Sri SAI RAM ENGINEERING COLLEGE
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Sairam INSTITUTIONS

SAIRAM DIGITAL RESOURCES

YEAR II

SEM IV

MA8391

PROBABILITY AND STATISTICS

(INFORMATION TECHNOLOGY)

UNIT II

TWO-DIMENSIONAL RANDOM VARIABLES

2.2 JOINT , MARGINAL, CONDITIONAL DISTRIUTION (CONTINUOUS CASE)

SCIENCE & HUMANITIES

CONTINUOUS RANDOM VARIABLES X AND Y

Joint Probability Density Function:

Let (X,Y) be a two-dimensional continuous random variable such that $P\left(x-\frac{dx}{2}\le X\le x+\frac{dx}{2}\right)=f(x,y)dxdy$, then $f(x,y)$ is called the joint probability density function (X,Y) if it satisfies the following conditions

$(i) f(x,y)\ge 0\ \forall\ (x,y)\in R$, where R is the range space.

$(ii)\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy=1$

Moreover if $(a,b),(c,d)\in R$, then

$$P(a\le X\le b, c\le Y\le d)=\int_c^d\int_a^b f(x,y)dxdy$$

Marginal Probability Distribution:

When (X, Y) is a two-dimensional continuous random variable, then the marginal density function of the random variable X is defined as

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\,dy$$

The marginal density function of the random variable Y is defined as

$$f_y(x) = \int_{-\infty}^{\infty} f(x, y)\,dx$$

Conditional Probability Function

If (X, Y) is two-dimensional continuous random variable, then

$f(x/y) = \frac{f(x,y)}{f(y)}$ is called the conditional probability function of X given Y and

$f(y/x) = \frac{f(x,y)}{f(x)}$ is called the conditional probability function of Y given X

1) If X and Y $f(x,y) = \begin{cases} x+y, & 0<x<1, \ 0<y<1 \\ 0, & otherwise \end{cases}$ check whether X and Y are independent.

Solution: Given the joint pdf of (X, Y) as

$$f(x,y) = \begin{cases} x+y, & 0<x<1, \ 0<y<1 \\ 0, & otherwise \end{cases}$$

If $f_X(x)$ and $f_Y(y)$ are the marginal functions of (X, Y) then X and Y are independent if $f(x, y) = f_X(x)\ .f_Y(y)$ for all x and y.

Now

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy$$

$$= \int_0^1 (x + y) dy$$

$$= \left[xy + \frac{y^2}{2} \right]_0^1$$

$$= x + \frac{1}{2}$$

$$= \frac{2x + 1}{2}$$

$\therefore$ The marginal pdf of X is $f_X(x) = \begin{cases} \frac{2x+1}{2}, 0 < x < 1 \\ 0, \quad elsewhere \end{cases}$

$$f_Y(x) = \int_{-\infty}^{\infty} f(x,y)dx$$

$$= \int_0^1 f(x,y)dx$$

$$= \int_0^1 (x+y)dx$$

$$= \left[\frac{x^2}{2} + yx\right]_0^1$$

$$= \frac{1}{2} + y$$

$$= \frac{1+2y}{2}$$

$\therefore$ The marginal pdf of X is

$$f_X(x) = \begin{cases} \frac{1+2y}{2}, 0 < y < 1 \\ 0, \quad elsewhere \end{cases}$$

$$f_X(x) \,.f_Y(y) = \left[\frac{2x+1}{2}\right]\left[\frac{1+2y}{2}\right]$$

$$= \frac{1}{4}[2x + 4xy + 1 + 2y]$$

$$= \frac{1}{4}[4xy + 2(x + y) + 1]$$

$$\neq x + y$$

$$\therefore f(x, y) \neq f_X(x) \,.f_Y(y)$$

$\therefore$ X and Y are not independent

2) The joint PDF of (X, Y) is given by $e^{-(x+y)}, 0 < x, y < \infty$. Are X and Y independent? Why?

Solution: If X and Y are independent, then

$$f(x,y) = f_X(x)\, .f_Y(y)$$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy$$

$$f_X(x) = \int_0^{\infty} e^{-x}e^{-y}dy$$

$$= e^{-x}\left[\frac{e^{-y}}{-1}\right]_0^{\infty}$$

$$= e^{-x}$$

$$f_Y(x) = \int_{-\infty}^{\infty} f(x,y)dx$$

$$f_X(x) = \int_0^{\infty} e^{-x}e^{-y}dx \Rightarrow e^{-y}\left[\frac{e^{-x}}{-1}\right]_0^{\infty} = e^{-y}$$

$$f_X(x)\,.f_Y(y) = e^{-x}e^{-y} = e^{-(x+y)} = f(x,y)$$

$\therefore$ They are independent.

3) Find K if the joint PDF of a bivariate random variable (X,Y) is given by

$$f(x,y) = \begin{cases} K(1-x)(1-y), & 0 < x < 4;\, 1 < y < 5 \\ 0, & otherwise \end{cases}$$

Solution: We know that if $f(x,y)$ is a joint PDF, then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dydx = 1$$

$$\int_{0}^{4}\int_{1}^{5} K(1-x)(1-y)\,dydx = 1$$

$$K\int_1^5\left[\frac{(1-x)^2}{-2}\right]_0^4(1-y)dy=1$$

$$-4K\int_1^5(1-y)dy=1$$

$$-4K\left[\frac{(1-y)^2}{-2}\right]_1^5=1$$

i.e $-4K\left[\frac{16}{-2}+0\right]=1\Rightarrow 32K=1\Rightarrow K=\frac{1}{32}$

4) The joint PDF of the random variable (X,Y) is given by $f(x,y)=Kxye^{-(x^2+y^2)}$, $x>0, y>0$. Find the value of K and prove also that X and Y are independent.

Solution: We know that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dydx = 1$$

$$\int_{0}^{\infty}\int_{0}^{\infty} Kxye^{-(x^2+y^2)}dx = 1$$

$$K\int_{0}^{\infty}\int_{0}^{\infty} xe^{-x^2}ye^{-y^2}dxdy = 1$$

$$K[\int_0^{\infty} xe^{-x^2}dx \int_0^{\infty} ye^{-y^2}dy] = 1 \ldots (1)$$

Now

$$\int_{0}^{\infty} xe^{-x^2}dx$$

$$t = x^2 \Rightarrow dt = 2xdx \Longrightarrow \frac{dt}{2} = xdx$$

$$x = 0 \Longrightarrow t = 0$$

$$x = \infty \Longrightarrow t = \infty$$

$$\int_0^\infty xe^{-x^2} dx = \int_0^\infty e^{-t} \frac{dt}{2} = \frac{1}{2} \int_0^\infty e^{-t}\, dt = \frac{1}{2}$$

Similarly we have $\int_0^\infty ye^{-y^2} dy = \frac{1}{2}$ in (1)

$$K\left(\frac{1}{2}.\frac{1}{2}\right) = 1$$

$$K = 4.$$

5. The joint PDF of the random variables (X, Y) is $f(x, y) = 8xy, 0 < x < 1,$ $0 < y < x$. Find the conditional density function. Find $(i) f_{\frac{Y}{X}}(y/x)$ $(ii) f_Y(y)$

Solution : Given $f(x, y) = 8xy, 0 < x < 1, 0 < y < x$

$$f(x) = \int_{-\infty}^{\infty} f(x, y)dy$$

$$= \int_0^x 8xydy$$

$$= 8x\left[\frac{y^2}{2}\right]_0^x = 8x.\frac{x^2}{2}$$

$$f(x) = 4x^3, 0 < x < 1$$

$$(i)) f_{\frac{Y}{X}}\left(\frac{y}{x}\right) = \frac{f(x,y)}{f(x)} = \frac{8xy}{4x^3} = \frac{2y}{x^2}$$

$$(ii)\ f(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_0^1 8xydx$$

$$= 8y\left[\frac{x^2}{2}\right]_0^1 = 8y\left(\frac{1}{2}\right) = 4y.$$

$$f(y) = 4y, \qquad 0 < y < 1$$

6) If the joint PDF of a two-dimensional random variable is given by

$$\boldsymbol{f(x) = \begin{cases} 2, & 0 < x < 1,\ 0 < y < x \\ 0, & Otherwise \end{cases}}$$ **. Find the marginal density function of X and Y.**

Solution: The marginal distribution of X is given by

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Solution: The marginal distribution of X is given by

$$f(x) = \int_{-\infty}^{\infty} f(x,y)dy$$

$$= \int_{0}^{x} 2dy = 2[y]_0^x = 2x$$

$$f(x) = 2x, \qquad 0 < x < 1$$

The marginal distribution of Y is given by

$$f(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_{y}^{1} 2dx = 2[x]_y^1 = 2(1-y)$$

$$f(y) = 2(1-y), \qquad 0 < y < 1$$

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7) If $f(x,y) = \begin{cases} xe^{-x(y+1)}, & x \geq 0 \\ 0, & otherside \end{cases}$ is the joint PDF of a two-dimensional random variable (X,Y). Find the marginal and conditional density function.

Solution : The marginal densities of X and Y are given by

$$f(x) = \int_0^\infty xe^{-x(y+1)}dy$$

$$= xe^{-x}\left[\frac{e^{-xy}}{-x}\right]_0^\infty = [-e^{-x}(0-1)] = e^{-x}$$

$$f(x) = e^{-x}, x \geq 0$$

$$f(y) = \int_0^\infty xe^{-x(y+1)}dx$$

$$= \left[x\left(\frac{e^{-x(y+1)}}{y+1}\right) - (1)\left(\frac{e^{-x(y+1)}}{(y+1)^2}\right)\right]_0^{\infty}$$

$$f(y) = \frac{1}{(1+y)^2}, \qquad y \geq 0$$

The conditional density function is given by

$$f(x/y) = \frac{f(x,y)}{f(y)} = (y+1)^2 x e^{-x(y+1)}, x \geq 0, y \geq 0$$

$$f(y/x) = \frac{f(x,y)}{f(x)} = x^2 e^{-xy}, x \geq 0, y \geq 0$$

8) If the joint PDF of X and Y is given by

$$f(x,y) = \begin{cases} \frac{1}{8}(6-x-y) & ,0<x<2, 2<y<4 \\ 0, & otherwise \end{cases}$$

Find (i) $P(X<1 \cap Y<3)$ $(ii) P(X<1/Y<3)$

Solution:

$(i)\ P(X<1 \cap Y<3) = \int_0^1 \int_2^3 f(x,y)\,dydx$

$$= \int_0^1 \int_2^3 \frac{1}{8}(6-x-y)\,dydx$$

$$= \frac{1}{8}\int_0^1 \left[6y - xy - \frac{y^2}{2}\right]_2^3 dx$$

$$= \frac{1}{8}\int_0^1 \left[18 - 3x - \frac{9}{2}\right] - \left[12 - 2x - \frac{4}{2}\right] dx$$

$$= \frac{1}{8}\int_0^1 \left[6 - x - \frac{5}{2}\right] dx$$

$$= \frac{1}{8}\int_0^1 \left[\frac{7}{2} - x\right] dx$$

$$= \frac{1}{8}\left[\frac{7}{2}x - \frac{x^2}{2}\right]_0^1 = \frac{3}{8}$$

$$(ii)\ P(X < 1/Y < 3) = \frac{P(X<1, Y<3)}{P(Y<3)} \ldots(1)$$

$$= \int_0^2 \int_2^3 \frac{1}{8}(6 - x - y)\,dy\,dx$$

$$= \frac{1}{8}\int_0^2 \left[6y - xy - \frac{y^2}{2}\right]_2^3 dx$$

$$= \frac{1}{8}\int_0^2 \left[18 - 3x - \frac{9}{2}\right] - [12 - 2x - 2]\,dx$$

$$= \frac{1}{8}\int_0^2 \left(\frac{7}{2} - x\right) dx$$

$$\therefore P(Y<3)=\frac{5}{8}.$$

Substituting in (1) we get

$$P(X<1/Y<1)=\frac{\frac{3}{8}}{\frac{5}{8}}=\frac{3}{5}$$

9) The joint PDF of two-dimensional random variable (X,Y) is given by $f(x,y)=xy^2+\frac{x^2}{8},\ 0\le x\le 2,\ 0\le y\le 1.$ Find

$(i)\ P(X>1)$, $(ii)\ P\left(Y<\frac{1}{2}\right)$, $(iii)\ P\left(X>1/Y<\frac{1}{2}\right)$,

$(iv)\ P\left(Y<\frac{1}{2}/X>1\right)$ $(v)\ P(X<Y)$ $(iv)\ P(X+Y\le 1)$

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Solution:

$$(i)\ P(X>1)=\int_0^2\int_1^2\left(xy^2+\frac{x^2}{8}\right)dxdy$$

$$=\int_0^1\left[\frac{x^2y^2}{2}+\frac{x^3}{24}\right]_1^2 dy$$

$$=\int_0^1\left[\frac{4y^2}{2}+\frac{8}{24}-\frac{y^2}{2}-\frac{1}{24}\right]dy$$

$$=\int_0^1 2y^2+\frac{7}{24}-\frac{y^2}{2}dy$$

$$= \left[\frac{2y^3}{3} + \frac{7y}{24} - \frac{y^3}{6}\right]_0^1$$

$$= \frac{2}{3} + \frac{7}{24} - \frac{1}{6} = \frac{19}{24} \ldots (i)$$

$$(ii)\ P\left(\left(Y < \frac{1}{2}\right) = \int_0^{\frac{1}{2}} \int_0^2 \left(xy^2 + \frac{x^2}{8}\right) dxdy\right.$$

$$= \int_0^{\frac{1}{2}} \left[\frac{x^2y^2}{2} + \frac{x^3}{24}\right]_0^2 dy$$

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$$= \int_0^{\frac{1}{2}} \left[\frac{4y^2}{2} + \frac{8}{24}\right] dy$$

$$= \left[\frac{4y^3}{6} + \frac{8y}{24}\right]_0^{\frac{1}{2}}$$

$$= \frac{4\left(\frac{1}{2}\right)^3}{6} + 8\,\frac{\left(\frac{1}{2}\right)}{24} = \frac{1}{4} \;\dots (ii)$$

$$P\left(X > 1/Y < \frac{1}{2}\right) = \frac{P\left(X > 1 \cap Y < \frac{1}{2}\right)}{P\left(Y < \frac{1}{2}\right)}$$

$$= \frac{P\left(X > 1, Y < \frac{1}{2}\right)}{P\left(Y < \frac{1}{2}\right)} \;\dots (iii)$$

Now

$$P\left(X > 1/Y < \frac{1}{2}\right) = \int_0^{\frac{1}{2}} \int_1^2 \left(xy^2 + \frac{x^2}{8}\right) dx dy$$

$$= \int_0^{\frac{1}{2}} \left[\frac{x^2y^2}{2} + \frac{x^3}{24}\right]_1^2 dy$$

$$= \int_0^{\frac{1}{2}} \left[\frac{4y^2}{2} + \frac{8}{24} - \frac{y^2}{2} - \frac{1}{24}\right] dy$$

$$= \left[\frac{4y^3}{6} + \frac{8y}{24} - \frac{y^3}{6} - \frac{y}{24}\right]_0^{\frac{1}{2}}$$

$$= \frac{4\left(\frac{1}{2}\right)^3}{6} + 8\frac{\left(\frac{1}{2}\right)}{24} - \frac{\left(\frac{1}{2}\right)^3}{6} - \frac{\left(\frac{1}{2}\right)}{24} = \frac{5}{24} \dots (iv)$$

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Substituting in (iii) we get

$$P\left(X > 1/Y < \frac{1}{2}\right) = \frac{\frac{5}{24}}{\frac{1}{4}} = \frac{5}{6}$$

$$(iv) P\left(Y > \frac{1}{2}/X > 1\right) = \frac{P\left(Y > \frac{1}{2}, X > 1\right)}{P(X > 1)}$$

$$= \frac{\frac{5}{24}}{\frac{19}{24}} = \frac{5}{19}$$

$$(v)\ \ P(X < Y) = \int_0^1 \int_0^y \left[xy^2 + \frac{x^2}{8}\right] dydx$$

$$= \int_0^1 \left[\frac{x^2y^2}{2} + \frac{x^3}{24}\right]_0^y dy$$

$$= \int_0^1 \left[\frac{y^4}{2} + \frac{y^3}{24}\right] dy$$

$$= \left[\frac{y^5}{10} + \frac{y^4}{96}\right]_0^1 = \frac{1}{10} + \frac{1}{96} = \frac{53}{480}$$

$$(iv)\quad P(X + Y \leq 1) = \int_0^1 \int_0^{1-y} \left(xy^2 + \frac{x^2}{8}\right) dxdy$$

$$= \int_0^1 \left[\frac{x^2y^2}{2} + \frac{x^3}{24}\right]_0^{1-y} dy = \int_0^1 \frac{(1-y)^2y^2}{2} + \frac{(1-y)^3}{24} dy$$

$$= \left[\frac{1}{2}\left(\frac{y^3}{3} - \frac{2y^4}{4} + \frac{y^5}{5}\right) + \frac{1}{24}\left(y - \frac{3y^2}{2} + \frac{3y^3}{3} - \frac{y^4}{4}\right)\right]_0^1$$

$$= \frac{1}{2}\left(\frac{1}{3} - \frac{2}{4} + \frac{1}{5}\right) + \frac{1}{24}\left(1 - \frac{3}{2} + \frac{3}{3} - \frac{1}{4}\right)$$

$$= \frac{1}{60} + \frac{1}{96} = \frac{13}{480}$$

10) Given $f_{XY}(x,y) = \begin{cases} cx(x-y), & 0 < x < 2,\ -x < y < x \\ 0, & otherwise \end{cases}$

Evaluate $(i)\ c,\ (ii)\ f_X(x)\ \ (iii) f_{(Y/X)}(y/x)\ \ (iv) f_Y(y)$

Solution:

(i) To find the value of c, we know that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

$\int_{-\propto}^{\propto}\int_{-x}^{x} cx(x-y)dydx = 1\ldots(1)$

$$c\int_0^2\int_{-x}^{x}(x^2-xy)dydx = 1$$

$$c\int_0^2\left[x^2y-\frac{xy^2}{2}\right]_{-x}^{x}dx = 1$$

$$c\int_0^2\left(x^3-\frac{x^3}{2}+x^3+\frac{x^3}{2}\right)dx = c\int_0^2 2x^3dx = 1$$

$$2c\left(\frac{x^4}{4}\right)_0^2 = 2c.\frac{16}{4} = 1$$

$$8c = 1 \Rightarrow c = \frac{1}{8}$$

(ii) $f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy$

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$$f(x) = \frac{1}{8}\int_{-x}^{x}(x^2 - xy)dy$$

$$= \frac{1}{8}\left(x^3 - \frac{x^3}{2} + x^3 + \frac{x^3}{2}\right) = \frac{2x^3}{8} = \frac{x^3}{4}$$

$$f(x) = \frac{x^3}{4}, 0 < x < 2$$

$$(iii)\ f_{Y/X}(y/x) = \frac{f(x,y)}{f(x)}$$

$$\frac{\frac{x(x-y)}{8}}{\frac{x^3}{4}} = \frac{x-y}{2x^2}, -x < y < x$$

$$(iv)\ f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx$$

$$= \int_{-y}^{2} \frac{1}{8} x(x-y)dx, \qquad -2 \le y \le 0$$

$$\frac{1}{8}\left[\frac{x^3}{3} - \frac{x^2 y}{2}\right]_{-y}^{2} = \frac{1}{8}\left[\frac{8}{3} - 2y - \left(-\frac{y^3}{3} - \frac{y^2}{2}\right)\right]$$

$$= \frac{1}{3} - \frac{y}{4} + \frac{5y^3}{48}$$

$$f(y) = \int_{y}^{2} \frac{1}{8} x(x-y)dx, \qquad 0 \le y \le 2$$

$$= \frac{1}{8}\left(\frac{x^3}{3} - \frac{x^2 y}{2}\right)_{y}^{2} = \frac{1}{8}\left(\frac{8}{3} - \frac{4y}{2} - \frac{y^3}{3} + \frac{y^2}{2}\right)$$

$$= \frac{1}{3} - \frac{y}{4} + \frac{y^3}{48}$$

$$f(y) = \begin{cases} \frac{1}{3} - \frac{y}{4} + \frac{5y^4}{48}, & -2 < y < 0 \\ \frac{1}{3} - \frac{y}{4} + \frac{5y^4}{48}, & 0 < y < 2 \end{cases}$$

You tube

1) https://youtu.be/ipHbPkaUKEo

2) https://youtu.be/02wjNeQabaQ