Sri SAI RAM ENGINEERING COLLEGE
INSTITUTE OF TECHNOLOGY
West Tambaram, Chennai - 44

Sairam INSTITUTIONS

SAIRAM DIGITAL RESOURCES

YEAR II

SEM III

MA8351

DISCRETE MATHEMATICS
(COMMON TO CSE & IT)

UNIT 5
LATTICES AND BOOLEAN ALGEBRA

5.1 POSETS AND LATTICES

SCIENCE & HUMANITIES

4 QUALITY EDUCATION

NBA NATIONAL BOARD of ACCREDITATION

NAAC A+

nirf NATIONAL INSTITUTIONAL RANKING FRAMEWORK

Relation:

Relation *R* on a set A is a well-defined rule, which tells whether given two elements *x* and *y* of A are related or not. If *x* is related to *y*, we write *x R y*, otherwise $x \not R y$.

R- is Reflexive:

Let X be a set. R be a relation defined on X. Then R is said to be reflexive if it satisfies the following condition.

$$x\,R\,x \quad , \quad \forall\, x \in X \;\text{ i.e, } \{x/(x,x) \in R\}\ \forall\, x \ \in R$$

R- is Symmetric:

Let X be any set. R be a relation defined on X. Then R is said to be symmetric, if it satisfies the following condition.

$$x\,R\,y \ \Rightarrow \ y\,R\,x\ \forall\, x,y\ \in X \;\text{ i.e, } \{(y,x)/(x,y) \in R \Rightarrow (y,x) \in R\}\ \forall\, x,y\ \in X$$

Note: A relation which is not symmetric is called Asymmetric.

R- is Transitive:

Let X be any set, R be a relation defined on X. Then R is said to be transitive, if R satisfies the following condition.

$$x\,R\,y \,\&\, y\,R\,z \Rightarrow x\,R\,z\,\forall\; x,y,z\;\in X$$

i.e, $\{(x,z)/(x,y) \in\; R\; and\; (y,z) \in R \Rightarrow (x,z) \in R\} \qquad \forall\, x,y,z \in X.$

R- is Antisymmetric:

Let X be any set. R be a relation defined on X. Then R is said to be Antisymmetric, if it satisfies the following condition.

$$x\,R\,y \,\&\, y\,R\,x \;\Rightarrow x = y\;\forall\, x,y\;\in X.$$

Equivalence Relation:

Let X be any set. R be a relation defined on X. If R satisfies Reflexive, symmetric and transitive then the relation R is said to be an Equivalence relation.

Partial Order Relation:

Let X be any set. R be a relation defined on X. Then R is said to be a partial order relation if it satisfies reflexive, antisymmetric and transitive relation.

Example 1: Subset relation " $\subseteq$ " is a partial order relation.

Consider any 3 sets A,B C

(i) Since any set is a subset of itself, $A \subseteq A$, therefore " $\subseteq$ " is reflexive.
(ii) If $A \subseteq B$ and $B \subseteq A$ then A = B. Therefore " $\subseteq$ " is antisymmetric.
(iii) If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$. Therefore " $\subseteq$ " is transitive.

$\therefore$ " $\subseteq$ " is a partial order relation.

Example 2: Divides relation (/) is a partial order relation.

For Z_+ be the set of positive integers, $a, b, c \in Z_+$

(i) Since a/a, "/" is reflexive.
(ii) Since, a/b and b/a $\Rightarrow$ $a = b$, "/" is antisymmetric.
(iii) Since a/b and b/c $\Rightarrow$ a/c , "/" is transitive.

$\therefore$ Divides relation "/" is a partial order relation.

Partially ordered set or Poset:

A set together with a partial order relation defined on it is called partially ordered set or Poset.

Example: (i) Let $\mathbb{R}$ be the set of real numbers. The relation "less than or equal to" or " $\leq$ " is a partial order on $\mathbb{R}$. Therefore, $(\mathbb{R}, \leq)$ is a Poset.

(ii) Let $\wp(A)$ be the power set of A. The relation "$\subseteq$ " (subset relation or inclusion) on $\wp(A)$ is a partial order. Therefore, $(\wp(A), \subseteq)$ is a Poset.

Totally ordered set or linearly ordered set or chain:

A partially ordered set $(\wp, \leq)$ is said to be totally ordered set or linearly ordered set or chain if any two elements are comparable. i.e, given any two elements x and y of a Poset either $x \leq y$ or $y \leq x$.

Example : Show that $(N, \leq)$ is a partially ordered set where N is set of all positive integers and $\leq$ is defined by $m \leq n$ iff $n - m$ is a non-negative integer.

Solution: Given N is the set of all positive integer.

The given relation is, $m \leq n$ iff $n - m$ is a non-negative integer

Now, $\forall\, x \in N$, $x - x = 0$ is non- negative integer.

$\therefore\ x\, R\, x \quad \forall \quad x \in N \ \Rightarrow$ R is reflexive.

Consider $x\, R\, y$ & $y\, R\, x$,

Since $x\, R\, y \ \Rightarrow x - y$ is non negative integer.

$y\, R\, x \ \Rightarrow y - x$ is non negative integer

$\Rightarrow\ -(x - y)$ is non negative integer

$\Rightarrow\ x = y$

$\therefore\ R$ is Antisymmetric.

Assume $x\,R\,y\;\;\&\;\;y\,R\,z$

$x\,R\,y \Rightarrow x - y$ is a non-negative integer.

$y\,R\,z \Rightarrow y - z$ is a non-negative integer

$\Rightarrow (x - y) + (y - z)$ is a non-negative integer.

$\Rightarrow x - z$ is a non-negative integer.

$\Rightarrow x\,R\,z$

$\therefore\; x\,R\,y\;\;\&\;\;y\,R\,z \Rightarrow x\,R\,z$

R is transitive.

Therefore, R is partial order relation.

HASSE DIAGRAM:

Pictorial representation of a Poset is called Hasse diagram.

Problem 1: Draw Hasse diagram for $(\wp(A), \subseteq)$ where

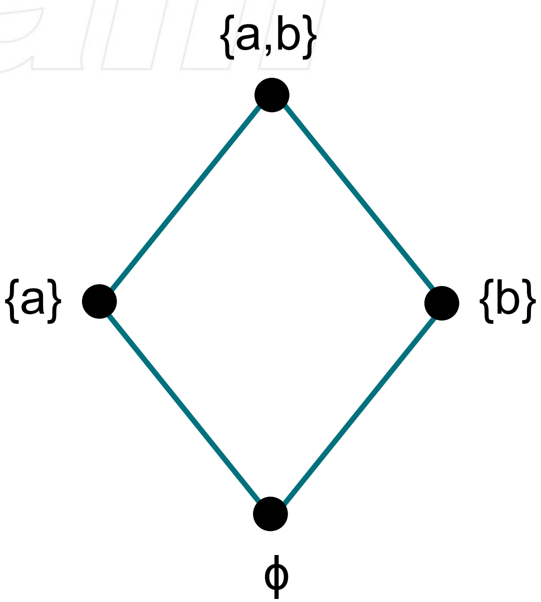
1. $A = \{a, b\}$ 2. $A = \{a, b, c\}$

Solution:

1. $A = \{a, b\}$

$$\wp(A) = \{\{a\}, \{b\}, \{a, b\}, \phi\}$$

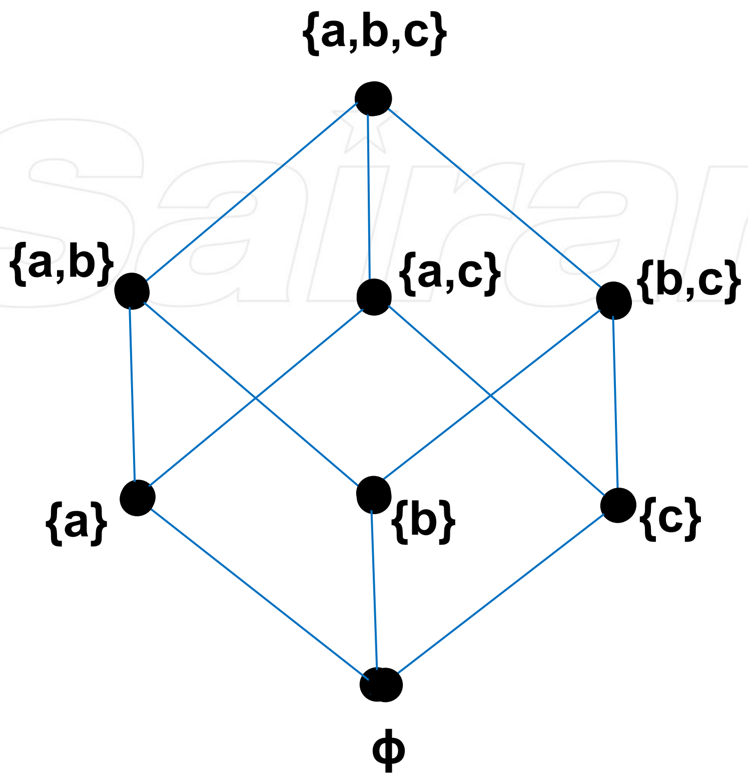
The diagram can be represented as



2. $A = \{a, b, c\}$

$$\wp(A) = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}\{a, c\}, \{a, b, c\}, \phi\}$$

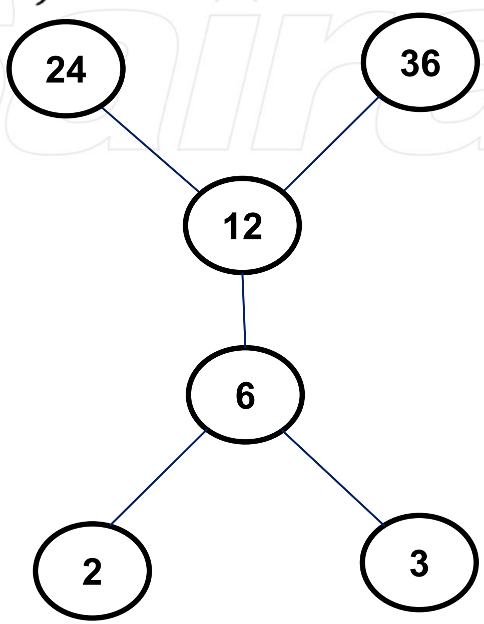
The diagram can be represented as



Problem 2: If $X = \{2,3,6,12,24,36\}$ and the relation R defined on X by $R = \{\langle a,b\rangle / \ a/b\}$. Draw Hasse diagram for (X,R).

Solution: The Relation $R = \left\{\begin{matrix} \langle 2,6\rangle, \langle 2,12\rangle, \langle 2,24\rangle, \langle 2,36\rangle, \langle 3,6\rangle, \langle 3,12\rangle \\ \langle 3,24\rangle, \langle 3,36\rangle, \langle 6,12\rangle, \langle 6,24\rangle, \langle 6,36\rangle, \\ \langle 12,24\rangle, \langle 12,36\rangle \end{matrix}\right\}$

The Hasse diagram for (X,R) is

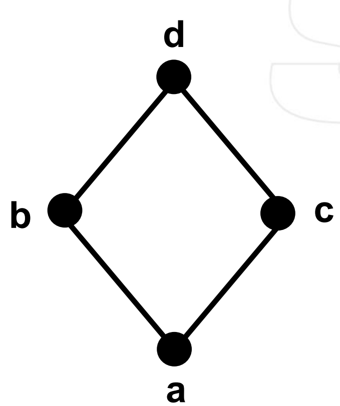


Definition:

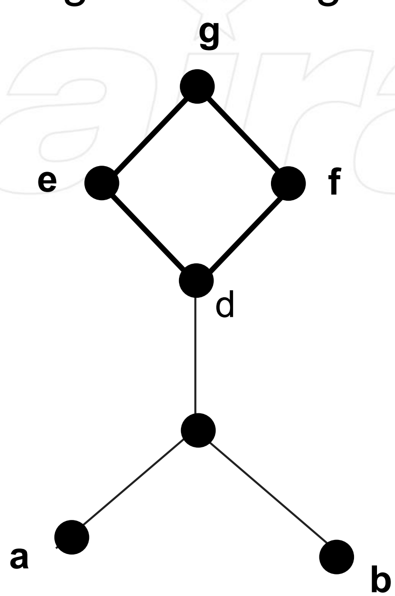
Let $(P, \leq)$ be a Poset. An element $a \in P$ is called Least element in P, if $a \leq x$ for all $x \in P$.

An element $b \in P$ is called the Greatest element in P, if $b \geq x$ for all $x \in P$.

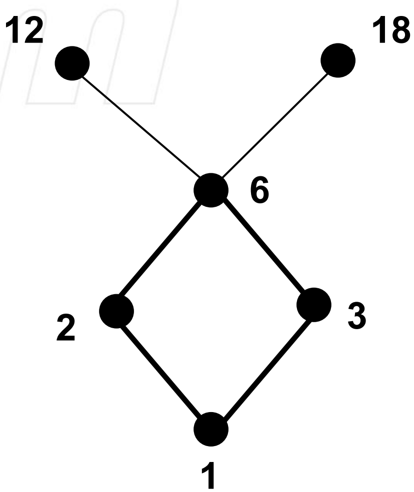
Example: Consider the following Hasse diagrams.



(i)



(ii)



(iii)

- In (i) “a” is the least element and “d” is the greatest element.
- In (ii) “g” is the greatest element and there is no least element.
- In (iii) “1” is the least element and there is no greatest element.

Definition:

Let $(P, \leq)$ be a Poset and A be any non-empty subset of P. An element $a \in P$ is an upper bound of A, if $a \geq x$ for all $x \in A$.

An element $b \in P$ is said to be lower bound for A, if $b \leq x$ for all $x \in A$.

Least Upper Bound (LUB):

Let $(P, \leq)$ be a Poset and $A \subseteq P$. An element $a \in P$ is said to be least upper bound (LUB) or supremum(sup) of A, if

1. “a” is a upper bound of A.
2. $a \leq c$, where c is any other upper bound of A.

Greatest Lower Bound (GLB):

Let $(P, \leq)$ be a Poset and $A \subseteq P$. An element $b \in P$ is said to be greatest lower bound (GLB) or infimum(inf) of A, if

1. "b" is a lower bound of A.
2. $b \geq d$, where "d" is any other lower bound of A.

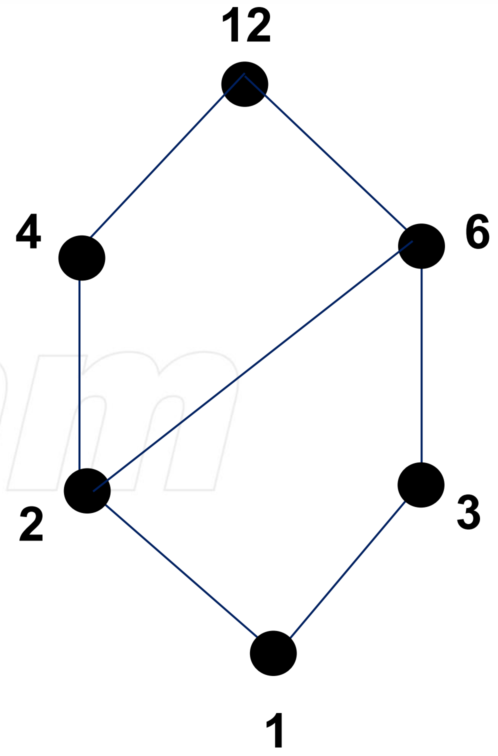
Example 1: Consider $X = \{1,2,3,4,6,12\}$. $R = \{\langle a, b\rangle / \ a/b\}$. Find Least upper bound and greatest lower bound for the Poset (X, R).

Solution: The Relation

$$R = \begin{Bmatrix} \langle 1,2\rangle\langle 1,3\rangle\langle 1,4\rangle\langle 1,6\rangle\langle 1,12\rangle\langle 2,4\rangle \\ \langle 2,6\rangle\langle 2,12\rangle\langle 3,6\rangle\langle 3,12\rangle\langle 4,12\rangle \end{Bmatrix}$$

The Hasse diagram for (X, R) is

1. UB {1,3} = {3,6,12}
   LUB {1,3} = {3}
2. UB {1,2,3} = {6,12}
   LUB {1,2,3} = {6}
3. UB {2,3} = {6,12}
   LUB {2,3} = {6}
4. UB {2,3,6} = {6,12}
   LUB {2,3,6} = {6}
5. UB {4,6} = {12}
   LUB {4,6} = {12}
6. LB {4,6} = {1,2}
   GLB {4,6} = {2}
7. LB {2,3,6} = {1}
   GLB {2,3,6} = {1}
8. LB {1,3} = {1}
   GLB {1,3} = {1}

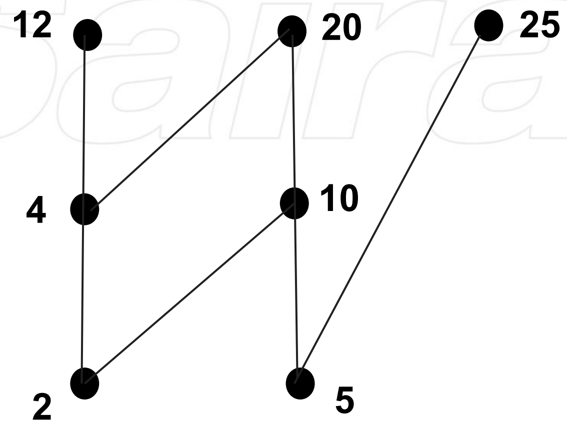


Example 2: Draw the Hasse diagram for the Poset $(\{2,4,5,10,12,20,25\},/)$. What are maximal and minimal elements of the Poset?

Solution: The relation R is

$$R = \left\{ \begin{matrix} \langle 2,4\rangle\langle 2,10\rangle\langle 2,12\rangle\langle 2,20\rangle\langle 4,12\rangle\langle 4,20\rangle \\ \langle 5,10\rangle\langle 5,20\rangle\langle 5,25\rangle\langle 10,20\rangle \end{matrix} \right\}$$

The Hasse diagram is



The maximal elements are 12,20 and 25.

The minimal elements are 2 and 5.

Example 3: Let D_{36} denote the set of divisors of 36. Let the relation $\leq$ be given by $a \leq b$ if a/b. Draw the Hasse diagram for the Poset $(D_{36},/)$.

Solution:

Let $D_{36} = \{1,2,3,4,6,9,12,18,36\}$

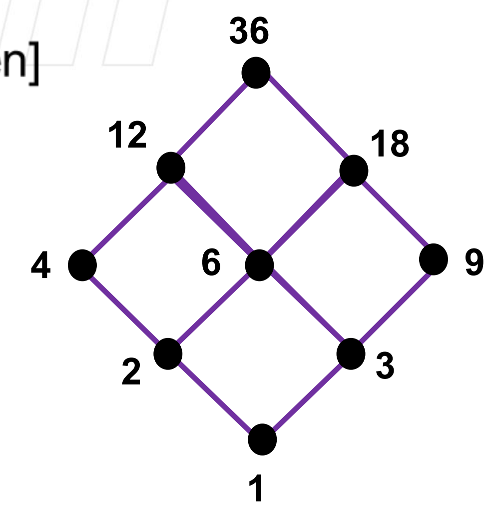
The immediate successors are1<<2, 1<<3, 2<<4, 2<<6 $[4 \nmid 6$, 4 cannot come between 2 and 6]

3<<6, 3<<9 $[\, 6 \nmid 9$, 6 cannot come between 3 and 9]

4<<12 $[4 \nmid 6, 4 \nmid 9$, so 6 and 9 cannot come between]

6<<12, 6<<18, 9<<18, 12<<36, 18<<36.

The Hasse diagram is



LATTICE:

A Lattice is a partially ordered set (Poset)$(L, \leq)$, in which for every pair of elements $a, b \in L$, both greatest lower bound (GLB) and least upper bound (LUB) exists.

Note: (i) The Greatest lower bound (GLB of inf) of a subset having 2 elements $\{a, b\} \subseteq L$ is denoted by $a * b$ or $a \wedge b$ and it is called the product or meet of a and b.

(ii) The Least upper bound (LUB or sup) of {a,b} is denoted by $a \oplus b$ or $a \vee b$ and is called the sum or join of a and b.

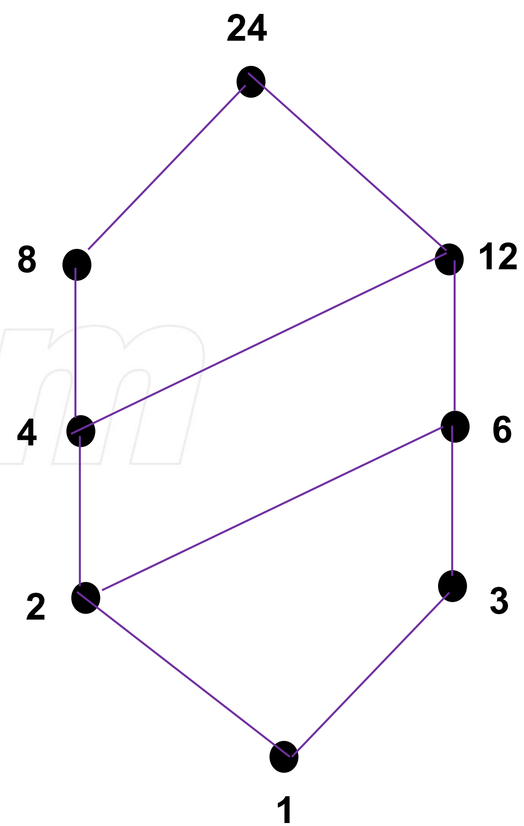
Example: Let S_{24} be the set of all divisor of 24 and D denote the relation of division. Then (S_{24}, D) is a Lattice.

Solution: Let $S_{24} = \{1,2,3,4,6,8,12,24\}$

$D = \{\langle a, b\rangle / \ a \ /b\}$. The relation D is

$$D = \begin{Bmatrix} \langle 1,2\rangle\langle 1,3\rangle\langle 1,4\rangle\langle 1,6\rangle\langle 1,8\rangle\langle 1,12\rangle\langle 1,24\rangle \\ \langle 2,4\rangle\langle 2,6\rangle\langle 2,8\rangle\langle 2,12\rangle\langle 2,24\rangle\langle 3,6\rangle\langle 3,12\rangle \\ \langle 3,24\rangle\langle 4,8\rangle\langle 4,12\rangle\langle 4,24\rangle\langle 6,12\rangle\langle 6,24\rangle \end{Bmatrix}$$

1. UB {1,3} = {6,12,24}
   LUB {1,3} = {6}
2. UB {1,2,3} = {6,12,24}
   LUB {1,2,3} = {6}
3. UB {6,8} = {24}
   LUB {6,8} = {24}
4. UB {4,6} = {12,24}
   LUB {4,6} = {12}
5. LB {4,6} = {2,1}
   GLB {4,6} = {2}
6. LB {8,6} = {2,1}
   GLB {8,6} = {2}
7. LB {2,3,6} = {1}
   GLB {2,3,6} = {1}
8. LB {8,12,24} = {4,2,1}
   GLB {8,12,24} = {4}



We can see that the GLB and LUB exists for every pair of elements.

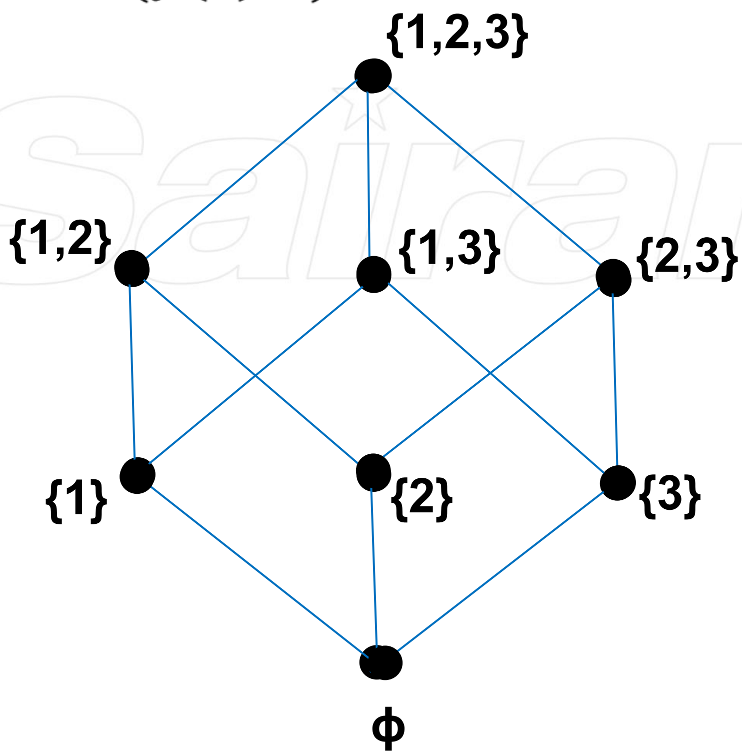
Hence, the given relation is a Lattice.

Example 2: Determine whether $(\wp(A), \subseteq)$ where $A = \{1,2,3\}$ is a Lattice.

Solution: Given $A = \{1,2,3\}$

$$\wp(A) = \{\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}, \phi\}$$

Now, the Hasse diagram for $(\wp(A), \subseteq)$ is



1.		
2.	LUB [{1} ,{2}] = {1,2}	
3.	LUB [{1},{3}] = {1,3}	
4.	LUB [{1}, {1,2}] = {1,2}	GLB [{1}, {1,2}] = {1}
5.	LUB [{1}, {1,3}] = {1,3}	GLB [{1}, {1,3}] = {1}
6.	LUB [{1}, {2,3}] = {1,2,3}	
7.	LUB [{1}, {1,2,3}] = {1,2,3}	GLB [{1}, {1,2,3}] = {1}

Similarly, we can observe that GLB and LUB exists for all pairs of elements.

Hence, $(\wp(A), \subseteq)$ is a Lattice.