Exercise 4.7.28:

(a) We can use the fact that L^2 is self-dual so that Hölder's inequality becomes

$$\left|\int_B K(x,y)f(y)dy\right| \leq \left(\int_B |K(x,y)|^2 dy\right)^{1/2} \left(\int_B |f(y)|^2 dy\right)^{1/2}$$

Applying the estimate $|K(x,y)| \leq A|x-y|^{\alpha-d}$ we see that

$$\begin{aligned}\left|\int_B K(x,y)f(y)dy\right| &\leq \left(\int_B A^2|x-y|^{2(\alpha-d)} dy\right)^{1/2} \left(\int_B |f(y)|^2 dy\right)^{1/2} \\ &\leq \left(A^2 2^{2(\alpha-d)} \int_B 1 dy\right)^{1/2} \left(\int_B |f(y)|^2 dy\right)^{1/2} \\ &= A2^{\alpha-d} m(B)^{1/2} \left(\int_B |f(y)|^2 dy\right)^{1/2}\end{aligned}$$

where $m(B)$ is the measure of the unit ball in $\mathbb{R}^d$. Because $f \in L^2(B)$ we have that the right hand side above is bounded and hence, if we square the above inequality and integrate (with respect to x) we get

$$\|Tf\|^2_{L^2(B)} \leq (A2^{\alpha-d}m(B)^{1/2})^2 \|f\|^2_{L^2(B)}$$

Equivalently,

$$\|Tf\|_{L^2(B)} \leq A2^{\alpha-d}m(B)^{1/2}\|f\|_{L^2(B)}$$

So we have that

$$\|T\| = \inf\{M \mid \|Tf\|_{L^1} \leq M\|f\|_{L^2}\} \leq A2^{\alpha-d}m(B)^{1/2}$$

Which means that T is a bounded operator on $L^2(B)$.

(b) Following the hint, define

$$K_n(x,y) = \begin{cases} K(x,y) & |x-y| \geq 1/n \\ 0 & \text{otherwise} \end{cases}$$

and

$$T_n(f)(x) = \int_B K_n(x,y)f(y)dy$$

We begin by verifying the following

Claim. *Each of the operators T_n is a compact operator.*

Proof. This is clear if we note that we just apply Exercise 4.7.26 to each of the T_n. this exercise was proved on the homework and gives that each operator is compact. □

We now want to show that $T_n \to T$ in the operator norm. Indeed,

$$\int_B |K_n(x,y) - K(x,y)|dy \leq \int_{|x-y|\leq 1/n} A|x-y|^{\alpha-d}dy$$

Then if we set

$$M_n = \int_{|z|\leq 1/n} \frac{1}{|z|^{d-\alpha}}dz$$

Then $1/|z|^{d-\alpha} \in L^1(\mathbb{R}^d)$ because the exponent is less that d. Consequently, the absolute continuity of the integral implies that $M_n \to 0$ as $n \to \infty$. Substituting this in the above gives that

$$\int_B |K_n(x,y) - K(x,y)| dy \leq AM_n$$

Which also goes to 0 as $n \to \infty$ and so $\|T_n - T\| \to 0$. Because each T_n is compact and $T_n \to T$ we can apply Proposition 6.1 to get that T is compact as well.

Exercise 6.7.8:

(a) We note that we can decompose Q_1 into a disjoint union of exactly n^d cubes of side length $1/n$. Each of these cubes is a translate of the cube $Q_{1/n}$. So because the union is disjoint we see that

$$\mu(Q_1) = \sum_{k=1}^{n^d} \mu(Q_{1/n}) = n^d \mu(Q_{1/n})$$

Dividing out n^d gives that

$$m(Q_{1/n}) = n^{-d}\mu(Q_1)$$

And if we let $\mu(Q_1) = c$ then we have that $\mu(Q_{1/n}) = cn^{-d}$.

(b) We want to see that $\mu \ll m$. So we need to see that for every Lebesgue measurable set E with measure 0, then $\mu(E) = 0$. Indeed, choose a μ-measurable set E with $m(E) = 0$. Then for any $\epsilon > 0$ we can find an open set $\mathcal{O}$ such that $m(\mathcal{O} - E) < \epsilon$. Consequently, we see that $m(\mathcal{O} < \epsilon$. Then we can decompose $\mathcal{O}$ into a countable union of almost disjoint cubes Q_j with side length $1/n$ for some integer n (Theorem 1.1.4). Then by the previous part $\mu(Q_j) = cm(Q_j)$ and so

$$\mu(\mathcal{O}) = \sum_{j=1}^{\infty} \mu(Q_j) = \sum_{j=1}^{\infty} cm(Q_j) = cm(\mathcal{O} < c\epsilon$$

Letting $\epsilon \to 0$ shows that $\mu(E) = 0$. Thus, $\mu \ll m$ and we can apply Theorem 6.4.3 to get that

$$\mu(E) = \int_E f dm$$

We then immediately get that f is locally integrable because it must be integrable on every μ-measurable set (Borel sets) and in particular, every ball.

(c) Let x be a point in the Lebesgue set of f and let Q_n be a collection of half-open dyadic rational cubes that contain x. Then the family Q_n shrinks regularly to x because the ratio of a cube Q of size length n to its circumscribing ball is given by

$$\frac{m(Q)}{m(B_Q)} = \frac{n^d}{r^d v_d m(B_1(0))} = \frac{n^d}{2^{-1/2} n^d v_d m(B_1(0))} = \frac{\sqrt{2}}{v_d m(B_1(0))}$$

where $B_1(0)$ is the unit ball about the origin and v_d is a constant depending only on d, and hence the ratio is constant. For each cube we have that

$$\frac{1}{m(Q_n)} \int_{Q_n} f dm = \frac{1}{m(Q_n)} \cdot \mu(Q_n) = \frac{cm(Q_n)}{m(Q_n)} = c$$

So $f(x) = c$ at every Lebesgue point of f. We then note that because f is locally integrable almost every point of $\mathbb{R}^d$ is in the Lebesgue set of f and so $f(x) = c$ almost everywhere.

Exercise 1.8.10:

(a) To see that $\mathbb{R}^d$ with Lebesgue measure is separable we set the collection of sets $\{E_k\}_{k=1}^{\infty}$ to be all finite unions of cubes with side length $1/n$ with rational endpoints for some integer n. Then for each measurable set E in $\mathbb{R}^d$ and $\epsilon > 0$ we can choose a family of cubes such that

$$E \subset \bigcup_{j=1}^{\infty} Q_j \text{ and } \sum_{j=1}^{\infty} m(Q_j) \leq m(E) + \epsilon/2$$

This choice can always be made because if E is measurable then we can find an open set $\mathcal{O} \supset E$ such that $m(\mathcal{O} - E) < \epsilon$ and we can choose a set of cubes Q_j such that $\sum_{j=1}^{\infty} m(Q_j) = m(\mathcal{O})$.

Since $m(E) < \infty$, the series converges and so we can find an N large enough that $\sum_{j>N} m(Q_j) < \epsilon/2$. Then if $F = \bigcup_{j=1}^{N} Q_j$ we see

$$\begin{aligned} m(E \triangle F) &= m(E - F) + m(F - E) \\ &\leq m\left(\bigcup_{j>N} Q_j\right) + m\left(\bigcup_{j \leq N} Q_j - E\right) \\ &\leq \sum_{j>N} m(Q_j) + \sum_{j=1}^{\infty} m(Q_j) - m(E) \\ &\leq \epsilon \end{aligned}$$

Letting $\epsilon \to 0$ gives the result.

(b) We need to find the countable dense subset in $L^p(X)$. Consider all functions of the form $r\,\chi_{E_k}(x)$ where r is a complex number with rational real and imaginary parts and E_k is the family of measurable sets in X such that $\mu(E) < \infty$ implies that $\mu(E \triangle E_{n_k}) \to 0$ as $k \to \infty$. I claim that the set of finite linear combinations of these functions is dense in $L^p(X)$.

Suppose that we have an $f \in L^p(X)$ and for each $n \geq 1$ the function

$$g_n(x) = \begin{cases} f(x) & \text{if } x \in E_n \text{ and } |f(x)| \leq n \\ 0 & \text{otherwise} \end{cases}$$

Then we have that $|f - g_n|^p \leq 2^p |f|^p$ so $f - g_n$ is in $L^p(X)$. Furthermore, $f \in L^p(X)$ implies that $f(x) \leq \infty$ almost everywhere and so $g_n(x) \to f(x)$ almost everywhere. We then apply the dominated convergence theorem to get that $\|f - g_n\|_{L^p(X)} \to 0$ as $n \to \infty$. Thus, for sufficiently large N we have $\|f - g_N\|_{L^p(X)} < \epsilon/2$. Let $g = g_N$ and observe that g is a bounded function supported on a set of finite measure, so $g \in L^1(X)$. In particular, g is measurable and can therefore be approximated by a sequence of increasing, simple functions $\varphi_k \nearrow g$ such that each for large enough M we have $\varphi_M \leq N$ and

$$\int_X |\varphi_M - g| d\mu \leq \frac{\epsilon^p}{4(2N)^{p-1}}$$

We then note that because $\varphi_M = \sum_{j=1}^{D} \alpha_j \chi_{A_j}$ where the $\alpha_j \in \mathbb{C}$ and A_j is measurable.

So we can find a simple function $\psi = \sum_{j=1}^{D} r_j \, \chi_{E_j}$ such that

$$\begin{aligned}
\int_X |\varphi(x) - \psi(x)| d\mu &\leq \sum_{j=1}^{D} \int_X \alpha_j \, \chi_{A_j}(x) - r_j \, \chi_{E_j}(x) d\mu \\
&\leq \sum_{j=1}^{D} \int_X \alpha_j \, \chi_{E_j}(x) - r_j \, \chi_{E_j}(x) + \alpha_j \, \chi_{A_j - E_j} \, d\mu \\
&\leq \sum_{j=1}^{D} \int_X (\alpha_j - r_j) \, \chi_{E_j}(x) + \alpha_j \, \chi_{A_j - E_j} \, d\mu
\end{aligned}$$

We then choose r_j such that

$$|\alpha_j - r_j| \leq \frac{\epsilon^p}{2^{p+1}(2N)^{p-1} \sum_{j=1}^{d} \mu(E_j)}$$

and choose the E_j such that if $\alpha = \max_{j \leq D} \alpha_j$ then

$$\mu(A_j - E_j) \leq \frac{\epsilon^p}{2^{p+1}(2N)^{p-1} D\alpha}$$

So that the above becomes

$$\begin{aligned}
\sum_{j=1}^{D} \int_X (\alpha_j - r_j) \, \chi_{E_j}(x) + \alpha_j \, \chi_{A_j - E_j} \, d\mu &\leq \sum_{j=1}^{D} |\alpha_j - r_j| \int_X \chi_{E_j} \, d\mu + \sum_{j=1}^{D} \alpha \int_X \chi_{A_j - E_j} \, d\mu \\
&\leq \frac{\epsilon^p}{2^{p+1}(2N)^{p-1} \sum_{j=1}^{D} \mu(E_j)} \sum_{j=1}^{D} \mu(E_j) + \alpha \sum_{j=1}^{d} \mu(A_j - E_j) \\
&\leq \frac{\epsilon^p}{2^{p+1}(2N)^{p-1}} + D\alpha \cdot \frac{\epsilon^p}{2^{p+1}(2N)^{p-1} D\alpha} \\
&= 2 \cdot \frac{\epsilon^p}{2^{p+1}(2N)^{p-1}} \\
&= \frac{\epsilon^p}{2^{p+1}(2N)^{p-1}}
\end{aligned}$$

Hence, we can approximate f as follows

$$\|f - \psi\|_{L^p(X)} \leq \|f - g\|_{L^p(X)} + \|g - \psi\|_{L^p(X)}$$

We have already shown that with our choice of g the first term is bounded by $\epsilon/2$. For the second term we compute

$$\begin{aligned}
\|g - \psi\|_{L^p(X)}^p &= \int_X |g - \psi|^p d\mu \\
&= \int_X |g - \psi|^{p-1} |g - \psi| d\mu \\
&\leq 2^{p-1} (|g|^{p-1} - |\psi|^{p-1}) |g - \psi| d\mu \\
&\leq 2(2N)^{p-1} \int_X |g - \psi| d\mu \\
&\leq 2(2N)^{p-1} \left(\int_X |g - \varphi| d\mu + \int_X |\varphi - \psi| d\mu \right) \\
&\leq 2(2N)^{p-1} \left(\frac{\epsilon^p}{2^{p+1}(2N)^{p-1}} + \frac{\epsilon^p}{2^{p+1}(2N)^{p-1}} \right) \\
&= \frac{\epsilon^p}{2^p}
\end{aligned}$$

Taking p^{th} roots gives $\|g - \psi\|_{L^p(X)} < \epsilon/2$. Plugging this back into the triangle inequality gives that

$$\|f - \psi\|_{L^p(X)} \leq \epsilon$$

So $L^p(X)$ is separable.

Exercise 1.8.16: We are given a finite set of functions $f_j \in L^{p_j}(X)$, where $\sum_{j=1}^N 1/p_j = 1$ whenever $p_j \geq 1$ for each j. We need to verify that

$$\| \prod_{j=1}^N f_j \|_{L^1(X)} \leq \prod_{j=1}^N \|f_j\|_{L^{p_j}(X)}$$

It will actually we be easier to prove the more general case when $\sum_{j=1}^N 1/p_j = r$ for some r and so

$$\| \prod_{j=1}^N f_j \|_{L^r(X)} \leq \prod_{j=1}^N \|f_j\|_{L^{p_j}(X)}$$

This is a slight abuse of notation because it may be the case that r is not actually a norm. In this case we still are referring to the quantity

$$\|f\|_{L^r(X)} = \left(\int_X |f|^r d\mu \right)^{1/r}$$

We will proceed by induction on N. The case $N = 1$ is the familiar fact that

$$\left| \int_X f d\mu \right|^r \leq \int_X |f|^r d\mu$$

For the case $N = 2$ we apply the usual Hölder inequality to $|f_1|^r$, $g = |f_2|^r$ and $p_1/r, p_2/r$ as the exponents. This gives that

$$\begin{aligned}
\|f_1 f_2\|_{L^r(X)}^r &= \int_X |f_1(x) f_2(x)|^r d\mu \\
&\leq \||f_1|^r\|_{L^{p_1/r}(X)} \cdot \||f_2|^r\|_{L^{p_2/r}(X)} \\
&= \left(\int_X |f_1|^{r(p_1/r)} d\mu \right)^{r/p_1} \left(\int_X |f_2|^{r(p_2/r)} d\mu \right)^{r/p_2} \\
&= \|f_1\|_{L^{p_1}(X)}^r \cdot \|f_2\|_{L^{p_2}(X)}^r
\end{aligned}$$

So the result is established in the $N = 2$ case. We then proceed to the inductive step and suppose that

$$\| \prod_{j=1}^{n-1} f_j \|_{L^r(X)} \leq \prod_{j=1}^{n-1} \|f_j\|_{L^{p_j}(X)}$$

To show that the inequality is valid for n we simply compute

$$\| \prod_{j=1}^n f_j \|_{L^1(X)} = \|(\prod_{j=1}^{n-1} f_j) \cdot |f_n| \|$$

We then apply the usual Hölder's inequality with $\prod_{j=1}^{n-1} f_j$ and f_n noting that $1/p_n + \sum_{j=1}^{n-1} 1/p_j = 1$ so if we let $r = \left(\sum_{j=1}^{n-1} 1/p_j \right)^{-1}$ then this becomes

$$\| \prod_{j=1}^n f_j \|_{L^1(X)} \leq \| \prod_{j=1}^{n-1} f_j \|_{L^r(X)} \cdot \|f\|_{L^{p_n}(X)}$$

We then apply the $n-1$ inequality to get the desired result.

Exercise 4.6.11:

Let $\mathcal{M}_{[a,b]}$ denote the set of continuous functions that are not monotonic in $[a,b]$ with $a,b \in \mathbb{Q}$. We need to show that each of these sets must be open in $\mathcal{C}([0,1])$. If we choose an $f \in \mathcal{M}_{[a,b]}$ then it measn that f is not monotonic in $[a,b]$ and therefore we can find points $x_0 < x_1 < x_2$ such that $f(x_0) < f(x_1) < f(x_2)$. If we compute the distance between f and some other function g we see that g is closer to f than the smaller of the numbers $f(y) - f(z)$ and $f(y) - f(z)$. As a result we also see that $g(x) < g(y)$ and $g(z) < g(y)$ so that g is not monotonic on $\mathcal{M}_{[a,b]}$ either and so $M_{[a,b]}$ is open.

We have seen that $\mathcal{M}_{[a,b]}$ is open and so it is everywhere dense.