Problem 3.5.12

It is clear by elementary calculus that the function

$$F(x) = \begin{cases} 0 & x = 0\\ x^2 \sin(1/x^2) & \text{otherwise} \end{cases}$$

is differentiable everywhere with derivative

$$F'(x) = \begin{cases} 0 & x = 0\\ 2x\sin(1/x^2) - \frac{2\cos(1/x^2)}{x} & \text{otherwise} \end{cases}$$

We now need to show that F' is not integrable on [-1,1]. First we note that F' is symmetric about the origin so that

$$\int_{-1}^{1} |F'(x)| dx = 2 \int_{0}^{1} |F'(x)|$$

We will show that the integral on the right diverges. The intuition is that near the origin $\sin(1/x^2)$ and $\cos(1/x)$ oscillate too rapidly and add infinitely much "area" under the curve. More formally, consider the set of points

$$x_j = \left\{ \frac{2}{(2j+1)\pi} \right\}_{j=1}^{\infty}$$

It is clear that $x_j \to 0$ as $j \to \infty$ and so it suffices to show that

$$\int_{x_N}^1 |F'(x)| dx \to \infty \text{ as } N \to \infty$$

We will break the domain into smaller intervals of the form $[x_j, x_{j+1}]$ and observe that

$$\int_{x_N}^1 = \sum_{j=1}^N \int_{x_{j+1}}^{x_j} |F'(x)| dx$$

Then note that

$$\int_{x_{j+1}}^{x_j} |F'(x)| dx = \int_{\sqrt{\frac{1}{2j\pi}}}^{x_j} F'(x) \pm \int_{x_{j+1}}^{\sqrt{\frac{1}{2j\pi}}} F'(x) = \pm 2 \int_{\sqrt{\frac{1}{2j\pi}}}^{x_j} F'(x)$$

Where the sign is chosen to make the value positive. We then evaluate each term in the partial sum

$$\int_{\sqrt{\frac{1}{2j\pi}}}^{x_j} F'(x) = F(x) \Big|_{\sqrt{\frac{1}{2j\pi}}}^{x_j} = \frac{1}{(2j+1)\pi}$$

So that

$$\int_{x_N}^1 |F'(x)| = \frac{4}{\pi} \sum_{j=1}^N \frac{1}{(2j+1)}$$

Which diverges as $N \to \infty$ by comparison with the harmonic series. Hence, F' is not integrable on [-1,1].

Problem 3.5.16

- (a)
- (b)

Problem 3.5.17

We proceed similarly to the proof of Theorem 3.21. We note that

$$|(f * K_{\epsilon})(x)| = \left| \int f(x-y)K_{\epsilon}(y)dy \right| \le \int |f(x-y)||K_{\epsilon}(y)|dy$$

We break the integral on the right up into

$$\int_{|y|<\epsilon} |f(x-y)||K_{\epsilon}(y)|dy + \sum_{k=1}^{\infty} \int_{|y|\in[2^{k-1}\epsilon,2^k\epsilon]} |f(x-y)||K_{\epsilon}(y)|dy$$

For the first term, we note that

$$\int_{|y|<\epsilon} |f(x-y)| |K_{\epsilon}(y)| dy \le A\epsilon^{-d} \int_{|x-y|<\epsilon} |f(y)| dy$$

We then note that $|x-y| < \epsilon$ describes precisely the ball with radius ϵ about the origin. As a result we see that

$$\sup_{\epsilon>0} \int_{|x-y|<\epsilon} |f(y)| \le m(B_{\epsilon}) \sup_{r\to 0} \frac{1}{m(B_r)} \int_{B_r} |f(y)| dy$$

This gives

$$\int_{|y| < \epsilon} |f(x - y)| |K_{\epsilon}(y)| dy \le Ac_d f^*(x)$$

Where c_d is a constant depending on the volume of the unit ball in \mathbb{R}^d . To estimate

$$\sum_{k=1}^{\infty} \int_{|y| \in [2^{k-1}\epsilon, 2^k\epsilon]} |f(x-y)| |K_{\epsilon}(y)| dy$$

We restrict our attention to the partial sums and observe that

$$\int_{|y| \in [2^{k-1}\epsilon, 2^k\epsilon]} |f(x-y)| |K_{\epsilon}(y)| dy \le \frac{2^{(1-k)(d+1)}}{\epsilon^{d+1}} \int_{|y| < 2^k\epsilon} |f(x-y)| dy$$

By the same reasoning as the first case (multiply by volume of appropriate ball and canceling powers), this gives

$$2^d 2^{1-k} \epsilon^{-1} c_d f^*(x)$$

As a result of this

$$\sum_{k=1}^{\infty} \int_{|y| \in [2^{k-1}\epsilon, 2^k\epsilon]} |f(x-y)| |K_{\epsilon}(y)| dy \le \sum_{k=1}^{\infty} 2^d 2^{1-k} \epsilon^{-1} c_d f^*(x)$$

$$\le 2^{d+1} \epsilon^{-1} c_d f^*(x)$$

So in total we have that

$$|(f * K_{\epsilon})(x)| \le (2^{d+2}c_d A)f^*(x)$$

as desired.

Problem 3.5.19

(a) Let $E \subset \mathbb{R}$ have measure 0. Pick some $\epsilon > 0$ and use the absolute continuity of f to see that

$$\sum_{k=1}^{N} |b_k - a_k| < \delta \text{ implies } \sum_{k=1}^{N} |f(b_k) - f(a_k)| < \epsilon$$

We then use the fact that E has measure 0 to find some open set $\mathcal{O} \supset E$ such that $m(\mathcal{O}) < \delta$. Because $\mathcal{O} \subset \mathbb{R}$ we can write it as a disjoint union of open intervals $I_k = (a_k, b_k)$ such that

$$\mathcal{O} = \bigcup_{k=1}^{\infty} I_k$$

and then we note that

$$m(\mathcal{O}) = \sum_{k=1}^{\infty} (b_k - a_k) < \delta$$

Moreover we have that the closure of \mathcal{O} also satisfies $m(\overline{\mathcal{O}}) < \delta$ because the boundary of \mathcal{O} is countable and hence has measure 0. Now we look at the images of these closed intervals under f. Consider the points $\{y_k\}_1^\infty$ and $\{Y_k\}_1^\infty$ which are defined to be the points such that $y_k = \inf_{x \in I_k} f(x)$ and $Y_k = \sup_{x \in I_k} f(x)$. Then set $\{x_k = \{f^{-1}(y_k)\}$ and $\{X_k = \{f^{-1}(Y_k)\}$. We have that $x_k, X_k \in \overline{I_k}$ because f is continuous and $\overline{I_k}$ is compact. So then we have that

$$f(\mathcal{O}) \subset \bigcup_{k} [f(x_k), f(X_k)]$$

Because next note that

$$\sum_{k=1}^{\infty} |X_k - x_k| \le \sum_{k=1}^{\infty} |b_k - a_k| < \delta$$

because $x_k, y_k \in I_k$ and so the absolute continuity of f implies that

$$m(\overline{\mathcal{O}}) = \sum_{k=1}^{\infty} |f(X_k) - f(x_k)| < \epsilon$$

So

$$m(E) < m(\overline{\mathcal{O}}) < \epsilon$$

Letting $\epsilon \to 0$ gives the result.

(b) Let E now be a measurable subset of \mathbb{R} . Then we know that E differs from an F_{σ} by a set of measure 0. So we can write

$$E = \left(\bigcup_{k=1}^{\infty} \mathcal{I}_k\right) \cup Z$$

Where Z is a set of measure 0 and each of the I_k is closed (and bounded and hence, compact). Then the image under f gives

$$f(E) = \bigcup_{k} f(\mathcal{I}_{k}) \cup f(Z)$$

The first element in the union is a countable union of measurable sets, and hence, measurable and f(Z) has measure 0 by the previous part. Hence, f(E) is measurable.

Problem 3.5.20

(a) The main hurdle to this exercise is that if F' = 0 on any interval, then F is constant a.e. on that interval, and hence not strictly increasing. As as result, we are led to try to find a set with positive measure, but which contains no intervals, and then define F as an integral over such a set. We constructed such a set (the "fat" Cantor set) in a previous problem set. Let \mathcal{C} denote such a set (translated and dilated to appropriately sit in [a, b]). Consider the function

$$D_{\mathcal{C}}(t) = d(t, \mathcal{C}) = \inf_{x \in \mathcal{C}} d(t, x)$$

It is clear that $D_{\mathcal{C}}(t)$ is continuous and satisfies $D_{\mathcal{C}}(t) \geq 0$ with equality iff $t \in \mathcal{C}$. As a result, the function

$$F(x) = \int_{-\infty}^{x} D_{\mathcal{C}}(t)dt$$

is well-defined. Moreover, F is absolutely continuous on [a,b] because of the absolute continuity of the integral. We can see that F satisfies the requirements because F' = 0 a.e. in \mathcal{C} , which is a set of positive measure. To see that F is strictly increasing, choose $x,y \in [a,b]$ with x < y, then we just set y = x + h for some h and note that

$$F(y) - F(x) = \int_{a}^{x+h} D_{\mathcal{C}}(t)dt - \int_{a}^{x} D_{\mathcal{C}}(t)dt = \int_{x}^{x+h} D_{\mathcal{C}}(t)dt$$

So we are integrating over some interval $I_x = [x, x + h]$. We then note that \mathcal{C} contains no intervals and so there must be some interval $\mathcal{I}_x \subset \mathcal{C}^{\downarrow}$ such that

$$\int_{T}^{x+h} D_{\mathcal{C}}(t)dt \ge \int_{T_{-}} D_{\mathcal{C}}(t)dt > 0$$

Because $D_{\mathcal{C}}(t) > 0$ when $t \notin \mathcal{C}$. Hence, F is strictly increasing and we are done.

(b) Define F as in part (a). Then we have that F is strictly increasing and absolutely continuous. We then set $\mathcal{O} = [a, b] - \mathcal{C}$. Then \mathcal{O} is open, as it is the complement of a closed set and so we can write

$$\mathcal{O} = \bigcup_{k=1}^{\infty} I_k$$

Where the I_k are a collection of disjoint open intervals. Because F is increasing $I_j \cap I_k = \emptyset$ implies that $F(I_j) \cap F(I_k) = \emptyset$ and so we have that

$$F(\mathcal{O}) = \bigcup_{k=1}^{\infty} F(I_k)$$

And so we have that

$$m(\mathcal{O}) = \sum_{k=1}^{\infty} m(I_k) = \int_{\mathcal{O}} D_{\mathcal{C}}(t)dt$$

We then note that $\int_{\mathcal{C}} D_{\mathcal{C}}(t)dt = 0$ so that

$$\int_{\mathcal{O}} d_{\mathcal{C}}(t)dt = \int_{[a,b]} D_{\mathcal{C}}(t)dt = F(b) - F(a)$$

So we have that $m(F(\mathcal{O})) = m(F([a,b]))$ and as a result $m(F(\mathcal{C})) = 0$. Consequently, if we take any other set $U \subset \mathcal{C}$ we must have that m(F(U)) = 0. However, we also have that because $m(\mathcal{C}) > 0$ that it contains a non-measurable subset, \mathcal{N} . But then $F(\mathcal{N}) \subset F(\mathcal{C})$ and so $m(F(\mathcal{N})) = 0$ however $F^{-1}(F(\mathcal{N})) = \mathcal{N}$ is non-measurable and we are done.

(c)

Problem 3.5.21

- (a)
- (b) Before we begin we need to prove the following fact

Lemma. Let f and q absolutely continuous. Then

$$(f \circ g)'(x) = (f' \circ g)(x)g'(x)$$

almost everywhere.

Proof. Because both functions are absolutely continuous we must have that they are differentiable almost everywhere. As a result, we can consider the set D where both f' and g' exist (it's complement has measure 0). Let

$$\Delta_h(u) = u(x+h) - u(x)$$

Then we note that

$$\Delta(f \circ g) = [(f \circ g) + o(\Delta(g))] \cdot \Delta_t(g)$$

Then divide by We then divide by t giving and apply the definition of the derivative to see that

$$(f \circ g)'(x) = (f' \circ g)(x)g'(x)$$

on D. And as a result, almost everywhere so we are done.

To complete the proof we define the following function

$$h(x) = \int_{-\infty}^{x} f(t)dt$$

We know h is absolutely continuous because f is integrable. We then compute

$$\int_a^b (h\circ F)'(x)dx = (h\circ F)(b) - (h\circ F)(a) = \int_{-\infty}^{F(b)} f(y)dy - \int_{-\infty}^{F(a)} f(y)dy = \int_A^B f(y)dy$$

Applying the lemma to this identity gives the desired result.