Parameter Estimation (102103417)

Let $X_1, X_2 \ldots$ be a random sample of size n taken from a normal population with parameter mean $= \theta_1$ & variance $= \theta_2$. Find the maximum likelihood estimates of these two parameters.

$$\rightarrow \quad \text{pmf}(x_i) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x_i-\mu_i)^2/2\sigma^2}$$

$$\mu = \theta_1 \qquad \sigma^2 = \theta_2$$

$$f(x_i|\theta_1,\theta_2) = \frac{1}{\sqrt{2\pi\theta_2}} e^{-(x_i-\theta_1)^2/2\theta_2}$$

now likelihood functn

$$L(\theta_1,\theta_2) = \prod_{i=1}^{n} f(x_i|\theta_1,\theta_2)$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta_2}} e^{-(x_i-\theta_1)^2/2\theta_2}$$

$$L(\theta_1,\theta_2) = \prod_{i=1}^{n} (\theta)^{-1/2} \prod_{i=1}^{n} (2\pi)^{-1/2} \prod_{i=1}^{n} e^{-(x_i-\theta_i)^2/2\theta_2}$$

$$L(\theta_1,\theta_2) = (\theta_2)^{-n/2} (2\pi)^{-n/2} e^{-\left(\sum_{i=1}^{n} (x_i-\theta_1)^2/2\theta_2\right)}$$

$$L(\theta_1,\theta_2) = \frac{\theta_2}{2}^{-n/2} (2\pi)^{-n/2} e^{-\frac{1}{2\theta_2}\sum_{i=1}^{n}(x_i-\theta_1)^2}$$

$$\ln(L(\theta_1,\theta_2)) = \ln\left[\theta_2^{-n/2}(2\pi)^{-n/2} e^{-1/2\theta_2 \sum_{i=1}^{n}(x_i-\theta_1)^2}\right]$$

$$= \ln(\theta_2)^{-n/2} + \ln(2\pi)^{-n/2} + \ln\left(e^{-1/2\theta_2\sum_{i=1}^{n}(x_i-\theta_1)^2}\right)$$

$$\ln L(\theta_1,\theta_2) = -\frac{n}{2}\ln\theta_2 - \frac{n}{2}\ln 2\pi - \frac{1}{2\theta_2}\sum_{i=1}^{n}(x_i-\theta_1)^2$$

$$\frac{\partial L(\theta_1,\theta_2)}{\partial\theta_1} = \frac{1}{\theta_2}\sum_{i=1}^{n}(x_i-\theta_1)$$

$$\text{now, } \frac{\partial L(\theta_1,\theta_2)}{\partial\theta_1} = 0$$

$$\text{hence } \frac{1}{\theta_2}\sum_{i=1}^{n}(x_i-\theta_1) = 0$$

$$(1/\theta_2)\left(\sum_{i=1}^{n} x_i - n\theta_1\right) = 0$$

$$\sum_{i=1}^{n} x_i - n\theta_1 = 0$$

$$\theta_1 = \sum_{i=1}^{n} x_i/n \qquad \theta_{1\,MLE} = \overline{x}_n$$

$$\theta_1 = \overline{x}_n$$

$$\frac{\partial L(\theta_1,\theta_2)}{\partial \theta_2} = \frac{-n}{2\theta_2} - \frac{1}{2(\theta_2)^2}\sum_{i=1}^{n}(x_i-\theta_1)^2$$

Since $\frac{\partial L(\theta_1,\theta_2)}{\partial \theta_2} = 0$

$$\frac{-n}{2\theta_2} - \frac{1}{2\theta_2^2}\sum_{i=1}^{n}(x_i-\theta_1)^2 = 0$$

$$\frac{1}{2\theta_2^2}\sum_{i=1}^{n}(x_i-\theta_1)^2 = -n/2\theta_2$$

$$\theta_2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\theta_1)^2$$

$$\theta_2\ MLE = \frac{1}{n}\sum_{i=1}^{n}(x_i-\overline{x}_n)^2$$

Q2 Let $x_1, x_2 \dots x_n$ be a random sample from $B(m,\theta)$ distribution where $\theta \in (0,1)$ is unknown & 'm' is a known positive integer. Compute θ using MLE

$\to$ $pmf(x_i) = {}^{n}C_{x_i}\, p^{x_i}(1-p)^{n-x_i}$ $\quad n=m,\ p=\theta$

$$P(x_i|m,\theta) = {}^{m}C_{x_i}\,\theta^{x_i}(1-\theta)^{m-x_i}$$

$$L(P) = \prod_{i=1}^{n} P(x_i|m,\theta)$$

$$= \prod_{i=1}^{n}\left({}^{m}C_{x_i}\,\theta^{x_i}(1-\theta)^{m-x_i}\right)$$

$$= \prod_{i=1}^{n}{}^{m}C_{x_i}\prod_{i=1}^{n}\theta^{x_i}\prod_{i=1}^{n}(1-\theta)^{m-x_i}$$

$$= \prod_{i=1}^{n}{}^{m}C_{x_i}\ \theta^{\sum_{i=1}^{n}x_i}(1-\theta)^{nm-\sum_{i=1}^{n}x_i}$$

$$\ln(L(P)) = \ln\left(\prod_{i=1}^{n}{}^{m}C_{x_i}\right) + \ln\left(\theta^{\sum_{i=1}^{n}x_i}\right) + \ln\left((1-\theta)^{nm-\sum_{i=1}^{n}x_i}\right)$$

$$= \ln\left(\prod_{i=1}^{n}{}^{m}C_{x_i}\right) + \ln(\theta)\cdot\sum_{i=1}^{n}x_i + \ln(1-\theta)\left(nm-\sum_{i=1}^{n}x_i\right)$$

$$\frac{\partial L(P)}{\partial \theta} = \frac{1}{\theta}\sum_{i=1}^{n}x_i - \frac{1}{1-\theta}\left(nm-\sum_{i=1}^{n}x_i\right), \quad \frac{\partial L(P)}{\partial\theta} = 0$$

$$\frac{1}{\theta}\sum_{i=1}^{n}x_i = \frac{1}{1-\theta}\left(nm-\sum_{i=1}^{n}x_i\right)$$

$$\frac{1-\theta}{\theta} = \frac{nm-\sum_{i=1}^{n}x_i}{\sum_{i=1}^{n}x_i} \qquad \theta = \sum_{i=1}^{n}x_i/nm$$

$$\theta = \frac{\overline{x}_n}{m} \qquad \theta_{MLE} \in (0,1) = \frac{\overline{x}_n}{m}$$