Sri SAI RAM ENGINEERING COLLEGE
INSTITUTE OF TECHNOLOGY
West Tambaram, Chennai - 44

SAIRAM DIGITAL RESOURCES

YEAR II

SEM III

MA 8351

DISCRETE MATHEMATICS

UNIT II

COMBINATORICS

2.1 MATHEMATICAL INDUCTION AND STRONG INDUCTION

SCIENCE & HUMANITIES

MATHEMATICAL INDUCTION

INTRODUCTION:

Mathematical induction can be used to prove statements that assert that P(n) is true for all positive integers n, where P(n) is a propositional function.

Principle Of Mathematical Induction

To prove that P(n) is true for all positive integers n, where P (n) is a propositional function, we complete two steps:

Basis step: We verify that P(1) is true.

Inductive step: We show that the conditional statement $P(k) \rightarrow P(k + 1)$ is true for all positive integers k

Problem:1

Show that if n is a positive integer, then $1+2+..+n=\frac{n(n+1)}{2}$

Solution:

Let $p(n)=1+2+..+n=\frac{n(n+1)}{2}$

Now $p(1)=1=\frac{1(1+1)}{2}$

Assume

$$p(k)=1+2+..+k=\frac{k(k+1)}{2}$$

To prove $p(k+1)$ is true

$$p(k+1) = 1+2+....k+k+1 = \frac{(k+1)((k+1)+1)}{2}$$

$$\frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}$$

$$\frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$$

$$\frac{(k+1)(k+2)}{2} = \frac{(k+1)(k+2)}{2}$$

Hence $p(k+1)$ is true

Hence $1+2+..+n=\dfrac{n(n+1)}{2}$ is true for all n

Problem:2

Use mathematical Induction to prove that $(3^n + 7^n - 2)$is divisible by 8, for $n \geq 1$.

Solution:

Let P(n) : $(3^n + 7^n - 2)$is divisible by 8.

Now P(1) : $(3^1 + 7^1 - 2) = 8$ is divisible by 8, is true.

Assume P(k) :$(3^k + 7^k - 2)$ is divisible by 8 is true..... (1)

Claim: P(k+1)is true

$$P(k+1) = 3^{k+1} + 7^{k+1} - 2$$
$$= 3 \cdot 3^k + 7 \cdot 7^k - 2$$
$$= 3.3^k + 3 \cdot 7^k + 4 \cdot 7^k - 6 + 4$$

$= 3(3^k + 7^k - 2) + 4(7^k + 1)$

$\therefore$ $4(7^k + 1)$ is divisible by 8 and by (1) $3(3^k + 7^k - 2)$ is divisible by 8.

$P(k+1) = 3(3^k + 7^k - 2) + 4(7^k + 1)$ is divisible by 8 is true.

Problem: 3

Prove by mathematical induction that $6^{n+2} + 7^{2n+1}$ is divisible by 43 for each positive integer n.

Solution:

S(1): Inductive step: for $n = 1$,

$6^{1+2} + 7^{2+1}$ =559, which is divisible by 43

So S(1) is true.

Assume S(k) is true

(i.e) $6^{k+2} + 7^{2k+1} = 43\,m$ for some integer m.

To prove S(k+1) is true.

Now $6^{k+3} + 7^{2k+3} = 6^{k+3} + 7^{2k+1} . 7^{2}$

$= 6\,(6^{k+2} + 7^{2k+1}) + 4\,3\,.7^{2k+1}$

$= 6\,(6^{k+2} + 7^{2k+1}) + 43 \cdot 7^{2k+1}$

$= 6 \cdot 43\,m + 43 \cdot 7^{2k+1}$

$= 43\,(6m + 7^{2k+1})$

Which is divisible by 43.

So S(k+1) is true.

By Mathematical Induction ,S(n) is true for all integer n

Problem: 4

Using mathematical induction,

prove that $2 + 2^{2} + 2^{3} + \text{.........} + 2^{n} = 2^{n+1} - 2$

Solution:

Let $p(n) = 2 + 2^{2} + 2^{3} + \ldots + 2^{n}$.

Assume $p(1)$: $2^{1} = 2^{1+1} - 2$ is true.

Assume $p(k)$: $2 + 2^{2} + 2^{3} + \text{.........} + 2^{k} = 2^{k+1} - 2$ is true

Claim $p(k+1)$ is true.

$P(k+1)$: $2 + 2^{2} + 2^{3} + \ldots + 2^{k} + 2^{k+1}$

$= 2^{k+1} - 2 + 2^{k+1}$

$= 2 \cdot 2^{k+1} - 2 = 2^{k+2} - 2$

$P(k+1)$ is true.

Hence it is true for all n.

Problem: 5

Use mathematical induction to prove the inequality $n < 2^n$

Solution:

Let P (n) be the proposition that $n < 2^n$.

Now P (1) is true, because $1 < 2^1 = 2$.

Assume that p(k) is true, $k<2^k$

Now $k+1 < 2^k+1 < 2^k+2^k = 2.2^k = 2^{k+1}$

Hence p(k+1) is true.

Problem: 6

Use mathematical induction to prove that $2^n < n!$ for every integer n with $n \geq 4$.

(Note that this inequality is false for n = 1, 2, and 3.)

Solution:

Let P (n) be the proposition that $2^n < n!$.

To prove the inequality for $n \geq 4$ requires that the basis step be P (4). Note that P (4) is true, because $2^4 = 16 < 24 = 4!$

For the inductive step, we assume that P (k) is true for an arbitrary integer k with $k \geq 4$. That is, we assume that $2^k < k!$ for the positive integer k with $k \geq 4$.

We must show that under this hypothesis, P (k + 1) is also true. That is, we must show that if $2^k < k!$ for an arbitrary positive integer k where $k \geq 4$, then $2^{k+1} < (k + 1)!$.

$$2^{k+1} = 2 \cdot 2^k$$

$$< 2 \cdot k!$$

$$< (k + 1)k!$$

$$= (k + 1)!$$

Hence p(k+1) is true.

Problem: 7

Use mathematical induction to prove that $n^3 - n$ is divisible by 3 whenever n is a positive integer

Solution:

To construct the proof, let P (n) denote the proposition: "$n^3 - n$ is divisible by 3."

The statement P (1) is true because $1^{3-}1 = 0$ is divisible by 3

For the inductive hypothesis we assume that P (k) is true

we assume that $k^3 - k$ is divisible by 3 for an arbitrary positive integer k.

$$(k + 1)^3 - (k + 1) = (k^3 + 3k^2 + 3k + 1) - (k + 1)$$

$$= (k^3 - k) + 3(k^2 + k).$$

Using the inductive hypothesis, we conclude that the first term

$k^3 - k$ is divisible by 3. The second term is divisible by 3 because it is 3 times

an integer.

Hence $n^3 - n$ is divisible by 3.

Problem: 8

Using mathematical induction, show that $\sum_{r=0}^{n} 3^r = \frac{3^{n+1}-1}{2}$

Solution:

Let

i) which is true

ii) Assume that is true.

i.e. $3^0+3^1+3^2+\ldots+3^k = \frac{3^{k+1}-1}{2}$

iii) Consider the statement $p(k+1)$

Now

$$p(k+1) \quad 3^0+3^1+3^2+\ldots+3^k+3^{k+1} = \frac{3^{k+1}-1}{2}+3^{k+1}$$

$$= \frac{3^{k+1} - 1 + 2.3^{k+1}}{2}$$

$$= \frac{3.3^{k+1} - 1}{2}$$

$$= \frac{3^{k+2} - 1}{2}$$

Therefore $p(k+1)$ is true.

Hence $\sum_{r=0}^{n} 3^r = \frac{3^{n+1} - 2}{2}$ is true for all n ≥ 0

Strong Induction:

There is another form of mathematics induction that is often useful in proofs. In this form we use the basis step as before, but we use a different inductive step. We assume that p(j)is true for j=1…,k and show that p(k+1)must also be true based on this assumption . This is called strong Induction.

We summarize the two steps used to show that p(n)is true for all positive integers n.

Basis Step : The proposition P(1) is shown to be true

Inductive Step: It is shown that $[P(1) \wedge P(2) \wedge \ldots\ldots \wedge P(k)] > p(k+1)$

Problem: 1

Show that if n is an integer greater than 1, then n can be written as the product of primes.

Solution:

Let p(n) be the proportion that n can be written as the product of primes

Now p(2) is true since 2 can be written as the product of one prime

Assume that p(j) is positive for all integer j with $j \leq k$

Claim: p(k+1) is true

There are two cases to consider namely

i) When (k+1) is prime

ii) When (k+1) is composite

Case i: If (k+1) is prime then p(k+1) is true.

Case ii: If (k+1) is composite then it can be written as the product of two positive integers a and b with $2 \leq a < b \leq k+1$

By the induction hypothesis both a and b can be written as the product of primes, namely those primes in the factorization of a and those in the factorization of b.

1)What is well ordering principle?

Solution:

Every non empty set of non negative integers has a least element. The well ordering property can often be used directly in proofs.