Problem 1.8.25

Let $\mathcal{B}$ be a Banach space in which the parallelogram law holds. We can define an inner product on $\mathcal{B}$ by

$$4(f,g) = \|f+g\|^2 - \|f-g\|^2$$

Note that $(\cdot,\cdot)$ induces the norm because

$$\begin{aligned}(f,f) &= \frac{1}{4}\left(\|f+f\|^2 - f - f^2\right)\\ &= \frac{1}{4}(4\|f\|^2)\end{aligned}$$

So we have that $\|f\| = \sqrt{(f,f)}$. We need to show that $(\cdot,\cdot)$ is indeed an inner product because

$$\begin{aligned}4(f,g) &= \|f+g\|^2 - \|f-g\|^2\\ &= (\|f+g\| - \|f-g\|)(\|f+g\| + \|f-g\|)\\ &= (\|f+g\| + \|g-f\|)(\|f+g\| + \|f-g\|)\end{aligned}$$

Because the norm is non-negative, we have $(\cdot,\cdot)$ is the product of positive terms and hence, positive. Symmetry is also clear because

$$\begin{aligned}4(f,g) &= \|f+g\|^2 - \|f-g\|^2\\ &= \|g+f\|^2 - (-1)^2\|g-f\|^2\\ &= 4(g,f)\end{aligned}$$

We are only left to verify linearity. In the real case we compute

$$(f,g+h) = \|f+g+h\|^2 - \|f-(g+h)\|^2$$

We can decompose the right side into the following quantities

$$\begin{aligned}A &= \frac{1}{2}\|f+g+h\|^2 + \frac{1}{2}\|f+g-h\|^2 - \|h\|^2\\ B &= \frac{1}{2}\|f-g+h\| + \frac{1}{2}\|f-g-h\| - \|h\|^2\\ C &= \frac{1}{2}\|f+g+h\| + \frac{1}{2}\|f-g-h\|^2 - \|g\|^2\\ D &= \frac{1}{2}\|f+g-h\|^2 + \frac{1}{2}\|f-g-h\|^2 - \|g\|^2\end{aligned}$$

Then we have

$$\begin{aligned}\|f+g+h\|^2 - \|f-(g+h)\|^2 &= A - B + C - D\\ &= \left(\|f+g\|^2 - \|f-g\|^2\right) + \left(\|f+h\|^2 - \|f-h\|^2\right)\\ &= 4(f,g) + 4(f,h)\end{aligned}$$

So the operation is linear.

To get scalar multiplication we note that linearity immediately implies that

$$(nf,g) = (f+\cdots+f,g) = (f,g)+\cdots+(f,g) = n(f,g)$$

whenever $n \in \mathbb{Z}$. To get rational scalars we note that

$$(f,g) = (n(\frac{1}{n}f),g) = n(\frac{1}{n}f,g)$$

So for any $q \in \mathbb{Q}$ we must have $q(f,g) = (qf,g)$. To extend this to real scalars we define a function $\varphi : \mathbb{R} \to \mathbb{R}$ given by

$$t \mapsto t(f,g) - (tf,g)$$

This function is clearly continuous because it is the difference of continuous functions. Moreover, it is 0 on $\mathbb{Q}$, which is dense in $\mathbb{R}$ and therefore it must be zero on the whole line. Consequently, we must have that $(tf,g) = t(f,g)$ for every $t \in \mathbb{R}$.

For complex Banach spaces we define the inner product by the formula

$$4(f,g) = \left(\|f+g\|^2 - \|f-g\|^2\right) + i\left(\|f+ig\|^2 - \|f-ig\|^2\right)$$

The computation to show that this indeed an inner product in analogous to the one above, but considers the real and imaginary parts of the vectors separately.

To see that $L^p(\mathbb{R})$ is only a Hilbert space for $p=2$ consider the functions $f(x) = \chi_{[0,1]}(x)$ and $g(x) = \chi_{[1/2,1]}(x)$. In order for the parallelogram law to hold in $L^p(\mathbb{R})$ we must have that $2(2)^{2/p} = 1(1)^{2/p}$, which holds only when $p=2$.

Problem 1.8.30

It is clear that $\mathcal{B}/\mathcal{S}$ is a vector space. Let $[f]$ denote the equivalence class of f under the relation $\sim$. We define addition pointwise

$$[f] + [g] = [f+g]$$

and scalar multiplication via $\alpha[f] = [\alpha f]$. We then put the following norm on $\mathcal{B}/\mathcal{S}$,

$$\|[f]\|_{\mathcal{B}/\mathcal{S}} = \inf_{f' \sim f} \|f'\|_{\mathcal{B}}$$

To see that this is indeed a norm we note that $f' \sim f$ implies that $f = f' + s$ for some non-zero $s \in \mathcal{S}$. We check scalar multiplication by noting that if $\alpha \neq 0$ then

$$\begin{aligned}
\|\alpha[f]\|_{\mathcal{B}/\mathcal{S}} &= \|[\alpha f]\|_{\mathcal{B}/\mathcal{S}} \\
&= \inf_{s \in \mathcal{S}} \|\alpha f + s\|_{\mathcal{B}} \\
&= \inf_{s \in \mathcal{S}} \|\alpha f + \alpha s\|_{\mathcal{B}} \\
&= |\alpha| \inf_{s \in \mathcal{S}} \|f + s\|_{\mathcal{B}} \\
&= |\alpha| \|[f]\|_{\mathcal{B}/\mathcal{S}}
\end{aligned}$$

Equality still holds when $\alpha = 0$ because $[0] \in \mathcal{S}$ and $\inf_{s \in \mathcal{S}} \|s\| = 0$. Now we check the triangle inequality by computing

$$\begin{aligned}
\|[f] + [g]\|_{\mathcal{B}/\mathcal{S}} &= \|[f+g]\|_{\mathcal{B}/\mathcal{S}} \\
&= \inf_{s \in \mathcal{S}} \|f + g + s\|_{\mathcal{B}} \\
&= \inf_{s,s' \in \mathcal{S}} \|f + s + g + s'\|_{\mathcal{B}} \\
&= \inf_{s \in \mathcal{S}} \|f + s\|_{\mathcal{B}} + \|g + s\|_{\mathcal{B}} \\
&= \|[f]\|_{\mathcal{B}/\mathcal{S}} + \|[g]\|_{\mathcal{B}/\mathcal{S}}
\end{aligned}$$

Now we need only verify that $\|[f]\| = 0$ implies that $[f] = 0$. Let $f \in \mathcal{B}$ be such that $\|[f]\| = 0$. Then we must have that $\inf_{s \in \mathcal{S}} \|f + s\|_{\mathcal{B}} = 0$. So for each $n > 0$ we can sind an $s_n \in \mathcal{S}$ such that $\|f + s_n\|_{\mathcal{B}} < 1/n$. Consequently, we must have that $-s_n \to f$ as $n \to \infty$. Because $\mathcal{S}$ is closed we must have that $f \in \mathcal{S}$ and therefore $[f] = 0$ in $\mathcal{B}/\mathcal{S}$.

Now we need to show that $\mathcal{B}/\mathcal{S}$ is complete. Let $\sum_n F_n$ be an absolutely convergent series in $\mathcal{B}/\mathcal{S}$. By definition of $\|\cdot\|_{\mathcal{B}/\mathcal{S}}$ we can find f_n such that

$$\|f_n\|_{\mathcal{B}} \leq \|F_n\|_{\mathcal{B}/\mathcal{S}} + 2^{-n}$$

It is then clear that $\sum_n |f_n|$ is bounded and hence converges and therefore $\sum_n f_n$ converges because $\mathcal{B}$ is a Banach space.

Let $f = \sum_n f_n$ and let $S_N(f)$ denote the N^{th} partial sum. We then see that

$$\begin{aligned} \|[S_N(f)] - [f]\|_{\mathcal{B}/\mathcal{S}} &= \|S_N([f_n - f])\|_{\mathcal{B}/\mathcal{S}} \\ &= \inf_{s \in \mathcal{S}} \|S_N(f) - f + s\|_{\mathcal{B}} \\ &\leq \|S_N(f) - f + S_N(s)\|_{\mathcal{B}} \\ &= \|S_N(f + m) - f\|_{\mathcal{B}} \end{aligned}$$

The last quantity clearly goes to 0 as $N \to \infty$. Thus, we have that $S_N([f]) \to [f]$ as $N \to \infty$ and so $\mathcal{B}/\mathcal{S}$ is complete.

Problem 1.8.32
If $\mathcal{B}^*$ is separable then we can find a countable dense subset $\{\varphi_1, \varphi_2, \ldots\}$. Now choose a set of unit vectors $f \in \mathcal{B}$ such that $|\varphi_n(x_n)| \geq \frac{1}{2}\|\varphi_n\|$. Let $\mathcal{C}$ be the set of linear combinations of the x_n. We need to show that $\mathcal{C}$ is dense in $\mathcal{B}$. Suppose not, then we have that $\overline{\mathcal{C}}$ is a proper closed subspace of $\mathcal{B}$. So we can find some non-zero bounded linear functional $L \in \mathcal{B}^*$ such that $L(\overline{\mathcal{C}}) = 0$. Since $L \in \mathcal{B}^*$ and $\{\varphi_n\}$ is dense in $\mathcal{B}^*$ we must have some sequence $\varphi_{n_k} \to L$ as $n_k \to \infty$. Hence $\|L - \varphi_{n_k}\| \to 0$ as $n_k \to \infty$. But,

$$\begin{aligned} \|L - \varphi_{n_k}\| &\geq |(L - \varphi_{n_k})(x_{n_k})| \\ &= |\varphi_{n_k}(x_{n_k})| \\ &\geq \frac{1}{2}\|\varphi_{n_k}\| \end{aligned}$$

So we must have that $\|\varphi_{n_k}\| \to 0$ as $n_k \to \infty$. However, $\varphi_{n_k} \to L$ and so $\|\varphi_{n_k}\| \to \|L\|$, which means that $\|L\| = 0$. This is a contradiction, and therefore we have that $\mathcal{C}$ is dense in $\mathcal{B}$.

Problem 1.8.33
Following the hint, we take $u = \mathrm{Re}(\ell_0)$. We then apply Theorem 1.5.2 to extend u to a linear functional $U : V \to \mathbb{R}$ such that $U(f_0) = u(f_0)$ for every $f_0 \in V_0$ and $U(f) \leq p(f)$ for every $f \in V$. We then define

$$\ell(f) = U(f) - iU(if)$$

Then it is clear that $\ell(f_0) = \ell_0(f_0)$ whenever $f_0 \in V_0$. Fix a $g \in V$ and find a complex z such that $|z| = 1$ and $z\ell(g) = |\ell(g)|$. Hence,

$$|\ell(g)| = z\ell(g) = \ell(zg)$$

which means that $\ell(zg) \in \mathbb{R}$ and therefore $\ell(zg) = U(zg)$. As a result we see

$$|\ell(g)| = U(zg) \leq p(zg) = |z|p(g) = p(g)$$

And therefore ℓ is the functional with the desired properties.

Problem 1.8.34
Consider the functional $\ell : V \to \mathbb{R}$ defined by

$$\ell(f) = \frac{\|f - p(f)\|}{\|f_0 - p(f_0)\|}$$

where $p(f)$ is the canonical projecction of f onto $\mathcal{S}$. If $f_0 \notin \mathcal{S}$ then the mapping is well-defined because $f_0 \neq p(f_0)$ so the denominator is non-zero. Moreover, it is clear that ℓ s continuous because the norm is continuous and the denominator is a non-zero scalar. If

$f \in \mathcal{S}$ then we have that $f = p(f)$ and so $\ell(f) = 0$. To check the other conditions we compute

$$\ell(f_0) = \frac{\|f_0 - p(f_0)\|}{\|f_0 - p(f_0)\|} = 1$$

So $\ell(f_0) = 1$.

Now we need to verify that $\|\ell\| = 1/d$ where d is the distance from f_0 to $\mathcal{S}$. We begin by computing

$$\begin{aligned}
\|\ell\| &= \sup_{f\neq 0} \frac{|\ell(f)|}{\|f\|} \\
&= \sup_{f\neq 0} \frac{\|f - p(f)\|}{\|f\|} \cdot \frac{1}{d} \\
&= \sup_{f\neq 0} \left(1 - \frac{\|p(f)\|}{\|f\|}\right) \cdot \frac{1}{d} \\
&\leq \frac{1}{d}
\end{aligned}$$

But we can find a case where equlity holds by choosing $f \in \ker p$. Then we have

$$\begin{aligned}
\frac{|\ell(f)|}{\|f\|} &= \sup_{f\neq 0} \frac{\|f - p(f)\|}{\|f\|} \cdot \frac{1}{d} \\
&= \sup_{f\neq 0} \frac{\|f - 0\|}{\|f\|} \cdot \frac{1}{d} \\
&= \frac{1}{d}
\end{aligned}$$

Which means that $\|\ell\| = 1/d$ as desired.