Problem 1.6.30

Fix an $\epsilon \in (0,1)$. We apply the result of Exercise 28 to guarantee the existence of two intervals I_1, I_2 such that $m(-E \cap I_1) > (1-\epsilon)m(I_1)$ and $m(F \cap I_2) > (1-\epsilon)m(I_2)$. Without loss of generality suppose that $m(I_1) \leq m(I_2)$. This means that we can translate I_1 by some amount h so that it lies inside of I_2. Let $\delta = m(I_1)$, so that for $0 < |\alpha| < \epsilon\delta$ we see that the interval

$$I_{1,|\alpha|} = (I_1 + h + \alpha) \cap I_2$$

must have $m(I_{1,|\alpha|}) = \delta - \alpha > (1-\epsilon)\delta$. As a result of this, we must have that $I_{1,|\alpha|} \cap I_2 \neq \emptyset$. Pick some $x \in (I_{1,|\alpha|} \cap I_2)$ and observe that this means that there is some $u \in E$ and $v \in F$ such that

$$x = v = -u + h + \alpha$$

So this means that $u + v = (h + \alpha)$. So $E + F$ must contain the interval $(h - \epsilon\delta, h + \epsilon\delta)$ and we are done.

Problem 2.5.20
We want to show that if $E \subset \mathbb{R}^2$ is a Borel set, then for any y the slice E^y is (a parallel translate of) a Borel set in $\mathbb{R}$. We begin by considering the following

$$\mathcal{C} = \{E \subset \mathbb{R}^2 \,|\, \forall y, E^y \text{ is Borel}\}$$

We first prove the following

Claim. *$\mathcal{C}$ is a σ-algebra.*

Proof. It is clear that $\mathcal{C}$ is non-empty; the open unit disk is in $\mathcal{C}$, for instance. Next we will see that $\mathcal{C}$ is closed under taking complements. Let $E \in \mathcal{C}$, then we know that for any y the set E^y is Borel and that $(E^c)^y = (E^y)^c$ must also be Borel because the Borel sets are a σ-algebra. So $E^c \in \mathcal{C}$. Next, suppose that $\{E_k\}_{k=1}^{\infty}$ is a countable collection of sets in $\mathcal{C}$. We see that

$$\left(\bigcup_{k=1}^{\infty} E_k\right)^y = \bigcup_{k=1}^{\infty} E_k^y$$

Then we use the fact that each of the E_k^y is Borel, and so their union must be as well because the Borel sets are closed under countable union. Thus, $\mathcal{C}$ is a σ-algebra. □

Now to complete the proof we need to prove the following

Proposition. *If $E \subset \mathbb{R}^2$ is open then $E \in \mathcal{C}$.*

Proof. We begin with the proof in the case of E being an open cube. We note here that we must have $E = (a, b) \times (a + h, b + h)$ for some $h > 0$. We then take the slice E^y which is the same as $(a, b) + y$, where $y \in \mathbb{R}$. Geometrically, we simply have the line segment (a, b) translated in the y-direction a distance y. Clearly, this is and open interval and hence, Borel. We then prove the claim for the family of closed cubes. The proof is the same, we note that the slices E^y of a closed cube is now the closed interval, which can be written as the countable intersection of intervals of the form $\bigcap_n (a - 1/n, b + 1/n)$, and therefore is Borel. We then use the fact that every open set in $\mathbb{R}^2$ can be written as the countable union of almost disjoint closed cubes to see that we can write $E = \bigcup_{j=1}^{\infty} Q_k$, where each of the Q_k is a closed cube. We then use the fact that each of the $Q_k \in \mathcal{C}$ and that $\mathcal{C}$ is closed under countable unions to see that $E \in \mathcal{C}$. □

Finally, we recall that the Borel sets is the smallest σ-algebra containing the open sets. We have shown that $\mathcal{C}$ contains the open sets. And therefore, it contains the Borel sets. This completes the proof.

Problem 3.5.6
For functions $f:\mathbb{R}\to\mathbb{R}$ we define

$$f_+^*(x) = \sup_{h>0} \frac{1}{h}\int_x^{x+h} |f(y)|dy$$

We want to show that the set

$$E_\alpha^+ = \{x\in\mathbb{R}\,|\, f_+^*(x) > \alpha\}$$

is measurable with measure

$$m(E_\alpha^+) = \frac{1}{\alpha}\int_{E_\alpha^+} |f(y)|dy$$

Following the hint, we wish to apply Lemma 3.5 (Rising Sun Lemma) to the function

$$F(x) = \int_0^x |f(y)|dy - \alpha x$$

We first observe that if $x\in E_\alpha^+$ then there is some $h>0$ such that

$$\frac{1}{h}\int +x^{x+h}|f(y)|dy > \alpha$$
$$\int +x^{x+h}|f(y)|dy > h\alpha$$

So on the right side we consider

$$h\alpha = h\alpha + x\alpha - x\alpha = (x+h)\alpha - x\alpha$$

and note that because $\int |f(y)|$ is increasing have $\int_x^{x+h}|f(y)|dy = \int_0^{x+h}|f(y)|dy - \int_0^x |f(y)|dy$. So the inequality becomes

$$\int_0^{x+h}|f(y)|dy - \int_0^x |f(y)|dy > (x+h)\alpha - x\alpha$$
$$\int_0^{x+h}|f(y)|dy - (x+h)\alpha > \int_0^x |f(y)|dy - x\alpha$$

Consequently, we must have

$$E_\alpha^+ = \{x\in\mathbb{R}\,|\,\exists h>0, F(x+h) > F(x)\}$$

If f is integrable then F is absolutely continuous by the absolute continuity of the integral. We then apply Lemma 3.5 to write E_α^+ as the disjoint union of intervals so

$$E_\alpha^+ = \bigcup_{k=1}^\infty (a_k, b_k)$$

such that $F(a_k) = F(b_k)$. As a result, we see that

$$\begin{aligned}\int_0^{a_k} |f(y)|dy - \alpha a_k &= \int_0^{b_k} |f(y)|dy - \alpha b_k \\ \alpha b_k - \alpha a_k &= \int_0^{b_k} |f(y)|dy - \int_0^{a_k} |f(y)|dy \\ \alpha(b_k - a_k) &= \int_{a_k}^{b_k} |f(y)|dy\end{aligned}$$

Moreover, because all of the intervals are disjoint we have that

$$m(E_\alpha^+) = \sum_{k=1}^{\infty}(b_k - a_k)$$

We then observe that

$$\int_{E_\alpha^+} |f(y)|dy = \sum_{k=1}^{\infty} \int_{a_k}^{b_k} |f(y)|dy = \alpha \sum_{k=1}^{\infty}(b_k - a_k) = \alpha m(E_\alpha^+)$$

So that

$$m(E_\alpha^+) = \frac{1}{\alpha} \int_{E_\alpha^+} |f(y)|dy$$

And we are done.

Problem 3.5.10
Let $\{r_n\}_{n=0}^{\infty}$ be an enumeration of the rationals. Consider the function

$$f(x) = \sum_{r_n < x} 2^{-n}$$

I claim that this function is increasing on $\mathbb{R}$,continuous on $\mathbb{R} - \mathbb{Q}$ and discontinuous elsewhere (i.e. only on $\mathbb{Q}$). First we verify that f is increasing. Pick $x, y \in \mathbb{R}$ such that $x \neq y$ and assume without loss of generality that $x < y$. Because $\mathbb{Q}$ is dense in $\mathbb{R}$ we can find some rational r_m such that $x < r_m < y$. So we can see that

$$f(y) = \sum_{r_n < y} 2^{-n} > 2^{-m} + \sum_{r_n < x} 2^{-n} = 2^{-m} + f(x) > f(x)$$

Now we will show that f is discontinuous on $\mathbb{Q}$. To do this we examine the left and right hand limits of f at a point x. So consider

$$f^+(x) = \inf_{t > x} f(t) \text{ and } f^-(x) = \sup_{t < x} f(t)$$

Then f is continuous at x iff $f^+(x) - f^-(x) = 0$. For any rational r_m we have

$$\begin{aligned}
f^+(r_m) - f^-(r_m) &= \inf_{r_p > r_m} \sum_{r_n < r_p} 2^{-n} - \sup_{r_q < r_m} \sum_{r_n < r_m} 2^{-n} \\
&= \inf_{r_p > r_m} \sum_{r_m < r_k \leq r_p} 2^{-n} + f(r_m) - \sup_{r_q < r_m} \sum_{r_n < r_m} 2^{-n} \\
&= \inf_{r_p > r_m} \sum_{r_m < r_k \leq r_p} 2^{-n} \\
&= 2^{-m}
\end{aligned}$$

So f is discontinuous on $\mathbb{Q}$. We will now show that f is continuous on $\mathbb{R} - \mathbb{Q}$. We note that for $x \notin \mathbb{Q}$ we have that $x \neq r_n$ for all n. So the same computation shows that $f^+(x) - f^-(x) = 0$ because the last line of the computation will never have any terms in the sum because $r_p \neq x$ for all p.

Problem 3.5.32
We need to verify that the following

Lemma. *A function $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz with constant M if and only if f has the following two properties:*

(i) f is absolutely continuous.

(ii) $|f'(x)| \leq M$ for almost every x.

Proof. In the forward diretion we suppose that f is Lipschitz. This means that

$$|f(x) - f(y)| \leq M|x - y|$$

for some M and all $x, y \in \mathbb{R}$. Pick an $\epsilon > 0$ and observe that $\delta = \epsilon/M$ immediately gives that

$$\sum_{j=0}^{N} |b_j - a_j| < \delta \text{ implies } \sum_{j=1}^{N} |f(b_j) - f(a_j)| < \epsilon$$

so f is absolutely continuous. We then use this fact to apply Theorem 3.11 and get that f must be differentiable almost everywhere. If x is a point for which f' exists we observe that for any $h > 0$ we must have that

$$|\frac{f(x+h) - f(x)}{h}| \leq M$$

We then take the limit as $h \to 0$ to see that $|f'(x)| \leq M$ whenever f' exists (i.e. almost everywhere).

In the reverse direction, we have that f is absolutely continuous and $|f'(x)| \leq M$ almost everywhere. We apply Theorem 3.11 again to see that for any $x, y \in \mathbb{R}$ with $x \leq y$ we have

$$f(x) - f(y) = \int_x^y f'(t)dt$$

Taking the absolute value gives

$$|f(x) - f(y)| = \left| \int_x^y f'(t)dt \right| \leq \int_x^y |f'(t)|dt \leq \int_x^y M dt = M|x - y|$$

So f satisfies a Lipschitz condition and we are done. $\square$

Problem 4.6.4
We are considering the space

$$\ell^2(\mathbb{Z}) = \left\{ (\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots) \mid a_i \in \mathbb{C}, \sum_{k=-\infty}^{\infty} |a_k|^2 < \infty \right\}$$

when endowed with the inner product and norm

$$(a, b) = \sum_{k=-\infty}^{\infty} a_k \overline{b_k} \text{ and } \|a\| = \left(\sum_{k=-\infty}^{\infty} |a_k|^2 \right)^{1/2}$$

We will verify that ℓ^2 is a Hilbert space. It is clear that ℓ^2 is a vector space with pointwise addition and multiplication by scalars because if $a, b \in \ell^2$ then

$$\sum_{k=-\infty}^{\infty} |a_k + b_k|^2 \le 4 \sum_{k=-\infty}^{\infty} |a_k|^2 + 4 \sum_{k=-\infty}^{\infty} |b_k|^2 < \infty$$

Note that we have used the inequality $|a_k + b_k| \le 2 \max\{|a_k|, |b_k|\}$. Furthermore, for any $\lambda \in \mathbb{C}$

$$\sum_{k=-\infty}^{\infty} |\lambda a_k|^2 = \sum_{k=-\infty}^{\infty} |\lambda|^2 |a_k|^2 = |\lambda|^2 \sum_{k=-\infty}^{\infty} |a_k|^2 < \infty$$

So ℓ^2 is indeed a vector space.

We then check the properties of the inner product that we gave ℓ^2. It is clear that for fixed b the map $a \mapsto (a, b)$ is linear, it follows from the linearity of the sum because for $x, y \in \ell^2$ we have

$$\sum_{k=-\infty}^{\infty} (x_k + y_k)\overline{b_k} = \sum_{k=-\infty}^{\infty} x_k \overline{b_k} + \sum_{k=-\infty}^{\infty} y_k \overline{b_k} = (x, b) + (y, b)$$

The fact that $(a, b) = \overline{(b, a)}$ follows from the symmetry in the inner product. To see that $(a, a) \ge 0$ for every $a \in \ell^2$, observe that

$$(a, a) = \sum_{k=-\infty}^{\infty} a_k \overline{a_k} = \sum_{k=-\infty}^{\infty} |a_k|^2 \ge 0$$

Because the last term is a sum of non-negative values. Now we check that $\|a\| = (a, a)^{1/2}$, which is straightforward because

$$(a, a)^{1/2} = \left(\sum_{k=-\infty}^{\infty} a_k \overline{a_k} \right)^{1/2} = \left(\sum_{k=-\infty}^{\infty} |a_k|^2 \right)^{1/2} = \|a\|$$

The next thing we have to verify is that $\|a\| = 0$ if and only if $a = 0$. This is clear because $\left(\sum_{k=-\infty}^{\infty} |a_k|^2 \right)^{1/2} = 0$ implies that $\sum_{k=-\infty}^{\infty} |a_k|^2 = 0$, which is only possible if each of the terms $|a_k|^2 = 0$. This immediately gives that $|a_k| = 0$, so that means $a_k = 0$ for every k. So $a = 0$.

Now we will prove the Cauchy-Schwartz inequality in ℓ^2. Choose $a, b \in \ell^2$. In the case that a and b are linearly dependent we have equality. So if a and b are linearly independent we define

$$c = a - \frac{(a,b)}{(b,b)}b$$

We then observe that

$$(c,b) = \left(a - \frac{(a,b)}{(b,b)}b, b\right) = (a,b) - \frac{(a,b)}{(b,b)}(b,b) = 0$$

So this means that c is orthogonal to b. We then apply the Pythagorean theorem to $a = \frac{(a,b)}{(b,b)}b + c$ to see

$$\|a\|^2 = \left|\frac{(a,b)}{(b,b)}\right|^2 \|b\|^2 + \|c\|^2 = \frac{(a,b)^2}{\|b\|^2} + \|c\| \geq \frac{(a,b)^2}{\|b\|^2}$$

So we multiply by $\|b\|$ and take square roots to see that $|(a,b)| \leq \|a\|\|b\|$. The triangle inequality follows immediately, the proof is exactly the same as the one given on the bottom of page 158 in Stein and Shakarchi modulo a change of variable, so we forgo it here.

We will now show that ℓ^2 is complete in the metric induced by the norm. Let $\{a_i\}_{i\in\mathbb{Z}}$ be a Cauchy sequence in ℓ^2. We can imagine that the elements of each sequence a_i are the rows of an infinite matrix with entries $a_{i,j}$. We know that the rows of this matrix all converge to unique limits in $\mathbb{C}$. We also can get the convergence of the columns by noting that for fixed k

$$|a_{i,k} - a_{j,k}|^2 \leq \|a_i - a_j\|^2 \to 0$$

when we send $\min\{i,j\} \to \infty$ because $\{a_i\}$ is Cauchy. This means that each column $\{a_{i,j}\}_{i\in\mathbb{Z}}$ converges to some unique limit $b_j \in \mathbb{C}$. We then set $b = \{b_j\}_{j\in\mathbb{Z}}$. It is clear from above that $d(a_i, b) \to 0$ as $i \to \infty$. We now need to show that $b \in \ell^2$. To show this, we need the following

Lemma. *Every Cauchy sequence in ℓ^2 is bounded in the norm.*

Proof. Let $\{x_n\}_{n\in\mathbb{Z}}$ be a Cauchy sequence in ℓ^2. Then for any $\epsilon > 0$ we can find $N > 0$ such that $|n| > N$ implies $\|x_{|n|} - x_N\| < \epsilon$. We then see that

$$\|x_{|n|}\| \leq \|x_{|n|} - x_N\| + \|x_N\| < \|x_N\| + \epsilon$$

We then note that if $|n| < N$ that if

$$M = \max\{\|x_{-N+1}\|, \ldots, \|x_0\|, \|x_1\|, \ldots, \|x_{N-1}\|\}$$

As a result we have that $\|x_n\| \leq \max\{M, \|x_N\| + \epsilon\} < \infty$, so $\{x_n\}$ is bounded in the norm. ∎

With the lemma in hand we can proceed to see that for any fixed j we have an M such that
$$\sum_{k=-\infty}^{\infty} |a_{k,j}|^2 \leq M^2$$
So we can see that for any finite N we have that
$$\sum_{|n|<N} |b_n|^2 = \lim_{m\to\infty} \sum_{|n|<N} |a_{m,n}|^2 < M^2$$
Taking $N \to \infty$ gives the result.

Now we show that ℓ^2 is separable. We construct the natural basis for ℓ^2 consisting of the vectors $e_k = (\ldots, 0, 1, 0, \ldots)$, which have a 1 in the k^{th} position. We need to show that finite linear combinations of the e_k are dense in ℓ^2. For each element $x \in \ell^2$ we consider the combination
$$S_N(x) = \sum_{|k|\leq N} x_k e_k$$
It is clear that $S_N(x)$ and x agree for all indices $|k| \leq N$. We then see that
$$\|S_N(x) - x\| = \sum_{|k|>N} |x_k|$$
Because $x \in \ell^2$ we have that $\sum_k |x_k|^2$ converges and so $\sum_k |x_k|$ must converge as well. We apply the definition of convergence to see that $\sum_{|k|>N} |x_k| \to 0$ as $N \to \infty$. Thus, finite linear combinations of the e_k are dense in ℓ^2.