Module 4

Generating functions and Recurrence Relations

Generating function.

Let $a_0, a_1, a_2, \ldots$ be a sequence of real numbers. The function

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

$$= \sum_{i=0}^{\infty} a_i x^i$$

is called the generating function for the given sequence.

Example: Consider the sequence $nC_0, nC_1, nC_2, \ldots nC_n, 0, 0, \ldots$

Corresponding generating function is

$$\boxed{\binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \cdots + \binom{n}{n} x^n = (1+x)^n}$$

Thus $(1+x)^n$ is the generating function for the sequence $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$

2. For $n \in \mathbb{Z}^+$

$$\frac{1-x^{n+1}}{1-x} = 1+x+x^2+\cdots+x^n.$$

Thus $\frac{1-x^{n+1}}{1-x}$ is the generating function for the sequence $1, 1, \ldots, 1, 0, 0, \ldots$

ie, the first $(n+1)$ terms are 1 and all other elements are 0

3. $$\frac{1}{1-x} = 1+x+x^2+x^3+\cdots = \sum_{n=0}^{\infty} x^n$$

Thus $\frac{1}{1-x}$ is the generating function of the sequence $1, 1, 1, \ldots$ called fundamental generating function

4. $$\frac{1}{1-x^2} = 1+2x+3x^2+4x^3+5x^4+\cdots$$

Thus $\frac{1}{1-x^2}$ is the generating function of the sequence $1, 2, 3, \ldots$

5. $$\boxed{\frac{x}{(1-x)^2} = 0 + 1x + 2x^2 + 3x^3 + 4x^4 + \cdots}$$

Thus $\frac{x}{(1-x)^2}$ is the generating function of the sequence $0, 1, 2, 3, \ldots$

6. $$\boxed{\frac{x+1}{(1-x)^3} = 1 + 2^2x + 3^2x^2 + 4^2x^3 + \cdots}$$

Thus $\frac{x+1}{(1-x)^3}$ is the generating function of $1^2, 2^2, 3^3, 4^2, \ldots$

7. $$\boxed{\frac{x(x+1)}{(1-x)^3} = x + 2^2x^2 + 3^2x^3 + \cdots}$$

Thus $\frac{x(x+1)}{(1-x)^3}$ is the generating function of $0^2, 1^2, 2^2, 3^2, \ldots$

Rough

$$\frac{1}{1-x} = 1+x+x^2+x^3+x^4+\dots$$

* $\frac{d}{dx}\left[\frac{1}{1-x}\right] = 0+1+2x+3x^2+4x^3+\dots$

$\Rightarrow \frac{-1}{(1-x)^2}(-1) = 1+2x+3x^2+\dots$

$\Rightarrow \frac{1}{(1-x)^2} = 1+2x+3x^2+\dots$

* multiply by x on both sides

$\Rightarrow \frac{x}{(1-x)^2} = x+2x^2+3x^3+\dots$

* $\frac{d}{dx}\left[\frac{x}{(1-x)^2}\right] = 1+2^2x+3^2x^2+4^2x^3+\dots$

$\Rightarrow x\times\left[\frac{-2}{(1-x)^3}\times -1\right]+\left[\frac{1}{(1-x)^2}\right](1) = 1+2^2x+3^2x^2+4^2x^3+\dots$

$\Rightarrow \frac{2x}{(1-x)^3}+\frac{1}{(1-x)^2} = 1+2^2x+3^2x^2+\dots$

$\Rightarrow \frac{2x+(1-x)}{(1-x)^3} = 1+2^2x+3^2x^2+\dots$

$$\Rightarrow \frac{x+1}{(1-x)^3} = 1 + 2^2 x + 3^2 x^2 + 4^2 x^3 + \dots$$

* multiply both sides by x

$$\Rightarrow \frac{x(x+1)}{(1-x)^3} = x + 2^2 x^2 + 3^2 x^3 + 4^2 x^4 + \dots$$

* Ex: 8 Consider $\frac{1}{1-x} = 1+x+x^2+x^3+\cdots$

Put $x = 2y$

$$\Rightarrow \frac{1}{1-2y} = 1+2y+2^2y^2+2^3y^3+2^4y^4+\cdots$$

$\therefore \frac{1}{1-2y}$ is the generating function of

the sequence $1, 2, 2^2, 2^3, 2^4, \ldots$

$$\Rightarrow 2^0, 2^1, 2^2, 2^3, \ldots$$

9. Thus $\frac{1}{1-ax} = 1+ax+a^2x^2+a^3x^3+\cdots$

Thus $\frac{1}{1-ax}$ is the generating function of

the sequence $a^0, a^1, a^2, a^3, \ldots$

10. $\frac{1}{1-x} = 1+x+x^2+x^3+\cdots$

$$a\left[\frac{1}{1-x}\right] = a+ax+ax^2+ax^3+\cdots$$

Thus $\frac{a}{1-x}$ is the generating function of

the sequence $a, a, a, \ldots$

$$* \quad \frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots$$

$\therefore \frac{1}{1+x}$ is the generating function of the sequence $1, -1, 1, -1, 1, -1, \ldots$

Table 1 *Useful Generating Functions.*

$G(x)$	a_k
$(1+x)^n = \sum_{k=0}^{n} C(n,k)x^k = 1 + C(n,1)x + C(n,2)x^2 + \cdots + x^n$	$C(n,k)$
$(1+ax)^n = \sum_{k=0}^{n} C(n,k)a^k x^k = 1 + C(n,1)ax + C(n,2)a^2x^2 + \cdots + a^n x^n$	$C(n,k)a^k$
$(1+x^r)^n = \sum_{k=0}^{n} C(n,k)x^{rk} = 1 + C(n,1)x^r + C(n,2)x^{2r} + \cdots + x^{rn}$	$C(n, k/r)$ if $r \mid k$; 0 otherwise
$\frac{1-x^{n+1}}{1-x} = \sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n$	1 if $k \le n$; 0 otherwise
$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots$	1
$\frac{1}{1-ax} = \sum_{k=0}^{\infty} a^k x^k = 1 + ax + a^2x^2 + \cdots$	a^k
$\frac{1}{1-x^r} = \sum_{k=0}^{\infty} x^{rk} = 1 + x^r + x^{2r} + \cdots$	1 if $r \mid k$; 0 otherwise
$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} (k+1)x^k = 1 + 2x + 3x^2 + \cdots$	$k+1$
$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)x^k = 1 + C(n,1)x + C(n+1,2)x^2 + \cdots$	$C(n+k-1,k) = C(n+k-1,n-1)$
$\frac{1}{(1+x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)(-1)^k x^k = 1 - C(n,1)x + C(n+1,2)x^2 - \ldots$	$(-1)^k C(n+k-1,k) = (-1)^k C(n+k-1,n-1)$
$\frac{1}{(1-ax)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)a^k x^k = 1 + C(n,1)ax + C(n+1,2)a^2x^2 + \cdots$	$C(n+k-1,k)a^k = C(n+k-1,n-1)a^k$
$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	$1/k!$
$\ln(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k} x^k = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	$(-1)^{k+1}/k$

Q Find the generating function of the Sequence 0, 0, 0, 1, 1, 1, 1, ...

ie, $a_n = \begin{cases} 0, & n = 0, 1, 2, \\ 1, & n \geq 3 \end{cases}$

Corresponding generating function will be

$g(x) = 0 + 0x + 0x^2 + 1x^3 + 1x^4 + \cdots$

$= x^3 + x^4 + x^5 + \cdots$

$= x^3 [1 + x + x^2 + x^3 + \cdots]$

$$g(x) = x^3 \left[\frac{1}{1-x}\right]$$

$$= \frac{x^3}{1-x}$$

Q. Find the generating function of the sequence $1, -1, 1, -1, 1, -1, \ldots$ ie, $a_n = \{(-1)^n\}_{n=0}^{\infty}$

Soln

$$g(x) = 1 - x + x^2 - x^3 + x^4 + \ldots$$

$$= \sum_{n=0}^{\infty} (-1)^n x^n$$

$$= \sum_{n=0}^{\infty} (-x)^n$$

since $\frac{1}{1-x} = \sum_{n=0}^{\infty} (x)^n$

$$= \frac{1}{1+x}$$

Q Find the generating function for $1, 0, 1, 0, 1, 0, \ldots$

ie, $a_n = \begin{cases} 1, & n \text{ is even} \\ 0, & n \text{ is odd} \end{cases}$

Here $f(x) = 1 + 0x + 1x^2 + 0x^3 + 1x^4 + 0x^5 + \cdots$

$$= (1 + x^2 + x^4 + x^6 + \cdots)$$

$$= \sum_{n=0}^{\infty} x^{2n}$$

$$= \sum_{n=0}^{\infty} (x^2)^n$$

$$= \frac{1}{1-(x^2)}$$

5) Find the generating function for the sequence $0, 2, 4, 8, 16, \ldots$

Here $f(x) = 0 + 2x + 4x^2 + 8x^3 + 16x^4 + \cdots$

$$= 2x + (2x)^2 + (2x)^3 + \cdots]$$

$$= \left[\frac{1}{1-2x}\right]$$ is the generating function

NOTE

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

NOTE

* With $n, r \in \mathbb{Z}^+$, $n \geq r > 0$.

$$\boxed{\binom{n}{r} = \frac{n!}{(n-r)!\,r!}} \quad \text{--- (1)}$$

$$= \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$$

* If $n \in \mathbb{R}$, we use.

$$\binom{n}{r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!} \quad \rightarrow r \text{ terms}$$

$$\Rightarrow \binom{-n}{r} = \frac{(-n)(-n-1)(-n-2)\cdots(-n-r+1)}{r!}$$

$$= \frac{(-1)^r\left[n(n+1)(n+2)\cdots(n+r-1)\right]}{r!}$$

$$\boxed{\binom{-n}{r} = \frac{(-1)^r\,(n+r-1)!}{(n-1)!\,r!}} \quad \text{--- (2)}$$

$$= (-1)^r \binom{n+r-1}{r}$$

also $\boxed{\binom{n}{0} = 1 \quad \forall\, n \in \mathbb{R}}$

* $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n = \sum_{k=0}^{n}\binom{n}{k}x^k$

* $(1+x)^{-n} = \binom{-n}{0} + \binom{-n}{1}x + \cdots = \sum_{k=0}^{\infty}\binom{-n}{k}x^k$

* $\Rightarrow (1+x)^{-n} = \sum_{k=0}^{\infty}\frac{(-1)^k(n+k-1)!}{(n-1)!\,k!}x^k$

* $(1-x)^{-n} = \sum_{k=0}^{\infty}\frac{(-1)^k(n+k-1)!}{(n-1)!\,k!}(-x)^k$

$= \sum_{k=0}^{\infty}\frac{(n+k-1)!}{(n-1)!\,k!}x^k$

Thus coefficient of x^k in $(1-x)^{-n}$

is $\dfrac{(n+k-1)!}{(n-1)!\,k!}$

1 Q Find the Coefficient of x^5 in $(1-2x)^{-7}$

$$\boxed{(1-x)^{-n} = \sum_{k=0}^{\infty} \frac{(n+k-1)!}{(n-1)!\,k!} x^k}$$

$n \rightsquigarrow 7$

$x \rightsquigarrow 2x$

$k \rightsquigarrow 5$

$$\therefore (1-(2x))^{-7} = \sum_{k=0}^{\infty} \frac{(7+k-1)!}{(7-1)!\,k!} (2x)^k$$

$\therefore$ Coefficient of x^5 ie; when $k=5$

$$\text{Coefficient} \Rightarrow \frac{(7+5-1)!}{(7-1)!\,5!} 2^5$$

$$= \left[\frac{(7+5-1)!}{(7-1)!\,5!}\right](32)$$

$$= \frac{[11!]}{6!\,5!}(32)$$

$$= \frac{32 \times 11!}{6!\,5!}$$

$$= 32 \times \frac{7 \times \cancel{8}^{2} \times \cancel{9}^{3} \times \cancel{10} \times 11}{1 \times \cancel{2} \times \cancel{3} \times \cancel{4} \times \cancel{5}} = 32 \times 7 \times 6 \times 11$$

$$= \underline{14,784}$$

2Q: Determine the coeffient of x^{15} in $(x^2 + x^3 + x^4 + \dots)^4$.

$$(x^2 + x^3 + x^4 + \dots) = x^2[1 + x + x^2 + x^3 + \dots]$$

$$= x^2\left[\frac{1}{1-x}\right]$$

$$\therefore (x^2 + x^3 + x^4 + \dots)^4 = \left[\frac{x^2}{1-x}\right]^4$$

$$= x^8[1-x]^{-4}$$

Thus, the coeffient of x^{15} in $x^8[1-x]^{-4}$ can be obtined by finding the coefficient of x^7 in the expansion of $(1-x)^{-4}$

$$\boxed{(1-x)^{-n} = \sum_{k=0}^{\infty} \frac{(n+k-1)!}{(n-1)!\,k!} x^k}$$

$$(1-x)^{-4} = \sum_{k=0}^{\infty} \frac{(4+k-1)!}{(4-1)!\,k!} (x)^k$$

$k \rightsquigarrow 7$

$n \rightsquigarrow 4$

Coefficient of x^k is $\dfrac{(n+k-1)!}{(n-1)!\,k!}$

$\therefore$ coefficient of x^7 is $\dfrac{(4+7-1)!}{(4-1)!\,7!}$

$$= \frac{[10!]}{3!\,7!} = \frac{\overset{4}{\cancel{8}} \times \overset{3}{\cancel{9}} \times 10}{1 \times \cancel{2} \times \cancel{3}}$$

$= 120$

NOTE

$$(1+x+x^2+\cdots)^n = \frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k$$

$\therefore$ Coeffuent of x^k in the expansion of $\frac{1}{(1-x)^n}$ is $\binom{n+k-1}{k}$

3Q Determine the coefficient of x^8 in $\frac{1}{(x-3)(x-2)^2}$

$$\frac{1}{(x-3)(x-2)^2} = \frac{A}{x-3} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

$\Rightarrow 1 = A(x-2)^2 + B(x-3)(x-2) + C(x-3)$

$\boxed{\text{Put } x=2}$

$\Rightarrow 1 = 0+0+C(-1) \Rightarrow C=-1$

$\boxed{\text{Put } x=3}$

$\Rightarrow 1 = A(3-2)^2 + 0 + 0 \Rightarrow A=1$

$\boxed{\text{Put } x=0}$

$$\Rightarrow 1 = A(-2)^2 + B(-3)(-2) + C(-3)$$

$$1 = 4 + 6B + 3$$

$$\Rightarrow 6B = -6$$

$$\Rightarrow B = -1$$

NOTE
$A = 1$ &
$C = -1$

$$\therefore \frac{1}{(x-3)(x-2)^2} = \frac{1}{x-3} - \frac{1}{x-2} - \frac{1}{(x-2)^2}$$

$$= \frac{-1}{3-x} + \frac{1}{2-x} - \frac{1}{(2-x)^2}$$

$$= \frac{-1}{3\left[1-\frac{x}{3}\right]} + \frac{1}{2\left[1-\frac{x}{2}\right]} - \frac{1}{2^2\left[1-\frac{x}{2}\right]^2}$$

We know $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$

$$= \left(\frac{-1}{3}\right)\left[\frac{1}{1-\frac{x}{3}}\right] + \frac{1}{2}\left[\frac{1}{1-\frac{x}{2}}\right] - \frac{1}{4}\left[\frac{1}{1-\frac{x}{2}}\right]^2$$

$$= \frac{-1}{3}\sum_{k=0}^{\infty}\left(\frac{x}{3}\right)^k + \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{x}{2}\right)^k - \frac{1}{4}\sum_{k=0}^{\infty}\frac{(2+k-1)!}{(2-1)!\,k!}\left(\frac{x}{2}\right)^k$$

$\therefore$ Coeffcl. of x^8 will be.

$$= \frac{-1}{3}\left[\text{Coeff. of } x^8 \text{ in the expression of } \sum_{k=0}^{\infty}\left(\frac{x}{3}\right)^k\right]$$

$$+\frac{1}{2}\left[\text{coeff. } x^8 \text{ in } \sum_{k=0}^{\infty}\left(\frac{x}{2}\right)^k\right] - \frac{1}{4}\left[\text{coeff. of } x^8 \text{ in } \sum_{k=0}^{\infty}\frac{(k+1)!}{1!\,k!}\left(\frac{x}{2}\right)^k\right] =$$

$\boxed{k=8}$

$$= \frac{-1}{3}\left[\frac{1}{3^8}\right] + \frac{1}{2}\left[\frac{1}{2^8}\right] - \frac{1}{4}\left[\frac{(8+1)!}{8!}\left(\frac{1}{2}\right)^8\right]$$

$$= \frac{-1}{3^9} + \frac{1}{2^9} - \frac{1}{4}\left[\frac{9}{2^8}\right]$$

$$= \frac{-1}{3^9} + \frac{1\times 2}{2^9\times 2} - \frac{9}{2^{10}}$$

$$= \frac{-1}{3^9} - \frac{7}{2^{10}}$$

4. Determine the constant (ie; coefficient of x^0) in $\left(4x^3 - \frac{5}{x}\right)^{16}$

We have the result.

$$\boxed{\begin{array}{l} (x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}. \\ \text{ie; coefficient of } x^k \text{ is } \binom{n}{k} y^{n-k} \end{array}}$$

$$\left(4x^3 - \frac{5}{x}\right)^{16} = \sum_{k=0}^{16} \binom{16}{k} (4x^3)^k \left(\frac{-5}{x}\right)^{16-k}$$

$\therefore$ coeffient of x^0 means

here $3k - (16-k) = 0$

Since / here $\sum_{k=0}^{16} \binom{16}{k} 4^k x^{3k} (-5)^{16-k} (x)^{-(16-k)}$

ie; $3k - 16 + k = 0$

$\Rightarrow 4k - 16 = 0$

$4k = 16$

$k = 4$

When $k =$ we get the coeffient of x^0

Thus coeffient of x^0

$= \binom{16}{4} 4^4 (-5)^{16-4}$

$= \frac{16!}{12!\,4!} \times 4^4 \times (5)^{12}$

5. Find the coeffcient of x^{60} in $(x^8 + x^9 + x^{10} + \dots)^7$

$$x^8 + x^9 + x^{10} + \dots = x^8\left[1 + x + x^2 + \dots\right]$$

$$\therefore \left[x^8 + x^9 + \dots\right]^7 = \left[x^8\left[1 + x + x^2 + \dots\right]\right]^7$$

$$= x^{56}\left[\frac{1}{1-x}\right]^7$$

$$= x^{56}\left[(1-x)^{-7}\right]$$

We need the coeffcient of x^{60}

Now, Coeffcient of x^4 in $(1-x)^{-7}$

$$\boxed{\text{Coefficient of } x^k \text{ in the expansion of } (1+x)^{-n} \text{ is } \frac{(n+k-1)!}{(n-1)!\,k!}}$$

Here $n = 7$, $k = 4$

Coeffcient is $\dfrac{(7+4-1)!}{(7-1)!\,4!} = \dfrac{10!}{6!\,4!}$

$$= \frac{7 \times \cancel{8} \times \cancel{9}^{3} \times 10}{1 \times \cancel{2} \times \cancel{3} \times \cancel{4}}$$

$$= 210$$

$\therefore$ coeffnt. of x^{60} in $(x^8 + x^9 + \ldots)^7$

is 210

7. Find the sequence generated by

$$f(x) = \frac{x}{(1-x)^2}$$

$$= x\left[\frac{1}{(1-x)^2}\right]$$

$$= x\left[(1-x)^{-2}\right]$$

$$= x\left[\sum_{k=0}^{\infty} \frac{(2+k-1)!}{(2-1)!\,k!} x^k\right]$$

$$= \sum_{k=0}^{\infty} \frac{(k+1)!}{k!} x^{k+1}$$

$$= \sum_{k=0}^{\infty} (k+1) x^{k+1}$$

$$= 1 \cdot x + 2 \cdot x^2 + 3x^3 + 4x^4 + \dots$$

$\therefore$ Corresponding sequence is $0, 1, 2, 3, \dots$

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10 Find the sequence generated by $f(x) = \frac{1}{3-x}$

thw

$$f(x) = \frac{1}{3-x}$$

$$= \frac{1}{3\left[1 - \frac{x}{3}\right]}$$

$$= \frac{1}{3}\left[1 + \left(\frac{x}{3}\right) + \left(\frac{x}{3}\right)^2 + \left(\frac{x}{3}\right)^3 + \dots\right]$$

$$= \frac{1}{3} + \frac{1}{3^2}x + \frac{1}{3^3}x^2 + \dots$$

$\therefore$ Corresponding sequence is $\frac{1}{3}, \frac{1}{3^2}, \frac{1}{3^3}, \dots$

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11. Obtain the generating sequence of

Hw

$$f(x) = \frac{x^3}{1-x^2}$$

$$f(x) = \frac{x^3}{1-x^2}$$

$$= x^3\left[\frac{1}{1-x^2}\right]$$

$$= x^3\left[1+(x^2)+(x^2)^2+(x^2)^3+\cdots\right]$$

$$= x^3+x^5+x^7+x^9+\cdots$$

$\therefore$ Sequence is $0, 0, 0, 1, 0, 1, 0, 1, 0, \ldots$

8. Determine the sequence generated by the function $f(x) = \frac{2}{1-4x^2}$

$$f(x) = \frac{2}{1-4x^2}$$

$$= \frac{2}{(1-2x)(1+2x)}$$

$$\frac{2}{(1-2x)(1+2x)} = \frac{A}{1-2x} + \frac{B}{1+2x}.$$

$$\Rightarrow 2 = A(1+2x) + B(1-2x)$$

$\Rightarrow$ put $x = \frac{-1}{2}$

$$2 = A(0) + B\left[1-(2)\left(\frac{-1}{2}\right)\right]$$

$$= B\left[1+1\right]$$

$$\Rightarrow \underline{\underline{B = 1}}$$

put $x = \frac{1}{2}$

$$\Rightarrow 2 = A\left[1+1\right] + 0$$

$$\Rightarrow \underline{\underline{A = 1}}$$

Thus $f(x) = \frac{1}{(1-2x)} + \frac{1}{(1+2x)}$

$$= \left[1+(2x)+(2x)^2+(2x)^3+\cdots\right] + \left[1-(2x)+(2x)^2-\cdots\right]$$

$$= 2+2(2x)^2+2(2x)^4+\cdots$$

$$= 2\left[1+(2x)^2+(2x)^4+\cdots\right]$$

$$= 2 + 2^3 x^2 + 2^5 x^4 + 2^7 x^6 + \dots$$

Correspondg sequence $a_0 = 2$

$a_1 = 0$

$a_2 = 2^3$

$a_3 = 0$

$a_4 = 2^5$

$\vdots$

$\{2, 0, 2^3, 0, 2^5, 0, \dots\}$

* Show that $(1-4x)^{-1/2}$ generates the sequence $\binom{2n}{n}$

9 Determine the sequence generated by $n \in N$

$(1-4x)^{-\frac{1}{2}}$

We know

$$(1+x)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} x^k$$

$$(1-4x)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} (-4x)^k$$

$$= \sum_{k=0}^{\infty} \binom{-1/2}{k} (-4)^k x^k$$

Now coefficient of x^k will be

$$\binom{-\frac{1}{2}}{k}(-4)^k = \frac{\overbrace{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\cdots\left(-\frac{1}{2}-k+1\right)}^{k \text{ terms}}}{k!}(-4)^k$$

$$= \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{1-2k+2}{2}\right)}{k!}(-4)^k$$

$-2k+1$

$$= \frac{(-1)^k\left[\frac{1}{2}\right]\left[\frac{3}{2}\right]\cdots\left[\frac{2k-1}{2}\right]}{k!}[-4]^k$$

$$= \frac{(-1)^k\, 1\cdot 3\cdot 5\cdots(2k-1)}{2^k\, k!}(-1)^k(4)^k$$

$$= \frac{1\cdot 3\cdot 5\cdots(2k-1)(2^2)^k}{2^k\, k!}$$

$$= \frac{\left[1\cdot 3\cdot 5\cdots(2k-1)\cdot 2^k\, k!\right]}{k!\; k!}$$

when $k=1 \rightarrow 1\cdot 2^1 1! = 2!$

$k=2 \rightarrow 1\cdot 3\cdot 2^2\cdot 2! = 1\cdot 2\cdot 3\cdot 4 = 4!$

$k=3 \rightarrow 1\cdot 3\cdot 5\cdot 2^3\cdot 3! = 1\cdot 2\cdot 3\times 4\times 5\times 6$

$=$

$$= \frac{(2k)!}{k!\,k!}$$

$$= \binom{2k}{k} \quad \text{where } k \in N$$

Thus the sequence generated by $(1-4x)^{-\frac{1}{2}}$ is $\left\{ \binom{2n}{n},\ n \in N \right\}$

ie; $\binom{2}{1}, \binom{4}{2}, \binom{6}{3}, \ldots$

NOTE

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repetition allowed

$$f(x) = \underbrace{(x^0+x+x^2+\cdots)}_{\text{object 1}}\underbrace{(x^0+x+x^2+\cdots)}_{\text{object 2}}\cdots\underbrace{(x^0+x+x^2+\cdots)}_{\text{object } n}$$

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need to select 2 items.

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ie; $\frac{5!}{3!\,2!} = 10$ ways

To solve this problem using generating function

[keeping in mind we can choose 0 or 1 of each item ie, either leave that or choose that item]

Consider the generating function

$$g(x) = (x^0+x^1)(x^0+x^1)(x^0+x^1)(x^0+x^1)(x^0+x^1)$$

$$= (1+x)(1+x)(1+x)(1+x)(1+x)$$

$$= (1+x)^5$$

We need the no: of ways of

Choosing 2 items

That is nothing but the coefficient of x^2 in the above generating function

Thus $g(x) = (1+x)^5$

$$= \sum_{k=0}^{\infty} \binom{5}{k} x^k$$

$$= \binom{5}{0} x^0 + \binom{5}{1} x + \binom{5}{2} x^2 + \binom{5}{3} x^3 + \ldots$$

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Thus for 4 officers, the resulting generating function will be

$$f(x) = (x^3 + x^4 + \dots + x^8)^4$$

We need to distribute 24 rifle shells

$\Rightarrow$ we need to find the coefficient of x^{24} in $f(x)$

Here $f(x) = [x^3 + x^4 + x^5 + x^6 + x^7 + x^8]^4$

$$= \left(x^3[1 + x + x^2 + x^3 + x^4 + x^5]\right)^4$$

$$= x^{12}\left[1+x+x^2+x^3+x^4+x^5\right]^4$$

$$= x^{12}\left[\frac{1-x^{5+1}}{1-x}\right]^4$$

$$= x^{12}\left[\frac{1-x^6}{1-x}\right]^4 \qquad \frac{1-x^{n+1}}{1-x} = 1+x+\cdots+x^n$$

We need to get the coefficient of x^{24}

ie, here, it is enough to find the coefficient of x^{12} in $\left(\frac{1-x^6}{1-x}\right)^4$.

ie, in $(1-x^6)^4[1-x]^{-4}$

$$\boxed{\begin{aligned}(1+x)^n &= \binom{n}{0}x^0+\binom{n}{1}x+\binom{n}{2}x^2+\cdots+\binom{n}{n}x^n \\ (1+x)^{-n} &= \binom{-n}{0}x^0+\binom{-n}{1}x+\binom{-n}{2}x^2+\cdots\end{aligned}}$$

$$= \left[\binom{4}{0}+\binom{4}{1}(-x^6)+\binom{4}{2}(-x^6)^2+\binom{4}{3}(-x^6)^3+\binom{4}{4}(-x^6)^4\right]$$

$$\left[\binom{-4}{0}+\binom{-4}{1}(-x)+\binom{-4}{2}(-x)^2+\binom{-4}{3}(-x)^3+\cdots\right]$$

$x \to -x^6$ $\qquad\qquad$ $x \to -x$

$$= \left[1 - \binom{4}{1}x^6 + \binom{4}{2}x^{12} - \binom{4}{3}x^{18} + x^{24}\right]$$

$$\left[1 - \binom{-4}{1}x + \binom{-4}{2}x^2 - \binom{-4}{3}x^3 + \dots\right]$$

Thus the term of x^{12} will be

$$\left[\binom{-4}{12}(-1)^{12}x^{12} - \binom{4}{1}x^6\binom{-4}{6}(-1)^6x^6 + \binom{4}{2}x^{12}\right]$$

$$\text{ie; } \left[\binom{-4}{12}(-1)^{12} - \binom{4}{1}\binom{-4}{6}(-1)^6 + \binom{4}{2}\right]x^{12}$$

Thus coefficient of x^{12}

$$\Rightarrow \binom{-4}{12} - 4\binom{-4}{6} + \binom{4}{2}$$

$$\binom{-n}{r} = \frac{(-1)^r(n+r-1)!}{(n-1)!\,r!}$$

$$\Rightarrow \frac{(-1)^{12}[4+12-1]!}{(4-1)!\,12!} - 4\left[\frac{(-1)^6(4+6-1)!}{(4-1)!\,6!}\right] + \frac{4!}{2!\,2!}$$

$$\Rightarrow \frac{\overset{13\times14\times15}{15!}}{3!\,12!} - \frac{4\times \overset{7\times8\times9}{9!}}{3!\,6!} + \frac{4!}{2!\,2!}$$

$$\Rightarrow \frac{13\times \overset{7}{\cancel{14}}\times \overset{5}{\cancel{15}}}{1\times\cancel{2}\times\cancel{3}} - \frac{7\times \overset{4}{\cancel{8}}\times \overset{3}{\cancel{9}}\times 4}{1\times\cancel{2}\times\cancel{3}} + \frac{3\times \overset{2}{\cancel{4}}}{1\times\cancel{2}}$$

$$\Rightarrow 455 - 336 + 6$$

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3 Use generating functions to determine how many four element subset of $S = \{1, 2, 3, \ldots, 15\}$ contain no consecutive integers?

Let $S = \{1, 2, 3, \ldots, 15\}$

We need to select a 4 element subset of S, which contain no consecutive integers.

Let $\{1, 3, 7, 10\}$ be such a subset

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$$1-1 = 0$$
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Sum of those differences $= 0+2+4+3+5$
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Again consider $\{1, 3, 6, 8\}$

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Thus, we can say that there is a one-one correspondence between the four element subsets to be counted and the integer solutions to

$$c_1 + c_2 + c_3 + c_4 + c_5 = 14$$

where c_i's are the corresponding differences, $c_1, c_5 \geq 0$ and $c_2, c_3, c_4 \geq 2$ which guarantee that there are no consecutive integers in the subset.

Thus, we can form the generating function as

$$f(x) = \underbrace{(x^0 + x^1 + x^2 + \cdots)}_{c_1} \underbrace{(x^2 + x^3 + \cdots)}_{c_2}$$

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$$\underbrace{(x^0 + x^1 + x^2 + \cdots)}_{c_5}$$

$$= (x^0+x^1+x^2+\cdots)(x^2+x^3+x^4+\cdots)^3(x^0+x+x^2+\cdots)$$

$$= (1+x+x^2+\cdots)^2(x^2+x^3+x^4+\cdots)^3$$

$$= \left(\frac{1}{1-x}\right)^2\left[(x^2)\left[1+x+x^2+\cdots\right]\right]^3$$

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We need to find the coeffient of x^{14} in $f(x)$

So it is enough to find the coeffient of x^8 in $(1-x)^{-5}$

Coeffient of x^k in $(1-x)^{-n}$ is given by $\frac{(n+k-1)!}{(n-1)!\,k!}$

Coefficient of x^8 in $(1-x)^{-5}$

$n=5$

$k=8$

$$= \frac{(5+8-1)!}{(5-1)!\,8!}$$

$$= \frac{12!}{4!\,8!} = \frac{9 \times \cancel{10}^{5} \times 11 \times \cancel{12}^{2}}{1 \times \cancel{2} \times \cancel{3} \times \cancel{4}}$$

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4. In how many ways can 15 identical chocolates be distributed among 4 children, so that each child gets atleast 2?

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Eg:

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For n distinct objects, the series $1+x+x^2+x^3+\dots$ represents the possible Choices for the objects (repetition allowed) (namely none, one, two, three, ...). Considering all the n distinct objects the generating function for this case will be

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$$\underbrace{(x^0 + x^1 + x^2 + \cdots)}_{c_5}$$

$$= (x^0+x^1+x^2+\cdots)(x^2+x^3+x^4+\cdots)^3(x^0+x+x^2+\cdots)$$

$$= (1+x+x^2+\cdots)^2(x^2+x^3+x^4+\cdots)^3$$

$$= \left(\frac{1}{1-x}\right)^2\left[(x^2)\left[1+x+x^2+\cdots\right]\right]^3$$

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$= x^8\left[\frac{1}{1+x}\right]^4$

$= x^8(1+x)^{-4}$ $k=7, n=4$

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$$= 120$$

The Exponential generating function

For a sequence $a_0, a_1, a_2, \ldots$

of real numbers,

$$f(x) = a_0 + a_1 x + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + \cdots$$

$$= \sum_{k=0}^{\infty} a_k \frac{x^k}{k!}$$ is called

the exponential generating function for the given sequence.

1. Find the exponential generating function of the sequence 1, 1, 1, ...

Soln: exponential generating function of 1, 1, 1, ...

is $f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$= e^x.$

Thus e^x is the exponential generating function of the sequence 1, 1, 1, ...

NOTE.

e^x is the generating function of the sequence $1, 1, \frac{1}{2!}, \frac{1}{3!}, \ldots$

* $$\boxed{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots}$$

* $$\boxed{e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots}$$

are the Maclaurin series expansion of e^x and e^{-x} respectively

* $e^{x}+e^{-x}=\left[1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right]+$

$$\left[1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\cdots\right]$$

$$=2+2\frac{x^2}{2!}+2\frac{x^4}{4!}+\cdots$$

$$=2\left[1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right]$$

Thus $\boxed{\frac{e^{x}+e^{-x}}{2}=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots}$

Also $e^{x}-e^{-x}=\left[1+x+\frac{x^2}{2!}+\cdots\right]-\left[1-x+\frac{x^2}{2!}-\cdots\right]$

$$=2x+2\frac{x^3}{3!}+2\frac{x^5}{5!}+\cdots$$

$$=2\left[x+\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots\right]$$

Thus $\boxed{\frac{e^{x}-e^{-x}}{2}=x+\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots}$

2. Determine the exponential generating function of the sequence a) 1, −1, 1, −1, ...

$$f(x) = a_0 + a_1 x + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + \cdots$$

$$= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots$$

$$= e^{-x}$$

b) $1, 2, 2^2, 2^3, \ldots$

$$f(x) = 1 + 2x + 2^2 \frac{x^2}{2!} + 2^3 \frac{x^3}{3!} + \cdots$$

$$= 1 + (2x) + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \cdots$$

$$= e^{2x}$$

c) $1, -a, a^2, -a^3, a^4, \ldots, \quad a \in \mathbb{R}$

$$f(x) = 1 - ax + \frac{a^2x^2}{2!} - \frac{a^3x^3}{3!} + \frac{a^4x^4}{4!} - \cdots$$

$$= 1 + (-ax) + \frac{(-ax)^2}{2!} + \frac{(-ax)^3}{3!} + \cdots$$

$$= e^{-ax}$$

d) $1, a^2, a^4, a^6, \ldots, \quad a \in \mathbb{R}$

$$f(x) = 1 + a^2 x + a^4 \frac{x^2}{2!} + a^6 \frac{x^3}{3!} + \cdots$$

$$= 1 + (a^2 x) + \frac{(a^2 x)^2}{2!} + \frac{(a^2 x)^3}{3!} + \cdots$$

$$= e^{a^2 x}$$

e) $a, a^3, a^5, a^7, \ldots, \quad a \in \mathbb{R}$

$$f(x) = a + a^3 x + a^5 \frac{x^2}{2!} + a^7 \frac{x^3}{3!} + \cdots$$

$$= a\left[1 + a^2 x + a^4 \frac{x^2}{2!} + a^6 \frac{x^3}{3!} + \cdots\right]$$

$$= a\left[1 + (a^2 x) + \frac{(a^2 x)^2}{2!} + \cdots\right]$$

$$= a e^{a^2 x}$$

f) $0, 1, 2(2), 3(2^2), 4(2^3), \ldots$

$$f(x) = 0 + x + 2(2)\frac{x^2}{2!} + 3(2^2)\frac{x^3}{3!} + 4(2^3)\frac{x^4}{4!} + \ldots$$

$$= x\left[1 + \frac{(2^2)x}{2!} + 3\frac{(2^2)x^2}{3!} + 4\frac{(2^3)x^3}{4!} + \ldots\right]$$

$$= x\left[1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \ldots\right]$$

$$= x e^{2x}$$

II Determine the sequence generated by each of the following exponential generating functions.

1. $f(x) = 5e^{5x}$

$$= 5\left[1 + (5x) + \frac{(5x)^2}{2!} + \frac{(5x)^3}{3!} + \frac{(5x)^4}{4!} + \ldots\right]$$

$$= 5 + 5^2 x + 5^3 \frac{x^2}{2!} + 5^4 \frac{x^3}{3!} + \cdots$$

Correspondiy Sequence $5, 5^2, 5^3, \cdots$

---

2. $f(x) = 7e^{8x} - 4e^{3x}$

$$= 7\left[1 + (8x) + \frac{(8x)^2}{2!} + \frac{(8x)^3}{3!} + \cdots\right]$$

$$- 4\left[1 + (3x) + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \cdots\right]$$

$$= \left[7 + 7(8)x + 7(8^2)\frac{x^2}{2!} + \cdots\right]$$

$$+ \left[(-4) + (-4)3x + (-4)3^2\frac{x^2}{2!} + \cdots\right]$$

Correspondiy sequence

$(7-4), 7(8)-4(3), 7(8^2)-4(3^2), 7(8^3)-4(3^3), \cdots$

$= 3, 44, 412, 3476, \cdots$

3. $f(x) = 2e^{x} + 3x^{2}$.

$$= 2\left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right] + 3x^2$$

$$= 2 + 2x + \left(\frac{2+(3)2!}{2!}\right)\frac{x^2}{2!} + \frac{2x^3}{3!} + \dots$$

∴ corresponding sequence

$$2, 2, \cancel{4}\,5, 2, \dots$$

4. $f(x) = e^{3x} - 28x^3 - 6x^2 + 9x$

?

$$= \left(1 + (3x) + \frac{(3x)^2}{2!} + \dots\right) - 28x^3 - 6x^2 + 9x$$

$$= 1 + (3+9)x + \left[\frac{3^2}{2!} - 6\right]x^2 + \left(\frac{3^3}{3!} - 28\right)x^3 + \dots$$

$$= 1 + 12x + \left(\frac{3^2 - 6(2!)}{2!}\right)x^2 + \left(\frac{27 - 28(3!)}{3!}\right)x^3 + \frac{(3x)^4}{4!} + \dots$$

$$= 1 + 12x + [9 - 12]\frac{x^2}{2!} + [27 - 168]\frac{x^3}{3!} + 3^4\frac{x^4}{4!} + \dots$$

Sequence is 1, 12, -3, 141, 3^4, 35, ...

Recurrence Relation.

Consider a sequence of numbers

5, 15, 45, 135, . . .

Let $a_0 = 5$

$a_1 = 15 = 3 \times 5$

$a_3 = 45 = 3 \times 15$

$a_4 = 135 = 3 \times 45$

⋮

Generally $a_{n+1} = 3a_n$ defines a recurrence relation. Thus we can calculate the value of a_n from a_{n-1}, value of a_{n-1} using a_{n-2} and so on. Provided that the value of the function at one or more points are given. Those given points are called boundary points.

Example: Consider the fibonacci sequence of numbers.

1, 1, 2, 3, 5, 8, 13, 21, ...

ie, Sequence with $a_0 = 1$

$a_1 = 1$

(boundary conditions)

Generally we can write the relation

$$a_r = a_{r-1} + a_{r-2} \quad r \geq 2$$

with $a_0 = 1$ & $a_1 = 1$

Linear Recurrence relation with Constant Coefficient.

First Order linear recurrence relation with constant Coefficients

(Homogeneous)

* The general first order linear recurrence relation with constant

coefficients has the form

$$a_{n+1} + c\, a_n = f(n) \,, \quad n \geq 0$$

where c is a constant and $f(n)$ is a function on the set $\mathbb{N}$ of non negative integers.

* When $f(n) = 0$, $\forall n \in \mathbb{N}$ the relation is called homogeneous

* Otherwise it is called Non Homogeneous

* $a_{n+1} + c\, a_n = f(n)$ is called a recurrence relation of 1st order, since the value of a_{n+1} depends only on its immediate predecessor.

First order Homogeneous recurrence relation with constant coefficient.

The unique solution of the recurrence relation $a_{n+1} = d\, a_n$ where $n \geq$ ... d is a constant & $a_0 = A$

is given by $a_n = Ad^n$, $n \geq 0$

Thus the solution $a_n = Ad^n$, $n \geq 0$ defines a discrete function whose domain is the set of all non negative integers.

1. Solve the recurrence relation

$$a_n = 7a_{n-1} \text{ where } n \geq 1, a_2 = 98$$

Soln

Given $a_n = 7a_{n-1}$, $n \geq 1$, $a_2 = 98$

$\Rightarrow a_{n+1} = 7a_n$, $n \geq 0$, $a_2 = 98$ — (1)

Thus Here $d = 7$

$a_2 = 98$

ie, $n = 1$ in (1)

$\Rightarrow 7a_1 = 98$ Using (1)

$\Rightarrow a_1 = \frac{98}{7}$

$\Rightarrow 7a_0 = \frac{98}{}$ $n = 0$

$\Longrightarrow a_0 = \frac{98}{7^2}$

$= 2$

Thus $d = 7$ and $a_0 = 2 = A$

Thus the unique solution of the given sequence relation is $a_n = A d^n$

$$a_n = 2(7)^n, \quad n \geq 0$$

ie, 2, 14, 98, 686, ..., is the solution

2. Find a unique solution for the sequence relation $a_{n+1} - (3) a_n = 0$, $n \geq 0$, $a_0 = 5$

Soln given $a_{n+1} - (3) a_n = 0$

$\Longrightarrow a_{n+1} = (3) a_n$, $n \geq 0$, $a_0 = 5$

Here $d = 3$ $A = 5$

$\therefore$ Unique solution is given by

$$a_n = 5(3^n), \quad n \geq 0$$

ie, 5, 15, 45, 135, ..., is the solution

3. Solve $a_{n+1} = 3a_n, n \geq 0$ & $a_1 = 21$

HW

Here $a_1 = 21$

$\Rightarrow 3a_0 = 21$

$\Rightarrow a_0 = \frac{21}{3} = 7$

ie; $A = 7$

$d = 3$

$\therefore$ Unique solution is given by

$a_n = A\, d^n$

$= 7(3^n), n \geq 0.$

ie, 7, 21, 63, . . . is the unique soln

4. Find the solution of

$a_{n+1} - 1.5\, a_n = 0, n \geq 0$

Let $a_0 = A$

here $d = 1.5$

$\therefore$ Solution is given by

$a_n = A d^n$

$= A(1.5)^n$ is the solution

$n \geq 0$

5. find the unique solution of

$3a_{n+1} - 4a_n = 0 \quad n \geq 0, \; a_1 = 5$

Soln

$3a_{n+1} - 4a_n = 0.$

$3a_{n+1} = 4a_n$

$a_{n+1} = \frac{4}{3} a_n, \; n \geq 0.$

$n=0 \Rightarrow a_1 = \frac{4}{3} a_0$

$\Rightarrow a_0 = \frac{3}{4} a_1$

$= \frac{3}{4}(5) = \frac{15}{4}$

$A = \frac{15}{4}$

$d = \frac{4}{3}$

Unique solution is given by

$a_n = A d^n$

$= \frac{15}{4}\left(\frac{4}{3}\right)^n, \; n \geq 0$

Q: find a recurrence relation with initial condition that uniquely determines each of the following sequence that begin with the given terms

a) $3, 7, 11, 15, 19, \ldots$

Here $a_0 = 3$

$$a_1 = 7 = 3 + 4 = a_0 + 4$$

$$a_2 = 11 = 7 + 4 = a_1 + 4$$

$$a_3 = 15 = 11 + 4 = a_2 + 4$$

$$\vdots$$

$$a_{n+1} = a_n + 4, \; n \geq 0$$

is the recurrence relation

b) $8, \frac{24}{7}, \frac{72}{49}, \frac{216}{343}, \ldots$

Here $a_0 = 8$

$$a_1 = \frac{24}{7} = \frac{3}{7}(8) = \frac{3}{7} a_0$$

$$a_2 = \frac{72}{49} = \frac{3}{7}\left[\frac{24}{7}\right] = \frac{3}{7} a_1$$

⋮

Thus $a_{n+1} = \frac{3}{7} a_n \ , n \geq 0.$

Q: Find the unique solution of the recurrence relation. $6a_n - 7a_{n-1} = 0 \ , n \geq 1$

and $a_3 = 343.$

$$6a_n - 7a_{n-1} = 0 \quad n \geq 1$$

$$\Rightarrow 6a_{n+1} - 7a_n = 0, \ n \geq 0.$$

$$6a_{n+1} = 7a_n$$

$$\Rightarrow a_{n+1} = \frac{7}{6} a_n$$

here $d = \frac{7}{6}$

$a_3 = 343$

ie; $\frac{7}{6} a_2 = 343$

$\Rightarrow \left(\frac{7}{6}\right)\left(\frac{7}{6} a_1\right) = 343$

$$\Rightarrow \left(\frac{7}{6}\right)^3 a_0 = 343$$

$$\Rightarrow a_0 = 343 \times \frac{6^3}{7^3}$$

$$= 6^3 = 216$$

ie, $A = 216$

$d = \frac{7}{6}$

$\therefore$ Unique solution is $a_n = A\, d^n$

$$\Rightarrow a_n = 216 \cdot \left[\frac{7}{6}\right]^n, \; n \geq 0$$

Q: A person invests Rs 100,000 at 12 % interest compounded annually

a) find the amount at the end of 1st, 2nd and 3rd year

b) write the general explicit formula

c) How long will it take to double the investment.

Let a_n denote the amount at the end of n years.

Then $a_n = a_{n-1} + (0.12)a_{n-1}$

$a_n = 1.12\, a_{n-1} \quad n \geq 1$

where $a_{n-1}(0.12)$ is the interest received

a). $a_0 = 100,000$

$a_1 = a_0 + (0.12)\, a_0$

$= 100,000 + (0.12)100,000$

$= (1.12)(100,000)$

$= 112,000$

$a_2 = 1.12(a_1)$

$= 1.12(112,000)$

$= 125,440$

$a_3 = a_2(1.12)$

$= 140,492.8$

The amounts at the end of 1st, 2nd & 3rd year are given by a_1, a_2 and a_3 respectively

b)

$$a_n = (1.12)\, a_{n-1}$$

$$a_{n-1} = (1.12)\, a_{n-2}$$

$$a_{n-2} = (1.12)\, a_{n-3}$$

$$\vdots$$

$$a_3 = (1.12)\, a_2$$

$$a_2 = (1.12)\, a_1$$

$$a_1 = (1.12)\, a_0$$

n terms

Thus $a_n = (1.12)(1.12)\cdots(1.12)\, a_0$

$a_n = (1.12)^n a_0$ is the

general explicit formula

c) $a_0 = 100,000$

Let $a_n = 2(100,000) = 200,000$

ie, Let the amount will double in n years, we need to find n.

Thus $a_n = (1.12)^n a_0$

$$\Rightarrow 200,000 = (1.12)^n \, 100,000$$

$$\Rightarrow 2 = (1.12)^n$$

$$\ln 2 = \ln (1.12)^n$$

$$\ln 2 = n \ln(1.12)$$

$$\Rightarrow n = \frac{\ln 2}{\ln(1.12)}$$

$$= 6.116$$

$$= 6.12 \text{ years}$$

Q: Solve $a_n = n \cdot a_{n-1}$, where $n \geq 1$.

and $a_0 = 1$.

Soln Thus $a_n = n\, a_{n-1}$ $\quad n \geq 1$

$$\Longrightarrow a_{n+1} = (n+1)\, a_n \quad n \geq 0$$

here $d = (n+1)$, a variable not a constant

$$a_0 = 1 = A$$

~~Thus unique solution~~

~~is $a_n = A(d^n)$~~

~~$= (n+1)^n$~~

$a_0 = 1$

$a_1 = 1 \cdot a_0 = 1 \quad = 1!$

$a_2 = 2 \cdot a_1 = 2 \quad = 2!$

$a_3 = 3 \cdot a_2 = 6 = 3!$

$a_4 = 4 \cdot a_3 = 24 = 4!$

$\vdots$

Thus $a_n = n!$ $n \geq 0$ is the

unique solution

Q: Find a_{12}, if $a_{n+1}^2 = 5a_n^2$ where

$a_n > 0$ for $n \geq 0$ and $a_0 = 2$

Here $a_{n+1}^2 = 5a_n^2$ is not linear — ①

but let $\boxed{a_n^2 = b_n}$ then ①

$\Rightarrow$ $b_{n+1} = 5b_n$ which is a linear

homogeneous recurrence relation

Here $d = 5$

$b_0 = A = a_0^2 = 4$

Thus unique solution of

$b_{n+1} = 5b_n$ is given by

$b_n = A(d^n)$

$\Rightarrow b_n = 4(5)^n, n \geq 0$

Thus $a_n^2 = 4(5)^n$

Taking square root on both sides

$$\Rightarrow a_n = 2(\sqrt{5})^n \quad n \geq 0$$

Thus $a_{12} = 2(\sqrt{5})^{12}$

$= 31,250$

Q: A bank pays 6% (annual) interest on savings, compounding the interest monthly. If Bonnie deposits 1000/- on the first day of May. How much will this deposit be worth a year later?

Annual interest $= 6\%$

$\therefore$ monthly interest $= \frac{6\%}{12}$

$= 0.5\%$

$= 0.005$

For $0 \le n \le 12$, let P_n denote the value of Bonnie's deposit at the end of n months.

Then $P_{n+1} = P_n + 0.005\, P_n$

where $0.005\,P_n$ is the interest earned on P_n during the $(n+1)^{th}$ month for $0 \le n \le 11$.

Now $P_0 = 1000$

$$P_{n+1} = (1.005)\,P_n \quad n \ge 0.$$

Thus here $A = 1000$

$$d = 1.005$$

$\therefore$ Unique solution

$$P_n = A(d)^n$$

$$= (1000)(1.005)^n, \; n \ge 0$$

Consequently, at the end of one year.

ie; $P_{12} = 1000(1.005)^{12}$

$$= 1061.67781 \text{ Rs}$$

Hw solve $4a_n - 5a_{n-1} = 0$, $n \geq 1$

$2a_n - 3a_{n-1} = 0$ $n \geq 1$ $a_4 = 8$

Second order linear recurrence relation with constant coefficient

(Homogeneous)

General form of a second order linear homogeneous recurrence relation with constant coefficient is given by

$$C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} = 0, \quad n \geq 2 \qquad \text{(1)}$$

Step 1 Substitute $a_n = cr^n$ $c \neq 0$ $r \neq 0$

$$(1) \Rightarrow C_0 c r^n + C_1 c r^{n-1} + C_2 c r^{n-2} = 0$$

$$\Rightarrow C_0 r^n + C_1 r^{n-1} + C_2 r^{n-2} = 0$$

divide through out by smallest power of r

$$\Rightarrow \frac{C_0 r^n + C_1 r^{n-1} + C_2 r^{n-2}}{r^{n-2}} = 0$$

$$\Rightarrow \boxed{C_0 r^2 + C_1 r + C_2 = 0}$$ is

called the characteristic equation of (1), which is a quadratic equation. Let r_1 and r_2 be the roots of ✱

Now the solution of (1) depends on the nature of r_1 and r_2

Case I [Distinct and real roots]

If r_1 and r_2 are distinct real roots then the corresponding solution is given by

$$\boxed{a_n = c_1 r_1^n + c_2 r_2^n}$$

Now using the given boundary conditions find c_1 and c_2

1. Solve the recurrence relation $a_n + a_{n-1} - 6a_{n-2} = 0$ where $n \geq 2$ $a_0 = -1$ $a_1 = 8$

Soln $a_n + a_{n-1} - 6a_{n-2} = 0$

Corresponding characteristic equation is

$$r^n + r^{n-1} - 6r^{n-2} = 0$$

divide throughout by least power of 'r'

$$r^2 + r - 6 = 0$$

which is a quadratic equation

$$r = \frac{-1 \pm \sqrt{1-(4)(1)(-6)}}{2} = \frac{-1 \pm \sqrt{25}}{2}$$

$$= \frac{-1 \pm 5}{2}$$

$$\Longrightarrow r_1 = -3 \quad \text{and} \quad r_2 = 2$$

ie, real and distinct roots

$\therefore$ general solution

$$a_n = c_1 r_1^n + c_2 r_2^n$$

$$\Longrightarrow a_n = c_1(-3)^n + c_2(2)^n \quad \text{——— (1)}$$

Now using the boundary conditions

$a_0 = -1$ and $a_1 = 8$

find c_1 and c_2

Consider $a_0 = -1$

ie; put $n = 0$ in (1)

$$a_0 = c_1(-3)^0 + c_2(2)^0$$

$$-1 = c_1 + c_2 \quad \text{---} \quad (2)$$

Put $n=1$ in (1) $\qquad a_1 = 8$

$$a_1 = c_1(-3)^1 + c_2(2)^1$$

$$\Rightarrow 8 = -3c_1 + 2c_2 \quad \text{---} \quad (3)$$

Solve equations (2) & (3)

$$(2) \times -3 \Rightarrow 3 = -3c_1 - 3c_2$$

$$(3) \Rightarrow 8 = -3c_1 + 2c_2 \quad -$$

$$-5 = 0 - 5c_2$$

$$\Rightarrow \underline{\underline{c_2 = 1}}$$

Eqn (2) $\Rightarrow c_1 + c_2 = -1$

$$\Rightarrow c_1 + 1 = -1$$

$$\Rightarrow c_1 = -2$$

∴ The required solution is

$$a_n = c_1(-3)^n + c_2(2)^n$$

$$= -2(-3)^n + (2)^n, \; n \geq 0$$

which is the unique solution of the given recurrence relation

2. Solve the recurrence relation

$F_{n+2} = F_{n+1} + F_n$ where $n \geq 0$ and

$F_0 = 0, \; F_1 = 1$

Soln $F_{n+2} = F_{n+1} + F_n$

Corresponding characteristic equation will be $\Rightarrow r^{n+2} = r^{n+1} + r^n$

divide throughout by r^n

$\Rightarrow r^2 = r + 1$

$\Rightarrow r^2 - r - 1 = 0$ is the characteristic eqn

$$\therefore r = \frac{-1 \pm \sqrt{(-1)^2 - (4)(1)(-1)}}{2}$$

$$= \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$$r_1 = \frac{1+\sqrt{5}}{2} \text{ and } r_2 = \frac{1-\sqrt{5}}{2}$$

are real and distinct roots

$\therefore$ general solution is given by

$$F_n = c_1 (r_1)^n + c_2 (r_2)^n$$

$$F_n = c_1 \left[\frac{1+\sqrt{5}}{2}\right]^n + c_2 \left[\frac{1-\sqrt{5}}{2}\right]^n \quad \text{—— (1)}$$

$n \geq 0$

$F_0 = 0$

Put $n=0$ in (1)

$$F_0 = c_1 \left[\frac{1+\sqrt{5}}{2}\right]^0 + c_2 \left[\frac{1-\sqrt{5}}{2}\right]^0$$

$$\Rightarrow 0 = c_1 + c_2 \quad \text{--- (2)}$$

$F_1 = 1$ put $n=1$ in (1)

$$\Rightarrow F_1 = c_1\left[\frac{1+\sqrt5}{2}\right] + c_2\left[\frac{1-\sqrt5}{2}\right]$$

$$\Rightarrow 1 = \frac{c_1(1+\sqrt5) + c_2(1-\sqrt5)}{2}$$

$$\Rightarrow 2 = c_1(1+\sqrt5) + c_2(1-\sqrt5) \quad \text{--- (3)}$$

Solving eqns (2) & (3) we get

$$(2) \times (1+\sqrt5) \Rightarrow c_1(1+\sqrt5) + c_2(1+\sqrt5) = 0$$

$$(3) \Rightarrow c_1(1+\sqrt5) + c_2(1-\sqrt5) = 2$$

$$0 + c_2\left[\cancel{1}+\sqrt5 - \cancel{1}+\sqrt5\right] = -2$$

$$\Rightarrow 2\sqrt5\, c_2 = -2$$

$$\Rightarrow c_2 = \frac{-1}{\sqrt5}$$

Now ② $\Rightarrow c_1 + c_2 = 0$

$$\Rightarrow c_1 = -c_2$$

$$\Rightarrow c_1 = \frac{1}{\sqrt{5}}$$

Thus the unique solution of the given recurrence relation is

$$f_n = \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2}\right]^n - \frac{1}{\sqrt{5}}\left[\frac{1-\sqrt{5}}{2}\right]^n$$

$n \geq 0$

Hw. Solve $a_n = 5a_{n-1} - 6a_{n-2}$, $n \geq 2$

$a_0 = 1$ and $a_1 = 0$

$$a_n = c_1 2^n + c_2 3^n$$

$$c_2 = -2 \;\&\; c_1 = 3$$

3. Solve $a_n = 10a_{n-1} + 29a_{n-2}$, $n \geq 2$

$a_0 = 0$ and $a_1 = 10$

Soln $a_n = 10a_{n-1} + 29a_{n-2}$

Ch. eqn

$r^n = 10r^{n-1} + 29r^{n-2}$

divide throughout by r^{n-2}

$$\begin{array}{r|l} 2 & 216 \\ 2 & 108 \\ 2 & 54 \\ 3 & 27 \\ 3 & 9 \\ & 3 \end{array}$$

$\Longrightarrow r^2 - 10r - 29 = 0$

$r = \dfrac{10 \pm \sqrt{100-(4)(-29)}}{2} = \dfrac{10 \pm \sqrt{100+116}}{2}$

$= \dfrac{10 \pm 6\sqrt{6}}{2} = 5 \pm 3\sqrt{6}$

$r_1 = 5+3\sqrt{6} \qquad r_2 = 5-3\sqrt{6}$

Soln

$$a_n = c_1(5+3\sqrt{6})^n + c_2(5-3\sqrt{6})^n \quad \text{— (1)}$$

$a_0 = 0 \qquad n=0$ in (1)

$\Rightarrow c_1 + c_2 = 0$ — (2)

$a_1 = 10 \qquad n=1$ in (1)

$$\Rightarrow c_1(5+3\sqrt{6}) + c_2(5-3\sqrt{6}) = 10 \quad \text{— (3)}$$

$(2) \times (5+3\sqrt{6}) \Rightarrow (5+3\sqrt{6})c_1 + (5+3\sqrt{6})c_2 = 0$

$$(5+3\sqrt{6})c_1 + (5-3\sqrt{6})c_2 = 10$$

$$0 + (\cancel{5}+3\sqrt{6} - \cancel{5} + 3\sqrt{6})c_2 = -10$$

$$\Rightarrow 6\sqrt{6}\, c_2 = -10$$

$$\Rightarrow c_2 = \frac{-10}{6\sqrt{6}}$$

$$= \frac{-5}{3\sqrt{6}}$$

Now ② $\Rightarrow c_1 = -c_2$

$$= \frac{5}{3\sqrt{6}}$$

Thus unique solution is given by

$$a_n = \frac{5}{3\sqrt{6}}(5+3\sqrt{6})^n - \frac{5}{3\sqrt{6}}(5-3\sqrt{6})^n$$

$n \geq 0$

---

4. Solve $P_n = 2P_{n-2}$ $n \geq 3$, $P_1 = 1$ and $P_2 = 2$

Soln $P_n - 2P_{n-2} = 0$.

Ch. eqn. $r^n - 2r^{n-2} = 0$.

$\Rightarrow r^2 - 2 = 0$ is the characteristic eqn

$r^2 = 2$

$r = \pm\sqrt{2}$

$r_1 = \sqrt{2}$ and $r_2 = -\sqrt{2}$

$$\therefore P_n = c_1(\sqrt{2})^n + c_2(-\sqrt{2})^n \quad n \geq 0$$

$$\text{———— } (1)$$

Now $P_1 = 1$ $\quad n = 1$ in (1)

$$\Rightarrow c_1\sqrt{2} + c_2(-\sqrt{2}) = 1$$

$$\Rightarrow \sqrt{2}c_1 - \sqrt{2}c_2 = 1 \text{ —— } (2)$$

also $P_2 = 2$ $\quad n = 2$ in (1)

$$2 = c_1(\sqrt{2})^2 + c_2(-\sqrt{2})^2$$

$$\Rightarrow 2c_1 + 2c_2 = 2$$

$$\Rightarrow c_1 + c_2 = 1 \text{ —— } (3)$$

Solving (2) & (3)

$$(3) \times \sqrt{2} \Rightarrow \sqrt{2}c_1 + \sqrt{2}c_2 = \sqrt{2}$$

$$(2) \Rightarrow \sqrt{2}c_1 - \sqrt{2}c_2 = 1 \quad +$$

$$2\sqrt{2}c_1 + 0 = (\sqrt{2} + 1)$$

$$\Rightarrow c_1 = \frac{\sqrt{2}+1}{2\sqrt{2}} = \frac{1}{2} + \frac{1}{2\sqrt{2}}$$

$$③ \Rightarrow c_2 = 1 - c_1$$

$$= 1 - \frac{1}{2} - \frac{1}{2\sqrt{2}}$$

$$= \frac{1}{2} - \frac{1}{2\sqrt{2}}$$

Thus $P_n = \left(\frac{1}{2} + \frac{1}{2\sqrt{2}}\right)(\sqrt{2})^n + \left[\frac{1}{2} - \frac{1}{2\sqrt{2}}\right](-\sqrt{2})^n$

where $n \geq 0$ is the

unique solution

HW Solve $a_n = a_{n-1} + a_{n-2}$, $n \geq 3$

$a_1 = 2$ and $a_2 = 3$

HW solve the recurrence relation

10.28

$$2a_n = 7a_{n-1} - 3a_{n-2}, \quad a_0 = 2$$

and $a_1 = 5$.

Soln: $r_1 = 3 \quad r_2 = \frac{1}{2}$

$$c_1 = \frac{8}{5} \quad c_2 = \frac{2}{5}$$

Case III [Complex roots]

1. De Moivre's Theorem

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta, \; n \geq 0$

Let $r_1 = x+iy$ and $r_2 = x-iy$

then the general solution will be of the form $a_n = c_1(x+iy)^n + c_2(x-iy)^n$

where $n \geq 0$, c_1 and c_2 can be obtained using the given boundary conditions

*

If $z = x + iy$ then it can be expressed as

$$z = r[\cos\theta + i\sin\theta] \text{ where}$$

$x = r\cos\theta$

$y = r\sin\theta$ and $r = \sqrt{x^2+y^2}$

* If

$z = yi$ form then it can be expressed as

$$z = yi\left[\sin\frac{\pi}{2}\right]$$

$$= y\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right]$$

1. Solve the recurrence relation

$$a_n = 2(a_{n-1} - a_{n-2}) \quad n \geq 2, a_0 = 1$$

and $a_1 = 2$, final answer should not involve complex numbers.

Sol

Here $a_n = 2a_{n-1} - 2a_{n-2}$

$$\Rightarrow r^n = 2r^{n-1} - 2r^{n-2}$$

divide throughout by r^{n-2}

$$\Rightarrow r^2 = 2r - 2$$

$$\Rightarrow$$

$r^2 - 2r + 2 = 0$ is the Characteristic equation

$$r = \frac{2 \pm \sqrt{4 - (4 \times 2)}}{2} = \frac{2 \pm \sqrt{-4}}{2}$$

$$= \frac{2 \pm 2i}{2}$$

$$= 1 \pm i$$

$r_1 = 1 + i$ and $r_2 = 1 - i$

General solution is given by

$$a_n = c_1 (1+i)^n + c_2 (1-i)^n \quad \text{—— (1)}$$

$1+i \Rightarrow x=1 \ , y=1$

$1+i$
$\Rightarrow (1,1)$

$$r=\sqrt{x^2+y^2}=\sqrt{2}$$

$$\theta=\tan^{-1}\left(\frac{y}{x}\right)=\tan^{-1}(1)=\frac{\pi}{4}$$

$\therefore \ x+iy = r\left[\cos\theta + i\sin\theta\right]$

$\Rightarrow 1+i = \sqrt{2}\left[\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right]$

$1-i \Rightarrow x=1 \quad y=-1$

$1-i$
$\Rightarrow (1,-1)$

$$r=\sqrt{x^2+y^2}=\sqrt{2}$$

$$\theta=\tan^{-1}\left(\frac{y}{x}\right)=\tan^{-1}(-1)$$
$$=-\frac{\pi}{4}$$

$1-i \Rightarrow \sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right]$

Now,

$$1+i = \sqrt{2}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]$$

$$1-i = \sqrt{2}\left[\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right]$$

Thus.

$$a_n = c_1(1+i)^n + c_2(1-i)^n$$

$$= c_1\left[\sqrt{2}\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]^n +$$

$$c_2\left[\sqrt{2}\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right]^n$$

Using De Moivre's Theorem

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

$$= c_1\left[(\sqrt{2})^n\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right)\right] +$$

$$c_2\left[(\sqrt{2})^n\left[\cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right]\right]$$

$$= \left[c_1 \left[(\sqrt{2})^n\right] \cos \frac{n\pi}{4} + c_2 (\sqrt{2})^n \cos \frac{n\pi}{4} \right]$$

$$+ \left[c_1 i (\sqrt{2})^n \sin \frac{n\pi}{4} + c_2(-i) \sin \frac{n\pi}{4} \right]$$

$$= (\sqrt{2})^n \left[(c_1 + c_2) \cos \frac{n\pi}{4} + (c_1 - c_2) i \sin \frac{n\pi}{4} \right]$$

Let $k_1 = c_1 + c_2$

$k_2 = (c_1 - c_2) i$

$$\therefore a_n = (\sqrt{2})^n \left[k_1 \cos \frac{n\pi}{4} + k_2 \sin \frac{n\pi}{4} \right] \quad \text{—} (1)$$

Now $a_0 = 1$ and $a_1 = 2$

Put $n = 0$ in (1)

$$\Rightarrow 1 = (\sqrt{2})^0 \left[k_1 \cos 0 + k_2 \sin 0 \right]$$

$$\Rightarrow k_1 = 1 \quad \text{—} (2)$$

Put $n = 1$ in (1)

$$a_1 = (\sqrt{2})^1\left[k_1 \cos\frac{\pi}{4} + k_2 \sin\frac{\pi}{4}\right]$$

$$2 = \sqrt{2}\left[k_1 \frac{1}{\sqrt{2}} + k_2 \frac{1}{\sqrt{2}}\right]$$

$$\Rightarrow \quad k_1 + k_2 = 2 \quad \text{—} \ (3)$$

$$\Rightarrow \quad 1 + k_2 = 2 \quad (\text{using } 2)$$

$$\Rightarrow \quad \underline{\underline{k_2 = 1}}$$

Thus the unique solution of the given recurrence relation is:

$$a_n = (\sqrt{2})^n\left[\cos\left(\frac{n\pi}{4}\right) + \sin\left(\frac{n\pi}{4}\right)\right]$$

$$n \geq 0$$

---

2 Solve $a_{n+2} + a_n = 0$, $n \geq 0$

$a_0 = 0$ $a_1 = 2$ ... find

answer should not involve complex numbers?

$$a_{n+2} + a_n = 0$$

$\Rightarrow r^{n+2} + r^n = 0$

divide throughout by r^n

$\Rightarrow r^2 + 1 = 0$ is the characteristic equation

$$r = \frac{-0 \pm \sqrt{0-(4)(1)(1)}}{2} = \frac{\pm\sqrt{-4}}{2}$$

$$= \frac{\pm 2i}{2}$$

$r_1 = i$ and $r_2 = -i$

(0,1) (0,-1)

$\therefore$ general solution is given by

$$a_n = c_1 [i]^n + c_2 [-i]^n$$

$$\boxed{yi = yi \left[\sin \frac{\pi}{2}\right]}$$

Now

$$i = i \left[\sin \frac{\pi}{2}\right]$$

$$= \left[\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right]$$

$$(i)^n = \left[\cos\frac{n\pi}{2} + i\sin\frac{n\pi}{2}\right]^n$$

Similarly

$$\left[-i\right]^n = \left[\cos\frac{n\pi}{2} - i\sin\frac{n\pi}{2}\right]^n$$

Thus

$$a_n = c_1\left[\left(\cos\frac{n\pi}{2} + i\sin\frac{n\pi}{2}\right)\right] +$$

$$c_2\left[\left[\cos\frac{n\pi}{2} - i\sin\frac{n\pi}{2}\right]\right]$$

$$= \left[(c_1+c_2)\cos\frac{n\pi}{2} + (c_1-c_2)i\sin\frac{n\pi}{2}\right]$$

Let $k_1 = c_1 + c_2$

$k_2 = (c_1 - c_2)i$

$$\Rightarrow a_n = \left[k_1\cos\frac{n\pi}{2} + k_2\sin\frac{n\pi}{2}\right] \quad \text{—— (1)}$$

Now $a_0 = 0$ and $a_1 = 3$

Put $n = 0$ in (1)

$$\Rightarrow 0 = \left[k_1\cos 0 + k_2\sin 0\right]$$

$$\Rightarrow k_1 = 0 \quad \text{—— (2)}$$

put $n=1$ in ①

$$a_1 = \left[k_1 \cos\frac{\pi}{2} + k_2 \sin\frac{\pi}{2} \right]$$

$$3 = \left[0 + k_2 \right]$$

$$\Rightarrow k_2 = 3$$

$$k_2 = 3$$

$$\therefore a_n = \left[3 \sin\frac{n\pi}{2} \right], n \geq 0$$

---

3 Solve $a_n + 2a_{n-1} + 2a_{n-2} = 0$

Hw $n \geq 2$, $a_0 = 1$ $a_1 = 3$

$$r^n + 2r^{n-1} + 2r^{n-2} = 0$$

$\Rightarrow r^2+2r+2=0.$

$$r = \frac{-2 \pm \sqrt{4-(4)(1)(2)}}{2} = \frac{-2 \pm 2i}{2}$$

$$= -1 \pm i$$

$-1+i \Rightarrow x=-1 \quad y=1$

$$r = \sqrt{x^2+y^2} = \sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}(-1) = \frac{-\pi}{4}, \frac{3\pi}{4}$$

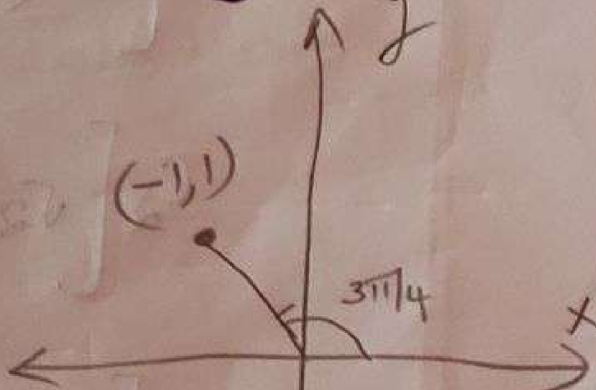
here $-1+i$ is a point on second quadrant

$-1+i \Rightarrow r[\cos\theta + i\sin\theta]$ $\quad \therefore \theta = \frac{3\pi}{4}$

$$\Rightarrow \sqrt{2}\left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right]$$

$z = x+iy$

Corresponds to the point (x,y) in the x-y plane.



$$\tan\left(\frac{3\pi}{4}\right) = -\tan\left(\frac{\pi}{4}\right) = -1$$

$\frac{225}{45}$

$\pi + \frac{\pi}{4}$

$\frac{5\pi}{4}$

Now,

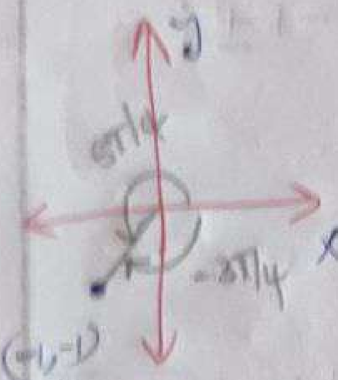
$-1-i \Rightarrow x=-1 \quad y=-1$

$r = \sqrt{2}$

$(-1,-1)$

$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}(1)$

$= \frac{\pi}{4}, \frac{5\pi}{4}, -\frac{3\pi}{4}$



$= \left(-\frac{3\pi}{4}\right)$

$\therefore a_n = c_1 r_1^n + c_2 r_2^n$

$= c_1\left[\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)\right]^n +$

$c_2\left[\sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right)\right]^n$

$= c_1\left[(\sqrt{2})^n\left(\cos\frac{3n\pi}{4} + i\sin 3n\frac{\pi}{4}\right) +\right.$

$\left.(\sqrt{2})^n c_2\left[\cos\left(3n\frac{\pi}{4}\right) - i\sin\left(3n\frac{\pi}{4}\right)\right]\right]$

$$a_n = (\sqrt{2})^n \left[(c_1 + c_2) \cos \frac{3n\pi}{4} + (c_1 - c_2) i \sin \frac{3n\pi}{4} \right]$$

$n \geq 0$

$$\Rightarrow a_n = (\sqrt{2})^n \left[k_1 \cos \frac{3n\pi}{4} + k_2 \sin \frac{3n\pi}{4} \right]$$

$n \geq 0$ —— (1)

$a_0 = 1$ put $n = 0$ in (1)

$$\Rightarrow 1 = (\sqrt{2})^0 \left[k_1 \cos 0 + 0 \right]$$

$$\Rightarrow k_1 = 1$$

$a_1 = 3$ put $n = 1$ in (1)

$$\Rightarrow a_1 = \sqrt{2} \left[\cos \frac{3\pi}{4} + k_2 \sin \frac{3\pi}{4} \right]$$

$\cos 3\pi/4 = \frac{-1}{\sqrt{2}}$ and $\sin \frac{3\pi}{4} = \frac{1}{\sqrt{2}}$

$$\Rightarrow 3 = \sqrt{2}\left[-\frac{1}{\sqrt{2}} + k_2\left(\frac{1}{\sqrt{2}}\right)\right]$$

$$\Rightarrow 3 + 1 = k_2 \Rightarrow k_2 = 4$$

$$\therefore a_n = (\sqrt{2})^n\left[\cos\left(\frac{3n\pi}{4}\right) + 4\sin\left(\frac{3n\pi}{4}\right)\right]$$

$n \geq 0$

Q: Determine $(1+\sqrt{3}i)^{10}$

Here $1+\sqrt{3}i \Rightarrow x = 1$

$y = \sqrt{3}$

$$\therefore r = \sqrt{x^2+y^2} = \sqrt{1+3} = 2$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left[\frac{\sqrt{3}}{1}\right]$$

$$= \frac{\pi}{3}$$

$$\therefore (1+\sqrt{3}i)^{10} = \left[2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right]^{10}$$

* $x+iy = r[\cos\theta + i\sin\theta]$ where $r=\sqrt{x^2+y^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$$= (2)^{10}\left[\cos\frac{10\pi}{3} + i\sin\frac{10\pi}{3}\right]$$

$$= 2^{10}\left[\left(-\frac{1}{2}\right) - i\left[\frac{\sqrt{3}}{2}\right]\right]$$

$$= 2^{10}\left[-\frac{1}{2}\right] - i\,2^{10}\left[\frac{\sqrt{3}}{2}\right]$$

$$= (2)^{9}\left[-i\sqrt{3}(2)^{9} - 1\right]$$

$$= (-2)^{9}\left[i\sqrt{3}+1\right]$$

$$\cos\frac{10\pi}{3} = -\frac{1}{2}$$

$$\sin\frac{10\pi}{3} = \frac{-\sqrt{3}}{2}$$

Case III [Repeated Real roots]

If $r_1 = r_2 = r$, then the solution is given by

$$\boxed{a_n = c_1 r^n + c_2 n r^n}$$

Now using the given boundary conditions find c_1 and c_2.

1. Solve the recurrence relation

$$a_{n+2} = 4a_{n+1} - 4a_n \, , \; n \geq 0$$

$$a_0 = 1 \text{ and } a_1 = 3$$

Soln $a_{n+2} = 4a_{n+1} - 4a_n$

$\Rightarrow r^{n+2} = 4r^{n+1} - 4r^n$

divide throughout by r^n

$\Rightarrow r^2 - 4r + 4 = 0$ is the characteristic equation

$\therefore r = \frac{4 \pm \sqrt{16-16}}{2} = 2, 2$

$r_1 = r_2 = 2$

$\therefore$ solution is given by

$a_n = c_1 (2)^n + c_2 . n . (2)^n \quad , n \geq 0$ —— (1)

$a_0 = 1$ put $n=0$ in (1)

$\Rightarrow 1 = c_1 + 0$

$\Rightarrow c_1 = 1$

$a_1 = 3$ put $n=1$ in (1)

$\Rightarrow 3 = 2^1 + c_2 (1)(2)$

$\Rightarrow 3 = 2 + 2c_2$

$\Rightarrow 1 = 2c_2$

$\Rightarrow c_2 = \frac{1}{2}$

∴ The unique solution is given by

$$a_n = 2^n + \left(\frac{1}{2}\right) n (2)^n$$

$$= 2^n + n(2)^{n-1}, \quad n \geq 0$$

2. Solve $a_n - 6a_{n-1} + 9a_{n-2} = 0$

$n \geq 2$, $a_0 = 5$, $a_1 = 12$

$r^n - 6r^{n-1} + 9r^{n-2} = 0$

$\Rightarrow r^2 - 6r + 9 = 0$ is the characteristic equation

$$r = \frac{6 \pm \sqrt{36 - (4)(9)}}{2} = 3, 3$$

$r_1 = r_2 = 3$

$\therefore a_n = C_1 (3)^n + C_2 n (3)^n$, $n \geq 0$ —— (1)

$a_0 = 5$ put $n = 0$

$5 = C_1$ —— (2)

$a_1 = 12$ put $n = 1$

$12 = 3C_1 + 3C_2$ —— (3)

$\Rightarrow 12 = 15 + 3c_2$

$\Rightarrow -3 = 3c_2$

$\Rightarrow c_2 = -1$

Ktunotes.in

$\therefore a_n = 5(3^n) + (-1)n\,3^n$

$= (5-n)3^n, \quad n \geq 0$

is the unique solution.

Non homogeneous Recurrence relation

General solution for a non homogeneous Recurrence relation with Constant coefficient [1st and IInd Order] is given by

$$\boxed{a_n = a_n^h + a_n^p}$$

where a_n^h is the general solution of corresponding to the homogeneous equation and a_n^p is the Particular solution of the Corresponding non homogeneous part. To find a_n^p we use the form of $f(n)$ in the given relation

Type $f(n)$	a_n^p
C, a constant	A, nA, n^2A, ...
n	$A_1 n + A_0$
n^2	$A_2 n^2 + A_1 n + A_0$
n^t, $t \in Z^+$	$A_t n^t + A_{t-1} n^{t-1} + \dots + A_0$
r^n, $r \in R$	$A r^n$ $A n r^n$, $A n^2 r^n$, ...
$\sin n\theta$	$A \sin n\theta + B \cos \theta n$
$\cos n\theta$	$A \sin n\theta + B \cos n\theta$
$r^n n^t$	$r^n [A_t n^t + A_{t-1} n^{t-1} + \dots + A_0]$
$r^n \sin n\theta$	$A r^n \sin n\theta + B r^n \cos n\theta$
$r^n \cos n\theta$	$A r^n \sin n\theta + B r^n \cos n\theta$

Problems.

1. Solve the recurrence relation

$$a_{n+2} - 4a_{n+1} + 3a_n = -200 \quad (1)$$

$n \geq 0$, $a_0 = 3000$, $a_1 = 3300$.

Here the required solution is of the form $\boxed{a_n = a_n^h + a_n^p}$

* To find a_n^h

Consider $a_{n+2} - 4a_{n+1} + 3a_n = 0$

$$r^{n+2} - 4r^{n+1} + 3r^n = 0$$

$\Longrightarrow r^2 - 4r + 3 = 0$ is the characteristic equation

$$\therefore r = \frac{4 \pm \sqrt{16-(4)(3)}}{2} = \frac{4 \pm 2}{2} = 3, 1$$

$\therefore \quad a_n^h = c_1 r_1^n + c_2 r_2^n$

$= c_1(3)^n + c_2(1)^n$

$= c_1(3)^n + c_2$

* Now to find a_n^p

Here $f(n) = -200$, a constant

So let $a_n = A$

$a_n = A$
$\Rightarrow a_{n+2} = A$
$a_{n+1} = A$

put $a_n = A$ in ①

$\Rightarrow A - 4A + 3A = -200$

$\Rightarrow -3A + 3A = -200$

$\Rightarrow 0 = -200$ ← false statement

So put $a_n = nA$ in ①

$(n+2)A - 4(n+1)A + 3(nA) = 200$

$\Rightarrow \cancel{nA} + 2A - \cancel{4nA} - 4A + \cancel{3nA} = 200$

$\Rightarrow -2A = 200$

$\Rightarrow A = -100$

$\therefore \quad a_n^P = nA$

$= n(100)$

Thus the required solution is

$$a_n = c_1(3^n) + c_2 + 100n \quad \text{—— (2)}$$

with $a_0 = 3000$, $a_1 = 3300$

Put $n=0$ in (2)

$\Rightarrow a_0 = c_1(3)^0 + c_2 + 100(0)$

$\Rightarrow 3000 = c_1 + c_2$ —— (3)

Put $n=1$ in (2)

$\Rightarrow 3300 = c_1(3) + c_2 + 100$

$\Rightarrow 3c_1 + c_2 = 3200$ —— (4)

Solve (3) & (4)

(4) $-$ (3) $\Rightarrow 2c_1 = 200$

$\Rightarrow c_1 = 100$

3. HW Solve $a_n = a_{n-1} + a_{n-2} + 1,\ n \geq 2$

$a_0 = 1,\ a_1 = 1$

Given $a_n = a_{n-1} + a_{n-2} + 1$ —— (1)

Consider $a_n = a_{n-1} + a_{n-2}$

$\Longrightarrow a_n - a_{n-1} - a_{n-2} = 0$

$\Longrightarrow r^n - r^{n-1} - r^{n-2} = 0$

$\Longrightarrow r^2 - r - 1 = 0$

$$r = \frac{1 \pm \sqrt{1-(4)(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$$\therefore r_1 = \frac{1+\sqrt{5}}{2} \ \& \ r_2 = \frac{1-\sqrt{5}}{2}$$

$$\therefore a_n^h = C_1\left[\frac{1+\sqrt{5}}{2}\right]^n + C_2\left[\frac{1-\sqrt{5}}{2}\right]^n$$

$n \geq 0$ —— (2)

Now $f(x) = 1$

Let $a_n^p = A$, a constant

put $a_n = A$ in ①

$\Rightarrow A - A - A = 1$

$\Rightarrow -A = 1$

$\Rightarrow A = -1$

Thus $a_n^p = -1$

Thus $a_n = C_1\left[\frac{1+\sqrt{5}}{2}\right]^n + C_2\left[\frac{1-\sqrt{5}}{2}\right]^n - 1$

$n \geq 0$ ——— ③

$a_0 = 0$ & $a_1 = 0$

$n=0$ in ③

$0 = C_1 + C_2 - 1$

$\Rightarrow C_1 + C_2 = 1$ ——— ④

$n=1$ in ③

$0 = C_1\left[\frac{1+\sqrt{5}}{2}\right] + C_2\left[\frac{1-\sqrt{5}}{2}\right] - 1$

$$\Rightarrow c_1\left[\frac{1+\sqrt{5}}{2}\right] + c_2\left[\frac{1-\sqrt{5}}{2}\right] = 1$$

$$\Rightarrow c_1(1+\sqrt{5}) + c_2(1-\sqrt{5}) = 2 \quad \text{--- (5)}$$

Solving (4) & (5)

$$\Rightarrow (4) \times (1+\sqrt{5}) \Rightarrow \cancel{(1+\sqrt{5})}c_1 + (1+\sqrt{5})c_2 = 1+\sqrt{5}$$

$$(5) \Rightarrow \cancel{(1+\sqrt{5})}c_1 + (1-\sqrt{5})c_2 = 2$$

$$(1+\sqrt{5}-1+\sqrt{5})c_2 = 1+\sqrt{5}-2$$

$$\Rightarrow 2\sqrt{5}\, c_2 = \sqrt{5}-1$$

$$\Rightarrow c_2 = \frac{\sqrt{5}-1}{2\sqrt{5}}$$

$$= \frac{1}{2} - \frac{1}{2\sqrt{5}}$$

Now $c_1 = 1 - c_2$

$$= 1 - \frac{1}{2} + \frac{1}{2\sqrt{5}}$$

$$= \frac{1}{2} + \frac{1}{2\sqrt{5}} = \frac{1+\sqrt{5}}{2\sqrt{5}} = \frac{\sqrt{5}+1}{2\sqrt{5}}$$

Thus $c_1 = \frac{(1+\sqrt{5})}{2\sqrt{5}}$ & $c_2 = \frac{\sqrt{5}-1}{2\sqrt{5}}$

Thus

$$a_n = \left[\frac{1+\sqrt{5}}{2\sqrt{5}}\right]\left[\frac{1+\sqrt{5}}{2}\right]^n + \left[\frac{\sqrt{5}-1}{2\sqrt{5}}\right]\left[\frac{1-\sqrt{5}}{2}\right]^n$$

$$= \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2}\right]^{n+1} - \frac{1}{\sqrt{5}}\left[\frac{1-\sqrt{5}}{2}\right]^{n+1}$$

$n \geq 0$

is the required solution

Now ③ $\Rightarrow c_2 = 3000 - 100$

$= 2900$

$\therefore$ Unique solution of the given non homogeneous recurrence relation is given by $a_n = 100(3^n) + 2900 + 100n$

$n \geq 0$

Q) Solve $a_{n+1} - 2a_n = 5$, $n \geq 0$, $a_0 = 1$
H/W

$$a_n = a_n^h + a_n^p$$

* Consider $a_{n+1} - 2a_n = 0$ / 1st order homogeneous

$$a_{n+1} = 2a_n$$

$$d = 2$$

$$\therefore a_n^h = A(d^n) = A2^n \quad , n \geq 0.$$

* To find a_n^p.

here $f(n) = 5$

Let $a_n = A$.

We have $a_{n+1} - 2a_n = 5$ — (1)

$a_n = A$

$\Rightarrow A - 2A = 5$

$\Rightarrow -A = 5$

$\Rightarrow A = -5$

$\therefore a_n^p = -5$

Thus the required solution is

$$a_n = a_n^h + a_n^p$$

$$a_n = A2^n - 5, \quad n \geq 0$$

$a_0 = 1 \Rightarrow 1 = A(2^0) - 5$

$\Rightarrow A = 1$

$\therefore$ ...

Result

Consider the non homogeneous second order relation of the form

$$a_n + c_1 a_{n-1} + c_2 a_{n-2} = k r^n$$

where k is a constant. then

- $a_n^p = A r^n$, for a constant A

  (If r^n is not a solution of the associated homogeneous relation),

- $a_n^p = A n r^n$, where A is a constant

  (If $r_1 = r$ and $r_2 \neq r$ where r_1 and r_2 are roots of the corresponding characteristic equation of homogeneous relation) ie; If $a_n^h = c_1 r^n + c_2 r_1^n$

- $a_n^p = A n^2 r^n$, where A is a constant

  (If $r_1 = r_2 = r$ are the roots of the corresponding homogeneous eqn) ie, when $a_n^h = (c_1 + c_2 n) r^n$

4. Solve $a_n - 3a_{n-1} = 5(7^n)$ $n \geq 1$

$$a_0 = 2$$

Here $a_n - 3a_{n-1} = 5(7^n)$ — (1)

Consider $a_n - 3a_{n-1} = 0$

$a_n = 3a_{n-1}, \; n \geq 1$

$\Longrightarrow a_{n+1} = 3a_n \quad n \geq 0$

Here $d = 3$

$A = a_0 = 2$

$\therefore a_n^h = A(d^n)$,

$= A(3^n), \; n \geq 0$

Now $f(n) = 5(7^n)$ $\boxed{\text{Since } a_n^h = A(3^n) \; r \neq 3 \text{ here}}$

$\therefore$ let $a_n^p = B(7^n)$,

$\therefore$ ① $\Longrightarrow B(7^n) - 3B(7^{n-1}) = 5(7^n)$

$\Longrightarrow B\,7^n - \dfrac{3B7^n}{7} = 5(7^n)$

divide throughout by 7^n

$\Longrightarrow B - \dfrac{3}{7}B = 5$

$\Longrightarrow \dfrac{4}{7}B = 5$

$\Longrightarrow B = \dfrac{35}{4}$ //

$$\therefore a_n^p = A(7^n)$$

$$= \frac{35}{4}(7^n) = \frac{5\times 7}{4}(7)^n$$

$$= \frac{5}{4}(7)^{n+1}$$

Thus required solution is

$$a_n = a_n^h + a_n^p$$

$$= A(3)^n + \frac{5}{4}(7)^{n+1} \quad n \geq 0 \qquad (2)$$

Now by using the boundary condition

ie, $a_0 = 2$

Put $n=0$ in

$$2 = A(3)^0 + \frac{5}{4}(7)^{0+1}$$

$$2 = A + \frac{35}{4}$$

$$\Longrightarrow A = 2 - \frac{35}{4} = \frac{8-35}{4} = \frac{-27}{4}$$

$$\therefore a_n = \left(\frac{-27}{4}\right)3^n + \left(\frac{5}{4}\right)7^{n+1}$$

$$= \frac{-3^3}{4}(3^n) + \frac{5}{4}(7^{n+1})$$

$$= \left(\frac{-1}{4}\right)3^{n+3} + \left(\frac{5}{4}\right)7^{n+1} \qquad n \geq 0$$

5. Solve the recurrence relation

$$a_n - 3a_{n-1} = 5(3^n) \quad n \geq 1 \quad a_0 = 2$$

Here $a_n = a_n^h + a_n^p$

$r^{n+1} = 3r^n$

$r = 3$

* **To find a_n^h**

Consider $a_n - 3a_{n-1} = 0$

$\Longrightarrow a_{n+1} = 3a_n \qquad d = 3$

$\Longrightarrow a_n^h = A(d)^n$

$= A(3)^n$

* To find a_n^p

Here $f(n) = 5(3)^n$

Let $a_n^p = B(3)^n$

we have

$$a_n - 3a_{n-1} = 5(3)^n \quad \text{--- (1)}$$

put $\boxed{a_n = B(3)^n}$ in (1)

$$\Rightarrow B(3)^n - 3B(3)^{n-1} = 5(3)^n$$

$$\Rightarrow B(3)^n - B(3)^n = 5(3)^n$$

$$\Rightarrow 0 = 5(3)^n \quad n \geq 0$$

not a true statement

So here $\boxed{a_n = Bn(3)^n}$ in (1)

Since $a_n^h = A(3)^n$

$$\Rightarrow Bn3^n - 3B(n-1)3^{n-1} = 5 3^n$$

$\div 3^{n-1}$

$$\Rightarrow Bn(3) - 3B(n-1) = 5(3)$$

$\Rightarrow 3Bn - 3Bn + 3B = 15$

$\Rightarrow B = \frac{15}{3} = 5$

$\therefore$ Here $a_n^p = 5n(3)^n$

Thus the required solution

$$a_n = a_n^h + a_n^p$$

$$= A(3)^n + 5n(3)^n \quad n \geq 0 \quad \text{—— (2)}$$

Now $a_0 = 2$

Put $n=0$ in (2)

$\Rightarrow 2 = A(3)^0 + 5(0)$

$\Rightarrow A = 2$

$\therefore$ Solution is $a_n = 2(3)^n + 5n(3)^n$

$= (2+5n)3^n, \; n \geq 0$

$\therefore$ Required solution is given by

$$a_n = \left(\frac{11}{6}\right) 3^n - n - \frac{3}{2}, \; n \geq 0$$

6 Solve the recurrence relation

$$a_{n+2} - 8a_{n+1} + 16a_n = 8(5^n) + 6(4^n) \quad \text{---} (1)$$

where $n \geq 0$ and $a_0 = 12$, $a_1 = 5$.

Here (1) is a second order non-homogeneous linear recurrence relation

$$\therefore a_n = a_n^h + a_n^p$$

* To find a_n^h

Consider $a_{n+2} - 8a_{n+1} + 16a_n = 0$

$$r^{n+2} - 8r^{n+1} + 16r^n = 0$$

$\Rightarrow r^2 - 8r + 16 = 0$ is the characteristic eqn

$\therefore r_1 = 4, r_2 = 4$

Case III, real & repeated roots

$\therefore a_n^h = c_1 r^n + c_2 n r^n$

$= c_1 4^n + c_2 n(4)^n$

* To find a_n^p

Here $f(n) = 8(5^n) + 6(4)^n$

$\therefore a_n^p = a_n^{p_1} + a_n^{p_2}$

* where $a_n^{p_1}$ corresponds to $8(5)^n$

and $a_n^{p_2}$ corresponds to $6(4)^n$

$a_n^h = c_1 4^n + c_2 n(4)^n$

Let $a_n^p = A(5^n) + B(n^2)(4)^n$

Since $a_n^h = c_1 4^n + c_2 n 4^n$

So put $a_n = A(5^n) + Bn^2 4^n$ in

(1)

Thus $a_{n+2} - 8a_{n+1} + 16a_n = 8(5^n) + 6(4^n)$

$$\Rightarrow \left[A 5^{n+2} + B(n+2)^2 4^{n+2}\right] -$$

$$8\left[A5^{n+1} + B(n+1)^2 4^{n+1}\right] +$$

$$16\left[A5^n + Bn^2 4^n\right] = 8(5^n) + 6(4)^n$$

Compare the coefficients of

5^n on both sides

$A(5^2) - 8A(5) + 16A = 8$

$\Rightarrow 25A - 40A + 16A = 8$

$\Rightarrow A = 8$

Compare the coeffients of 4^n on both sides

$16B(4) - 32B = 6$

$64B - 32B = 6$

$32B = 6$

$B = \frac{6}{32} = \frac{3}{16}$

LHS

$\left[B(n+2)^2 4^n 4^2 - 8(B)(n+1)^2 4^n \cdot 4 + 16Bn^2 4^n\right]$

$\Rightarrow (B[n^2+4n+4]4^2)4^n - 32B(n^2+2n+1)4^n + 16Bn^2 4^n$

coeff of 4^n

$B(4)(4^2) - 32B$

RESULT

7,8,9

* If $f(n) = n^t,\ t \in Z^+$ then

$$a_n^p = A_t n^t + A_{t-1} n^{t-1} + \dots + A_0$$

* $t = 1$, if $f(n) = n$

$$a_n^p = A_1 n + A_0$$

* $t = 2$, if $f(n) = n^2$

$$a_n^p = A_2 n^2 + A_1 n + A_0.$$

⋮

Use boundary conditions & find the constants $A_0, A_1, \dots, A_t$.

If arrived is false statement then $a_n^p = n\left[A_t n^t + A_{t-1} n^{t-1} + \dots + A_0\right]$

again, if it fails to find the constants then $a_n^p = n^2\left[A_t n^t + A_{t-1} n^{t-1} + \dots + A_0\right]$

8) For $n \geq 2$, suppose that there are n people at a party and that each of these people shakes hands (exactly one time) with all the other people there (and no one shakes hand with himself)

Solve the corresponding recurrence relation?

Let a_n denotes the count of total number of handshakes of n people.

If $(n+1)$th person arrives, then

$$a_{n+1} = a_n + (n) \quad n \geq 2$$

$$a_2 = 1$$

Thus $a_{n+1} - a_n = n$, $a_2 = 1$, $n \geq 2$ — 1

is the required generating function

which is a 1st order non homogeneous recurrence relation

$$a_n = a_n^h + a_n^p.$$

* To find a_n^h

Consider $a_{n+1} = a_n$

ie; $d = 1$

$$\therefore a_n^h = A(d^n)$$

$$= A(1)^n$$

* To find a_n^p

here $f(n) = n$

$$a_n^p = A_1 n + A_0.$$

Put $a_n = A_1 n + A_0$ in (1)

Thus $a_{n+1} - a_n = n$

$$\Rightarrow \left(A_1(n+1) + A_0\right) - \left(A_1 n + A_0\right) = n$$

$\Rightarrow A_1 n + A_1 + A_0 - A_1 n - A_0 = n$

Compare the coefficients of n

$\Rightarrow A_1 - A_1 = 1$

$0 = 1$, false statement

$\therefore$ Choose $a_n^{P} = n[A_1 n + A_0]$

put $a_n = n[A_1 n + A_0]$ in (1)

(1) $\Rightarrow (n+1)[A_1(n+1) + A_0] - n[A_1 n + A_0] = n$

$\Rightarrow A_1(n+1)^2 + A_0(n+1) - A_1 n^2 - A_0 n = n$

$\Rightarrow A_1 n^2 + 2A_1 n + A_1 + A_0 n + A_0 - A_1 n^2 - A_0 n = n$

Compare the like terms

* Coefficient of n^2

$\Rightarrow A_1 - A_1 = 0$

$\Rightarrow A_1 = A_1$ ✓

Coeffnt. of n

$2A_1 + A_0 - A_0 = 1$

$\Rightarrow 2A_1 = 1$

$A_1 = \frac{1}{2}$

Coeffnt. of n^0 (ie; constnt. terms)

$A_1 + A_0 = 0$

$A_0 = -A_1$

$= \frac{-1}{2}$

Then $a_n^P = n[A_1 n + A_0]$

$= n\left[\frac{1}{2}(n) - \frac{1}{2}\right]$

$= \left(\frac{1}{2}\right)n^2 - \left(\frac{1}{2}\right)n$

Thus the general solution is given by

$$a_n = a_n^h + a_n^p$$

$$= A(1)^n + \left(\frac{1}{2}\right)n^2 - \left(\frac{1}{2}\right)n \quad \text{--- (2)}$$

where $a_2 = 1$

Put $n=2$ in (2)

$$\Rightarrow 1 = A(1)^2 + \frac{1}{2}(2)^2 - \left(\frac{1}{2}\right)2$$

$$= A + 2 - 1$$

$$1 = A + 1$$

$$\Rightarrow A = 0$$

$\therefore$ Required solution is given by

$$a_n = \left(\frac{1}{2}\right)n^2 - \left(\frac{1}{2}\right)n$$

$$= \left(\frac{1}{2}\right)n\left[n-1\right], \quad n \geq 2$$

9. Derive the formula for the sum of the cubes of the first n natural numbers using a recurrence relation?

We need $\sum_{i=0}^{n} i^3$

Let $a_n = \sum_{i=0}^{n} i^3$

Thus $a_{n+1} = a_n + (n+1)^3, n \geq 0$ ——— (1)

and $a_0 = 0$.

Which is a non homogenous recurrence relation.

$\therefore a_n = a_n^h + a_n^p$

* To find a_n^h

Consider $a_{n+1} - a_n = 0$

$a_{\ldots} = a_{\ldots}$

$d=1$

$$\Rightarrow \quad a_n = A(d^n)$$

$$= A(1)^n$$

$$= A$$

* **To find a_n^p**

here $f(n) = (n+1)^3$

$$= n^3 + 3n^2 + 3n + 1$$

$$\therefore a_n^p = A_3 n^3 + A_2 n^2 + A_1 n + A_0$$

Put $a_n = A_3 n^3 + A_2 n^2 + A_1 n + A_0$

in (1)

$$\Rightarrow \left[A_3(n+1)^3 + A_2(n+1)^2 + A_1(n+1) + A_0\right]$$

$$-\left[A_3 n^3 + A_2 n^2 + A_1 n + A_0\right] = (n+1)^3$$

Compare the coeffnt. of n^3

$$\Rightarrow \quad A_3 - A_3 = 1$$

$\Rightarrow a = 1$, false statement

$\therefore$ Choose $a_n^{p} = n\left[A_3 n^3 + A_2 n^2 + A_1 n + A_0\right]$

Put $a_n = n\left[A_3 n^3 + A_2 n^2 + A_1 n + A_0\right]$ in (1)

$$\Rightarrow (n+1)\left[A_3(n+1)^3 + A_2(n+1)^2 + A_1(n+1) + A_0\right] - n\left[A_3 n^3 + A_2 n^2 + A_1 n + A_0\right] = (n+1)^3$$

Comparing the like coefficients on both sides

ie

$$A_3(n+1)^4 + A_2(n+1)^3 + A_1(n+1)^2 + A_0(n+1) - A_3 n^4 - A_2 n^3 - A_1 n^2 - A_0 n = (n+1)^3 \quad \text{--- (2)}$$

coefficient $(n^4) \rightarrow A_3 - A_3 = 0$

$(2) \Rightarrow A_3[n^4+4n^3+6n^2+4n+1]+A_2[n^3+3n^2+3n+1]+A_1[n^2+2n+1]+A_0(n+1)-A_3n^4-A_2n^3-A_1n^2-A_0n$

$= n^3+3n^2+3n+1$

Coeffnt. of n^3

$\Rightarrow 4A_3+A_2-A_2=1$

$\Rightarrow 4A_3=1 \Rightarrow A_3=\frac{1}{4}$

Coeffnt. of n^2

$\Rightarrow A_3(6)+A_2(3)+A_1-A_1=3$

$\Rightarrow 6A_3+3A_2=3$

$\Rightarrow \frac{6}{4}+3A_2=3$

$\Rightarrow A_2=\frac{\left(3-\frac{3}{2}\right)}{3}=\frac{6-3}{6}=\frac{+1}{2}$

Coeffnt. of n

$\Rightarrow 4A_3 + 3A_2 + 2A_1 + A_0 - A_0 = 3$

$\Rightarrow 4A_3 + 3A_2 + 2A_1 = 3$

$\Rightarrow 4\left(\frac{1}{4}\right) + 3\left(\frac{1}{2}\right) + 2A_1 = 3$

$\Rightarrow 1 + \frac{3}{2} + 2A_1 = 3$

$\Rightarrow \frac{5}{2} + 2A_1 = 3$

$\Rightarrow 2A_1 = 3 - \frac{5}{2}$

$\Rightarrow 2A_1 = \frac{1}{2}$

$\Rightarrow A_1 = \frac{1}{4}$

comparing the constant terms

$\Rightarrow A_3 + A_2 + A_1 + A_0 = 1$

$\Rightarrow \frac{1}{4} + \frac{1}{2} + \frac{1}{4} + A_0 = 1$

$$= \frac{1+2+1}{4} + A_0 = 1$$

$$\Rightarrow \underline{\underline{A_0 = 0}}$$

Thus $a_n^p = n\left[\frac{1}{4}n^3 + \frac{1}{2}n^2 + \frac{1}{4}n + 0\right]$

$$= \underline{\underline{\frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2}}$$

Thus $a_n = a_n^h + a_n^p$

$$= A + \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2 \quad (3)$$

$a_0 = 0$

$n \geq 0$

Put $n = 0$ in (3)

$$0 = A$$

Thus the required solution is given by

$$a_n = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2, \ n \geq 0$$

$$= \frac{n^2}{4}\left[n^2 + 2n + 1\right]$$

$$= \frac{n^2}{4}(n+1)^2$$

$$= \left[\frac{n}{2}(n+1)\right]^2$$

Thus $\displaystyle\sum_{n=0}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

7

HW

find the solution of $a_n = 3a_{n-1} + 2n$

where $a_1 = 3$.

Here $a_n - 3a_{n-1} = 2n$ is ——— (1)

the given sequence relation

$$a_n = a_n^h + a_n^p$$

* To find a_n^h

Consider $a_n - 3a_{n-1} = 0$

$$\Longrightarrow a_{n+1} = 3a_n$$

$$\Longrightarrow a_n = A(d^n)$$

$$d = 3$$

$$\Longrightarrow a_n = A(3)^n, \ ^1$$

* To find a_n^p

Here $f(n) = 2n$

$f(n) = n \Rightarrow a_n^p = A_1 n + A_0$

here Choose $a_n = A_1 n + A_0$.

Substitute in (1)

$\Rightarrow (A_1 n + A_0) - 3(A_1(n-1) + A_0) = 2n$

$\Rightarrow A_1 n + A_0 - 3A_1 n + 3A_1 - 3A_0 = 2n$

$\Rightarrow -2A_1 n - 2A_0 + 3A_1 = 2n$

$\Rightarrow (-2A_1) n + (-2A_0 + 3A_1) = 2n + 0$ —— (2)

Comparing terms of n in (2)

$\Rightarrow -2A_1 = 2$

$\Rightarrow A_1 = -1$

Comparing the constant terms in (2)

$\Rightarrow -2A_0 + 3A_1 = 0$

$\Rightarrow -2A_0 + 3(-1) = 0$

$-2A_0 = 3$

$A_0 = \frac{-3}{2}$

$\therefore a_n^p = (-1)n - \frac{3}{2}$

$\frac{4\times 4}{1\times 2\times 3\times 4}$

$= -n - \frac{3}{2}$

6

Thus the general solution is

$a_n = A(3)^n - n - \frac{3}{2}, n \geq 0$ — (3)

Now $a_1 = 3$

Put $n=1$ in (3)

$3 = A(3) - 1 - \frac{3}{2}$

$\Rightarrow A = \frac{3+1+\frac{3}{2}}{3} = \frac{4+\frac{3}{2}}{3}$

$= \frac{8+3}{6} = \frac{11}{6}$

$\therefore$ Required solution is given by

$$a_n = \left(\frac{11}{6}\right) 3^n - n - \frac{3}{2}, \quad n \geq 0$$

Q HW Solve $a_{n+1} - 2a_n = 5$, $n \geq 0$, $a_0 = 1$

$$a_n = a_n^h + a_n^p$$

* Consider $a_{n+1} - 2a_n = 0$, 1st order homogeneous

$$a_{n+1} = 2a_n$$

$$d = 2$$

$$\therefore a_n^h = A(d^n) = A2^n \quad , n \geq 0.$$

* To find a_n^p.

here $f(n) = 5$

Let $a_n = A$.

We have $a_{n+1} - 2a_n = 5$ — ①

$a_n = A$

$$\Rightarrow A - 2A = 5$$

$$\Rightarrow -A = 5$$

$$\Rightarrow A = -5$$

$$\therefore a_n^p = -5$$

Thus the required solution is

$$a_n = a_n^h + a_n^p$$

$$a_n = A2^n - 5, \ n \geq 0$$

$a_0 = 1 \Rightarrow 1 = A(2^0) - 5$

$$\Rightarrow A = 1$$

$\therefore$

Result

Consider the non homogeneous second-order relation of the form

$$a_n + c_1 a_{n-1} + c_2 a_{n-2} = kr^n$$

where k is a constant. then

- $a_n^p = Ar^n$, for a constant A

  (If r^n is not a solution of the associated homogeneous relation),

- $a_n^p = Anr^n$, where A is a constant

  (If $r_1 = r$ and $r_2 \neq r$ where r_1 and r_2 are roots of the corresponding characteristic equation of homogeneous relation) ie; If $a_n^h = c_1 r^n + c_2 r_1^n$

- $a_n^p = An^2 r^n$, where A is a constant

  (If $r_1 = r_2 = r$ are the roots of the corresponding homogeneous eqn) ie, when $a_n^h = (c_1 + c_2 n) r^n$

4. Solve $a_n - 3a_{n-1} = 5(7^n) \quad n \geq 1$

$$a_0 = 2$$

Here $a_n - 3a_{n-1} = 5(7^n)$ —— (1)

Consider $a_n - 3a_{n-1} = 0$

$a_n = 3a_{n-1},\ n \geq 1$

$\Longrightarrow a_{n+1} = 3a_n \quad n \geq 0$

Here $d = 3$

$A = a_0 = 2$

$\therefore a_n^h = A(d^n),$

$= A(3^n),\ n \geq 0$

Now $f(n) = 5(7^n)$ $\boxed{\text{Since } a_n^h = A(3^n),\ r \neq 3 \text{ here}}$

$\therefore$ let $a_n^p = B(7^n),$

$\therefore (1) \Longrightarrow B(7^n) - 3B(7^{n-1}) = 5(7^n)$

$\Longrightarrow B\,7^n - \frac{3B7^n}{7} = 5(7^n)$

divide throughout by 7^n

$\Longrightarrow B - \frac{3}{7}B = 5$

$\Longrightarrow \frac{4}{7}B = 5$

$\Longrightarrow B = \frac{35}{4}$ //

$$\therefore a_n^p = A(7^n)$$

$$= \frac{35}{4}(7^n) = \frac{5\times 7}{4}(7)^n$$

$$= \frac{5}{4}(7)^{n+1}$$

Thus required solution

$$a_n = a_n^h + a_n^p$$

$$= A(3)^n + \frac{5}{4}(7)^{n+1} \quad n\geq 0 \qquad \text{--- (2)}$$

Now by using the boundary condition

i.e., $a_0 = 2$

Put $n=0$ in

$$2 = A(3)^0 + \frac{5}{4}(7)^{0+1}$$

$$2 = A + \frac{35}{4}$$

$$\Longrightarrow A = 2 - \frac{35}{4} = \frac{8-35}{4} = \frac{-27}{4}$$

$\therefore a_n = \left(\frac{-27}{4}\right) 3^n + \left(\frac{5}{4}\right) 7^{n+1}$

$= \frac{-3^3}{4}(3^n) + \frac{5}{4}(7^{n+1})$

$= \left(\frac{-1}{4}\right) 3^{n+3} + \left(\frac{5}{4}\right) 7^{n+1} \quad n \geq 0$

5. Solve the recurrence relation

$a_n - 3a_{n-1} = 5(3^n) \quad n \geq 1 \quad a_0 = 2$

Here $a_n = a_n^h + a_n^p$

$r^{n+1} = 3r^n$

$r = 3$

* <u>To find a_n^h</u>

Consider $a_n - 3a_{n-1} = 0$

$\Longrightarrow a_{n+1} = 3a_n \qquad d = 3$

$\Longrightarrow a_n^h = A(d)^n$

$= A(3)^n$

* To find a_n^p

Here $f(n) = 5(3)^n$

Let $a_n^p = B(3)^n$

we have

$$a_n - 3a_{n-1} = 5(3)^n \quad \text{--- (1)}$$

put $\boxed{a_n = B(3)^n}$ in (1)

$$\Rightarrow B(3)^n - 3B(3)^{n-1} = 5(3)^n$$

$$\Rightarrow B(3)^n - B(3)^n = 5(3)^n$$

$$\Rightarrow 0 = 5(3)^n \quad n \geq 0$$

not a true statement

So here $\boxed{a_n = Bn(3)^n}$ in (1)

Since $a_n^h = A(3)^n$

$$\Rightarrow Bn3^n - 3B(n-1)3^{n-1} = 5 3^n$$

$\div 3^{n-1}$

$$\Rightarrow Bn(3) - 3B(n-1) = 5(3)$$

$\Rightarrow 3Bn - 3Bn + 3B = 15$

$\Rightarrow B = \frac{15}{3} = 5$

$\therefore$ Here $a_n^p = 5\,n(3)^n$

Thus the required solution

$$a_n = a_n^h + a_n^p$$

$$= A(3)^n + 5\,n(3)^n \quad n \geq 0 \quad \text{--- } (2)$$

Now $a_0 = 2$

Put $n=0$ in (2)

$\Rightarrow 2 = A(3)^0 + 5(0)$

$\Rightarrow A = 2$

$\therefore$ Solution is $a_n = 2(3)^n + 5\,n(3)^n$

$= (2+5n)3^n, \ n \geq 0$

$\therefore$ Required solution is given by

$$a_n = \left(\frac{11}{6}\right)3^n - n - \frac{3}{2}, \; n \geq 0$$

6 Solve the recurrence relation

$$a_{n+2} - 8a_{n+1} + 16a_n = 8(5^n) + 6(4^n) \quad \text{—— (1)}$$

where $n \geq 0$ and $a_0 = 12$, $a_1 = 5$.

Here (1) is a second order non-homogeneous linear recurrence relation

$$\therefore a_n = a_n^h + a_n^p$$

* To find a_n^h

Consider $a_{n+2} - 8a_{n+1} + 16a_n = 0$

$$r^{n+2} - 8r^{n+1} + 16r^n = 0$$

$\Longrightarrow r^2 - 8r + 16 = 0$ is the characteristic eqn

$\therefore r_1 = 4, r_2 = 4$

Case III, real & repeated roots

$$\therefore a_n^h = c_1 r^n + c_2 n r^n$$

$$= c_1 4^n + c_2 n(4)^n$$

* To find a_n^p

Here $f(n) = 8(5^n) + 6(4)^n$

$$\therefore a_n^p = a_n^{p_1} + a_n^{p_2}$$

* where $a_n^{p_1}$ corresponds to $8(5)^n$

and $a_n^{p_2}$ corresponds to $6(4)^n$

$a_n^h = c_1 4^n + c_2 n(4)^n$

Let $a_n^p = A(5^n) + B(n^2)(4)^n$

Since $a_n^h = c_1 4^n + c_2 n 4^n$

So put $a_n = A(5^n) + Bn^2 4^n$ in

(1)

Thus $a_{n+2} - 8a_{n+1} + 16a_n = 8(5^n) + 6(4^n)$

$$\Rightarrow \left[A5^{n+2} + B(n+2)^2 4^{n+2}\right] -$$

$$8\left[A5^{n+1} + B(n+1)^2 4^{n+1}\right] +$$

$$16\left[A5^n + Bn^2 4^n\right] = 8(5^n) + 6(4)^n$$

Compare the coefficients of

5^n on both sides

$A(5^2) - 8A(5) + 16A = 8$

$\Rightarrow 25A - 40A + 16A = 8$

$\Rightarrow A = 8$

$\begin{array}{r} \overset{2}{56} \times \\ 4 \\ \hline 224 \end{array}$

Compare the coeffients of 4^n on both sides

$16B(4) - 32B = 6$

$64B - 32B = 6$

$32B = 6$

$B = \frac{6}{32} = \frac{3}{16}$

LHS

$\left[B(n+2)^2 4^n 4^2 - 8(B)(n+1)^2 4^n \cdot 4 + 16Bn^2 4^n \right]$

$\Rightarrow (B[n^2+4n+4]4^2)4^n - 32B(n^2+2n+1)4^n + 16Bn^2 4^n$

coeff of 4^n

$B(4)(4^2) - 32B$

Thus

$a_n^p = 8(5^n) + \frac{3}{16} n^2 (4)^n$

$56 + \frac{3}{4} = \frac{227}{4}$

$5 - \frac{227}{4}$

Thus $a_n = a_n^h + a_n^p$

$= c_1 4^n + c_2 n 4^n + 8(5^n) + \frac{3}{16} n^2 (4)^n$ —— (2)

Now $a_0 = 12$ and $a_1 = 5$

$a_0 = 12$

Put $n = 0$, in (2)

$$12 = c_1 4^0 + c_2(0) + 8(5)^0 + \frac{3}{16}(0)$$

$$12 = c_1 + 8$$

$$c_1 = 12 - 8 = 4$$

Now $a_1 = 5$

Put $n = 1$, in (2)

$$5 = c_1(4)^1 + c_2(1)4^1 + 8(5) + \frac{3}{16}(4)$$

$$= 4c_1 + 4c_2 + 40 + \frac{3}{4}$$

$$= 4(4) + 4c_2 + 40 + \frac{3}{4}$$

$$= 56 + 4c_2 + \frac{3}{4}$$

$$5 - 56 - \frac{3}{4} = 4c_2$$

$$\frac{(-51)(4) - 3}{4} = 4c_2$$

$$\Rightarrow c_2 = \frac{(-51)(4) - 3}{16}$$

$$= \frac{-207}{16}$$

$$\therefore a_n = 4(4)^n + \left(\frac{-207}{16}\right) n 4^n + 8(5^n) + \frac{3}{16} n^2 (4)^n$$

$$n \geq 0$$

RESULT [1,8,9]

* If $f(n) = n^t,\ t \in Z^+$ then

$$a_n^p = A_t n^t + A_{t-1} n^{t-1} + \dots + A_0$$

* $t = 1$, if $f(n) = n$

$$a_n^p = A_1 n + A_0$$

* $t = 2$, if $f(n) = n^2$

$$a_n^p = A_2 n^2 + A_1 n + A_0.$$

⋮

Use boundary conditions & find the constants $A_0, A_1, \dots, A_t$.

If arrived is false statement then $a_n^p = n\left[A_t n^t + A_{t-1} n^{t-1} + \dots + A_0\right]$

again, if it fails to find the constants take $a_n^p = n^2\left[A_t n^t + A_{t-1} n^{t-1} + \dots + A_0\right]$

8) For $n \geq 2$, suppose that there are n people at a party and that each of these people shakes hands (exactly one time) with all the other people there (and no one shakes hand with himself)

Solve the corresponding recurrence relation ?

Let a_n denotes the count of total number of handshakes of n people.

If $(n+1)$th person arrives, then

$$a_{n+1} = a_n + (n) \quad n \geq 2$$

$$a_2 = 1$$

Thus $a_{n+1} - a_n = n$, $a_2 = 1$, $n \geq 2$ — 1

is the required generating function which is a 1st order non homogeneous recurrence relation

$$a_n = a_n^h + a_n^p.$$

* <u>To find a_n^h</u>

Consider $a_{n+1} = a_n$

ie; $d = 1$

$$\therefore a_n^h = A(d^n)$$

$$= A(1)^n$$

* <u>To find a_n^p</u>

here $f(n) = n$

$$a_n^p = A_1 n + A_0.$$

Put $a_n = A_1 n + A_0$ in (1)

Thus $a_{n+1} - a_n = n$

$$\Rightarrow \left(A_1(n+1) + A_0\right) - \left(A_1 n + A_0\right) = n$$

$$\Rightarrow A_1 n + A_1 + A_0 - A_1 n - A_0 = n$$

Compare the coeffients of n

$$\Rightarrow A_1 - A_1 = 1$$

$$0 = 1$$, false stetemele

$\therefore$ Choose $a_n^P = n[A_1 n + A_0]$

put $a_n = n[A_1 n + A_0]$ in (1)

$$(1) \Rightarrow (n+1)[A_1(n+1) + A_0] - n[A_1 n + A_0] = n$$

$$\Rightarrow A_1(n+1)^2 + A_0(n+1) - A_1 n^2 - A_0 n = n$$

$$\Rightarrow A_1 n^2 + 2A_1 n + A_1 + A_0 n + A_0 - A_1 n^2 - A_0 n = n$$

Compare the like terms

* Coefficent of n^2

$$\Rightarrow A_1 - A_1 = 0$$

$$\Rightarrow A_1 = A_1$$

Coeffnt. of n

$2A_1 + A_0 - A_0 = 1$

$\Rightarrow 2A_1 = 1$

$A_1 = \frac{1}{2}$

Coeffnt. of n^0 (ie; constnt. terms)

$A_1 + A_0 = 0$

$A_0 = -A_1$

$= \frac{-1}{2}$

Thus $a_n^P = n[A_1 n + A_0]$

$= n\left[\frac{1}{2}(n) - \frac{1}{2}\right]$

$= \left(\frac{1}{2}\right)n^2 - \left(\frac{1}{2}\right)n$

Thus the general solution is given by

$$a_n = a_n^h + a_n^p$$

$$= A(1)^n + \left(\frac{1}{2}\right)n^2 - \left(\frac{1}{2}\right)n \quad \text{--- } (2)$$

where $a_2 = 1$

Put $n=2$ in (2)

$$\Rightarrow 1 = A(1)^2 + \frac{1}{2}(2)^2 - \left(\frac{1}{2}\right)2$$

$$= A + 2 - 1$$

$$1 = A + 1$$

$$\Rightarrow \underline{A = 0}$$

$\therefore$ Required solution is given by

$$a_n = \left(\frac{1}{2}\right)n^2 - \left(\frac{1}{2}\right)n$$

$$= \left(\frac{1}{2}\right)n\left[n-1\right], \quad n \geq 2$$

Q. Derive the formula for the sum of the cubes of the first n natural numbers using a recurrence relation?

We need $\sum_{i=0}^{n} i^3$

Let $a_n = \sum_{i=0}^{n} i^3$

Thus $a_{n+1} = a_n + (n+1)^3, n \geq 0$ —— (1)

and $a_0 = 0$.

Which is a non homogenous recurrence relation.

$\therefore a_n = a_n^h + a_n^p$

* To find a_n^h

Consider $a_{n+1} - a_n = 0$

$a_{\ldots} = a_{\ldots}$

$d = 1$

$$\Rightarrow a_n = A(d^n)$$

$$= A(1)^n$$

$$= A$$

* To find a_n^P

here $f(n) = (n+1)^3$

$$= n^3 + 3n^2 + 3n + 1$$

$$\therefore a_n^P = A_3 n^3 + A_2 n^2 + A_1 n + A_0$$

Put $a_n = A_3 n^3 + A_2 n^2 + A_1 n + A_0$

in (1)

$$\Rightarrow \left[A_3(n+1)^3 + A_2(n+1)^2 + A_1(n+1) + A_0\right]$$

$$-\left[A_3 n^3 + A_2 n^2 + A_1 n + A_0\right] = (n+1)^3$$

Compare the coeffnt. of n^3

$$\Rightarrow A_3 - A_3 = 1$$

$\Rightarrow a = 1$, false statement

$\therefore$ Choose $a_n^{p} = n\left[A_3 n^3 + A_2 n^2 + A_1 n + A_0\right]$

Put $a_n = n\left[A_3 n^3 + A_2 n^2 + A_1 n + A_0\right]$

in (1)

$$\Rightarrow (n+1)\left[A_3(n+1)^3 + A_2(n+1)^2 + A_1(n+1) + A_0\right]$$

$$- n\left[A_3 n^3 + A_2 n^2 + A_1 n + A_0\right]$$

$$= (n+1)^3$$

Comparing the like coefficients on both sides

ie

$$A_3(n+1)^4 + A_2(n+1)^3 + A_1(n+1)^2 + A_0(n+1)$$

$$- A_3 n^4 - A_2 n^3 - A_1 n^2 - A_0 n = (n+1)^3 \quad \text{--- (2)}$$

coefficient

$(n^4) \rightarrow \quad A_3 - A_3 = 0$

(2) $\Rightarrow A_3[n^4+4n^3+6n^2+4n+1]+A_2[n^3+3n^2+3n+1]+A_1[n^2+2n+1]+A_0(n+1)-A_3n^4-A_2n^3-A_1n^2-A_0n$

$= n^3+3n^2+3n+1$

Coeffnt. of n^3

$\Rightarrow 4A_3+A_2-A_2=1$

$\Rightarrow 4A_3=1 \Rightarrow \underline{\underline{A_3=\frac{1}{4}}}$

Coeffnt. of n^2

$\Rightarrow A_3(6)+A_2(3)+A_1-A_1=3$

$\Rightarrow 6A_3+3A_2=3$

$\Rightarrow \frac{6}{4}+3A_2=3$

$\Rightarrow A_2 = \frac{\left(3-\frac{3}{2}\right)}{3} = \frac{6-3}{6} = \underline{\underline{+\frac{1}{2}}}$

Coeffnt. of n

$\Rightarrow 4A_3 + 3A_2 + 2A_1 + A_0 - A_0 = 3$

$\Rightarrow 4A_3 + 3A_2 + 2A_1 = 3$

$\Rightarrow 4\left(\frac{1}{4}\right) + 3\left(\frac{1}{2}\right) + 2A_1 = 3$

$\Rightarrow 1 + \frac{3}{2} + 2A_1 = 3$

$\Rightarrow \frac{5}{2} + 2A_1 = 3$

$\Rightarrow 2A_1 = 3 - \frac{5}{2}$

$\Rightarrow 2A_1 = \frac{1}{2}$

$\Rightarrow A_1 = \frac{1}{4}$

comparing the constant terms

$\Rightarrow A_3 + A_2 + A_1 + A_0 = 1$

$\Rightarrow \frac{1}{4} + \frac{1}{2} + \frac{1}{4} + A_0 = 1$

$$= \frac{1+2+1}{4} + A_0 = 1$$

$$\Rightarrow \underline{\underline{A_0 = 0}}$$

Thus $a_n^p = n\left[\frac{1}{4}n^3 + \frac{1}{2}n^2 + \frac{1}{4}n + 0\right]$

$$= \underline{\underline{\frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2}}$$

Thus $a_n = a_n^h + a_n^p$

$$= A + \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2 \quad \text{--- (3)}$$

$a_0 = 0$

$n \geq 0$

Put $n = 0$ in

$0 = A$

Thus the required solution is given by

$$a_n = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2, \ n \geq 0$$

$$= \frac{n^2}{4}\left[n^2 + 2n + 1\right]$$

$$= \frac{n^2}{4}(n+1)^2$$

$$= \left[\frac{n}{2}(n+1)\right]^2$$

Thus $\sum_{n=0}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

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HW

find the solution of $a_n = 3a_{n-1} + 2n$

where $a_1 = 3$.

Here $a_n - 3a_{n-1} = 2n$ is ——— (1)

the given sequence relation

$$a_n = a_n^h + a_n^p$$

* To find a_n^h

Consider $a_n - 3a_{n-1} = 0$

$\Rightarrow a_{n+1} = 3a_n$

$\Rightarrow a_n = A(d^n)$

$d = 3$

$\Rightarrow a_n = A(3)^n$,

* To find a_n^p

Here $f(n) = 2n$

$f(n) = n \Rightarrow a_n^p = A_1 n + A_0$

here Choose $a_n = A_1 n + A_0$.

Substitute in (1)

$\Rightarrow (A_1 n + A_0) - 3(A_1(n-1) + A_0) = 2n$

$\Rightarrow A_1 n + A_0 - 3A_1 n + 3A_1 - 3A_0 = 2n$

$\Rightarrow -2A_1 n - 2A_0 + 3A_1 = 2n$

$\Rightarrow (-2A_1) n + (-2A_0 + 3A_1) = 2n + 0$ — (2)

Comparing terms of n in (2)

$\Rightarrow -2A_1 = 2$

$\Rightarrow A_1 = -1$

Comparing the constant terms in (2)

$\Rightarrow -2A_0 + 3A_1 = 0$

$\Rightarrow -2A_0 + 3(-1) = 0$

$-2A_0 = 3$

$A_0 = \frac{-3}{2}$

$\therefore a_n^p = (-1)n - \frac{3}{2}$

$= -n - \frac{3}{2}$

$\frac{4 \times 4}{1 \times 2 \times 3 \times 4}$

6

Thus the general solution is

$a_n = A(3)^n - n - \frac{3}{2}, n \geq 0$ —— 3

Now $a_1 = 3$

Put $n=1$ in 3

$3 = A(3) - 1 - \frac{3}{2}$

$\Rightarrow A = \frac{3+1+\frac{3}{2}}{3} = \frac{4+\frac{3}{2}}{3}$

$= \frac{8+3}{6} = \frac{11}{6}$

$\therefore$ required solution is given by

$$a_n = \left(\frac{11}{6}\right) 3^n - n - \frac{3}{2}, \quad n \geq 0$$

8

Solution of recurrence relation using Generating function.

If $a_0, a_1, \ldots$ is a sequence of numbers then the corresponding generating function is given by $a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{k=0}^{\infty} a_k x^k$

ie, $\boxed{G(x) = a_0 + a_1 x + a_2 x^2 + \cdots}$

RESULT :

Generating function	a_k
$G(x) = \frac{1}{1-ax}$	a^k
$G(x) = \frac{1}{(1-x)^2}$	$(k+1)$
$G(x) = \frac{1}{(1-ax)^2}$	$(k+1)\, a^k$
$G(x) = \frac{x}{(1-x)^2}$	k
$G(x) = \frac{ax}{(1-ax)^2}$	$k a^k$
$G(x) = e^x$	$\frac{1}{k!}$
$G(x) = \ln(1+x)$	$\frac{(-1)^{k+1}}{k}$
$G(x) = \frac{1}{1-x}$	1

$$G(x) = \frac{1}{1+x}$$

$(-1)^k$

Method

Step 1.

Multiply both side of the given relation by x^r, if r is the largest subscript in the given relation

Step 2

Sum up all the values of r (given in the question)

Step 3 Take each term from the above relation and express it in the terms of $G(x)$ where

$$G(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

Step 4. Substitute those values in the expression obtained in step 2 Obtain $G(x)$, and then find a_k using above result

1. Solve the sequence relation $a_n - 3a_{n-1} = n$ $n \geq 1$ and $a_0 = 1$

Here $a_n - 3a_{n-1} = n$

Step 1

Multiply throughout by x^n [since n is the largest subscript]

$$\Longrightarrow a_n x^n - 3a_{n-1} x^n = n x^n$$

step 2 Here $n \geq 1$, so take summation on both sides for $n \geq 1$

$$\Longrightarrow \sum_{n=1}^{\infty} a_n x^n - 3 \sum_{n=1}^{\infty} a_{n-1} x^n = \sum_{n=1}^{\infty} n x^n \quad \text{—— (1)}$$

Step 3

here, $\sum_{n=1}^{\infty} a_n x^n = a_1 x + a_2 x^2 + a_3 x^3 + \dots$

$= [G(x) - a_0]$

$\sum_{n=1}^{\infty} a_{n-1} x^n = a_0 x + a_1 x^2 + a_2 x^3 + \dots$

$= [G(x)] x$

$$\boxed{\begin{aligned} \sum_{n=1}^{\infty} n x^n &= x + 2x^2 + 3x^3 + \dots \\ &= x[1 + 2x + 3x^2 + \dots] \\ &= x\left[\frac{1}{(1-x)^2}\right] \end{aligned}}$$

Step 4

$① \Rightarrow [G(x) - a_0] - 3G(x)x = \sum_{n=1} n x^n$

$a_0 = 1$

$\Rightarrow G(x) - 1 - 3x\,G(x) = \frac{x}{(1-x)^2}$

$\Rightarrow G(x)[1 - 3x] - 1 = \frac{x}{(1-x)^2}$

$$G(x) = \underbrace{\frac{x}{(1-x)^2(1-3x)}}_{\text{1st part}} + \frac{1}{(1-3x)}$$

Now, use partial fraction to decompose 1st part $G(x)$

Thus
$$\frac{x}{(1-x)^2(1-3x)} = \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{1-3x}.$$

$$\Rightarrow x = A(1-x)(1-3x) + B(1-3x) + C(1-x)^2$$

$\boxed{\text{Put } x=1}$

$$\Rightarrow 1 = 0 + B(-2) + 0$$

$$\Rightarrow B = \frac{-1}{2}$$

$\boxed{\text{Put } x = \frac{1}{3}}$

$$\Rightarrow \frac{1}{3} = 0 + 0 + C\left[1 - \frac{1}{3}\right]^2$$

$$= C\left[\frac{2}{3}\right]^2$$

$$\Rightarrow C = \frac{1}{3}\left(\frac{3}{2}\right)^2 = \frac{3}{4}$$

$\boxed{\text{Put } x=0}$

$$0 = A(1) + B(1) + C \Rightarrow A = -B - C$$

$$= \frac{1}{2} - \frac{3}{4} = \frac{-1}{4}$$

$$G(x) = \frac{\left(\frac{-1}{4}\right)}{(1-x)} + \frac{\left(\frac{-1}{2}\right)}{(1-x)^2} + \frac{\left(\frac{3}{4}\right)}{(1-3x)} + \frac{1}{(1-3x)}$$

$$G(x) = \frac{\left(\frac{-1}{4}\right)}{(1-x)} + \frac{\left(\frac{-1}{2}\right)}{(1-x)^2} + \frac{\left(\frac{7}{4}\right)}{(1-3x)}$$

Now the k^{th} term will be

$$\Rightarrow a_k = \left(\frac{-1}{4}\right)[1] - \left(\frac{1}{2}\right)[k+1] + \left(\frac{7}{4}\right)\left[3^k\right]$$

Thus the unique solution

$$a_n = \left(\frac{-1}{4}\right) - \frac{1}{2}(n+1) + \left(\frac{7}{4}\right)3^n$$

$$= -\frac{1}{4} - \frac{1}{2}n - \frac{1}{2} + \left(\frac{7}{4}\right)3^n$$

$$= \left(-\frac{1}{4} - \frac{1}{2}\right) - \frac{1}{2}n + \left(\frac{7}{4}\right)3^n$$

$$= \left(\frac{-3}{4}\right) - \frac{1}{2}n + \left(\frac{7}{4}\right)3^n, \; n \geq 0$$

Q. Solve the recurrence relation using generating function

$$a_{n+2} - 5a_{n+1} + 6a_n = 2, \quad n \geq 0, \quad a_0 = 3, \; a_1 = 7$$

Soln Consider $a_{n+2} - 5a_{n+1} + 6a_n = 2$

Multiply throughout by x^{n+2}

$$x^{n+2} a_{n+2} - 5a_{n+1}x^{n+2} + 6a_n x^{n+2} = 2x^{n+2}$$

Now here $n \geq 0$, summing up all the terms,

$$\sum_{n=0}^{\infty} a_{n+2}x^{n+2} - 5\sum_{n=0}^{\infty} a_{n+1}x^{n+2} + 6\sum_{n=0}^{\infty} a_n x^{n+2}$$

$$= 2\sum_{n=0}^{\infty} x^{n+2} \quad \text{--- (1)}$$

where $\sum_{n=0}^{\infty} a_{n+2}x^{n+2} = a_2x^2 + a_3x^3 + \cdots$

$$= G(x) - a_0 - a_1 x$$

$$\sum_{n=0}^{\infty} a_{n+1}x^{n+2} = a_1x^2 + a_2x^3 + a_3x^4 + \cdots$$

$$= [G(x) - a_0]x$$

$$\sum_{n=0}^{\infty} a_n x^{n+2} = a_0 x^2 + a_1 x^3 + a_2 x^4 + \cdots$$

$$= (G(x))x^2$$

$$\sum_{n=0}^{\infty} x^{n+2} = x^2 + x^3 + x^4 + \cdots$$

$$= x^2 [1 + x + x^2 + \cdots]$$

$$= x^2 \left(\frac{1}{1-x}\right)$$

$$= \frac{x^2}{1-x}$$

Substitute all these values in (1)

$$\Rightarrow [G(x) - a_0 - a_1 x] - 5[G(x) - a_0]x + 6[G(x)]x^2 = \frac{2x^2}{1-x}$$

$a_0 = 3 \quad a_1 = 7$

$$\Rightarrow G(x) - 3 - 7x - 5x\,G(x) + 15x + 6\,G(x)x^2 = \frac{2x^2}{1-x}$$

$$G(x)\left[1-5x+6x^2\right]-3-7x+15x=\frac{2x}{1-x}$$

$$G(x)\left[1-5x+6x^2\right)=3-8x+\frac{2x^2}{1-x}$$

$$=\frac{(3-8x)(1-x)+2x^2}{1-x}$$

$$=\frac{3-3x-8x+8x^2+2x^2}{1-x}$$

$$=\frac{3-11x+10x^2}{1-x}$$

$$\therefore G(x)=\frac{3-11x+10x^2}{(1-x)(1-5x+6x^2)}$$

$$=\frac{3-11x+10x^2}{(1-x)(2x-1)(3x-1)}$$

$$=\frac{3-11x+10x^2}{(1-x)(1-2x)(1-3x)}$$

$6x^2-5x+1$

Let α,β be the roots $\alpha\beta=1\times6$

$\alpha+\beta=-5$

$\therefore \alpha=-3$

$\beta=-2$

decompose

$\Rightarrow 6x^2-2x-3x+1$

$=2x[3x-1]-[3x-1]$

$(2x-1)(3x-1)$

Now use partial fraction to split $G(x)$

Also

$$\frac{3-11x+10x^2}{} = 10x^2 - 11x + 3$$

$$= 10x^2 - 5x - 6x + 3$$

$$= 5x[2x-1] - 3[2x-1]$$

$$= (2x-1)(5x-3)$$

$\alpha\beta = 30$

$\alpha + \beta = -11$

$\Rightarrow \alpha = -6 \quad \beta = -5$

Thus

$$G(x) = \frac{3-11x+10x^2}{(1-5x+6x^2)(1-x)} = \frac{\cancel{(2x-1)}^{-1}(5x-3)}{(1-x)\cancel{(1-2x)}(1-3x)}$$

$$= \frac{(3-5x)}{(1-x)(1-3x)}$$

Now

$$G(x) = \frac{3-5x}{(1-x)(1-3x)} = \frac{A}{1-x} + \frac{B}{1-3x}$$

$\Rightarrow 3-5x = A(1-3x) + B(1-x)$

Put $x = 1$

$\Rightarrow 3-5 = A(1-3) + 0$

$\Rightarrow -2 = A(-2) \Rightarrow \underline{\underline{A = 1}}$

Put $x = \frac{1}{3}$

$\Rightarrow 3 - \frac{5}{3} = A(0) + B\left(1 - \frac{1}{3}\right)$

$\Rightarrow \frac{4}{3} = \frac{2}{3}B \Rightarrow \underline{\underline{B = 2}}$

Thus $G(x) = \frac{1}{1-x} + \frac{2}{1-3x}$.

$\therefore \ a_k = (1) + 2\left[3^k\right],$

Thus the required unique solution

$a_n = 1 + 2(3^n), \ n \geq 0$

3. Solve the recurrence relation by generating function

a) $a_r - 2a_{r-1} - 3a_{r-2} = 0$, $r \geq 2$, $a_0 = 3$

$a_1 = 1$

Step 1

Multiply both sides by x^r [since r is the largest subscript here]

$$a_r x^r - 2a_{r-1} x^r - 3a_{r-2} x^r = 0$$

Step 2

Since $r \geq 2$, summing for all r, we get

$$\sum_{r=2}^{\infty} a_r x^r - \sum_{r=2}^{\infty} 2a_{r-1} x^r - \sum_{r=2}^{\infty} 3a_{r-2} x^r = 0 \quad \text{--- I}$$

Now

$$\sum_{r=2}^{\infty} a_r x^r = a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

$$= G(x) - a_0 - a_1 x \quad \text{--- (1)}$$

$$\sum_{r=2}^{\infty} a_{r-1} x^r = a_1 x^2 + a_2 x^3 + a_3 x^4 + \cdots$$

$$= [G(x) - a_0] x \quad \text{--- (2)}$$

$$\sum_{r=2}^{\infty} a_{r-2} x^r = a_0 x^2 + a_1 x^3 + a_2 x^4 + \cdots$$

$$= [G(x)]x^2 \quad \text{③}$$

Step 3 (I)

Substitute ①, ② and ③ in (I)

Thus

$$[G(x) - a_0 - a_1 x] - 2[G(x) - a_0]x$$

$$- 3[G(x)x^2] = 0$$

$\Rightarrow$ Now $a_0 = 3$ and $a_1 = 1$

$$\Rightarrow G(x) - 3 - x - 2G(x)x + 2(3)x$$

$$- 3G(x)x^2 = 0$$

Thus $G(x) - 3 + 5x - 2G(x)x - 3G(x)x^2$

$$= 0$$

$$\Rightarrow G(x)\left[1-2x-3x^2\right]=3-5x.$$

$$\Rightarrow G(x)=\frac{3-5x}{\left(1-2x-3x^2\right)}$$

Consider the denominator

$-3x^2 - 2x + 1$

Let α, β be the roots

$\alpha \times \beta = (-3)(1) = -3$

$\alpha + \beta = -2$

$\Longrightarrow \alpha = -3 \quad \beta = 1$

Now $-3x^2 - 2x + 1$

$= -3x^2 - 3x + x + 1$

$= -3x[x+1] + [x+1]$

$= [x+1][-3x+1]$

$\therefore G(x) = \dfrac{3-5x}{(1+x)(1-3x)}$

$= \dfrac{A}{1+x} + \dfrac{B}{1-3x}$

Thus $3-5x = A(1-3x)+B(1+x)$

Put $x=-1 \Rightarrow 3+5 = A[1+3]+0$

$\Rightarrow 8 = 4A$

$\Rightarrow \underline{\underline{A = 2}}$

Put $x=\frac{1}{3} \Rightarrow 3-\frac{5}{3} = 0+B\left[\frac{1}{3}+1\right]$

$\Rightarrow \frac{4}{3} = B\left[\frac{4}{3}\right]$

$\Rightarrow \underline{\underline{B = 1}}$

$\therefore G(x) = \frac{2}{1+x} + \frac{1}{1-3x}$

$\therefore$ Corresponding k^{th} term

$a_k = 2\left[(-1)^k\right] + \left[3^k\right]$

$\therefore$ Required unique solution is

$a_n = 2(-1)^n + 3^n \quad n \geq 0$