Sri SAI RAM ENGINEERING COLLEGE
INSTITUTE OF TECHNOLOGY
West Tambaram, Chennai - 44

SAIRAM DIGITAL RESOURCES

YEAR II

SEM III

MA 8351

DISCRETE MATHEMATICS (COMMON TO CSE & IT)

UNIT II COMBINATORICS

2.2 THE BASICS OF COUNTING - THE PIGEONHOLE PRINCIPLE

SCIENCE & HUMANITIES

THE BASICS OF COUNTING

Basic Counting Principles:

There are two basic counting principles:

1) The product rule
2) The sum rule

These rules can be used to solve many different counting problems.

THE PRODUCT RULE :

Suppose that a procedure can be broken down into a sequence of two tasks. If there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are n_1n_2 ways to do the procedure.

1) A new company with just two employees, Sanchez and Patel, rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees?

Solution:

The procedure of assigning offices to these two employees consists of assigning an office to Sanchez, which can be done in 12 ways, then assigning an office to Patel different from the office assigned to Sanchez, which can be done in 11 ways. By the product rule, there are $12 \cdot 11 = 132$ ways to assign offices to these two employees.

2) The chairs of an auditorium are to be labeled with an uppercase English letter followed by a positive integer not exceeding 100. What is the largest number of chairs that can be labelled differently?

Solution:

The procedure of labelling a chair consists of two tasks, namely, assigning to the seat one of the 26 uppercase English letters, and then assigning to it one of the 100 possible integers. The product rule shows that there are $26 \cdot 100 = 2600$ different ways that a chair can be labelled.Therefore, the largest number of chairs that can be labelled differently is 2600.

3)There are 32 microcomputers in a computer centre. Each microcomputer has 24 ports. How many different ports to a microcomputer in the centre are there?

Solution:

The procedure of choosing a port consists of two tasks, first picking a microcomputer and then picking a port on this microcomputer. Because there are 32 ways to choose the microcomputer and 24 ways to choose the port no matter which microcomputer has been selected the product rule shows that there are $32 \cdot 24 = 768$ ports

Note:

An extended version of the product rule is often useful. Suppose that a procedure is carried out by performing the tasks $T_1, T_2, \ldots, T_m$ in sequence. If each task T_i, $i = 1, 2, \ldots, n$, can be done in n_i ways, regardless of how the previous tasks were done, then there are $n_1 \cdot n_2 \cdot \cdots \cdot n_m$ ways to carry out the procedure. This version of the product rule can be proved by mathematical induction from the product rule for two tasks .

4) How many different bit strings of length seven are there?

Solution:

Each of the seven bits can be chosen in two ways, because each bit is either 0 or 1.Therefore, the product rule shows there are a total of 2^7 = 128 different bit strings of length seven.

5) How many different license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits (and no sequences of letters are prohibited, even if they are obscene)?

Solution:

There are 26 choices for each of the three uppercase English letters and ten choices for each of the three digits. Hence, by the product rule there are a total of $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17{,}576{,}000$ possible license plates.

6) How many functions are there from a set with *m* elements to a set with *n* elements?

Solution:

A function corresponds to a choice of one of the n elements in the co domain for each of the m elements in the domain. Hence, by the product rule there are $n \cdot n \cdot \cdot \cdot \cdot \cdot n = nm$ functions from a set with m elements to one with n elements. For example, there are 53 = 125 different functions from a set with three elements to a set with five elements.

THE SUM RULE:

If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task.

1) Suppose that either a member of the mathematics faculty or a student who is a mathematics major is chosen as a representative to a university committee. How many different choices are there for this representative if there are 37 members of the mathematics faculty and 83 mathematics majors and no one is both a faculty member and a student?

Solution:

There are 37 ways to choose a member of the mathematics faculty and there are 83ways to choose a student who is a mathematics major. Choosing a member of the mathematics faculty is never the same as choosing a student who is a mathematics major because no one is both a faculty member and a student. By the sum rule it follows that there are 37 + 83 = 120 possible ways to pick this representative.

PIGEONHOLE PRINCIPLE:

If m pigeons are assigned to n pigeonholes m > n ,then there must be a pigeonhole containing at least two pigeons.

Generalised pigeonhole principle:

If m pigeons are assigned to n pigeonholes, then there must be a pigeonhole containing at least $\left\lfloor \frac{m-1}{n} \right\rfloor + 1$ pigeons, where $\lfloor \; \rfloor$ is the greatest integer function.

1)From a college if we select 367 students, then at least two of them will have same birthday.

Solution:

Since there are 367 students and we know that there are maximum 366 days in a leap year and so there are 366 different birthdays possible. Treating 366 as pigeonholes and 367 students are pigeons, then by pigeonhole principle at least two students will have the same birthday.

2) Find the minimum number of students needed to make sure that 5 of them take the same Engineering course ECE,CSE,EEE and MECH.

Solution:

Consider the 5 students as pigeons and n=4 courses as pigeonholes.Let m be the minimum number of students needed so that 5 of them take the same course.

By generalised principle of pigeonhole,

$$\left\lfloor \frac{m-1}{4} \right\rfloor + 1 = 5$$

$$\left\lfloor \frac{m-1}{4} \right\rfloor = 4$$

$$\left\lfloor \frac{m-1}{4} \right\rfloor . \leq \frac{m-1}{4}$$

$$4 \leq \frac{m-1}{4}$$

$$16 \leq m-1$$

$$m \geq 17$$

So the minimum number of students must be 17

3) Find the minimum number of students need to guarantee that five of them belongs to the same subject, if there are five different major subjects

Solution:

By the pigeon hole principle, among 21 students five of them must be in the same subjects if there are five different subjects (hole).

4) A man hiked for 10 hours and covered a total distance of 45km. It is known that he hiked 6 km in the first hour and only 3 km in the last hour. Show that he must have hiked at least 9 km within a certain period of 2 consecutive hours.?

Solution:

Since the man hiked 6+3 = 9 km in the first and last hours , he must have hiked 45 -9=36 km during the period from second to ninth hours.

If we combine the second and third hours together, the fourth and fifth hours together,

etc, and the eight hour and ninth hours together, we have 4 time periods.
Let us now treat the 4 time periods as pigeonholes and 36 km as 36 pigeons.
By pigeonhole principle , the least number of pigeon accommodated in in one pigeonhole

$$= \left\lfloor \frac{36-1}{4} \right\rfloor + 1$$

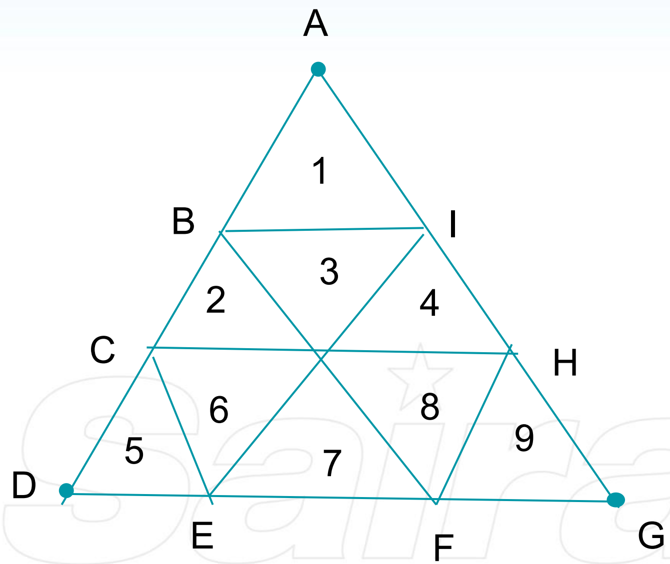
$$= \lfloor 8.75 \rfloor + 1$$

$$= 9$$

5) If we select 10 points in the interior of the equilateral triangle of side 1 ,show that there must be at least two points whose distance apart is less than 1/3.

Solution:

Let ADG be the given equilateral triangle. The pair of points B,C,E,F and H,I are the points of trisection of the sides AD, DG and GA respectively. We have divided the triangle ADG into 9 equilateral triangles each of side 1/3.



The 9 sub – triangles may be regarded as 9 pigeonholes and 10 interior points may be regarded as 10 pigeons.

Then by pigeonhole principle , at least one sub triangle must contain 2 interior points.

The distance between any two interior points of any sub triangle cannot exceed the length of the side ,namely 1/3

6) i) If n pigeonholes are occupied by (k n +1) pigeons, where k is a positive integer, prove that at least one pigeonhole is occupied by (k +1) or more pigeons.

ii) Hence , find the minimum number m of integers to be selected from $S = \{ 1, 2, 3, \ldots\ldots 9 \}$ so that

a) the sum of the m integers is even .

b) the difference of two of the m integers is 5. But there are (k n+ 1) pigeons.

Solution:

i) If at least one pigeonhole is not occupied by (k+1) or more pigeons , each pigeonhole contains at most k pigeons. Hence ,the total number of pigeons occupying the n pigeonholes is at most kn.

But there are (k n +1) pigeons. This result is a contradiction .Hence the result.

ii) a) Sum of two even integers or two odd integers is even.

Let us divide the set S into 2 subsets { 1, 3, 5, 7, 9 } and { 2, 4, 6, 8, } which may be treated as pigeonholes. Thus n = 2

At least 2 numbers must be chosen either from the first subset or from the second.

ie) at least one pigeonhole must contain 2 pigeons.

ie) k +1 =2 or k =1

Therefore the minimum number if pigeons required or the minimum number of integers to be selected is equal to

$$k\,n + 1 = 3$$

b) Let us divide the set S into 5 subsets {1, 6 }, { 2, 7 }, { 3, 8 }, { 4, 9 }, { 5 } which may be treated as pigeonholes. Thus n = 5

If m =6 , then 2 of integers of S will belong to one of the subsets and their difference is 5