5/09

MODULE - 2

PRINCIPLES OF COUNTING

$(x+y)^n = {}^nC_k x^k y^{n-k}$

Binomial theorem

If x and y are variables and n is a positive integer then, $(x+y)^n = \sum_{k=0}^{n} {}^nC_k x^k y^{n-k}$

${}^nC_k \Rightarrow \binom{n}{k} = \frac{n!}{r!(n-r)!}$

1. Determine the coefficient of x^5y^2 in the expansion of $(x+y)^7$.

${}^7C_5$

Soln: We have $(x+y)^n = \sum_{k=0}^{n} {}^nC_k x^k y^{n-k}$

i.e, the coefficient of $x^k y^{n-k}$ is ${}^nC_k$ (or $\binom{n}{k}$).

$\therefore$ The coefficient of x^5y^2 in the expansion of $(x+y)^7$ is ${}^7C_5 = 21$.

2. Determine the coefficient of x^9y^3 in the expansion of $(x+y)^{12}$.

Soln: The coefficient of x^9y^3 in the expansion of $(x+y)^{12}$ is ${}^{12}C_9 = \underline{220}$

${}^{12}C_r$ $\qquad \frac{12!}{9!\,3!} = \frac{10\times 11\times 12}{6}$

3. Coefficient of a^5b^2 in the expansion of $(2a-3b)^7$

Soln: We have $(x+y)^n = \sum_{k=0}^{n} {}^nC_k x^k y^{n-k}$ $\Rightarrow$ Let $x = 2a$, $y = -3b$

$\Rightarrow (2a-3b)^n = \sum_{k=0}^{n} {}^nC_k (2a)^k (-3b)^{n-k}$ $\qquad {}^7C_5\, 2^5 \cdot (-3)^2$

i.e, coefficient of $a^k b^{n-k}$ is ${}^nC_k \cdot 2^k \cdot (-3)^{n-k}$

$\therefore$ Coefficient of a^5b^2 in the expansion of $(2a-3b)^7$ is ${}^7C_5 \cdot 2^5 \cdot (-3)^2 = \underline{6048}$.

4. Coefficient of ~~exp~~ $x^9 y^3$ in the expansion of $(2x-3y)^{12}$

Soln: Coefficient of $x^k y^{n-k}$ in the expansion of $(2x-3y)^n$ is ${}^nC_k \cdot 2^k \cdot (-3)^{n-k}$

$\therefore$ The coefficient of $x^9 y^3$ is ${}^{12}C_9 \cdot 2^9 \cdot (-3)^3$ is

-3041280.

Multinomial theorem

For positive integers n, t the coefficient of $x_1^{n_1} x_2^{n_2} x_3^{n_3} \dots x_t^{n_t}$

* in the expansion of $(x_1 + x_2 + x_3 + \dots + x_t)^n$ is $\dfrac{n!}{n_1!\, n_2!\, n_3! \dots n_t!}$

where each n_i is an integer $0 \le n_i \le n$ $\forall\ 1 \le i \le t$ and $n_1 + n_2 + \dots + n_t = n$

* in the expansion of $(a_1 x_1 + a_2 x^2 + a_3 x^3 + \dots a_t x_t)^n$ is $\dfrac{n!\, a_1^{n_1} \cdot a_2^{n_2} \dots a_t^{n_t}}{n_1!\, n_2!\, n_3! \dots n_t!}$

1. Determine the coefficient of $x^2y^2z^3$, of xyz^5 and x^3z^4 in the expansion of $(x+y+z)^7$.

Soln: Coefficient of $x^2y^2z^3$ in the expansion of $(x+y+z)^7$ is $\dfrac{7!}{2!\,2!\,3!} = \underline{\underline{210}}$

Coefficient of xyz^5 is $\dfrac{7!}{1!\,1!\,5!} = \underline{\underline{42}}$

Coefficient of x^3z^4 is $\dfrac{7!}{3!\,4!} = \underline{\underline{35}}$.

2. Determine the coefficient of xyz^2 in the expansion of $(w+x+y+z)^4$.

Soln: The coefficient of xyz^2 in the expansion of $(w+x+y+z)^4$ is $\dfrac{4!}{1!\,1!\,2!} = \underline{\underline{12}}$

3 Coefficient of xyz^2 in the expansion of $(ax+by+cz)^4$

Soln: The coefficient of xyz^2 in the expansion of $(ax+by+cz)^4$ is $\dfrac{4! \times a^1 \times b^1 \times c^2}{1!\,1!\,2!} = \underline{12abc^2}$

4 Determine the coefficient of $w^3x^2yz^2$ in the expansion of $(2w-x+3y-2z)^8$.

Soln: The coefficient is $\dfrac{8! \times 2^3 (-1)^2 \cdot 3 \cdot (-2)^2}{3!\,2!\,1!\,2!}$

$= \underline{161280}$.

5. Determine the coefficient of $a^2b^3c^2d^5$ in the expansion of $(a+2b-3c+2d+5)^{16}$.

Soln: We know that the coefficient of $x_1^{n_1} x_2^{n_2} \cdots x_5^{n_5}$ in the expansion of $(x_1+x_2+x_3+x_4+x_5)^n$ is

$\dfrac{n!}{n_1!\,n_2! \cdots n_5!}$ provided $n_1+n_2+\cdots+n_5 = n$.

Let $x_1 = a$, $x_2 = 2b$, $x_3 = -3c$, $x_4 = 2d$, $x_5 = 5$. and

$n_1 = 2$, $n_2 = 3$, $n_3 = 2$, $n_4 = 5$, $n_5 = n-(n_1+n_2+n_3+n_4)$

$= 16-12 = \underline{4}$.

Hence the coefficient of $a^2b^3c^2d^5$ in the expansion of $(a+2b-3c+2d+5)^{16}$ is $\dfrac{16! \times 1^2 \times 2^3 \times -3^2 \times 2^5 \times 5^4}{2!\,3!\,2!\,5!\,4!}$

$= 4.358514 56 \times 10^{14}$

6 Determine the coefficient of $w^2x^2y^2z^2$ in the expansion of $(2w-x+3y+z-2)^{12}$.

Soln: Coefficient is $\dfrac{12! \times 2^2 \times (-1)^2 \times 3^2 \times 1^2 \times (-2)^4}{2!\,2!\,2!\,2!\,4!} = \underline{718502400}$

20/09

1. For every integer $n > 0$, P.T (i) $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} = 2^n$

(ii) $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \dots + (-1)^n \binom{n}{n} = 0$

$x=1,\ y=1 \quad 2^n = \sum_{k=0}^{n} {}^nC_k$

$x=-1,\ y=1 \quad (-1+1)^n = 0 = \sum_{k=0}^{n} {}^nC_k (-1)^k$

Soln: We have $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$ — (1)

(i) put $x=1$, $y=1$ in eq (1),

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$ i.e, The sum of all the coefficients in the expansion of $(x+y)^n$ is 2^n

(ii) Put $x=-1$, $y=1$ in eq (1)

$$0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k.$$

2. Determine the sum of all the coefficients in the expansion of $(x+y)^{10}$ $= (1+1)^{10}$ $\hookrightarrow 2^n$

Soln: The sum of all the coefficients in the expansion of $(x+y)^{10}$ is 2^{10}

~~(Note: For $(x+y)^n$, it is 2)~~
(Note: For $(x+y)^n$, it is 2^n)

Note: The sum of all the coefficients in the expansion of $(a_1x_1 + a_2x_2 + \dots a_tx_t)^n$ is $(a_1 + a_2 + a_3 + \dots + a_t)^n$

3. Determine the sum of all the coefficients in the expansion of

(i) $(w+x+y+z)^5$ $= (1+1+1+1)^5$

Soln: The sum of all the coefficients is 4^5.

(ii) $[2s - 3t + 5u + 6v - 11w + 3x + 2y]^{10}$

$= (2-3+5+6-11+3+2)^{10}$

4^{10}

Soln: $\underline{4^{10}}$.

Permutation

A permutation is an arrangement of objects in a particular order from a collection of n distinct objects. Any linear arrangement of these objects is called permutation of these collection.

1. The no. of permutations of n different objects taken all at a time is $\underline{n!}$

2. The no. of permutations of n different objects taken r at a time when $0 < r \le n$ is $\underline{{}^nP_r}$

$$= \frac{n!}{(n-r)!}$$

3. The no. of permutations of n different objects taken all at a time when repetitions of objects allowed is n^n.

4. The no. of permutations of n different objects taken r at a time when repetition allowed is $\underline{n^r}$.

Permutation of alike things

If there are n objects with n_1 identical objects of first type, n_2 identical objects of 2^{nd} type and n_r identical objects of r^{th} type where $n_1 + n_2 + \dots + n_r = n$ then there are $\frac{n!}{n_1! n_2! \dots n_r!}$ linear arrangements of the given n objects.

Circular Permutations

The permutations discussed so far are linear permutations, as the objects are arranged in a line. If the objects are arranged in a circle, it is a circular permutation.

The no. of different circular arrangements of n objects is $(n-1)!$

Q. 1 List all the permutations for the letters a, c, g

Soln: $n! = 3! = 6$

acg , cag , gac
agc , cga , gca

2. How many permutations are there for the 8 letters, a, c, f, g, i, t, w, x. How many of these (i) start with letter g, (ii) start with t and end with c

Soln: $n! = 8! = 40320$

(i) $7! = 5040$

(ii) $6! = 720$

3. In a class of 10 students 5 are to be chosen and seated in a row for a picture. How many such linear arrangements are possible.

Soln: ${}^{10}P_5 = 30240$

4. How many ways can we arrange the letter in the word computer. Find the no. of permutations of size two. If repetitions are allowed find the no. of possible 12 letter sequence n^r

Soln: (i) Total No. of ways $= 8! = 40320$

(ii) Of size two $= {}^8P_2 = 56$

(iii) When repetitions are allowed, no. of ways $= 8^{12}$

~~5. Find the pos~~

5. How many arrangements are there of all the letters in the word databases.

Soln: $\dfrac{9!\ (\text{total})}{\underset{(a)}{3!}\ \underset{(s)}{2!}} = \cancel{181440}\ 30240$.

6. How many arrangements are there for the word SOCIOLOGICAL. How many of these arrangements are A and G adjacent. " " " " " all the vowels adjacent.

Soln: (i) Total no. of arrangements $= \dfrac{12!}{3!\,2!\,2!\,2!} = 9979200$

(ii) AG adjacent $\Rightarrow \dfrac{11!}{3!\,2!\,2!\,2!} = 831600$

consider AG as one letter

(iii) All vowels adjacent $\Rightarrow \dfrac{7!}{2!\,2!} \times \dfrac{6!}{3!\,2!} = \underline{75600}$

Take vowels as one letter

permutations within vowels

Q.1 In how many ways can the letters in ~~[illegible]~~ UNUSUAL is arranged such that (a) there is no restriction (b) all the three U's together.

Soln: a) $\frac{7!}{3!} = 840$

b) $5! = 120$

2. In how many ways can the letters of the word MATHEMATICS ~~can~~ be arranged such that vowels must always come together?

Soln: $\frac{8!}{2!\,2!} \times \frac{4!}{2!} = 120960$

3. In how many ways can 8 men and 8 women sealed in a row if (a) any person may sit next to any other b) men and women must occupy alternate seats.

Soln: a) $16!$

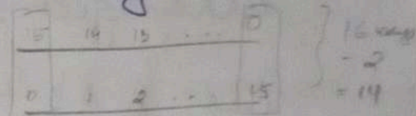
b) Consider the case with man sitting first. Then 8 men can sit in $8!$ ways and 8 women can sit in $8!$ ways. $\therefore$ No. of arrangements $= (8!)^2$

Consider the case with women sitting first ~~Then~~ Here also no. of arrangements is $8! \times 8!$.

Thus, the no. of ways men and women must occupy alternate seats $= 2 \times (8!)^2$

4. Pamela has 15 different books. In how many can she place her books on the two shelves so that there is atleast one book on each shelf.

Soln. Let T1 be the arrangement of 15 books in 2 shelves ~~so that there is at~~ and T2 be the arrangement of 15 books on 2 shelves so that there is atleast one book on each shelf. T1 can be performed in 15! different ways and T2 can be performed in 14 ways



$\therefore$ T1 and T2 can be performed in $14 \times 15!$ ways.

5. How many positive integers can we form using the digits 3, 4, 4, 5, 5, 6, 7, if we want to exceed 5,000,000?

Soln: $\dfrac{6!}{2!} + \dfrac{6!}{2! \times 2!} + \dfrac{6!}{2! \times 2!} = 720$.

6. 6 papers are set in an examination of which 2 distinct papers are mathematical. Only one examination will be conducted in a day (a) in how many different orders the paper can be arranged?

b) " " " " " " " so that 2 mathematical papers are consecutive?

Soln: a) $6!$ b) $2 \times 5!$

7. How many distinct five digit integers can ~~we~~ be make from the digits 1, 2, 3, 4 and 5? How many of them are even? How many of them are even and greater than 3,00,00?

Soln: a) $5! = 120$ b) $4! + 4! = 48$ c) $3! + 3! + 3! + 3! + 3! = 30$

8. In how many ways can the letters MASSASAUGA is arranged such that a) there is no restriction b) all the A's together

Soln: a) $\dfrac{10!}{4!\,3!} = 25200$ b) $\cancel{\dfrac{8!}{2!\,3!}}$ $\dfrac{7!}{3!} = 840$.

9. If 6 people, designated A, B, C, D, E, F are seated about a round table, how many different circular arrangements are possible? Of these 6 people ABC are female and DEF are male, find the no. of arrangements of the people around the table so that female is alternate.

(a) $(6-1)! = 5! = 120$. b) $2 \times 3! = 12$

10. In how many ways can the symbols a, b, c, d, e, e, e, e, e be arranged so that no e is adjacent to another e.

_ a _ b _ c _ d _

Soln: $4! = 24$

29/9

Combination

A combination is a selection of objects from a group where order is not relevant.

An unordered ~~selected~~ selection of r objects from a set of n objects is called a combination.

For example, if a committee is being selected and there will be a president, vice president and treasurer, then order matters, so it is a permutation problem.

But if a committee of 3 people is being formed without specific roles, then order does not matter and it is a combination problem.

The no. of selections of n distinct objects taking r at a time is the combination denoted by ${}^nC_r$ or $C(n,r) = \dfrac{n!}{r!(n-r)!}$ $\binom{n}{r} = {}^nC_r$

Q. 1 A student is to answer 7 out of 10 questions on an examination. In how many ways can he make this selection if (i) there are no restrictions.

(ii) he must answer the first 2 questions.

(iii) he must answer atleast 4 of the first 6 questions.

Soln: (i) ${}^{10}C_7 = \dfrac{10!}{7!(10-7)!} = 120$

(ii) ${}^8C_5 = \cancel{\dfrac{8!}{5!(8-7)!}}\ \dfrac{8!}{5!(8-5)!} = 56$

(iii) ${}^6C_4 \times {}^4C_3 + {}^6C_5 \times {}^4C_2 + {}^6C_6 \times {}^4C_1 = 100$

2. A committee of 8 is to be formed from 16 men and 10 women. In how many ways can the committee be formed if (i) there are no restrictions (ii) there must be 4 men and 4 women, (iii) there should be an even no. of women, (iv) more women than men (v) atleast 6 men.

Soln: (i) $^{26}C_8 = 1562275$.

(ii) $^{16}C_4 \times {}^{10}C_4 = 382200$

(iii) $^{10}C_2 \times {}^{16}C_6 + {}^{10}C_4 \times {}^{16}C_4 + {}^{10}C_6 \times {}^{16}C_2 + {}^{10}C_8 = 767805$

(iv) $^{10}C_6 \times {}^{16}C_2 + {}^{10}C_8 + {}^{10}C_5 \times {}^{16}C_3 + {}^{10}C_7 \times {}^{16}C_1 = 168285$

(v) $^{16}C_6 \times {}^{10}C_2 + {}^{16}C_7 \times {}^{10}C_1 + {}^{16}C_8 = 487630$

3. A choir director must select 6 hymns for a Sunday church service. She has 3 hymn books each containing 25 hymns (there are 75 different hymns in all). In how many ways can she select the hymns if she wishes to select (i) 2 hymns from each book (ii) atleast 1 hymn from each book.

Soln: (i) $^{25}C_2 \times {}^{25}C_2 \times {}^{25}C_2 = 27{,}000{,}000$

(ii) $3\left({}^{25}C_1 \times {}^{25}C_1 \times {}^{25}C_4\right) + \left({}^{25}C_1 \times {}^{25}C_2 \times {}^{25}C_3\right) \times 3!$

$+ \left({}^{25}C_2 \times {}^{25}C_2 \times {}^{25}C_2\right) = 154218750$

4. How many arrangements of the letters BALL has no consecutive Ls

Soln: Total no of permutations = $\frac{4!}{2!} = 12$

BALL, BLAL, ABLL, LLAB

LALB, LLBA, BLLA, ALLB

LBAL, LABL, LBLA ALBL

With no consecutive Ls = $\underline{\underline{6}}$

$\left(\frac{4!}{2!} - 3! = 6\right)$

OR

$2! \times {}^3C_2 = 6$

5. Find the no. of arrangements of letters in TALLAHASSEE

How many arrangements have no adjacent As.

Soln: Total no. of ways = $\frac{11!}{3! \times 2! \times 2! \times 2!}$ = $\underline{\underline{831600 \text{ ways}}}$

A L S E

~~With no adjacent As = 831600~~

(ii) If we remove all As the letter become TLLHSSEE

No. of arrangements of the word TLLHSSEE without

As $= \frac{8!}{2!\,2!\,2!} = 5040$ ways

We can place A in the following 9 places.

– T – L – L – H – S – S – E – E – . 3 of these location can be selected in ${}^9C_3$ ways = 84 ways. This is also possible for all the other 5039 arrangements

∴ Total no. of arrangements having no consecutive no adjacent As = 5040 × 84 = 423360

Q.1 An urn contains 15 balls, 8 of which are red, 7 are black. In how many ways, can 5 balls be chosen so that (a) all 5 are red (b) all 5 are black (c) 2 are red, 3 are black (d) 3 red, 2 black.

a) $^{8}C_5 = 56$ b) $^{7}C_5 = 21$ c) $^{8}C_2 \times {^{7}C_3} = 980$

d) $^{8}C_3 \times {^{7}C_2} = 1176$.

2. How many arrangements of the letter MISSISSIPPI have no consecutive S's?

_M_I_I_I_P_P_I_

No. of arrangements without S = $\frac{7!}{4!\,2!} = 105$

We can place S in 8 locations. 4 of these can be selected in $^{8}C_4$ ways = 70.

∴ Total no. of arrangements with no consecutive S = 7350.

3. In how many ways can 12 different books be distributed among 4 children so that a) each child gets 3 books? b) The 2 oldest children get 4 books each and the 2 youngest get 2 books each.

a) $^{12}C_3 \times {^{9}C_3} \times {^{6}C_3} \times {^{3}C_3} = 369600$

b) $^{12}C_4 \times {^{8}C_4} \times {^{4}C_2} \times {^{2}C_2} = 207900$

4. a) How many ways are there to pick a 5 person basketball team from 12 possible players? b) How many selections include the weakest and strongest players.

↳ consider them to be already in the team ∴ choose the remaining 3 players from 10

$^{10}C_3$

a) $^{12}C_5 = 792$

b) $^{10}C_3 = 120$

5. In the manufacture of a certain type ~~and~~ of automobile, 4 kinds of major defects and 7 kinds of minor defects can occur. For those situations in which defects do occur, in how many ways can there be twice as many minor defects as there are major ones.

$$^4C_1 \times {}^7C_2 + {}^4C_2 \times {}^7C_4 + {}^4C_3 \times {}^7C_6 = \underline{\underline{322}}$$

6. A physical education teacher must make up 4 volley ball teams A, B, C, D of 9 girls each from 36 girls in the class. In how many ways can she select these 4 teams?

$$^{36}C_9 \times {}^{27}C_9 \times {}^{18}C_9 \times {}^9C_9 = \underline{\underline{214.5 \times 10^{17}}} \text{ ways}$$

04/10

Combinations with repetition

We have seen that for n distinct objects an arrangement of size r can be obtained in n^r ways if repetitions are allowed.

We now consider the case of combinations of n objects taken r at a time if repetitions are allowed.

Suppose that there are n elements in a set A. We are asked to select r elements from the set given that each element can be selected multiple times, this is known as a combination with repetition.

The no. of combinations of n objects taken r at a time with repetition is $^{(n+r-1)}C_r$ which is equivalent to the no. of integer solutions of the equation $x_1 + x_2 + .. + x_n = r$, $x_i \geq 0$, $1 \leq i \leq n$.

Q.1 An ice cream vendor sells 3 flavours of ice cream, i.e. vanilla, chocolate, mango. 4 kids visit the shop and they all take the flavours of their choice. In how many ways can the vendor sell 4 ice creams of 3 flavours

$x_1 + x_2 + x_3$

C	V	M
0	0	4
0	4	0
4	0	0
0	3	1
0	1	3
1	0	3
1	3	0
3	0	1
3	1	0
0	2	2
2	0	2
2	2	0

C	V	M
1	1	2
1	2	1
2	1	1

$n = 3, \quad r = 4$

$x_1 + x_2 + x_3 = 4 \Rightarrow x_1 + x_2 + x_n = r$

$$^{(n+r-1)}C_r = {}^{(3+4-1)}C_4 = {}^6C_4 = \underline{\underline{15}}$$

Distribution of bananas & oranges are independent. ∴ No. of bananas and oranges cannot be added to determine r

2. In how many ways can we distribute 7 bananas and 6 oranges among 4 kids so that each child gets atleast one banana.

Soln: After giving 1 banana to each child the remaining 3 bananas can be distributed to 4 children with repetition r.

$n=4$, $r=3$ $\quad (x_1+x_2+x_3+x_4=3)$

$^{(n+r-1)}C_r = {}^6C_3 = 20$ ways.

6 oranges can be distributed to 4 kids in $(x_1+x_2+x_3+x_4=6)$

$n=4$, $r=6$ is $^{(4+6-1)}C_6 = {}^9C_6 = 84$ ways

$\therefore$ Total no. of ways $= {}^6C_3 \times {}^9C_6 = \underline{1680}$ ways.

3. Determine the no. of integer solutions of $x_1+x_2+x_3+x_4 = 32$

(a) $x_i \geq 0$, $1 \leq i \leq 4$

(b) $x_i > 0$, $1 \leq i \leq 4$ $\Rightarrow x_i \geq 1 \Rightarrow y_i = x_i - 1 \Rightarrow x_i = y_i + 1$

(c) $x_1, x_2 \geq 5$, $x_3, x_4 \geq 7$ $\quad y_1 = x_1 - 5$, $y_2 = x_2 - 5$, $y_3 = x_3 - 7$, $y_4 = x_4 - 7$

(d) $x_i \geq 8$, $1 \leq i \leq 4$ $\Rightarrow y_i = x_i - 8$

(e) $x_i \geq -2$, $1 \leq i \leq 4$ $\Rightarrow y_i = x_i + 2$

Soln: (a) The no. of integer solutions of the equation $x_1 + x_2 + \dots + x_n = r$; $x_i \geq 0$, $1 \leq i \leq n$ is given by

$^{(n+r-1)}C_r$. ~~where~~

$n=4$, $r=32$

$$^{35}C_{32} = \underline{\underline{6545}}$$

b) Given $x_i > 0$, $1 \leq i \leq 4$

$\Rightarrow x_i \geq 1$, $1 \leq i \leq 4$

let $y_i = x_i - 1$, $1 \leq i \leq 4 \Rightarrow y_i \geq 0$, $1 \leq i \leq 4$

$x_1 + x_2 + x_3 + x_4 = 32$

$y_1 + 1 + y_2 + 1 + y_3 + 1 + y_4 + 1 = 32 \qquad y_i \geq 0 \ \forall i$

$y_1 + y_2 + y_3 + y_4 = 28$

$\therefore n = 4, \quad r = 28$

$^{(n+r-1)}C_r = {}^{31}C_{28} = \underline{4495}$

(c) Let $x_1, x_2 \geq 5$ $\qquad$ $x_3, x_4 \geq 7$

$\Rightarrow$ ~~Let y_1, y~~ $\qquad$ Let $y_3 = x_3 - 7$

Let $y_1 = x_1 - 5$ $\qquad$ $y_4 = x_4 - 7$

$y_2 = x_2 - 5$

$x_1 + x_2 + x_3 + x_4 = 32$

$y_1 + 5 + y_2 + 5 + y_3 + 7 + y_4 + 7 = 32$

$y_1 + y_2 + y_3 + y_4 = 8$

$n = 4, \quad r = 8$

$^{(n+r-1)}C_r = {}^{11}C_8 = \underline{\underline{165}}$

d) $x_i \geq 8$

Let $y_i = x_i - 8$

$x_1 + x_2 + x_3 + x_4 = 32$

$y_1 + 8 + y_2 + 8 + y_3 + 8 + y_4 + 8 = 32$

$y_1 + y_2 + y_3 + y_4 = 0$

$n = 4, r = 0 \Rightarrow$ $^{(n+r-1)}C_r = {}^{3}C_0 = \underline{\underline{1}}$

e) $x_i \geq -2$

Let $y_i = x_i + 2$

$x_1 + x_2 + x_3 + x_4 = 32$

$y_1 - 2 + y_2 - 2 + y_3 - 2 + y_4 - 2 = 32$

$y_1 + y_2 + y_3 + y_4 = 40$

$n = 4, \quad r = 40 \Rightarrow$ $^{(n+r-1)}C_r = {}^{43}C_{40} = \underline{12341}$

06/10

Q. Determine the no. of integer solutions of $x_1+x_2+x_3+x_4+x_5 < 40$ where 1) $x_i \geq 0,\ 1 \leq i \leq 5$

2) $x_i \geq -3,\ 1 \leq i \leq 5$

Soln: Given $x_1+x_2+x_3+x_4+x_5 < 40$

Consider $x_6 > 0$ such that $x_1+x_2+x_3+x_4+x_5+x_6 = 40$

(1) The no. of integer solutions of $x_1+x_2+x_3+x_4+x_5 < 40$ when $x_i \geq 0,\ 1 \leq i \leq 5$ = No. of integer solutions of $x_1+x_2+x_3+x_4+x_5+x_6 = 40$, $x_i \geq 0\ \ 1 \leq i \leq 5$, $x_6 \geq 1$

Let $y_6 = x_6 - 1$. $y_6 \geq 0$

and $y_i = x_i$

$\Rightarrow y_1+y_2+y_3+y_4+y_5+y_6+1 = 40$, $y_i \geq 0$

$\Rightarrow y_1+y_2+\dots+y_6 = 39$

$n = 6,\ r = 39$

$^{(n+r-1)}C_r = {}^{44}C_{39} = \underline{1086008}$

(2) The no. of integer solutions of $x_1+x_2+x_3+x_4+x_5 < 40$ when $x_i \geq -3,\ 1 \leq i \leq 5$ = No. of integer solutions of $x_1+x_2+x_3+x_4+x_5+x_6 = 40$, $x_i \geq -3,\ 1 \leq i \leq 5$, $x_6 \geq 1$

Let $y_6 = x_6 - 1$, $y_6 \geq 0$

and $y_i = x_i + 3$

$\Rightarrow y_1 - 3 + y_2 - 3 + y_3 - 3 + y_4 - 3 + y_5 - 3 + y_6 + 1 = 40$

$y_i \geq 0, \ 1 \leq i \leq 6$

$\Rightarrow y_1 + y_2 + \cdots + y_6 = 54$

$n = 6, \ r = 54$

$^{(n+r-1)}C_r = {}^{59}C_{54} = 5006386$

Q. Determine the no. of integer solutions to the pairs of equations $x_1 + x_2 + \cdots + x_7 = 37$ and $x_1 + x_2 + x_3 = 6$
$x_i \geq 0, \ 1 \leq i \leq 7$

Soln: ~~No. of integer solutions~~

Soln:

$x_1 + x_2 + \cdots + x_7 = 37$

No. of integer solutions of $x_1 + x_2 + \cdots + x_7 = 37$,
$x_i \geq 0, \ 1 \leq i \leq 7 = {}^{(7+37-1)}C_{37} = {}^{43}C_{37} = 6096454.$

No. of integer solutions of $x_1 + x_2 + x_3 = 6$, $x_i \geq 0$,
$1 \leq i \leq 3 = {}^{(3+6-1)}C_6 = {}^{8}C_6 = 28.$

$\therefore$ No. of integer solutions of these pair of eqn

$= {}^{43}C_{37} \times {}^{8}C_6 = 170700712$

Q. Determine the no. of integer solutions to the pair of eqn $x_1 + x_3 + x_5 + x_7 = 5$, $x_1, x_3, x_5, x_7 > 0$
and $x_2 + x_4 + x_6 = 10$, $x_2, x_4, x_6 > 0$.

Soln: For eqn $x_1 + x_3 + x_5 + x_7 = 5$, ~~$n = 4$~~, $x_1, x_3, x_5, x_7 > 0$
$\Rightarrow x_1, x_3, x_5, x_7 \geq 1$

$y_1 + 1 + y_3 + 1 + y_5 + 1 + y_7 + 1 = 5$

$y_1 + y_3 + y_5 + y_7 = 5 - 4 = 1$

$\therefore n = 4, \ r = 1$

$^{(n+r-1)}C_r = {}^{(4+1-1)}C_1 = {}^4C_1$

For eqn. $x_2 + x_4 + x_6 = 10$, $x_2, x_4, x_6 > 0$

$y_2 + 1 + y_4 + 1 + y_6 + 1 = 10$ $\Rightarrow x_2, x_4, x_6 \geq 1$

$n = 3,\ r = 7$

Let $y_i = x_i - 1$

$i = 2, 4, 6$

$^{(n+r-1)}C_r = {}^{(3+7-1)}C_7 = {}^9C_7$

For the pair of eqns, no. of integer solns is

$^4C_1 \times {}^9C_7 = 144$

12/10

Principle of Inclusion and exclusion

Consider a set S with $|S| = N$ and conditions C_i, $1 \leq i \leq t$, each of which may be satisfied by some of the elements of S. The no. of elements of S that satisfy none of the conditions is denoted by $\overline{N} = N(\overline{C}_1 \overline{C}_2 \ldots \overline{C}_t)$

$= N - [N(C_1) + N(C_2) + \ldots + N(C_t)] + [N(C_1C_2) + N(C_1C_3) + \ldots + N(C_{t-1}C_t)]$

$- [N(C_1C_2C_3) + N(C_1C_2C_4) + \ldots + N(C_{t-2}, C_{t-1}, C_t)]$

$+ \ldots + (-1)^t N(C_1 C_2 \ldots C_t)$

Q.1 Determine the no. of +ve integers 'n' where $1 \leq n \leq 100$ and n is not divisible by 2, 3 ~~or~~ and 5.

Soln:

$S = \{1, 2, 3 \ldots, 100\}$, $N = 100$

For $n \in S$, n satisfy

(1) C_1 : if n is divisible by 2

(2) C_2 : if n is divisible by 3

(3) C_3 : if n is divisible by 5

By principle of inclusion and exclusion, the no of +ve integers 'n', $1 \leq n \leq 100$, n is not divisible by 2, 3 or 5.

$$\bar{N} = N - [N(C_1) + N(C_2) + N(C_3)] + [N(C_1C_2) + N(C_1C_3) + N(C_2C_3)] - N(C_1C_2C_3) \quad -(1)$$

$N(C_1) = \left\lfloor \frac{100}{2} \right\rfloor = 50$ (No. of members of S divisible by 2)

$N(C_2) = \left\lfloor \frac{100}{3} \right\rfloor = 33$ (" " " " " " 3)

$N(C_3) = \left\lfloor \frac{100}{5} \right\rfloor = 20$ (" " " " " " 5)

$$N(C_1C_2) = \left\lfloor \frac{100}{\text{lcm}(2,3)} \right\rfloor = 16$$

$$N(C_1C_3) = \left\lfloor \frac{100}{\text{lcm}(2,5)} \right\rfloor = 10$$

$$N(C_2C_3) = \left\lfloor \frac{100}{\text{lcm}(3,5)} \right\rfloor = 6$$

$$N(C_1C_2C_3) = \left\lfloor \frac{100}{\text{lcm}(2,3,5)} \right\rfloor = \cancel{30}\ 3$$

$$\bar{N} = 100 - (50 + 33 + 20) + (16 + 10 + 6) - 3$$

$$= 100 - 103 + 32 - 3$$

$$= \underline{26}$$

8. Find the no. of integers b/w 1 and 10,000 inclusive which are not divisible by 5, 6 ~~or~~ and 8

$S = \{1, 2, 3, \ldots, 10{,}000\}$ $N = 10{,}000$

$C_1 \Rightarrow$ divisible by 5

$C_2 \rightarrow$ " " 6

$C_3 \Rightarrow$ " " 8

By principle of inclusion & exclusion,

$$\bar{N} = N - [N(C_1) + N(C_2) + N(C_3)] + [N(C_1C_2) + N(C_1C_3) + N(C_2C_3)] - N(C_1C_2C_3)$$

$$N(C_1) = \left\lfloor \frac{10000}{5} \right\rfloor = \cancel{5000}\ 2000$$

$$N(C_2) = \left\lfloor \frac{10000}{6} \right\rfloor = 1666$$

$$N(C_3) = \left\lfloor \frac{10000}{8} \right\rfloor = 1250$$

$$N(C_1C_2) = \left\lfloor \frac{10000}{lcm(5,6)} \right\rfloor = 333$$

$$N(C_1C_3) = \left\lfloor \frac{10000}{lcm(5,8)} \right\rfloor = 250$$

$$N(C_2C_3) = \left\lfloor \frac{10000}{lcm(6,8)} \right\rfloor = 416$$

$$N(C_1C_2C_3) = \left\lfloor \frac{10000}{lcm(5,6,8)} \right\rfloor = 83$$

$$\bar{N} = 10{,}000 - (2000 + 1666 + 1250) + (333 + 250 + 416) - 83$$

$$= \underline{6000}$$

3. In how many ways can the 26 letters of the alphabet be permuted so that none of the patterns car, dog, pun and byte occurs.

Ans: Let S denote set of all permutation of 26 letters

$|S| = N = 26!$

For each permutations on S define the following conditions :-

(1) C_1 : permutations contains the pattern car

(2) C_2 : " " " dog

(3) C_3 : " " " pun

(4) C_4 : " " " byte

$$\bar{N} = N - [N(C_1) + N(C_2) + N(C_3) + N(C_4)] + [N(C_1C_2) + N(C_1C_3) + N(C_1C_4) + N(C_2C_3) + N(C_2C_4) + N(C_3C_4)] - [N(C_1C_2C_3) + N(C_1C_2C_4) + N(C_1C_3C_4) + N(C_2C_3C_4)] + N(C_1C_2C_3C_4)$$

$N(C_1) = 24!$ (consider car as single object)

$N(C_2) = 24!$

$N(C_3) = 24!$

$N(C_4) = 23!$

$N(C_1C_2) = 22!$ $\left(\underset{1}{\text{car}} + \underset{1}{\text{dog}} + \underset{20}{\underline{\qquad}} \right)$

$N(C_1C_3) = 22!$

$N(C_1C_4) = 21!$

$N(C_2C_3) = 22!$

$N(C_2C_4) = 21!$

$N(C_3C_4) = 21!$

$N(C_1C_2C_3) = 20!$

$N(C_1C_2C_4) = 19!$

$N(C_1C_3C_4) = 19!$

$N(C_2C_3C_4) = 19!$

$N(C_1C_2C_3C_4) = 17!$

$\bar{N} = 26! - (24! + 24! + 24! + 23!) + (22! + 22! + 21! + 22! + 21! + 21!)$

$- (20! + 19! + 19! + 19!) + 17!$

$= 4.014 \times 10^{26}$

18/10

Q. Consider the eqn $x_1 + x_2 + x_3 + x_4 = 18$, $x_i \geq 0$. Find the no. of integer solutions in which $x_i \leq 7$, $1 \leq i \leq 4$

Soln: Let S be the set of integer solutions of the eqn $x_1 + x_2 + x_3 + x_4 = 18$, $x_i \geq 0$, $1 \leq i \leq 4$.

Then $|S| = N = {}^{(n+r-1)}C_r = {}^{(4+18-1)}C_{18} = {}^{21}C_{18} = 1330$

Now we define some conditions as follows

(1) C_1 : The solutions of $x_1 + x_2 + x_3 + x_4 = 18$, $x_1 > 7$

(2) C_2 : The solns of $x_1 + x_2 + x_3 + x_4 = 18$, $x_2 > 7$

(3) C_3 : " " " " " " , $x_3 > 7$

(4) C_4 : " " " " " " " , $x_4 > 7$

The no. of integers solns in which $x_1 + x_2 + x_3 + x_4 = 18$, $x_i \leq 7$, $1 \leq i \leq 4$ $= \bar{N}$

$\bar{N} = N - [N(C_1) + N(C_2) + N(C_3) + N(C_4)] + [N(C_1C_2) + N(C_1C_3) + N(C_1C_4) + N(C_2C_3) + N(C_2C_4) + N(C_3C_4)] - [N(C_1C_2C_3) + N(C_1C_3C_4) + N(C_2C_3C_4)] + N(C_1C_2C_3C_4)$

$\bar{N} = 1330 - [{}^{(4+10-1)}C_{10} + {}^{13}C_{10} + {}^{13}C_{10} + {}^{13}C_{10}] + [{}^{(4+2-1)}C_2 + {}^{5}C_2$

${}^5C_2 + {}^5C_2 + {}^5C_2 + {}^5C_2 \,] - [\text{Not possible}] \rightarrow N(C_1C_2C_3)$

$\Rightarrow y_1+8+y_2+8+y_3+8+y_4 = 18$

$\Rightarrow \sum_{n=1}^{4} y_n = -6 \Rightarrow$ not possible

$= 1330 - (1144) + 60$

$= \underline{\underline{246}}$.

Q Determine how many integer solns there are to the eqn $x_1 + x_2 + x_3 = 11$ if $x_1 \leq 3$, $x_2 \leq 4$, $x_3 \leq 6$

Soln: Let S be the set of integer solns of eqn

$x_1 + x_2 + x_3 = 11, \quad x_i \geq 0$

$|S| = N = {}^{(3+11-1)}C_{11} = {}^{13}C_{11} = \underline{\underline{78}}$

Now we define some conditions as follows

(1) C_1 : Solns of $x_1 + x_2 + x_3 = 11$, ~~$x_1 \leq 3$~~ $x_1 > 3 \Rightarrow x_1 \geq 4$

$y_1 + 4 + y_2 + y_3 = 11$, $n = 3$, $r = 11 - 4 = 7$

(2) C_2 : Solns of $x_1 + x_2 + x_3 = 11$, $x_2 > 4 \Rightarrow x_2 \geq 5$

$n = 3$, $r = 6$

(3) C_3 : Solns of $x_1 + x_2 + x_3 = 11$, $x_3 > 6 \Rightarrow x_3 \geq 7$

$n = 3$, $r = 4$

The no. of integer solns in which $x_1 + x_2 + x_3 = 11$,

$x_1 \leq 3$, $x_2 \leq 4$, $x_3 \leq 6 = \bar{N}$

$y_1 + 4 + y_2 + 5 + y_3 = 11$, $n = 3$, $r = 11 - 4 - 5 = 2$

$\bar{N} = N - [N(C_1) + N(C_2) + N(C_3)] + [N(C_1C_2) + N(C_1C_3)$

$n = 3$, $r = 0$

$+ N(C_2C_3)] - N(C_1C_2C_3)$ Not possible

$n = 3$, $r = -1$ × not possible

$= 78 - [{}^{(3+7-1)}C_7 + {}^{(3+6-1)}C_6 + {}^{(3+4-1)}C_4] + [{}^{(3+2-1)}C_2 + {}^{(3+0-1)}C_0]$

$= 78 - (36 + 28 + 15) + (6 + 1)$

$= 78 - 79 + 7 = \underline{\underline{6}}$.

18/10

Derangements

Nothing in its right place.

Q. Consider no.s 1, 2, 3, 4, how many ways we can arrange this no.s such that 1 is not in the first position, 2 is not in 2nd place, 3 is not in the 3rd place. Similarly 4 is not in 4th place.

Soln:

1	2	3	4
2	3	4	1
2	4	1	3
3	1	4	2
3	4	1	2
3	4	2	1
4	1	2	3
4	3	1	2
4	3	2	1
2	1	4	3

The no. of derangements of a set with n elements is D_n

Derived from principle of exclusion

$$\boxed{D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + \frac{(-1)^n}{n!}\right] \quad \text{where } n > 1}$$

Here, $n = 4$

$$\therefore D_4 = 4!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right]$$

$$= 24\left[1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24}\right]$$

$$= 9.$$

Q. How many permutations of 1, 2, 3, 4, 5, 6, 7 are not derangements.

Soln: Total permutations $= 7! = 5040$

No. of derangements $= D_7$

$$D_7 = 7!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!}\right] = 1854$$

No of permutations that are not derangements

$= 7! - D_7$

$= 5040 - 1854 = \underline{\underline{3186}}$

Q. List all the derangements of the nos 1, 2, 3, 4, 5, 6 where the first 3 nos are 1, 2, 3 in some order

$2 \times 3! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!}\right)$

$= 4$

1	2	3	4	5	6
2	3	1	5	6	4
3	1	2	5	6	4
2	3	1	6	4	5
3	1	2	6	4	5

Q. There are 8 guests in a party. Each guest brings and each receives another ~~gues~~ gift in return. No one is allowed to receive a gift they brought. How many ways are there to ~~st~~ distribute the gifts.

Soln: $D_8 = 8!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \frac{1}{8!}\right]$

$= \cancel{14833}$ 14833

Pigeon hole principle

If m pigeons occupy n pigeon holes (m>n), then atleast one pigeonhole will contain more than 1 pigeon.

Generalization of Pigeon hole principle

If m pigeons occupy n pigeon holes with m sufficiently large than n, then one of the pigeon hole must contain atleast $\left\lfloor\frac{(m-1)}{n}\right\rfloor+1$ pigeon. where $\left\lfloor\frac{m-1}{n}\right\rfloor$ denote the greatest integer less than or equal to $\left(\frac{m-1}{n}\right)$, which is a real number.

Q.1 S.T if any 4 nos from 1 to 6 are chosen then at least 2 of them will add to 7.

Soln: Here the pigeons constitute any 4 nos from 1, 2, 3, 4, 5, 6 and the pigeon holes are the subsets $\{1,6\}$, $\{2,5\}$, $\{3,4\}$. Each of the 4 nos chosen from 1 to 6 must belong to one of these sets. Since pigeons are greater than pigeon holes by the pigeon hole principle we can conclude that 2 of the selected nos belong to the same sets whose sum is 7.

2. Given a group of 100 people, at minimum how many people were born in the same month?

Soln: Pigeons = m = 100

Pigeon holes = months = 12

$$\left\lfloor\frac{(m-1)}{n}\right\rfloor+1=\left\lfloor\frac{100-1}{12}\right\rfloor+1=\left\lfloor\frac{99}{12}\right\rfloor+1=8+1=9$$

OR

S.T among hundred people there are atleast 9 who were born in the same month.

Soln:

Here 100 peoples are pigeons, i.e, m is 100 and 12 months are pigeon holes, i.e., n is 12

By generalised pigeon hole principle atleast

$\left\lfloor \frac{m-1}{n} \right\rfloor + 1 = \left\lfloor \frac{99}{12} \right\rfloor + 1 = 8 + 1 = 9$ were born in the same month.

3. How many people must you to guarantee that atleast 9 of them will have birthdays ~~at~~ in the same day of the week.

Soln: Given $\left\lfloor \frac{m-1}{n} \right\rfloor + 1 = 9$, where $n = 7$.

$\therefore \left\lfloor \frac{m-1}{7} \right\rfloor + 1 = 9$

$\left\lfloor \frac{m-1}{7} \right\rfloor = 8$

$m - 1 = 8 \times 7 = 56$

$m = 57$