1 Measure and Integration Theory

Basically just a collection of Theorems and proofs from measure theory.

1.1 Measurable Sets

Theorem (Inner and Outer Regularity for Lebesgue Measure). *Suppose that $E_1, E_2, \ldots$ is a sequence of measurable sets in $\mathbb{R}^d$ then*

I. If $E_k \nearrow E$ then $\lim_{k\to\infty} m(E_k) = m(E)$.

I. If $E_k \searrow E$ and $m(E_k) < \infty$ for some k then $\lim_{k\to\infty} m(E_k) = m(E)$.

Proof. For I. we let $F_1 = E_1$ and then recursively define $F_n = E_n - E_{n-1}$. By construction we have that each of the F_n are measurable, disjoint and $\bigcup_n F_n = E$. So we have that

$$m(E) = \sum_{n=1}^{\infty} m(F_n) = \lim_{N\to\infty} \sum_{n=1}^{N} m(F_n) = \lim_{N\to\infty} m\left(\bigcup_{n=1}^{\infty} F_n\right) = \lim_{N\to\infty} m(E_n)$$

So we have the desired limit.

For II. we suppose without loss of generality that $m(E_1) < \infty$. We then do a similar construction letting $F_n = E_n - E_{n+1}$ so that we have

$$E_1 = E \cup \bigcup_{n=1}^{\infty} F_n$$

is a disjoint union of measurable sets. We then see that

$$\begin{aligned} m(E_1) &= m(E) + \lim_{N\to\infty} \sum_{n=1}^{N-1} m(E_n - E_{n+1} \\ &= m(E) + m(E_1) - \lim_{N\to\infty} m(E_N) \end{aligned}$$

Subtracting $m(E_1)$ from both sides gives the result. □

Theorem (Regularity Theorem for Lebesgue Measure). *Suppose that $E \subset \mathbb{R}^d$ is measurable. Then for every $\epsilon > 0$*

I. There exists an open set $\mathcal{O}$ with $E \subset \mathcal{O}$ and $m(\mathcal{O} - E) \leq \epsilon$.

II. There exists a closed set F with $F \subset E$ and $m(E - F) \leq \epsilon$.

III. If $m(E) < \infty$, there exists a compact set K with $K \subset E$ and $m(E - K) \leq \epsilon$.

IV. If $m(E) < \infty$, there exists a finite union $F = \bigcup_{j=1}^{N} Q_j$ of closed cubes such that $m(E\Delta F) \leq \epsilon$. Where Δ is the summetric difference operator.

Proof. The proof of I. is just the definition of measurability we chose (not Carathéodory measurability). To show II. we reduce to I. because if E is measurable then E^c is measurable and so there is some open set $\mathcal{O}$ that contains E^c and has $m(\mathcal{O} - E^c) \leq \epsilon$. We then note that $F = \mathcal{O}^c$ is closed and contained in E. Moreover, $E - F = \mathcal{O} - E^c \leq \epsilon$. So II. is proved.

For III. we use the usual trick to go from closed sets to compact sets on σ-finite measure spaces. We know by II. that there is a closed set $F \subset E$ such that $m(E - F) \leq \epsilon$. Define the decreasing sequence of sets $K_n = E \cap B_n$ where B_n is the ball of radius n centered at the origin. Then we have that $E - K_n$ is a decreasing sequence of measurable sets that decreases to $E - F$ and so we have that for sufficiently large n that $m(E - K_n) \leq \epsilon$. Note that it is critical that $m(E)$ is finite.

For the proof of IV. we choose a family of closed cubes Q_j such that

$$E \subset \bigcup_{j=1}^{\infty} Q_j \text{ and } \sum_{j=1}^{\infty} |Q_j| \leq m(E) + \epsilon/2$$

Because $m(E) < \infty$ we have that the series converges and so we can find $N > 0$ such that $\sum_{j=N+1}^{\infty} |Q_j| < \epsilon/2$. If we set $F = \bigcup_{j=1}^{N} Q_j$ then we see that

$$\begin{aligned} m(E\Delta F) &= m(E - F) + m(F - E) \\ &\leq m\left(\bigcup_{N+1}^{\infty} Q_j\right) + m\left(\bigcup_{j=1}^{\infty} Q_j - E\right) \\ &\leq \sum_{j=N+1}^{\infty} |Q_j| + \sum_{j=1}^{\infty} |Q_j| - m(E) \\ &\leq \epsilon \end{aligned}$$

So we are done. $\square$

Theorem (Decomposition of Measurable Sets)**.** *A subset E in $\mathbb{R}^d$ is measurable*

I. if and only if E differs from a G_δ set by a set of measure zero.

II. if and only if E differs from an F_σ set by a set of measure zero.

Proof. The forward direction is trivial because we know that if $E = A \cup B$ where A is either F_σ or G_δ and B has measure zero, then E is the union of measurable sets and hence, measurable.

In the reverse direction in I. we are given that E is measurable. For each n we know that there is an open set $\mathcal{O}_n$ that contains E and such that $m(\mathcal{O}_n - E) \leq 1/n$. Then we set $G = \bigcap_{n=1}^{\infty} \mathcal{O}_n$ is a G_δ set and we have that $(S - E) \subset (\mathcal{O}_n - E)$ for all n so $S - E$ has measure zero.

The reverse direction for II. is very similar. We use the Regularity Theorem to get a sequence of closed sets $\mathcal{C}_n$ such that each is contained in E and $m(E - \mathcal{C}_n) \leq 1/n$. We set $F = \bigcup_{n=1}^{\infty} \mathcal{C}_n$. Then F is clearly an F_σ and moreover $m(E - \mathcal{C}_n) \to 0$ as $n \to \infty$ and so $m(E - F) = 0$. $\square$

1.2 Measurable Functions

Theorem (Properties of Measurable Functions)**.** *We have the following:*

P1. The finite- valued function f is measurable iff for every open set $\mathcal{O}$ and every closed set F the sets $f^{-1}\mathcal{O}$ and $f^{-1}(F)$ are measurable.

P2. If f is continuous on $\mathbb{R}^d$, then f is measurable. If f is measurable and finite valued and Φ is continuous then $\Phi \circ f$ is measurable.

P3. If $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions then

$$\sup_n f_n(x), \inf_n f_n(x), \limsup_{n\to\infty} f_n(x), \text{ and } \liminf_n f_n(x)$$

are measurable.

P4. If $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions and $f_n \to f$ pointwise a.e. then f is measurable.

P5. If f, g are measurable and finite valued then $f+g, fg,$ and f^k are measurable.

P6. If f is measurable and $f = g$ a.e. then g is measurable.

Proof. The proof for $P1$ is clear. If f is measurable then we know that the sets $f^{-1}((-\infty, a))$ are measurable for any a. We can then construct every interval (a, b) by taking $(-\infty, b) \cap \bigcup_{n=1}^{\infty}(a - 1/n, \infty)$. This is a countable union of open sets and hence open. Furthermore, its inverse image is measurable. We then use the fact that the intervals are a basis for the topology on $\mathbb{R}$ to complete the proof.

For $P2$ we use the fact that if f is continuous then the inverse image of an open set is open, hence measurable. To see that Φ is measurable we note that it is continuous. Then we have that if f is measurable that $(\Phi \circ f)^{-1} = f^{-1} \circ \Phi^{-1}$ is clearly measurable because the inverse image of the interval under Φ is open and f is measurable.

To prove $P3$ we first note that $\{\sup_n f_n(x) > a\} = \bigcup_n \{f_n(x) > a\}$ is clearly measurable as the union of measurable sets. Similarly, $\inf_n f_n(x) = -\sup_n -f_n(x)$ is measurable. To get lim sup and lim inf we simply note that

$$\limsup_{n\to\infty} f_n(x) = \inf_k \{\sup_{n\geq k} f_n(x)\} \text{ and } \liminf_{n\to\infty} f_n(x) = \sup_k \{\inf_{n\geq k} f_n(x)\}$$

$P4$ is simply a consequence of $P3$ because we have that

$$f(x) = \limsup_{n\to\infty} f_n(x) = \liminf_{n\to\infty} f_n(x)$$

For $P5$ we can get that $f + g$ is measurable because we can write

$$\{(f+g)(x) > a\} = \bigcup_{r\in\mathbb{Q}} \{f(x) > a - r\} \cap \{g(x) > r\}$$

which is the union of measurable sets. We can then see that f^k is measurable because we have

$$\{f^(x) > a\} = \begin{cases} \{f(x) > a^{1/k}\} & k \text{ is even} \\ \{f(x) > a^{1/k}\} \cup \{f(x) < -a^{1/k}\} & k \text{ is odd} \end{cases}$$

Finally, for $P6$ we simply note that $f(x) = g(x)$ a.e. implies that the sets $\{f(x) < a\}$ and $\{g(x) < a\}$ differ by at most a set of measure zero. This immediately gives the result. $\square$

Theorem (Egorov). *Suppose $\{f_k\}_{k=1}^{\infty}$ is a sequence of measurable functions defined on a measurable set E with $m(E) < \infty$, and assume that $f_k \to f$ a.e. on E. Given $\epsilon > 0$ there exists a closed set $A_\epsilon \subset E$ such that $m(E - A_\epsilon) < \epsilon$ and $f_k \to f$ uniformly on A_ϵ.*

Proof. We assume without loss of generality that $f_k \to f$ on all of E because we can just replace E by the set $E - N$ if N are the points where f_n doesn't converge.

For each pair of integers of non-negative integers define

$$E_k^n = \{x \in E : |f_j(x) - f(x)| < 1/n, \text{ for all } j > k\}$$

If we fix n and note that $E_k^n \subset E_{k+1}^n$ and $E_n^n \nearrow E$ as $k \to \infty$. We then note that there is some k_n such that $m(E - E_{k_n}) < 2^{-n}$. Then we have that

$$|f_j(x) - f(x)| < 1/n \text{ whenever } j > k_n \text{ and } x \in E_{k_n}^n$$

Now we choose N large enough that $\sum_{n=N}^{\infty} 2^{-n} < \epsilon/2$, and let

$$\tilde{A}_\epsilon = \bigcap_{n \geq N} E_{k_n}^n$$

Observe that

$$m(E - \tilde{A}_\epsilon) \leq \sum_{n=N}^{\infty} m(E - E_{k_n}^n) < \epsilon/2$$

then if we have a $\delta > 0$ then we choose $n \geq N$ such that $1/n < \delta$ and note that $x \in \tilde{A}_\epsilon$ implies that $x \in E_{k_n}^n$. Thus, $|f_j(x) - f(x)| < \delta$ whenever $j > k_n$. So $f_k \to f$ uniformly on $\tilde{A}_\epsilon$. We then find a closed subset $A_\epsilon \subset \tilde{A}_\epsilon$ such that $m(\tilde{A}_\epsilon - A_\epsilon) < \epsilon/2$. Consequently, we see that $m(E - A_\epsilon) < \epsilon$ and so we are done. □

Theorem (Lusin)**.** *Suppose that f is measurable and finite-valued on E with $m(E) < \infty$. Then for any $\epsilon > 0$ there exists a closed set F_ϵ such that $F_\epsilon \subset E$ and $m(E - F_\epsilon) \leq \epsilon$ and such that $f|_{F_\epsilon}$ is continuous.*

Proof. Find a sequence of step function $\{f_n\}$ such that $f_n \to f$ a.e. Then we may find sets E_n such that $m(E_n) < 2^{-n}$ and f_n is continuous outside of E_n. We then apply Egorov's theorem to get a set $A_{\epsilon/3}$ such that $f_n \to f$ uniformly on $A_{\epsilon/3}$ and $m(E - A_{\epsilon/3}) \leq \epsilon/3$. Then we consider the set

$$\tilde{F} = A_{\epsilon/3} - \bigcup_{n \geq N} E_n$$

for N large enough that $\sum_{n \geq N} 2^{-n} < \epsilon/3$. Now we have that for every $n \geq N$ the function f_n is continuous on $\tilde{F}$. Thus, f is continuous on $\tilde{F}$ because $f_n \to f$ uniformly there. To complete the proof we simply find a closed set $F_\epsilon \subset \tilde{F}$ such that $m(\tilde{F} - F_\epsilon) < \epsilon/3$. □

1.3 Abstract Measure Theory

Theorem (Carathéodory)**.** *Given an outer measure μ_* on a set X, the collextion $\mathfrak{M}$ of Carathéodory measurable sets forms a σ-algebra. Moreover, μ_* restriced to $\mathfrak{M}$ is a measure.*

Proof. Clearly, $\emptyset$ and X are in $\mathfrak{M}$ becuase the definition of measurability is symmetric in complements.

Next we need to prove that we are closed under finite unions of disjoint sets and that μ_* is finitely additive on $\mathfrak{M}$. If we have $E_1, E_2 \in \mathfrak{M}$ and A is any subset of X then

$$\begin{aligned}
\mu_*(A) &= \mu_2(E_2 \cap A) + \mu_*(E_2^c \cap A) \\
&= \mu_*(E_1 \cap E_2 \cap A) + \mu_*(E_1^c \cap E_2 \cap A) + \mu_*(E_1 \cap E_2^c \cap A) + \mu_*(E_1^c \cap E_2^c \cap A) \\
&\geq \mu_*((E_1 \cup E_2) \cap A) + \mu_*((E_1 \cup E_2)^c \cap A)
\end{aligned}$$

The first line follows from the measurability condition on E_1 and E_2 and the last inequality is the sub-additivity of μ_*. Therefore we have that $E_1 \cup E_2 \in \mathfrak{M}$. If they are disjoint then we have that

$$\mu_*(E_1 \cup E_2) = \mu_*(E_1 \cap (E_1 \cup E_2)) + \mu_*(E_1^c \cap (E_1 \cup E_2)) = \mu_*(E_1) + \mu_*(E_2)$$

Now we need to show that $\mathfrak{M}$ is closed under countable unions of disjoint sets, i.e. μ_* is countably additive on $\mathfrak{M}$. Let $E_1, E_2, \ldots$ be a countable collection of disjoint sets in $\mathfrak{M}$. Define

$$G_n = \bigcup_{j=1}^{n} E_j \text{ and } G = \bigcup_{j=1}^{\infty} E_j$$

For each n we have that G_n is a finite union of sets in $\mathfrak{M}$ and so $G_n \in \mathfrak{M}$. Furthermore, for any $A \subset X$ we have that

$$\begin{aligned}
\mu_*(G_n \cap A) &= \mu_*(E_n \cap (G_n \cap A)) + \mu_*(E_n^c \cap (G_n \cap A)) \\
&= \mu_*(E_n \cap A) + \mu_*(G_{n-1} \cap A) \\
&= \sum_{j=1}^{\infty} \mu_*(E_j \cap A)
\end{aligned}$$

The last inequality is gotten by induction. We know that $G_n \in \mathfrak{M}$ and $G^c \subset G_n^c$ we see that

$$\mu_*(A) = \mu_*(G_n \cap A) + \mu_*(G_n^c \cap A) \geq \sum_{j=1}^{\infty} \mu_*(E_j \cap A) + \mu_*(G^c \cap A)$$

We then let $n \to \infty$ to see that

$$\mu_*(A) \geq \sum_{j=1}^{\infty} \mu_*(E_j \cap A) + \mu_*(G^c \cap A) \geq \mu_*(G \cap A) + \mu_*(G^c \cap A) \geq \mu_*(A)$$

Therefore all of the inequalities above must in fact be equalities and so we have that $G \in \mathfrak{M}$. Furthermore, if we take $A = G$ in the above we get that μ_* is countably additive on $\mathfrak{M}$ and so we are done. □

Theorem (Carathéodory Extension Theorem)**.** *Suppose that $\mathcal{A}$ is an algebra of sets in X, μ_0 is a premeasure on $\mathcal{A}$, $\mathfrak{M}$ is the σ-algebra generated by $\mathcal{A}$. Then there exists a measure μ on $\mathfrak{M}$ that extends μ_0. If μ is σ-finite then this extension is unique.*

Proof. The exterior μ_* induced by μ_0 defines a measure on the σ- algebra of Carathéodory measurable sets defined by

$$\mu_*(E) = \inf\left\{\sum_{j=1}^{\infty} \mu_0(E_j) : E \subset \bigcup_{j=1}^{\infty} E_j, \text{ where } E_j \in \mathcal{A} \text{ for all } j\right\}$$

(The proof is clear. The E_j play the role of cubes). Hence, μ is also a measure on $\mathfrak{M}$ that extends μ_0.

To show uniqueness whenever μ is σ-finite on the σ- algebra of Carathéodory measurable sets we suppose that there is another measure ν that agrees with μ_0 on $\mathcal{A}$. Suppose that $F \in \mathfrak{M}$ has finite measure. Then we see

$$\nu(F) \leq \sum_{j=1}^{\infty} \nu(E_j) = \sum_{j=1}^{\infty} \mu_0(E_j)$$

So we get for free that $\nu(F) \le \mu(F)$. Now we need the reverse inequality. Observe that if $E = \bigcup_j F_j$ then then fact that ν and μ agree on $\mathcal{A}$ gives that

$$\nu(E) = \lim_{n\to\infty} \nu\left(\bigcup_{j=1}^{n} E_j\right) = \lim_{n\to\infty} \mu\left(\bigcup_{j=1}^{n} E_j\right) = \mu(E)$$

If we pick the E_j such that $\mu(e) \le \mu(F) + \epsilon$, then the fact that $\mu(F) < \infty$ implies that $\mu(E - F) \le \epsilon$. So

$$\mu(F) \le \mu(E) = \nu(E) = \nu(F) + \nu(e - F) \le \nu(F) + \mu(E - F) \le \mu(F) + \epsilon$$

Since ϵ is arbitrary we have the reverse inequality is proved.

Now we can use the σ-finiteness. We write $X = \bigcup_j E_j$ where $E_1, E_2, \ldots$ is a countable collection of disjoint sets in $\mathcal{A}$ with finite measure. Then for any $F \in \mathfrak{M}$ we have that

$$\mu(F) = \sum_{j=1}^{\infty} \mu(F \cap E_j) = \sum_{j=1}^{\infty} \nu(F \cap E_j) = \nu(F)$$

□

1.4 Integration

Theorem (Monotone Convergence Theorem)**.** *Let $\{f_n\}$ be a sequence of measurable functions on X and suppose that*

1. *$f_n(x) \ge 0$ for every x in X.*
2. *$f_n(x) \to f(x)$ for every $x \in X$.*

Then f is measurable and

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$$

Proof. Since we know that $\int f_n d\mu \le \int f_{n+1} d\mu$ we have that there exists an $\alpha \in [0, \infty]$ such that

$$\lim_{n\to\infty} \int f_n d\mu \to \alpha$$

Since $f_n \le f$ for every n we see $\int f_n d\mu \le \int f d\mu$ for every n. So the above implies that $\alpha \le \int f d\mu$. Let s be sny simple measurable sunction such that $0 \le s \le f$ and let c be a constant such that $0 < c < 1$ and define

$$E_n = \{x : f_n(x) \ge cs(x)\}$$

Each E_n is measurable, $E_1 \subset E_2 \subset \cdots$, and $X = \bigcup_n E_n$. To see the equality consider a particular $x \in X$. If $f(x) = 0$ then $x \in E_1$ and if $f(x) > 0$ then $cs(x) < f(x)$ because $c < 1$. Hence, we have that $x \in E_n$ for some n. Also,

$$\int_X f_n d\mu \ge \int_{E_n} f_n d\mu \ge c \int_{E_n} s d\mu$$

for every n. We then let $n \to \infty$ and so $\alpha \ge c \int_X s d\mu$. Since this holds for every $c < 1$ we have that

$$\alpha \ge \int s d\mu$$

For every simple function $s \leq f$. Hence, we have that

$$\alpha \geq \int_X f d\mu$$

This proves the reverse inequality and we are done. ∎

Theorem (Fatou's Lemma). *If $\{f_n\}$ is a sequence of non-negative measurable functions on X then*

$$\int \liminf_{n\to\infty} f_n d\mu \leq \liminf_{n\to\infty} \int f_n d\mu$$

Proof. Let $g_k(x) = \inf_{i\geq k} f_i(x)$. Then we have that $g_k \leq f_k$, so we see that

$$\int g_k d\mu \leq \int f_k d\mu$$

Moreover we see that $0 \leq g_1 \leq g_2 \leq \cdots$, and each g_k is measurable with $g_k(x) \to \liminf f_n(x)$ as $k \to \infty$. The Monotone Convergence Theorem implies that the left side tends to the right side as $k \to \infty$ we are done. ∎

Theorem (Dominated Convergence Theorem). *Suppose that $\{f_n\}$ is a sequence of complex measurable functions on X such that*

$$\lim_{n\to\infty} f_n(x) = f(x)$$

exists for every $x \in X$. If there is a function $g \in L^1(\mu)$ such that $|f_n(x)| \leq g(x)$ for each $x \in X$ and all $n > 0$ then $f \in L^1(\mu)$ and

$$\lim_{n\to\infty} \int_X |f_n - f| d\mu = 0$$

as well as

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X (\lim_{n\to\infty} f_n) d\mu$$

Proof. It is clear that $f \in L^1(\mu)$ because $|f| \leq g$ and f is measurable. We also have that $|f_n - f| \leq 2g$ and so we can apply Fatou's lemma to then function $2g - |f_n - f|$ to see that

$$\begin{aligned}\int_X 2g d\mu &\leq \liminf_{n\to\infty} \int_X (2g - |f_n - f|) d\mu \\ &= \int_X 2g d\mu + \liminf_{n\to\infty} \left(- \int_X |f_n - f| d\mu \right) \\ &= \int_X 2g d\mu - \limsup_{n\to\infty} \int_X |f_n - f| d\mu\end{aligned}$$

Becuase $\int_X 2g d\mu$ is finite we can subtract it to see that

$$\limsup_{n\to\infty} \int_X |f_n - f| d\mu \leq 0$$

This immediately gives that

$$\lim_{n\to\infty} \int_X |f_n - f| d\mu = 0$$

We then recall that if $f \in L^1$ then $|\int_x f| \leq \int_X |f|$ so we get

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$$

∎

Theorem (Fubini's Theorem)**.** *Suppose that μ_1, μ_2 are σ-finite complete measures on X_1 and X_2, respectively and $f(x_1, x_2)$ is integrable on $(X_1 \times X_2, \mu_1 \times \mu_2)$ and is such that for every measurable set E we have that the slice E^{x_2} is μ_1-measurable and $\mu_1(E^{x_2})$ is defined for almost every $x_2 \in X_2$. Then*

1. *For almost every $x_2 \in X_2$, the slice $f^{x_2}(x_1) = f(x_1, x_2)$ is integrable on (X_1, x_1).*
2. *$\int_{X_1} f(x_1, x_2) d\mu_1$ is an integrable function on X_2.*
3. $$\int_{X_2} \left(\int_{X_1} f(x_1, x_2) d\mu_1 \right) d\mu_2 = \int_{X_1 \times X_2} f(x_1, x_2) d\mu_1 \times \mu_2$$

Proof. This one is a marathon. We have to start by building up the product measure . We begin with measurable rectangles $A \times B$ in $X_1 \times X_2$ as cartesian products of measurable sets and define the function

$$\mu_0(A \times B) = \mu_1(A)\mu_1(B)$$

It is clear that this is a premeasure on the algebra $\mathcal{A}$ of sets generated by the measurable rectangles (use the Monotone Convergence Theorem). We now show the following

Lemma. *If E belongs to $\mathcal{A}_{\sigma\delta}$, then E^{x_2} is μ_1- measurable for every x_2. Moreover, $\mu_1(E^{x_2})$ is a μ_2- measurable function and*

$$\int_{X_2} \mu_1(E^{x_2}) d\mu_2 = (\mu_1 \times \mu_2)(E)$$

Proof. First note that this is trivial when E is a measurable rectangle. Next, suppose that E is a set in $\mathcal{A}_\sigma$. Then we can decompose it into a countable collection of disjoint rectangles E_j. Then for each x_2 we have that $E^{x_2} = \bigcup_j E_j^{x_2}$, and then we observe that each of the $E_j^{x_2}$ are disjoint as well. Thus, we have that

$$\int_{X_2} \mu_1(E_j^{x_2}) d\mu_2 = (\mu_1 \times \mu_2)(E_j)$$

We then apply the Monotone Convergence Theorem to get the result for each $E \in \mathcal{A}_\sigma$.

Next, we take $E \in \mathcal{A}_{\sigma\delta}$ such that $(\mu_1 \times \mu_2)(E) < \infty$. Then there exists a sequence of sets E_j in $\mathcal{A}_\sigma$ such that $E_k \subset E_{k+1}$ $E_k \in \mathcal{A}_\sigma$ and $E = \bigcap_k E_k$. We then set $f_j(x_2) = \mu_1(E_j^{x_2}$ and $f(x_2) = \mu_1(E^{x_2})$. To see that E^{x_2} is measurable and f is well-defined observe that E^{x_2} is the decreasing limit of the sets $E_j^{x_2}$, which are all measurable by the preceding argument. Also, because $E_1 \in \mathcal{A}_\sigma$ and $(\mu_1 \times \mu_2)(E) < \infty$ we see that $f_j(x_2) \to f(x_2)$ as $j \to \infty$ for each x_2. Thus, we see that $f(x_2)$ is measurable. However, we also see that $\{f_j(x_2)\}$ is a decreasing sequence of non-negative functions and hence we have that

$$\int_{X_2} f(x_2) d\mu_2(x) = \lim_{j \to \infty} \int_{X_2} f_j(x_2) d\mu_2(x)$$

And therefore we are done in the case that $(\mu_1 \times \mu_2)(E) < \infty$.

When $(\mu_1 \times \mu_2)(E) = \infty$ we need to use the σ-finiteness of μ_1 and μ_2 to get the result to hold. Because both are σ-finite we can find sequence $\{F_j\}_1^\infty$ and $\{G_k\}_1^\infty$ such that $F_j \subset F_{j+1}$ and $G_k \subset G_{k+1}$ such that $\mu_1(F_j), \mu_2(G_k) < \infty$ for all j, k and $X_1 = \bigcup_j F_j$ and $X_2 = \bigcup_k G_k$. Then we simply replace E by $E \cap (F_j \times G_j)$ and let $j \to \infty$ to get the lemma. $\square$

Now we extend the previous lemma to arbitrary measurable sets in $X_1 \times X_2$. If we let $\mathfrak{M}$ be the σ-algebra generated by the measurable rectangles then we see the following

Lemma. *If E is an arbitrary measurable set in X then the conclusion of the previous lemma is still valid with the relaxed hypothesis that E^{x_2} is μ_1-measurable and $\mu_1(E^{x_2})$ is defined for almost every $x_2 \in X_2$.*

Proof. Consider first the case of $m(E) = 0$. Then by the previous lemma we have that there is some $F \in \mathcal{A}_{\sigma\delta}$ such that $E \subset F$ and $(\mu_1 \times \mu_2)(F) = 0$. Since $E^{x_2} \subset F^{x_2}$ for every x_2 and F^{x_2} has μ_1-measure zero for almost every x_2 (apply integral in previous lemma), the assumed completeness of the measure μ_2 immediately gives that E^{x_2} is measurable and has measure zero for those x_2. Thus, we have the right conclusion when E has measure zero.

For general E we find an $F \in A_{\sigma\delta}$ such that $F \supset E$ and $F - E = Z$ has measure zero (Give a proof that this is possible. Not hard). Since $F^{x_2} - E^{x_2} = Z^{x_2}$ we apply the case we just proved and observe that for almost all x_2 the set E^{x_2} is measurable and $\mu_1(E^{x_2}) = \mu_1(F^{x_2}) - \mu_1(Z^{x_2})$. This proves the lemma. □

Now we have the machinery to get the theorem fairly easily. We note that if the desired conclusions hold for finitely many functions, then they must also hold for their linear combinations. So we assume that f is non-negative (otherwise decompose into positive and negative parts). When $f = \chi_E$, with E a set of finite measure, what we need to proved is contained in the previous lemma. So the desired result holds for simple functions. Therefore, by the Monotone Convergence theorem we have that it holds for all non-negative functions. □

Theorem (Polar Coordinates Formula)**.** *Suppose that f is an integrable function on $\mathbb{R}^d$. then for almost every γ in S^{d-1} the slice defined by $f^\gamma(r) = f(r\gamma)$ is an integrable function with respect to the measure $r^{d-1}dr$. Moreover, $\int_0^\infty f^\gamma(r) r^{d-1} dr$ is integrable on S^{d-1} and the following holds*

$$\int_{\mathbb{R}^d} f(x) = \int_{S^{d-1}} \left(\int_0^\infty f(r\gamma) r^{d-1} dr \right) d\sigma(\gamma)$$

Proof. We consider the product measure $\mu = \mu_1 \times \mu_2$ on $X_1 \times X_2$ given by Fubini's theorem *3*. Since the space

$$X_1 \times X_2 = \{(r, \gamma) : 0 < r < \infty \text{ and } \gamma \in S^{d-1}\}$$

can be identified with $\mathbb{R}^d - \{0\}$, we can think of μ as a measure on the latter space and our main goal is to identify it with the (restriction of) the Lebesgue measure on that space. We first have the following

Claim. *If $E = E_1 \times E_2$ is a measurable set then we have that $m(E) = \mu(E)$ and $\mu(E) = \mu_1(E_1)\mu_1(E_2)$.*

Proof. This is clear when E is a measurable rectangle $E = E_1 \times E_2$ and in this case we see that $\mu(E) = \mu_1(E_1)\mu_1(E_2)$. We then extend it to the case that E_2 is an arbitrary measurable subset of S^{d-1} and $E_1 = (0, 1)$ by noting that E is the sector of $\tilde{E_2}$, while $\mu_1(E_1) = 1/d$.

Because of the relative dilation invariance of the Lebesgue measure we have the claim for sets $E = (0, b) \times E_2$ and $b > 0$. We then use a limiting argument to get half-open intervals of the form $E_1 = (0, a]$. We can then easily get the open intervals $E_1 = (a, b)$ by set subtraction and thus we get all open sets. So at this point we have $m(E_1 \times E_2)$ for all open sets E_1, and hence for all closed sets as well and consequently for all Lebesgue measurable sets. So we have established the claim for all measurable rectangles and as a result all finite unions of measurable rectangles. □

This is an algebra of sets and so we apply the Carathéodory Extension theorem to extend this to the σ-algebra genrated by the measurable rectangles. Hence, we have that whenever $E \in \mathfrak{M}$, the result of the theorem holds for χ_E.

To make the final extension, we note that any open set in $\mathbb{R}^d - \{0\}$ can be written as a countable union of rectangles $\bigcup_j A_j \times B_j$, where A_j and B_j are open in $(0, \infty)$ and S^{d-1} respectively (The proof of this fact is clear. Exercise). It follows that any open set is in $\mathfrak{M}$, and therefore so is any Borel set. So we have that the theorem is true for χ_E whenever E is a Borel set in $\mathbb{R}^d - \{0\}$. Then we clearly get the result for all characteristic functions of Lebesgue measurable sets because they can be decomposed into a Borel set and a set of measure zero. We then complete the proof by the familiar construction of going from characteristic functions, to simple functions, and then using the Monotone convergence theorem to get the general result. $\square$

Theorem (Lebesgue-Stieltjes Integral)**.** *Let F be an increasing function on $\mathbb{R}$ that is normalized. Then there exists a unique measure μ (also known as dF) on the Borel sets $\mathcal{B}$ on $\mathbb{R}$ such that $\mu((a, b]) = F(b) - F(a)$ if $a < b$. Conversely, if μ is a meaure on $\mathcal{B}$ that is finite on bounded intervals then F defined by $F(x) = \mu((-\infty, x), x > 0, F(0) = 0$ and $F(x) = -\mu(-x, 0]), x < 0$ is increasing and normalized.*

Proof. We begin by defining a function μ_* on the subsets of $\mathbb{R}$ via

$$\mu_*(E) = \inf \sum_{j=1}^{\infty} (F(b_j) - F(a_j))$$

where the infimum is taken over all coverings of E of the form $\bigcup_{j=1}^{\infty} (a_j, b_j]$.

It is straightforward to verify that μ_* is an outer measure on $\mathbb{R}$. We then observe that $\mu_*((a, b]) = F(b) - F(a)$ is $a < b$. Clearly, we see that $\mu_*((a, b]) \leq F(b) - F(a)$ because $(a, b]$ covers itself. Now suppose that $\bigcup_{j=1}^{\infty} (a_j, b_j]$ covers $(a, b]$. Then it must also cover $[a', b]$ for any $a < a' < b$. Because F is continuous we see that for any $\epsilon > 0$ we ccan always choose $b'_j > b_j$ such that $F(b'_j) < F(b_j) + \epsilon/2^j$. The union of such intervals covers $[a', b]$. By the compactness of this interval we can see that only finitely many of these are needed. So $\bigcup_{j=1}^{N} (a_j, b'_j)$ covers $[a', b]$ for some N. Because F is increasing we have that

$$F(b) - F(a') \leq \sum_{j=1}^{N} F(b'_j) - F(a_j) \leq \sum_{j=1}^{N} F(b_j) - F(a_j) + \epsilon/2^j) \leq \mu_*((a, b]) + \epsilon$$

We then let $a' \to a$ and use the fact that F is continuous to get that $F(b) - F(a) \leq \mu_*((a, b]) + \epsilon$. Because ϵ was arbitrary we have

$$F(b) - F(a) = \mu_*((a, b])$$

$\square$

2 Differentiation Theory

3 Hilbert Spaces

4 Banach Spaces

5 Applications of Banach Space Theory