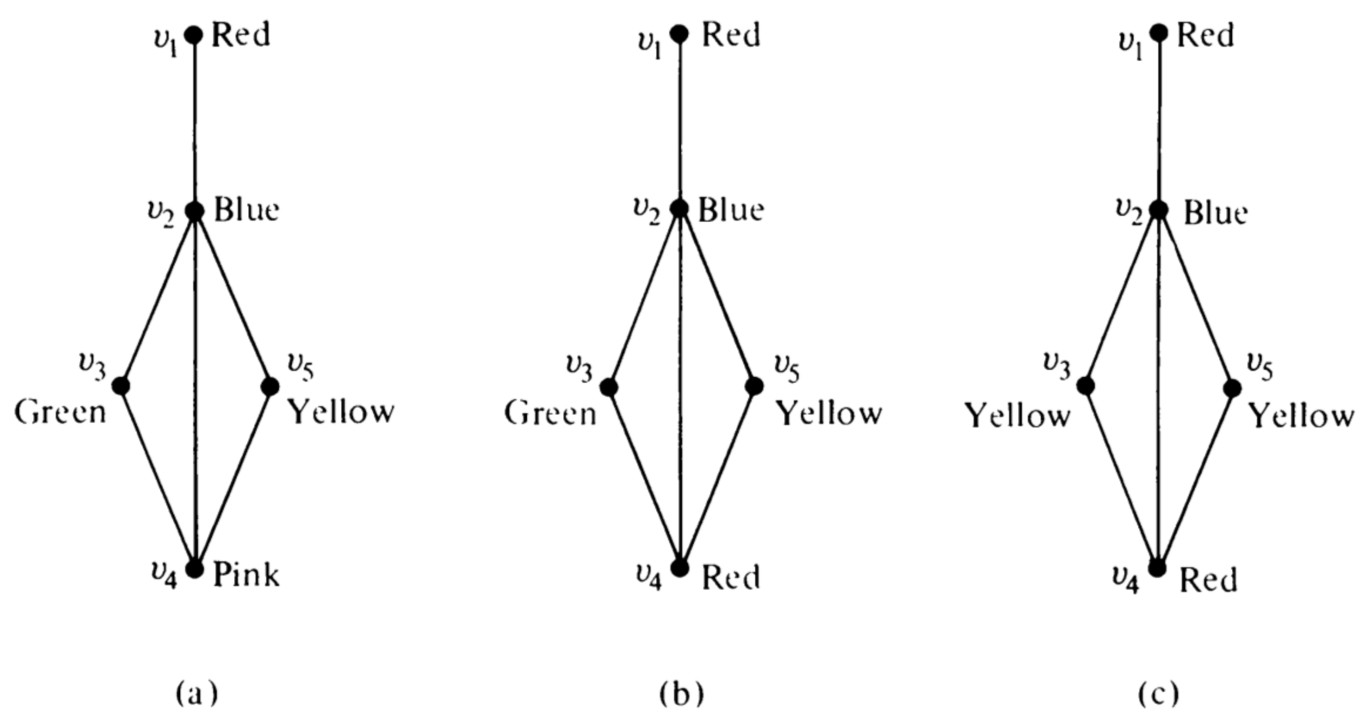
MODULE- 5

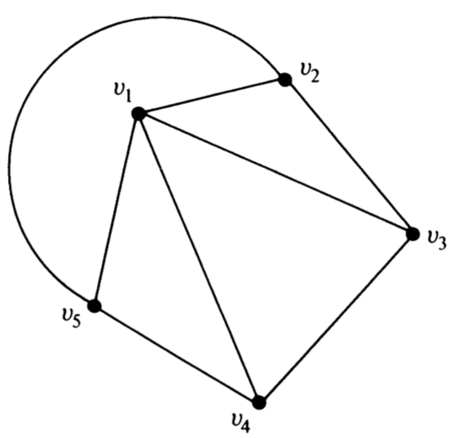
GRAPH REPRESENTATION AND VERTEX COLORING

proper coloring of a graph

Painting all the vertices of a graph with colors such that no two adjacent vertices have the same color is called the *proper coloring* (or sometimes simply *coloring*) of a graph. A graph in which every vertex has been assigned a color according to a proper coloring is called a *properly colored* graph. Usually a given graph can be properly colored in many different ways as follows



Proper colorings of a graph.

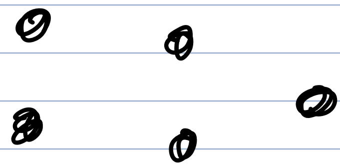


Chromatic Graph and Chromatic Number

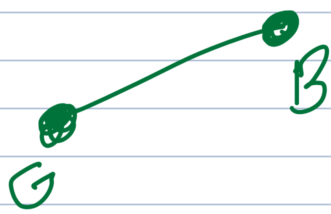
A graph G that requires κ different colors for its proper coloring, and no less, is called a κ-*chromatic* graph, and the number κ is called the *chromatic number* of G.

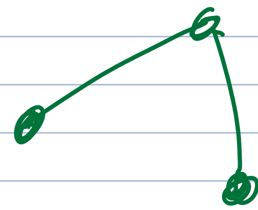
Observations

1. A graph consisting of only isolated vertices is 1-chromatic.

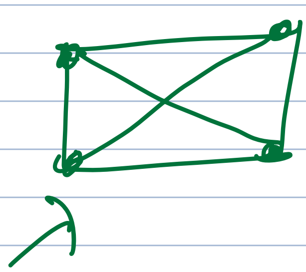


2. A graph with one or more edges (not a self-loop, of course) is at least *2-chromatic* (also called *bichromatic*).

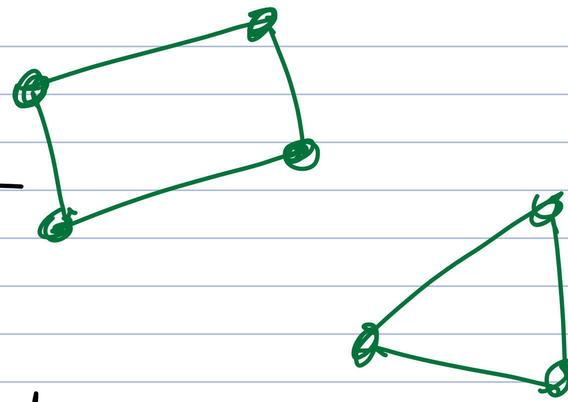




3. A complete graph of n vertices is n-chromatic, as all its vertices are adjacent.



4. A graph consisting of simply one circuit with $n \geq 3$ vertices is 2-chromatic if n is even and 3-chromatic if n is odd.



2014 [illegible]

1. Aanya Rajeev
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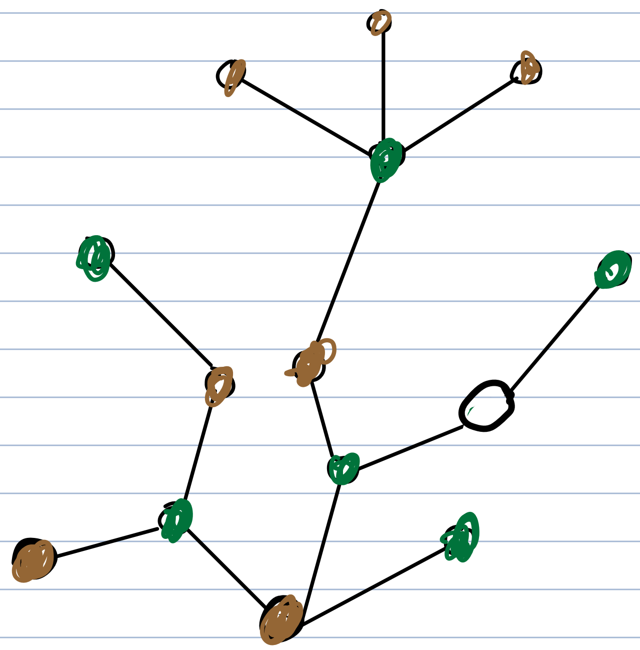
5. Asiya
6. Vaishnavi

THEOREM 1

Every tree with two or more vertices is 2-chromatic.



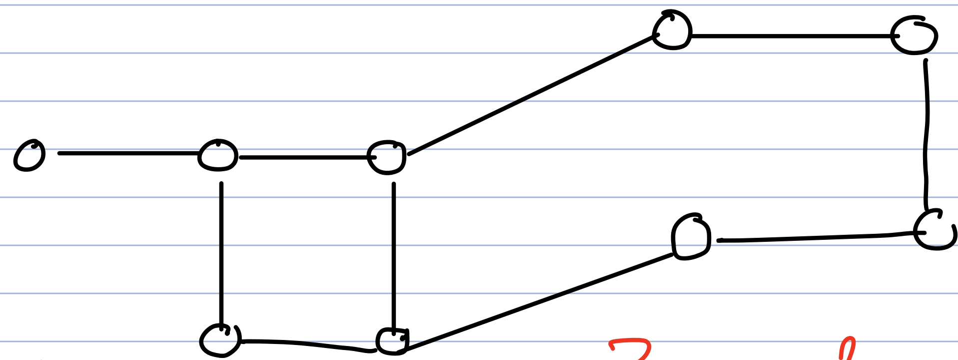
Proof: Select any vertex v in the given tree T. Consider T as a rooted tree at vertex v. Paint v with color 1. Paint all vertices adjacent to v with color 2. Next, paint the vertices adjacent to these (those that just have been colored with 2) using color 1. Continue this process till every vertex in T has been painted. Now in T we find that all vertices at odd distances from v have color 2, while v and vertices at even distances from v have color 1.

Now along any path in T the vertices are of alternating colors. Since there is one and only one path between any two vertices in a tree, no two adjacent vertices have the same color. Thus T has been properly colored with two colors. ■

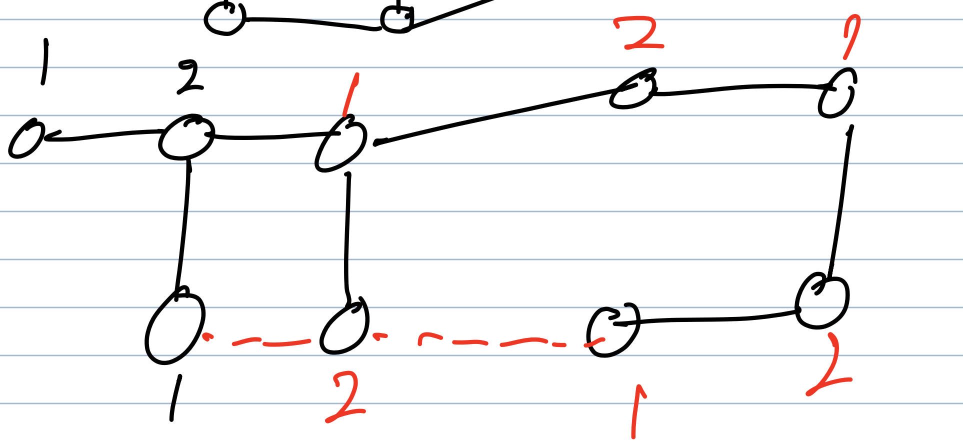
THEOREM -2

A graph with at least one edge is 2-chromatic if and only if it has no circuits of odd length.

G



G



Proof: Let G be a connected graph with circuits of only even lengths. Consider a spanning tree T in G. Using the coloring procedure and the result of Theorem 1, let us properly color T with two colors. Now add the chords to T one by one. Since G had no circuits of odd length, the end vertices of every chord being replaced are differently colored in T. Thus G is colored with two colors, with no adjacent vertices having the same color. That is, G is 2-chromatic.

Conversely, if G has a circuit of odd length, we would need at least three colors just for that circuit (observation 4). Thus the theorem. ■

CHROMATIC POLYNOMIAL

The value of the chromatic polynomial $P_n(\lambda)$ of a graph with n vertices gives the number of ways of properly coloring the graph, using λ or fewer colors.

Let c_i be the different ways of properly coloring G using exactly i different colors. Since i colors can be chosen out of λ colors in

$$\binom{\lambda}{i} \quad \text{different ways,}$$

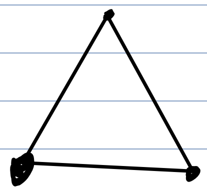
there are $c_i\binom{\lambda}{i}$ different ways of properly coloring G using exactly i colors out of λ colors.

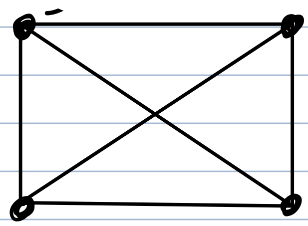
Since i can be any positive integer from 1 to n (it is not possible to use more than n colors on n vertices), the chromatic polynomial is a sum of these terms; that is,

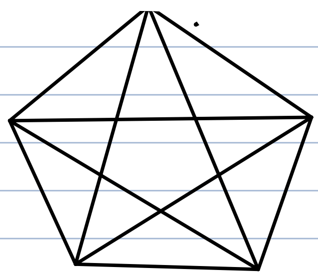
$$\begin{aligned} P_n(\lambda) &= \sum_{i=1}^{n} c_i\binom{\lambda}{i} \\ &= c_1\frac{\lambda}{1!} + c_2\frac{\lambda(\lambda-1)}{2!} + c_3\frac{\lambda(\lambda-1)(\lambda-2)}{3!} + \cdots \\ &\quad + c_n\frac{\lambda(\lambda-1)(\lambda-2)\cdots(\lambda-n+1)}{n!}. \end{aligned}$$

$c_i \rightarrow$ No. of ways graph can be colored with i colors

Chromatic polynomial for a complete graph.







THEOREM

A graph of n vertices is a complete graph if and only if its chromatic polynomial is

$$P_n(\lambda) = \lambda(\lambda - 1)(\lambda - 2) \dots (\lambda - n + 1).$$

Proof: With λ colors, there are λ different ways of coloring any selected vertex of a graph. A second vertex can be colored properly in exactly $\lambda - 1$ ways, the third in $\lambda - 2$ ways, the fourth in $\lambda - 3$ ways, . . . , and the nth in $\lambda - n + 1$ ways if and only if every vertex is adjacent to every other. That is, if and only if the graph is complete. ■

THEOREM 8-5

An n-vertex graph is a tree if and only if its chromatic polynomial

$$P_n(\lambda) = \lambda(\lambda - 1)^{n-1}.$$

<u>proof</u> by induction on n

When $n = 1$ (for isolated vertex)

$\therefore P_1(\lambda) = \lambda$

When $n = 2$.

$P_2(\lambda) = \lambda(\lambda - 1)$

Assume the result is true for all the trees with no. of vertices $n-1$

$P_{n-1}(\lambda) = \lambda(\lambda-1)^{n-2}$

$\because$ Every ~~tree~~ tree has minimum 2 Pendant vertices with $n \geq 2$

Now Consider ~~the~~ a tree with n vertices

Remove one pendant vertex along with edge from the tree. ~~So~~ Then the tree left with $n-1$ vertices

$\therefore$ We can coloured the tree ~~with~~ in $\lambda(\lambda-1)^{n-2}$ ways

Now after colouring $n-1$ vertices, attach the removed pendant vertex to the tree along with edge.

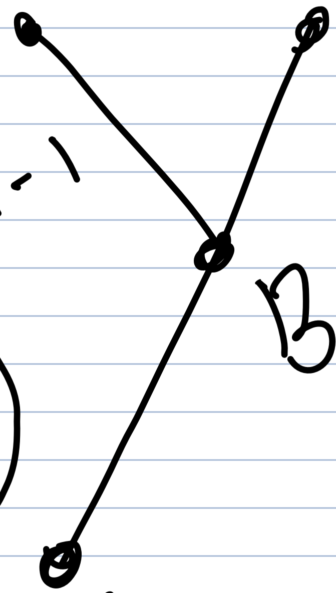
So we cannot give the same colour of other end vertex.

$\therefore$ We can color the vertex $\lambda-1$ ways.

By the rule of product, Total ways of colouring is $\lambda(\lambda-1)^{n-2}(\lambda-1)$

$$= \lambda(\lambda-1)^{n-1} = P_n(\lambda)$$

$\therefore$ This is true for n vertices



4 vertices

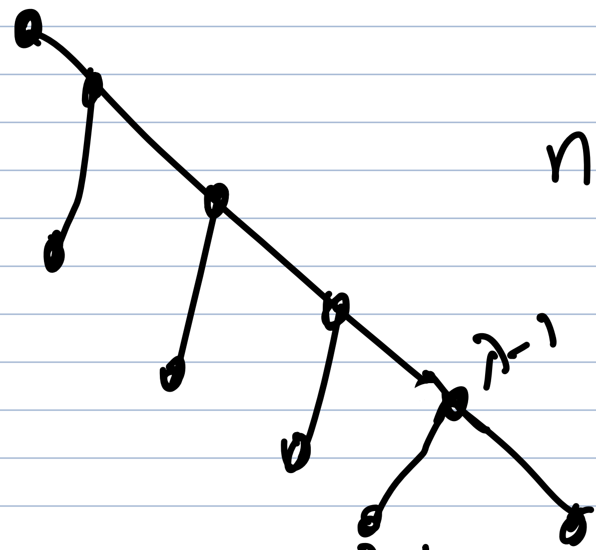
λ   n

GBRW

$\lambda(\lambda-1)^{n-1}$ GA

$\lambda(\lambda-1)^3$

$\lambda(\lambda-1)^{n-1}$



n vertices

n − 1 vertices

$\lambda(\lambda-1)^{n-2} \quad \lambda-1 \quad \lambda$

$= \lambda(\lambda-1)^{n-1}$

$= \boxed{P_n(\lambda)}$

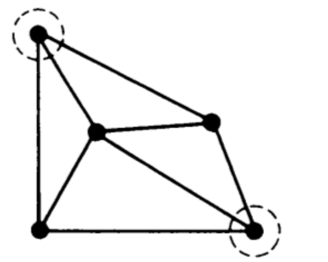
Theorem (Possible ways of coloring graph)

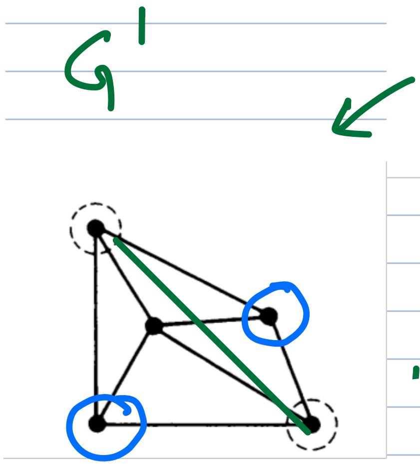
Let a and b be two nonadjacent vertices in a graph G. Let G' be a graph obtained by adding an edge between a and b. Let G'' be a simple graph obtained from G by fusing the vertices a and b together and replacing sets of parallel edges with single edges. Then

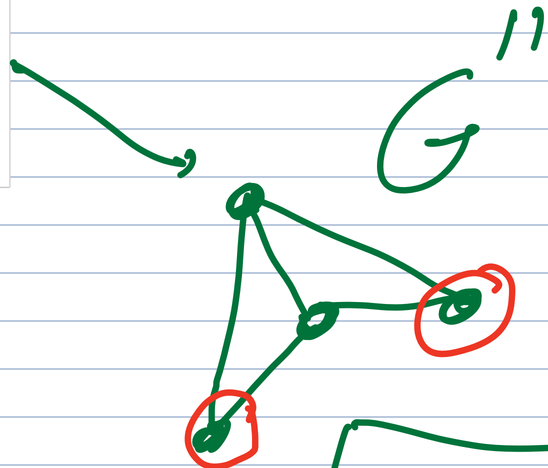
$$P_n(\lambda) \text{ of } G = P_n(\lambda) \text{ of } G' + P_{n-1}(\lambda) \text{ of } G''.$$

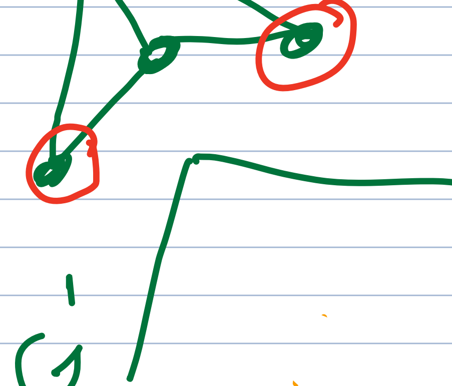
Proof: The number of ways of properly coloring G can be grouped into two cases, one such that vertices a and b are of the same color and the other such that a and b are of different colors. Since the number of ways of properly coloring G such that a and b have different colors = number of ways of properly coloring G', and

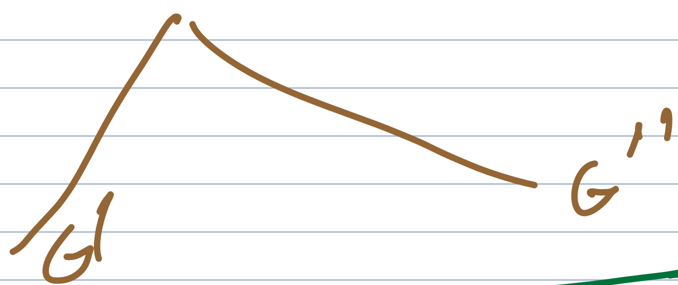
number of ways of properly coloring G such that a and b have the same color

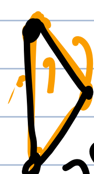




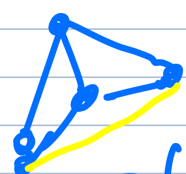




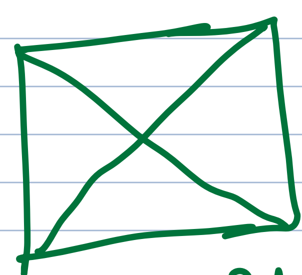




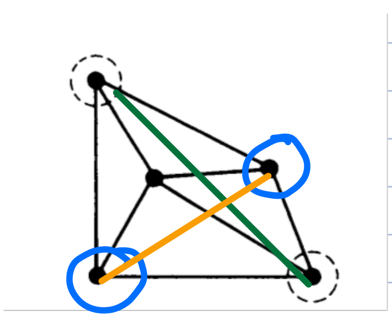
$\lambda(\lambda-1)(\lambda-2)$



$\lambda(\lambda-1)(\lambda-2)(\lambda-3)$



$\lambda(\lambda-1)(\lambda-2)(\lambda-3)$



$\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)$

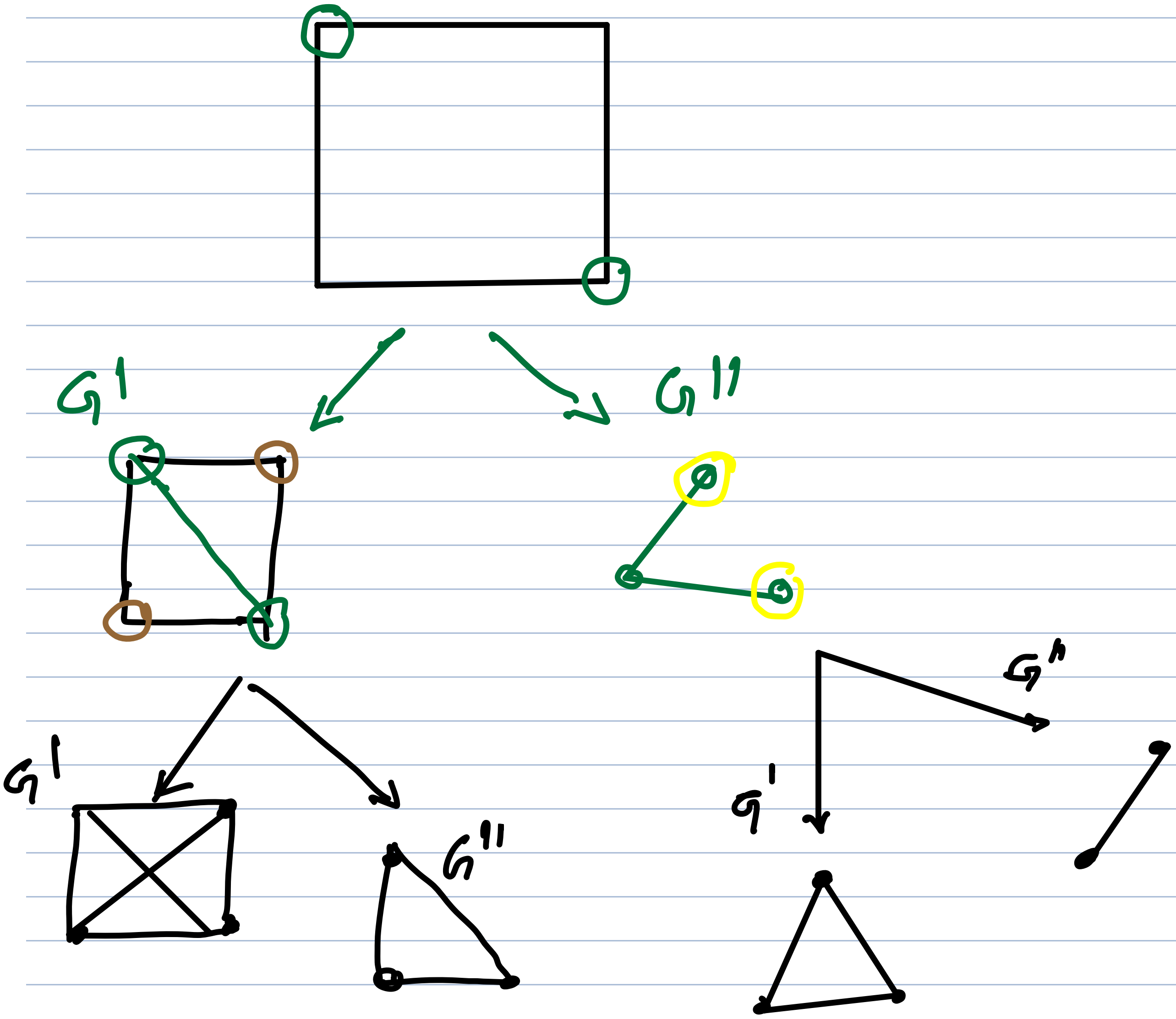
$$\therefore P_n(\lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4) +$$
$$\lambda(\lambda-1)(\lambda-2)(\lambda-3) +$$
$$\lambda(\lambda-1)(\lambda-2)(\lambda-3) +$$
$$\lambda(\lambda-1)(\lambda-2)$$

$$= \lambda(\lambda-1)(\lambda-2)\left[(\lambda-3)(\lambda-4)+2(\lambda-3)+1\right]$$

$$= \lambda(\lambda-1)(\lambda-2)\left[\lambda^2-7\lambda+12+2\lambda-6+1\right]$$

$$= \underline{\underline{\lambda(\lambda-1)(\lambda-2)\left[\lambda^2-5\lambda+7\right]}}$$

Chromatic polynomial of cyclic graph on 4 vertices



$$P_n(\lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda-3) + 2\lambda(\lambda-1)(\lambda-2)$$
$$+ \lambda(\lambda-1)$$
$$= \lambda(\lambda-1)\{(\lambda-2)(\lambda-3) + 2(\lambda-2) + 1\}$$
$$= \lambda(\lambda-1)[\lambda^2 - 5\lambda + 6 + 2\lambda - 4 + 1]$$

$$= \lambda(\lambda-1)[\lambda^2 - 3\lambda + 3]$$

Find chromatic polynomial for the following graph