Problem 4.6.1

(a) We need to show that $U = \bigcap_n U_n$ is a generic set of measure 0. To see that U is generic we will show that U^c is meager. Indeed, we have that $U^c = \bigcup_n U_n^c$. Each of the U_n is open and dense because it can be written as the union of intervals about the rationals, which are dense. Consequently, we have that U_n^c is closed and meager. But then U^c can be written as the countable union of meager sets, and is therefore meager.

To see that U has measure 0 we observe that

$$m(U) \leq m(U_n) = \frac{1}{n}\sum_{j=1}^{\infty} 2^{j-1} = \frac{2}{n}$$

for every n. And therefore $m(U) = 0$.

(b) Recall the construction on the Cantor-like set. At each iteration of the construction we remove a centrally situated interval of length ℓ_k from each remaining interval in the construction subject to the restriction that $\sum_{k=1}^{\infty} \ell_k 2^{-k} < 1$. If we let $\mathcal{C}$ denote a Cantor-like set, we are tasked to show that $\mathcal{C}$ is first category. Suppose not, then we have that the $\text{Int}(\overline{\mathcal{C}})$ contains an interval. We know that $\mathcal{C}$ is closed so $\mathcal{C} = \overline{\mathcal{C}}$ and therefore we have that $\text{Int}(\mathcal{C})$ contains an interval. This is impossible by Book III Exercise 1.6.4 (proved in problem set 1) and so we have that $\mathcal{C}$ is first category.

Problem 4.6.2

(a) In the forward direction we have that F is closed and first category. This means that we can write

$$F = \bigcup_{n=1}^{\infty} E_n$$

To see that $\text{Int}(F) = \emptyset$ we suppose to the contrary. Then we must be able to find some closed ball $\overline{B} \subset F$. We then note that $\overline{B}$ is a complete metric space in its own right and apply the category theorem to get that $\overline{B}$ is second category in itself. Hence, is non-empty and open. If $x \in \text{Int}(\overline{B}$ then we can find some open ball about x of radius δ that is wholly contained in $\text{Int}(\overline{B}$. But then we have that

$$\overline{B_\delta(x)} = \bigcup_{n=1}^{\infty} (\overline{B_\delta(x)} \cap E_n)$$

And so $E_n \cap \overline{B_\delta(x)}$ must be non-empty for some n. This implies that the interior of one of the E_n is non-empty and therefore $E_n$cannot be nowhere dense. This contradiction proves the forward direction.

In the reverse direction we have that F is closed and has empty interior. Hence $\text{Int}(\overline{F}) = \text{Int}(F) = \emptyset$ and so F is nowhere dense and therefore first category.

(b) For this part we appeal to part **(a)**. In the forward direction we must have that $\mathcal{O}$ is open and first category. If we write $\mathcal{O} = \bigcup_n E_n$ with each of the E_n nowhere dense. then we must have that $\mathcal{O}^c = \bigcap_n E_n^c$ is a closed, second category set. Furthermore, each on the E_n^c is dense and so we have that $\mathcal{O}^c$ is dense as well. Then $\mathcal{O}^c$ is a closed, dense set in X and therefore equals the whole of X. So $(\mathcal{O}^c)^c = \mathcal{O} = \emptyset$. The reverse direction is trivial because the empty set is its own closure and therefore it is nowhere dense and consequently first category.

(c) Suppose that F is generic. Then we have that F^c must be meager and this happens iff $F^c = \emptyset$ because F^c is open, which implies that $F = X$. The reverse direction is the category theorem.

If $\mathcal{O}$ is generic, then we have that $\mathcal{O}^c$ is meager and therefore has empty interior. The reverse direction is the same, we have that $\mathcal{O}^c$ having no interior means that it is of first category. So its complement, $\mathcal{O}$ is of the second category.

Problem 4.6.5

(a) If Y is a dense G_δ set then we have that $Y = \bigcap_{n=1}^\infty U_n$ where each of the U_n is open and dense (otherwise Y could not be dense). We need to show that Y is generic. Suppose to the contrary that Y is first category and so we also have that $Y = \bigcup_{n=1}^\infty W_n$ where each of the W_n is nowhere dense. We then note that each of the sets U_n^c is nowhere dense and then write

$$X = Y^c \cup Y = \bigcup_{n=1}^\infty U_n^c \cup \bigcup_{n=1}^\infty W_n$$

But then X is a countable union of nowhere dense sets, and so it is of first category. But this violates the Baire theorem and so we must have had Y was generic.

(b) We can write any countable dense set as a union of singletons, each of which are closed and nowhere dense. Hence, any countable dense set is an F_σ. However, the previous part says that it cannot be a G_δ because if it were then it would be generic, which is not possible.

(c) Let E be a generic set in X. Then we have that E^c is of the first category. So we can write $E^c = \bigcup_{n=1}^\infty W_n$ with each W_n nowhere dense. Hence, we can write $E = \bigcup_{n=1}^\infty W_n^c$ and each of the W_n^c is dense. We then set $E_0 = \cap_{n=1}^\infty \mathrm{Int}(W_n^c)$. Clearly, $E_0 \subset E$ because $\mathrm{Int}(W_n^c \subset W_n^c$. Additionally, each of the $\mathrm{Int}(W_n^c)$ is an open dense set. Hence, E_0 is a dense G_δ contained in E.

Problem 4.6.6

The hard work for this fact was done in the proof of Theorem 4.1.3. They showed that the set

$$E_\epsilon = \{x \,|\, \mathrm{osc}(f)(x) < \epsilon\}$$

is open. This fact, coupled with the observation (also shown in the text) that $\mathrm{osc}(f)(x) = 0$ iff f is continuous at x gives that we can always write the set of points of continuity of f, say $\mathcal{C}_f$ as

$$\mathcal{C}_f = \bigcap_{n=1}^\infty E_{1/n}$$

This is a G_δ set. We then apply the previous exercise to the countable, dense set $\mathbb{Q}$ to see that it cannot be a G_δ and therefore cannot be the set of continuity points of any function $\mathbb{R} \to \mathbb{R}$.

Problem 4.6.8

Suppose towards a contradiciton that the Banach space X had a countable Hamel basis say $\{f_n\}_{n=1}^\infty$. If we let $S_m = \mathrm{span}(f_{k_1}, f_{k_2}, \ldots, f_{k_m}$ be some finite dimensional subspace, then we can see that S must be closed in X. Moreover, we can see that a finite dimensional subspace S has non-empty interior iff $S = X$. Indeed, it $S = X$ then $\mathrm{Int}(S) = X$. Additionally, observe that if $\mathrm{Int}(S) \neq \emptyset$ then there would be some $s \in \mathrm{Int}(S)$. Let δ be such that $\overline{B_\delta(s)} \subset S$. Then we have that $\overline{B_\delta(0)} = \overline{B_\delta(s)} - s \subset S$. Let $x \in X$ so that if $x = 0$ then $x \in S$ and otherwise $\frac{\delta x}{\|x\|} \in \overline{B_\delta(0)} \subset S$ and so $x \in S$ as well.

We have that S_m^c is open in X. Since f_N is not in S_m for sufficiently large N we see that $S_m \neq X$ and hence $\mathrm{Int}(S_m)$ is empty by the above. This means that S_m^c must be dense in X. Because X is complete, we have that

$$D = \bigcap_{m=1}^\infty S_m^c$$

must be dense in X by the category theorem. However, because the span of the f_i is all of X we have that $D = \emptyset$ and we have a contradiction. So X cannot have a countable basis.

Problem 4.6.12

For each $x \in X$ we define the linear operator $B_x(y) = T(x, y)$ and similarly we have that

$B_y(x) = T(x, y)$. Because each of these functions is continuous, both of these operators are bounded. Hence,

$$\|T(x, y)\| \leq C_y\|x\|$$

In terms of B_x we have that

$$\|B_x\| \leq C_y\|x\|$$

Hence, we have that the family of norms $\{\|B_x(y)\| \,|\, x \in X, \|x\| = 1\}$ is bounded for each fixed y. In the same way he have that the $\{\|B_y(x)\| \,|\, y \in Y, \|y\| = 1\}$ is bounded in the norm. this gives that $\|T(x, y)\| \leq C_xC_y$ for each x and y. Then we apply the uniform boundedness principle to get that $\|T(x, y)\| \leq C$ for all x, y. By scaling the unit vectors we get that $\|T(x, y)\| \leq C\|x\|\|y\|$.

Problem 4.6.13
We define the sequence of linear operators

$$\ell_n(g) = \int_X f_n(x)g(x)d\mu(x)$$

We apply the uniform boundedness principle to see that $\{\ell_n\}_{n=1}^{\infty}$ is either uniformly bounded, or unbounded on a G_δ set. If the sequence f_n converges weakly (or is weakly bounded) then for a fixed g we must have that

$$\lim_{n\to\infty} |\ell_n(g)| = \lim_{n\to\infty} \left|\int_X f_n(x)g(x)d\mu(x)\right| < \infty$$

in particular the limit must exist. A convergent sequence of reals is bounded and so the set of g for which

$$\sup_n \int_X f_n(x)g(x)d\mu(x) = \infty$$

is empty. Hence, we must have a uniform bound in the norm. This means that

$$\|\ell_n\| = \sup_{g\neq 0} \frac{|\ell_n(g)|}{\|g\|} < M, \forall n$$

By Cauchy Schwartz we immediately have that

$$\sup_n \|f_n\|_{L^p} < \infty$$