Sri SAI RAM ENGINEERING COLLEGE

INSTITUTE OF TECHNOLOGY

West Tambaram, Chennai - 44

Sairam INSTITUTIONS

SAIRAM DIGITAL RESOURCES

YEAR II

SEM III

UNIT III

GRAPHS

3.5 EULER AND HAMILTON PATHS

MA8351

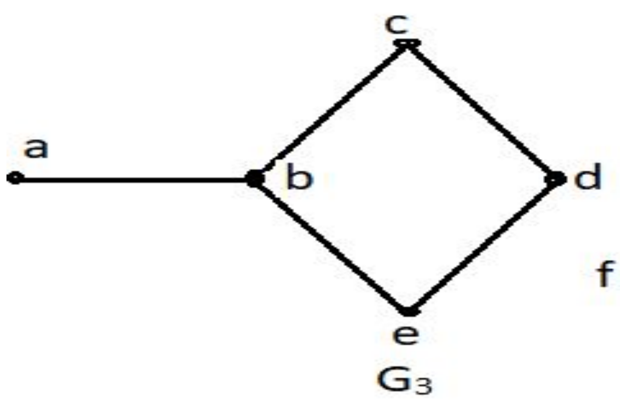
DISCRETE MATHEMATICS

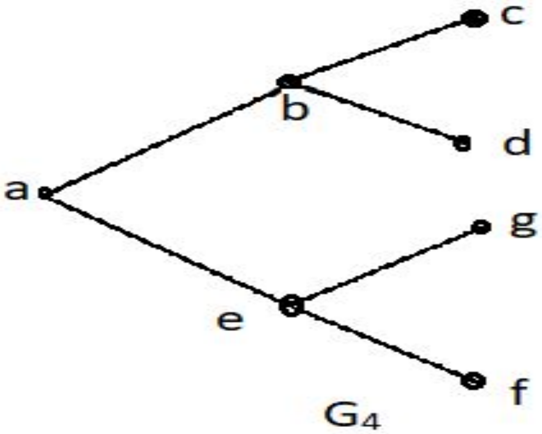
(Common to CSE & IT)

SCIENCE & HUMANITIES

4 QUALITY EDUCATION

The graph G_1 is Eulerian because it contains an Euler circuit *a,b,c,e,f,g,c,d,a* it contains edge only once.
The graph G_2 is Eulerian because it contains an Euler circuit *a.c.e.d.c.b.a.* It contains each edge only once.





The graph G_3 is not all Eulerian because it does not contain Euler circuit, but it contain an Euler path *a,b,c,d,e,b* containing every edge only once.

G_4 has no Euler path or Euler circuit.

EULER AND HAMILTON PATHS

Euler path and circuits

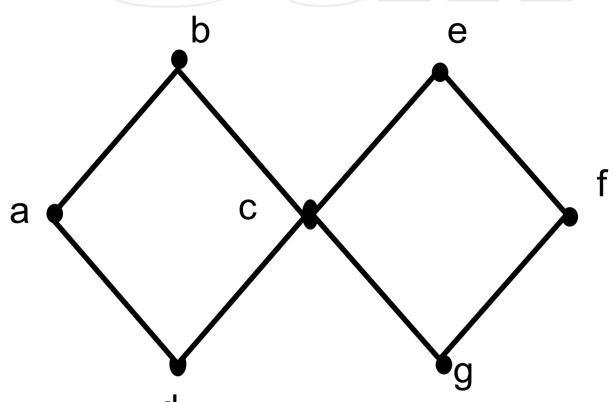
Definition

An Euler circuit or cycle in a graph G is a simple circuit containing every edge of G.

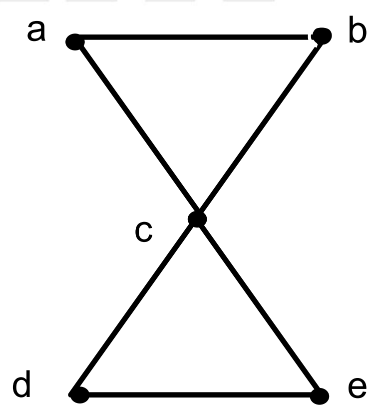
An Euler path in a graph G is a simple path containing every edge of G.

A connected graph with an Euler circuit is called an Euler graph for Eulerian graph.

Examples



G_1



G_2

Theorem:

A connected graph G is an Euler graph iff all the vertices of G are of even degree.

Proof:

Necessary part:

Let G be a graph which has an Euler circuit.
To prove that all the vertices are of even degree.
We assume that G has an Euler circuit . Hence, there is a closed path which passes through all the edges exactly once. Let the Euler circuit begins with a vertex '*a*' and continues with an edge incident with '*a*' (i.e) (*a,b*) . This edge contributes one to deg(*a*). Since the path traces each edge exactly once, hence every time when we visit a vertex, we need two new edges one toenter and another to exit. These two edges contribute 2 to the vertex's degree.
This procedure is repeated and finally the circuit terminates where it started contributing 1 to deg (*a*).

∴deg(a) is even and all the intermediate vertices will also be of even degree.
∴All the vertices of the graph G has even degree.

Sufficient Part:

Let us assume that the graph G be a connected graph with all the vertices of even degree.

To prove: G has an Euler circuit.

i.e. To prove that G has a closed path passing through all the edges of G exactly once.

We construct a path passing through all the edges of G exactly once. First we arbitrarily choose a vertex v_0 , we choose an edge (v_0,v_1) incident with v_0 .

We continue to trace a path as long as possible. The path terminates at some stage, since the graph has a finite number of edges. If all the edges of G are covered by this closed path, then this will be the required Euler circuit.

Suppose if this circuit does not contain all the edges of G, consider a sub graph H of G by deleting the edges already used in the circuit.

Since G is connected, H has at least one vertex in common with the circuit that has been deleted. Let it be the vertex w. Note that every vertex in H has even degree. We repeat that same procedure in this graph H. Continue this process until all the edges has been used. The union of these paths forms the required Euler circuit.

Hence the theorem.

Theorem:

A connected graph has an Euler path but not an Euler Circuit if and only if it has exactly two vertices of odd degree.

Proof:

Given G is a connected graph.

Suppose it has an Euler path from v_0 to v_n , say $v_0, e_1, v_1, e_2, \ldots\ldots, e_n, v_n$.

The edges $e_1$and e_n contribute 1 to the degrees of v_0 and v_n respectively.

Every time the path passes through a vertex, it contributes 2 to its degree.

Its true for v_0 and v_n also.

So, the degree of v_0 and v_n are always odd and the degrees of each internal vertices remain even.

Thus the graph contains exactly two vertices of degree.

Conversely, let the connected graph G contains graph G contains two vertices of odd degree , say v_0 and v_n.

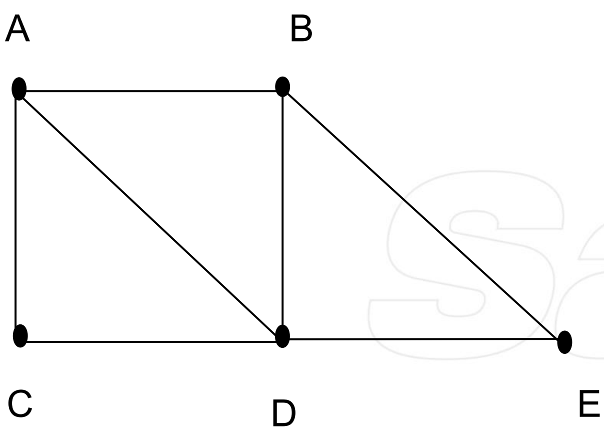
Adding a new edge $e = v_0 v_n$ to G we get a graph $G_1$with all even degree vertices.

Therefore by previous theorem G_1 is Eulerian.

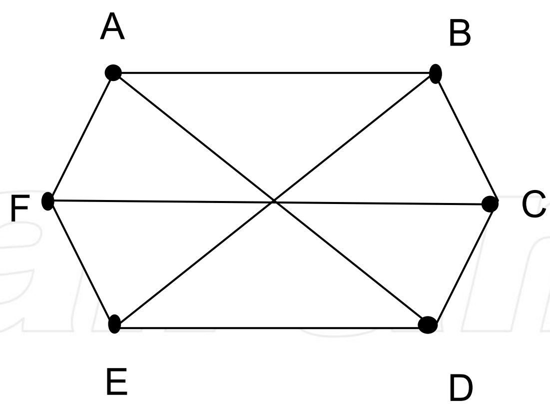
Removing $e = v_0 v_n$ from G_1, we get G containing an Euler path from v_0 to v_n.

Problems:

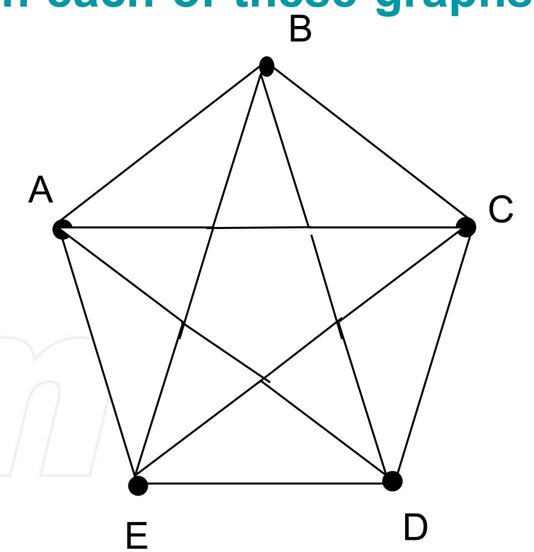
1. **Find an Euler path or an Euler circuit, if it exits in each of these graphs below. If it does not exist, explain why?**



G_1



G_2



G_3

Solution:

In G_1, there are only two vertices namely A and B of degree 3 and other vertices are of even degree.

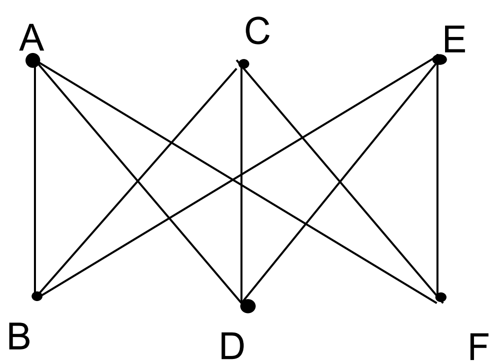
Hence, there exists an Euler path between A and B. The actual path is $A-B-E-D-A-C-D-B$. This is an Eulerian path, as it includes each of the 7 edges exactly once.

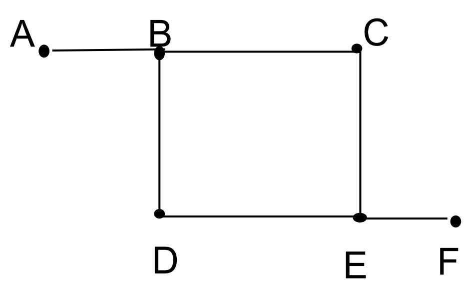
In G_2, there are 6 vertices of odd degree. Hence, G_2 contains neither an Euler path nor an Euler circuit.

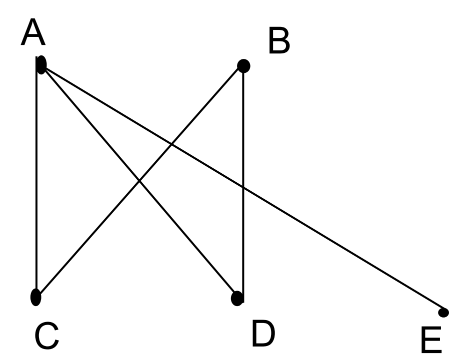
In G_3, all the vertices of even degree. Hence, there exist an Euler circuit in G_3.

It is $A-B-C-D-E-A-C-E-B-D-A$. This circuit is Eulerian, since it includes each of the 10 edges exactly once.

2. Find a Hamiltonian path or a Hamiltonian circuit, if it exists in each of these graphs. If it does not exist, explain why?







Solution:

G_1 contains a Hamiltonian circuit, for example

$A-B-C-D-E-F-A$. In fact there are 5 more Hamiltonian circuits in G_1, namely $A-B-C--F-E-D-A$, $A-B-E-D-C-F-A$,

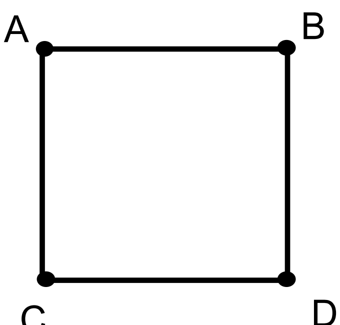
$A-B-E-F-C-D-A$, $A-D-C-B-E-F-A$ and $A-D-E-B-C-F-A$.

G_2 contains neither a Hamiltonian path nor a Hamiltonian circuit. Since, any path containing all the vertices must contain one of the edges $A-B$ and $E-F$ more than once.

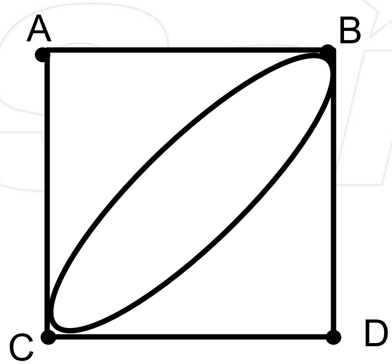
G_3 contains 2 Hamiltonian paths from **C** to ***E*** and from ***D*** to ***E***, namely $C-B-D-A-E$ and $D-E-C-A-E$ but no Hamiltonian circuits.

3. Give an example of a graph which contains

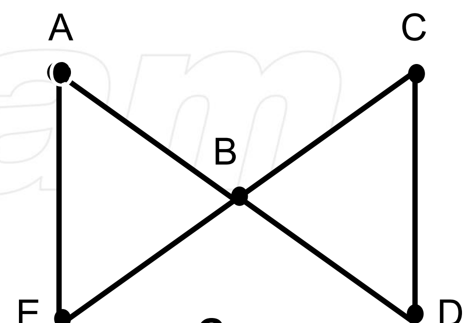
i. An Eulerian circuit that is also a Hamiltonian circuit.
ii. An Eulerian circuit and a Hamiltonian circuit that are distinct.
iii. An Eulerian circuit, but not a Hamiltonian circuit.
iv. An Hamiltonian circuit, but not an Eulerian circuit.
v. Neither an Eulerian circuit nor a Hamiltonian circuit.



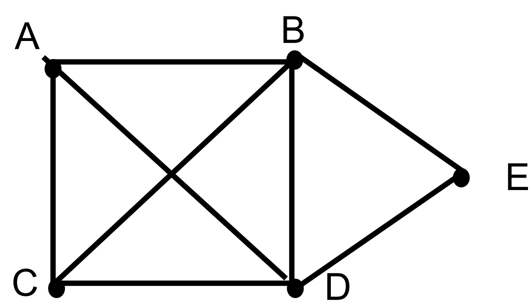
G_1



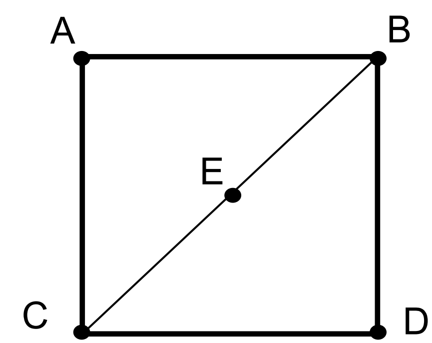
G_2



G_3



G_4



G_5

i. The circuit $A-B-C-D-A$ in G_1 consists of all edges and all vertices, each exactly once.

$\therefore$ G_1 contains a circuit that is both Eulerian and Hamiltonian.

ii. The graph G_2 contains the Eulerian circuit $A-B-D-B-C-D-A$ and the Hamiltonian circuit $A-B-C-D-A$, but the two circuits are different.

iii. The graph G_3 contains the Eulerian circuit $A-B-C-D-B-E-A$, but this circuit is not Hamiltonian as the vertex ***B*** repeated twice.

iv. The graph G_4 contains the Hamiltonian circuit $A-B-C-D-E-A$. However, it does not contain Eulerian circuit as there are 4 vertices each of degree 3.

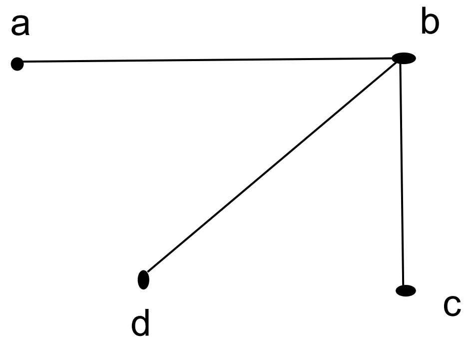
v. In G_5, degree of ***B*** and degree of ***D*** each equal to 3. Hence, there is no Euler circuit in it. Also no circuit passes through each of the vertices exactly once.

4. Give an example of a connected graph that has

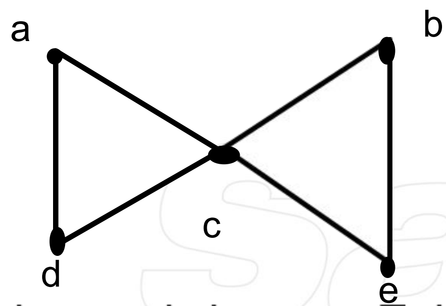
a) Neither an Euler circuit nor a Hamiltonian cycle.
b) An Euler circuit but no Hamiltonian cycle.
c) An Euler path but no Euler circuit.
d) A Hamiltonian cycle but no Euler circuit.
e) Both a Hamiltonian cycle and an Euler circuit.
f) A Hamiltonian path but no Hamiltonian circuit.

Solution:

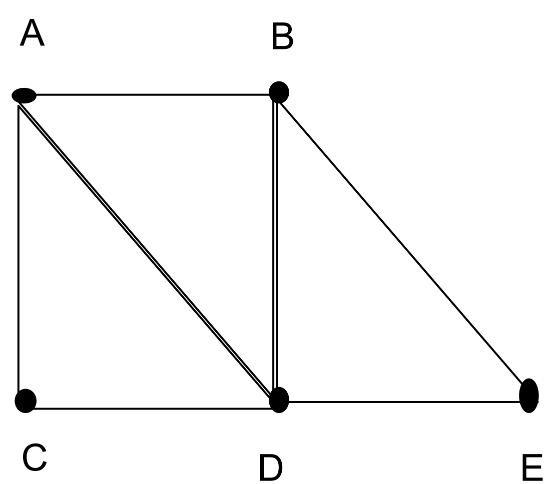
a) This contains no cycle (circuit) neither an Euler circuit nor a Hamiltonian cycle without traversing an edge more than once or traversing a vertex more than once.



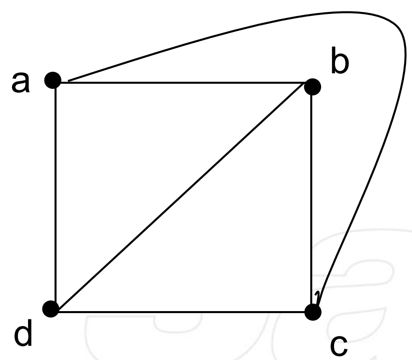
b) This graph has an Euler circuit a, c, e, b, c, d, a in which every edge is traversed exactly once but has no Hamiltonian cycle contains every vertex exactly once. (Here vertex $\boldsymbol{c}$ will appear twice)



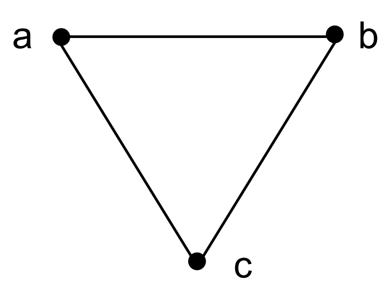
c) This graph has an Euler path a, c, d, e, b, d, a, b but has no Euler circuit. (because for a cycle an edge is to be traversed more than once)



d) This graph has a Hamiltonian cycle $\boldsymbol{a, b, c, d, a}$ but has no euler circuit covering all edges exactly once



e) This graph has Hamiltonian cycle $\boldsymbol{a, b, c, a}$ (touching each vertex exactly once) and has a Euler circuit $\boldsymbol{a, b, c, a}$ (traversing each edge exactly once)



f) This graph has Hamiltonian path d, c, b, a (with every vertex exactly once) but has no Hamiltonian circuit since any circuit containing every vertex must contain the edge $\{a, b\}$ twice.