Problem 4.7.11

(a) This is clear if we recall that for a closed subspace $S \subset \mathcal{H}$ we have that $\mathcal{H} = S \oplus S^{\perp}$. As a result if we take $f \in \mathcal{H}$ we can find $g \in S$ and $h \in S^{\perp}$. Then we compute

$$P^2 f = P^2(g+h) = P(P(g+h)) = P(Pg + Ph) = Pg = P(g+h) = Pf$$

which gives $P^2 = P$. Similarly, if we choose another vector $f' \in g' + h'$ with $g' \in S$ and $h' \in S^{\perp}$ then we see that

$$(Pf, f') = (Pg + Ph, g' + h') = (g, g' + h') = (g, g') = (g + h, g') = (f, Pg' + Ph') = (f, Pf)$$

So we have $P = P^*$.

(b) To show the converse we choose the natural choice of the subspace, namely $P(\mathcal{H}) = \mathrm{im}(P)$. We know that P is a linear operator and so $P(\mathcal{H})$ is a linear subspace of $\mathcal{H}$. We begin by showing that $P(\mathcal{H})$ is closed. Indeed, if $\{f_n\}$ is a sequence in $P(\mathcal{H})$ converging to x then we have that because P is bounded, and therefore continuous, that $Pf_n \to Pf$. But because each of the $f_n \in P(\mathcal{H})$ that $f_n \to Pf$, meaning that $f = Pf$ so that $f \in P(\mathcal{H})$. We are left to verify that $Pg = 0$ if $g \in P(\mathcal{H})^{\perp}$. Observe that if $g \in P(\mathcal{H})^{\perp}$ then

$$(f, Pg) = (Pf, g) = (f, g) = 0$$

So this means that $Pg \in P(\mathcal{H})^{\perp}$. However, because $Pg \in P(\mathcal{H})$ we must have that $g = 0$. By Proposition 4.2 this means that $\mathcal{H}$ admits the decomposition $\mathcal{H} = P(\mathcal{H}) \oplus P(\mathcal{H})^{\perp}$. So for any $h \in \mathcal{H}$ we have $f \in P(\mathcal{H})$ and $g \in P(\mathcal{H})^{\perp}$ such that $h = f + g$ and so $Ph = Pf + Pg = Pf$. So P is the projection operator onto $P(\mathcal{H})$.

(c) Suppose that $\mathcal{H}$ is a separable Hilbert space and $\mathcal{S}$ a closed subspace. Let the map $P_{\mathcal{S}}$ be the orthogonal projection onto $\mathcal{S}$. Because $\mathcal{H}$ is separable we can find a countable dense subset $\{x_n\}$ in $\mathcal{H}$. We will see that $\{P_{\mathcal{S}} x_n\}$ is dense in $\mathcal{S}$. Because $\mathcal{S} \subset \mathcal{H}$ we can find a subsequence $x_{n_k} \to x$ for any $x \in \mathcal{S}$. Then we see that

$$P_{\mathcal{S}}(x_{n_k} - x) = P_{\mathcal{S}} x_{n_k} - P_{\mathcal{S}} x = P_{\mathcal{S}} x_{n_k} - x$$

We then observe that $\|P_{\mathcal{S}} y\| \leq \|y\|$ for any $y \in \mathcal{H}$ because $P_{\mathcal{S}}$ is an orthogonal projection. So we have that for any $\epsilon > 0$

$$\|P_{\mathcal{S}} x_{n_k} - x\| \leq \|x_{n_k} - x\| < \epsilon$$

So $P_{\mathcal{S}} x_{n_k} \to x$ and therefore $\{P_{\mathcal{S}} x_n\}$ is dense in $\mathcal{S}$.

Problem 4.7.13

In the forward direction suppose that $P_1 P_2$ is an orthogonal projection. By the previous problem, this means that $(P_1 P_2)^2 = P_1 P_2$ and $(P_1 P_2)^* = P_1 P_2$. Because P_1 and P_2 are orthogonal they also have $P_1^* = P_1$ and $P_2^* = P_2$ so

$$P_2 P_1 = P_2^* P_1^* = (P_1 P_2)^* = P_1 P_2$$

So P_1 and P_2 commute. Conversely, suppose that P_1 and P_2 commute. Then we have that

$$(P_1 P_2)^2 = P_1^2 P_2^2 = P_1 P_2$$

because $P_1^2 = P_1$ and $P_2^2 = P_2$. Similarly,

$$(P_1 P_2)^* = P_2^* P_1^* = P_2 P_1 = P_1 P_2$$

So $P_1 P_2$ is an orthogonal projection.

Now we need to show that $\mathrm{im}(P_1 P_2) = S_1 \cap S_2$. Observe that for any $f \in \mathcal{H}$ we have that

$$P_1 P_2 f = P_1(P_2 f) \in S_1$$

Analogously we see that

$$P_1P_2f = P_2P_1f = P_2(P_1f) \in S_2$$

So $\mathrm{im}(P_1P_2) \subseteq S_1 \cap S_2$. For the reverse inclusion choose any vector $f \in S_1 \cap S_2$. Then we have that

$$P_1P_2f = P_1(P_2f) = P_1f = f$$

So that $f \in \mathrm{im}(P_1P_2)$.

Problem 4.7.19
We first recall Lemma 5.1 to see that

$$\|T\| = \sup\{|(Tf, g)| : \|f\| \leq 1, \|g\| \leq 1\}$$

We begin by showing that $\|T^*T\| = \|T\|^2$. Note that this will also show that $\|TT^*\| = \|T^*\|^2$ because the equality is symmetric in T^* (just replace T with T^* in the equality). We compute

$$\|T^*T\| = \sup_{\|f\|,\|g\|\leq 1} |(T^*Tf, g)| = \sup_{\|f\|,\|g\|\leq 1} |(Tf, Tg)|$$

We then apply the Cauchy-Schwartz inequality to get

$$\sup_{\|f\|,\|g\|\leq 1} |(Tf, Tg)| \leq \sup_{\|f\|,\|g\|\leq 1} \|Tf\|\|Tg\| = \sup_{\|f\|\leq 1} \|Tf\| \sup_{\|g\|\leq 1} \|Tg\| = \|T\|^2$$

So $\|T^*T\| \leq \|T\|^2$. We now proceed to get the reverse inequality. Pick a sequence $\{f_n\}$ such that $\|Tf_n\| \to \|T\|$. Then we see that $(Tf_n, Tf_n) \to \|T\|^2$ so that

$$\|T^*T\| = \sup_{\|f\|,\|g\|\leq 1} (Tf, Tg) \geq \sup(Tf, Tf) \geq \|T\|^2$$

So $\|T^*T\| \geq \|T\|^2$, and therefore we must have equality. We then apply Proposition 5.4 to get $\|T\|^2 = \|T*\|^2$ which gives

$$\|T\|^2 = \|T^*T\| = \|TT^*\| = \|T^*\|^2$$

Problem 4.7.22
(a) We apply the polarization identity to compute

$$\begin{aligned}
(Tf, Tg) &= \frac{1}{4}(\|Tf + Tg\|^2 - \|Tf - Tg\|^2 + i\|Tf + iTg\|^2 - i\|Tf - iTg\|^2) \\
&= \frac{1}{4}(\|T(f + g)\|^2 - \|T(f - g)\|^2 + i\|T(f + ig)\|^2 - i\|T(f - ig)\|^2) \\
&= \frac{1}{4}(\|(f + g)\|^2 - \|(f - g)\|^2 + i\|(f + ig)\|^2 - i\|(f - ig)\|^2) \\
&= (f, g)
\end{aligned}$$

Consequently we have that

$$(f, g) = (Tf, Tg) = (f, T^*Tg)$$

So $T^*T = I$.

(b) We are given that T is an isometry and surjective. To verify that T is unitary we are only left to establish that T is injective. To see this note that $\ker(T) = \{0\}$ because if we have that $Tf = 0$ then $\|f\| = \|0\| = 0$ because T is an isometry, so $f = 0$. Thus, T is unitary. By the previous part we have that $T^*T = I$. Because T is linear and bijective its left inverse must also be its right inverse. This gives that $TT^* = I$.

(c) It is clear from the previous part that we must look for a map that is an isometry by is not surjective. Consider the space $\ell^2(\mathbb{N})$ together with the map $R : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ given by

$$R(x_0, x_1, x_2, \ldots) = (0, x_0, x_1, x_2, \ldots)$$

This map is clearly not surjective because the element $(1, x_0, x_1, \ldots)$ has no pre-image. However, we can see that this is an isometry because

$$\|(0, x_0, x_1, \ldots)\|^2 = \sum_{n=1}^{\infty} |x_{n-1}|^2 = \sum_{n=0}^{\infty} |x_n|^2 = \|(x_0, x_1, \ldots)\|^2$$

So $\|R\vec{x}\| = \|\vec{x}\|$

(d) We are given that T^*T is a unitary map. So we begin by estimating

$$\|Tf\|^2 = (Tf, Tf) = (f, T^*Tf) \leq \|f\|\|T^*Tf\| = \|f\|^2$$

To show the reverse inequality we have observe that

$$\|f\|^2 = \|T * Tf\|^2 = (T^*Tf, T^*Tf) = (Tf, TT^*Tf) \leq \|Tf\|\|T(T^*Tf)\|$$

Applying the fact that $\|Tf\| \leq \|f\|$ we get that $\|T(T^*Tf)\| \leq \|T^*Tf\|$ so that

$$\|Tf\|\|T(T^*Tf)\| \leq \|Tf\|\|T^*Tf\| = \|Tf\|\|f\|$$

Dividing by $\|f\|$ gives that $\|f\| \leq \|Tf\|$. So we have equality and T must be an isometry.

Problem 4.7.25
Following the hint, we begin by showing the reverse direction. Suppose that $\lambda_k \to 0$. We let P_n be the projection onto the subspace spanned by $\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$. Then we have that P_nT is also a diagonal operator and that

$$\|P_nT - T\| = \sup_{\varphi_k} P_nT\varphi_k - T\varphi_k = \sup_{k>n} |\lambda_k|$$

So $\|P_nT - T\| \to 0$ as $n \to \infty$. Because each of the P_nT is a finite rank operator we can apply Proposition 6.1 to see that T is compact. To prove the reverse direction we will instead show the converse. Suppose that $\lambda_n \not\to \infty$. Then we have that $\limsup |\lambda_k| > 0$ and so we can find a subsequence $\{\lambda_{k_j}\}$ such that $|\lambda_{k_j}| > \delta/2$ for sufficiently large j. So we see that because $T\varphi_{k_j} = \lambda_{k_j}\varphi_{k_j}$ and so

$$\|T\varphi_{k_j} - T\varphi_{k_\ell}\| = \sqrt{\lambda_{k_j}^2 + \lambda_{k_\ell}^2} > 2^{-1/2}\delta$$

Thus, the $\{T\varphi_{k_j}\}$ are uniformly separated from each other, and it can have no convergent subsequence. Because all of these points lie in $T(B)$, where B is the unit ball, we must have that $T(B)$ is not compact. Hence, T is not a compact operator. This completes the proof.

Problem 4.27.26
Take any $f \in L^2$ and compute

$$\|Tf\|^2 = \int \left| \int K(x,y)f(y)dy \right|^2 dx \leq \int \left(\int |K(x,y)||f(y)|dy \right)^2 dx$$

We then estimate the inner term

$$\begin{aligned}
\left(\int |K(x,y)||f(y)|dy \right)^2 &= \left(\int |K(x,y)||f(y)|w(y)^{1/2}w(y)^{-1/2}dy \right)^2 \\
&= \left[\int \left(\sqrt{|K(x,y)|}w(y)^{1/2} \right) \left(\sqrt{|K(x,y)|}|f(y)|w(y)^{-1/2} \right) dy \right]^2 \\
&\leq \left[\left(\int \sqrt{|K(x,y)|}w(y)^{1/2}dy \right) \left(\int \sqrt{|K(x,y)|}|f(y)|w(y)^{-1/2}dy \right) \right]^2 \\
&\leq \left(\int |K(x,y)|w(y)dy \right) \left(\int |K(x,y)||f(y)|^2w(y)^{-1}dy \right)
\end{aligned}$$

We then recall that

$$\int |K(x,y)|w(y)dy \leq Aw(x) \text{ a.e.}$$

So we have that

$$\left(\int |K(x,y)||f(y)|dy\right)^2 \leq Aw(x)\int |K(x,y)||f(y)|^2 w(y)^{-1}dy$$

So we have

$$\|Tf\|^2 \leq \int \left(\int |K(x,y)||f(y)|dy\right)^2 dx \leq \int Aw(x)\left(\int |K(x,y)||f(y)|^2 w(y)^{-1}dy\right)dx$$

Then we apply Tonelli's theorem to get

$$\int Aw(x)\left(\int |K(x,y)||f(y)|^2 w(y)^{-1}dy\right)dx = A\int |f(y)|^2 w(y)^{-1}\left(\int |K(x,y)|w(x)dx\right)dy$$

Applying the estimate

$$\int |K(x,y)|w(x)dx \leq Aw(y) \text{ a.e.}$$

gives

$$\begin{aligned}
\|Tf\|^2 &\leq A\int |f(y)|^2 w(y)^{-1}\left(\int |K(x,y)|w(x)dx\right)dy \\
&\leq A\int A|f(y)|^2 w(y)^{-1}w(y)dy \\
&= A^2\int |f(y)|^2 dy \\
&= A^2\|f\|^2
\end{aligned}$$

So

$$\|T\| = \inf\{M : \|Tf\| \leq M\|f\|\} \leq A$$

and we are done.