Problem 3.5.2
We follow the same basic idea as the proof of Theorem 2.1. We observe that

$$(f * K_\delta)(x) = \int f(x-y)K_\delta(y)dy$$

We then need to show that

$$\int |f(x-y)|K_\delta(y)dy \to 0$$

We take the same approximation as in the proof of he theorem by breaking up the integral as follows

$$\int |f(x-y)|K_\delta(y)dy = \int_{|y|\leq\delta} |f(x-y)|K_\delta(y)dy + \sum_{k=0}^{\infty} \int_{2^k\delta<|y|\leq 2^{k+1}\delta} |f(x-y)|K_\delta(y)dy$$

We then have an easier time than in Theorem 2.1 because we appeal to the property of good kernels to see that

$$\int_{|y|\geq\delta} |K_\delta(x)|dx \to 0$$

So we can choose each of the terms in the sum to be less that $\epsilon/2^k$ for any $\epsilon > 0$. For the first term we use the fact that $\int_{\mathbb{R}} K_\delta = 0$ and the estimates given by the two properties of good kernels to see that

$$\int_{|y|\leq delta} |f(x-y)||K_\delta(y)|dy \leq CA\delta/\epsilon \int |f(x-y)|$$

By taking δ small we can force this term to zero as well. This gives that the total sum goes to zero as $\delta \to \infty$, which completes the proof.

Problem 3.5.4
Following the hint, we use the fact that f is not a.e. 0 to see that there must be some ball B such that $\int_B |f| > 0$. Without loss of generality we can assume that B is centered at the origin with radius 1 because the integral is translation invariant and a dilation is a linear change of variable. Suppose that we have that $\int_B f = c > 0$. Then pick any x with $|x| \geq 1$ and observe that the ball of radius $|x|$ centered at the origin has measure $v_n|x|^n$. This gives that

$$f^*(x) \geq \int_{|y|\leq|x|} f \geq \frac{c}{v_n|x|^n}$$

Combining the constant terms we get that $f * (x) \geq c'/|x|^d$. Now we apply linearity of the integral to get that

$$\int f^* \geq \int |f| \geq c \int 1/|x|^d$$

The integral on the right is unbounded and so f^* is not integrable. To get the "best inequality", we note that the inequality above is the reverse inequality of the weak inequality. And if we take $\int |f| = 1$, then in the above argument we are left with

$$m(\{f^* > \alpha\}) \leq c'/\alpha$$

Where c' is a fixed constant not depending on α This is the desired result.

Problem 3.5.5
(a) We consider

$$f(x) = \begin{cases} \frac{1}{|x|(\log 1/|x|)^2} & \text{if } |x| \leq 1/2 \\ 0 & \text{otherwise} \end{cases}$$

The main observation is that f is symmetric about the origin, so we can compute

$$\int_{\mathbb{R}} |f(x)|dx = \int_{|x|\leq 1/2} |f(x)|dx = 2\int_0^{1/2} \frac{1}{x \log 1/x} dx$$

We then change variables with $u = \log 1/x$ to see that

$$\int \frac{1}{x \log 1/x} dx = \int \frac{-du}{u^2} = \frac{1}{u} = \frac{1}{\log 1/x}$$

Substituting we get

$$\int |f(x)|dx = \frac{2}{\log 1/x}\bigg|_0^{1/2} = \frac{2}{\log 2}$$

So $\int |f| < \infty$ and f is integrable.

(b) In $\mathbb{R}$ balls containing x are just intervals. So if we choose a ball of radius $2r$ centered about x then we have that

$$f^*(x) \geq \frac{1}{2r}\int_0^{2r} |f(y)|dy$$

We then take $|x| \leq 1/2$ and set $r = 2|x|$ we get that

$$f^*(x) \geq \frac{1}{2|x|}\int_0^{2|x|} |f(y)|dy \geq \frac{1}{2|x|} \cdot \frac{1}{\log 1/|x|}$$

So taking the supremum of all balls we have

$$f^*(x) \geq \frac{c}{|x| \log(1/|x|)}$$

Integrating on any ϵ-neighborhood of 0 we see

$$\int_{-\epsilon}^{\epsilon} |f^*(x)|dx \geq \int_{-\epsilon}^{\epsilon} \frac{c}{|x| \log(1/|x|)} dx = -c \log\log(1/x)\bigg|_0^{\epsilon}$$

Which goes to ∞. So f^* is not locally integrable.

Problem 3.5.8

Yes we can find such a sequence. Following the hint given in the text we are led to consider the Lebesgue density set of A. By Theorem 1.4 in the text we see that we can always find the desired epsilon and I_ϵ with

$$m(A \cap I_\epsilon) \geq (1-\epsilon)m(I_\epsilon)$$

Next, we can actually construct the sequence as follows. Let $\{r_n\}$ be an enumeration of $\mathbb{Q}$ and let d be a point of Lebesgue density of A. We set $t_k = r_k - d$. Now we let E be all the translates of A by t_k or more precisely set

$$E = \bigcup_k A + t_k$$

We will show that $m(E^c) = 0$. Note that if we decompose the set into pieces

$$E^c = \bigcap_n (E^c \cap [n, n+1))$$

it suffices to show that $E^c_n = E^c \cap [n, n+1)$ has measure zero for every $n \in \mathbb{Z}$. We further restrict our attention to $E^c \cap [0,1]$ because the measure is translation is translation invariant and we we can translate to the rest of $\mathbb{R}$.

Now we extract from the sequence of $k = 1, 2, 3\ldots$ an integer n_k such that

$$m(A \cap B_{n_k^{-1}}(r)) \geq (1 - 1/n_k)m(B_{n_k^{-1}}(r))$$

Each n_k must exist because x is in the Lebesgue set of A and r is a parallel translate of x (by construction). If we look closely at these open balls we can enumerate a subset of them via

$$B^m_{n_k^{-1}} = B_{n_k^{-1}}(j/2n_k)$$

for each $m \leq 2n_k$. Then

$$[0,1] \subset \bigcup_m B^m_{n_k^{-1}}$$

So

$$E^c \cap [0,1] \subseteq \bigcup_{m=1}^{2n_k} B^m_{n_k^{-1}} \cap [0,1]$$

Which means

$$\begin{aligned} m(E^c \cap [0,1]) &\leq \sum_{m=1}^{2n_k} m(B^m_{n_k^{-1}} \cap [0,1]) \\ &\leq \sum_{m=1}^{2n_k} m(B^m_{n_k^{-1}})/m \\ &= 2n_k \cdot \frac{1}{m} \cdot \frac{1}{n_k} \\ &= 4/m \end{aligned}$$

This inequality holds for all m meaning that $m(E^c \cap [0,1]) = 0$. Therefore $m(E^c) = 0$ and we are done.

Problem 3.5.9
Consider the function

$$\delta(x) = \inf\{|x-y| : y \in F\}$$

We want to show that $\delta(x+y)/|y| \to 0$ almost everywhere on F. The quantity $\delta(x+y)/|y|$ would lead us to consider the derivative the function $\delta(x)$. Consider δ on any interval $I = (a,b)$. We can see that for any $y, z \in \mathbb{R}$

$$|\delta(y) - \delta(z)| \leq |y - z|$$

Restricting values to I we get that $T_F(a,b) \leq |b-a|$. This means that δ is of bounded variation and so it is differentiable almost everywhere, and in particular at almost every $x \in F$. We then observe that by definition $\delta \geq 0$ and is identically 0 on F. And so δ has local minima on all of F, and because δ is differentiable its derivative on F is zero almost everywhere. We then use the definition of the derivative to see that $\delta(x+y)/|y| \to 0$ a.e.