X > 2.5] = P[X > 1] \text{ , by memoryless property}$$
$$= P[1 < X < \infty]$$
$$= \int_1^{\infty} f(x)dx$$
$$= \frac{1}{3}\int_1^{\infty} e^{-\frac{1}{3}x}dx \quad = \frac{1}{3}\left[\frac{e^{-\frac{1}{3}x}}{-\frac{1}{3}}\right]_1^{\infty} \quad = e^{-\frac{1}{3}}$$

5. The daily consumption of milk in a city in excess of 20,000 litres is exponentially distributed. The average excess in consumption of milk is 3,000 litres. The city has a daily stock of 35,000 litres. What is the probability that a day is selected at random and the stock is insufficient for that day.

Solution: Let X be the random variable of daily consumption of milk in excess of 20,000 litres.

Now X follows exponential distribution with mean 3000 litres

But exponential distribution has mean $\frac{1}{\lambda}$. Therefore $\frac{1}{\lambda} = 3000 \Rightarrow \lambda = \frac{1}{3000}$.

The pdf of exponential distribution is $f(x) = \lambda e^{-\lambda x} = \frac{1}{3000} e^{-\frac{1}{3000}x}, 0 < x < \infty$

Let Y be the daily consumption of milk. Then $X = Y - 20000$

Probability for the stock is insufficient happens if the daily consumption is more than the stock 35000 litres.

$$P[Y > 35000] = P[X + 20000 > 35000]$$
$$= P[X > 15000]$$
$$= P[15000 < X < \infty]$$
$$= \int_{15000}^{\infty} f(x)dx$$
$$= \frac{1}{3000}\int_{15000}^{\infty} e^{-\frac{1}{3000}x}dx$$
$$= \frac{1}{3000}\left[\frac{e^{-\frac{1}{3000}x}}{-\frac{1}{3000}}\right]_{15000}^{\infty}$$
$$= \frac{1}{3000}\left[0 - \frac{e^{-5}}{-\frac{1}{3000}}\right] \quad = e^{-5}$$

Normal Distribution

A normal distribution is a continuous distribution given by $y = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ where X is a Continuous normal variate distributed with density function $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ with mean μ and standard deviation σ.

Moment generating function, mean, variance of Normal distribution:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

The m.g.f

$$M_X(t) = E[e^{tx}]$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x) dx$$

$$= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Put $z = \left(\frac{x-\mu}{\sigma}\right) \Rightarrow \sigma z = x - \mu \Rightarrow \sigma dz = dx\ ,\ x = \sigma z + \mu$

$$x \to -\infty \Rightarrow z \to -\infty\ ;\quad x \to \infty \Rightarrow z \to \infty$$

$$M_X(t) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\sigma z+\mu)}\ e^{-\frac{1}{2}(z)^2}\ \sigma\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t\sigma z+\mu t-\frac{1}{2}(z)^2}\, dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2t\sigma z)} dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2t\sigma z + \sigma^2 t^2)} e^{\frac{1}{2}\sigma^2 t^2)} dz$$

$$= \frac{e^{(\mu t + \frac{1}{2}\sigma^2 t^2)}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z - \sigma t)^2} dz$$

Put $u = z - \sigma t \Rightarrow du = dz$ $, z \to -\infty \Rightarrow u \to -\infty$ $, z \to \infty \Rightarrow u \to \infty$

$$= \frac{e^{(\mu t + \frac{1}{2}\sigma^2 t^2)}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(u)^2} du = \frac{e^{\left(\mu t + \frac{1}{2}\sigma^2 t^2\right)}}{\sqrt{2\pi}} \sqrt{2\pi}$$

$$M_X(t) = e^{\left(\mu t + \frac{1}{2}\sigma^2 t^2\right)} \qquad \because \int_{-\infty}^{\infty} e^{-\frac{1}{2}(u)^2} du = \sqrt{2\pi}$$

$$\frac{d}{dt} M_X(t) = e^{\left(\mu t + \frac{1}{2}\sigma^2 t^2\right)} (\mu + t\sigma^2)$$

$$\frac{d^2}{dt^2} M_X(t) = e^{\left(\mu t+\frac{1}{2}\sigma^2 t^2\right)}[\sigma^2] + (\mu + t\sigma^2)\left[e^{\left(\mu t+\frac{1}{2}\sigma^2 t^2\right)}(\mu + t\sigma^2)\right]$$

$$\text{Mean} = E[X] = \left[\frac{d}{dt}[M_X(t)]\right]_{t=0} = \mu$$

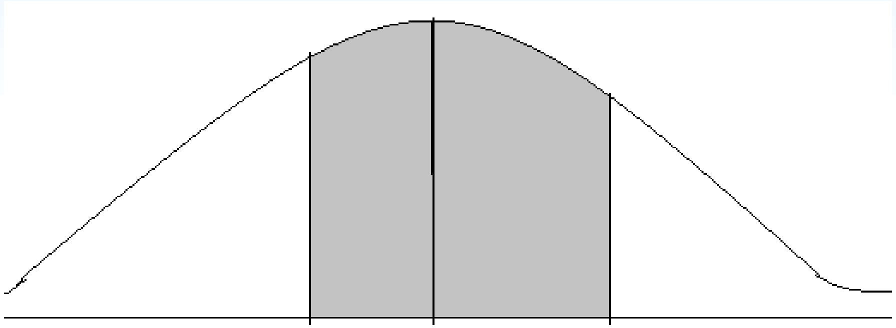
$$E[X^2] = \left[\frac{d^2}{dt^2}[M_X(t)]\right]_{t=0} = \sigma^2 + \mu^2$$

$$Var(X) = E[X^2] - \left[E[X]\right]^2 = \mu^2 + \sigma^2 - \mu^2 = \sigma^2$$

1. If X is a normal variate with $\mu = 30$ and $\sigma = 5$, find $(i) P(26 \le X \le 40)$ $(ii) P(X \ge 45)$ $(iii) P(|X - 30| \ge 5)$

Solution: Given X is a normal variate with $\mu = 30$ and $\sigma = 5$.

Then $Z = \frac{X-\mu}{\sigma} = \frac{X-30}{5}$



$$(i)\ P(26 \leq X \leq 40) = P\left[\frac{X-30}{5} < Z < \frac{X-30}{5}\right]$$

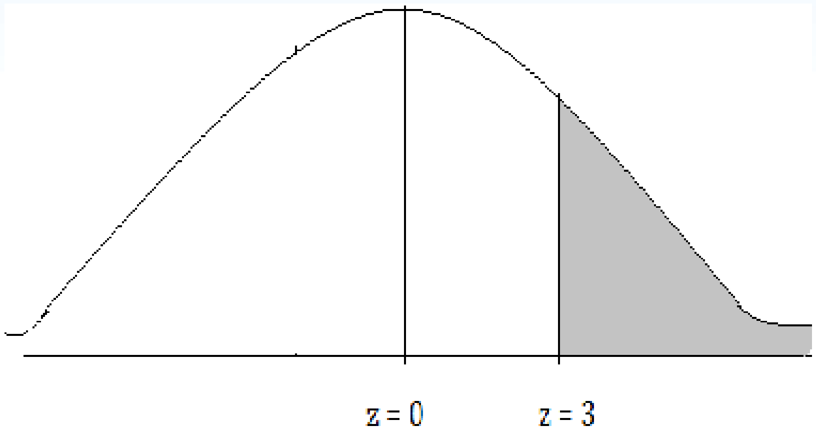
$$= P\left[\frac{26-30}{5} < Z < \frac{40-30}{5}\right]$$

$$= P[-0.8 < Z < 2]$$

$$= P[0 < Z < 0.8] + P[0 < Z < 2]$$

$$= 0.2881 + 0.4772$$

$$= 0.7653$$



$$(ii)\ P(X \geq 45) = P\left[Z \geq \frac{X-30}{5}\right]$$

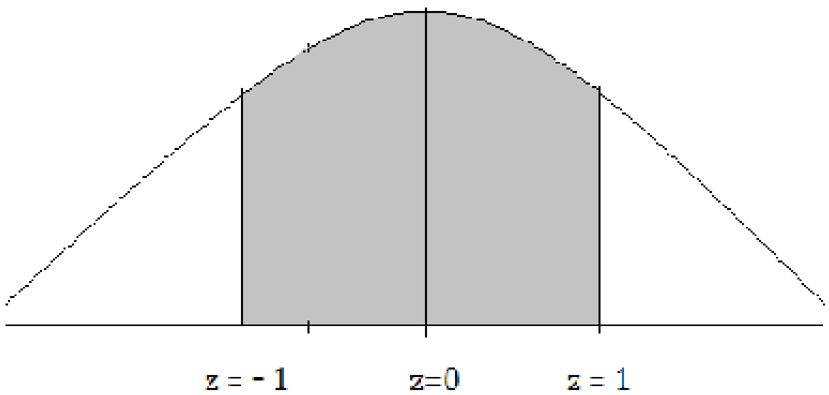
$$= P\left[Z \geq \frac{45-30}{5}\right]$$

$$= P[Z \geq 3]$$

$$= 0.5 - P[0 < Z < 3]$$

$$= 0.5 - 0.4987$$

$$= 0.0013$$



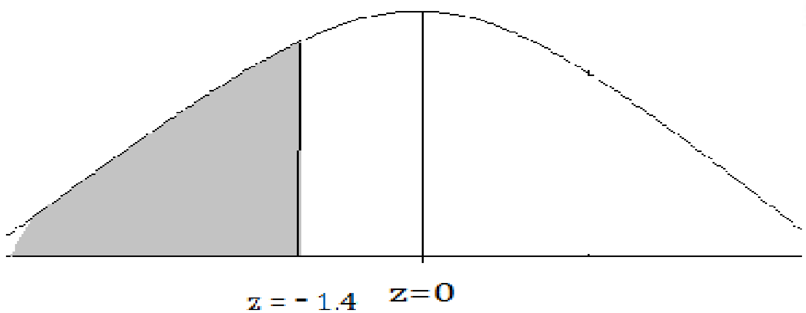
$$(iii)\ P(|X-30|>5) = 1 - P(|X-30|<5)$$
$$= 1 - P[-5 < X - 30 < 5]$$
$$= 1 - P[25 < X < 35]$$
$$= 1 - P\left[\frac{X-30}{5} < Z < \frac{X-30}{5}\right]$$
$$= 1 - P\left[\frac{25-30}{5} < Z < \frac{35-30}{5}\right]$$
$$= 1 - P[-1 < Z < 1]$$
$$= 1 - 2P[0 < Z < 1]$$
$$= 1 - 2(0.3413) \quad = 0.3174$$

2. A certain type of storage battery lasts on the average 3 years with standard deviation 0.5 year. Assuming that the battery lives are normally distributed, find the probability that the given battery will last less than 2.3 years.

Solution : Given X is a normal variate with $\mu = 3$ and $\sigma = 0.5$

Then $Z = \frac{X-\mu}{\sigma} = \frac{X-3}{0.5}$

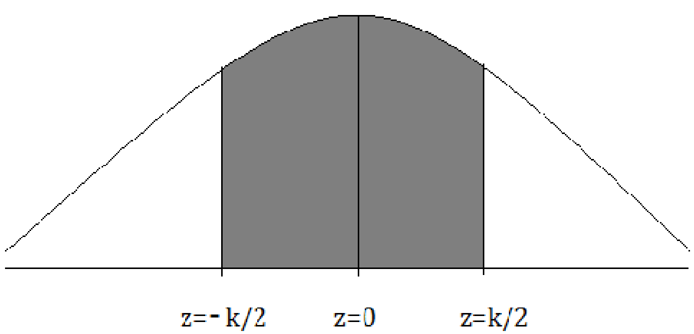
To find the probability that the battery will last less than 2.3 years



$$P[X < 2.3] = P\left[Z < \frac{X-3}{0.5}\right]$$
$$= P\left[Z < \frac{2.3-3}{0.5}\right]$$
$$= P[Z < -1.4]$$
$$= 0.5 - P[0 < Z < 1.4]$$
$$= 0.5 - 0.4192 \quad = 0.0808$$

3. If X is N(3,4), find 'k' so that $P(|X-3| > k) = 0.05$

Solution : Given X is a normal variate with $\mu = 3$ and $\sigma = 2$ Then $Z = \frac{X-\mu}{\sigma} = \frac{X-3}{2}$



Given that $P[|X-3|>k]=0.05$

$$1-P[|X-3|<k]=0.05$$

$$P[|X-3|<k]=1-0.05$$

$$P[-k<X-3<k]=0.95$$

$$P[-k<X-3<k]=0.95$$

$$P[3-k<X<k+3]=0.95$$

$$P\left[\frac{X-3}{2}<Z<\frac{X-3}{2}\right]=0.95$$

$$P\left[\frac{3-k-3}{2}<Z<\frac{k+3-3}{2}\right]=0.95$$

$$P\left[\frac{-k}{2}<Z<\frac{k}{2}\right]=0.95$$

$$2P\left[0<Z<\frac{k}{2}\right]=0.95$$

$P\left[0<Z<\frac{k}{2}\right]=0.475 \therefore \frac{k}{2}=1.96$ from normal table

$$i.e.\, k=3.92$$