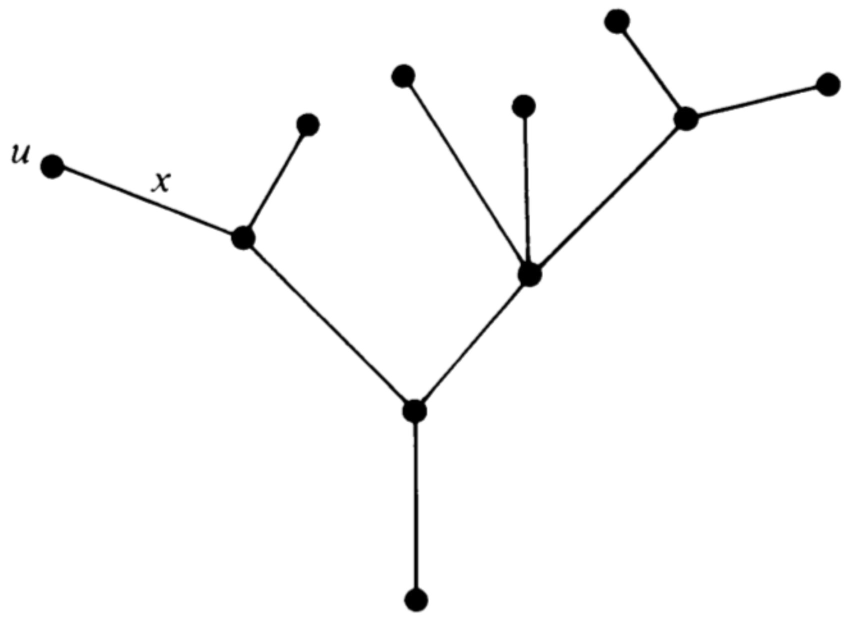
Module- 3

Trees and Graph Algorithm

Trees – properties, Pendant vertex, Distance and centres in a tree - Rooted and binary trees, counting trees, Prim's algorithm and Kruskal's alagorithm. Dijkstra's shortest path algorithm, Floyd-Warshall shortest path algorithms.

TREES

A *tree* is a connected graph without any circuits.



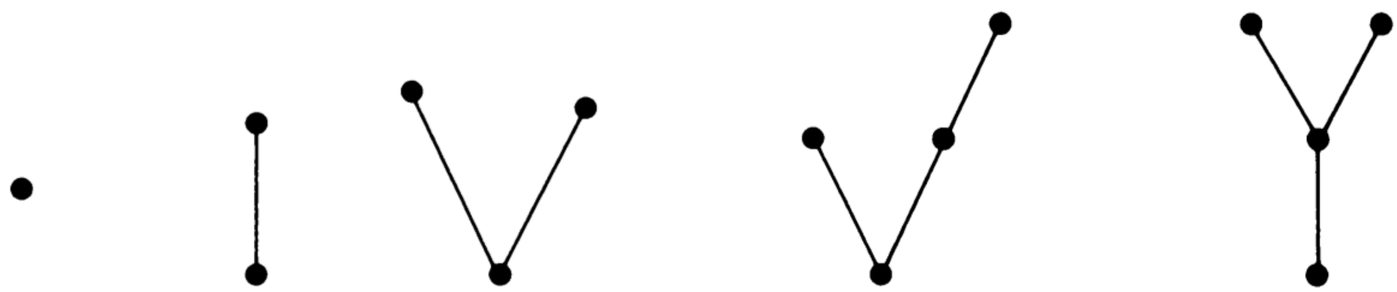


Fig. 3-2 Trees with one, two, three, and four vertices.

Note: ·

we are considering only finite graphs, our trees are also finite.

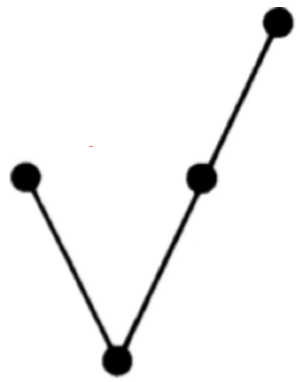
A river with its tributaries and subtributaries can be represented by a tree.

PROPERTIES OF TREES

THEOREM -1

There is one and only one path between every pair of vertices in a tree, T.

Proof: Since T is a connected graph, there must exist at least one path between every pair of vertices in T. Now suppose that between two vertices a and b of T there are two distinct paths. The union of these two paths will contain a circuit and T cannot be a tree. ■



THEOREM -2

If in a graph G there is one and only one path between every pair of vertices, G is a tree.

Proof: Existence of a path between every pair of vertices assures that G is connected. A circuit in a graph (with two or more vertices) implies that there is at least one pair of vertices a, b such that there are two distinct paths between a and b. Since G has one and only one path between every pair of vertices, G can have no circuit. Therefore, G is a tree. ■

Note: From theorems 1 and 2,

If G is a tree iff ∃ exactly one path between every pair of vertices.

THEOREM 3

A tree with n vertices has $n - 1$ edges.

~~10 edg~~
~~11 edg~~
~~8 edg~~
$n = 10$
9 edges

Proof: The theorem will be proved by induction on the number of vertices.

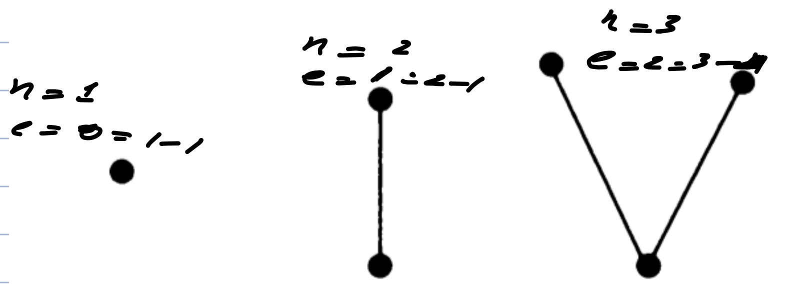
It is easy to see that the theorem is true for $n = 1$, 2, and 3. Assume that the theorem holds for all trees with fewer than n vertices.

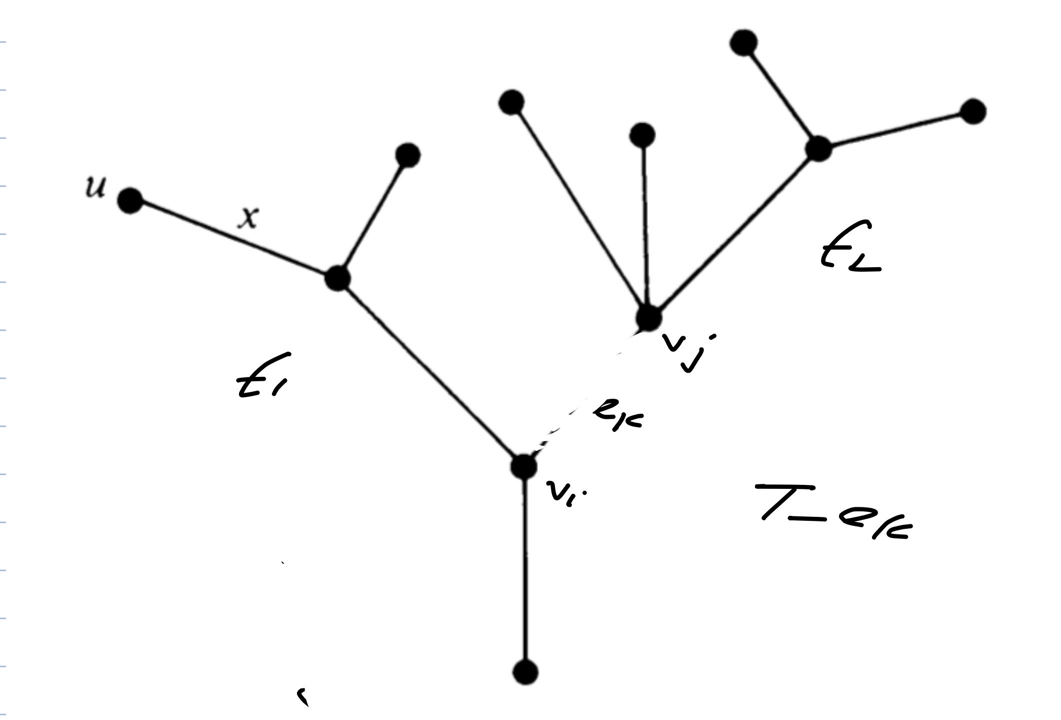
Let us now consider a tree T with n vertices. In T let e_k be an edge with end vertices v_i and v_j. According to Theorem 1, there is no other path between v_i and v_j except e_k. Therefore, deletion of e_k from T will disconnect the graph. Furthermore, $T - e_k$ consists of exactly two components, and since there were no circuits in T to begin with, each of these components is a tree. Both these trees, t_1 and t_2, have fewer than n vertices each, and therefore, by the induction hypothesis, each contains one less edge than the number of vertices in it. Thus $T - e_k$ consists of $n - 2$ edges (and n vertices). Hence T has exactly $n - 1$ edges. ■

$t_1 \to n$ $\quad T - e_k \quad t_1 \quad t_2 \quad < n$

$(n-1) - 1 = n - 2 \quad (n-1) \quad n - 1 - 1 = n - 2$

The thm is true for $n = 1, 2, 3$





$n = 11$

t_1 t_2

5 6

$4 = 5 - 1$ $5 = 6 - 1$

$n - 1$ $n - 1$

$n - 2$ $n - 1 - 1$

THEOREM 1

There is one and only one path between every pair of vertices in a tree, T.

THEOREM 4

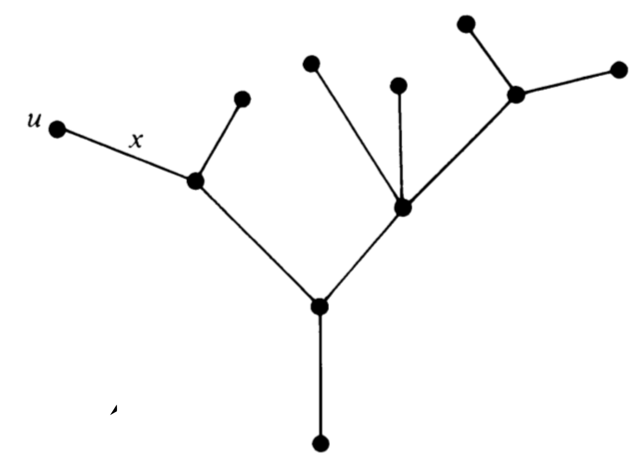
Any connected graph with n vertices and $n - 1$ edges is a tree.

Suppose that a connected graph G with n vertices and $n-1$ edges is not a tree

$\therefore \exists$ atleast one circuit in this connected graph.

Remove one of the edge ~~w~~ within this circuit.

This leaves a connected graph on n' vertices and $n-2$ edges, which is impossible. Because by a connected ~~[illegible]~~ graph with ~~on~~ n' vertices must have ~~at~~least $n-1$ edges.



Note:

A connected graph is minimally connected, if removal of any edge from it disconnects the graph.

Theorem

A graph is a tree if and only if it is minimally connected.

THEOREM

A graph G with n vertices, $n - 1$ edges, and no circuits is connected.

Proof: Suppose there exists a circuitless graph G with n vertices and $n - 1$ edges which is disconnected. In that case G will consist of two or more circuitless components. Without loss of generality, let G consist of two components, g_1 and g_2. Add an edge e between a vertex v_1 in g_1 and v_2 in g_2. Since there was no path between v_1 and v_2 in G, adding e did not create a circuit. Thus $G \cup e$ is a circuitless, connected graph (i.e., a tree) of n vertices and n edges, which is not possible, because of Theorem . ■ A tree with n vertices has $n - 1$ edges.

a graph G with n vertices is called a tree if

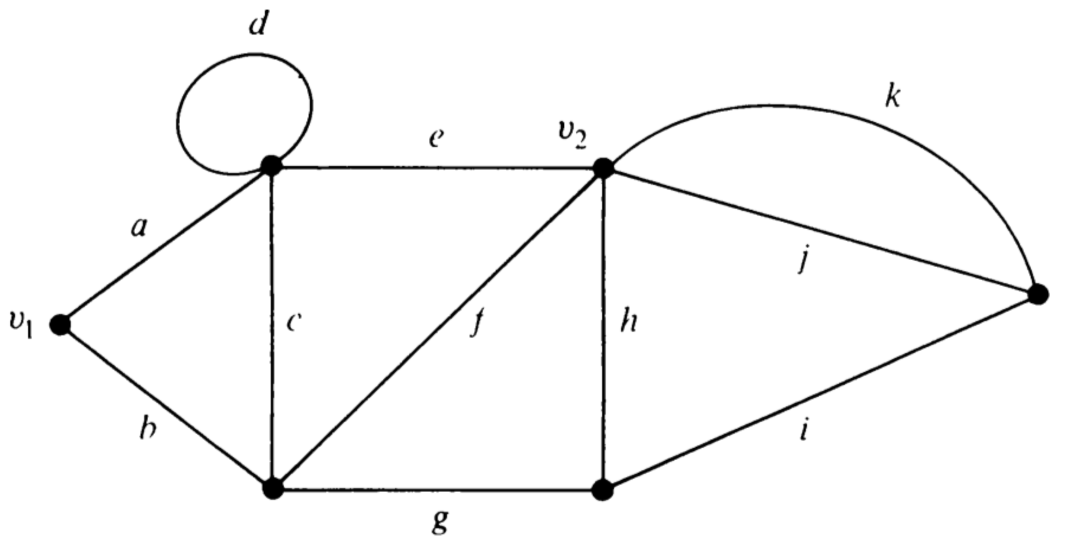
1. G is *connected* and is *circuitless*, or
2. G is *connected* and has $n - 1$ *edges*, or
3. G is *circuitless* and has $n - 1$ *edges*, or
4. There is *exactly one path* between every pair of vertices in G, or
5. G is a *minimally connected* graph.

THEOREM 3-7

In any tree (with two or more vertices), there are at least two pendant vertices.

DISTANCE AND CENTERS IN A TREE

In a connected graph G, the *distance* $d(v_i, v_j)$ between two of its vertices v_i and v_j is the length of the shortest path (i.e., the number of edges in the shortest path) between them.



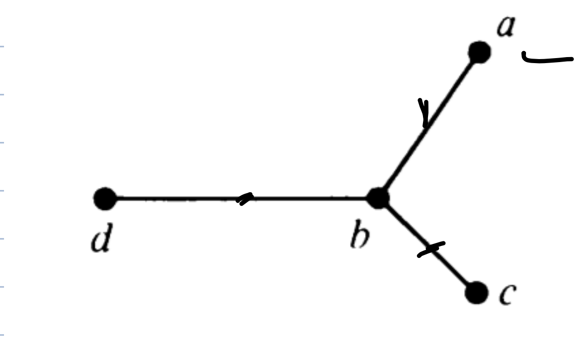
$d(v_1, v_2) = 2$

path b/w v_1, v_2 = $\overset{2}{(a, e)}$, $\overset{3}{(b, c, e)}$, $\overset{2}{(b, f)}$,
$\underset{3}{(a, c, f)}$

$\underline{\underline{d(v_1, v_2)}} = \underline{\underline{2}}$

The eccentricity $E(v)$ of a vertex v in a graph G is the distance from v to the vertex farthest from v in G; that is,

$$E(v) = \max_{v_i \in G} d(v, v_i).$$



$E(a) = d(a,b) = 1$

$d(a,c) = 2$

$d(a,d) = 2$

$E(a) = \text{Max } d(a, v_i) = 2$

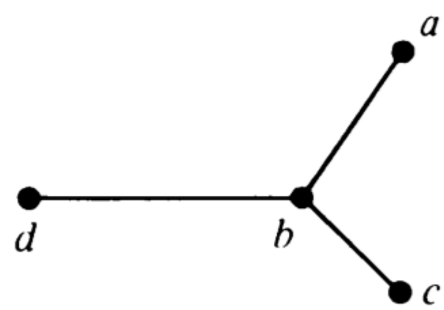
$E(b) = 1$

$E(c) = 2$

$E(d) = 2$

Center of a connected graph

A vertex with minimum eccentricity in graph G is called a *center* of G.



$E(a) = 2$

$E(b) = 1 \Rightarrow$ Minimum ecc

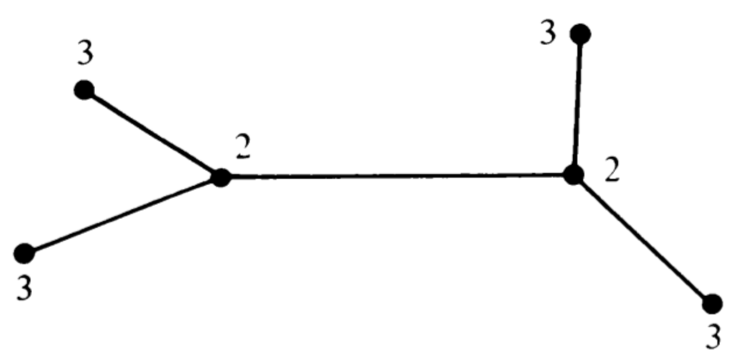
$E(\cancel{d}) = 2$

$E(d) = 2$

Center $= \underline{\underline{b}}$

Note

In general, a graph has many centers.



A graph with two centers.

In a circuit, every vertex is a center.