Module - I

Fundamentals of Logic

statement or propositions.

statements are declarative sentences that are either true or false but not both.

Eg: 1. $2+3=5$

2. $2>3$

3. In 2003 George W Bush was the president of the United states

4. Fifteen is an even number

5. If Jennifer is late for the party, then her cousin Zachary will be quite angry.

The following are not a statement.

1. $x+3$ is a positive integer.

2. What time is it?

3. What a beautiful Evening?

4. Get up and do your exercise?

Note: Usually we denote smaller case letters to represent a statement.

ie $p: 2+3 \neq 5$ etc.

Primitive statement

Primitive statements are statements which cannot be break into smaller statement.

New statement can be obtained from existing ones in two ways

① Transform a given statement p into the statement ¬p which denotes its negation and is read "Not p"

Eg: p: fifteen is an Even number

~p: fifteen is not an even number or

¬p: fifteen is an odd number.

② Combining two or more statements into a compound statement, using logical connectives.

1. Conjunction : Conjunction of two statement p and q is denoted by $p \wedge q$ and is read as p and q.

Truth table for $p \wedge q$

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Truth table for Negation

p	~p
T	F
F	T

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1. Let p, q, r, s denote the following statements

p : I finish writing my computer programme before lunch

q : I shall play tennis in the afternoon.

r : The sun is shining

s : The humidity is low.

Write the following in symbolic form:

1. If the sun is shining, I shall play tennis this afternoon.
2. Finishing the writing of my computer programme before lunch is necessary for my playing tennis this afternoon.
3. Low humidity and sunshine are sufficient for me to play tennis this afternoon.

Ans : ① $r \rightarrow q$ ② $q \rightarrow p$ ③ $(s \wedge r) \rightarrow q$.

2. <u>Disjunction</u>: Disjunction of two statements p and q is denoted by $p \vee q$ and is read as p or q.

Truth table for $p \vee q$

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

3. <u>Implication</u>: We say p implies q and is denoted by $p \rightarrow q$ and is read as ① if p then q ② p is sufficient for q ③ p is a sufficient condition for q ④ q is necessary for p ⑤ p only if q ⑥ q is a necessary condition for p.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

In $p \rightarrow q$ p is called a hypothesis and q is called the conclusion of the implication.

4. <u>BiConditional</u>: The biconditional of two statement p and q is denoted by $p \leftrightarrow q$ and is read as p if and only if q

Truth table for Conditional and Biconditional

p	q	$p \rightarrow q$	$p \leftrightarrow q$
T	T	T	T
T	F	F	F
F	T	T	F
F	F	T	T

2. Let p, q, r denote the following statement about a particular triangle ABC

p: Triangle ABC is isosceles
q: Triangle ABC is equilateral.
r: Triangle ABC is equiangular.

Translate the following into English sentences.

a) $q \to p$ b) $\neg p \to \neg q$ c) $q \to r$ d) $p \wedge \neg q$
e) $r \to p$

Ans: a) If Triangle ABC is equilateral then it is isosceles

b). If Triangle ABC is not isosceles then it is not equilateral

c) If Triangle ABC is equilateral then it is equiangular

d) Triangle ABC is isosceles but not equilateral.

e) If Triangle ABC is equiangular then it is isosceles.

③ Determine the truth value of the following implications.

a) If $3+4=12$ then $3+2=6$

b) if $3+3=6$ then $3+4=9$

c) If Thomas Jefferson was the third president of United states then $2+3=5$

Ans: a) T b) F c) T.

④ Construct the truth table for the following Compound statement.

H.W

a) $\sim(p \vee \sim q) \rightarrow \sim p$

b) $p \rightarrow (q \rightarrow r)$

c) $(p \rightarrow q) \rightarrow r$

d) $(p \rightarrow q) \rightarrow (q \rightarrow p)$

e) $[p \wedge (p \rightarrow q)] \rightarrow q$

f) $q \leftrightarrow (\sim p \vee \sim q)$

g) $(p \wedge q) \rightarrow p$.

5 Determine all the truth value assignment, if any, for the ~~truth table of the compound statement~~ primitive statement p, q, r, s, t that make each of the following compound statements false.

a) $[(p \wedge q) \wedge r] \rightarrow (s \vee t)$

Ans: $(p \wedge q) \wedge r \rightarrow (s \vee t)$ is false when $(p \wedge q) \wedge r$ is true and $s \vee t$ is false

ie p, q, r are must be true and s, t are must be false.

b)

Tautology and Contradiction

A compound statement is called Tautology if it is true for all truth value assignments for its compound statements. If a compound statement is false for all such assignments, then its called a contradiction.

Eg: $p \wedge (\neg p \wedge q)$ is a contradiction.

p	q	$\neg p$	$\neg p \wedge q$	$p \wedge (\neg p \wedge q)$
T	T	F	F	F
T	F	F	F	F
F	T	T	T	F
F	F	T	F	F

Logical Equivalence

Two statements s_1, s_2 are said to be logically equivalent, and we write $s_1 \Longleftrightarrow s_2$, when the statement s_1 is true (false) iff the statement s_2 is true (false). s_1 and s_2 have same truth value

① S.T $p \to q \Longleftrightarrow \neg p \vee q$

p	q	$\neg p$	$\neg p \vee q$	$p \to q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

② S.T $p \leftrightarrow q \Longleftrightarrow (p \to q) \wedge (q \to p)$

p	q	$p \leftrightarrow q$	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

3 a) $\neg(p \wedge q) \Longleftrightarrow \neg p \vee \neg q$

b) $\neg(p \vee q) \Longleftrightarrow \neg p \wedge \neg q$

p	q	$p \wedge q$	$p \vee q$	$\neg(p \wedge q)$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$\neg p \vee \neg q$
T	T	T	T	F	F	F	F	F	F
T	F	F	T	T	F	F	T	F	T
F	T	F	T	T	F	T	F	F	T
F	F	F	F	T	T	T	T	T	T

The Law of Logic

1. $\neg\neg p \Longleftrightarrow p$ Double negation

2. $\neg(p \vee q) \Longleftrightarrow \neg p \wedge \neg q$
   $\neg(p \wedge q) \Longleftrightarrow \neg p \vee \neg q$ } Demorgan's law

3. $p \vee q \Longleftrightarrow q \vee p$
   $p \wedge q \Longleftrightarrow q \wedge p$ } Commutative law

4. $p \vee (q \vee r) \Longleftrightarrow (p \vee q) \vee r$
   $p \wedge (q \wedge r) \Longleftrightarrow (p \wedge q) \wedge r$ } Associative law

5. $p \vee (q \wedge r) \Longleftrightarrow (p \vee q) \wedge (p \vee r)$
   $p \wedge (q \vee r) \Longleftrightarrow (p \wedge q) \vee (p \wedge r)$ } Distributive law

6. $p \vee p \Longleftrightarrow p$, $p \wedge p \Longleftrightarrow p$ — idempotent law

7. $p \vee F \Longleftrightarrow p$, $p \wedge T \Longleftrightarrow p$ identity law

8) $p \vee \neg p \Leftrightarrow T$, $p \wedge \neg p \Leftrightarrow F$ Inverse law

9) $p \vee T \Leftrightarrow T$, $p \wedge F \Leftrightarrow F$ Domination law

10) $p \vee (p \wedge q) \Leftrightarrow p$
$p \wedge (p \vee q) \Leftrightarrow p$ } Absorption law.

Dual of a statement

Dual of a stament 's' is denoted by s^d and is obtained by replacing '$\wedge$' by '$\vee$' '$\vee$' by '$\wedge$' and T&F by F and T respectively.

Eg: Write the Dual of $s: (p \wedge \neg q) \vee (r \wedge T)$

The Dual is $s^d: (p \vee \neg q) \wedge (r \vee F)$

Principle of Duality

Let s and t be statements that contains no logical connectives other than $\wedge$ and $\vee$. If $s \Leftrightarrow t$ then $s^d \Leftrightarrow t^d$

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Show that $p \rightarrow q \Longleftrightarrow \neg p \vee q$

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Show that $p \rightarrow q \Longleftrightarrow \neg q \rightarrow \neg p$

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg q \rightarrow \neg p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Inverse, Converse and Contrapositive

The inverse of $p \rightarrow q$ is $\neg p \rightarrow \neg q$

The Converse of $p \rightarrow q$ is $q \rightarrow p$

The Contrapositive of $p \rightarrow q$ is $\neg q \rightarrow \neg p$

Let p, q, r denote primitive sentences. Use truth tables to verify the following logical equivalences.

1. $p \to (q \wedge r) \Leftrightarrow (p \to q) \wedge (p \to r)$

p	q	r	$p \to q$	$p \to r$	$q \wedge r$ $\cancel{(p \to q)}$	$p \to (q \wedge r)$	$(p \to q) \wedge (p \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	F	T	F	F	F
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	T	T	F	T	T
F	F	T	T	T	F	T	T
F	F	F	T	T	F	T	T

HW 1. $[(p \vee q) \to r] \Leftrightarrow [(p \to r) \wedge (q \to r)]$

2. $[p \to (q \vee r)] \Leftrightarrow [\neg r \to (p \to q)]$

2. Negate and Simplify the compound statement $(p \vee q) \to r$.

$(p \vee q) \to r \Leftrightarrow \sim(p \vee q) \vee r$

$\sim[(p \vee q) \to r] \Leftrightarrow \sim[\sim(p \vee q) \vee r]$; $p \to q \Leftrightarrow \neg p \vee q$

$\Leftrightarrow \sim\sim(p \vee q) \wedge \sim r$; Demorgans law

$\Leftrightarrow (p \vee q) \wedge \sim r$; law of double negation

3 Simplify the following Compound statement

$(p \vee q) \wedge \neg(\neg p \wedge q)$

$(p \vee q) \wedge \neg(\neg p \wedge q) \Leftrightarrow \cancel{(p \vee q) \wedge (p \vee \neg q)}$

$\Leftrightarrow (p \vee q) \wedge \neg\neg p \vee \neg q$, Demorgan's law

$\Leftrightarrow (p \vee q) \wedge p \vee \neg q$, double negation

$\Leftrightarrow p \vee (q \wedge \neg q)$, distributive law

$\Leftrightarrow p \vee F$, Inverse law

$\Leftrightarrow p$, identity law.

4 $\neg[\neg[(p \vee q) \wedge r] \vee \neg q]$

$\neg[\neg[(p \vee q) \wedge r] \vee \neg q] \Leftrightarrow \neg[\neg[(p \vee q) \wedge r] \wedge \neg\neg q]$, Demorgans law

$\Leftrightarrow \neg\neg[(p \vee q) \wedge r] \wedge q$, Demorgans law & Double negation

$\Leftrightarrow (p \vee q) \wedge (r \wedge q)$, Double negation

$\Leftrightarrow (p \vee q) \wedge (q \wedge r)$, Commutative law

$\Leftrightarrow [(p \vee q) \wedge q] \wedge r$, Associative law

$\Leftrightarrow q \wedge r$, Absorption law

$(p \vee q) \wedge (r \wedge q)$

$(p \vee q) \wedge (q \wedge r)$

$[(p \vee q) \wedge q] \wedge r$

(3) Negate and Simplify the following statement.

a) $(p \wedge q) \to r$

$$\neg[(p \wedge q) \to r] \iff \neg[\neg(p \wedge q) \vee r] \quad ; \; p \to q \iff \neg p \vee q$$

$$\iff \neg\neg(p \wedge q) \wedge \neg r \quad ; \text{ Demorgan's law}$$

$$\iff (p \wedge q) \wedge \neg r \quad ; \text{ Double negation.}$$

b) $p \to (\neg q \wedge r)$

$$\neg[p \to (\neg q \wedge r)] \iff \neg[\neg p \vee (\neg q \wedge r)] \quad ; \; p \to q \iff \neg p \vee q$$

$$\iff \neg\neg p \wedge \neg(\neg q \wedge r) \quad ; \text{ Demorgans law}$$

$$\iff p \wedge (\neg\neg q \vee \neg r) \quad ; \text{ Double negation \& Demorgans law}$$

$$\iff \underline{\underline{p \wedge (q \vee \neg r)}} \quad ; \text{ Double negation.}$$

c) $(p \vee q) \vee (\neg p \wedge \neg q \wedge r)$

$$\neg[(p \vee q) \vee (\neg p \wedge \neg q \wedge r)]$$

$$\iff \neg(p \vee q) \wedge \neg[(\neg p \wedge \neg q) \wedge r] \quad ; \text{ De morgan's law}$$

$$\iff (\neg p \wedge \neg q) \wedge \neg[(\neg p \wedge \neg q) \wedge r]$$

$$\iff (\neg p \wedge \neg q) \wedge$$

$\sim[(p\vee q)\vee((\neg p\wedge\neg q)\wedge r)]$

$\Leftrightarrow \sim[(p\vee q)\vee((\sim(p\vee q)\wedge r))]$

$\Leftrightarrow \sim[((p\vee q)\vee\sim(p\vee q))\wedge(p\vee q)\vee r]$; $p\vee\sim p\Leftrightarrow T$

$\Leftrightarrow \sim[T\wedge(p\vee q)\vee r]$

$\Leftrightarrow \sim[(p\vee q)\vee r]$

$\Leftrightarrow \sim(p\vee q)\wedge\sim r$

$\Leftrightarrow \underline{\underline{\sim p\wedge\sim q\wedge\sim r}}$

d) $p\wedge(q\vee r)\wedge(\sim p\vee\sim q\vee r)$

$\sim[p\wedge(q\vee r)\wedge(\sim p\vee\sim q\vee r)]$

$\Leftrightarrow [\sim p\vee\sim(q\vee r)]\vee\sim[(\sim p\vee\sim q)\vee r]$; Demorgan's law

$\Leftrightarrow [\sim p\vee\sim(q\vee r)]\vee(p\wedge q\wedge\sim r)$

$\Leftrightarrow [\sim p\vee(\sim q\wedge\sim r)]\vee(p\wedge q\wedge\sim r)$

$\Leftrightarrow (\sim q\wedge\sim r)\vee\sim p\vee[(p\wedge q\wedge\sim r)]$; Assosiative law

$\Leftrightarrow (\sim q\wedge\sim r)\vee[(\sim p\vee p)\wedge\sim p\vee(q\wedge\sim r)]$, Distributive law

$\Leftrightarrow (\sim q\wedge\sim r)\vee[T\wedge\sim p\vee(q\wedge\sim r)]$

$\Leftrightarrow (\sim q\wedge\sim r)\vee[\sim p\vee(q\wedge\sim r)]$

$\Leftrightarrow \sim p\vee[(\sim q\wedge\sim r)\vee(q\wedge\sim r)]$, Assosiative law

$\Leftrightarrow \sim p\vee((\sim q\vee q)\wedge\sim r)$, Distributive law

$\Leftrightarrow \sim p\vee(T\wedge\sim r)$ $\Leftrightarrow$ $\underline{\underline{\sim p\vee\sim r}}$

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(6) If p and q are primitive statement then

HW show that $(\neg p \vee q) \wedge (p \wedge (p \wedge q)) \Leftrightarrow p \wedge q$.

(Using truth table)

(7) Write the dual of the following statement.

(1) $(\neg p \vee q) \wedge p \wedge (p \wedge q) \Leftrightarrow p \wedge q$

The dual is $(\neg p \wedge q) \vee (p \vee (p \vee q)) \Leftrightarrow p \vee q$.

(2) $q \to p$

$q \to p \Leftrightarrow \neg q \vee p$.

$\boxed{{}^{*} p \to q \Leftrightarrow \neg p \vee q}$

$\therefore (q \to p)^d \Leftrightarrow \neg q \wedge p$.

(3) $p \to (q \wedge r)$

$p \to (q \wedge r) \Leftrightarrow \sim p \vee (q \wedge r)$

$\therefore (p \to (q \wedge r))^d \Leftrightarrow \sim p \wedge (q \vee r)$

(4) $p \leftrightarrow q$ $\cancel{\Rightarrow}$

$p \leftrightarrow q \Leftrightarrow (p \to q) \wedge (q \to p)$

$\Leftrightarrow (\neg p \vee q) \wedge (\neg q \vee p)$

$\Leftrightarrow (\neg p \wedge q) \vee (\neg q \wedge p)$

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8. For the primitive statements p, q

(a) Verify that $p \to (q \to (p \wedge q))$ is a tautology.

(b) Verify that $(p \vee q) \to (q \to q)$ is a tautology by using the result from part (a).

(a)

p	q	$p \wedge q$	$q \to (p \wedge q)$	$p \to (q \to (p \wedge q))$
T	T	T	T	T
T	F	F	T	T
F	T	F	F	T
F	F	F	T	T

b) Replace p by $p \vee q$ in (a) we get

$$(p \vee q) \to (q \to (p \vee q) \wedge q)$$

$$\Leftrightarrow (p \vee q) \to (q \to q) \quad \text{, absorption law}$$

$$\Leftrightarrow T$$

9. Simplify the following compound statement.

(a) $[(p \vee q) \wedge (p \vee \neg q)] \vee q$

Ans

$$[(p \vee q) \wedge (p \vee \neg q)] \vee q \Leftrightarrow [p \vee (q \wedge \neg q)] \vee q \text{ ; distributive property}$$

$$\Leftrightarrow (p \vee F) \vee q \text{ , Inverse law}$$

$$\Leftrightarrow p \vee q \text{ , } \sim\sim\text{Demorgan's law}\sim\sim \text{ identity law}$$

b) $(p \to q) \wedge [\neg q \wedge (r \vee \neg q)]$

$(p \to q) \wedge [\neg q \wedge (r \vee \neg q)]$

$\Leftrightarrow (p \to q) \wedge \neg q$, Absorption law

$\Leftrightarrow (\neg p \vee q) \wedge \neg q$, $p \to q \Leftrightarrow \neg p \vee q$.

$\Leftrightarrow (\neg p \wedge \neg q) \vee (q \wedge \neg q)$; Distributive law

$\Leftrightarrow \neg (p \vee q) \vee F$; Demorgan's law & Inverse law

$\Leftrightarrow \neg (p \vee q)$; identity law.

Without Using truth table Establish the following logical Equivalences:

a) $p \vee [p \wedge (p \vee q)] \Leftrightarrow p$.

$p \vee [p \wedge (p \vee q)] \Leftrightarrow p \vee p$; absorption law

$\Leftrightarrow p$; idempotent law.

b) $p \vee q \vee [\neg p \wedge \neg q \wedge r] \Leftrightarrow p \vee q \vee r$

$p \vee q \vee [\neg p \wedge \neg q \wedge r] \Leftrightarrow (p \vee q) \vee [\neg (p \vee q) \wedge r]$; Demorgan's law

$\Leftrightarrow [(p \vee q) \vee \neg (p \vee q)] \wedge [(p \vee q) \vee r]$; Distributive law

$\Leftrightarrow T \wedge [(p \vee q) \vee r]$, inverse law

$\Leftrightarrow p \vee q \vee r$, identity law.

Logical Implication : Rules of Inference.

A set of arguments $p_1, p_2, \ldots p_n$ are said to be valid, Consider the implications $(p_1 \wedge p_2 \wedge \ldots p_n) \rightarrow q$ where q is the Conclusion for the argument. if $p_1, p_2, \ldots p_n$ is true then the Conclusion q is true.

Note: If any one of $p_1, p_2, \ldots p_n$ is false then the hypothesis $p_1 \wedge p_2 \wedge \ldots \wedge p_n$ is false and the implication $(p_1 \wedge p_2 \ldots \wedge p_n) \rightarrow q$ is automatically true, regardless of the truth value of q.

$\therefore$ To Establish the validity of a given argument is to show that the statement $(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \rightarrow q$ is a tautology.

Problems

Check the validity of the following argument (Using Truth table)

① If Roger studies, then he will pass discrete mathematics. If Roger doesn't play cricket then he will study. Roger failed discrete mathematics. Therefore Roger ~~passes discrete mathematics~~ plays cricket.

<u>Ans</u> :

p : Roger studies

q : Roger plays Cricket

r : Roger passes discrete mathematics.

$P_1 : p \to r$

$P_2 : \neg q \to p$

$P_3 : \neg r$

$c : q$

p	q (c)	r	$\neg q$	$\neg r$ (P_3)	$p \to r$ (P_1)	$\neg q \to p$ (P_2)	$((p\to r) \wedge (\neg q \to p) \wedge \neg r) \to q$ ($(P_1 \wedge P_2 \wedge P_3) \to c$)
T	T	T	F	F	T	T	T
T	T	F	F	T	F	T	T
T	F	T	T	F	T	T	T
T	F	F	T	T	F	T	T
F	T	T	F	F	T	T	T
F	T	F	F	T	T	T	T
F	F	T	T	F	T	F	T
F	F	F	T	T	T	F	T

(a) Check the validity for the following arguments

$P_1 : p \quad ; \quad P_2 : (p \wedge r) \to s \quad c : r \to s.$

P_1				P_2	c	$(P_1 \wedge P_2) \to c$
p	r	s	$p \wedge r$	$(p \wedge r) \to s$	$r \to s$	$p \wedge ((p \wedge r) \to s) \to r \to s$
T	T	T	T	T	T	T
T	T	F	T	F	F	T
T	F	T	F	T	T	T
T	F	F	F	T	T	T
F	T	T	F	T	T	T
F	T	F	F	T	F	T
F	F	T	F	T	T	T
F	F	F	F	T	T	T

Hence P_1 & P_2 are valid argument from the conclusion c

If p and q are arbitrary statements such that $p \to q$ is a tautology, then we say that p logically implies q and written as $p \Rightarrow q$

Eg: $p \Rightarrow p \vee q$ Law of Disjunction.

$q \Rightarrow p \vee q$

$p, p \to q \Rightarrow q$ – Modus Ponens or Rule of Detachment

$p \to q, q \to r \Rightarrow p \to r$ – Law of syllogism.

Check the Validity of the followings Arguments

(1) Rita is baking a cake. If Rita is baking a cake then she is not practicing her piano. If Rita is not practicing her piano then her father will not buy her a car. Therefore Ritas father will not buy her a car.

Ans :

p : Rita is baking ~~ba~~ a cake

q : Rita is not practicing her piano

r : Rita's father will not buy her a car.

The arguments are

$P_1 : p$

$P_2 : p \to q$

$P_3 : q \to r$

$c : r$

step		Reason
1.	p	premise
2.	$p \to q$	premise
3.	q	Modus ponens (1) & (2)
4.	$q \to r$	premise
5.	r	Modus ponens (3) & (4)

2. Show that the following argument is valid from the primitive statements p, r, s, t and u.

$p \to r,\ r \to s,\ t \vee \neg s,\ \neg t \vee u,\ \neg u \Rightarrow \neg p.$

Step		Reason.
1.	$p \to r$	premise
2.	$r \to s$	premise
3	$p \to s$	Law of syllogism 2 & 3
4.	$t \vee \neg s$	premise
5.	$\neg s \vee t$	Commutative property
6.	$s \to t$	Using 4 $p \to q \Leftrightarrow \neg p \vee q$
7.	$p \to t$	Law of syllogism 3 & 6
8.	$\neg t \vee u$	premise.
9	$t \to u$	$p \to q \Leftrightarrow \neg p \vee q$.
10.	$p \to u$	Law of syllogism 7 & 9
11	$\neg u \to \neg p$	Contrapositive law 10
12.	$\neg u$	premise.
13	$\neg p$	Modus ponens 10 & 12

③ check the validity of the following Arguments

$p \to r$
$\neg p \to q$
$q \to s$

$\therefore \neg r \to s$

step		Reason
1.	$p \to r$	premise
2.	$\neg p \to q$	premise.
3.	$q \to s$	premise
4.	$\neg p \to s$	Law of syllogism ② & ③
5.	$\neg s \to p$	Law of Contrapositive $p \Rightarrow q \Leftrightarrow \neg q \to \neg p$
6.	$\neg s \to r$	Law of syllogism ① & ⑤
7.	$\neg r \to s$	Law of Contrapositive

④ Establish the validity of the argument

$p \to q$
$q \to (r \wedge s)$
$\neg r \vee (\neg t \vee u)$
$p \wedge t$

$\therefore u$

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	step	Reason
1.	$p \to q$	premise
2.	$q \to (r \wedge s)$	premise.
3.	$p \to (r \wedge s)$	Law of syllogism
4.	$p \wedge t$	premise.
5.	p	Conjuctive Simplification.
6.	$r \wedge s$	Modus ponens (3) & (5)
7.	r	Conjuctive Simplification.
8.	$\neg r \vee (\neg t \vee u)$	premise
9.	$r \to (\neg t \vee u)$	$p \to q \Leftrightarrow \neg p \vee q$
10.	$\neg t \vee u$	Law of syllogism (7) & (9)
11.	$t \to u$	$p \to q \Leftrightarrow \neg p \vee q$
12.	t	Conjuctive Simplification
13.	u	Modus ponens (11) & (12)

⑤ If the band couldn't play rock music or the refreshments were not delivered on time then the New Year's party would have been canceled and Alicia would have been angry. If the party were cancelled, then refund would have had to be made. No refund were made. Therefore the band could play rock music.

p: The band could play rock music
q: The refreshment were delivered on time.
r: The New Years party was cancelled
s. Alicia was angry.
t: Refunds had to be made.

P_1: $(\neg p \vee \neg q) \rightarrow (r \wedge s)$

P_2: $r \rightarrow t$

P_3: $\neg t$

$\therefore p$

Step	Reason
1. $r \to t$	premise.
2. $\neg t$	premise
3. $\neg t \to \neg r$	Contrapositive.
4. $\neg r$	Modus ponens 2 & 3
5. $(\neg p \vee q) \to (r \wedge s)$	premise.
6. $\neg r \vee \neg s$	Disjunctive amplification.
7. $\neg(r \wedge s)$	Demorgan's law
8. $\neg(r \wedge s) \to \neg(\neg p \vee q)$	Contrapositive. 5
9. $\neg(\neg p \vee q)$	Modus ponens. 7 & 8
10. $p \wedge \neg q$	Demorgan's law. 9
11. p	Conjunctive Simplification. 10

6 Using Method of Contradiction check the validity of the following Arguments.

$$\begin{array}{c} \neg p \leftrightarrow q \\ q \to r \\ \neg r \\ \hline \therefore p \end{array}$$

Step	Reasons
1. $\neg p \leftrightarrow q$	premise.
2. $(\neg p \to q) \wedge (q \to \neg p)$	$p \leftrightarrow q \Leftrightarrow (p \to q) \wedge (q \to p)$ ①
3. $\neg p \to q$	Conjunctive Simplification
4. $q \to r$	premise.
5. $\neg p \to r$	Syllogism ③ & ④
6. $\neg p$	Assumed premise.
7. $\neg r$	premise.
8. $\neg r \to p$	Contrapositive ⑤
9. $\cancel{p}\, r$	Modus ponens ⑤ & ⑥
10. $r \wedge \neg r \ (\Leftrightarrow F)$	Rules of Conjunction.
11 $\therefore p$	Method of proof by Contradiction.

⑦ Establish the validity of the following argument

$$\begin{array}{l} u \to r \\ (r \wedge s) \to (p \vee t) \\ q \to u \wedge s \\ \neg \cancel{s} t \\ \hline \therefore q \to p \end{array}$$

Step		Reason
1	q	Assumed premise
2	$q \to u \wedge s$	premise
3	$u \wedge s$	Modus ponens 1 & 2
4	u	Conjunctive Simplification
5	$u \to r$	premise
6	r	Modus ponens 5 & 6
7	s	Conjunctive Simplification 3
8	$r \wedge s$	Rule of Conjunction 6 & 7
9	$(r \wedge s) \to (p \vee t)$	premise
10	$p \vee t$	Modus ponens.
11	$\neg t$	premise
12	$\therefore p$	Disjunctive syllogism

8. Establish the Validity of the following arguments

$$
\begin{array}{l}
p \\
p \vee q \\
q \to (r \to s) \\
t \to r \\
\hline
\therefore \neg s \to \neg t
\end{array}
$$

	Step	Reason
1	p	premise
2	$p \vee q$	premise
3	$\neg p \to q$	$p \to q \Leftrightarrow \neg p \vee q$ (2)
4	$q \to (r \to s)$	premise.
5	$\neg p \to (r \to s)$	Syllogism (3) & (4)
6	$p \vee (r \to s)$	$p \to q \Leftrightarrow \neg p \vee q$
7	$p \vee (\neg r \vee s)$	$p \to q \Leftrightarrow \neg p \vee q$
8	$(p \vee \neg r) \vee s$	Associative property
9	$\neg s$	Assumed premise.
10	$p \vee \neg r$	Disjunctive syllogism
11	$\neg r$	Disjunctive syllogism (1) & (10)
12	$t \to r$	premise.
13	$\therefore \neg t$	Modus tollens

(8) Show that the following arguments are valid

$$\begin{array}{l} p \\ p \to q \\ s \vee r \\ r \to \neg q \\ \hline \therefore s \vee t \end{array}$$

Step		Reason
(1)	p	premise
(2)	$p \to q$	premise
(3)	q	Modus ponens (1) & (2)
(4)	$s \vee r$	premise.
(5)	s	Disjunctive simplification
(6)	$s \vee t$	Disjunctive amplification.

The Use of Quantifiers

The number $x+2$ is an even integer is not necessarly true or false unless we know that what value is substituted for x. If we restrict our choices to integers, then x is replaced by $-5, -2, 4,$ for instance etc then resulting statement is false. Also it is false whenever x is replaced by any odd integer. When an even integer is substituted for x, the resulting statement is true

Open statement.

A declarative sentence is an open statement if

① it contains one or more variables.

② it is not a statement but

③ it becomes a statement when the variables in it are replaced by certain allowable choices.

The allowable choices constitute what is called the universe or universe of discourse for the open statement.

Eg: "The number $x+2$ is an even integer" is denoted by $p(x)$ or $q(x)$ etc

$\neg P(x)$ may be read as $x+2$ is not an even integer

$q(x, y)$: The number $y+2$, $x-y$ and $x+2y$ are even integers.

$q(4, 2)$: The numbers 4, 2, 8 are even integers (True)

$P(5)$: The number $5+2=7$ is an even integer (False)

Therefore we can make the following true statements ① for some x, $P(x)$

② for some x, y $q(x, y)$

Note that in this situation, the statements "for some x, $\neg P(x)$" and "for some x, y, $\neg q(x, y)$" are also true

Since the statements "for some x, $P(x)$" and "for some x, $\neg P(x)$" are true, The second statement is not the negation of first statement.

The phrases "For some x" and "For some x, y" are said to quantify the open statement $p(x)$ and $q(x, y)$ respectively.

Free Variable and bound variable

In the open statement $p(x)$, the variable x is called a free variable. As x varies over the universe for an open statement, the truth value of the statement may vary

Eg: $P(x)$: x is an even integer

$P(5)$ is false and $P(6)$ is true

$r(x)$: $2x$ is an even integer.

Here the statement is true for every integer x.

In the open statement $P(x)$ the statement $\exists x\, P(x)$ has a fixed truth value true.

In $\exists x\, P(x)$ the variable x is said to be a bound variable.

Also $\forall x\, r(x)$, $\forall x\, \neg r(x)$, in each case x is a bound variable. ~~for~~

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Universal Quantifier and Existential Quantifier

"For some x" can also be expressed as "For atleast one x" or "There exist x such that" This quantifier is written in symbolic form as $\exists x$ called Existential quantifier.

For some x $P(x)$ can be written as $(\exists x)\ P(x)$.

The Universal quantifier is denoted by $\forall x$ and is read as "for all x", "for any x"

From the previous example $\forall x\ P(x)$ is a false statement.

Eg: $p(x)$: "$2x$ is an even integer" with some universe (of all integers), then the statement $\forall x\ P(x)$ is a true statement. Also $\exists x\ P(x)$ is a true statement. where as $\forall x\ \neg p(x)$ and $\exists x\ \neg P(x)$ are both false.

Eg: Consider the following Open statement $p(x)$, $q(x)$, $r(x)$ and $s(x)$ given by

$x = 4, -1$

$p(x): x \geq 0$ $\quad q(x): x^2 \geq 0$ $\quad r(x): x^2 - 3x - 4 = 0$

$s(x): x^2 - 3 > 0$. The following statements are true.

1. $\exists x\, [p(x) \wedge r(x)]$
2. $\forall x\, (p(x) \rightarrow q(x))$
3. $\exists x\, [p(x) \rightarrow q(x)]$

The statement $\forall x\, (p(x) \rightarrow q(x))$ also read as for every real number x, if $x \geq 0$ then $x^2 \geq 0$

The statement

$\forall x\, [q(x) \rightarrow s(x)]$. is false.

$q(1) \rightarrow s(1)$ is false $\quad q(1): 1^2 > 0$

$s(1): 1^2 - 3 > 0$

$q(1)$ is true and $s(1)$ is false

$\therefore q(1) \rightarrow s(1)$ is false.

$\forall x\, [r(x) \vee s(x)]$ is a false statement.

$r(1): 1^2 - 3 - 4 \neq 0$ is false

$s(1): 1^2 - 3 = 1 - 3 = -2 > 0$ is false

$\therefore s(1) \vee r(1)$ is false.

~~$\exists x$~~ $\exists x\,(r(x) \vee s(x))$ is true

is $\forall x\, P(x) \Rightarrow \exists x\, P(x)$. but $\exists x\, P(x)$ doesn't logically imply $\forall x\, P(x)$

Eg: a) Consider the universe of all real numbers and examine the sentences

1) If a number is rational, then it is a real number
2) If x is rational, then x is real

$P(x)$: x is a rational number
$q(x)$: x is a real number

$\forall x\,(P(x) \rightarrow q(x))$ is true

b) For the Universe of all triangles in the plane, the sentence "An equilateral triangle has three angles of 60°, and conversely"

$e(t)$: Triangle t is equilateral
$a(t)$: Triangle t has three angles of 60°

is $\forall t\,(e(t) \leftrightarrow a(t))$

c) $\sin^2 x + \cos^2 x = 1$.

$\forall x\,(\sin^2 x + \cos^2 x = 1)$

d) Consider the Universe of all positive integers and the sentence "The integer 41 is equal to sum of two perfect squares".

$\exists x\,\exists y\,[x^2 + y^2 = 41]$

e) Consider the open statement $p(x): x^2 \geq 1$, if the Universe consists of all positive integers then the quantified statement $\forall x\, p(x)$ is true. For the Universe of all positive real numbers $\forall x\, p(x)$ is false. Hence $\exists x\, P(x)$ is true.

Let $P(x)$ and $q(x)$ be open statements defined for a given universe then $P(x)$ and $q(x)$ are called logically equivalent if $\forall x\,[P(x) \Leftrightarrow q(x)]$ when the biconditional $P(a) \Leftrightarrow q(a)$ is true for each replacement of a

Inverse, Converse and Contrapositive

For Open statements $p(x)$, $q(x)$ - defined for a prescribed universe - and the universally quantified statement $\forall x\,[p(x) \rightarrow q(x)]$

The inverse of $\forall x\,[p(x) \rightarrow q(x)]$ is $\forall(x)\,[\neg p(x) \rightarrow \neg q(x)]$

The Converse of $\forall x\,[p(x) \rightarrow q(x)]$ is $\forall x\,[q(x) \rightarrow p(x)]$

The Contrapositive of $\forall x\,[p(x) \rightarrow q(x)]$ is $\forall x\,[\neg q(x) \rightarrow \neg p(x)]$

Eg: For the Universe of all quadrilaterals in the plane let $s(x)$ and $e(x)$ denote the open statements

$s(x)$: x is a square
$e(x)$: x is equilateral.

The statement $\forall(x)\,(s(x) \rightarrow e(x))$ is a true statement and is logically equivalent to its Contrapositive $\forall x\,[\neg e(x) \rightarrow \neg s(x)]$

The statement $\forall(x)\,(e(x) \rightarrow s(x))$ is a false statement and its Converse $\forall x.\,(s(x) \rightarrow e(x))$ is true.

Eg: The Open statements $p(x)$ and $q(x)$ given below

$p(x) : |x| > 3$ and $q(x) : x > 3$ and the Universe consists of all real numbers then

(a) $\forall x \, (p(x) \rightarrow q(x))$ is a false statement

if $x = -5$ $p(-5)$ is true and $q(-5)$ is false

$\therefore$ $p(-5) \rightarrow q(-5)$ is false

The Converse of $\forall x \, (p(x) \rightarrow q(x))$ is
$\forall x \, (q(x) \rightarrow p(x))$, a true statement.

The inverse of $\forall x \, (p(x) \rightarrow q(x))$ is
$\forall x \, (\neg p(x) \rightarrow \neg q(x))$ is a true statement.

The Contrapositive of $\forall x \, (p(x) \rightarrow q(x))$ is
$\forall x \, (\neg q(x) \rightarrow \neg p(x))$ is a false statement

$x = -4$, $\neg q(-4)$ is true and $\neg p(-4)$ is false

Hence $\neg q(-4) \rightarrow \neg p(-4)$ is false.

The Universe Consists of all the integers ~~then~~ and the Open statements $r(x)$ and $s(x)$ defined by

$r(x): 2x+1=5 \qquad s(x): x^2=9$

Here $\exists x\,[r(x) \wedge s(x)]$ is false because there is no integer a such that

$2a+1=5$ and $a^2=9$

$a=2$ and $a=\pm 3$.

ie $r(2)$ is true and $s(3)$ is true

Hence $\exists x(r(x)) \wedge \exists x(s(x))$ is true

Hence $\exists x\,[r(x) \wedge s(x)] \not\Leftrightarrow \exists x(r(x)) \wedge \exists x\, s(x)$

ie whenever $\exists x\,(p(x) \wedge q(x))$ is true it follows that $\exists x\,(p(x) \wedge q(x)) \Rightarrow \exists x\, p(x) \wedge \exists x\, q(x)$

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Logical Equivalence and Implications for Quantified statement in one variable.

$$\exists x\,(p(x) \wedge q(x)) \Rightarrow \exists x\, p(x) \wedge \exists x\, q(x)$$

$$\exists x\,(p(x) \vee q(x)) \Leftrightarrow \exists x\, p(x) \vee \exists x\, q(x)$$

$$\forall x\,(p(x) \wedge q(x)) \Leftrightarrow \forall x\, p(x) \wedge \forall x\, q(x)$$

$$\cancel{\forall x\, p(x) \vee \forall x\, q(x)}$$

$$\forall x\, p(x) \vee \forall x\, q(x) \Rightarrow \forall x\,(p(x) \vee q(x))$$

Rules for Negating statements with One Quantifier

$$\neg \forall x\, p(x) \Leftrightarrow \exists x\, \neg p(x)$$

$$\neg \exists x\, p(x) \Leftrightarrow \forall x\, \neg p(x).$$

$$\neg \forall x\, \neg p(x) \Leftrightarrow \exists x\, p(x)$$

$$\neg \exists x\, \neg p(x) \Leftrightarrow \forall x\, p(x).$$

Eg. Let $p(x)$ and $q(x)$ be given by (the universe is set of all integers)

$p(x)$: x is odd $q(x)$: x^2-1 is even.

The statement "if x is odd then x^2-1 is even" can be symbolized as $\forall x\,[p(x) \rightarrow q(x)]$

The negation of the statement is given by

$$\neg \forall x\,(p(x) \to q(x))$$

$$\Longleftrightarrow \exists x\, \neg (p(x) \to q(x))$$

$$\Longleftrightarrow \exists x\, \neg (\neg p(x) \vee q(x))$$

$$\Longleftrightarrow \exists x\, (\neg\neg p(x) \wedge \neg q(x))$$

$$\Longleftrightarrow \exists x\, (p(x) \wedge \neg q(x))$$

ie there exist an integer x such that x is odd and $x^2 - 1$ is ~~even~~ odd (the statement is false)

Eg: Let $r(x)$ and $s(x)$ be two open statements where x is the universe of integers defined by $r(x): 2x+1 = 5$

$s(x): x^2 = 9$.

The statement $\exists x\,(r(x) \wedge s(x))$ is false

Its negation $\neg \exists x\,(r(x) \wedge s(x))$

$$\Longleftrightarrow \forall x\, \neg (r(x) \wedge s(x))$$

$$\Longleftrightarrow \forall x\, (\neg r(x) \vee \neg s(x))$$

The negation is a true statement.

ie for every integer x, $2x+1 \neq 5$ or $x^2 \neq 9$

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Eg: For the Universe of integers, Consider the statement " There exist integers x, y such that $x+y=6$"

It can be represented by $\exists x \exists y\,(x+y=6)$

Eg: For the Universe of all integers and let $p(x,y)$ denote the open statement " $x+y=17$", then the statement $\forall x \exists y\, p(x,y)$ says that " For every integer x, there exist an integer y such that $x+y=17$". The statement is true, once we select x, the integer $y=17-x$ does exist. Also the statement $\exists y \forall x\, p(x,y)$ is false $\because$ Once an integer y is selected, the only value of x can have is $17-y$.

Eg: Write the negation of the statement

$\forall x \exists y\, [(p(x,y) \wedge q(x,y)) \rightarrow r(x,y)]$

$\neg[\forall x \exists y\, [(p(x,y) \wedge q(x,y)) \rightarrow r(x,y)]]$

$\Leftrightarrow \exists x \neg [\exists y [p(x,y) \wedge q(x,y)] \rightarrow r(x,y)]$

$\Leftrightarrow \exists x \forall y\, [\neg(p(x,y) \wedge q(x,y)) \rightarrow r(x,y)]$

$$\Leftrightarrow \exists x \forall y \neg [\neg (p(x,y) \wedge q(x,y)) \vee r(x,y)]$$

$$\Leftrightarrow \exists x \forall y [\neg \neg ((p(x,y) \wedge q(x,y)) \wedge \neg r(x,y)]$$

$$\Leftrightarrow \exists x \forall y [p(x,y) \wedge q(x,y) \wedge \neg r(x,y)]$$

The Rule of Universal Specification (US) and Rule of Universal Generalization.

$\forall x\, P(x) \Rightarrow P(a)$ (U.S)

$P(a) \Rightarrow \forall x\, P(x)$ (U.G)

$\exists x\, P(x) \Rightarrow P(a)$ (E.S)

$P(a) \Rightarrow \exists x\, P(x)$ (E.G)

Eg: If $p(x)$, $q(x)$ and $r(x)$ be open statements then show that the following arguments are valid

$$\forall x [p(x) \rightarrow q(x)],\ \forall x [q(x) \rightarrow r(x)] \Rightarrow \forall x\, p(x) \rightarrow r(x)$$

	Step	Reasons.
1.	$\forall x [p(x) \rightarrow q(x)]$	premise.
2.	$p(a) \rightarrow q(a)$	U.S
3.	$\forall x [q(x) \rightarrow r(x)]$	premise.
4.	$q(a) \rightarrow r(a)$	U.S
5.	$p(a) \rightarrow r(a)$	Law of the syllogisms 2 & 4
6.	$\forall x (p(x) \rightarrow r(x))$	U.G. 5

Check the validity of the following argument. (23)

(1)

$\forall x\,[P(x) \vee q(x)]$

$\forall x\,[[\neg p(x) \wedge q(x)] \rightarrow r(x)]$

$\therefore \forall x\,[\neg r(x) \rightarrow p(x)]$

p

$c: p \rightarrow q$

$p \vee q$

$\neg p$

	Step	Reason.
1.	$\forall x\,[P(x) \vee q(x)]$	premise.
2.	$P(a) \vee q(a)$	U.S
3.	$\forall x\,[(\neg p(x) \wedge q(x)) \rightarrow r(x)]$	premise.
4.	$\neg p(a) \wedge q(a) \rightarrow r(a)$	U.S
5.	$\neg r(a) \rightarrow \neg(\neg p(a) \wedge q(a))$	Contrapositive (4)
6.	$\neg r(a)$	Additional premise
7.	$\neg(\neg p(a) \wedge q(a))$	Modus ponens (5) & (6)
8.	$p(a) \vee \neg q(a)$	Demorgan's law
9.	$(P(a) \vee q(a)) \wedge (p(a) \vee \neg q(a))$	Rule of Conjuction.
10.	$P(a) \vee [q(a) \wedge \neg q(a)]$	Distributive law
11.	$P(a) \vee F \Leftrightarrow P(a)$	Inverse law
12.	$\therefore \forall x\,[\neg r(x) \rightarrow P(x)]$	U.G Using (6)

② $\forall x\, [P(x) \rightarrow (q(x) \wedge r(x))]$
$\forall x\, [P(x) \wedge s(x)]$

$\therefore \forall x\, [r(x) \wedge s(x)]$

	Step	Reason
1.	$\forall x\, [p(x) \rightarrow q(x) \wedge r(x)]$	premise.
2.	$p(a) \rightarrow q(a) \wedge r(a)$	U.S ①
3.	$\forall x\, [P(x) \wedge s(x)]$	premise.
4.	$p(a) \wedge s(a)$	U.S
5.	$p(a)$	Conjuctive Simplification. ④
6.	$q(a) \wedge r(a)$	Modus ponens ② & ⑤
7.	$r(a)$	Conjuctive Simplification ⑥
8.	$s(a)$	Conjuctive Simplification ④
9.	$r(a) \wedge s(a)$	Rule of Conjuction.
10	$\forall x\, (r(x) \wedge s(x))$	U.G.

(3) $\forall x\,[p(x) \vee q(x)]$

$\exists x\, \neg p(x)$

$\forall x\,[\neg q(x) \vee r(x)]$

$\forall x\,[s(x) \rightarrow \neg r(x)]$

$\therefore \exists x\, \neg s(x)$.

	step	Reason.
1.	$\forall x\,[p(x) \vee q(x)]$	premise.
2.	$p(a) \vee q(a)$	U.S
3.	$\neg p(a) \rightarrow q(a)$	$p \rightarrow q \Leftrightarrow \neg p \vee q$
4.	$\exists x\, \neg p(x)$	premise
5.	$\neg p(a)$	E.S
6.	$q(a)$	Modus ponens (3) & (5)
7.	$\forall x\,[\neg q(x) \vee r(x)]$	premise
8.	$\neg q(a) \vee r(a)$	U.S
9.	$r(a)$	Disjunctive syllogism
10.	$\forall x(s(x) \rightarrow \neg r(x))$	premise.
11.	$s(a) \rightarrow \neg r(a)$	U.S
12.	$\neg s(a)$	Modus tollens
13.	$\exists x\, \neg s(x)$	E.G.

(4) Prove the following Implication

$\forall x\,(P(x) \to Q(x)),\ \forall x\,(R(x) \to \neg Q(x)) \Rightarrow \forall x\,[R(x) \to \neg P(x)]$

	Step	Reason
1.	$\forall x\,(P(x) \to Q(x))$	premise
2.	$P(c) \to Q(c)$	U.S (1)
3.	$\forall x\,(R(x) \to \neg Q(x))$	premise
4.	$R(c) \to \neg Q(c)$	U.S (3)
5.	$\neg Q(c) \to \neg P(c)$	Contrapositive (2)
6.	$R(c) \to \neg P(c)$	Law of syllogism (4) & (5)
7.	$\forall x\,(R(x) \to \neg P(x))$	U.G

(5) Prove that $\exists z\, Q(z)$ is a Valid Conclusion from the premises $\forall x\,[P(x) \to Q(x)]$, $\exists y\, P(y)$

	Step	Reason
1.	$\forall x\,[P(x) \to Q(x)]$	premise
2.	$P(c) \to Q(c)$	U.S
3.	$\exists y\, P(y)$	premise
4.	$P(c)$	E.S
5	$Q(c)$	Modus ponens (2) & (4)
6.	$\exists z\, Q(z)$	E.G.

(7) Symbolize "if x is odd then x^2-1 is even".

$p(x)$: x is odd

$q(x)$: x^2-1 is even.

$\forall_x \left(p(x) \rightarrow q(x)\right)$

(2) Let $p(x)$, $q(x)$ denote the following open statements $p(x)$: $x \leq 3$; $q(x)$: $x+1$ is odd. If the Universe consist of all integers, what are the truth value of the following statements?

(a) $q(1)$ b) $\neg p(3)$ c) $p(7) \vee q(7)$ d) $p(3) \wedge q(4)$ ~~e)~~

e) $\neg p(-4) \vee q(-3)$ f) $\neg p(-4) \wedge \neg q(-3)$.

Ans : $p(x)$: $x \leq 3$; $q(x)$: $x+1$ is odd.

a) $q(1)$: $1+1$ is odd - False.

b) $\neg p(3)$: ~~x~~ $x > 3$ ie $3 > 3$ - False.

c) $p(7) \vee q(7)$: $7 \leq 3$ or $7+1$ is odd - False.

d) $p(3) \wedge q(4)$: $3 \leq 3$ and $4+1$ is odd - True

e) $\neg p(-4) \vee q(-3)$: $-4 > 3$ or $-3+1$ is odd - False.

f) $\neg p(-4) \wedge \neg q(-3)$: $-4 > 3$ and $-3+1$ is even - False.

③ Let $r(x)$ be the open statement "$x > 0$" and $p(x), q(x)$ defined in question ② Determine the truth values of the following statements.

a) $p(3) \vee [q(3) \vee \neg r(3)]$ — True

b) $p(2) \rightarrow [q(2) \rightarrow r(2)]$ — True

c) $(p(2) \wedge q(2)) \rightarrow r(2)$ — True

d) $p(1) \rightarrow [-q(-1) \leftrightarrow r(1)]$ — True

Determine all values of x for which

$[p(x) \wedge q(x)] \wedge r(x)$ result in true statement.

Ans: The only value of x that makes the statement $[p(x) \wedge q(x)] \wedge r(x)$ is true is 2

④ Let $p(x)$ be the open statement "$x^2 = 2x$" where the universe comprises all integers. Determine whether each of the following statement is true or false.

a) $p(0)$ b) $p(1)$ c) $p(2)$ d) $p(-2)$ e) $\exists x\, p(x)$

f) $\forall (x)\, p(x)$

Ans: a) True b) False c) True d) False e) True

f) False.

(5) Consider the Universe of all polygons with three or four sides, and define the following open statements for this Universe

$a(x)$: all exterior angles of x are equal.

$e(x)$: x is an equilateral triangle

$h(x)$: all sides of x are equal.

$i(x)$: x is an isosceles triangle.

$p(x)$: x has an exterior angle that exceed $180°$

$q(x)$: x is a quadrilateral.

$s(x)$: x is a square

$t(x)$: x is a triangle.

Ans Translate each of the following into English sentence and determine whether the statement is true or false.

a) $\forall x\, [q(x) \underline{\vee} t(x)]$

Ans: Every polygon is a quadrilateral or a triangle
- True for this Universe.

b) $\forall x\, [i(x) \rightarrow e(x)]$

Every Isosceles triangle is equilateral - ~~True~~ False

c) $\exists x\, [t(x) \wedge p(x)]$

There exist a triangle where exterior angle exceed $180°$ - False.

d) $\forall x [a(x) \wedge t(x)] \leftrightarrow e(x)$

A triangle has all of its exterior angles are equal if and only if it is an equilateral triangle — True

e) $\exists x [q(x) \wedge \neg r(x)]$

There exist a quadrilateral that is not a rectangle — True

f) $\exists x [r(x) \wedge \neg s(x)]$

There exist a rectangle that is not a square — True

g) $\forall x [h(x) \rightarrow e(x)]$

if all sides of a polygon are equal then the polygon is an equilateral triangle — False.

h) $\forall x [t(x) \rightarrow \neg p(x)]$

No triangle has an exterior angle that exceed 180° — True

i) $\forall x [s(x) \leftrightarrow a(x) \wedge h(x)]$

A polygon is a square if and only if all of its exterior angles are equal and all of its sides are equal — False.

j) $\forall x [t(x) \rightarrow (a(x) \leftrightarrow h(x))]$

A triangle has all interior angles equal iff all of its sides are equal — True

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5) Professor Carlson's class is mechanics is comprised of 29 students of which exactly

1. three physics majors are juniors.
2. two electrical engineering majors are juniors
3. Four mathematics majors are juniors.
4. twelve physics majors are juniors.
5. Four electrical engineering majors are seniors.
6. two electrical engineering majors are ~~seniors.~~ graduate students.
7. two mathematics majors are graduate students.

Consider the following Open statements.

$c(x)$: studenta x is in the class

$j(x)$: student x is a junior

$s(x)$: student x is a senior

$g(x)$: student x is a graduate student.

$p(x)$: student x is a physics major

$e(x)$: student x is an electrical engineering major

$m(x)$: student x is a mathematics major.

Write each of the following statement in terms of quantifiers and the open statement $c(x), j(x), s(x), g(x), p(x), e(x), m(x)$ and determine whether the statements is true or false. Here the Universe Comprises of all of the 12,500 students enrolled at the University where Professor Carlson teaches. Further more, at this University each student has only one major

a) There is a mathematics major in the class whose is a junior

b) There is a senior in the class whose is not a mathematics major.

c) Every student in the class is majoring in mathematics or physics.

d) No graduate student in the class is a physics major.

e) Every seniors in the class is majoring in either physics or electrical engineering.

Ans

a) $\exists x[m(x) \land c(x) \land j(x)]$ True

b) $\exists x[s(x) \land c(x) \land \neg m(x)]$ True

c) $\forall x[c(x) \rightarrow m(x) \underline{\lor} p(x)]$ False.

d) $\forall x[(g(x) \land c(x)) \rightarrow \neg p(x)]$ True

or

$\forall x[p(x) \land c(x) \rightarrow \neg g(x)]$

or

$\forall x[g(x) \land p(x) \rightarrow \neg c(x)]$

e) $\forall x[s(x) \land c(x) \rightarrow p(x) \underline{\lor} e(x)]$ True

6. Let $p(x,y)$, $q(x,y)$ denote the following open statement $p(x,y): x^2 \geq y$. $q(x,y): x+2<y$
If the Universe for each of x and y consists of all real numbers determine the truth value for each of the following statements.

a) $p(2,4)$ — True
b) $q(1,\pi)$ — True
c) $p(-3,8) \wedge q(1,3)$ - False.
d) $p(\frac{1}{2},\frac{1}{3}) \vee \neg q(-2,-3)$ - True
e) $p(2,2) \rightarrow q(1,1)$ — False.
f) $p(1,2) \leftrightarrow \neg q(1,2)$ — False.

7. For the Universe of all integers let $p(x)$, $q(x)$, $r(x)$, $s(x)$ and $t(x)$ be the following open statements

$p(x): x > 0$
$q(x)$: x is even
$r(x)$: x is a perfect square
$s(x)$: x is divisible by 4.
$t(x)$: x is divisible by 5

a) Write the following statements in symbolic form
1. Atleast one integer is even.
2. There exist a positive integer that is even.

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③ If x is even, then x is not divisible by 5

④ No even integer is divisible by 5

⑤ There exist an even integer divisible by 5

⑥ If x is even and x is a perfect square, then x is divisible by 4.

Ans.

① $\exists x\ q(x)$

② $\exists x\ [p(x) \wedge q(x)]$

③ $\forall x\ [q(x) \rightarrow \neg t(x)]$

④ $\forall x\ [q(x) \rightarrow \neg t(x)]$

⑤ $\exists x\ [q(x) \wedge t(x)]$

⑥ $\forall x\ [(q(x) \wedge r(x)) \rightarrow s(x)]$

b) Determine each of the above six statements in part a is true or false. For each false statement provide a counter example

c) Express each of the following symbolic representation in Words

1) $\forall(x)\ [r(x) \rightarrow p(x)]$ ② $\forall(x)\ [s(x) \rightarrow q(x)]$

3) $\forall x\ [s(x) \rightarrow \neg t(x)]$ ④ $\exists x\ [s(x) \wedge \neg r(x)]$.

d) provide a Counter example for each false statement in part c

b) ① True ② False ③ False ④ True

⑤ True ⑥ True.

c) 1) If x is a perfect square, then $x \geq 0$

2) If x is divisible by 4, then x is even.

3) If x is divisible by 4, then x is not divisible by 5

4) There exist an integer that is divisible by 4 but it is not a perfect square.

d) ① $x = 0$ ③ $x = 20$

⑧ Let $p(x)$, $q(x)$ and $r(x)$ denote the following open statements

$$p(x): x^2 - 8x + 15 = 0$$
$$q(x): x \text{ is odd}$$
$$r(x): x > 0$$

For the Universe of all integers, determine the truth or falsity of each of the following statements. If the statement is false give a counter example.

a) $\forall x\,[p(x) \rightarrow q(x)]$ – True

b) $\forall x\,[q(x) \rightarrow p(x)]$ – False $x = 1$ $q(x)$ is true, $p(x)$ is False.

c) $\exists x\,[p(x) \rightarrow q(x)]$ – True

d) $\exists x\,[q(x) \rightarrow p(x)]$ – True

e) $\exists x\,[r(x) \rightarrow p(x)]$ – True

f. $\forall x\, [\neg q(x) \rightarrow \neg p(x)]$ – True

g. $\exists x\, [p(x) \rightarrow [q(x) \wedge r(x)]]$ – True

h: $\forall x\, [(p(x) \vee q(x)) \rightarrow r(x)]$ – False

$x = -1$

$p(-1) \vee q(-1)$ is true

$r(-1)$ is false.

(9) Let $p(x)$, $q(x)$ and $r(x)$ be the following statements

$p(x): x^2 - 7x + 10 = 0$

$q(x): x^2 - 2x - 3 = 0$

$r(x) : x < 0$

a) Determine truth or falsity of the following statements, where the Universe is all integers. If a statement is false, provide a counter example or explanation

(1) $\forall x\, [p(x) \rightarrow \neg r(x)]$ (2) $\forall x\, [q(x) \rightarrow r(x)]$

(3) $\exists x\, [q(x) \rightarrow r(x)]$ (4) $\exists x\, [p(x) \rightarrow r(x)]$

b) Find the answers to part (a) when the Universe consists of all positive integers.

c) Find all answers to part (a) when the Universe consists only the integers 2 and 5.

9 Ans a) ① True ② False – $x = 3$

③ True ④ True ⑤

b) ① True ② False – $x = 3$.

③ True ④ True

c) ① True ② True ③ True ④ False – $x = 3$.

10) Identify the bound variables and the free variables in each of the following expressions. In both cases the universe comprises all real numbers

a) $\forall y \exists z [\cos(x+y) = \sin(z-x)]$

Ans x is a free variable and y, z are bounded variable

b) $\exists x \exists y [x^2 - y^2 = z]$

Here the variables x, y are bound; the variable z is free.

11) Let $p(x, y)$ denote the open statement "x divides y". Where the universe for each of the variables x, y comprises all integers. Determine the truth value of each of the following statements; if a quantified statement is false, provide a ~~countable~~ Counter

example.

① $p(3,7)$ ② $p(3,27)$ ③ $\forall y\, p(1,y)$ ④ $\forall x\, p(x,0)$
⑤ $\forall x\, p(x,x)$ ⑥ $\forall y\, \exists x\, p(x,y)$ ⑦ $\exists y\, \forall x\, p(x,y)$
⑧ $\forall x \forall y\, [(p(x,y) \wedge p(y,x)) \rightarrow (x=y)]$

Ans b) Determine which of the eight statements in part ⓐ will change in truth value if the Universe for each of the variable x, y ~~whose~~ were restricted to just the positive integers.

c) Determine the truth value of each of the following statements. If the statement is false, provide an explanation or a Counter example

i) $\forall x\, \exists y\, p(x,y)$ ii) $\forall y\, \exists x\, p(x,y)$
iii) $\exists x\, \forall y\, p(x,y)$ iv) $\exists y\, \forall x\, p(x,y)$.

Ans a) ① False ② True ③ ~~False~~ True ~~if~~ ④ False if $x=0$
⑤ False if $x=0$ ⑥ True ⑦ False if $y=0$. ⑧ $x \neq 0$; $y \neq 0$, $x = 2y$
⑧ False. Let $x=3$ and $y=-3$

b) The statement 4, 5 and 8 change to True because the change of Universe.

c) i) True ii) True iii) True iv) False

For any y Consider $x = 2y$.

13) Suppose that $p(x,y)$ is an open statement where the universe for each of x, y consists of only three integers 2, 3, 5. Then the quantified statement $\exists y\, p(2,y)$ is logically equivalent to $p(2,2) \vee p(2,3) \vee p(2,5)$. The quantified statement $\exists x \forall y\, p(x,y)$ is logically equivalent to $[p(2,2) \wedge p(2,3) \wedge p(2,5)] \vee [p(3,2) \wedge p(3,3) \wedge p(3,5)] \vee [p(5,2) \wedge p(5,3) \wedge p(5,5)]$. Use conjunctions and/or disjunctions to express the following statements without quantifiers

a) $\forall x\, p(x,3)$ b) $\exists x \exists y\, p(x,y)$ c) $\forall y \exists x\, p(x,y)$

Ans: a) $P(2,3) \wedge P(3,3) \wedge p(5,3)$

b) $[p(2,2) \vee p(2,3) \vee p(2,5)] \vee [p(3,2) \vee p(3,3) \vee p(3,5)] \vee [p(5,2) \vee p(5,3) \vee p(5,5)]$

c) $[p(2,2) \vee p(3,2) \vee p(5,2)] \wedge [p(2,3) \vee p(3,3) \vee p(5,3)] \wedge$

$[p(2,5) \vee p(3,5) \vee p(5,5)]$.

(13) Let $p(n)$, $q(n)$ represent the open statement

$p(n)$: n is odd $q(n)$: n^2 is odd for the Universe of all integers. which of the following statements are logically equivalent to each other.

a) If the square of an integer is odd then the integer is odd?

b) $\forall n\,[p(n)$ is necessary for $q(n)]$

c) The square of an odd integer is odd.

d) There are some integers whose square is odd.

e) Given an integer whose square is odd that integer is likewise odd.

f) $\forall n\,[\neg p(n) \rightarrow \neg q(n)]$ g) $\forall n\,[p(n)$ is sufficient for $q(n)]$

Ans a, b, e, f are logically equivalent

statement (c) and (g) are logically equivalent.

statement (d) is not logically equivalent to any of the other statements.

14) Let the Universe for the variables in the following statements consists of all real numbers. In each case negate and simplify the given statement.

ⓐ $\forall x \forall y [(x > y) \to (x - y > 0)]$

ⓑ $\forall x \forall y [(x < y) \to \exists z (x < z < y)]$

ⓒ $\forall x \forall y [(|x| = |y|) \to (y = \pm x)]$

Ans :-

a) $\exists x \exists y [(x > y) \wedge x - y \leq 0]$ $\quad p \to q \Leftrightarrow \neg p \vee q$

$\neg(p \to q) \Leftrightarrow p \wedge \neg q$

b) $\exists x \exists y [(x < y) \wedge \forall x [x \geq z \geq y]]$

c) $\exists x \exists y [(|x| = |y|) \wedge y \neq \pm x]$

15) Negate and Simplify each of the following

a) $\exists x [p(x) \vee q(x)]$ b) $\forall x [p(x) \wedge \neg q(x)]$

c) $\forall x [p(x) \to q(x)]$ d) $\exists x [(p(x) \vee q(x)) \to r(x)]$

Ans : a) $\forall x [\neg p(x) \wedge \neg q(x)$

b) $\exists x [\neg p(x) \vee q(x)]$

c) $\exists (x) [p(x) \wedge \neg q(x)]$

d) $\forall (x) [(p(x) \vee q(x)) \wedge \neg r(x)]$.

16) Write the negation of each of the following true statements. Here the Universe consists of all integers.

a) For all integers n, if n is not divisible by 2, then n is odd

b) If k, m, n are any integers where $k-m$ and $m-n$ are odd, then $k-n$ is even.

c) If x is a real number where $x^2 > 16$, then $x < -4$ or $x > 4$.

d) For all real numbers x if $|x-3| < 7$ then $-4 < x < 10$.

Ans

a) $\forall n\,(p(n) \longrightarrow q(n))$ negation is $\exists n\,(p(n) \wedge \neg q(n))$

i.e. There exist an integer n such that it is not divisible by 2 but n is even.

b) There exist integers k, m, n such that $k-m$ and $m-n$ are odd and $k-n$ is ~~said to be~~ odd

c) For some real number x, $x^2 > 16$ but $-4 \le x \le 4$.

d) $\exists$ a real number x such that $|x-3| < 7$ and either $x \le -4$ or $x \geq 10$.

17) For the following statements the Universe comprises all non-zero integers. Determine the truth value of each statement.

I

a) $\exists x \exists y [xy = 1]$ b) $\exists x \forall y [xy = 1]$ c) $\forall x \exists y [xy = 1]$

d) $\exists x \exists y [2x + y = 5) \wedge x - 3y = -8]$

e) $\exists x \exists y [(3x - y = 7) \wedge (2x + 4y = 3)]$.

II If the Universe comprises of all non zero real numbers determine the truth value of the above statement.

Ans: I a) True b) False c) False d) True e) False

II a) True b) False c) True d) True e) True

18) Rewrite the following statement to if then form. Then write the converse, inverse and contrapositive. For each result in part (a) and (c) give the truth value for the implication and the truth value of its converse, inverse and contrapositive

a) (The Universe comprises of all positive integers) Divisibility by 21 is a sufficient condition for divisibility by 7.

b) [The Universe comprises all snakes presently slithering about the jungles of Asia]

Being a cobra is a sufficient condition for a snake to be dangerous.

e) [The Universe consists of all complex numbers]

For every complex number z, z being real is necessary for z^2 to be real.

Ans: a) Implication

if a positive integer is divisible by 21, then it is divisible by 7 – True

Converse

if a positive integer is divisible by 7 then it is divisible by 21 – False (Eg: 14)

Inverse

if a positive integer is not divisible by 21 then it is not divisible by 7 – False (Eg: 14)

Contrapositive

if a positive integer is not divisible by 7 then it is not divisible by 21. – True

b) Implication : if a snake is a cobra then it is dangerous.

Converse

If a snake is dangerous then it is a cobra

~~Inverse~~ Contrapositive

If a snake is not dangerous then it is not a cobra

Inverse

If a snake is not a cobra then it is not dangerous

c) Implication

For each complex number z, if z^2 is real then z is real

Converse For each complex number z, if z is real then z^2 is real – True

– False

Inverse: if z^2 is not real then z is not real – ~~False~~ True

Contrapositive: if z is not real then z^2 is not real – False

($z = i$)

18) For each of the following statement state the Converse, inverse and Contrapositive. Also determine the truth value for each given statement as well as the Converse, inverse and Contrapositive

a) [The Universe Comprises all positive integers]

if $m > n$ then $m^2 > n^2$

b) [The Universe Comprises of all integers]

If m divides n and n divides p, then m divides p

c) [The Universe Comprises of all integers]

if $a > b$ then $a^2 > b^2$

d) [The Universe Comprises of all real numbers]

$\forall x\, [(x > 3) \rightarrow (x^2 \geq 9)]$

e) [The Universe Comprises of all real numbers]

For all real numbers x if $x^2 + 4x - 21 > 0$ then $x > 3$ or $x < -7$

Ans a) TRUE

Converse: for all positive integers m, n if $m^2 > n^2$ then $m > n$ – true

Inverse: if $m \leq n$ then $m^2 \leq n^2$ – True

Contrapositive if $m^2 \leq n^2$ then $m \leq n$ - True

b) statement : if m divides n and n divides p then m divides p – True

Converse : if m divides p then m divides n and n divides p – False (m = 1, n = 2, p = 3)

Inverse : if m doesnot divide n or doesnot divide p then m doesnot divide p – False (m = 1, n = 2, p = 3)

Contrapositive : If m doesnot divide p then m doesnot divide n or n doesnt divide p – True

c) ~~statement : for all integers a, b if $a^2 > b^2$ then a > b – False (a = −2, b = 1)~~

~~Converse : if a > b then $a^2 > b^2$~~

c) statement : for all integers a, b if a > b then $a^2 > b^2$ – False (a = 1, b = −2)

Converse : if $a^2 > b^2$ then a > b – False (a = −5, b = 3)

Inverse : if a ≤ b then $a^2 \leq b^2$ – False (a = −5, b = 3)

Contrapositive : if $a^2 \leq b^2$ then a ≤ b – False (a = 1, b = −2)

d) statement: $\forall x\,[(x > 3) \rightarrow (x^2 > 9)]$ - True

Converse: $\forall x\,[x^2 > 9 \rightarrow (x > 3)]$ - False $(x = -5)$

Inverse: $\forall x\,[x \leq 3 \rightarrow x^2 \leq 9]$ - False $(x = -5)$

Contrapositive: $\forall x\,[x^2 \leq 9 \rightarrow x \leq 3]$ - True

e) statement: $\forall x\,[(x^2 + 4x - 21 > 0) \rightarrow [(x > 3) \vee (x < -7)]]$

- True

Converse: $\forall x\,[(x > 3) \vee (x < -7) \rightarrow (x^2 + 4x - 21 > 0)]$

- True

Inverse $\forall x\,[x^2 + 4x - 21 \leq 0 \rightarrow (x \leq 3 \wedge x \geq -7)]$ - True

Contrapositive $\forall x\,[(x \leq 3) \wedge (x \geq -7)] \rightarrow (x^2 + 4x - 21 \leq 0]$ - True

(Additional Problems) Problem Questions ①

Establish the validity of the following Arguments.

① $[(p \wedge \neg q) \wedge r] \rightarrow [(p \wedge r) \vee q]$

	Step	Reason
1.	$p \wedge \neg q$	premise.
2.	p	Conjunctive Simplification.
3.	r	premise.
4.	$p \wedge r$	Rule of Conjunction. ② & ③
5.	$(p \wedge r) \vee q$	Disjunctive amplification. ④

②

$[p \wedge (p \rightarrow q)] \wedge (\neg q \vee r)] \rightarrow r$

	Step	Reason
1.	p	premise.
2.	$p \rightarrow q$	premise
3.	q	Modus ponens ① & ②
4.	$\neg q \vee r$	premise.
5.	$q \rightarrow r$	$p \rightarrow q \Leftrightarrow \neg p \vee q$
6	r	Modus ponens ③ & ⑤
7.		

(3)

$p \to q$

$\neg q$

$\neg r$

$\therefore \neg(p \vee r)$

	Step	Reason
1.	$p \to q$	premise
2.	$\neg q$	premise.
3.	$\neg q \to \neg p$	Contrapositive. $p \to q \Leftrightarrow \neg q \to \neg p$
4	$\neg p$	Modus ponens
5.	$\neg r$	premise.
6.	$\neg p \wedge \neg r$	Rule of Conjuction (4) & (5)
7	$\neg(p \vee r)$	De morgan's law, step (6)

(4)

$p \to (q \to r)$

$\neg q \to \neg p$

p

$\therefore r$

	Step	Reason
1.	p	premise.
2.	$\neg q \to \neg p$	premise.
3.	$p \to q$	Contrapositive. (2)
4.	q	Modus ponens (1) & (3)
5.	$p \to (q \to r)$	premise.
6.	$q \to r$	Modus ponens (1) & (5)
7.	r	Modus ponens (4) & (6)

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Problem Question (2)

(5)

$p \to q$

$r \to \neg q$

r

$\therefore \neg p$

	step	Reason
1.	r	premise
2.	$r \to \neg q$	premise.
3.	$\neg q$	Modus ponens (1) & (2)
4.	$p \to q$	premise
5.	$\neg q \to \neg p$	Contrapositive (4)
6	$\neg p$	Modus ponens (3) & (5)

(6)

$p \wedge q$

$p \to (r \wedge q)$

$r \to (s \vee t)$

$\neg s$

$\therefore t$

	step	
1.	$p \wedge q$	premise.
2.	p	Conjunctive simplification.
3.	$p \to (r \wedge q)$	premise.
4.	$r \wedge q$	Modus ponens (2) & (3)
5.	r	Conjunctive simplification (4)
6.	$r \to (s \vee t)$	premise
7.	$s \vee t$	Modus ponens (5) & (6)
8.	$\neg s$	premise.
9.	$\therefore t$	Disjunctive syllogism.

7. $p \to (q \to r)$

$p \vee s$

$t \to q$

$\neg s$

$\neg r \to \neg t$

	Step	Reason
1.	$\neg s$	premise
2.	$p \vee s$	premise
3.	p	Disjuctive syllogism 1 & 2
4.	$p \to (q \to r)$	premise.
5.	$q \to r$	Modus poneris 3 & 4
6.	$t \to q$	premise
7.	$t \to r$	Syllogism law 5 & 6
8.	$\neg r \to \neg t$	Contrapositive law 7

8. $p \vee q$

$\neg p \vee r$

$\neg r$

$\therefore q$

	Step	Reason
1.	$p \vee q$	premise.
2.	$\neg p \vee r$	premise.
3.	$p \to r$	$p \to q \Leftrightarrow \neg p \vee q$ 2
4.	$\neg r$	premse.
5.	$\neg r \to \neg p$	Contrapositive 3
6.	$\neg p$	Modus ponens 4 & 5
7	$\therefore q$	Disjuctive syllogism

Problem Question - ③

Write each of the following arguments in symbolic form, then establish the validity of the argument.

a) If Rochelle gets the supervisor's position and works hard then she will get a raise. If she ~~will~~ gets the raise, then she will buy a new car. She has not purchased a new car. Therefore either Rochelle did not get the supervisor's position or she did not work hard.

Ans : p : Rochelle gets the supervisors position.

q : Rochelle works hard

r : Rochelle gets a raise.

s : Rochelle buys a new car.

$$(p \wedge q) \longrightarrow r$$
$$r \longrightarrow s$$
$$\neg s$$
$$\therefore \neg p \vee \neg q$$

	Step	Reason.
1.	$r \rightarrow s$	premise.
2.	$\neg s$	premise.
3	$\neg r$	Modus tollens 1 & 2
4.	$(p \wedge q) \rightarrow r$	premise.
5.	$\neg(p \wedge q)$	Modus tollens 3 & 4
6.	$\therefore \neg p \vee \neg q$	Demorgans law 5

b) If Domenic goes to racetrack, then Helen will be mad. If Ralph plays cards all right, then Carmela will be mad. If either Helen or Carmela gets mad, then Veronica will be notified. Veronica has not heard from either of these two clients. Consequently, Domenic didn't make it to the racetrack and Ralph didn't play cards all right.

Ans: p: Domenic goes to the racetrack.
q: Helen gets mad
r: Ralph plays Cards all right.
s: Carmela gets mad
t: Veronica is notified.

Problem Question ④

$p \to q$

$r \to s$

$(q \vee s) \to t$

$\neg t$

$\therefore \neg p \wedge \neg r$

	Step	Reason
1.	$\neg t$	premise
2.	$(q \vee s) \to t$	premise
3.	$\neg(q \vee s)$	Modus tollens ① & ②
4.	$\neg q \wedge \neg s$	Demorgans law
5.	$\neg q$	Conjunctive Simplification ④
6.	$p \to q$	premise.
7.	$\neg p$	Modus tollens. ⑤ & ⑥
8.	$\neg s$	Conjunctive Simplification ④
9.	$r \to s$	premise.
10	$\neg r$	Modus tollens
11.	$\neg p \wedge \neg r$	Conjunctive amplification ⑦ & ⑩

(c) If there is a chance of rain or her red headband is missing, then Lois will not mow her lawn. Whenever the temperature is over 80°F, there is no chance for rain. Today the temperature is 85°F and Lois is wearing her red headband. Therefore Lois will mow her lawn.

Ans:

p: There is a chance of rain.

q: Lois's red headband is missing.

r: Lois doesnot mow her lawn.

s: The temperature is over 80°F.

$$(p \vee q) \rightarrow r$$
$$s \rightarrow \neg p$$
$$s \wedge \neg q$$
$$\therefore \neg r$$

	Step	Reason
1.	$s \wedge \neg q$	Premise.
2.	s	Conjunctive Simplification.
3.	$s \rightarrow \neg p$	premise.
4.	$\neg p$	Modus ponens.
5.	$(p \vee q) \rightarrow r$	premise.
6.	$\neg(p \vee q) \vee r$	$p \rightarrow q \Leftrightarrow \neg p \vee q$.
7.	$\neg q$	Conjunctive Simplification

Problem Question ⑤

8.	$\neg p \wedge \neg q$	Rule of Conjunctions ④ & ⑦
9.	$\neg(p \vee q)$	Demorgans law
10.	$\neg[(p \vee q) \wedge \neg r]$	Demorgans law
11.	$\neg r$	⑨ & ⑩

H.W show that $[(p \vee q) \wedge (\neg p \vee r)] \rightarrow (q \vee r)$ is a tautology. (with Using Truth table) without Using truth table.

	step	Reason
1.	$p \vee q$	premise.
2.	$\neg p \vee r$	premise.
3.	$p \rightarrow r$	$p \rightarrow q \Leftrightarrow \neg p \vee q$. ②
4.	$\neg q \rightarrow p$	$p \rightarrow q \Leftrightarrow \neg p \vee q$ ①
5.	$\neg q \rightarrow r$	Syllogism law ③ & ④
6	$q \vee r$	$p \rightarrow q \Leftrightarrow \neg p \vee q$.

Note

$(p \vee q) \wedge (\neg p \vee r) \Longleftrightarrow q \vee r$, This rule is called ~~rule of~~ resolution. and the conclusion $q \vee r$ is called resolvent.

Establish the validity of the following Arguments, Using resolution.

(1)

$$\begin{array}{l} p \vee (q \wedge r) \\ p \rightarrow s \\ \hline \therefore r \vee s \end{array}$$

	Step	Reason
1.	$p \vee (q \wedge r)$	premise
2.	$(p \vee q) \wedge (p \vee r)$	Distributive property
3.	$p \rightarrow s$	premise.
4	$\neg p \vee s$	$p \rightarrow q \Leftrightarrow \neg p \vee q$.
5.	$p \vee r$	Conjunctive Simplification
6.	$r \vee s$	Using (4) & (5) and (2) law of resolution.

Problem Question 6

2.

p

$p \leftrightarrow q$

$\therefore q$

	Step	Reason
1.	p	premise.
2.	$p \leftrightarrow q$	premise.
3.	$(p \to q) \wedge (q \to p)$	2 $p \leftrightarrow q \Leftrightarrow (p \to q) \wedge (q \to p)$
4.	$p \to q$	Conjuctive Simplification.
5.	$\neg p \vee q$	$p \to q \Leftrightarrow \neg p \vee q$ 4
6.	$p \vee q$	1 Disjuctive amplification
7.	$(p \vee q) \wedge (\neg p \vee q)$	Rule of Conjuction,
8.	$q \vee q$	Resolution 7
9.	q	Idempotent law.

3.

$p \vee q$

$p \to r$

$r \to s$

$\therefore q \vee s$

	Step	Reason
1.	$p \vee q$	premise.
2.	$p \to r$	premise.
3.	$r \to s$	premise.
4.	$\neg p \vee r$ ~~$p \to s$~~	$p \to q \Leftrightarrow \neg p \vee q$ 2
5.	$(p \vee q) \wedge (\neg p \vee r)$	Rule of Conjuction.
6	$q \vee r$	Resolution.
7.	$\neg r \vee s$	3 $p \to q \Leftrightarrow \neg p \vee q$.
8.	$(q \vee r) \wedge (\neg r \vee s)$	Rule of Conjuction.
9	$q \vee s$	Resolution.

(4) $\neg p \vee q \vee r$

$\neg q$

$\neg r$

$\therefore \neg p$

	Step	Reason
1.	$\neg p \vee q \vee r$	premise.
2.	$q \vee (\neg p \vee r)$	Commutative & Associative law
3.	$\neg q$	premise.
4.	$\neg q \vee (\neg p \vee r)$	Law of Disjunction.
5.	$[q \vee (\neg p \vee r)] \wedge [\neg q \vee (\neg p \vee r)]$	Conjunctive amplification.
6.	$\neg p \vee r$	Resolution
7.	$\neg r$	premise.
8.	$\neg r \vee \neg p$	Disjunctive amplification.
9.	$(\neg p \vee r) \wedge (\neg p \vee \neg r)$	Rule of Conjunction.
10	$\neg p$	Resolution.