Problem 2.5.6
(a) Consider the function f such that for each $n \geq 1$

$$f(x) = \begin{cases} n & x \in [n, n + 1/n^3) \\ 0 & \text{otherwise} \end{cases}$$

Pictorially, f consists of rectangles with width $1/n^3$ and height n on each interval $[n, n+1)$. We can see that f is integrable because if we consider the interval $I_n = [n, n+1]$, then

$$\int_{I_n} f = n/n^3 = 1/n^2$$

Then note that

$$\int_{\mathbb{R}} f = \sum_{n=1}^{\infty} \inf_{I_n} f = \sum_{n=1}^{\infty} 1/n^2 = \frac{\pi^2}{6}$$

However, we can see that

$$\limsup_{x \to \infty} f(x) = \lim_{x \to \infty} (\sup_{y \geq x} f(y)) = \infty$$

because the "rectangles" get arbitrarily high and so for every x we can find $y > x$ such that $f(y) > M$ for any M.

(b) We will prove the contrapositive. Namely, if f is uniformly continuous and $f \not\to 0$ as $|x| \to \infty$ then f is not Lebesgue integrable. Let $\epsilon > 0$ be given and find $\delta_0 > 0$ such that $|x - y| < \delta_0$ implies $|f(x) - f(y)| < \epsilon/2$, then set $\delta = \min\{\delta_0, 1/2\}$. Because $f \not\to 0$ we can find some x_0 such that $f(x_0) > \epsilon$. But then in some δ-neighborhood of x_0, $|f| > \epsilon/2$. We iterate this process, because we know we can always find an $x_{n+1} > x_n + 1$ such that $f(x_{n+1}) > \epsilon$ and so in some δ-neighborhood of x_{n+1} we have that $|f| > \epsilon/2$. Furthermore, each of these neighborhoods, $\{N_n\}$ are disjoint because $|x_{n+1} - x_n| > 1$ and $\delta > 1/2$. This means that

$$\int_{\mathbb{R}} |f| \geq \int_{\cup N_n} |f| \geq \sum_{n=1}^{\infty} \frac{\epsilon}{2}(2\delta) = \sum_{n=1}^{\infty} \epsilon\delta$$

The sum on the right diverges and so f is not integrable. This verifies the contrapositive, and completes the proof.

Problem 2.5.8
Suppose that f is integrable and let $\epsilon > 0$ be given. By Proposition 1.12 (ii) there is some $\delta > 0$ such that

$$\int_E |f| < \epsilon \text{ whenever } m(E) < \delta$$

Now choose $h < \delta$ so that $|(x+h) - x| = h < \delta$ and let $E = [x, x+h]$ so that $m(E) = m([x, x+h]) = h < \delta$

$$\left| \int_{-\infty}^{x+h} f - \int_{-\infty}^{x} f \right| = \left| \int_E f \right| < \epsilon$$

Hence, $F(x) = \int_{-\infty}^{x} f(t)dt$ is uniformly continuous.

Problem 2.5.9
To see this inequality observe that if fix $\alpha > 0$ and define

$$E_\alpha = \{x \mid f(x) > \alpha\}$$

Then we have that

$$\int_{E_\alpha} f \geq \int_{\mathbb{R}^n} f\chi(E_\alpha) \geq \alpha m(E_\alpha)$$

Re-ordering the terms gives

$$m(E_\alpha) \leq \frac{1}{\alpha} \int f$$

Problem 2.5.10
Let f be a non-negative function and define

$$E_{2^k} = \{x \mid f(x) > 2^k\} \text{ and } F_k = \{x \mid 2^k < f \leq 2^{k+1}\}$$

If if is finite a.e. then

$$\bigcup_{-\infty}^{\infty} F_k = \{f > 0\}$$

Moreover, each of the F_k are disjoint. Then

Proposition. *The following are equivalent*

1. *f is integrable*
2. $\sum_{k=-\infty}^{\infty} 2^k m(F_k) < \infty$
3. $\sum_{k=-\infty}^{\infty} 2^k m(E_{2^k}) < \infty$

Proof. To see that 1. $\Rightarrow$ 2. observe that if f is integrable then $\int f < \infty$. Next we note that because $f \geq 0$ we must have that

$$\int_{\mathbb{R}} f = \int_{\cup_k F_k} f$$

Then because each of the F_k are disjoint we get that

$$\int_{\cup_k F_k} f = \sum_{k=-\infty}^{\infty} \int_{F_k} f = \sum_{k=-\infty}^{\infty} \int f\chi_{F_k} \geq \sum_{k=-\infty}^{\infty} 2^k m(F_k)$$

So because $\int f < \infty$ we have that $\sum_{k=-\infty}^{\infty} 2^k m(F_k) < \infty$, and we are done. Now we will show that 2. $\Rightarrow$ 3. Note that because $\sum_{k=-\infty}^{\infty} 2^k m(F_k) < \infty$ we have that for any $\epsilon > 0$ we can find an N such that

$$\sum_{|k|>N} 2^k m(F_k) < \epsilon$$

Furthermore, we have that $E_k = \bigcup_{|\ell|>|k|} F_k$, and that the union is disjoint. Then we see that

$$\sum_{|k|>N} 2^k m(E_2^k) = \sum_{|k|>N} 2^k \sum_{|\ell|>|k|} m(F_k) = \sum_{|k|>N} \sum_{\ell>k} 2^k m(F_k)$$

We then take $\sum_{|\ell|>|k|} 2^k m(F_k) < \epsilon/2^k$ So that we have that

$$\sum_{|k|>N} 2^k m(E_{2^k}) \leq \sum_{|k|>N} \epsilon/2^k \leq \epsilon$$

We now proceed to 3. $\Rightarrow$ 1. To see this note that

$$f \leq \sum_{k=-\infty}^{\infty} 2^k \chi_{E_{2^k}} \leq 2f$$

This is because $\mathbb{R}^+ \subset \bigcup_k E_k$ and $f(x) < 2^k$ for $x \in E_{2^\ell}$ with $\ell < k$. Then

$$\int f < \int \sum_{k=-\infty}^{\infty} 2^k \chi_{E_2^k} = \sum_{k=-\infty}^{\infty} 2^k m(E_k) < \infty$$

Which means that f is integrable by the dominated convergence theorem. This gives 1. $\Rightarrow$ 2. $\Rightarrow$ 3. $\Rightarrow$ 1. and so we are done. $\square$

Next we consider the function

$$f(x) = \begin{cases} |x|^{-\alpha} & \text{if } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

For f we get that the sets $E_{2^k}(f) = \{f(x) > 2^k\}$ depend on $|k|$ because if $k \leq 0$ we have that $2^{|k|} > |x|^\alpha$ so $|x| \leq 1$. And if $k \geq 1$ then we see that $|x| \leq 2^{k/\alpha}$. We then compute

$$m(E_{2^k}(f)) = \begin{cases} 2^d & k \leq 0 \\ 2^{d-kd/\alpha} & k \geq 1 \end{cases}$$

By the Proposition we have that f is integrable iff the sum $\sum_{k=-\infty}^{\infty} 2^k m(E_{2^k}(f))$ converges. For this to happen we must have that

$$\sum_{k=\infty}^{\infty} 2^k m(E_{2^k}(f)) = \sum_{k=-\infty}^{0} 2^k(2^d) + \sum_{k=1}^{\infty} 2^k 2^(d - kd/\alpha)$$
$$= 2^{d+1} + 2^d \sum_{k=-\infty}^{\infty} 2^{(1-d/\alpha)k}$$

Which will converge given that $1 - d/\alpha$, which is precisely when $a < d$. For the function

$$g(x) = \begin{cases} |x|^{-\beta} & \text{if } |x| > 1 \\ 0 & \text{otherwise} \end{cases}$$

In this case we compute the sets $E_{2^k}(g)$. Note that if $k \geq 1$ then $2^k > 1$ and $|x|^{-\beta} < 1$ so $E_{2^k}(g)$ is empty. If $k \leq 0$ then we have that $g(x) > 2^k$ iff $|x| < 2^{k/\beta}$. In this case we have that measure of $E_{2^k}(g)$ is $2^d 2^{-kd/\beta}$. And so we proceed as with f by applying the proposition to get that g is integrable iff

$$\sum_{k=-\infty}^{\infty} 2^k m(E_{2^k}(g)) = \sum_{k=-\infty}^{0} 2^k 2^d 2^{-kd/\beta} = 2^d \sum_{k=-\infty}^{0} 2^{1-d/\beta}$$

converges. This means that $1 - d/\beta > 0$ or that $\beta > d$, and we are done.

Problem 2.5.11
Let $f : \mathbb{R}^d \to \mathbb{R}$ be integrable and suppose that $\int_E f(x)dx \geq 0$ for every measurable E. We want to show that $f \geq 0$ a.e. Suppose not, then the set $F = \{x \,|\, f(x) < 0\}$ has positive measure. Next, define the collection of sets $\{F_n\} = \{x \,|\, f(x) < -1/n\}$ and observe that

$$F = \bigcup_{n=1}^{\infty} F_n$$

By subadditivity we have that

$$0 < m(F) < \sum_{n=1}^{\infty} m(F_n)$$

Hence, there must be at least one n such that $m(F_n) > 0$ and so

$$\int_{F_n} f(x)dx \leq \int_{F_n} \frac{-1}{n} dx \leq \frac{-1}{n} m(F_n) < 0$$

But F_n is a measurable set by definition, so we have a contradiction. Hence, $f \geq 0$ a.e.

As a result of this fact, if $\int_E f(x)dx = 0$ for every measurable E, then $\int_E f(x) \geq 0$ and $\int_E -f(x)dx \geq 0$ for every measurable E. By above we see that $f \geq 0$ a.e. and $-f \geq 0$ a.e. which means that $f = 0$ a.e.

Problem 2.5.15
We begin by computing the integral of f on $\mathbb{R}$. We first note that f is only non-zero on $(0,1)$ so $\int_{\mathbb{R}} f(x)dx = \int_{(0,1)} f(x)dx$ and note that we can write $(0,1) = \bigcup_{n=1}^{\infty}(\frac{1}{n+1}, \frac{1}{n})$, and the union is disjoint. This gives that

$$\int_{(0,1)} f = \int_{\cup_n(\frac{1}{n+1},\frac{1}{n})} f = \sum_{n=1}^{\infty} \int_{(\frac{1}{n+1},\frac{1}{n})} f$$

We can take the sum on the right as a limit of its partial sums to see that

$$\sum_{n=1}^{\infty} \int_{(\frac{1}{n+1},\frac{1}{n})} f = \lim_{m\to\infty} \sum_{n=1}^{m} \int_{(\frac{1}{n+1},\frac{1}{n})} f = \lim_{m\to\infty} \int_{m+1}^{1} f = \lim_{t\to 0} \int_{t}^{1} f$$

We then integrate in the usual (Riemann) way because the integrals must be equal to get

$$\lim_{t\to 0} \int_{t}^{1} \frac{1}{\sqrt{x}} dx = \lim_{t\to 0} 2\sqrt{x}\Big|_t^1 = \lim_{t\to 0} 2(1-\sqrt{t}) = 2$$

Next we use translation invariance of the integral to see that for each r_n

$$\int_{\mathbb{R}} f(x - r_n)dx = 2$$

Because $f \geq 0$ on $\mathbb{R}$ we have that the partial sums of F are monotone increasing and converge to F. We can then use the Monotone Convergence Theorem to see that

$$\int F dx = \int \sum_{n=1}^{\infty} 2^{-n} f(x-r_n)dx = \sum_{n=1}^{\infty} \int 2^{-n} f(x-r_n)dx = \sum_{n=1}^{\infty} 2^{-n} \int f(x-r_n)dx$$

But we saw that $\int f(x - r_n)dx = 2$ so this becomes

$$\int F dx = \sum_{n=1}^{\infty} 2^{1-n} = 2 < \infty$$

And so F is integrable. Furthermore, because F is integrable it must be bounded a.e. on $\mathbb{R}$. We now need to verify the following

Claim. *The following hold about the function F as defined above,*

1. *F is unbounded on every interval.*
2. *Any function $\tilde{F}$ such that $\tilde{F} = F$ a.e. is unbounded on every interval.*

Proof. First we will prove 1. Let I be any interval and choose $r_k \in I$. Then for any $M > 0$ we can see that $f(x - r_k) > M$ whenever

$$x \in (r_k - 1/M^2, r_k + 1/M^2)$$

because if $f(x - r_k) = \frac{1}{\sqrt{x - r_k}} > M$ then

$$r_k - \frac{1}{m^2} < x < r_k + \frac{1}{M^2}$$

So x is in the interval $I_{r_k} = (r_k - 1/M^2, r_k + 1/M^2)$. Furthermore, we can see that $I \cap I_{r_k} \neq 0$ and in fact has positive measure (it is an interval). So F is unbounded on each interval because we can make each term of the sum, $\sum_{n=1}^{\infty} 2^n f(x - r_n)$ arbitrarily large on I.

Next, take any function $\tilde{F}$ which equals F almost everywhere. We then focus our attention to the values of $\tilde{F}$ on I_{r_k}. Because $\tilde{F} = F$ almost everywhere we have that $\tilde{F} > M$ at almost every $x \in I_{r_k}$, which means that $\tilde{F}$ is unbounded on every interval. □