Problem 3.5.12
It is clear by elementary calculus that the function

$$F(x) = \begin{cases} 0 & x = 0 \\ x^2 \sin(1/x^2) & \text{otherwise} \end{cases}$$

is differentiable everywhere with derivative

$$F'(x) = \begin{cases} 0 & x = 0 \\ 2x \sin(1/x^2) - \frac{2\cos(1/x^2)}{x} & \text{otherwise} \end{cases}$$

We now need to show that F' is not integrable on $[-1, 1]$. First we note that F' is symmetric about the origin so that

$$\int_{-1}^{1} |F'(x)| dx = 2 \int_{0}^{1} |F'(x)|$$

We will show that the integral on the right diverges. The intuition is that near the origin $\sin(1/x^2)$ and $\cos(1/x)$ oscillate too rapidly and add infinitely much "area" under the curve. More formally, consider the set of points

$$x_j = \left\{ \frac{2}{(2j+1)\pi} \right\}_{j=1}^{\infty}$$

It is clear that $x_j \to 0$ as $j \to \infty$ and so it suffices to show that

$$\int_{x_N}^{1} |F'(x)| dx \to \infty \text{ as } N \to \infty$$

We will break the domain into smaller intervals of the form $[x_j, x_{j+1}]$ and observe that

$$\int_{x_N}^{1} = \sum_{j=1}^{N} \int_{x_{j+1}}^{x_j} |F'(x)| dx$$

Then note that

$$\int_{x_{j+1}}^{x_j} |F'(x)| dx = \int_{\sqrt{\frac{1}{2j\pi}}}^{x_j} F'(x) \pm \int_{x_{j+1}}^{\sqrt{\frac{1}{2j\pi}}} F'(x) = \pm 2 \int_{\sqrt{\frac{1}{2j\pi}}}^{x_j} F'(x)$$

Where the sign is chosen to make the value positive. We then evaluate each term in the partial sum

$$\int_{\sqrt{\frac{1}{2j\pi}}}^{x_j} F'(x) = F(x) \Big|_{\sqrt{\frac{1}{2j\pi}}}^{x_j} = \frac{1}{(2j+1)\pi}$$

So that

$$\int_{x_N}^{1} |F'(x)| = \frac{4}{\pi} \sum_{j=1}^{N} \frac{1}{(2j+1)}$$

Which diverges as $N \to \infty$ by comparison with the harmonic series. Hence, F' is not integrable on $[-1, 1]$.

Problem 3.5.16
(a)

(b)

Problem 3.5.17
We proceed similarly to the proof of Theorem 3.21. We note that

$$|(f * K_\epsilon)(x)| = \left| \int f(x-y) K_\epsilon(y) dy \right| \leq \int |f(x-y)||K_\epsilon(y)| dy$$

We break the integral on the right up into

$$\int_{|y|<\epsilon} |f(x-y)||K_\epsilon(y)| dy + \sum_{k=1}^{\infty} \int_{|y| \in [2^{k-1}\epsilon, 2^k \epsilon]} |f(x-y)||K_\epsilon(y)| dy$$

For the first term, we note that

$$\int_{|y|<\epsilon} |f(x-y)||K_\epsilon(y)| dy \leq A\epsilon^{-d} \int_{|x-y|<\epsilon} |f(y)| dy$$

We then note that $|x-y| < \epsilon$ describes precisely the ball with radius ϵ about the origin. As a result we see that

$$\sup_{\epsilon>0} \int_{|x-y|<\epsilon} |f(y)| \leq m(B_\epsilon) \sup_{r \to 0} \frac{1}{m(B_r)} \int_{B_r} |f(y)| dy$$

This gives

$$\int_{|y|<\epsilon} |f(x-y)||K_\epsilon(y)| dy \leq A c_d f^*(x)$$

Where c_d is a constant depending on the volume of the unit ball in $\mathbb{R}^d$. To estimate

$$\sum_{k=1}^{\infty} \int_{|y| \in [2^{k-1}\epsilon, 2^k \epsilon]} |f(x-y)||K_\epsilon(y)| dy$$

We restrict our attention to the partial sums and observe that

$$\int_{|y| \in [2^{k-1}\epsilon, 2^k \epsilon]} |f(x-y)||K_\epsilon(y)| dy \leq \frac{2^{(1-k)(d+1)}}{\epsilon^{d+1}} \int_{|y| \leq 2^k \epsilon} |f(x-y)| dy$$

By the same reasoning as the first case (multiply by volume of appropriate ball and canceling powers), this gives

$$2^d 2^{1-k} \epsilon^{-1} c_d f^*(x)$$

As a result of this

$$\sum_{k=1}^{\infty}\int_{|y|\in[2^{k-1}\epsilon,2^k\epsilon]}|f(x-y)||K_\epsilon(y)|dy \leq \sum_{k=1}^{\infty}2^d2^{1-k}\epsilon^{-1}c_df^*(x)$$
$$\leq 2^{d+1}\epsilon^{-1}c_df^*(x)$$

So in total we have that

$$|(f*K_\epsilon)(x)| \leq (2^{d+2}c_dA)f^*(x)$$

as desired.

Problem 3.5.19
(a) Let $E\subset\mathbb{R}$ have measure 0. Pick some $\epsilon>0$ and use the absolute continuity of f to see that

$$\sum_{k=1}^{N}|b_k-a_k|<\delta \text{ implies } \sum_{k=1}^{N}|f(b_k)-f(a_k)|<\epsilon$$

We then use the fact that E has measure 0 to find some open set $\mathcal{O}\supset E$ such that $m(\mathcal{O})<\delta$. Because $\mathcal{O}\subset\mathbb{R}$ we can write it as a disjoint union of open intervals $I_k=(a_k,b_k)$ such that

$$\mathcal{O}=\bigcup_{k=1}^{\infty}I_k$$

and then we note that

$$m(\mathcal{O})=\sum_{k=1}^{\infty}(b_k-a_k)<\delta$$

Moreover we have that the closure of $\mathcal{O}$ also satisfies $m(\overline{\mathcal{O}})<\delta$ because the boundary of $\mathcal{O}$ is countable and hence has measure 0. Now we look at the images of these closed intervals under f. Consider the points $\{y_k\}_1^\infty$ and $\{Y_k\}_1^\infty$ which are defined to be the points such that $y_k=\inf_{x\in I_k}f(x)$ and $Y_k=\sup_{x\in I_k}f(x)$. Then set $\{x_k=\{f^{-1}(y_k)\}$ and $\{X_k=\{f^{-1}(Y_k)\}$. We have that $x_k,X_k\in\overline{I_k}$ because f is continuous and $\overline{I_k}$ is compact. So then we have that

$$f(\mathcal{O})\subset\bigcup_k[f(x_k),f(X_k)]$$

Because next note that

$$\sum_{k=1}^{\infty}|X_k-x_k|\leq\sum_{k=1}^{\infty}|b_k-a_k|<\delta$$

because $x_k,y_k\in I_k$ and so the absolute continuity of f implies that

$$m(\overline{\mathcal{O}})=\sum_{k=1}^{\infty}|f(X_k)-f(x_k)|<\epsilon$$

So

$$m(E) < m(\overline{\mathcal{O}}) < \epsilon$$

Letting $\epsilon \to 0$ gives the result.

(b) Let E now be a measurable subset of $\mathbb{R}$. Then we know that E differs from an F_σ by a set of measure 0. So we can write

$$E = \left(\bigcup_{k=1}^{\infty} \mathcal{I}_k \right) \cup Z$$

Where Z is a set of measure 0 and each of the I_k is closed (and bounded and hence, compact). Then the image under f gives

$$f(E) = \bigcup_k f(\mathcal{I}_k) \cup f(Z)$$

The first element in the union is a countable union of measurable sets, and hence, measurable and $f(Z)$ has measure 0 by the previous part. Hence, $f(E)$ is measurable.

Problem 3.5.20

(a) The main hurdle to this exercise is that if $F' = 0$ on any interval, then F is constant a.e. on that interval, and hence not strictly increasing. As as result, we are led to try to find a set with positive measure, but which contains no intervals, and then define F as an integral over such a set. We constructed such a set (the "fat" Cantor set) in a previous problem set. Let $\mathcal{C}$ denote such a set (translated and dilated to appropriately sit in $[a, b]$). Consider the function

$$D_{\mathcal{C}}(t) = d(t, \mathcal{C}) = \inf_{x \in \mathcal{C}} d(t, x)$$

It is clear that $D_{\mathcal{C}}(t)$ is continuous and satisfies $D_{\mathcal{C}}(t) \geq 0$ with equality iff $t \in \mathcal{C}$. As a result, the function

$$F(x) = \int_a^x D_{\mathcal{C}}(t)dt$$

is well-defined. Moreover, F is absolutely continuous on $[a, b]$ because of the absolute continuity of the integral. We can see that F satisfies the requirements because $F' = 0$ a.e. in $\mathcal{C}$, which is a set of positive measure. To see that F is strictly increasing, choose $x, y \in [a, b]$ with $x < y$, then we just set $y = x + h$ for some h and note that

$$F(y) - F(x) = \int_a^{x+h} D_{\mathcal{C}}(t)dt - \int_a^x D_{\mathcal{C}}(t)dt = \int_x^{x+h} D_{\mathcal{C}}(t)dt$$

So we are integrating over some interval $I_x = [x, x + h]$. We then note that $\mathcal{C}$ contains no intervals and so there must be some interval $\mathcal{I}_x \subset \mathcal{C}^\rfloor$ such that

$$\int_x^{x+h} D_{\mathcal{C}}(t)dt \geq \int_{\mathcal{I}_x} D_{\mathcal{C}}(t)dt > 0$$

Because $D_{\mathcal{C}}(t) > 0$ when $t \notin \mathcal{C}$. Hence, F is strictly increasing and we are done.

(b) Define F as in part (**a**). Then we have that F is strictly increasing and absolutely continuous. We then set $\mathcal{O} = [a, b] - \mathcal{C}$. Then $\mathcal{O}$ is open, as it is the complement of a closed set and so we can write

$$\mathcal{O} = \bigcup_{k=1}^{\infty} I_k$$

Where the I_k are a collection of disjoint open intervals. Because F is increasing $I_j \cap I_k = \emptyset$ implies that $F(I_j) \cap F(I_k) = \emptyset$ and so we have that

$$F(\mathcal{O}) = \bigcup_{k=1}^{\infty} F(I_k)$$

And so we have that

$$m(\mathcal{O}) = \sum_{k=1}^{\infty} m(I_k) = \int_{\mathcal{O}} D_{\mathcal{C}}(t)dt$$

We then note that $\int_{\mathcal{C}} D_{\mathcal{C}}(t)dt = 0$ so that

$$\int_{\mathcal{O}} d_{\mathcal{C}}(t)dt = \int_{[a,b]} D_{\mathcal{C}}(t)dt = F(b) - F(a)$$

So we have that $m(F(\mathcal{O})) = m(F([a, b]))$ and as a result $m(F(\mathcal{C})) = 0$. Consequently, if we take any other set $U \subset \mathcal{C}$ we must have that $m(F(U)) = 0$. However, we also have that because $m(\mathcal{C}) > 0$ that it contains a non-measurable subset, $\mathcal{N}$. But then $F(\mathcal{N}) \subset F(\mathcal{C})$ and so $m(F(\mathcal{N})) = 0$ however $F^{-1}(F(\mathcal{N})) = \mathcal{N}$ is non-measurable and we are done.

(c)

Problem 3.5.21
(a)

(b) Before we begin we need to prove the following fact

Lemma. *Let f and g absolutely continuous. Then*

$$(f \circ g)'(x) = (f' \circ g)(x)g'(x)$$

almost everywhere.

Proof. Because both functions are absolutely continuous we must have that they are differentiable almost everywhere. As a result, we can consider the set D where both f' and g' exist (it's complement has measure 0). Let

$$\Delta_h(u) = u(x + h) - u(x)$$

Then we note that

$$\Delta(f \circ g) = [(f \circ g) + o(\Delta(g))] \cdot \Delta_t(g)$$

Then divide by We then divide by t giving and apply the definition of the derivative to see that

$$(f \circ g)'(x) = (f' \circ g)(x)g'(x)$$

on D. And as a result, almost everywhere so we are done. $\square$

To complete the proof we define the following function

$$h(x) = \int_{-\infty}^{x} f(t)dt$$

We know h is absolutely continuous because f is integrable. We then compute

$$\int_a^b (h \circ F)'(x)dx = (h \circ F)(b) - (h \circ F)(a) = \int_{-\infty}^{F(b)} f(y)dy - \int_{-\infty}^{F(a)} f(y)dy = \int_A^B f(y)dy$$

Applying the lemma to this identity gives the desired result.