Problem 2.5.4

We begin by noting that because we can always write $f = f^+ - f^-$ we can assume without loss of generality that f is non-negative (otherwise we would just look at each non-negative part separately). We then consider the function

$$g(x) = \int_x^b \frac{f(t)}{t} dt$$

which is defined on the interval $I = (0, b]$. More generally, let $I_x = (x, b]$. We want to integrate g, so we observe that

$$\int_I g(x)dx = \int_I \int_{I_x} \frac{f(t)}{t} dtdx$$

This would lead us to consider the function

$$h(x,t) = \frac{f(t)}{t} \chi_{I_x}$$

Note that h is measurable because it is the quotient of $f(t)$ and t, which are both measurable on I_x, multiplied by χ_{I_x}, which is clearly measurable. Furthermore, because we took f to be non-negative, h is also non-negative. We then rewrite the above to see

$$\int_I g(x)dx = \int_I \int_{I_x} \frac{f(t)}{t} dtdx = \int_{\mathbb{R}} \int_{\mathbb{R}} h(x,t)\, \chi_I \, dtdx$$

We then note that we satisfy the hypotheses for Fubini's theorem, so we can exchange the order of integration to get

$$\int_{\mathbb{R}} \int_{\mathbb{R}} h(x,t)\, \chi_I \, dtdx = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} h(x,t)\, \chi_I \, dx \right) dt$$

Simplifying we get

$$\begin{aligned}
\int_{\mathbb{R}} \left(\int_{\mathbb{R}} h(x,t)\, \chi_I \, dx \right) dt &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{f(t)}{t} \chi_I \, dx \right) \chi_{I_x} \, dt \\
&= \int_0^b \left(\int_0^t \frac{f(t)}{t} dx \right) dt \\
&= \int_0^b t \frac{f(t)}{t} \\
&= \int_0^b f(t)dt
\end{aligned}$$

Because f is integrable on $(0, b]$, we have that g must also be integrable and that

$$\int_0^b g(x)dx = \int_0^b f(t)dt$$

Problem 2.5.5

(a) We need to show that δ is continuous. We will in fact show more, that δ in Lipschitz with constant 1 and therefore continuous with setting $\delta = \epsilon$ and observing that

$$|\delta(x) - \delta(y)| \leq |x-y| < \delta = \epsilon$$

We proceed by choosing an $\epsilon > 0$ and a $z \in F$ such that $|y - z| < \delta(y) + \epsilon$. Then we see that

$$\delta(x) \leq |x - z| \leq |x - y| + |y - z| < |x - y| + \delta(y) + \epsilon$$

Which means that

$$\delta(x) - \delta(y) < |x - y| + \epsilon$$

We then note that this is symmetric in x and y, so the same process for x gives

$$\delta(y) - \delta(x) < |y - x| + \epsilon = |x - y| + \epsilon$$

These two facts together implies that

$$|\delta(x) - \delta(y)| < |x + y| + \epsilon$$

Letting $\epsilon \to 0$ gives the result.

(b) We are given that F is closed and that $x \notin F$. This means that $\delta(x)$ must be greater than zero because otherwise we could find a sequence of points in F that converge to x and that would imply x is a limit point of F, a contradiction. We set $\delta = \delta(x)$ then use the Lipschitz condition in the previous part to see that if we pick a $y \in N_{\delta/2}(x)$ then

$$|x - y| < \delta/2 \text{ and } |\delta(y) - \delta| < \delta/2$$

Which means that $\delta(y) \geq \delta/2$. So we can compute

$$I(x) = \int_{\mathbb{R}} \frac{\delta(y)}{|x-y|^2} dy \geq \int_{x-\delta/2}^{x+\delta/2} \frac{\delta(y)}{|x-y|^2} dy \geq \frac{\delta}{2} \int_{x-\delta/2}^{x+\delta/2} \frac{1}{|x-y|^2} dy$$

We then use a translation by x to see that

$$\frac{\delta}{2} \int_{x-\delta/2}^{x+\delta/2} \frac{1}{|x-y|^2} dy = \frac{\delta}{2} \int_{x-\delta/2}^{x+\delta/2} \frac{1}{y^2} dy$$

We then observe that the integral on the right diverges to ∞. This precisely gives that $I(x) = \infty$ for $x \notin F$.

(c) We proceed in a similar manner to the previous exercise. We observe that

$$\int_F I(x) dx = \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\delta(y)}{|x-y|^2} \chi_F(x) dy dx$$

We then note that because $I \geq 0$ and measurable on F as it is the quotient of two measurable functions, we can apply Theorem 3.2 from the text to see that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\delta(y)}{|x-y|^2} \chi_F(x) dy dx = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{\delta(y)}{|x-y|^2} \chi_F(x) dx \right) dy$$
$$= \int_{\mathbb{R}} \delta(y) \left(\int_{\mathbb{R}} \frac{1}{|x-y|^2} \chi_F(x) dx \right) dy$$

We then note that $F \subset \{x \,|\, |x-y| > \delta(y)\}$ and recall that $x \not\in F$ so that

$$\int_F \frac{1}{|x-y|^2} dx \leq \int_{|\delta(y)|}^{\infty} \frac{dx}{x^2} dx = 2 \int_{\delta(y)}^{\infty} \frac{dx}{x^2} = \frac{2}{\delta(y)}$$

Using this inequality above yields

$$\int_{\mathbb{R}} \delta(y) \left(\int_{\mathbb{R}} \frac{1}{|x-y|^2} \chi_F(x) dx \right) dy \leq \int_{F^c} \delta(y) \cdot \frac{2}{\delta(y)} = 2m(F^c)$$

We assumed that $m(F^c) < \infty$, so we have that $\int_{\mathbb{R}} I(x) dx < \infty$ which means that $I(x) < \infty$ a.e on F^c.

Problem 2.5.7
First we need to show that Γ is measurable. this is clear because we know that for each pair $(x, y) \in \mathbb{R}^d \times \mathbb{R}$, we have $\{x \,|\, x = f^{-1}(y)\}$ is a measurable set. Then we apply Proposition 3.6 in the text which says that the product of two measurable sets is measurable. Then we write for any

$$\Gamma = \bigcup_{k=1}^{\infty} (f^{-1}(I_k) \times I_k)$$

where I_k is any countable collection of intervals covering $\mathbb{R}$. Γ is the countable union of measurable sets, and is therefore measurable.

As usual we will assume that f is non-negative, because we can always just individually consider f^+ and f^-. Applying Theorem 3.2 from the text we see that each of the vertical slices of Γ are measurable and of the form

$$\Gamma_x = \{y \,|(x, y) \in \Gamma\}$$

Moreover, we have that $m(\Gamma_x) = 0$ for every x because each Γ_x contains only a single point. We then apply Theorem 3.2 again to express $m(\Gamma)$ as an integral

$$m(\Gamma) = \int \int \chi_{\Gamma_x}(x, y) dy dx = 0$$

which concludes the proof.

Problem 2.5.12
Following the hint given in the book, we will begin by constructing a sequence of measurable sets $I_n \subset \mathbb{R}$ such that $m(I_n) \to 0$. This construction will be based

on the harmonic series $\sum_{k=1}^{\infty} 1/k$. We construct a sequence of integers $b_1, b_2, \ldots$ such that for each N the partial sum

$$\sum_{k=1}^{b_N} 1/k > 10^N$$

We know that this is possible because the harmonic series is divergent. Then for each $N \in \mathbb{Z}^+$ we set $B_{b_{N+1}}$ to be the interval of length $1/b_{N+1}$. We then complete the sequence of B_ℓ as follows:

1. Find N such that $b_N < \ell \leq b_{N+1}$.
2. Set B_ℓ to be the interval centered at the origin such that

$$m(B_\ell) = m(B_{\ell-1}) + 1/\ell$$

We now use the intervals B_ℓ to construct the decreasing sequence of intervals by setting

$$I_n = \begin{cases} B_n & \text{if } \exists N, n = b_{N+1} \\ B_n - B_{n-1} & \text{otherwise} \end{cases}$$

We can now see that $m(I_n) \leq 1/n$ because if $n = b_{N+1}$ for some N, then it is true by definition, and for any other n then $m(I_n) = m(B_n) - m(B_{n-1}) < 1/n - 1/(n-1)$. As a result, we have that $m(I_n) \to 0$. However, we have that the union

$$m\left(\bigcup_{n=b_N}^{b_{N+1}} I_n\right) > 10^N$$

And because each interval is centered at the origin, we will eventually cover all of $\mathbb{R}$. We then set $f(x) = 0$ define $f_n(x) = \chi_{I_n}(x)$. Observe that by the above

$$\int_{\mathbb{R}} f_n \leq 1/n$$

And so

$$\|f_n - f\| = \int |f_n - f| = \int f_n \leq 1/n$$

And so $f_n \to f$ in the norm. However, we have that for any x there are infinitely many I_n such that $x \in I_n$, because their union is unbounded and so for sufficiently large N we have that $n > N$ implies $x \in \bigcup_{n=b_N}^{b_{N+1}} I_n$. this meas that there are infinitely many n with $f_n(x) = 1$. So f_n cannot converge to f pointwise for any x.

Problem 2.5.17

(a) We are given the function

$$f(x,y) = \begin{cases} a_n & (x,y) \in [n, n+1) \times [n, n+1) \\ -a_n & (x,y) \in [n, n+1) \times [n, n+2) \\ 0 & \text{otherwise} \end{cases}$$

Where the a_n are the partial sums of the convergent series $\sum_{j=1}^{\infty} b_i$ with b_j non-negative for every j. That each slice f^y and f_x are integrable is integrable is clear if we look at the graph of f. It is constant on each box in the lattice $\mathbb{Z}^2$, and takes non-zero values only on the two "diagonals" for $y = x$ and $y = x + 1$. So geometrically we have that the slices are constant on intervals contained in each box. More precisely we see that

$$f^y(x) = \begin{cases} -a_{\lfloor y \rfloor - 1} & y \geq 1 \text{ and } (\lfloor y \rfloor - 1) \leq x < \lfloor y \rfloor \\ a_{\lfloor y \rfloor} & y \geq 1 \text{ and } \lfloor y \rfloor \leq x < (\lfloor y \rfloor + 1) \\ a_0 & 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Similarly,

$$f_x(y) = \begin{cases} a_{\lfloor x \rfloor} & \lfloor x \rfloor \leq y < (\lfloor x \rfloor + 1) \\ -a_{\lfloor x \rfloor} & (\lfloor x \rfloor + 1) \leq y < (\lfloor x \rfloor + 2) \\ 0 & \text{otherwise} \end{cases}$$

To show that each slice $f^y(x)$ is integrable we simply compute

$$\int f^y(x)dx = \int_{\lfloor y \rfloor - 1}^{\lfloor y \rfloor} -a_{\lfloor y \rfloor - 1}dx + \int_{\lfloor y \rfloor}^{\lfloor y \rfloor + 1} a_{\lfloor y \rfloor}dx = a_{\lfloor y \rfloor} - a_{\lfloor y \rfloor - 1} = b_{\lfloor y \rfloor}$$

This quantity is bounded because the sequence b_n converges (so in fact it goes to 0). In the case that $0 \leq x < 1$, we have that that $\int f^y(x)dx = a_0$. For the slice $f_x(y)$ we have that

$$\int f_x(y) = \int_{\lfloor x \rfloor}^{\lfloor x \rfloor + 1} a_{\lfloor x \rfloor}dx + \int_{\lfloor x \rfloor + 1}^{\lfloor x \rfloor + 2} -a_{\lfloor x \rfloor}dx = a_{\lfloor x \rfloor} - a_{\lfloor x \rfloor} = 0$$

And so

$$\int\int f(x,y)dydx = \int \left(\int f_x(y)dy \right) dx = \int 0dx = 0$$

(b) We computed the values of each of he integral of each slice $f^y(x)$ in the previous part. We then compute

$$\int_{\mathbb{R}}\int_{\mathbb{R}} f(x,y)dxdy = \sum_{n=0}^{\infty} \int_n^{n+1} \left(\int_{\mathbb{R}} f^y(x)dx \right) dy = \sum_{n=0}^{\infty} b_n = s$$

(c) We then use the fact that $0 \leq |f(x,y)| < \infty$ and apply Theorem 3.2 to see that

$$\begin{aligned} \int\int |f(x,y)| &= \int \left(\int f_x(y)dy \right) dx \\ &= \sum_{n=0}^{\infty} \int_n^{n+1} \left(\int f_x(y)dx \right) dx \\ &= \sum_{n=0}^{\infty} 2a_n \end{aligned}$$

Because each of the $a_n \geq a_0 > 0$, this sum diverges and so

$$\int_{\mathbb{R}\times\mathbb{R}} |f(x,y)| = \infty$$

Problem 2.5.18
The plan is to use Fubini on the function $g(x,y) = |f(x) - f(y)|$. By hypothesis we have that g is integrable on $[0,1]\times[0,1]$, and so we can apply Fubini's theorem to get that each slice $g^y(x)$ is integrable for a.e. $y \in [0,1]$. Choose one of these y and note that because

$$f(x) - f(y) \leq |f(x) - f(y)|$$

we can use the monotonicity of the integral to see that

$$\int_{[0,1]} f(x) - f(y)dx \leq \int_{[0,1]} |f(x) - f(y)|dx \leq \infty$$

Which means that

$$\begin{aligned}\int_{[0,1]} f(x)dx &\leq \int_{[0,1]} f(y)dx + \int_{0,1} |f(x) - f(y)|dx \\ &= f(y) + \int_{0,1} |f(x) - f(y)|dx\end{aligned}$$

Each of these terms is finite, and so the sum is finite, which means that $f(x)$ is integrable.

Problem 2.5.19
The key observation is that

$$m(E_\alpha) = \int_{\mathbb{R}^d} \chi_{E_\alpha}(x)dx$$

So if we set

$$f(\alpha, x) = \chi_{E_\alpha}(x)$$

It is clear that f is non-negative and measurable and so we can apply Theorem 3.2 to see that

$$\begin{aligned}\int_0^\infty m(E_\alpha)d\alpha &= \int_0^\infty \int_{\mathbb{R}^d} f(\alpha,x)dxd\alpha \\ &= \int_{\mathbb{R}^d} \left(\int_0^\infty \chi_{E_\alpha}(x)d\alpha\right) dx \\ &= \int_{\mathbb{R}^d} |f(x)|dx\end{aligned}$$

And so we are done.