Sri SAI RAM ENGINEERING COLLEGE
INSTITUTE OF TECHNOLOGY
West Tambaram, Chennai - 44

SAIRAM DIGITAL RESOURCES

YEAR II SEM III

MA8351

DISCRETE MATHEMATICS

(COMMON TO CSE & IT)

UNIT-V

LATTICES AND BOOLEAN ALGEBRA

5.5 SOME SPECIAL LATTICES

SCIENCE & HUMANITIES

4 QUALITY EDUCATION

SOME SPECIAL LATTICES

BOUNDED LATTICE: $(\boldsymbol{L},\wedge,\vee,\mathbf{0},\mathbf{1})$

Let $(L,\wedge,\vee)$ BE A GIVEN Lattice. If it has both the elements '0' and '1' then it is said to be bounded lattices.

Eg: $(P(A),\subseteq)$ is a bounded lattice.

COMPLIMENT OF AN ELEMENT:

Let $(\boldsymbol{L},\wedge,\vee,\mathbf{0},\mathbf{1})$ be given bounded lattices. Let $a \in L$.

The element $a' \in L$ is called complement of a if $a \wedge a' = 0$ and $a \vee a' = 1$.

COMPLEMENTED LATTICE:

A bounded lattice $(\boldsymbol{L},\wedge,\vee,\mathbf{0},\mathbf{1})$ is said to be complemented lattice if every element of L has atleast one compliment.

COMPLETE LATTICE:

A lattice $(L,*,\oplus)$ is said to be complete if every non-empty subset has a least upper bound and a greatest lower bound.

DISTRIBUTIVE LATTICE:

A lattice $(L,*,\oplus)$ is called distributive lattice if the operations $*$ and $\oplus$ are distributive over each other.

(i) $a * (b \oplus c) = (a * b) \oplus (a * c)$

(ii) $a \oplus (b * c) = (a \oplus b) * (a \oplus c),\ \ \forall a, b, c \in L.$

MODULAR LATTICE:

A lattice $(L,*,\oplus)$ is called modular lattice if for any $a, b, c \in L, a \leq c \Longrightarrow\ a \oplus (b * c) = (a \oplus b) * c.$

Theorem:

Every chain is a distributive lattice.

Proof:

Let $(L, \leq)$ be a chain and $a, b, c \in L$. It is enough to verify one of the conditions of distributive lattice.

Since L is a chain either $b \leq c$ or , $c \leq b$, because of symmetry of roles of b *and* c.

Let us assume one of the case $b \leq c \Longrightarrow b \oplus c = c$.

We have the following cases:

case(i): $a \leq b \leq c$

Then $a * b = a$ and $a * c = a$

$a * (b \oplus c) = a * c = a$ and

$(a * b) \oplus (a * c) = a \oplus a = a.$

Therefore, $a * (b \oplus c) = (a * b) \oplus (a * c)$.

case(ii): $b \leq c \leq a$

Then $a * b = b$ and $a * c = c$

$a * (b \oplus c) = a * c = c$ and

$(a * b) \oplus (a * c) = b \oplus c = c$.

Therefore, $a * (b \oplus c) = (a * b) \oplus (a * c)$.

case(iii): $b \leq a \leq c$

Then $a * b = b$ and $a * c = a$

$a * (b \oplus c) = a * c = a$ and

$(a * b) \oplus (a * c) = b \oplus a = a$.

Therefore, $a * (b \oplus c) = (a * b) \oplus (a * c)$.

So, the distributive law holds in all the cases.

Therefore, the chain $(L, \leq)$ is a distributive lattice.

Theorem:

In a distributive lattice $(L, *, \oplus)$

if for any $a, b, c \in L$ $a * b = a * c$ and $a \oplus b = a \oplus b$, then $b = c$.

Proof:

Given $(L, *, \oplus)$ is a distributive lattice, we have

$(a * b) \oplus c = (a \oplus c) * (b \oplus c)$

$(a * b) \oplus c = (a * c) \oplus c = c$ an

$$(a \oplus c) * (b \oplus c) = (b \oplus a) * (b \oplus c)$$

$$= b \oplus (a * c) = b \oplus (a * b) = b.$$

Therefore, $b = c$.

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Theorem:

Every distributive lattice is modular.

Proof:

Let $(L,*,\oplus)$ be a distributive lattice. To prove that it is modular.

To prove that for any $a, b, c \in L, a \leq c \Longrightarrow a \oplus (b * c) = (a \oplus b) * c$.

Since L is distributive for any $a, b, c \in L$,

we have $a \oplus (b * c) = (a \oplus b) * (a \oplus c)$.

If $a \leq c$ then $a \oplus c = c$.

Therefore, $a \oplus (b * c) = (a \oplus b) * c$.

Thus, the modularity condition is satisfied.

Hence L is modular.

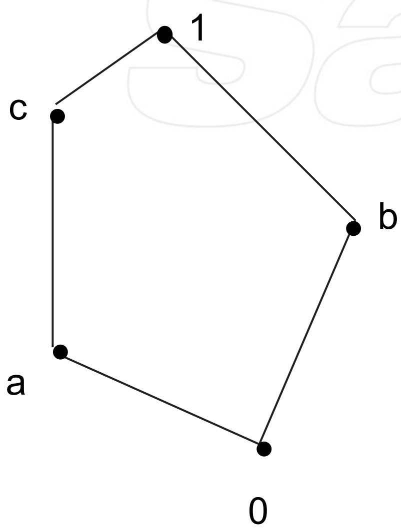
Note:

Every Modular lattice need not be distributive.

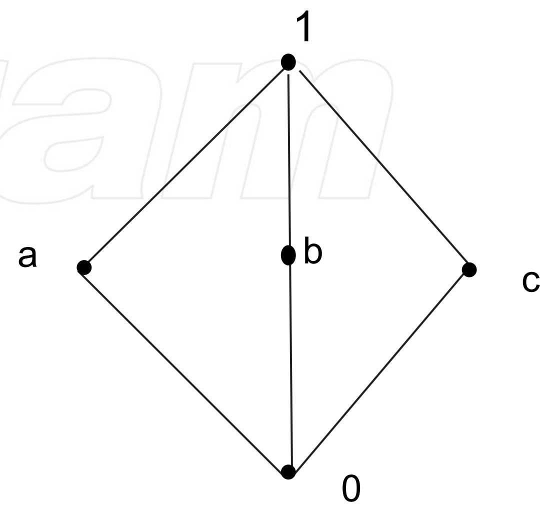
(i) Diamond lattice M_5 is Modular but it is not distributive.

(ii) $(\rho(A), \cup, \cap)$ is an example of both Modular Lattice and Distributive Lattice.

(iii) Pentagon lattice N_5 is an example of both non-modular and non-distributive lattice.



N_5



M_5

Problem

Check the pentagon lattice or N_5 Lattice is modular or not.

Solution:

Consider (a, b, c)

Clearly, $a \leq c$,

Now LHS = $a \vee (b \wedge c) = a \vee (0) = a$

RHS = $(a \vee b) \wedge c = (1) \wedge c = c$

LHS $\neq$ RHS

Therefore, if , $a \leq c, a \vee (b \wedge c) \neq (a \vee b) \wedge c$.

The pentagon lattice is not modular

Problem:

In a distributive lattice prove that complement of an element, if it exists, is unique.

Solution:

Let $(L,*,\oplus)$ be a distributive lattice. Let $a \in L$ be an element. If a_1 *and* a_2 are compliments of a, then

$$a * a_1 = 0, \quad a \oplus a_1 = 1$$

$$a * a_2 = 0, \quad a \oplus a_2 = 1$$

Therefore, $a * a_1 = a * a_2 \text{ and } a \oplus a_1 \text{ and } a \oplus a_2 \Longrightarrow a_1 = a_2$.

So, the compliment of a is unique.

Problem:

Show that a chain with three or more elements is not complemented.

Solution:

Let $(L, \leq)$ be a chain with three or more elements.

Since L is a chain, it is a totally ordered lattice.

So, any two elements are comparable with a least element 0 and greatest element 1.

Let $a \in L$ be an element then $0 \leq a \leq 1$.

Further $0 * a = 0,\ \ 0 \oplus a = a$ and $a * 1 = a,\ \ a \oplus 1 = 1$.

These relations show that a has no complement and hence is not complemented lattice.