Sri SAI RAM ENGINEERING COLLEGE
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Sairam INSTITUTIONS

SAIRAM DIGITAL RESOURCES

YEAR II | SEM IV

MA8391

PROBABILITY AND STATISTICS

(INFORMATION TECHNOLOGY)

UNIT I

PROBABILITY AND RANDOM VARIABLES

1.4 MOMENTS , MOMENT GENERATING FUNCTION

SCIENCE & HUMANITIES

Moments

Definition: The r-th moment about origin of a RV X is defined as the expected value of the r-th power of X.

Moments about origin(Raw Moments)

Discrete: $\mu_r' = E(X^r) = \sum_i x_i^r p_i \ , \ n \geq 1,$

Continuous: $\mu_r' = E\{(X)^r = \int_{-\infty}^{\infty} x^r f(x)dx, \quad n \geq 1$

Moments about Mean

Discrete: $\mu^r = E(X - \overline{X})^r = \sum_i (x_i - \bar{X})^r p_i$

Continuous: $\mu^r = E(X - \overline{X})^r = \int_{-\infty}^{\infty} (x - \bar{X})^r f(x)dx \quad .$

Relation between moments about origin and moment

$$\mu_r = \mu_r' - rC_1\mu_{r-1}' + rC_2\mu^2\mu_{r-2}' - \cdots$$

Hence, $\mu_1 = 0$,

$$\mu_2 = \mu_2' - \left(\mu_1'\right)^2,$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3,$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\left(\mu_1'\right)^2 - 3(\mu_1')^4$$

Moment generating function

Definition : Moment generating function of a random variable about the origin is defined as

Discrete: $M_X(t) = E(e^{tX}) = \sum_x e^{tx} p(x)$,

Continuous: $M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x) dx$.

Moment generating function

$$M_X(t) = 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \cdots + \frac{t^r}{r!}\mu_r'$$

Where μ_r' = coefficient of $\frac{t^r}{r!}$ in $M_X(t)$

Note: 1. $\mu_r' = \frac{d^r}{dt^r}[M_X(t)]_{t=0}$

2 . $M_{CX}(t) = M_X(Ct)$, C being a constant.

3. $M_{X=a}(t) = e^{-at}M_X(t)$

4. If $X_1, X_2, \ldots X_n$ are n independent RVs, then

$$M_{X_1+X_2+\cdots X_n}(t) = M_{X_1}(t)M_{X_2}(t)\ldots M_{X_n}(t)$$

.

Example 1) If X represents the outcome, when a fair die is tossed find the MGF of X and hence find $E(X)$ and $Var(X)$.

Solution:

The probability distribution of X is given by

$$p_i = p(X = i) = \frac{1}{6}\ , i = 1,2,\dots 6$$

$$M(t) \ = \sum_i e^{tx_i} p_i = \sum_{i=1}^{6} e^{ti} p_i$$

$$= \frac{1}{6}[e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t}]$$

$$M'(t) = \frac{1}{6}[e^t + 2e^{2t} + 3e^{3t} + 4e^{4t} + 5e^{5t} + 6e^{6t}]$$

$$E(X) = [M'(t)]_{t=0} = \frac{7}{2}$$

. $E(X^2) = [M''(t)]_{t=0}$

$$= \frac{1}{6}[e^t + 4e^{2t} + 9e^{3t} + 16e^{4t} + 25e^{5t} + 36e^{6t}]_{t=0} = \frac{91}{6}$$

$$Var(X) = \frac{91}{6} - \frac{49}{4} = \frac{35}{12}.$$

Example 2) If a RV X has the MGF $M(t) = \frac{3}{3-t}$, obtain the standard deviation of X

Solution:

$$M(t) = \frac{3}{3-t},$$

$$M(t) = \frac{3}{3\left(1-\frac{t}{3}\right)} = \left(1 - \frac{t}{3}\right)^{-1} = 1 + \frac{t}{3} + \frac{t^2}{9} + \cdots. +\infty \ldots\ldots\ldots\ldots (1)$$

$E(X)$ =coefficient of $\frac{t}{1!}$ ln $(1) = \frac{1}{3}$

$E(X^2)$ =coefficient of $\frac{t^2}{2!}$ ln $(1) = \frac{2}{9}$

$Var(X) = E(X^2) - [E(X)]^2 = \frac{1}{9}$, $\sigma_X = \frac{1}{3}$

Example 3) A random variable X has the probability function $f(x) = \frac{1}{2^x}$, $x = 1, 2, 3, \dots$ Find its (i)M.G.F, (ii) Mean.

Solution:

X is a discrete random variable,

M.G.F= $M_X(t) = E(e^{tx})$

$$= \sum_{x=1}^{\infty} e^{tx} f(x)$$

$$= \sum_{x=1}^{\infty} e^{tx} \frac{1}{2^x}$$

$$= \sum_{x=1}^{\infty} \left(\frac{e^t}{2}\right)^x$$

$$= \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \left(\frac{e^t}{2}\right)^3 + \cdots$$

$$= \frac{e^t}{2}\left[1 + \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \left(\frac{e^t}{2}\right)^3 + \cdots\right] = \frac{e^t}{2}\left(1 - \frac{e^t}{2}\right)^{-1}$$

$[\because (1-x)^{-1} = 1 + x + x^2 + \cdots]$

$$= \frac{e^t}{2}\left(\frac{2-e^t}{2}\right)^{-1} = \frac{e^t}{2-e^t}$$

$$M_X(t) = \frac{e^t}{2-e^t}$$

Mean $= \bar{X} = \frac{d}{dt}\left(\frac{e^t}{2-e^t}\right)_{t=0}$

$$= \left(\frac{(2-e^t)e^t - e^t(-e^t)}{(2-e^t)^2}\right)_{t=0}$$

$$= \frac{(2-1)+1}{(2-1)^2} = 2$$

Example 4) Find the m.g.f., mean and variance of the distribution whose p.m.f is $p(x) = \begin{cases} q^x p, & x = 0,1,2,.. \\ 0, & otherwise \end{cases}$ **given** $p + q = 1$

Solution:

$$M_X(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} p(x) = \sum_{x=0}^{\infty} e^{tx} q^x p$$

$$= p\sum_{x=0}^{\infty}(qe^t)^x = p[1 + qe^t + (qe^t)^2 + (qe^t)^3 + ..]$$

$$= p[1 - qe^t]^{-1} = \frac{p}{[1 - qe^t]}$$

$\mu_1' = \frac{d}{dt} M_X(t)$ at $t = 0$

$$= \frac{d}{dt}[p[[1 - qe^t]^{-1}]_{t=0}$$

$$= -p[(1 - qe^t)^{-2}(-qe^t)]_{t=0}$$

$= p(1 - q)^{-2}(q) = pqp^{-2}$ since $p + q = 1$

$$= \frac{pq}{p^2} = \frac{q}{p}$$

$$\mu_2' = \frac{d}{dt}[pqe^t(1-qe^t)^{-2}]_{t=0}$$

$$= pq[-2e^t(1-qe^t)^{-3}(-qe^t) + (1-qe^t)^{-2}e^t]_{t=0}$$

$$= pq[2q(1-q)^{-3} + (1-q)^{-2}]$$

$$= pq\left[\frac{2q}{p^3} + \frac{1}{p^2}\right] \quad \text{since } p+q=1$$

$$= \frac{2q^2}{p^2} + \frac{q}{p^2}$$

$\therefore$ Mean $= \mu_1' = \frac{q}{p}$

Variance $= \mu_2' - (\mu_1')^2 = \frac{2q^2}{p^2} + \frac{q}{p} - \frac{q^2}{p^2} = \frac{q^2}{p^2} + \frac{q}{p}$

$$= \frac{q^2+pq}{p^2} = \frac{q(q+p)}{p^2} = \frac{q}{p^2}$$

Example 5) The moment generating function of a r.v X is given by

$\boldsymbol{M_X(t) = \frac{3}{15}e^t + \frac{4e^{3t}}{15} + \frac{2e^{4t}}{15} + 4\frac{e^{5t}}{15}}$ **. Find the probability density function of X.**

Solution:

For a discrete r.v, $M_X(t) = E[e^{tX}] = \sum_i e^{tx_i}p(x_i)$

Hence the probability function is given by

x:	1	3	4	5
$p(x)$:	$\frac{3}{15}$	$\frac{4}{15}$	$\frac{2}{15}$	$\frac{4}{15}$

Example 6) Find m.g.f of the r.v whose moments are $\mu'_r = (r+1)!\,3^r$, and hence find its mean.

Solution:

$\mu'_r = (r+1)!\, 3^r$

The m.g.f is $M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu'_r$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!}(r+1)!\, 3^r$$

$$= \sum_{r=0}^{\infty} \frac{(3t)^r}{r!}(r+1)!$$

$$= 1 + 2(3t) + 3(3t^2) + \cdots$$

$$= \frac{1}{(1-3t)^2}$$

Mean$= \left(\frac{d}{dt}(M_X(t)\right)_{t=0} = \left[\frac{(-2)(-3)}{(1-3t)^2}\right]_{t=0} = 6$

Mean$= 6$ (or) Mean $= \mu'_1$ =Coefficient of $\frac{t^1}{1!} = 6$

Example 7) Find the m.g.f for the following function given by

X:	0	1	2	3	4	5	6
$p(X)$:	$\frac{1}{49}$	$\frac{3}{49}$	$\frac{5}{49}$	$\frac{7}{49}$	$\frac{9}{49}$	$\frac{11}{49}$	$\frac{13}{49}$

Solution:

$$M_X(t) = \sum_{x=0}^{\infty} e^{tx} p(x)$$

$$= e^0\left(\frac{1}{49}\right) + e^t\left(\frac{3}{49}\right) + e^{2t}\left(\frac{5}{49}\right) + e^{3t}\left(\frac{7}{49}\right) + e^{4t}\left(\frac{9}{49}\right) + e^{5t}\left(\frac{11}{49}\right) + e^{6t}\left(\frac{13}{49}\right)$$

$$M_X(t) = \frac{1}{49}[1 + 3e^t + 5e^{2t} + 7e^{3t} + 9e^{4t} + 11e^{5t} + 13e^{6t}]$$

Example 8) Find the m.g.f of the r.v X having the pdf

$$f(x) = \begin{cases} x, & 0 \le x < 1 \\ 2 - x, & 1 \le x < 2 \\ 0, & otherwise \end{cases}$$

Solution:

Given p.d.f $f(x) = \begin{cases} x, & 0 \le x < 1 \\ 2 - x, & 1 \le x < 2 \\ 0, & otherwise \end{cases}$

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx$$

$$= \int_0^1 x e^{tx} dx + \int_1^2 (2 - x) e^{tx} dx$$

Example 8) Find the m.g.f of the r.v X having the pdf

$$f(x) = \begin{cases} x, & 0 \le x < 1 \\ 2 - x, & 1 \le x < 2 \\ 0, & otherwise \end{cases}$$

Solution:

Given p.d.f $f(x) = \begin{cases} x, & 0 \le x < 1 \\ 2 - x, & 1 \le x < 2 \\ 0, & otherwise \end{cases}$

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx$$

$$= \int_0^1 x e^{tx} dx + \int_1^2 (2 - x) e^{tx} dx$$

$$= \left\{ \left[(x)\frac{e^{tx}}{t} \right]_0^1 - \left[(1)\frac{e^{tx}}{t^2} \right]_0^1 \right\} + \left\{ \left[(2-x)\frac{e^{tx}}{t} \right]_1^2 - \left[(-1)\frac{e^{tx}}{t^2} \right]_1^2 \right\}$$

$$= \frac{e^t}{t} - \left[\frac{e^t}{t^2} - \frac{1}{t^2} \right] + \left[0 - (2-1)\frac{e^t}{t} \right] + \left[\frac{e^{2t} - e^t}{t^2} \right]$$

$$= \frac{e^t}{t} - \frac{e^t}{t^2} + \frac{1}{t^2} - \frac{e^t}{t} + \frac{e^{2t}}{t^2} - \frac{e^t}{t^2}$$

$$= \frac{1 - 2e^t + e^{2t}}{t^2}$$

$$= \frac{(e^t - 1)^2}{t^2}$$

Example 9) Find the m.g.f and r^{th} moment for the distribution whose p.d.f is $f(x) = Ke^x : 0 \le x \le \infty$. Also find the mean and standard deviation.

Solution:

Given $f(x) = Ke^{-x} : 0 \le x \le \infty$

We know that $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\int_0^\infty Ke^{-x}dx = 1$$

$$K\left(\frac{e^{-x}}{-1}\right)_0^\infty = 1$$

$$\therefore K\left(\frac{0-1}{-1}\right) = 1 \Rightarrow K = 1.$$

The p.d.f is $f(x) = e^{-x}, 0 \le x \le \infty$

M.G.F= $M_X(t)$

$$M_X(t) = \int_0^\infty e^{-x} e^{tx} dx$$

$$= \int_0^\infty e^{-(1-t)x} dx$$

$$= \left(\frac{e^{-(1-t)x}}{-(1-t)}\right)_0^\infty = \frac{0-1}{-(1-t)} = \frac{1}{1-t}$$

$$M_X(t) = \frac{1}{1-t}$$

$$M_X(t) = (1-t)^{-1}$$

$$= 1 + t + t^2 + t^3 + \cdots + t^r + \cdots \infty$$

μ_r' =Coefficient of $\frac{t^r}{r!}$ In $M_X(t) = r!$

Example10) Let X be a random variable with PDF $f(x) = \begin{cases} \frac{1}{3}e^{\frac{-x}{3}}, x > 0 \\ 0, \quad otherwise \end{cases}$

Find $(i) P(X > 3)$ $(ii) MGF\ of\ X$ $iii)\ E(X)\ and\ Var(X)$.

Solution:

$$(i)\ P(X > 3) = \int_3^\infty f(x)dx = \int_3^\infty \frac{1}{3}e^{\frac{-x}{3}}dx$$

$$= \frac{1}{3}\left[\frac{e^{\frac{-x}{3}}}{\frac{-1}{3}}\right]_3^\infty = -(0 - e^{-1}) = e^{-1} = \frac{1}{e}$$

$$(ii)\ M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx} f(x)dx$$

$$= \int_0^\infty e^{tx}\frac{1}{3}e^{\frac{-x}{3}}dx$$

$$= \frac{1}{3}\int_0^{\infty} e^{\left(t-\frac{1}{3}\right)x}\,dx$$

$$= \frac{1}{3}\int_0^{\infty} e^{-\left(\frac{1}{3}-t\right)x}\,dx$$

$$= \frac{1}{3}\left[\frac{e^{-\left(\frac{1}{3}-t\right)x}}{-\left(\frac{1}{3}-t\right)}\right]_0^{\infty}$$

$$= \frac{1}{3}\left[0-\frac{1}{-\left(\frac{1}{3}-t\right)}\right] = \frac{1}{3}\left[\frac{1}{\left(\frac{1-3t}{3}\right)}\right]$$

$$M_X(t) = \frac{1}{1-3t} = (1-3t)^{-1}$$

$$\frac{d}{dt}[M_X(t) = -(1-3t)^{-2}(-3)] = 3(1-3t)^{-2}$$

(iii) $\frac{d^2}{dt^2}[M_X(t)] = -6(1-3t)^{-3}(-3) = 18(1-3t)^{-3}$

$$E(X^2) = \left[\frac{d^2}{dt^2}M_X(t)\right]_{t=0} = 18$$

$$Var(X) = E(X^2) - [E(X)]^2 = 18 - 3^2 = 9$$

Example11) Find the m.g.f of the random variable X whose p.d.f is

$f(x) = \begin{cases} \frac{x}{4}e^{\frac{-x}{2}}, & x > 0 \\ 0, & elsewhere \end{cases}$. **Also find the first four moments about the origin.**

Solution:

$$M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx} f(x)dx$$

$$\int_0^\infty e^{tx}\frac{x}{4}e^{\frac{-x}{2}}dx = \int_0^\infty \frac{x}{4}e^{-\left(\frac{1}{2}-t\right)x}dx$$

$$=\left[\frac{x}{4}.\frac{e^{-\left(\frac{1}{2}-t\right)x}}{-\left(\frac{1}{2}-t\right)}-\frac{1}{4}\frac{e^{-\left(\frac{1}{2}-t\right)x}}{\left(\frac{1}{2}-t\right)^2}\right]_0^\infty$$

$$=0-0+0+\frac{1}{4}.\frac{1}{\left(\frac{1}{2}-t\right)^2}$$

$$=\frac{1}{4}.\frac{1}{\left(\frac{1}{2}-t\right)^2}=\frac{1}{4}.\frac{4}{(1-2t)^2}=\frac{1}{(1-2t)^2}$$

$$M_X(t)=\frac{1}{(1-2t)^2}$$

$$\int_0^\infty e^{tx}\frac{x}{4}e^{\frac{-x}{2}}dx = \int_0^\infty \frac{x}{4}e^{-\left(\frac{1}{2}-t\right)x}dx$$

$$= \left[\frac{x}{4}\cdot\frac{e^{-\left(\frac{1}{2}-t\right)x}}{-\left(\frac{1}{2}-t\right)} - \frac{1}{4}\frac{e^{-\left(\frac{1}{2}-t\right)x}}{\left(\frac{1}{2}-t\right)^2}\right]_0^\infty$$

$$= 0 - 0 + 0 + \frac{1}{4}\cdot\frac{1}{\left(\frac{1}{2}-t\right)^2}$$

$$= \frac{1}{4}\cdot\frac{1}{\left(\frac{1}{2}-t\right)^2} = \frac{1}{4}\cdot\frac{4}{(1-2t)^2} = \frac{1}{(1-2t)^2}$$

$$M_X(t) = \frac{1}{(1-2t)^2}$$

$$(1-2t)^{-2} = 1 + 2(2t) + 3(2t)^2 + 4(2t)^3 + 5(2t)^4 + \cdots.$$

$$= 1 + 4t + 12t^2 + 32t^3 + 80t^4 + \cdots$$

$$=1+t(4)+\frac{t^2}{2!}(24) + \frac{t^3}{3!}(192) + \frac{t^4}{4!}(1920) + \cdots$$

Now μ_r'=Coefficient of $\frac{t^r}{r!}$

$\mu_1' = 4, \mu_2' = 24, \mu_3' = 192, \mu_4' = 1920$

Example12) A continuous random variable X has p.d.f $f(x) = kx^2e^{-x}\ x \geq 0$. Find k, r^{th} raw moment, mean and variance.

Solution:

$$\int_0^\infty kx^2e^{-x}dx = 1$$

$$k\int_0^\infty e^{-x}x^{3-1}dx = 1$$

k$\Gamma 3 = 1$ since $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$

$$i.e., 2k = 1, \qquad \therefore k = \frac{1}{2}$$

$$\therefore f(x) = \frac{1}{2} x^2 e^{-x}, x \geq 0$$

$$\mu_r' = \int_0^\infty x^r f(x) dx$$

$$\mu_r' = \int_0^\infty x^r \frac{1}{2} x^2 e^{-x} dx$$

$$= \frac{1}{2} \int_0^\infty e^{-x} x^{r+2} dx$$

$$= \frac{1}{2} \int_0^\infty e^{-x} x^{r+3-1} dx$$

$$= \frac{1}{2}\Gamma(r+3) = \frac{1}{2}(r+2)! \quad [since\ \ \Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx]$$

$$\mu_1' = \frac{1}{2}(3!) = \frac{1}{2}.6 = 3$$

$$\mu_2' = \frac{1}{2}(4!) = \frac{1}{2}.24 = 12$$

Mean= $\mu_1' = 3$

Variance= $\mu_2' - (\mu_1')^2 = 12 - 9 = 3.$

YOU TUBE

1.https://youtu.be/cbmfYoepHPk

2. https://youtu.be/dVRWBmykncQ