Sri SAI RAM ENGINEERING COLLEGE
INSTITUTE OF TECHNOLOGY
West Tambaram, Chennai - 44

Sairam INSTITUTIONS

SAIRAM DIGITAL RESOURCES

UNIT III

GRAPHS

3.1 GRAPHS AND GRAPH MODELS

YEAR II

SEM III

MA8351

DISCRETE MATHEMATICS
(Common to CSE & IT)

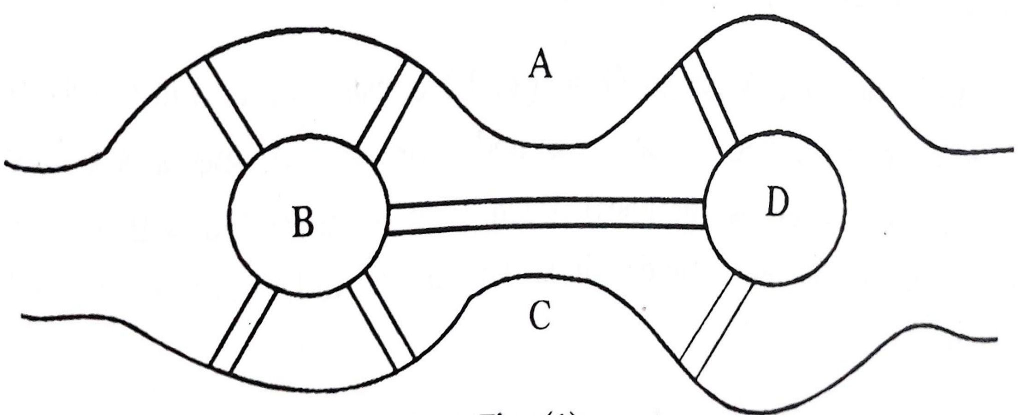
SCIENCE & HUMANITIES

4 QUALITY EDUCATION

GRAPHS AND GRAPH MODELS

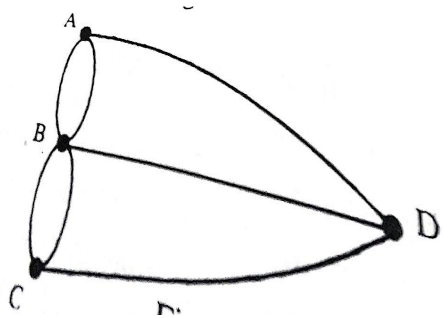
Introduction:

Graph theory originated with an interesting physical problem, the celebrated Konigsberg Bridge Problem. The Prussian city of Konigsberg lies on the Pregel river.



It consists of the two river banks A and C and the two islands B and D. Seven bridges connect the four land areas of the city. It was a pastime of the people of Konigsberg to check up the possibility of starting from one land area of the city, walking through each bridge only once and coming back to the starting point. They could not verify as the maximum number of possible paths is 7! = 5040

In 1736, the great Swiss Mathematician Euler proved that such a walk is not possible. Euler abstracted the problem representing the four land areas by four points and the seven bridges by seven lines are arcs joining the points.



Now the Konigsberg bridge problem is the same as the problem of drawing the graph, starting from any point and without lifting the pen from the paper and without retracing any line and coming back to the starting point.

For solving this problem Euler developed some of the fundamental concepts of Graph Theory and thus laying the foundation for a new branch of Mathematics. Euler became the father of Graph Theory.

Definition:

Graph

A graph G = (V, E) consists of V, a non empty set of vertices (or nodes) and E, a set of edges. Each edge has either one or two vertices associated with it, called its endpoints. An edge is said to connect its endpoints.

Remark:

The set of vertices *V* of a graph *G* may be infinite. A graph with an infinite vertex set or an infinite number of edges is called an infinite graph, and in comparison, a graph with a finite vertex set and a finite edge set is called a finite graph.

Definition:

Simple graphs and Multi graphs

A graph in which each edge connects two different vertices and where no two edges connect the same pair of vertices is called a simple graph. Graphs that may have multiple edges connecting the same vertices are called multigraphs.

Remark:

Edges that connect a vertex to itself. Such edges are called loops. Graphs that may include loops, and possibly multiple edges connecting the same pair of vertices or a vertex to itself, are sometimes called pseudo graphs.

Definition:

Directed graph

A directed graph (or digraph) (V, E) consists of a nonempty set of vertices V and a set of directed edges (or arcs) E. Each directed edge is associated with an ordered pair of vertices. The directed edge associated with the ordered pair (u, v) is said to start at u and end at v.

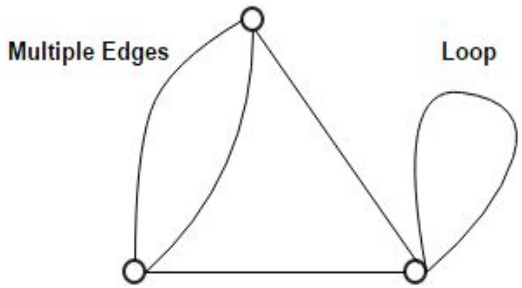
When a directed graph has no loops and has no multiple directed edges, it is called a simple directed graph.

Definition:

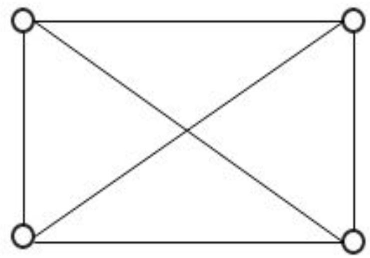
Regular graph

A simple graph is called regular if every vertices of the graph has the same degree.

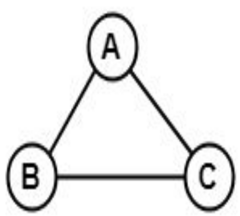
Examples of graph:

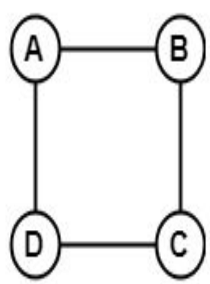


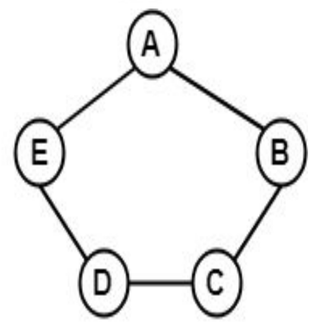
Not a Simple Graph



Simple Graph







Note:

- |V| denote the number of vertices of G.
- The Edge set E may be empty, There need not exist edges joining every pair of vertices.

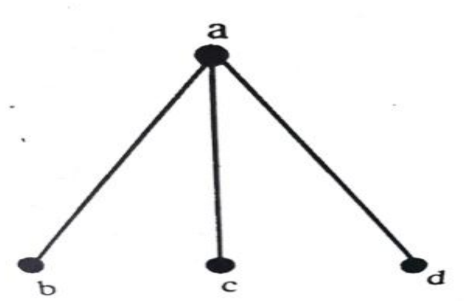
If e is associated with {u,v} we write e = u v

Example:

Let G=(V,E) where V={a,b,c,d}

$E = \{e_1=ab, e_2=ac, e_3=ad\}$

Then this graph can be represented by



Definition:

Incidence and Adjacency

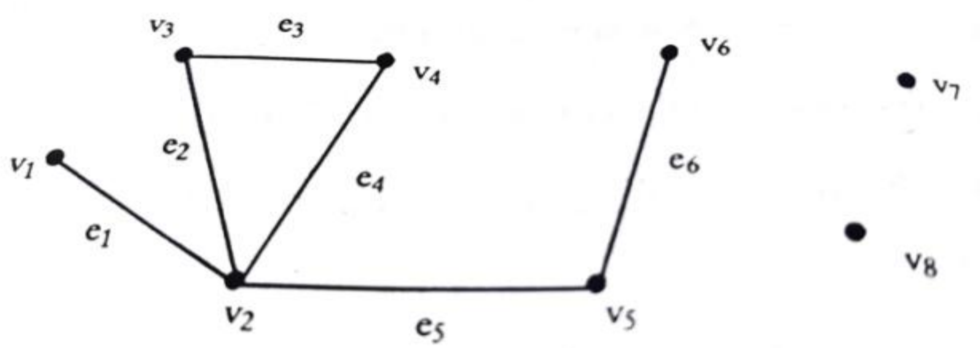
- Let G= (V,E) be a graph .If e=u v is an edge of G then u and v are called adjacent in G. e is incident with u and v.
- A vertex which is not adjacent to every other vertex is called an isolated vertex.
- A graph having only isolated vertices is called a null graph.

If two are more edges are incident with the same vertex, then they are said to be adjacent edges.

A vertex of a graph is called a pendant vertex if only one edge is incident with it.

Example:

Consider the graph given below and indicate all types of vertices and edges.



v_1, v_2 are adjacent vertices; v_1, v_3 are non adjacent vertices.

v_2, v_3; v_3, v_4; v_2, v_4; v_2, v_5; v_5, v_6 are pairs of adjacent vertices.

v_1 and v_6 are pendant vertices.

v_7, v_8 are isolated vertices.

e_1, e_2, e_4, e_5 are adjacent edges as they are all incident with v_2.

Definition:

Degree of a vertex

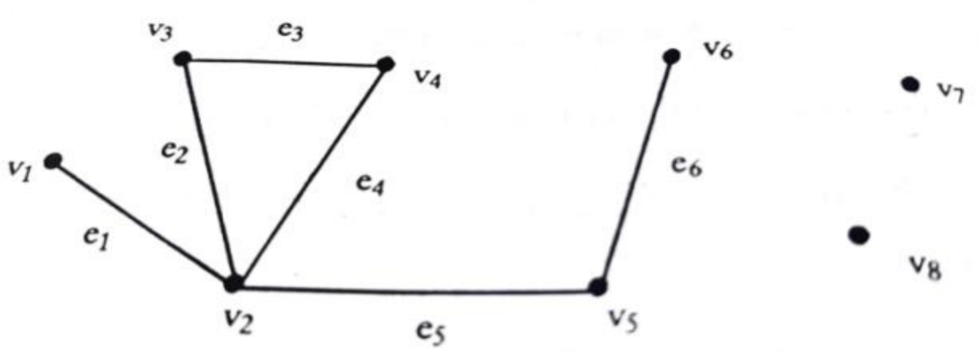
The degree of a vertex in a graph G is the number of edges incident with it.

Note:

- If v is an isolated vertex then deg(v)=0
- If v is a pendant vertex then deg(v)=1

Problem:1

Find the degree of all vertices of the given graph:



Solution:

$deg(v_1) = 1$; $deg(v_2) = 4$; $deg(v_3) = 2$; $deg(v_4) = 2$; $deg(v_5) = 2$; $deg(v_6) = 1$; $deg(v_7) = 0$ and $deg(v_8) = 0$

Theorem: 1

Let G= (V, E) be a graph and let e denote the number of edges and let $v_1, v_2, \ldots, v_n$ be n vertices then $\sum_{i=1}^{n} \deg(v_i) = 2e$

Proof:

Every non loop edge is incident with two vertices and so contributes 2 to the degree. Every loop edge contributes 2 to the degree.

Every edge (loop or not) contributes 2 to the sum of degrees of the vertices. So all the e edges contribute 2e degrees.

Sum of degrees of vertices = 2e

$$\Rightarrow \sum_{i=1}^{n} \deg(v_i) = 2e$$

Note:

(i) The sum of degrees of vertices of a graph is always even.

(ii) The above theorem is sometimes referred to as 'the handshaking theorem' because of the analogy between an edge and with two end points and a handshake involving two hands'.

Theorem: 2

The number of vertices of odd degree is even.

Proof:

Let G= (V,E) be a graph.

Let V_1 and V_2 be the set of vertices of even degree and the set of vertices of odd degree respectively in G.

Then $\sum \deg(v) = 2e$

$$\Rightarrow \sum_{v \in V_1} \deg(v) + \sum_{v \in V_2} \deg(v) = 2e$$

Since $\deg(v)$ is even if $v \in V_1$ then $\sum_{v \in V_1} \deg(v)$ is even = 2k

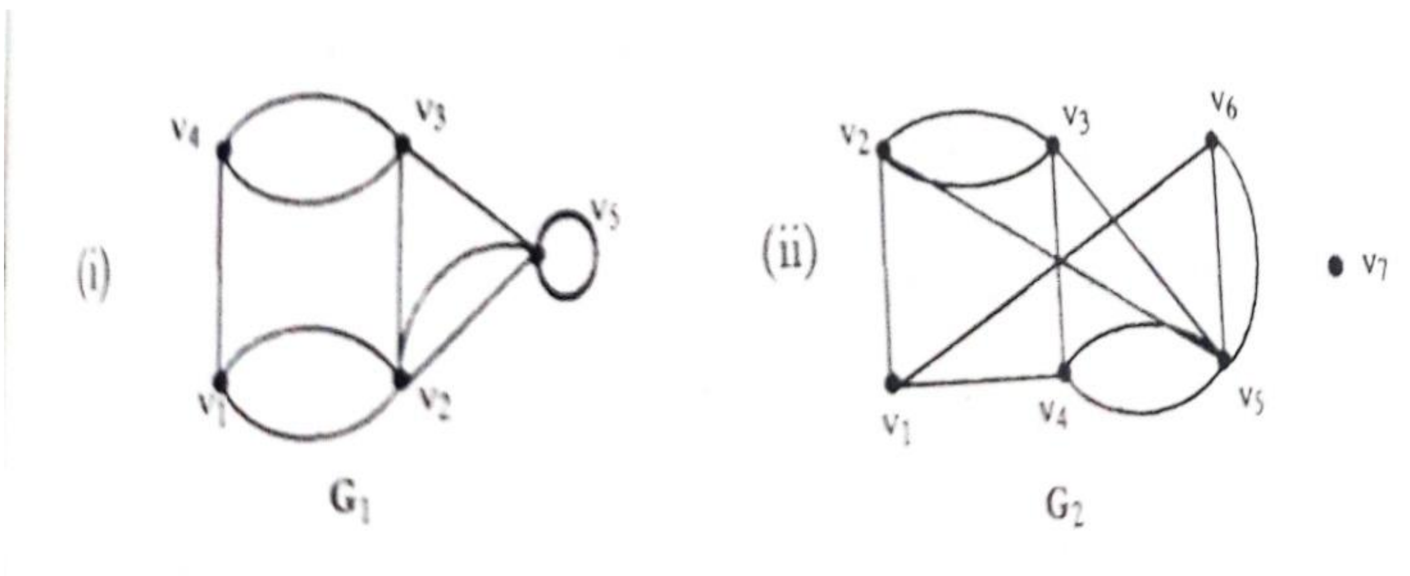
$$\therefore \sum_{v \in V_2} \deg(v) = 2e - 2k = 2(e-k) = even$$

Since each $\deg(v)$ is odd, and their sum is even and hence the number of terms must be even.

Hence there are even number of vertices of odd degree.

Problem: 2

Find the number of vertices ,number of edges and the degree of each vertex. Verify the handshaking theorem.



Solution:

(i) Here |V| = 5, |E| = 10

$$\deg(v_1) = 3, \deg(v_2) = 5, \deg(v_3) = 4, \deg(v_4) = 3, \deg(v_5) = 5$$

$$\sum \deg(v) = 3 + 5 + 4 + 3 + 5$$
$$= 20 = 2e, where\ e = |E| = 10$$

(ii) Here |V| = 7, |E| = 12

$$\deg(v_1) = 3, \deg(v_2) = 4, \deg(v_3) = 4, \deg(v_4) = 4, \deg(v_5) = 6, \deg(v_6)$$
$$= 3, \deg(v_7) = 0$$

$$\sum \deg(v) = 3 + 4 + 4 + 4 + +6 + 3 + 0$$
$$= 24 = 2e, where\ e = |E| = 12$$

Problem: 3

Determine |V| for the following graphs

(i) G has 9 edges and all vertices have degree 3.

(ii) G is regular with 15 edges (iii) G has 10 edges with two vertices of degree 4 and all others of degree 3. (iv) G is regular of degree 4 with 10 edges.

Solution:

Let |V| = n. Given each vertex has degree 3

$$\sum_{v \in V} \deg(v) = 3n$$

By handshaking theorem

$$\sum \deg(v) = 2e$$

$$\Rightarrow 3n = 2.9 \Rightarrow n = 6$$

$$\therefore \textit{number of vertices } |V| = 6$$

(ii) Given G is regular all the vertices have same degree r.

$$\sum \deg(v) = nr$$

Since e = 15, by handshaking theorem

n.r = 2.15 = 30

$\therefore$ n=1 or 2 or 3 or 5 or 6 or 10 or 15 or 30

(iii) Let |V| = n. Given e = 10

2 vertices are of degree 4 their total degree = 2.4 = 8

Each of the remaining (n-2) vertices are of degree 3

their total = 3(n-2)

By handshaking theorem $\sum \deg(v) = 2e$

$$\Rightarrow 8 + 3(n-2) = 2.10$$

$$\Rightarrow 3n + 2 = 20 \Rightarrow 3n = 18 \Rightarrow n = 6$$

(iv)Let |V| =n then $\sum \deg(v) = 4n$

$$\Rightarrow 2.10 = 4n \Rightarrow n = 5$$

Theorem: 3

The maximum degree of any vertex in a simple graph with n vertices is n-1

Proof:

Let G be a graph with n vertices and let v be any vertex of G. If v is adjacent with all other n-1 vertices of V(G) , then at most we get the degree for v as n-1. Since loops and parallel edges are not allowed in a simple graph.

Theorem: 4

The maximum number of edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$

Proof:

We can prove this by the method of induction on n.

For n=1 , a simple graph with one vertex has no edges.

The result is true for n=1

For n=2 , a simple graph with two vertices may have at most one edge.

$$\therefore \frac{2(2-1)}{2} = 1$$

The result is true for n=2

Assume that the result is true for n=k , i.e., a graph with k vertices has at most $\frac{k(k-1)}{2}$ edges.

When n= k+1 , let G be a graph having n vertices and G' be the graph obtained from G by deleting one vertex say $v \in V(G)$.

Since G' have k vertices , then by the hypothesis G' have at most $\frac{k(k-1)}{2}$ edges. Now add the vertex v to G'. v may adjacent to all k vertices of G'

$\therefore$ The total number of edges in G are , $\frac{k(k-1)}{2} + k = \frac{k^2 - k + 2k}{2}$

$$= \frac{k(k+1)}{2} = \frac{(k+1)(k+1-1)}{2}$$

$$= \frac{n(n-1)}{2}$$

The result is true for n= k+1. Hence the maximum number of edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$

MULTIPLE CHOICE QUESTIONS:

1. In a simple graph, the number of edges is equal to twice the sum of the degrees of the vertices.

   a) True

   b) False

Answer: b

Explanation: The sum of the degrees of the vertices is equal to twice the number of edges.

2.Which of the following is true?

a) A graph may contain no edges and many vertices

b) A graph may contain many edges and no vertices

c) A graph may contain no edges and no vertices

d) A graph may contain no vertices and many edges

Answer: b

Explanation: A graph must contain at least one vertex.

3. A graph with all vertices having equal degree is known as a __________

a) Multi Graph

b) Regular Graph

c) Simple Graph

d) Complete Graph

Answer: b

Explanation: The given statement is the definition of regular graphs.

4. The degree of any vertex of graph is .... ?

a) The number of edges incident with vertex

b) Number of vertex in a graph

c) Number of vertices adjacent to that vertex

d**)** Number of edges in a graph

Answer: The number of edges incident with vertex

5.The maximum degree of any vertex in a simple graph with n vertices is

a) n–1

b) n+1

c) 2n–1

d) n

Answer: n–1