Problem 1.6.19

(a) Without loss of generality suppose that A is open and choose $x \in A + B$. This means that there is an $a \in A$ and a $b \in B$ such that $a+b = x$. Because A is open we can choose a $\delta > 0$ such that $B_\delta(a) \subset A$. I claim that $B_\delta(x) \subset A + B$. We can see this because any $y \in B_\delta(x)$ satisfies $y = x + \vec{\epsilon}$, where $|\vec{\epsilon}| < \delta$. Then $y = a + b + \vec{\epsilon} = (a + \vec{\epsilon}) + b$. We note that $(a + \vec{\epsilon}) \in A$ so $y \in A + B$. Hence, $B_\delta(x) \subset A + B$ and so $A + B$ is open because x was arbitrary.

(b) Suppose that $A, B \subset \mathbb{R}^d$ are closed. We want to show that $A + B$ is measurable. To do this, note that it will suffice to prove the special case of A, B compact because we can write

$$A_k = \bigcup_{k=1}^{\infty} A \cap B_k(\vec{0}) \text{ and } B_k = \bigcup_{j=1}^{\infty} B \cap B_j(\vec{0})$$

Where $B_i(\vec{0})$ is the ball of radius i centered at the origin. Then A_k, B_j are compact for every k, j and therefore can then write

$$A + B = \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} A_k + B_j$$

Hence, if each of the $A_k + B_j$ are measurable, then $A + B$ will. So we have reduced the problem to the following

Claim. *If X and Y are compact subsets of $\mathbb{R}^d$, then the set $A + B$ is compact.*

Proof. Recall that a set in $\mathbb{R}^d$ is compact if and only if every sequence has a convergent subsequence. Consider any sequence $\{z_n\}_{n=1}^{\infty}$ in $X + Y$. Then by the definition of $X + Y$ we have that each of the z_n can be written $z_n = x_n + y_n$ where $x_n \in X$ and $y_n \in Y$. Because X, Y are compact we can find convergent subsequences $x_{n_k} \to x$ in X and $y_{n_k} \to y$ in Y. Because X and Y are closed we know that $x \in X$ and $y \in Y$ so we can see that for any $\epsilon > 0$ and sufficiently large n, k

$$|(x + y) - (x_{n_k} + y_{n_k})| = |(x - x_{n_k}) + (y - y_{n_k})| < |x - x_{n_k}| + |y - y_{n_k}| < \epsilon$$

Hence, $z_n = x_n + y_n$ has a convergent subsequence in $X + Y$ which shows that $X + Y$ is compact. $\square$

Going back to the original problem, we have that $A + B$ can be written as a countable union of compact, and hence closed, sets. This means that $A + B$ is not only measurable, but actually $\mathcal{F}_\sigma$.

(c) To find an example of two closed sets whose sum is not closed, we look to $\mathbb{R}^2$. Define

$$A = \{(x, y) \in \mathbb{R}^2 \mid y \geq b - mx, b > 0, m > 0\}$$

and

$$B = \{(x, y) \in \mathbb{R}^2 \mid y \geq -b - mx, b > 0, m > 0\}$$

Then we have that A^c and B^c are open, so A and B are closed. But

$$A + B = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$$

which is open.

Problem 1.6.20
(a) Let A denote the standard Cantor set, $\mathcal{C}$ and let $B = \mathcal{C}/2$. Recall that A consists of precisely those real numbers whose ternary expansions contain only the digits $0, 2$, and as a result the elements of B consist of the numbers whose ternary expansions contain only 0 and 1. Now take any $x \in [0, 1]$ and let its ternary decimal expansion be of the form

$$x = 0.d_1 d_2 d_3 \dots$$

We then construct two numbers $c_1 \in A$ and $c_2 \in B$ such that if $c_1 = 0.\alpha_1\alpha_2 \dots$ then

$$\alpha_k = \begin{cases} 2 & d_k = 2 \\ 0 & \text{otherwise} \end{cases}$$

Similarly, if $c_2 = \beta_1\beta_2 \dots$ then

$$\beta_k = \begin{cases} 1 & d_k = 1 \\ 0 & \text{otherwise} \end{cases}$$

Then we have that $d_k = \alpha_k + \beta_k$ for every k, and therefore $x = c_1 + c_2$. Then $[0, 1] \subset A+B$ and $A+B$ is measurable because A, B are closed. So monotonicity implies that $m(A + B) > 0$.

(b) We begin by showing that the set $A = I \times \{0\}$ and $B = \{0\} \times I$ have measure 0. Choose any $\epsilon > 0$ and cover A by cubes of dimension $1/n \times h$. Then the total volume of the cubes is h and so we can just choose any $h < \epsilon$. Letting $\epsilon \to 0$ gives the desired result. The proof that B has measure 0 is analogous. Now choose any $(x, y) \in I \times I$ and observe that

$$(x, y) = (x, 0) + (0, y)$$

And $(x, 0) \in A$ and $(0, y) \in B$. So $I \times I \subseteq A + B$. We then use monotonicity of the measure to see that $m(A + B) > 0$.

Problem 1.6.23
Let $f : \mathbb{R}^2 \to \mathbb{R}$ be separately continuous. We will modify the construction in Theorem 4.1 to construct a sequence of functions that converges pointwise to f. Fix y and partition $\mathbb{R}$ into dyadic intervals of length 2^{-n}. Then for each $x \in \mathbb{R}, n \in \mathbb{Z}^+$ we define $\alpha_{x,n} = \max_k\{k/2^n < x\}$. We then define

$$f_n(x, y) = f(\alpha_{x,n}, y)$$

We begin by showing that $f_n \to f$ pointwise. Pick an $\epsilon > 0$. Then for any $x, y \in \mathbb{R}$ we know that f is continuous in x. So we can find a neighborhood $[(x-\delta, y), (x+\delta, y)]$ such that for any $z \in (x-\delta, x+\delta)$

$$|f(x,y) - f(z,y)| < \epsilon$$

For large n, the approximation of x by dyadic rationals becomes arbitrarily good, that is $|x - \alpha_{x,n}| < \delta$ and so

$$|f(x,y) - f_n(x,y)| = |f(x,y) - f(\alpha_{n,k}, y)| < \epsilon$$

Which means that $f_n \to f$ pointwise.

To show that f is measurable, it suffices to show that for each n, f_n is measurable because f is the pointwise limit of the f_n. Fix n, and consider the set $M_a = \{(x,y) \in \mathbb{R}_2 \,|\, f_n(x,y) > a\}$. If we can show that M_a is measurable for each a, then we will have that f_n is measurable. Then notice that

$$\begin{aligned} M_a &= \bigcup_{k \in \mathbb{Z}} \{(x,y) \,|\, k/2^n \le x < (k+1)/2^n, f(k/2^n, y) > a\} \\ &= \bigcup_{k \in \mathbb{Z}} [k/2^n, (k+1)/2^n) \times \{f(k/2^n, y) > a\} \end{aligned}$$

It is clear that $[k/2^n, (k+1)/2^n)$ is measurable. We know that because f is continuous in y that $\{y \,|\, f(k/2^n, y) > a\}$ is open. Then each term in the union is measurable, and so M_a is the countable union of measurable sets, and hence measurable. So f_n is a measurable function for each n and we are done.

Problem 1.6.26

Suppose that A, B are measurable sets, and that $A \subset E \subset B$. We want to prove that E is measurable. First we note that because $A \subset E$ that we can write

$$E = A \cup (E - A)$$

Because A is measurable, it will suffice to show that $E - A$ is measurable, because then E will be a union of two measurable sets and, therefore, measurable. Observe that because $A \subset E \subset B$ that $(E - A) \subset (B - A)$. We use monotonicity of the measure to get

$$m_*(E - A) \le m_*(B - A) = m(B - A) = m(B) - m(A) = 0$$

So $E - A$ is a subset of a set with measure 0, and is measurable as a result. This immediately gives that E is measurable.

Problem 1.6.28

Let E be a subset of $\mathbb{R}$ with positive outer measure and fix an $\alpha \in (0,1)$. Because E has positive outer measure, we can find a covering of E by closed, almost disjoint interval I_j such that

$$\sum_j |I_j| < m_*(E) + \epsilon/2$$

We can expand each of these I_j to an open cube I'_j such that

$$m_*(I'_j - Q_j) < \epsilon/2^{k+1}$$

and set $\mathcal{O} = \bigcup_j Q'_j$. So $\mathcal{O}$ is an open set containing E and so we can write

$$E = E \cap \mathcal{O} = \bigcup_j E \cap I'_j$$

By monotonicity we can see that $m_*(E) \leq \sum_j m_*(E \cap I'_j)$.

Now suppose towards a contradiction that for every $j \in \mathbb{Z}^+$ we have that $m_*(E \cap I'_j) < \alpha m_*(I'_j)$. Then

$$m_*(E) \leq \sum_j m_*(E \cap I'_j) < \alpha_j \sum_j m_*(I'_j) < \alpha(m_*(E) + \epsilon)$$

But, if we take

$$\epsilon < \frac{1-\alpha}{\alpha} m_*(E)$$

Then we would get that $m_*(E) < m_*(E)$, which is impossible. Hence, we must be able to find some j such that

$$m_*(E \cap I'_j) \geq \alpha m_*(I)$$

Problem 1.6.37

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous. Let $I = [a, b]$ with $a < b$and consider

$$\Gamma_I = \{(x, f(x)) \in \mathbb{R}^2 \mid x \in I\}$$

Note that I is compact and f is continuous, and so f is uniformly continuous on I. Let $\epsilon > 0$ and set $\Delta = \epsilon/2(b-a)$. Because f is uniformly continuous we can find a δ such that $|f(x) - f(y)| < \Delta$ whenever $|x - y| < \delta$. We then Partition I into n intervals $[x_j, x_{j+1}]$ such that $\max_j\{x_{j+1} - x_j\} < \delta$. We then construct a set of n almost disjoint rectangles $R_1, R_2, \ldots, R_n$ where

$$R_j = [x_j, x_{j+1}] \times [(f(x_j) - \Delta), (f(x_j) + \Delta)]$$

Because we chose $|x_{j+1} - x_j| < \delta$ we have $|f(x) - f(x_j)| < \Delta$ for $x \in [x_j, x_{j+1}]$. This immediately gives that $\Gamma_I \subseteq \bigcup_{j=1}^n R_j$. We then have that

$$m(\Gamma_I) \leq m\left(\bigcup_{j=1}^n R_j\right) = \sum_{j=1}^n |R_j| = \sum_{j=1}^n 2\Delta(b-a) = \epsilon$$

We then let $\epsilon \to 0$ to see that $m(\Gamma_I) = 0$. Then letting $I \to R$ gives the result.