Module 1

Fundamentals of Logic

1

Proposition (Statement)

A proposition (statement) is a declarative sentence i.e., either true or false but not both.

Ex: p: Margret wrote "Gone with the Wind".

q: $2+3=8$

r: Combinatorics is a required course for sophomores.

We use the lowercase letters of the alphabet (such as p, q, ..) to represent a statement.

" What a beautiful evening! "

" Get up and do your exercises. "

are not considered as proposition or statement, since it has no truth value (true or false)

1. Determine whether each of the following sentences is a statement.

a) In 2003 George W. Bush was the president of the United States.

b) $x + 3$ is a positive integer.

c) Fifteen is an even number.

d) If Jennifer is late for the party, then her cousin Zachary will be quite angry.

e) What time is it?

f) As of June 30, 2003, Christine Marie Evert had won the French Open a record seven times.

Basic Connectives and Truth table

New statements can be obtained from existing ones as follows

(1.) Negation

Let 'p' be a proposition, the negation of P, denoted by $\neg p$ (not p) is the statement "It is not the case that p"

Ex: p: Combinatorics is a required course for sophomores.

$\neg p$: Combinatorics is not a required course for sophomores.

The truth value of $\neg p$ is the opposite of the truth value of p.

* For constructing truth table we use "0" for false and "1" for true.

Truth Table

p	$\neg p$
0	1
1	0

(2). Combine two or more statements into a Compound statement using the following logical Connectives

(1) Conjunction. (and)

The Conjunction ~~of the~~ statements p, q is denoted by $p \wedge q$ which is read as "p and q".

Ex: p: $2+3=5$

q: $2+5=7$

$p \wedge q$: $2+3=5$ and $2+5=7$

Truth value of $p \wedge q$ is 1 when both p and q are of truth value 1. Otherwise $p \wedge q$ has truth value 0.

Truth Table.

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

② Disjunction (or).

Let p and q be two statements then disjunction of p and q is denoted by $p \vee q$. Truth value of $p \vee q$ is 1 when atleast one of p or q is of truth value 1.

Ex: p: Ram is a boy

q: Sita is a girl

$p \vee q$: Ram is a boy or Sita is a girl

Truth Table

p	q	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

The exclusive or is denoted by $p \veebar q$. The compound statement

$p \veebar q$ is true if one or the other of p, q is true but not both

Ex: p: Combinatorics is a required course for sophomores

q: Margaret Mitchell wrote "Gone with the Wind".

$p \veebar q$: Combinatorics is a required course for sophomores, or Margaret Mitchell wrote Gone with the Wind, but not both

Truth value of $p \veebar q$ is 1 if one or the other of p, q is having truth value 1 but not both

Truth Table

p	q	$p \veebar q$
0	0	0
0	1	1
1	0	1
1	1	0

(3) Implication

Let p and q be two statements then the Implication / conditional proposition denoted by $p \rightarrow q$, to designate the statement

1) which is the implication of q by p.

2) If p, then q

3) p is sufficient for q

4) p is a sufficient condition for q

5) q is necessary for p

6) p only if q

7) q is a necessary condition for p.

Ex: $p \rightarrow q$: If Combinatorics is a required Course for sophomores, then Margret Mitchell wrote 'Gone with the wind'.

Truth Table

p	q	p → q
0	0	1
0	1	1
1	0	0
1	1	1

ie, truth value of $p \rightarrow q$ is false (0) when p is true (1) and q is false (0)

Here, In the Implication $p \rightarrow q$

p is called the hypothesis and

q is called the Conclusion.

④ **Biconditional statement**

The biconditional of two statements p, q is denoted by $p \leftrightarrow q$ which is read as "p if and only if q" or "p is necessary and sufficient for q".

Ex: $p \leftrightarrow q$

Combinatorics is a required course for Sophomores if and only if Majority

Mitchell wrote Gone with the wind.

* P if and only if q
* P iff q.

Truth Table

p	q	$p \leftrightarrow q$
0	0	1
0	1	0
1	0	0
1	1	1

Construct the truth table for the following compound propositions

1. $(p \rightarrow q) \wedge [(q \wedge \neg r) \rightarrow (p \vee r)]$

p	q	r	$p \rightarrow q$	$\neg r$	$q \wedge \neg r$	$p \vee r$	$(q \wedge \neg r) \rightarrow (p \vee r)$	$(p \rightarrow q) \wedge [(q \wedge \neg r) \rightarrow (p \vee r)]$
0	0	0	1	1	0	0	1	1
0	0	1	1	0	0	1	1	1
0	1	0	1	1	1	0	0	0
0	1	1	1	0	0	1	1	1
1	0	0	0	1	0	1	1	0
1	0	1	0	0	0	1	1	0
1	1	0	1	1	1	1	1	1
1	1	1	1	0	0	1	1	1

2. $p \vee (q \wedge r)$

p	q	r	$(q \wedge r)$	$p \vee (q \wedge r)$
0	0	0	0	0
0	0	1	0	0
0	1	0	0	0
0	1	1	1	1
1	0	0	0	1
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

3. $(p \vee q) \wedge r$

p	q	r	$(p \vee q)$	$(p \vee q) \wedge r$
0	0	0	0	0
0	0	1	0	0
0	1	0	1	0
0	1	1	1	1
1	0	0	1	0
1	0	1	1	1
1	1	0	1	0
1	1	1	1	1

4 $p \rightarrow (p \vee q)$ and $p \wedge (\neg p \wedge q)$

p	q	$p \vee q$	$\neg p$	$(\neg p \wedge q)$	$p \rightarrow (p \vee q)$	$p \wedge (\neg p \wedge q)$
0	0	0	1	0	1	0
0	1	1	1	1	1	0
1	0	1	0	0	1	0
1	1	1	0	0	1	0

Tautology and Contradiction

A compound statement is called a tautology if it is true for all truth value assignments for its component statements.

A compound statement is called a contradiction if it is false for all truth values assignments of its component statements.

Ex: $p \vee \neg p$ is a tautology

$p \wedge \neg p$ is a contradiction

p	$\neg p$	$\neg p \vee p$	$p \wedge \neg p$
0	1	1	0
0	1	1	0
1	0	1	0
1	0	1	0

Tautology Contradiction

Ex:2 $p \rightarrow (p \vee q)$ is a tautology

$p \wedge (\neg p \wedge q)$ is a contradiction

Hw: Construct the truth table for the following:

a) $q \wedge (\neg r \rightarrow p)$

b) $(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$

c) $\neg (p \wedge q) \leftrightarrow (\neg p \vee \neg q)$

d) $(p \rightarrow q) \rightarrow (q \rightarrow p)$

e) $(p \vee q) \rightarrow (p \wedge q)$

f) $(p \vee \neg q) \rightarrow (p \wedge q)$

g) $(p \vee \neg q) \rightarrow q$

Q. Show that $\neg (p \rightarrow q) \rightarrow \neg q$ is a tautology

p	q	$p \rightarrow q$	$\neg q$	$\neg (p \rightarrow q)$	$\neg (p \rightarrow q) \rightarrow \neg q$
0	0	1	1	0	1
0	1	1	0	0	1
1	0	0	1	1	1
1	1	1	0	0	1

Since the truth value is always true, given statement is a tautology.

S.T

* $(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$ is a tautology

p	q	$p \rightarrow q$	$\neg q$	$\neg p$	$\neg q \rightarrow \neg p$	$(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$
0	0	1	1	1	1	1
0	1	1	0	1	1	1
1	0	0	1	0	0	1
1	1	1	0	0	1	1

Since the truth value of $(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$ is true always, it is a tautology.

2.1 Let s, t, and u denote the following primitive statements:

s: Phyllis goes out for a walk.

t: The moon is out.

u: It is snowing.

The following English sentences provide possible translations for the given (symbolic) compound statements.

a) $(t \wedge \neg u) \rightarrow s$: If the moon is out and it is not snowing, then Phyllis goes out for a walk.

b) $t \rightarrow (\neg u \rightarrow s)$: If the moon is out, then if it is not snowing Phyllis goes out for a walk. [So $\neg u \rightarrow s$ is understood to mean $(\neg u) \rightarrow s$ as opposed to $\neg(u \rightarrow s)$.]

c) $\neg(s \leftrightarrow (u \vee t))$: It is not the case that Phyllis goes out for a walk if and only if it is snowing or the moon is out.

Now we will work in reverse order and examine the logical (or symbolic) notation for three given English sentences:

d) "Phyllis will go out walking if and only if the moon is out." Here the words "if and only if" indicate that we are dealing with a biconditional. In symbolic form this becomes $s \leftrightarrow t$.

e) "If it is snowing and the moon is not out, then Phyllis will not go out for a walk." This compound statement is an implication where the hypothesis is also a compound statement. One may express this statement in symbolic form as $(u \wedge \neg t) \rightarrow \neg s$.

f) "It is snowing but Phyllis will still go out for a walk." Now we come across a new connective — namely, *but*. In our study of logic we shall follow the convention that the connectives *but* and *and* convey the same meaning. Consequently, this sentence may be represented as $u \wedge s$.

4. Let p, q, r, s denote the following statements:

p: I finish writing my computer program before lunch.
q: I shall play tennis in the afternoon.
r: The sun is shining.
s: The humidity is low.

Write the following in symbolic form.

a) If the sun is shining, I shall play tennis this afternoon.

b) Finishing the writing of my computer program before lunch is necessary for my playing tennis this afternoon.

c) Low humidity and sunshine are sufficient for me to play tennis this afternoon.

4.

a) If the sun is shining, I shall play tennis this afternoon

$r \to q$

b) Finishing the writing of my computer program before lunch is necessary for my playing tennis this afternoon

$\left(\begin{array}{c} q \text{ is necessary for } p \\ p \to q \end{array} \right)$ $q \to p$

c) low humidity and sunshine are sufficient for me to play tennis this after-noon

p is suffi- for q $p \to q$

$(s \wedge r) \longrightarrow q$

Logical Equivalence : The laws of logic

A compound proposition that have the same truth value in all possible cases are called logically equivalent. $(p \equiv q)$ denotes that p and q are logically equivalent. or $p \Leftrightarrow q$

* The compound proposition p and q are logically equivalent if $p \Leftrightarrow q$ is a tautology.

Q1 S.T $(\neg p \vee q)$ and $(p \to q)$ are logically equivalent

p	q	$\neg p$	$\neg p \vee q$	$p \to q$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	0
1	1	0	1	1

From the last two columns in the above truth table, we can see that the truth value of $\neg p \vee q$ and $p \rightarrow q$ is same for all the cases

$\therefore (\neg p \vee q) \equiv (p \rightarrow q)$

2. S.T $(p \rightarrow q) \wedge (q \rightarrow p) \equiv p \leftrightarrow q$

p	q	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$	$p \leftrightarrow q$
0	0	1	1	1	1
0	1	1	0	0	0
1	0	0	1	0	0
1	1	1	1	1	1

Truth value of $(p \rightarrow q) \wedge (q \rightarrow p)$ and $p \leftrightarrow q$ is same for all the cases $\therefore$ They are logically equivalent statements

Q: Show that $\neg(p \wedge q) \equiv \neg p \vee \neg q$

$\neg(p \vee q) \equiv \neg p \wedge \neg q$ } De Morgan's Laws

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$	$p \vee q$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
0	0	0	1	1	1	1	0	1	1
0	1	0	1	1	0	1	1	0	0
1	0	0	1	0	1	1	1	0	0
1	1	1	0	0	0	0	1	0	0

Q: S.T $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

$\Big\}$ Distributive Laws

$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$ The Distributive Law of $\wedge$ over $\vee$

$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$ The Distributive Law of $\vee$ over $\wedge$

p	q	r	$p \wedge (q \vee r)$	$(p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r)$	$(p \vee q) \wedge (p \vee r)$
0	0	0	0	0	0	0
0	0	1	0	0	0	0
0	1	0	0	0	0	0
0	1	1	0	0	1	1
1	0	0	0	0	1	1
1	0	1	1	1	1	1
1	1	0	1	1	1	1
1	1	1	1	1	1	1

3 S.T $q \wedge \neg r$ and $p \vee (q \wedge \neg r)$ are not logically equivalent.

p	q	r	$\neg r$	$q \wedge \neg r$	$p \vee (q \wedge \neg r)$
0	0	0	1	0	0
0	0	1	0	0	0
0	1	0	1	1	1
0	1	1	0	0	0
1	0	0	1	0	1
1	0	1	0	0	1
1	1	0	1	1	1
1	1	1	0	0	1

last two columns are not identical, ie, truth value of $q \wedge \neg r$ is not the same as that of $p \vee (q \wedge \neg r)$. They are not logically equivalent statements

H.W Show that $p \wedge (\neg q \vee r)$ and $p \vee (q \wedge \neg r)$ are logically not equivalent. (Check)?

The Laws of Logic

For any primitive statements p, q, r, any tautology T_0, and any contradiction F_0,

1)	$\neg\neg p \Leftrightarrow p$	*Law of Double Negation*
2)	$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$ $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$	*DeMorgan's Laws*
3)	$p \vee q \Leftrightarrow q \vee p$ $p \wedge q \Leftrightarrow q \wedge p$	*Commutative Laws*
4)	$p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r^{\dagger}$ $p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$	*Associative Laws*
5)	$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$	*Distributive Laws*
6)	$p \vee p \Leftrightarrow p$ $p \wedge p \Leftrightarrow p$	*Idempotent Laws*
7)	$p \vee F_0 \Leftrightarrow p$ $p \wedge T_0 \Leftrightarrow p$	*Identity Laws*
8)	$p \vee \neg p \Leftrightarrow T_0$ $p \wedge \neg p \Leftrightarrow F_0$	*Inverse Laws*
9)	$p \vee T_0 \Leftrightarrow T_0$ $p \wedge F_0 \Leftrightarrow F_0$	*Domination Laws*
10)	$p \vee (p \wedge q) \Leftrightarrow p$ $p \wedge (p \vee q) \Leftrightarrow p$	*Absorption Laws*

Logical equivalences Involving conditional and biconditional statements

* $p \rightarrow q \equiv \neg p \vee q$
* $p \rightarrow q \equiv \neg q \rightarrow \neg p$
* $p \vee q \equiv \neg p \rightarrow q$
* $p \wedge q \equiv \neg(q \rightarrow \neg p)$
* $\neg(p \rightarrow q) \equiv p \wedge \neg q$
* $(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$
* $(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$
* $(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$
* $(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$
* $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
* $p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
* $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
* $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

The Implication ($p \to q$)

Let p and q be two statements then the Implication of p and q is denoted as $p \to q$

Ex: p: Jeff is concerned about his Cholesterol levels

q: Jeff walks atleast two miles three times a week.

$p \to q$: If Jeff is concerned about his cholesterol levels, then he will walk atleast two miles three times a week.

The Contrapositive ($\neg q \to \neg p$)

Contrapositive of the Implication $p \to q$ is defined as $\neg q \to \neg p$

Ex: $\neg q \to \neg p$: If Jeff does not walk 3 atleast two miles 3 times a week then he is not concerned about his Cholesterol levels

The converse $(q \rightarrow p)$

The converse statement of $p \rightarrow q$ is defined a $q \rightarrow p$

Ex: If Jeff walks atleast two miles 3 times a week then he is concerned about his Cholesterol levels.

The Inverse $(\neg p \rightarrow \neg q)$

The Inverse of $p \rightarrow q$ is defined as $\neg p \rightarrow \neg q$

Ex: If Jeff is not concerned about his cholestrol levels, then he will not walk atleast two miles 3 times a week.

Dual of a statement

Let 's' be a statement. If s contains no logical connectives other than $\wedge$ and $\vee$, then the dual of s, denoted by s^d, is the statement obtained from s replacing each occurrence of $\wedge$ and $\vee$ by $\vee$ and $\wedge$ respectively and each occurrence of T_0 and F_0 by F_0 and T_0 respectively.

Ex: $s: (p \wedge \neg q) \vee (r \wedge T_0)$

then $s^d: (p \vee \neg q) \wedge (r \vee F_0)$

The principle of Duality

Let s and t be statements that contain no logical connectives other than $\vee$ and $\wedge$, $\boxed{\text{If } s \Leftrightarrow t, \text{ then } s^d \Leftrightarrow t^d}$

Q: write the dual of the logical equivalence

$(\neg p \vee q) \wedge (p \wedge (p \wedge q)) \Leftrightarrow (p \wedge q)$

after proving the logical equivalence.

p	q	$\neg p$	$\neg p \vee q$	$p \wedge q$	$p \wedge (p \wedge q)$	$(\neg p \vee q) \wedge (p \wedge (p \wedge q))$
0	0	1	1	0	0	0
0	1	1	1	0	0	0
1	0	0	0	0	0	0
1	1	0	1	1	1	1

From columns 5 and 7 it is

Clear that $(\neg p \vee q) \wedge (p \wedge (p \wedge q)) \equiv (p \wedge q)$

Now to find the dual.

Dual of the given logical equivalence is as follows:

$(\neg p \wedge q) \vee (p \vee (p \vee q)) \equiv (p \vee q)$

Q Negative and simplify the compound Statements $(p \vee q) \rightarrow r$

a)

given $(p \vee q) \rightarrow r$

$\neg [p \vee q \rightarrow r]$ $\quad p \rightarrow q \equiv \neg p \vee q$

$\Leftrightarrow \neg [\neg (p \vee q) \vee r]$ $\quad \neg (p \vee r) \equiv \neg p \wedge \neg r$

$\Leftrightarrow \neg \neg (p \vee q) \wedge \neg r$ $\quad \neg \neg p \equiv p$

$\Leftrightarrow (p \vee q) \wedge \neg r$

Thus $\neg((p\vee q)\rightarrow \neg r) \equiv (p\vee q)\wedge \neg r$

b) $p\wedge(q\vee r)\wedge(\neg p\vee\neg q\vee r)$

$\neg\left[p\wedge(q\vee r)\wedge(\neg p\vee\neg q\vee r)\right]$

$\neg(p\wedge q)\equiv\neg p\vee\neg q$

$\Longleftrightarrow \neg p\vee\neg(q\vee r)\vee\neg(\neg p\vee\neg q\vee r)$

$\Longleftrightarrow \neg p\vee\neg q\wedge\neg r\vee\neg\neg p\wedge\neg\neg q\wedge\neg r$

$\Longleftrightarrow \neg p\vee(\neg q\wedge\neg r)\vee(p\wedge q\wedge\neg r)$

$(p\vee q)\vee r\equiv p\vee(q\vee r)$

$\Longleftrightarrow (\neg q\wedge\neg r)\vee\left[\neg p\vee(p\wedge q\wedge\neg r)\right]$

distributive law

$\Longleftrightarrow (\neg q\wedge\neg r)\vee\left[(\neg p\vee p)\wedge(\neg p\vee(q\wedge\neg r))\right]$

$\Longleftrightarrow (\neg q\wedge\neg r)\vee\left[T_0\wedge(\neg p\vee(q\wedge\neg r))\right]$

Inverse law

$p\vee\neg p\equiv T_0$

$\Longleftrightarrow (\neg q\wedge\neg r)\vee\left[\neg p\vee(q\wedge\neg r)\right]$

$p\wedge T_0\equiv p$, Identity law

$\Longleftrightarrow (\neg p)\vee(\neg q\wedge\neg r)\vee(q\wedge\neg r)$

$\Longleftrightarrow \neg p\vee\left[(\neg q\vee q)\wedge\neg r\right]$ Distributive law

$\Longleftrightarrow \neg p \vee [T_0 \wedge \cancel{r}] \neg r$

$q \vee \neg q \equiv T_0$

$\Longleftrightarrow \neg p \vee \neg r \equiv \neg(p \wedge r)$

$T_0 \wedge r \equiv r$

c. $(p \wedge q) \rightarrow r$

we need to find $\neg((p \wedge q) \rightarrow r)$

$\neg((p \wedge q) \rightarrow r)$

$p \rightarrow q \equiv \neg p \vee q$

$\Longleftrightarrow \neg[\neg(p \wedge q) \vee r]$

$\neg(p \vee r) \equiv \neg p \wedge \neg r$

$\Longleftrightarrow \neg\neg(p \wedge q) \wedge \neg r$

$\neg\neg p \equiv p$

$\Longleftrightarrow (p \wedge q) \wedge \neg r$

d. $p \rightarrow (\neg q \wedge r)$

we need to find $\neg(p \rightarrow (\neg q \wedge r))$

$\neg(p \rightarrow (\neg q \wedge r))$

$p \rightarrow q \equiv \neg p \vee q$

$\Longleftrightarrow \neg[\neg p \vee (\neg q \wedge r)]$

$\neg(p \vee r) \equiv \neg p \wedge \neg r$

$\Longleftrightarrow \neg\neg p \wedge \neg(\neg q \wedge r)$

$\Longleftrightarrow p \wedge (\neg\neg q \vee \neg r)$

$\Longleftrightarrow p \wedge (q \vee \neg r)$

e. $p \vee q \vee (\neg p \wedge \neg q \wedge r)$

$\neg [p \vee q \vee (\neg p \wedge \neg q \wedge r)]$

$\neg(p \vee q) \equiv \neg p \wedge \neg q$

$\Longleftrightarrow \neg p \wedge \neg q \wedge \neg(\neg p \wedge \neg q \wedge r)$

$\Longleftrightarrow \neg p \wedge \neg q \wedge (\neg\neg p \vee \neg\neg q \vee \neg r)$ — De Morgan

$\Longleftrightarrow \neg p \wedge \neg q \wedge (p \vee q \vee \neg r)$ — Dist. law

$\Longleftrightarrow \neg(p \vee q) \wedge ((p \vee q) \vee \neg r)$

$\Longleftrightarrow [(\neg(p \vee q)) \wedge ((p \vee q))] \vee [\neg(p \vee q) \wedge \neg r]$

$\Longleftrightarrow F_0 \vee [\neg p \wedge \neg q \wedge \neg r]$

$p \wedge \neg p \equiv F_0$

$F_0 \vee p \equiv p$

$\Longleftrightarrow \neg p \wedge \neg q \wedge \neg r$

$\neg(p \vee q) = \neg p \wedge \neg q$

f. find $\neg(p \to q)$

$$\neg(p \to q) \equiv \neg(\neg p \vee q)$$

$$\equiv \neg\neg p \wedge \neg q$$

$$\equiv p \wedge \neg q$$

Let p: Joan goes to lake George

q: Mary pays for Joan's shopping spree

Negate $p \to q$ and simplify ?

$$\neg(p \to q) \equiv p \wedge \neg q$$

$p \wedge \neg q$: John goes to lake George but Mary does not pays for Joan's shopping spree.

Show the following implication without constructing truth table:

a) $(p \vee q) \wedge \neg(\neg p \wedge q) \Leftrightarrow p$.

$(p \vee q) \wedge \neg(\neg p \wedge q) \equiv (p \vee q) \wedge (\neg\neg p \vee \neg q)$

DeMorgan's law

$\equiv (p \vee q) \wedge (p \vee \neg q)$

$\neg\neg p \equiv p$.

$\equiv p \vee (q \wedge \neg q)$

Distributive law

$\equiv p \vee F_0$, Inverse law

$q \wedge \neg q \equiv F_0$

$\equiv p$, Identity law

Thus $(p \vee q) \wedge \neg(\neg p \wedge q) \equiv p$

b) $\neg(p \rightarrow q) \equiv p \wedge \neg q$

$\neg(p \rightarrow q) \equiv \neg(\neg p \vee q)$ $\quad p \rightarrow q \equiv \neg p \vee q$

$\equiv \neg\neg p \wedge \neg q$ $\quad \neg(p \vee q) \equiv \neg p \wedge \neg q$

$\equiv p \wedge \neg q$ $\neg\neg p \equiv p$

c) $\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg q$

$\neg(p \vee (\neg p \wedge q)) \equiv (\neg p) \wedge \neg(\neg p \wedge q)$

De morgan's law

$\equiv \neg p \wedge (\neg\neg p \vee \neg q)$ $\neg\neg p \equiv p$

$\equiv \neg p \wedge (p \vee \neg q)$ Distributive law

$\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q)$

$\equiv F_0 \vee (\neg p \wedge \neg q)$ $F_0 \vee p \equiv p$

$\equiv \neg p \wedge \neg q$

d) $\neg[\neg[(p \vee q) \wedge r] \vee \neg q] \equiv (q \wedge r)$

$\neg[\neg[(p \vee q) \wedge r] \vee \neg q] \Leftrightarrow \neg(\neg[(p \vee q)) \wedge r] \wedge \neg\neg q$

$\Leftrightarrow [(p \vee q) \wedge r] \wedge q$

$\Leftrightarrow [(p \vee q) \wedge r] \wedge q$

$\Longleftrightarrow (p \vee q) \wedge (r \wedge q)$ Associative law of $\wedge$

$(s \wedge t) \wedge u \equiv s \wedge (t \wedge u)$

$\Longleftrightarrow (p \vee q) \wedge (q \wedge r)$ Commutative law $(\wedge)$

$r \wedge q \equiv q \wedge r$

$\Longleftrightarrow [(p \vee q) \wedge q] \wedge r$ Associative law of $\wedge$

$\Longleftrightarrow (q \wedge (p \vee q)) \wedge r$ Commutative law of $\wedge$

$\Longleftrightarrow (q \wedge (q \vee p)) \wedge r$ Absorption law

$q \wedge (q \vee p) \equiv q$

$\Longleftrightarrow (q \wedge r)$

Hence the proof.

2 Show that $(p \wedge q) \longrightarrow (p \vee q)$ is a tautology Using substitution method

ie; it is enough to prove

$$(p \wedge q) \longrightarrow (p \vee q) \equiv T$$

Consider

$(p \wedge q) \longrightarrow (p \vee q) \equiv \neg(p \wedge q) \vee (p \vee q)$ $\quad p \to q \equiv \neg p \vee q$

$\Leftrightarrow (\neg p \vee \neg q) \vee (p \vee q)$ De morgan's law

$\Leftrightarrow \neg p \vee (\neg q \vee (p \vee q))$ Associative law of 'v'

$\Leftrightarrow \neg p \vee (p \vee q) \vee \neg q$, Commutative law of v

$\Leftrightarrow (\neg p \vee p) \vee (q \vee \neg q)$ Associative law

$\Leftrightarrow T_0 \vee T_0$ Inverse law

$\Leftrightarrow T_0$ Domination law

ie, $(p \wedge q) \rightarrow (p \vee q)$ is a tautology

3 Using logical equivalence show that $\neg(p \vee \neg(p \wedge q))$ is a contradiction

It is enough to prove

$$\neg(p \vee \neg(p \wedge q)) \equiv F$$

$\neg(p \vee \neg(p \wedge q)) \Leftrightarrow \neg p \wedge \neg\neg(p \wedge q)$ De morgan law

$\Leftrightarrow \neg p \wedge (p \wedge q)$ Double negation

$\Longleftrightarrow (\neg p \wedge p) \wedge q$ Associative law of $\wedge$

$\Longleftrightarrow F_0 \wedge q$ Inverse law

$\Longleftrightarrow F_0$ Domination law

Hence $\neg(p \vee \neg(p \wedge q))$ is a contradiction

Hw S.T $\neg p \wedge (p \vee q) \longrightarrow q$ is a Tautology?

4. Use the substitution rules to verify that each of the following is a tautology.

a) $[p \vee (q \wedge r)] \vee \neg [p \vee (q \wedge r)]$

b) $[(p \vee q) \longrightarrow r] \longleftrightarrow [\neg r \longrightarrow \neg (p \vee q)]$

$[p \vee (q \wedge r)] \vee \neg [p \vee (q \wedge r)]$ —— ①

We have by Inverse law

$p \vee \neg p \equiv T_0$

Let $s = p \vee (q \wedge r)$

∴ ① will be of the form $s \vee \neg s$, which is always a tautology

since $S \vee \neg S \equiv T_0$

$$\Rightarrow (p \vee (q \wedge r)) \vee \neg(p \vee (q \wedge r)) \equiv T_0$$

2) $[(p \vee q) \rightarrow r] \longleftrightarrow [\neg r \rightarrow \neg(p \vee q)]$

We have the result

$$p \rightarrow q \Longleftrightarrow \neg q \rightarrow \neg p.$$

Substitute $s = (p \vee q)$

$t = r$

We know, $s \rightarrow t \Longleftrightarrow \neg t \rightarrow \neg s$

ie; $(p \vee q) \rightarrow r \Longleftrightarrow \neg r \rightarrow \neg(p \vee q)$

ie; $(p \vee q) \rightarrow r \Longleftrightarrow \neg r \rightarrow \neg(p \vee q)$ is

a Tautology

HW Verify Absorption law by means of a truth table.

5. Use the substitution rules to show that

$[p \to (q \vee r)] \Leftrightarrow [(p \wedge \neg q) \to r]$

$[p \to (q \vee r)] \Leftrightarrow [\neg(q \vee r) \to \neg p]$ $\quad p \to q \equiv \neg q \to \neg p$

$\Leftrightarrow [(\neg q \wedge \neg r) \to \neg p]$

$\Leftrightarrow \neg(\neg q \wedge \neg r) \vee \neg p$ $\quad p \to q \equiv \neg p \vee q$

$\Leftrightarrow (q \vee r) \vee \neg p$

Commutative on $\vee$

$\Leftrightarrow \neg p \vee (q \vee r)$

Associativity

$\Leftrightarrow (\neg p \vee q) \vee r$

$\Leftrightarrow \neg[p \wedge \neg q] \vee r$ $\quad \neg(p \wedge \neg q) \equiv \neg p \vee q$

$\Leftrightarrow (p \wedge \neg q) \to r$

$p \to q \equiv \neg p \vee q$

Hence the proof.

③ S.T $p \rightarrow (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

$p \rightarrow q \equiv \neg p \vee q$

$$p \rightarrow (q \rightarrow r) \equiv \neg p \vee (q \rightarrow r)$$

$$\equiv \neg p \vee (\neg q \vee r)$$

Associtivity

$$\equiv (\neg p \vee \neg q) \vee r$$

$\neg (p \wedge q) \equiv \neg p \vee \neg q$

$$\equiv \neg (p \wedge q) \vee r$$

$p \rightarrow q \equiv \neg p \vee q$

$$\equiv (p \wedge q) \rightarrow r$$

(5) $(p \to q) \to (\neg q \to \neg p) \equiv T$

$p \to q \equiv \neg q \to \neg p$

$(p \to q) \to (\neg q \to \neg p) \equiv (p \to q) \to (p \to q)$

$\equiv \neg(p \to q) \vee (p \to q)$

$\equiv T$ Negation laws.

$p \vee \neg p \equiv T$

(6) $p \to (q \to p) \equiv \neg p \to (p \to q)$

Consider

LHS $p \to (q \to p) \equiv p \to (\neg q \vee p)$ $\quad q \to p \equiv \neg q \vee p$

$\equiv \neg p \vee (\neg q \vee p)$,,

Associativity

$\equiv (\neg p \vee p) \vee \neg q$

$\equiv T_0 \vee \neg q$ $\quad p \vee \neg p \equiv T_0$

$\equiv T_0$

Now $\neg p \to (p \to q) \equiv \neg p \to (\neg p \vee q)$ $\quad p \to q \equiv \neg p \vee q$

$\equiv \neg\neg p \vee (\neg p \vee q)$,,

$\equiv p \vee (\neg p \vee q)$ $\quad\quad \neg\neg p \equiv p.$

$\equiv (p \vee \neg p) \vee q$ $\quad\quad$ Associativity

$\equiv T_0 \vee q$

$\equiv T_0$

Thus $p \rightarrow (q \rightarrow p) \equiv \neg p \rightarrow (p \rightarrow q)$

Q: PT $p \rightarrow (q \vee r) \equiv (p \rightarrow q) \vee (p \rightarrow r)$

$$p \rightarrow (q \vee r) \equiv \neg p \vee (q \vee r)$$
$$\equiv (\neg p \vee q) \vee r$$
$$\equiv (p \rightarrow q) \vee r$$

$$(p \rightarrow q) \vee (p \rightarrow r) \equiv (\neg p \vee q) \vee (\neg p \vee r)$$
$$\equiv q \vee (\neg p \vee \neg p) \vee r$$
$$\equiv q \vee \neg p \vee r$$

Associativity and Commutativity

$$\equiv (\neg p \vee q) \vee r$$
$$\equiv (p \rightarrow q) \vee r$$

$p \rightarrow q \equiv \neg p \vee q$

∴ They are equivalent

6. Show that

$$(p \to q) \wedge [\neg q \wedge (r \vee \neg q)] \equiv \neg (q \vee p)$$

Logical Implication : Rules of Inference

An argument in propositional logic is a sequence of propositions. All except the final proposition/statement are called the premises and the final proposition is called the conclusion.

An argument is valid if the truth of all its premises implies that the conclusion is true.

Let $p_1, p_2, \ldots, p_n$ be n premises and q be the conclusion then the argument will be of the form

$$(p_1 \wedge p_2 \wedge p_3 \wedge \ldots \wedge p_n) \longrightarrow q$$

Thus in order to prove the validity of a given argument, it is enough to prove the statement

$p_1 \wedge p_2 \wedge \ldots \wedge p_n \longrightarrow q$ is a tautology

Method 1

Using truth table.

I) Check the validity of the following arguments

a) $P \wedge P \rightarrow q \longrightarrow P$

b) $(P \rightarrow q) \wedge q \longrightarrow P$

HW c) $P \rightarrow \neg q \wedge r \rightarrow q \wedge r \longrightarrow \neg P$

d) $P \vee q \wedge P \rightarrow r \wedge q \rightarrow r \longrightarrow r$

a)

P	q	$P \rightarrow q$	$P \wedge (P \rightarrow q)$	$P \wedge (P \rightarrow q) \rightarrow P$
0	0	1	0	1
0	1	1	0	1
1	0	0	0	1
1	1	1	1	1

From the last column, all the truth value is 1. We can conclude that

$p \wedge (p \to q) \to p$ is a tautology and hence the given argument is valid

b.

p	q	$p \to q$	$(p \to q) \wedge q$	$(p \to q) \wedge q \to p$
0	0	1	0	1
0	1	1	1	0
1	0	0	0	1
1	1	1	1	1

ie, $(p \to q) \wedge q \to p$ is not a tautology

$\therefore$ it is not a valid argument

II S.T the argument $[p \wedge (p \wedge r) \to s] \to (r \to s)$ is valid ?

there $p_1 : p$

$p_2 : (p \wedge r) \to s$

$q : r \to s$

We need to prove $[p \wedge (p \wedge r) \to s] \to r \to s$ is a tautology

Table 2.19

p_1				p_2	q	$(p_1 \wedge p_2) \rightarrow q$
p	r	s	$p \wedge r$	$(p \wedge r) \rightarrow s$	$r \rightarrow s$	$[(p \wedge ((p \wedge r) \rightarrow s)] \rightarrow (r \rightarrow s)$
0	0	0	0	1	1	1
0	0	1	0	1	1	1
0	1	0	0	1	0	1
0	1	1	0	1	1	1
1	0	0	0	1	1	1
1	0	1	0	1	1	1
1	1	0	1	0	0	1
1	1	1	1	1	1	1

EXAMPLE 2.23

Let p, q, r denote the primitive statements given as

p: Roger studies.

q: Roger plays racketball.

r: Roger passes discrete mathematics.

Now let p_1, p_2, p_3 denote the premises

p_1: If Roger studies, then he will pass discrete mathematics.

p_2: If Roger doesn't play racketball, then he'll study.

p_3: Roger failed discrete mathematics.

We want to determine whether the argument

$$(p_1 \land p_2 \land p_3) \to q$$

is valid. To do so, we rewrite p_1, p_2, p_3 as

$$p_1: \quad p \to r \qquad p_2: \quad \neg q \to p \qquad p_3: \quad \neg r$$

and examine the truth table for the implication

$$[(p \to r) \land (\neg q \to p) \land \neg r] \to q$$

Table 2.18

			p_1	p_2	p_3	$(p_1 \wedge p_2 \wedge p_3) \rightarrow q$
p	q	r	$p \rightarrow r$	$\neg q \rightarrow p$	$\neg r$	$[(p \rightarrow r) \wedge (\neg q \rightarrow p) \wedge \neg r] \rightarrow q$
0	0	0	1	0	1	1
0	0	1	1	0	0	1
0	1	0	1	1	1	1
0	1	1	1	1	0	1
1	0	0	0	1	1	1
1	0	1	1	1	0	1
1	1	0	0	1	1	1
1	1	1	1	1	0	1

Rules of Inference

1. p

$p \to q$

$\therefore q$

Rule of Detachment (Modus Ponens)

* To check the validity of the rule of detachment.

$p \wedge (p \to q) \longrightarrow q$ is the argument

p	q	$p \to q$	$p \wedge (p \to q)$	$p \wedge (p \to q) \to q$
0	0	1	0	1
0	1	1	0	1
1	0	0	0	1
1	1	1	1	1

From the last column, truth value of $p \wedge (p \to q) \to q$ is **1** always

$\Rightarrow$ Rule of detachment is valid

	Rule of Inference	Related Logical Implication	Name of Rule
1)	p $p \to q$ $\therefore q$	$[p \wedge (p \to q)] \to q$	Rule of Detachment (Modus Ponens)
2)	$p \to q$ $q \to r$ $\therefore p \to r$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$	Law of the Syllogism
3)	$p \to q$ $\neg q$ $\therefore \neg p$	$[(p \to q) \wedge \neg q] \to \neg p$	Modus Tollens
4)	p q $\therefore p \wedge q$		Rule of Conjunction
5)	$p \vee q$ $\neg p$ $\therefore q$	$[(p \vee q) \wedge \neg p] \to q$	Rule of Disjunctive Syllogism
6)	$\neg p \to F_0$ $\therefore p$	$(\neg p \to F_0) \to p$	Rule of Contradiction
7)	$p \wedge q$ $\therefore p$	$(p \wedge q) \to p$	Rule of Conjunctive Simplification
8)	p $\therefore p \vee q$	$p \to p \vee q$	Rule of Disjunctive Amplification
9)	$p \wedge q$ $p \to (q \to r)$ $\therefore r$	$[(p \wedge q) \wedge [p \to (q \to r)]] \to r$	Rule of Conditional Proof
10)	$p \to r$ $q \to r$ $\therefore (p \vee q) \to r$	$[(p \to r) \wedge (q \to r)] \to [(p \vee q) \to r]$	Rule for Proof by Cases
11)	$p \to q$ $r \to s$ $p \vee r$ $\therefore q \vee s$	$[(p \to q) \wedge (r \to s) \wedge (p \vee r)] \to (q \vee s)$	Rule of the Constructive Dilemma
12)	$p \to q$ $r \to s$ $\neg q \vee \neg s$ $\therefore \neg p \vee \neg r$	$[(p \to q) \wedge (r \to s) \wedge (\neg q \vee \neg s)] \to (\neg p \vee \neg r)$	Rule of the Destructive Dilemma