Problem 4.7.2
We need to show that $|(f,g)| = \|f\| \cdot \|g\|$ iff f and g are linearly dependent, so $f = cg$ for some $c \in \mathbb{R}$. Following the hint, we take

$$\|f\| = \|g\| = 1 \text{ and } (f,g) = 1$$

Then it is clear that

$$(f-g, g) = (f,g) - (g,f) = 1 - 1 = 0$$

So $f-g$ and g are orthogonal. We then write $f = (f-g)+g$ to see that

$$\begin{aligned}
\|f\|^2 &= \|(f-g)+g\|^2 \\
&= ((f-g)+g, (f-g)+g) \\
&= (f-g, f-g) + (f-g, g) + (g, f-g) + (g,g) \\
&= \|f-g\|^2 + \|g\|^2
\end{aligned}$$

So we must have that $\|f-g\|^2 = 0$, because we have equality in Cauchy-Schwartz. As a result of this we note that this implies $f-g=0$ or $f=g$. For general f,g, we simply take $\hat{f} = f/\|f\|$ and $g/\|g\|$, which reduces to the case above, and gives that $f = \frac{\|f\|}{\|g\|} g$ and we are done.

Problem 4.7.5
(a) For the first inclusion we observe that if we have the functions $f, g : \mathbb{R}^n \to \mathbb{R}$ given by

$$f(x) = |x|^{-\alpha} \chi_{|y| \geq 1}(x)$$

and

$$g(x) = |x|^{-\alpha/2} \chi_{|y| \leq 1}(x)$$

Then if we take $\alpha = n$ we have that f^2 is integrable but f is not. Moreover, we have that g is integrable but g^2 is not (c.f. Stein and Shakarchi Exercise 2.5.10; proved on Pset 3). So $f \in L^2(R^n)$ but $f \notin L^1(\mathbb{R}^N)$ and $f \in L^1(\mathbb{R}^d)$ but $g \notin L^2(\mathbb{R}^d)$.

(b) Duly noted, we simply compute

$$\begin{aligned}
\|f\|_{L^1}^2 &= \int |f(x)| \chi_E(x) dx \\
&\leq \left(\int |f(x)| dx \right)^{1/2} \left(\int \chi_E(x) dx \right)^{1/2} \\
&= m(E)^{1/2} \|f\|_{L^2}
\end{aligned}$$

(c) We note that because $f \in L^1(\mathbb{R}^d)$ that $\int |f| < \infty$ so we use $|f| < M$ to see that

$$\begin{aligned}
\|f\|_{L^2}^2 &= \int |f(x)|^2 dx \\
&\leq M \int |f(x)| dx
\end{aligned}$$

So that

$$\|f\|_{L^2} \leq M^{1/2}\|f\|_{L^1}^{1/2}$$

Problem 4.7.7

First we observe that the $\varphi_{ij} \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$ because we compute

$$\begin{aligned}
\|\varphi_{ij}\|^2_{L^2(\mathbb{R}^d \times \mathbb{R}^d)} &= \int_{\mathbb{R}^d \times \mathbb{R}^d} |\varphi_{ij}(x,y)|^2 \\
&= \int_{\mathbb{R}^d \times \mathbb{R}^d} \int |\varphi_i(x)\varphi_j(y)|^2 dxdy \\
&= \int_{\mathbb{R}^d} |\varphi_i(x)|^2 \left(\int_{R^d} |\varphi_j(y)|^2 dy \right) dx \\
&= \left(\int_{\mathbb{R}^d} |\varphi_i(x)|^2 dx \right) \left(\int_{R^d} |\varphi_j(y)|^2 dy \right) \\
&= \|\varphi_i\|_{L^2(\mathbb{R}^d)} \|\varphi_j\|_{L^2(\mathbb{R}^d)} \\
&\leq \infty
\end{aligned}$$

We then verify they the set of $\{\varphi_{ij}\}$ is orthonormal. We begin by establishing normality, observe that

$$\begin{aligned}
\|\varphi_{ij}\|^2_{L^2(\mathbb{R}^d \times \mathbb{R}^d)} &= \int_{\mathbb{R}^d \times \mathbb{R}^d} |\varphi_i(x)\varphi_j(y)|^2 dxdy \\
&= \int_{\mathbb{R}^d} |\varphi_i(x)|^2 \left(\int_{\mathbb{R}^d} |\varphi_j(y)|^2 dy \right) dx \\
&= \left(\int_{\mathbb{R}^d} |\varphi_i(x)|^2 dx \right) \left(\int_{\mathbb{R}^d} |\varphi_j(y)|^2 dy \right) \\
&= \|\varphi_i\|_{L^2(\mathbb{R}^d)} \|\varphi_j\|_{L^2(\mathbb{R}^d)}
\end{aligned}$$

And because each of the terms in the above product is 1, we are done. Again, we use Fubini's theorem to compute

$$\begin{aligned}
(\varphi_{ij}, \varphi_{kl}) &= \int_{\mathbb{R}^d \times \mathbb{R}^d} \varphi_{ij}(x,y)\overline{\varphi_{kl}}(x,y) dxdy \\
&= \int_{\mathbb{R}^d \times \mathbb{R}^d} \varphi_i(x)\overline{\varphi_j}(y)\varphi_k(x)\overline{\varphi_l}(y) dxdy \\
&= \int_{\mathbb{R}^d} \varphi_i(x)\overline{\varphi_k}(x) \left(\int_{\mathbb{R}^d} \varphi_j(y)\overline{\varphi_l}(y) dy \right) dx \\
&= \left(\int_{\mathbb{R}^d} \varphi_i(x)\overline{\varphi_k}(x) dx \right) \left(\int_{\mathbb{R}^d} \varphi_j(y)\overline{\varphi_l}(y) dy \right)
\end{aligned}$$

And because the $\{\varphi_i\}$ are orthonormal, we have that the above quantity is zero and so the above quantity is 0, meaning that $\{\varphi_{ij}\}$ is orthonormal as well. Now we need to verify that the $\{\varphi_{ij}\}$ are indeed a basis. Take any $f \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$

and suppose that $(f, \varphi_{ij}) = 0$ for every j and k, then we see that

$$\begin{aligned} 0 &= \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x,y)\overline{\varphi_{ij}}(x,y)dxdy \\ &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} f(x,y)\overline{\varphi_j}(y)dy \right) \overline{\varphi_i}(x)dx \end{aligned}$$

We then define (as in the hint)

$$f_j(x) = \int_{\mathbb{R}^d} f(x,y)\overline{\varphi_j}(y)dy$$

So this means that

$$\int_{\mathbb{R}^d} f_j(x)\overline{\varphi_i}(x)dx = 0$$

Because the $\{\varphi_i\}$ form an orthonormal basis, this means that $f_j(x) = 0$ for all j. We then apply this fact again in the definition of f_j to see that $f(x,y) = 0$ for all j. So because f is orthogonal to all of the $\{\varphi_{ij}\}$ implies that $f = 0$, we must have that the $\{\varphi_{ij}\}$ are an orthonormal basis.

Problem 4.7.8

(a) It is clear that $\mathcal{H}_\eta$ is a vector space because for $f, g \in \mathcal{H}_\eta$

$$\begin{aligned} \int_a^b |(f+g)(t)|^2\eta(t)dt &\leq \int_a^b (|f(t)|^2 + 2\max\{|f(t)|, |g(t)|\}^2 + |g(t)|^2)\eta(t)dt \\ &= \int_a^b |f(t)|^2\eta(t)dt + 2\int_a^b \max\{|f(t)|, |g(t)|\}^2\eta(t)dt + \int_a^b |g(t)|^2\eta(t)dt \\ &< \infty \end{aligned}$$

Also

$$\int_a^b |\alpha f(t)|^2\eta(t)dt = \alpha^2 \int_a^b |f(t)|^2\eta(t)dt < \infty$$

We simply define the norm of an element to be $\|f\|_\eta = (\int_a^b |f(t)|^2\eta(t)dt)^{1/2}$ and then verify that $\|f\|_\eta = 0$ iff $f = 0$, so that

$$\|f\|_\eta^2 = \int_a^b |f(t)|^2\eta(t) = 0$$

Because η is strictly positive and increasing, we must have that $|f(t)|^2 = 0$ so $f = 0$. The converse is clear. Next we verify the Cauchy-Schwartz inequality and triangle inequalities. Take $f, g \in \mathcal{H}_\eta$ so that

$$\begin{aligned}
|(f,g)|^2 &= \left|\int_a^b f(t)\overline{g(t)}\eta(t)\right|^2 \\
&= \left|\int_a^b f(t)\overline{g(t)}\eta(t)\right| \cdot \left|\int_a^b \overline{f(t)}g(t)\eta(t)\right| \\
&\le \left|\int_a^b f(t)\overline{f(t)}g(t)\overline{g(t)}\eta^2(t)\right| \\
&= \left|\int_a^b |f(t)|^2\eta(t)|g(t)|^2\eta(t)dt\right| \\
&\le \left(\int_a^b |f(t)|^2\eta(t)dt\right)\left(\int_a^b |g(t)|^2\eta(t)dt\right) \\
&= (f,f)\cdot(g,g)
\end{aligned}$$

Taking square roots gives the result. For the triangle inequality

$$\begin{aligned}
\|f+g\|_\eta^2 &= \int_a^b |(f+g)|^2\eta(t)dt \\
&\le \int_a^b (|f(t)|^2 + |g(t)|^2)\eta(t)dt \\
&= \int_a^b |f(t)|^2\eta(t)dt + \int_a^b |g(t)|^2)\eta(t)dt \\
&= \|f\|_\eta\|g\|_\eta
\end{aligned}$$

Now we show that $\mathcal{H}_\eta$ is complete in the metric. This will follow from the completeness of $L^2([a,b])$ because $\mathcal{H}_\eta \subset L^2([a,b])$ so we take $f_n \to f$ in L^2 and note that this still converges in $\mathcal{H}_\eta$ because

$$\|f-f_n\|_\eta^2 = \int_a^b |f(t)-f_n(t)|^2\eta(t)dt \le \|f-f_n\|_{L^2}\|\eta\|_{L^2}$$

Which goes to 0 as $n \to \infty$. Furthermore, we can also see that $\mathcal{H}_\eta$ is separable because it is a subspace of a separable space $L^2([a,b])$ so the dense subset $D \subset L^2([a,b])$ will yield a new set $D' = D \cap \mathcal{H}_\eta$, which is dense in $\mathcal{H}_\eta$. Now we need to check that the map $U : f \mapsto \eta^{1/2}f$ is unitary. We can see that U is injective because its kernel is 0 because $\eta(t) > 0$ for all t. to see that the map is surjective we can see that $U^{-1}g = \eta^{-1/2}g$ is well defined for all g because η is positive and increasing. Lastly, we need to show that $\mathcal{U}$ preserves the norm. Observe that

$$\|Uf\|_{L^2}^2 = \int_a^b |\eta(t)^{1/2}f(t)|^2dt = \int_a^b |f(t)|\eta(t)dt = \|f\|_\eta^2$$

Where equality holds in the middle step because $|\eta(t)| = \eta(t)$ (i.e $\eta(t) > 0$). So U is a unitary map and we are done.

Problem 4.7.9
(a) We compute

$$\begin{aligned}\int_{\mathbb{R}} |f(x)|^2 dx &= \int_{\mathbb{R}} \frac{1}{\pi|i+x|^2} \left| F\left(\frac{i-x}{i+x}\right)\right|^2 dx \\ &= \int_{\mathbb{R}} \frac{1}{\pi(1+x^2)} \left| F\left(\frac{i-x}{i+x}\right)\right|^2 dx\end{aligned}$$

We make the change of variable

$$\theta = 2\tan^{-1}(x)$$

so that $x = \tan(\theta/2)$. Similarly we have $e^{i\theta} = (i-x)/(1+x)$ and $1 + x^2 = \sec^2(\theta/2)$. Taking derivatives we see that

$$dx = \frac{1}{2}\sec^2(\theta/2)d\theta$$

We then see that

$$\begin{aligned}\int_{\mathbb{R}} \frac{1}{\pi(1+x^2)} \left| F\left(\frac{i-x}{i+x}\right)\right|^2 dx &= \int_{-\pi}^{\pi} \frac{1}{\pi \sec^2(\theta/2)} |F(e^{i\theta})| \frac{1}{2}\sec^2(\theta/2)d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |F(e^{i\theta})| d\theta\end{aligned}$$

So we have that $\|f\|_{\mathcal{H}_2} = \|F\|_{\mathcal{H}_1}$, and we are done.

(b) We use the above and the fact that $\{e^{in\theta}\}$ are an orthonormal basis for $L^2([-\pi,\pi])$ to observe that the family

$$\{U(e^{in\theta})\}_{n=1}^{\infty} = \left\{\frac{1}{\sqrt{\pi}}\left(\frac{i-x}{i+x}\right)^n \frac{1}{i+x}\right\}_{n=1}^{\infty}$$

must also be an orthonormal basis for $L^2(\mathbb{R})$ because U is unitary and we can apply the Riesz Representation Theorem.

Problem 4.7.10
We proceed by a familiar construction. Let $\mathcal{C}$ be the collection of closed sets containing S, and set $\overline{S} = \bigcap_{C\in\mathcal{C}} C$. then we see that $\mathcal{S}$ is clearly closed as t is the intersection of closed subspaces and furthermore, it is a subspace, for the same reason. So what we need to show is that $(S^{\perp})^{\perp} = \overline{S}$. We know that $(S^{\perp})^{\perp}$ is closed (c.f. pg 177) so we must have that $\overline{S} \subset (S^{\perp})^{\perp}$. Thus, we need only show the reverse inclusion. We begin with the following claim,

Claim. *For any subspace S in a Hilbert space we have that $\overline{S}^{\perp} = S^{\perp}$*

Proof. Clearly we have that $\overline{S}^{\perp} \subset S^{\perp}$ because $S \subset \overline{S}$. So we need only show the reverse inclusion, Pick $p \in S^{\perp}$ and some $s \in \overline{S}$. Because $\overline{S}$ is closed we can find a sequence of points $\{s_n\}_{n=1}^{\infty}$ such that $s_n \to s$ as $n \to \infty$. We then use the fact that the inner product is continuous to see that

$$\lim_{n\to\infty}(s_n, p) = (s, p)$$

Because $(s_n, p) = 0$ for every n we must have that $(s, p) = 0$. Thus, $s \perp p$ and therefore $p \in \overline{S}^{\perp}$ and we are done. $\square$

With the claim in hand we apply Proposition 4.2 to see that

$$\mathcal{H} = \overline{S} \oplus S^{\perp}$$

Now choose $x \in (S^{\perp})^{\perp}$. Then we can write $x = u + v$ with $u \in \overline{S}$ and $v \in S^{\perp}$. As a result we see that

$$(x, v) = (u, v) + (v, v)$$

We know that $u, v = 0$ by construction and moreover we know that $(x, v) = 0$ because $c \in S^{\perp}$ and $x \in (S^{\perp})^{\perp}$. So this means that $(v, v) = 0$ which implies that $v = 0$. Hence, $x \in \overline{S}$ and we are done.