Problem 6.7.2
(a) We will first verify that $\overline{\mathcal{M}}$ is indeed a σ-algebra. It is clear that $\emptyset, X \in \overline{\mathcal{M}}$ because $\mathcal{M} \subset \overline{\mathcal{M}}$. Now we will see that $\overline{\mathcal{M}}$ is closed under countable unions. Let $\{E_n \cup Z_n\}_{n=1}^{\infty}$ be a collection of sets in $\overline{\mathcal{M}}$ where each of the $E_n \in \mathcal{M}$ and each Z_n is contained in some set $F_n \in \mathcal{M}$ such that $\mu(F_n) = 0$ for all n. So,

$$\bigcup_{n=1}^{\infty}(E_n \cup Z_n) = \left(\bigcup_{n=1}^{\infty} E_n\right) \cup \left(\bigcup_{n=1}^{\infty} Z_n\right)$$

Clearly, the first term on the right is in $\mathcal{M}$, because $\mathcal{M}$ is closed under countable unions. We then observe that because each of the $Z_n \subset F_n$ that we have $\bigcup_n Z_n \subset \bigcup_n F_n$ and that

$$\mu\left(\bigcup_n F_n\right) \leq \sum_{n=1}^{\infty} \mu(F_n) = 0$$

So that

$$\bigcup_{n=1}^{\infty}(E_n \cup Z_n) \in \overline{\mathcal{M}}$$

We now need to check closure under complements. Pick some set $E \cup Z \in \overline{\mathcal{M}}$, such that $Z \subset F$ and $F \in \mathcal{M}$. So

$$\begin{aligned}(E \cup Z)^c &= E^c \cap Z^c \\ &= E^c \cap (F^c \cup (F - Z)) \\ &= (E^c \cap F^c) \cup (E^c \cap (F - Z))\end{aligned}$$

Clearly, $(E^c \cap F^c) \in \mathcal{M}$. So we are left to verify that $E^c \cap (F - Z) \subset G$ where $G \in \mathcal{M}$ has measure zero. Note that $(E^c \cap F^c) \in \mathcal{M}$ and that $E^c \cap (F - Z) \subset F$ because $(F - Z) \subset F$ and the intersection preserves the superset, F. Hence, we take $G = F$ and we are done.

(b) Now we need to verify that $\bar{\mu}$ is a measure. It is clear that $\bar{\mu}(\emptyset) = \mu(\emptyset) = 0$. Now we observe that

$$\begin{aligned}\bar{\mu}\left(\bigcup_{n=1}^{\infty}(E_n \cup Z_n)\right) &= \bar{\mu}\left((\cup_n E_n) \cup (\cup_n Z_n)\right) \\ &\leq \bar{\mu}\left((\cup_n E_n) \cup (\cup_n F_n)\right) \\ &= \mu\left((\cup_n E_n) \cup (\cup_n F_n)\right) \\ &= \mu(\cup_n E_n) \\ &= \sum_{n=1}^{\infty} \mu(E_n) \\ &= \sum_{n=1}^{\infty} \bar{\mu}(E_n \cup Z_n)\end{aligned}$$

So $\bar{\mu}$ is a measure. We will now show that $\bar{\mu}$ is complete. Indeed, take $E \cup Z \in \overline{\mathcal{M}}$ such that $\bar{\mu}(E \cup Z) = 0$. For $Z \subset F$ with $\mu(F) = 0$ we have that $E \cup Z \subset E \cup F$. Because we have

$$\bar{\mu}(E \cup Z) = 0 = \mu(E)$$

we have that E is a measure zero set in $\mathcal{M}$. So for $E' \in \mathcal{M}$ with $E' \subset E \cup Z$ we have $E' \subset E \cup F$ and

$$\bar{\mu}(E')\bar{\mu}(\emptyset \cup E') = \mu(\emptyset) = 0$$

So $\bar{\mu}$ is complete.

Problem 6.7.3
Suppose that E is Carathéodory measurable and $m(E) < \infty$. Fix an $\epsilon > 0$ and find a $G \in G_\delta$ such that $m_*(E) = m_*(G)$. Because E is measurable we can see

$$m_*(G - E) = m_*(G) - m_*(G \cap E) = m_*(G) - m_*(E) < \epsilon$$

To establish the infinite case we use the standard process of intersecting E with the balls of radius n about the origin, and measuring those. More precisely, define $E_n = B_n(0) \cap E$, where $B_n(0)$ is the ball of radius n centered at the origin. Because each of the E_n is measurable (intersection of measurable sets) we can find a set $G_n \in G_\delta$ such that $m_*(G_n - E_n) < 2^{-n}\epsilon$. So if we set $G = \bigcup_{n=1}^{\infty} G_n$ then we have that $G \supset E$ and

$$m_*(G - E) = m_*\left(\left(\bigcup_n G_n\right) - E\right) \leq \sum_{n=1}^{\infty} m_*(G_n - E) \leq \sum_{n=1}^{\infty} m_*(G_n - E_n) \leq \epsilon$$

Which shows that E is Lebesgue measurable (in the sense of Chapter 1).

Now we suppose that E is Lebesgue measure (in the sense of Chapter 1) and deduce that it is Carathéodory measurable. Indeed, fix an $A \subset \mathbb{R}^n$ and choose an open set G such that $m_*(G - E) < \epsilon$. So

$$A \cap E^c = (A \cap G^c) \cup (A \cap (G - E))$$

And because G is open and therefore measurable we have

$$\begin{aligned} m_*(A \cap E) + m_*(A \cap E^c) &\leq m_*(A \cap E) + m_*(A \cap G^c) + m_*(A \cap (G - E)) \\ &\leq m_*(A \cap G) + m_*(A \cap G^c) + m_*(G - E) \\ &\leq m_*(A) + \epsilon \end{aligned}$$

Because ϵ was arbitrary, we can let it tend to zero yielding

$$m_*(A) \geq m_*(A \cap E) + m_*(A \cap E^c)$$

And so E is Carathéodory measurable.

Problem 6.7.4
Take some subset $E \subset S^{d-1}$. By definition we have that $\sigma(E) = d \cdot m(\tilde{E})$ where

$$\tilde{E} = \{x \in \mathbb{R}^d \mid x/\|x\| \in E, 0 < \|x\| < 1\}$$

Then if r is a rotation of $\mathbb{R}^d$ then $\sigma(rE) = d \cdot m(r\tilde{E})$. So we must have that $x \in r\tilde{E}$ means that $x = \gamma\psi$ for some $\gamma \leq 1$ and $\psi \in rE$. Equivalently we must have that $x = \gamma r(\alpha)$ for that same γ and $\alpha \in E$. Because rotations are linear we see $x = r(\gamma\alpha)$, so $x \in r(\tilde{E})$. As a result we have

$$m(rE) = d \cdot m(r\tilde{E}) = m(E)$$

Meaning that r preserves measure on the sphere.

Problem 6.7.10
(a) We have that $\nu_1 \perp \mu$ and $\nu_2 \perp \mu$. This means we can find two pairs of disjoint sets A_1, B_1 and A_2, B_2 such that $\nu_1(E) = \nu_1(A_1 \cap E)$ and $\mu(E) = \mu(A_1 \cap E)$, and likewise for A_2 and B_2. We then take $A = A_1 \cup A_2$ and $B = B_1 \cap B_2$ and propose that $(\nu_1 + \nu_2)(E) = (\nu_1 + \nu_2)(A \cap E)$ and $\mu(E) = \mu(B \cap E)$. Note that this will imply that $(\nu_1 + \nu_2) \perp \mu$ if A and B are disjoint. We begin by verifying that $A \cap B = \emptyset$. This is clear because $A_1 \cap B \subset A_1 \cap B_1 = \emptyset$ and $A_2 \cap B \subset A_2 \cap B_2 = \emptyset$. So for measurable E we have

$$\mu(E) = \mu(E \cap B_1) = \mu(E \cap B) + \mu(E \cap (B_1 - B_2))$$

And also

$$\mu(B_1 - B_2) = \mu((B_1 - B_2) \cap B_2) = 0$$

Which gives that $\mu(E) = \mu(E \cap B)$. We then compute

$$\nu_1(E) = \nu_1(E \cap A_1) = \nu_1(E \cap A) - \nu_1(E \cap (A - A_1))$$

We use the same trick as able to see that

$$\nu_1(A - A_1) = \nu_1((A - A_1) \cap A_1) = 0$$

So that $\nu_1(E) = \nu_1(E \cap A)$. In the same vein, we see that $\nu_2(E) = \nu_2(E \cap A)$. This means that $(\nu_1 + \nu_2)(E) = (\nu_1 + \nu_2)(E \cap A)$. Hence, $\nu_1 + \nu_2 \perp \mu$

(b) This one is straightforward. We are given that $\nu_1 << \mu$ and $\nu_2 << \mu$. Hence, if $\mu(E) = 0$ then we must have that $\nu_1(E) = \nu_2(E) = 0$. So clearly $(\nu_1 + \nu_2)(E) = 0$.

(c) Because $\nu_1 \perp \nu_2$ we can find disjoint A, B such that $\nu_1(E) = \nu_1(E \cap A)$ and $\nu_2(E) = \nu_2(E \cap B)$. Then we see that

$$|\nu_1|(E) = \sup \sum_j |\nu_1(E_j)| = \sup \sum_j |\nu_1(E_j \cap A)| = |\nu_1|(E \cap A)$$

The same argument holds to get $|\nu_2|(E) = |\nu_2|(E \cap B)$. So $|\nu_1|$ and $|\nu_2|$ are supported on disjoint sets and $|\nu_1| \perp |\nu_2|$

(d) Suppose that $|\nu|(E) = 0$. Then we must have that $\sup \sum_j |\nu(E_j)| = 0$ and so $\nu(E_j) = 0$ for every j. And so $\nu(E) = 0$.

(e) We have that $\nu \perp \mu$ and $\nu << \mu$. By the first condition we can find disjoint A, B such that $\nu(E) = \nu(E \cap A)$ and $\mu(E) = \mu(E \cap B)$. So for measurable E we have that $\mu(E \cap A) = \mu((E \cap A) \cap B) = 0$ because $A \cap B = \emptyset$. So $\nu(E) - \nu(E \cap A) = 0$ because $\mu(E \cap A) = 0$ and $\nu << \mu$ yielding that $\nu = 0$.

Problem 6.7.16
(a) The first thing that we verify is that μ is translation invariant because

$$\begin{aligned}
\mu(R + x) &= \mu((R_1 + x) \times \cdots \times (R_d + x_d)) \\
&= \mu(R + x_1) \cdots \mu(R_d + x_d) \\
&= \mu(R_1) \cdots \mu(R_d) \\
&= \mu(R)
\end{aligned}$$

Whenever R is a measurable rectangle in the above. So the outer measure μ_* generated by coverings by rectangles is translation invariant. Hence, μ_* restricted to $\mathcal{M}$ is translation invariant. So μ is a multiple of the Lebesgue measure . So $\mu(\mathbb{T}^d) = m(Q)$ modulo the correspondence between these two spaces.

(b) This part is straightforward. Suppose that f is measurable. Then $f^{-1}(U)$ is measurable whenever U is measurable. This means that $\tilde{f}(U)\,\mathbb{R}^d$ is measurable and so $\tilde{f}$ is measurable. Note that $\tilde{f}^{-1}(U) = f^{-1}(U) + \mathbb{Z}^d$ is measurable if and only if $f^{-1}(U)$ is because of translation invariance. The proof above is the same for continuous functions except you replace the word "measurable" by "continuous".

(c) Observe that

$$\begin{aligned}
\int_{\mathbb{T}^d} |(f * g)(x)| &= \int_{\mathbb{T}^d} \left| \int_{\mathbb{T}^d} f(x-y) g(y) dy \right| dx \\
&\leq \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} |f(x-y)||g(y)| dy dx \\
&= \int_{\mathbb{T}^d} |f(x-y)| dx \int_{\mathbb{T}^d} |g(y)| dy
\end{aligned}$$

Because both functions are integrable we see that $\int_{\mathbb{T}^d} |(f*g)(x)| < \infty$, so that $f*g<\infty$ almost everywhere. We then set $u = x-y$ to get

$$(f*g)(x) = \int_{\mathbb{T}^d} f(x-y)g(y)dy = \int_{\mathbb{T}^d} f(u)g(x-u)du = (g*f)(x)$$

and we are done.

(d) We saw above that $(f*g)$ is integrable and we know that $|e^{-2\pi inx}| = 1$ and so the function $(f*g)(x)e^{-2\pi inx}$ is also integrable. We compute

$$\begin{aligned}
\int_{\mathbb{T}^d} (f*g)(x)e^{-2\pi inx}dx &= \int_{\mathbb{T}^d} e^{-2\pi inx}\left(\int_{\mathbb{T}^d} f(x-y)g(y)dy\right)dx \\
&= \int_{\mathbb{T}^d} g(y)\left(\int_{\mathbb{T}^d} f(x-y)e^{-2\pi inx}dx\right)dy \\
&= \int_{\mathbb{T}^d} g(y)(e^{-2\pi iny})(e^{-2\pi iny})\left(\int_{\mathbb{T}^d} f(x-y)e^{-2\pi inx}dx\right)dy \\
&= \int_{\mathbb{T}^d} g(y)e^{-2\pi iny}\left(\int_{\mathbb{T}^d} f(x-y)e^{-2\pi in(x-y)}dx\right)dy \\
&= \int_{\mathbb{T}^d} a_n g(y)e^{-2\pi iny} \\
&= a_n b_n
\end{aligned}$$

Which is precisely the same as

$$f*g \sim \sum a_n b_n e^{-2\pi inx}$$

(e) Normality is obvious. We need to show that each of these functions are orthogonal. To see this note that

$$\int_{\mathbb{T}^d} e^{-2\pi inx}e^{-2\pi imx}dx = \int_{\mathbb{T}^d} e^{2\pi i(m-n)x}dx = \prod_{j=1}^{d}\int_{\mathbb{T}} e^{2\pi i(m_j-n_j)x_j}dx_j$$

We then note that

$$\int_{\mathbb{T}} e^{2\pi i(m_j-n_j)x_j dx_j} = \delta_{m_j}^{n_j}$$

Where δ is the Kronecker delta function. This gives that

$$\int_{\mathbb{T}^d} e^{-2\pi inx}e^{-2\pi imx}dx = \prod_{j=1}^{d}\delta_{m_j}^{n_j} = \prod_{j=1}^{d}\delta_m^n$$

We now need to show that the space is complete. We reduce to the case where $d=1$ by Fubini's theorem because f is integrable and $|e^{-2\pi in_k x_k}| = 1$. So we see that if $(f, e^{2\pi inx}) = 0$ then

$$\begin{aligned}
(f, e^{2\pi inx}) &= \int_{\mathbb{T}^d} f(x)e^{-2\pi inx}dx \\
&= \int_{\mathbb{T}} e^{-2\pi in_1x_1}\int_{\mathbb{T}} e^{-2\pi in_2x_2}\cdots\int_{\mathbb{T}} e^{-2\pi in_dx_d}f(x)dx_d dx_{d-1}\cdots dx_1 \\
&= 0
\end{aligned}$$

Then define

$$F_1(x_1) = \int_{\mathbb{T}} e^{-2\pi in_2x_2}\cdots\int_{\mathbb{T}} e^{-2\pi in_dx_d}f(x)dx_d dx_{d-1}\cdots dx_3$$

then

$$\int_{\mathbb{T}} F_1(x_1)e^{-2\pi i n_1 x_1}dx_1 = 0$$

for every x_1 and so $F_1(x_1) = 0$ almost everywhere because $\{e^{2\pi i n x}\}$ is complete in the single dimensional case. Now we inductively define

$$F_2(x_1, x_2) = \int_{\mathbb{T}} e^{-2\pi i n_3 x_3} \cdots \int_{\mathbb{T}} e^{-2\pi i n_d x_d} f(x)dx_d dx_{d-1}\cdots dx_3$$

Then at any x_1 we have that $F_2(x_1, x_2)$ is a function of x_2 with the property

$$\int_{\mathbb{T}} F_1(x_1, x_2)e^{-2\pi i n_2 x_2}dx_2 = F_1(x_1) = 0$$

Where the equalities hold almost everywhere. We continue this process d times to see that $f(x_1, x_2, \ldots, x_d) = 0$ almost everywhere. We then apply the fact that $(f, e_n) = 0$ for all n implies that $f = 0$ iff $\{e_n\}$ is an orthonormal basis to see that the set $\{e^{-2\pi i n x}\}$ is an orthonormal basis for $L^2(\mathbb{T}^d)$.

(f) We begin by defining

$$g(x) = \begin{cases} g_\epsilon(x) = \epsilon^{-d} & 0 < x_j \leq \epsilon, j = 1, \ldots, n \\ 0 & \text{elsewhere in } Q \end{cases}$$

Then we clearly have $\int g(x)dx = \int g_\epsilon(x)dx = 1$. Hence

$$\begin{aligned} |f(x) - (f * g_\epsilon)(x)| &= \left| f(x)\int g_\epsilon(y)dy - \int f(x-y)g_\epsilon(y)\right| dy \\ &\leq \int |f(x) - f(x-y)||g_\epsilon(y)|dy \end{aligned}$$

Because f is continuous on a compact set, it is uniformly continuous. So this means that given $\tilde{\epsilon} > 0$ there is a $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$. So for $\epsilon < \delta$ we have

$$|f(x) - (f * g_\epsilon(x))| \leq \int \tilde{\epsilon}|g_\epsilon|dy = \tilde{\epsilon}$$

Which gives that $(f * g_\epsilon)(x) \to f(x)$ uniformly as $\epsilon \to 0$. Now suppose that $f \sim \sum a_n e^{2\pi i n x}$ and $g_\epsilon \sim \sum b_n^\epsilon e^{2\pi i n x}$. Then we must have that $\sum |a_n|^2 < \infty$ and $\sum |b_n^\epsilon|^2 < \infty$ because $f, g_\epsilon \in L^2$. We then use Cauchy-Schwartz to get $\sum |a_n b_n| < \infty$. This gives that $f * g_\epsilon \in L^1$ because $f \hat{*} g_\epsilon \in L^1$ and we can use Fourier inversion. So we see that

$$(f * g_\epsilon)(x) = \sum_{n \in \mathbb{Z}^d} a_n b_n e^{2\pi i n x}$$

where the equality is almost everywhere. Now we can choose an ϵ such that $|f - (f * g_\epsilon)| < \tilde{\epsilon}/2$. For this choice of ϵ we have that $\sum_{|n|>N} a_n b_n^\epsilon \to 0$ for sufficiently large N. For this N we look at the truncated series $\sum_{|n|\leq N} |a_n b_n|$ so that $\sum_{|n|>N} |a_n b_n^\epsilon| < \tilde{\epsilon}/2$. So that

$$\begin{aligned} \left| f(x) - \sum_{|n|\leq N} a_n b_n e^{2\pi i n x} \right| &\leq |f(x) - (f * g_\epsilon)(x)| + \left| (f * g_\epsilon)(x) - \sum_{|n|\leq N} e^{2\pi i n x} \right| \\ &= |f(x) - (f * g_\epsilon)(x)| + \left| \sum_{|n|\leq N} a_n b_n^\epsilon e^{2\pi i n x} \right| \\ &\leq |f(x) - (f * g_\epsilon)(x)| + \sum_{|n|>N} |a_n b_n^\epsilon| \\ &\leq \tilde{\epsilon}/2 + \tilde{\epsilon}/2 \\ &= \tilde{\epsilon} \end{aligned}$$

Which shows that f can be uniformly approximated by finite linear combinations of the exponential functions in the basis.