10 Chapter Ten

3-D Transformation

Chapter Outline

- Introduction
- 3D-Scaling
- 3D Shearing
- Coordinate Transformations
- 3-D Translation
- Rotation
- Multiple Transformations
- Instance Transformations

10.1. INTRODUCTION

In the previous chapter, we have seen two dimensional objects (zero thickness) and the operations which we can perform on them. The present chapter is an extension of the previous one in the sense that it now takes z-coordinate apart from the x, y co-ordinates. To represent 3D object we need 3 parameters, x co-ordinate which represents width, y co-ordinate represents height and z represents depth. In 3D, we are having 3 axis *i.e.*, x, y and z. These three axis are arranged in such a way that they are normal to each other. There are two different orientations for z co-ordinate system.

- Right handed system (RHS)
- Left handed system (LHS)

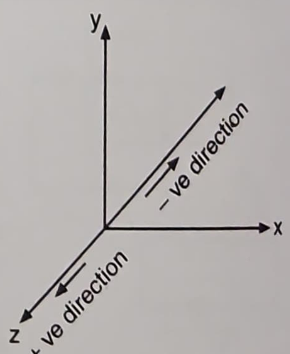

Fig. 10.1. Right handed system

In right handed system (RHS) the z-axis pointing towards the viewer.

If our right hand's four fingers *i.e.*, other than thumb are curling from x to y direction, then the thumb of right hand indicates positive z direction.

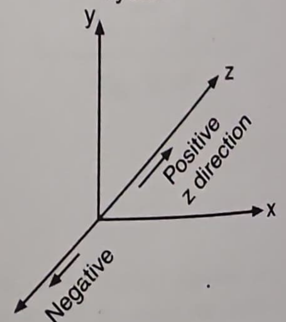

Fig. 10.2. Left handed system

A left handed system has the z-axis pointing away from the viewer.

If our left hand's four fingers (*i.e.*, other than thumb) are curling from z to y direction, then the thumb of left hand points positive z direction.

As done in case of 2D we express 3D object transformation in matrix form and any sequence of transformations is represented as a single matrix.

Just as 2D any 3D transformation can be represented by

$$[X'] = [T]\,[X]$$

where [X′] represents transformed homogeneous coordinates

$$\begin{bmatrix} x' \\ y' \\ z' \\ h \end{bmatrix} \text{ or } [x' \; y' \; z' \; h]$$

and [X] represents the untransformed coordinates

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \text{ or } [x \; y \; z \; 1]$$

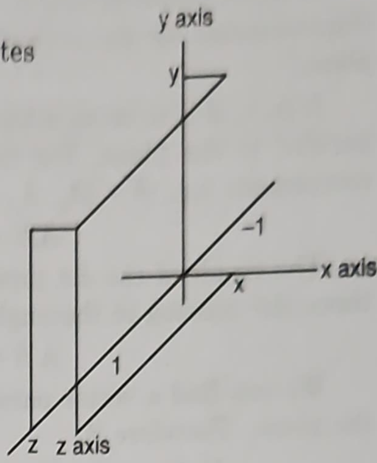

Fig. 10.3.

3-d gemetry. In 3 dimension we need an additional coordinate axis for the third dimension (*i.e.*, three axis in all one for height, one for width, and a third for depth)

In three dimensions as we move along the line, both. y and z cordinates will change proportionately to x.

A line in 3-dmensions is given by a pair of equations

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\frac{z - z_1}{x - x_1} = \frac{z_2 - z_1}{x_2 - x_1}$$

If requires the co-ordinates of two points to form these equations: (x_1, y_1, z_1) and (x_2, y_2, z_2). A more symmetrical expression for a line is the parametric from, where each of the co-ordinates is expessed in terms of a parameter u.

$$x = (x_2 - x_1)u + x_1$$
$$y = (y_2 - y_1)u + y_1$$
$$z = (z_2 - z_1)u + z_1$$

A plane is specified by a single equation of the form

$$Ax + By + Cz + D = 0$$

or $\quad x + B_1 y + C_1 z + D_1 = 0$

When $\qquad B_1 = \dfrac{B}{A} \quad C_1 = \dfrac{C}{A} \quad D_1 = \dfrac{D}{A}$

Therefore requires only three constants B_1, C_1 & D_1 to specify the plane. The equation for a particular plane may be determined if we know the co-ordinates of three points which lie within it. Let three points are (x_1, y_1, z_1) (x_2, y_2, z_2) & (x_3, y_3, z_3). We can determine the equation in the following manner

$$x_1 + B_1 y_1 + C_1 z_1 + D_1 = 0$$
$$x_2 + B_1 y_2 + C_1 z_2 + D_1 = 0$$
$$x_3 + B_1 y_3 + C_1 z_3 + D_1 = 0$$

Ths distance between a point (x, y, z) and the plane is given by

$$L = |A_2 x + B_2 y + C_1 z + D_2|$$

where $\qquad A_2 = A/d \quad B_2 = B/d \quad C_2 = C/d$ and $D_2 = D/d$

and $$d = (A^2 + B^2 + C^2)^{1/2}$$

A vector perpendicular to a plane is called a **normal vector**. We can call $[N_x, N_y, N_z]$ the displacements for the normal vector and (x_p, y_p, z_p) are the co-ordinates of a point on the plane.

If (x, y, z) is to be an arbitrary point in the plane then $[(x - x_p)\ (y - y_p)\ (z - z_p)]$ is a vector parallel to the plane. The dot product of two vectors is the sum of their corresponding components e.g., $A = [A_x, A_y, A_z]$ and $B = [B_x, B_y, B_z]$ then their dot product is

$$A.B = A_x B_x + A_y B_y + A_z B_z$$

The result of the dot product is equal to the product of the lengths of the two vectors times the causing of the angle between them.

$$A.B = |A|\ |B| \cos\theta.$$

We can find a vector parallel to the plane by taking the difference of two points within the plane. Therefore if

$$N_x (x - x_p) + N_y (y - y_p) + N_z (z - z_p) = 0$$

is a true equation then the vector formed by the difference between (x, y, z) and (x_p, y_p, z_p) must be parallel to the plane.

3D Primitives: As in the case of two dimensions, we picture an imaginary pen or printer which the user commands to move from point to point. We save this pen's current position is some global variables, only now three variables are required: DF-PEN-X, DF-PEN-Y and DF-PEN-Z, one for each of the three coordinates. The three dimensional LINE and MOVE algorithms are as follows:

1. Move-ABS3 (X, Y, Z)

X, Y, Z: World ocoordinates of the point to move the pen DF-PEN-X, DF-PEN-Y, DF-PEN-Z current pen position

```
BEGIN
    DF-PEN-X ← X;
    DF-PEN-Y ← Y;
    DF-PEN-Z ← Z;
    DISPLAY-FILE-ENTER (1);
    RETURN;
END;
```

2. LINE-ABS-3 (X, Y, Z)

```
BEGIN
    DF-PEN-X ← X;
    DF-PEN-Y ← Y;
    DF-PEN-Z ← Z;
    DISPLAY-FILE-ENTER (2);
    RETURN;
END;
```

3. Line-REL-3 (DX, DY, DZ)

DX, DY, DZ displacements over which a line to be drawn.

```
BEGIN
    DF-PEN-X ← DF-PEN-X + DX;
    DF-PN-Y ← DF-PEN-Y + DY;
```

```
        DF-PEN-Z ← DF-PEN-Z + DZ;
        DISPLAY-FILE-ENTER (Z);
        RETURN;
    END;
```

4. Polygon-ABS-3 (AX, AY, AZ, N)

N: No. of polygon sides.

AX, AY, AZ arrays of the coordinates of the vertices.

```
    BEGIN
        If N < 3 then return Error "Size error";
        DF-PEN-X ← AX [N];
        DF-PEN-y ← AY [N];
        DF-PEN-Z ← AZ [N];
        DISPLAY-FILE-ENTER [N];
        FOR I = 1 TO N DO LINE-ABS-3 [AX [I], AY [I], AZ [I]];
        RETURN;
    END;
```

10.2. 3-D TRANSLATION

Translation means shifting a point or moving whole object. In 3D we need 3 translation factors. So the transformation matrix in 3D will be

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ t_x & t_y & t_z & 1 \end{bmatrix}$$

For translation, we have to perform multiplication of a point with this translation matrix.

$$[x' \; y' \; z' \; 1] = [x \; y \; z \; 1] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ t_x & t_y & t_z & 1 \end{bmatrix}$$

where

$$x' = x + t_x$$
$$y' = y + t_y$$
$$z' = z + t_z$$

or

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

10.3. 3D-SCALING

Scaling changes size of object. Scaling in 3D is much identical to scaling in 2D. The matrix representation for scaling transformation, relative to the co-ordinate origin, will be

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{pmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$x' = S_x\, x$$
$$y' = S_y\, y$$
$$z' = S_z\, z$$

where S_x, S_y, S_z are scale factors in x, y and z directions respectively.

A uniform scaling can be done without matrix, just multiply each vector with the scale factor. If the scale factor is larger than 1, the object will become bigger, if it's smaller than 1, the object will become smaller. If the scale factor is negative, the object will be mirrored.

> **Note:** If we want to scale along another axis than x, y or z. We first rotate the object so the we axis we want has the direction of the z-axis, then scale and then rotate back.

EXAMPLE 10.1.

Consider the effect of a translation in the x y z direction by – 2, – 4 and – 6 respectively on the homogeneous co-ordinate position vectors [1, 6, 4].

Solution: We know that translation matrix is

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ t_x & t_y & t_z & 1 \end{bmatrix}$$

Here

$$t_x = -2$$
$$t_y = -4$$
$$t_z = -6$$

So $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & -4 & -6 & 1 \end{bmatrix} [1\ 6\ 4\ 1] = [-1\ 2\ -2\ 1]$

i.e., new co-ordinates are [–1, 2, – 2]

EXAMPLE 10.2

Given a rectangular parallelopiped which is unit distance on z-axis, 2-distance on x-axis and 3-distance on y-axis. What is the effect of scaling when scaling factor $S_x = 1/2$, $S_y = 1/3$ and $S_z = 1$?

Solution: The matrix representation for rectangular parallelopiped is

$$\begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \end{array}\begin{bmatrix} 0 & 0 & 1 & 1 \\ 2 & 0 & 1 & 1 \\ 2 & 3 & 1 & 1 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 2 & 3 & 0 & 1 \\ 0 & 3 & 0 & 1 \end{bmatrix}$$

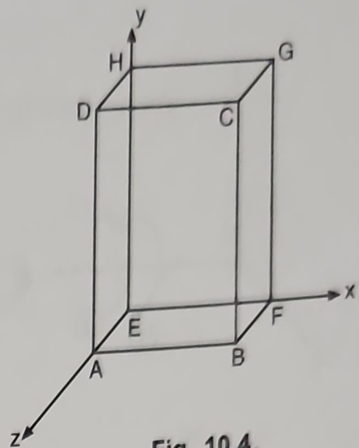

Fig. 10.4.

The transformation matrix for scaling is

$$= \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where $S_x = ½\ S_y = 1/3$ and $S_z = 1$

Now,

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 2 & 0 & 1 & 1 \\ 2 & 3 & 1 & 1 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 2 & 3 & 0 & 1 \\ 0 & 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

10.4. ROTATION

We have seen that any 2D-rotation transformation is uniquely defined by specifying a centre of rotation and amount of angular rotation. Rotation in 3D is slightly different than the rotation in 2D because these two parameters do not uniquely define a rotation in 3D space. The reason is that an object can rotate along different circular paths centering a given rotation centre and thus forming different planes of rotation. We need to fix the plane of rotation and that is done by specifying an axis of rotation instead of a centre of rotation. The radius of rotation path is always perpendicular to the axis of rotation.

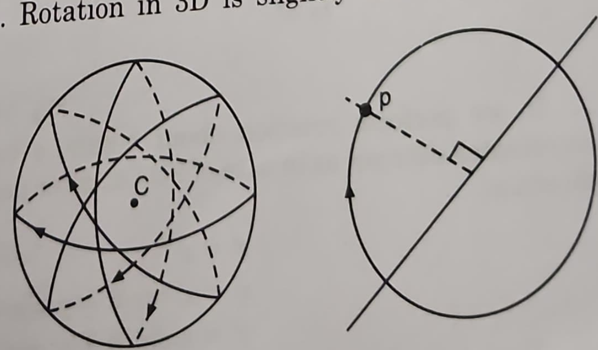

Fig. 10.5.

In 3D there are 3 axis, therefore there are 3 different planes xy, xz and yz planes. So in 3D we have to specify axis and by what angle we want to rotate the object. In 2D it is easy to find clockwise and anticlockwise dorections, but in 3D to detect clockwise and anticlockwise directions, we are looking from positive half of the axis towards the origin.

By convention positive rotation angles produced anticlockwise rotations and negative rotation angles produces clockwise rotations.

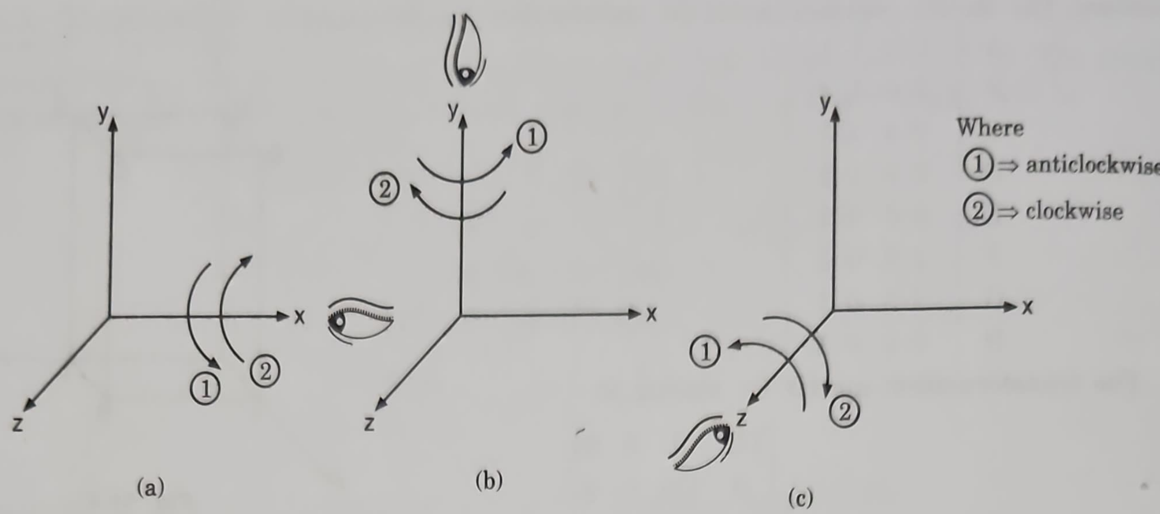

Fig. 10.6.

10.4.1. Rotation about z-axis

If we want to rotate a 2D point say (x, y) by angle θ in anticlockwise direction, then transformation will be

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

i.e.

$$x' = x \cos\theta - y \sin\theta$$
$$y' = x \sin\theta + y \cos\theta$$

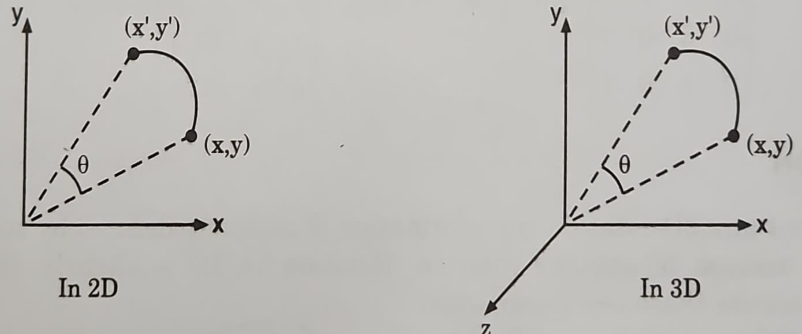

Fig. 10.7.

In we perform rotation about z-axis, z co-ordinate remains unchanged and x and y coordinates changes as in case of 2D *i.e.*, after performing rotation about z-axis in anticlockwise direction.

$$z' = z$$
$$\left.\begin{aligned} x' &= x \cos\theta - y \sin\theta \\ y' &= x \sin\theta + y \cos\theta \end{aligned}\right\} \text{Same as in 2D.}$$

Thus transformation matrix for rotation about z-axis in anticlock wise direction in homogeneous co-ordinates is

$$[x \;\; y \;\; z \;\; 1] * \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = [x' \; y' \; z' \; 1]$$

Therefore, the transformation matrix will be

$$R_z = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ For anticlockwise}$$

and

$$R_z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ for clockwise}$$

10.4.2. Rotation about x-axis

Here rotation takes place in planes perpendicular to x-axis hence x co-ordinate does not change after rotation while y and z co-ordinates are transformed.

The rotation about x-axis can be obtained from transformation matrix of rotation about z-axis with a cyclic permutation at the co-ordinate parameters x, y and z.

We replace z axis with x axis and the other two axis accordingly, maintaining right handed co-ordinate system.

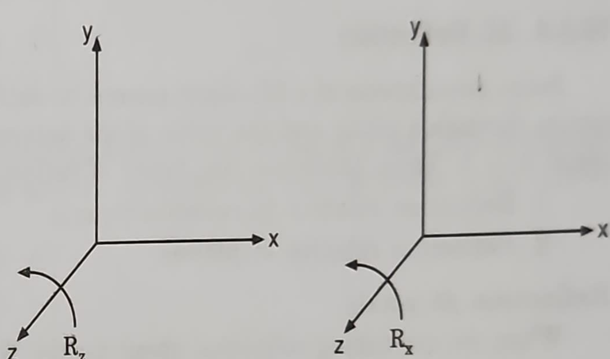

Fig. 10.8.

$$x' = x$$
$$y' = y \text{ Cos } \theta - z \text{ Sin } \theta$$
$$z' = y \text{ Sin } \theta + z \text{ Cos } \theta$$

i.e., we replace z by x, y by z and x by y in z-axis rotation. Therefore the transformation matrix in 4×4 form.

$$[x' \ y' \ z' \ 1] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & Cos\theta & Sin\theta & 0 \\ 0 & -Sin\theta & Cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} * [x \ y \ z \ 1]$$

or

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & Cos\theta & -Sin\theta & 0 \\ 0 & Sin\theta & Cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} * \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

10.4.3. Rotation about y-axis

For rotation about y-axis we will make use of transformation matrix of rotation about x-axis with cyclic permutation of the co-ordinate parameters x, y and z.

We replace $x \rightarrow y \rightarrow z \rightarrow x$

i.e., x by y, y by z and z by x

So, $y' = y$

$z' = z \text{ Cos } \theta - x \sin \theta$

$x' = z \text{ Sin } \theta + x \text{ Cos } \theta$

Now, lets drive transformation matrix for rotation about y-axis in anticlockwise direction.

$$[x\ y\ z\ 1] * \begin{bmatrix} \text{cos è} & 0 & -\text{sin è} & 0 \\ 0 & 1 & 0 & 0 \\ \text{sinè} & 0 & \text{cosè} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = [x'\ y'\ z'\ 1]$$

or

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} * \begin{bmatrix} \text{cos è} & 0 & \text{sin è} & 0 \\ 0 & 1 & 0 & 0 \\ -\text{sinè} & 0 & \text{cosè} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix}$$

10.4.4. 3D Reflection

Some orientations of a 3D object cannot be obtained using pure rotations. In 3D reflection occurs through a plane and the value of the determinant for pure reflection matrix is always equal to – 1. Thus, there are two types of reflection transformation

1. Reflection relative to coordinate axis.
2. Reflection relative to planes.

Reflection at *y*-axis

When we are taking reflection about y-axis. We have to keep the magnitudes of *x*, *y* and *z* co-ordinates as it is. We have to change only signs of *x* and *z* co-ordinates. Its transformation matrix will be.

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Reflection at *x*-axis

Value of *x* is not changed and y and *z*'s sign is changed.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Reflection at *z*-axis

Value of *z* is not changed and *x* and *y*'s sign is chnaged. so, the transformation matrix is

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(*b*) **Reflection through plane**

Reflection through *xy* plane: In a reflection through the *xy* plane only the *z*-coordinate values of the object's position vectors change *i.e.*, they are reversed in sign. Thus, the transformation matrix for reflection through the *xy* plane is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Transformation matrix for reflection through *yz* plane: Here *y* and *z* is unchanged and sign of *x* is changed

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Reflection about *xz* plane: For reflecting object about *xz* plane *x* and *z* is unchanged and sign of *y* is changed. Transformation matrix for reflection aout *xz* plane is as follows:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

10.5. 3D SHEARING

The off-diagonal terms in the upper left 3 × 3 submatrix of the general 4 × 4 transformation matrix produce shear in 3D that is

$$[x' \; y' \; z' \; 1] = [x \; y \; z \; 1] \begin{bmatrix} 1 & a & b & 0 \\ c & 1 & d & 0 \\ e & f & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

or $$[x' \; y' \; z' \; 1] = [x + yc + ze \quad ax + y + zf \quad bx + dy + z \quad 1]$$

EXAMPLE 10.3.

A rectangular parallelopiped is given having length on *x*-axis, *y*-axis and *z*-axis as 3, 2 and 1 respectively. Perform a rotation by an angle – 90° about *x*-axis and an angle 90° about y-axis.

Solution: The matrix representation for parallelopiped is

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 3 & 0 & 1 & 1 \\ 3 & 2 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 1 \\ 3 & 2 & 0 & 1 \\ 0 & 2 & 0 & 1 \end{bmatrix}$$

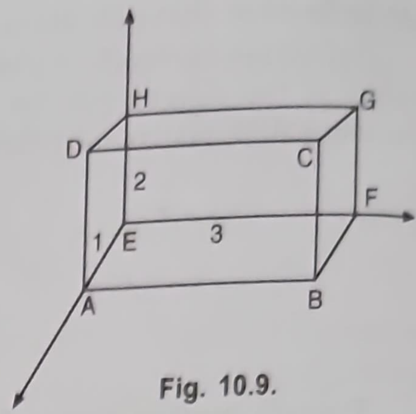

Fig. 10.9.

Rotation matrix about x-axis by angle $-90°$ is given as

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(-90°) & \sin(-90°) & 0 \\ 0 & -\sin(90°) & \cos(-90°) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 3 & 0 & 1 & 1 \\ 3 & 2 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 1 \\ 3 & 2 & 0 & 1 \\ 0 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 3 & 1 & 0 & 1 \\ 3 & 1 & -2 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 1 \\ 3 & 0 & -2 & 1 \\ 0 & 0 & -2 & 1 \end{bmatrix}$$

So, the new co-ordinates of rectangular parallelopiped as
(0, 1 ,0), (3, 1, 0), (3, 1, – 2), (0, 1, – 2), (0, 0, 0), (3, 0, 0), (3, 0, – 2), (0, 0, – 2)
Now, rotation matrix about y-axis by an angle 90°

$$\begin{bmatrix} \cos 90° & 0 & -\sin 90° & 0 \\ 0 & 1 & 0 & 0 \\ \sin 90° & 0 & \cos 90° & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

So,

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 3 & 0 & 1 & 1 \\ 3 & 2 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 1 \\ 3 & 2 & 0 & 1 \\ 0 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & -3 & 1 \\ 1 & 2 & -3 & 1 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 1 \\ 0 & 2 & -3 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

EXAMPLE 10.4.

For the given matrix

$$\begin{bmatrix} 2 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 \\ 4 & 0 & 1 & 0 \\ 0 & 3 & 6 & 1 \end{bmatrix}$$

first apply a rotation of 45° about the y-axis followed by a rotation of 45° about x-axis

Solution: Rotation matrix about y-axis at an angle 45° is

$$\begin{bmatrix} \cos 45^\circ & 0 & -\sin 45^\circ & 0 \\ 0 & 1 & 0 & 0 \\ \sin 45^\circ & 0 & \cos 45^\circ & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ 0 & 1 & 0 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

So,
$$\begin{bmatrix} 2 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 \\ 4 & 0 & 1 & 0 \\ 0 & 3 & 6 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ 0 & 1 & 0 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 3 & -1/\sqrt{2} & 0 \\ 5/\sqrt{2} & 0 & -3/\sqrt{2} & 0 \\ 6/\sqrt{2} & 3 & 6/\sqrt{2} & 1 \end{bmatrix}$$

Now, rotation about x-axis at angle 45°

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 45^\circ & \sin 45^\circ & 0 \\ 0 & -\sin 45^\circ & \cos 45^\circ & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

So, the final transformation matrix

$$\begin{bmatrix} 3/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 3 & -1/\sqrt{2} & 0 \\ 5/\sqrt{2} & 0 & -3/\sqrt{2} & 0 \\ 6/\sqrt{2} & 3 & 6/\sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3/\sqrt{2} & 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & \dfrac{(3\sqrt{2}+1)}{2} & \dfrac{(3\sqrt{2}-1)}{2} & 0 \\ 5/\sqrt{2} & 3/2 & -3/2 & 0 \\ 6/\sqrt{2} & \dfrac{(3-3\sqrt{2})}{2} & \dfrac{3+3\sqrt{2}}{2} & 1 \end{bmatrix}$$

EXAMPLE 10.5.

Perform reflection of unit cube about the xy plane.

Solution: The matrix representation of unit cube is

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

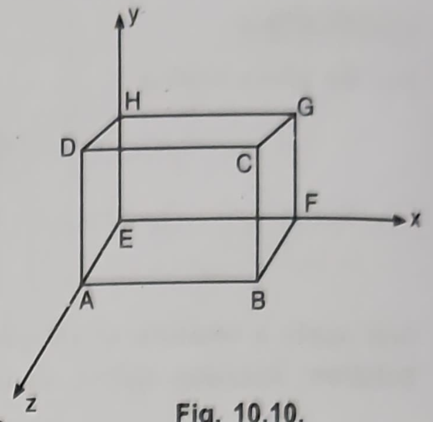

Fig. 10.10.

The transformation matrix for reflection about xy plane.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

After reflection about xy plane

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 1 & 0 & -1 & 1 \\ 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

So, the final co-ordinates are
(0, 0, – 1), (1, 0 – 1), (1, 1, – 1), (0, 1, – 1) (0, 0, 0) (1, 0, 0), (1, 1, 0), (0, 1, 0)

10.6. MULTIPLE TRANSFORMATIONS

Successive transformation can be combined or concatenated into a single 4 × 4 transformation that yields the same result. Since, matrix multiplication is non-commutative *i.e.*, the order of multiplication is important. In general

$$[A]\ [B] \neq [B]\ [A]$$

the order is determined by the position of the individual transformation matrix relative to the position vector matrix. The nearest to the position vector matrix generates the first individual transformation and the farthest generates the last transformation matrix.

10.6.1. Mirror Reflection with Respect to an Arbitrary Plane

Let the plane of reflection be specified by a normal vector N and a reference point P_0 (x_0, y_0, z_0). The mirror reflection with respect to xy plane can be done by the following steps

1. Translate P_0 to the origin
2. Align the normal vector N with the vector k normal to the xy plane.
3. Perform the mirror reflection in the xy plane.

4. Reverse steps 1 and 2.

So with translation vector $V = -x_0\hat{i} - y_0\hat{j} - z_0\hat{k}$, reflection matrix

$$\boxed{T_v^{-1} A_N^{-1} M A_N T_v}$$

Here A_N is the alignment matrix.

If the vector $N = n_1\hat{i} + n_2\hat{j} + n_3\hat{k}$ then

$$|N| = \sqrt{n_1^2 + n_2^2 + n_3^3} \text{ and } \lambda = \sqrt{n_2^2 + n_3^2}\text{, we find}$$

$$A_N = \begin{bmatrix} \frac{\lambda}{|N|} & \frac{-n_1n_2}{\lambda|N|} & \frac{-n_1n_3}{\lambda|N|} & 0 \\ 0 & \frac{n_3}{\lambda} & \frac{-n_2}{\lambda} & 0 \\ \frac{n_1}{|N|} & \frac{n_2}{|N|} & \frac{n_3}{|N|} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } A_N^{-1} = \begin{bmatrix} \frac{\lambda}{|N|} & 0 & \frac{n_1}{|N|} & 0 \\ \frac{-n_1n_2}{\lambda|N|} & \frac{n_3}{\lambda} & \frac{n_2}{|N|} & 0 \\ \frac{-n_1n_3}{\lambda|N|} & \frac{-n_2}{\lambda} & \frac{n_3}{|N|} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and
$$T_v = \begin{bmatrix} 1 & 0 & 0 & -x_0 \\ 0 & 1 & 0 & -y_0 \\ 0 & 0 & 1 & -z_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and
$$T_v^{-1} = \begin{bmatrix} 1 & 0 & 0 & x_0 \\ 0 & 1 & 0 & y_0 \\ 0 & 0 & 1 & z_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and homogeneous form of M is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

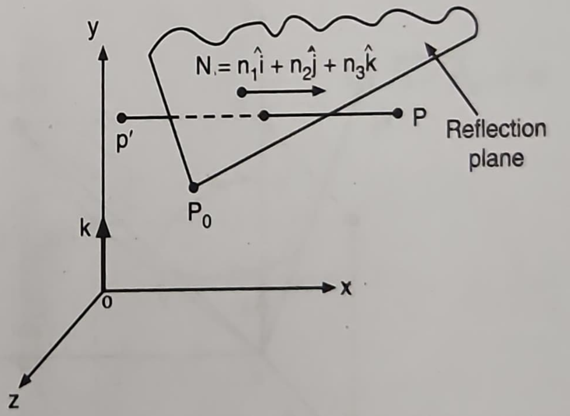

Fig. 10.11. Mirror reflection w.r.t. an arbitrary plane

EXAMPLE 10.6.

Define tilting transformation. Does the order of performing the rotation matter?

Solution: Tilting is a special transmormation, where the object first rotate about x-axis and then about y-axis. We can find the required transformation by composing two rotation matices, *i.e.,*

$$\begin{bmatrix} \cos\theta_y & 0 & \sin\theta_y & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x & 0 \\ 0 & \sin\theta_x & \cos\theta_x & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta_y & \sin\theta_y\sin\theta_x & \sin\theta_y\cos\theta_x & 0 \\ 0 & \cos\theta_x & -\sin\theta_x & 0 \\ -\sin\theta_y & \cos\theta_y\sin\theta_x & \cos\theta_y\cos\theta_x & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now, we multiply $R_{\theta_x} \cdot R_{\theta_y}$, we get

$$\begin{bmatrix} \cos\theta_y & 0 & \sin\theta_y & 0 \\ \sin\theta_x \sin\theta_y & \cos\theta_x & -\sin\theta_x \cos\theta_y & 0 \\ -\cos\theta_x \sin\theta_y & \sin\theta_x & \cos\theta_x \cos\theta_y & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This is not the same matrix, as $R_{\theta_y} \cdot R_{\theta_x}$. Thus ths order of rotation matrix is important.

EXAMPLE 10.7.