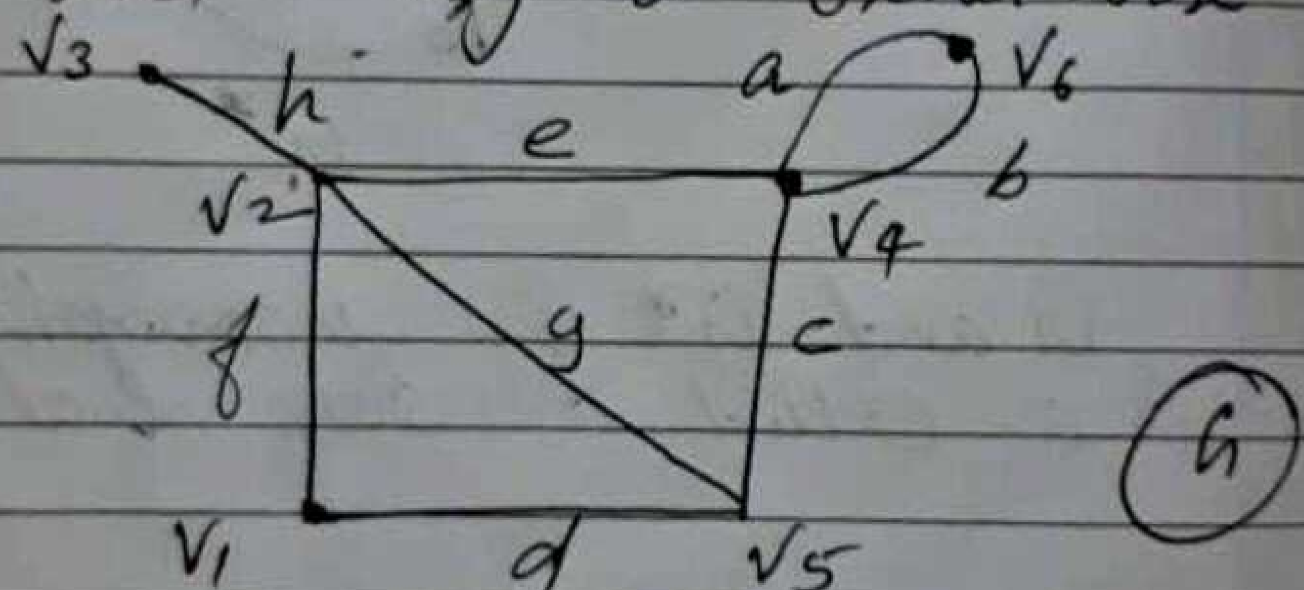
Module 5:

Graph Representations and Vertex Colouring

Incidence matrix.

Let G be a graph with n vertices, e edges and no self-loops. Define an n by e matrix $A = [a_{ij}]$ where n-rows correspond to the 'n' vertices and the e columns correspond to the ~~e-edges~~ e-edges as follows: The matrix element $a_{ij} = 1$ if jth edge e_j is incident on ith vertex v_i and $a_{ij} = 0$ otherwise

Eg:



$$
A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array}
\begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\
\begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \end{array}
$$

Such a matrix A is called the vertex - edge incidence matrix, or simply incidence matrix. Matrix A of a graph G is sometimes denoted by $A(G)$

The incidence matrix contains only two elements 0 and 1. Such a matrix is called a <u>binary matrix</u> or <u>(0,1) matrix</u>.

<u>Rank of the incidence matrix.</u>

<u>Theorem.</u> If $A(G)$ is an incidence matrix of a connected graph G with n vertices, the rank of $A(G)$ is $n-1$.

<u>Proof.</u> Let $A(G)$ be the incidence matrix of a connected graph G. Let A_1 be the vector in the first row, A_2 be the vector in the second row, and so on. Then

$$A(G) = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix}.$$

Since every edge is incident on exactly two vertices, each columns of $A(G)$ has exactly two 1's. Also $A_1, A_2 \ldots A_n$

are vectors over $GF(2)$ [Galois field modulo 2]. is the vector space of graph G. Since there are exactly two ~~1~~ 1's in every column of A, the sum of all these vectors is 0 (this being a modulo 2 sum of the corresponding entries). Thus vectors $A_1, A_2, \ldots A_n$ are linearly dependent. Therefore ~~rank A ≤ n-1~~. rank $A < n$.

i.e. rank $A(G) \leq n-1$. — (1)

Consider the sum of any m of these row vectors, ~~m ≤ n-1~~ $m \leq n-1$. Since G is connected $A(G)$ cannot be partitioned in the form

$$A(G) = \begin{bmatrix} A(G_1) & 0 \\ 0 & A(G_2) \end{bmatrix}$$

Such that $A(G_1)$ has m rows and $A(G_2)$ has $n-m$ rows. Thus there exists no $m \times m$ Submatrix of $A(G)$ for $m \leq n-1$ such that the modulo 2 sum of these m rows is equal ~~to~~ zero.

As there are only two elements 0 and 1 in this field, the additions of all vectors taken m at a time for $m = 1, 2 \ldots n-1$ gives all possible linear combinations of $n-1$ row vectors. Thus no linear combinations of m row vectors of A for $m \le n-1$ is zero.

Therefore ~~rank $A(G) \le n-1$~~

$$n-1 \le \text{rank } A(G) \quad \text{--- (2)}$$

from (1) & (2) rank $A = n-1$.

Circuit Matrix.

Let the number of different circuits in a graph G be q and the number of edges in G be e. Then a circuit matrix $B = [b_{ij}]$ of G is a q by e, $(0,1)$ matrix defined as follows:

$$b_{ij} = \begin{cases} 1 & \text{if } i\text{th circuit includes } j\text{th edge and} \\ 0 & \text{otherwise.} \end{cases}$$

The circuit matrix of G is also be written as $B(G)$.

Eg: Consider the graph G in the above example. G has four different circuits, $1 = \{a, b\}$, $2 = \{c, e, g\}$, $3 = \{d, f, g\}$ and $4 = \{c, d, f, e\}$

Hence the circuit matrix of G is a 4 by 8 $(0,1)$ matrix as shown below.

$$B(G) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Theorem. Let B and A be respectively, the circuit matrix and the incidence matrix (of a self-loop free graph) whose columns are arranged using the same order of edges. Then every row of B is orthogonal to every row of A. i.e.

$$AB^T = B^T A = 0 \pmod 2$$

where T denotes the transposed matrix.

Proof. Consider a vertex v and a circuit Γ in the graph G. Either v is in Γ or it is not. If v is not in Γ, there is no edge in the circuit Γ that is incident on v. On the other hand, if v is in Γ, the number of edges in the circuit Γ that are incident on v is exactly two.

With this remark in ~~mem~~ mind, consider the i^{th} row in A and the

j^{th} row in B. Since the edges are arranged in the same order, the non-zero entries in the corresponding positions occur only if the particular edge is incident on the i^{th} vertex and is also in the j^{th} circuit.

If the i^{th} vertex is not in the j^{th} circuit, there is no such non-zero entry, and the dot product of the two rows is zero. If the i^{th} vertex is in the j^{th} circuit, there will be exactly two 1's in the sum of the products of individual entries. Since $1+1=0$ (mod 2), the dot product of the two arbitrary rows - one from A and the other from B - is zero. Hence the theorem.

<u>Eg:</u> Let us multiply the incidence matrix and transposed circuit of the graph of the above example.

$$AB^T = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \pmod 2$$

Path Matrix.

A path matrix is defined for a specific pair of vertices in a graph, say (x, y) and is written as $P(x, y)$. The rows in $P(x, y)$ correspond to different paths between vertices x and y, and the columns correspond to the edges in G. That is the path matrix for (x, y) vertices is $P(x, y) = [p_{ij}]$ where

$$p_{ij} = \begin{cases} 1 & \text{if } j^{th} \text{ edge lies } i^{th} \text{ path} \\ 0 & \text{otherwise.} \end{cases}$$

Eg: Consider the ~~the~~ graph G in the above example. Consider all paths between v_3 and v_4. There are three different paths: $1 = \{h, e\}$ $2 = \{h, g, c\}$ and $3 = \{h, f, d, c\}$. Then we get the 3 by 8 path matrix $P(v_3, v_4)$

$$P(v_3, v_4) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

<u>Theorem.</u> If the edges of a connected graph are arranged in the same order for the columns of the incidence matrix A and the path matrix $P(x, y)$, then the product (mod 2)

$A \cdot P^T(x, y) = M$, where the matrix M has 1's in two rows x and y and the rest of the $n-2$ rows are all 0's.

<u>Eg:</u>

$$A \cdot P^T(v_3, v_4) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

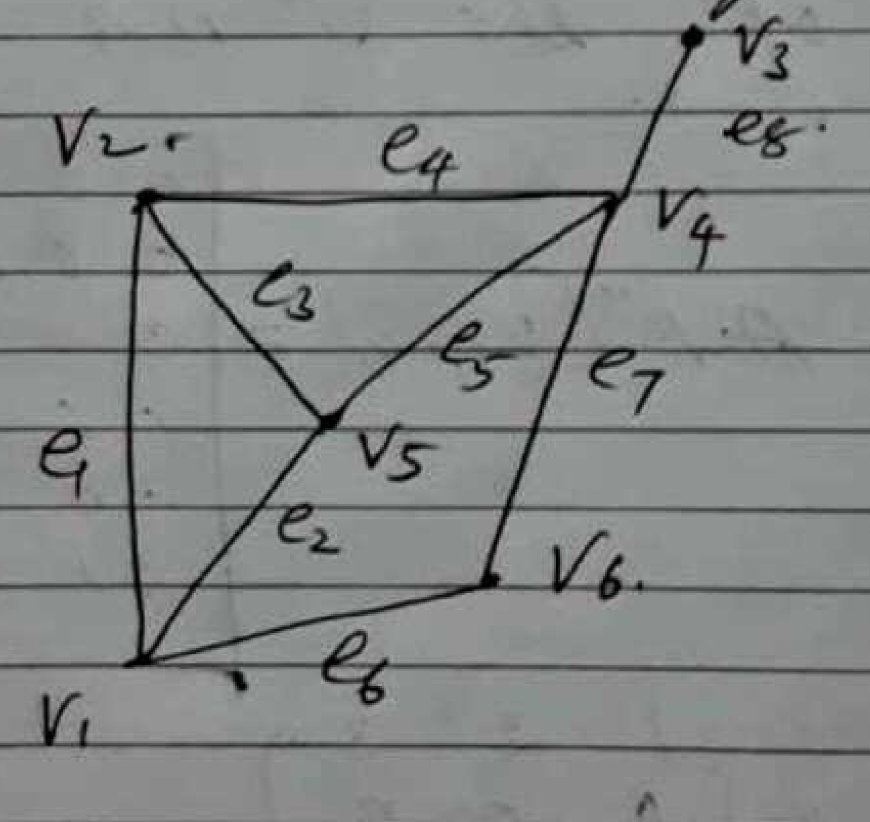
$$= \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \pmod 2$$

Adjacency Matrix

The adjacency matrix of a graph G with n vertices and no parallel edges is an n by n symmetric binary matrix $X = [x_{ij}]$ defined over the ring of integers such that

$$x_{ij} = \begin{cases} 1 & \text{if there is an edge between } i\text{th and } j\text{th vertices} \\ 0 & \text{if there is no edge between them.} \end{cases}$$

Eg:



$$X = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

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Relationship between $A(G)$ and $X(G)$

Let $A(G)$ be the incidence matrix and $X(G)$ be the adjacency matrix of the graph G and let d be the degree of the given vertex then

~~$AA^T = X +$~~

$$A A^T = X + \begin{bmatrix} d(v_1) & 0 & 0 & \cdots & 0 \\ 0 & d(v_2) & 0 & \cdots & 0 \\ 0 & 0 & d(v_3) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & d(v_n) \end{bmatrix}$$

~~Atob~~.

Q) Prove that the rank of a circuit matrix of a connected graph G with e edges and n vertices is $e-n+1$.

Let A be an incidence matrix of G. Then by a result, $A.B^T = 0$ (mod 2). Then by a ~~result~~ Theorem,

rank A + rank $B \le e$.

ie rank of $B \le e -$ rank of A.

Since rank of $A = n-1$,

rank of $B \le e-n+1$

But rank of $B \ge e-n+1$.

Therefore, we must have

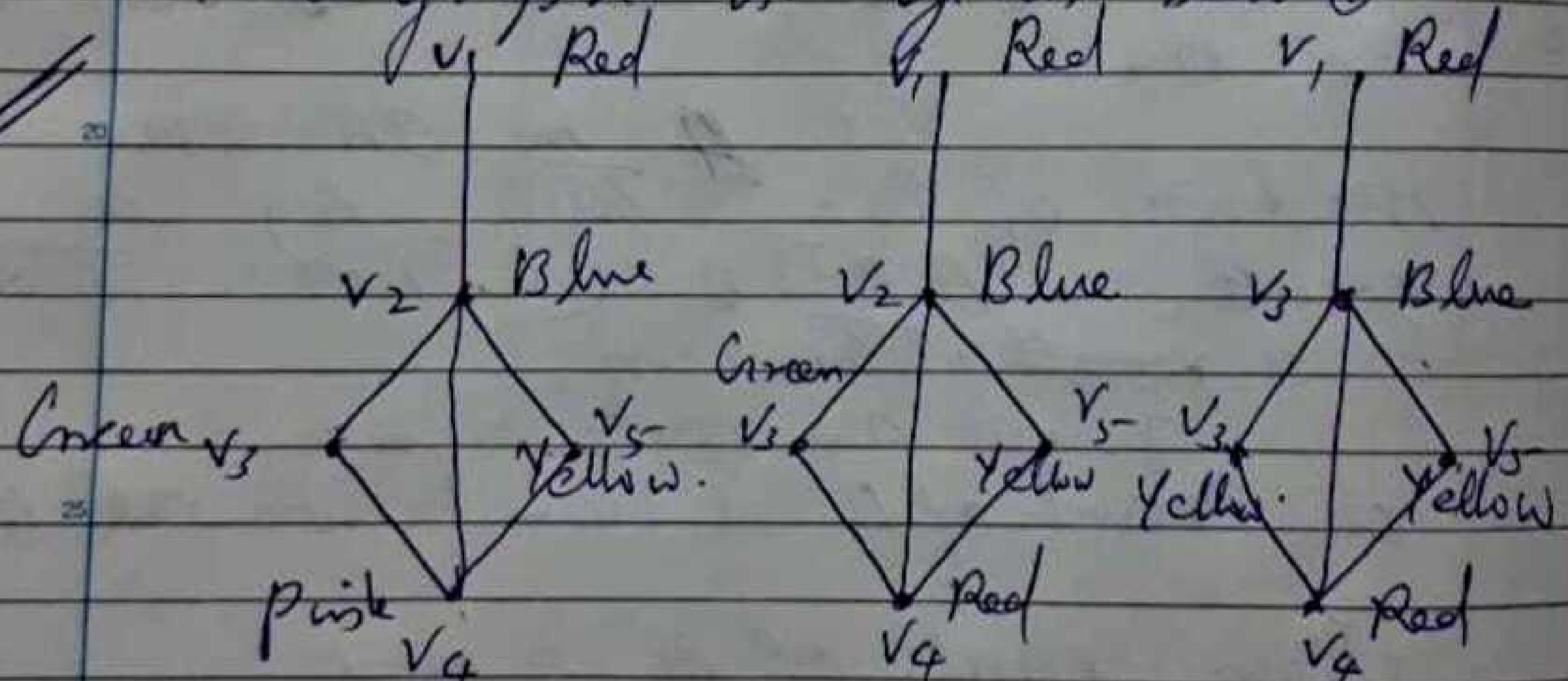
rank of $B = e-n+1$

Chromatic Number.

Painting all the vertices of a graph with colours such that no two adjacent vertices have the same colour is called the proper colouring (simply colouring) of a graph.

A graph in which every vertex has been assigned a colour according to a proper colouring is called a properly coloured graph.

Usually a given graph can be properly coloured in many different ways. three different proper colouring of a graph is given below.



A graph G that requires k different colours for its proper colouring, and no less, is called a k-chromatic graph, and the number k is called the chromatic number of G.

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The above graph is a ~~2~~-chromatic.

<u>Note</u>: For colouring problems we need to consider only simple, connected graphs.

<u>Observations from the definition of k-chromatic graph.</u>

1. A graph consisting of only isolated vertices is 1-chromatic

2) A graph with one or more edges (not a self-loop) is atleast 2-chromatic (or bichromatic)

3. A complete graph of n vertices is n-chromatic, as all its vertices are adjacent. Hence a ~~complete~~ graph containing a complete graph of r vertices is atleast r-chromatic.

Eg: Every graph having a triangle is atleast 3-chromatic

4. A graph consisting of simply one circuit with $n \geq 3$ vertices is 2-chromatic if n is even and 3-chromatic if n is odd.

<u>Theorem.</u> ~~Every~~ Every tree with two or more vertices is 2-chromatic.

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Proof Select any vertex v in the given tree T. Consider T as a rooted tree at vertex v. Paint v with colour 1. Paint all vertices adjacent to v with colour 2. Next paint the vertices adjacent to these (those that just have been coloured with 2) using colour 1. Continue this process till every vertex in T has been painted. Now in T we find that all vertices at odd distances from v have colour 2, while v and vertices at even distances from v have colour 1.

Now along any path in T the vertices are of alternating colours. Since there is one and only one path between any two vertices in a tree, no two adjacent vertices have the same colour. Thus T has been properly coloured with two colours. One colour would not have been enough.

Theorem. A graph with atleast one edge is 2-chromatic if and only if it has no circuits of odd lengths.

Proof Let G be a connected graph with circuits of only even lengths. Consider a spanning tree T in G. Using the colouring procedure and the result of above theorem, let us properly colour T with two colours. Now add the chords to T one by one. Since G had no circuits of odd length, the end vertices of every ~~chord~~ chord being replaced are differently coloured in T. Thus G is coloured with two colours, with no adjacent vertices having the same colour. That is G is 2-chromatic.

Conversely, if G has a circuit ~~off~~ of odd length, we would need atleast three colours just for that ~~for~~ circuit. Thus the theorem.

Result 1. If d_{max} is the maximum degree of the vertices in a graph G,

Chromatic number of $G \leq 1 + d_{max}$

2. Every tree is a bipartite graph.

3. Every 2-chromatic graph is bipartite.

~~4.~~ Def. A graph G is called p-partite if its vertex set can be decomposed into p-disjoint subsets $V_1, V_2, \ldots V_p$ such that no edge in G joins the vertices in the same subset.

A k-chromatic graph is p-partite if and only if $k \le p$.

Chromatic Polynomial.

A graph G of n vertices can be properly coloured in many different ways using a sufficiently large number of colours. This property of a graph is expressed elegantly by means of a polynomial. This polynomial is called chromatic polynomial of G and is defined as follows.

The value of the chromatic polynomial $P_n(\lambda)$ of a graph with n-vertices gives the number of ways of properly colouring the graph, using λ or ~~fewer~~ fewer colours.

Let c_i be the different ways of properly colouring G using exactly i different colours. Then the chromatic polynomial is

$$P_n(\lambda) = c_1 \frac{\lambda}{1!} + c_2 \frac{\lambda(\lambda-1)}{2!} + c_3 \frac{\lambda(\lambda-1)(\lambda-2)}{3!} + \cdots + c_n \frac{\lambda(\lambda-1)\cdots(\lambda-n+1)}{n!}$$

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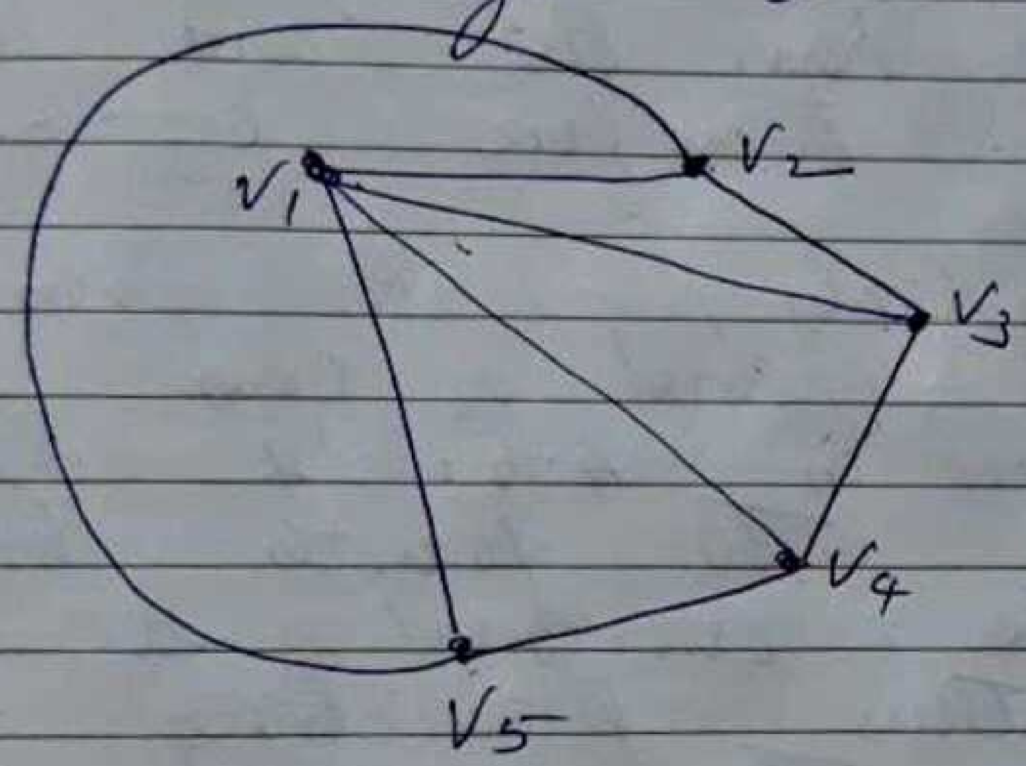
Each c_i has to be evaluated individually for the given graph. For example, any graph with one edge requires atleast two colours for proper colouring and hence $c_1 = 0$.

A graph with n vertices and using n different colours can be properly coloured in $n!$ ways. ie $c_n = n!$

Eg.



$$P_5(\lambda) = c_1 \lambda + c_2 \frac{\lambda(\lambda-1)}{2!} + c_3 \frac{\lambda(\lambda-1)(\lambda-2)}{3!}$$

$$+ c_4 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!} +$$

$$c_5 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{5!}$$

Since the given graph has a triangle, it will require atleast three different colours for proper colouring. Hence $c_1 = c_2 = 0$. & $c_5 = 5!$

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To evaluate c_3, suppose that we have three colours x, y and z. These three colours can be assigned properly to vertices v_1, v_2, v_3 in $3! = 6$ ~~ways~~ different ways. Having done that, we have no ~~more~~ more choices left, because vertex v_5 must have the same colour as v_3 and v_4 must have the same colour as v_2. Therefore $c_3 = 6$.

Similarly, with four colours v_1, v_5 and v_3 can be properly coloured in $4 \cdot 6 = 24$ different ways. The fourth colour can be assigned to v_4 or v_5; thus providing two choices. The fifth vertex provides no additional choice. Therefore $c_4 = 24 \cdot 2 = 48$.

Substituting these coefficients in $P_5(\lambda)$

$$P_5(\lambda) = \lambda(\lambda-1)(\lambda-2) + 2\lambda(\lambda-1)(\lambda-2)(\lambda-3) + \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)$$

$$P_5(\lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda^2 - 5\lambda + 7)$$

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Theorem. A graph of n vertices is a complete graph if and only if its chromatic polynomial is

$$P_n(\lambda) = \lambda(\lambda-1)(\lambda-2)\cdots(\lambda-n+1)$$

Proof. With λ colours, there are λ different ways of colouring any selected vertex of a graph. A second vertex can be coloured properly in exactly $\lambda-1$ ways, the third in $\lambda-2$ ways, the fourth in $\lambda-3$ ways ... and the nth is $\lambda-(n-1) = \lambda-n+1$ ways if and only if every vertex is adjacent to every other. That is, if and only if, the graph is complete.

Result An n-vertex graph is a tree if and only if its chromatic polynomial

$$P_n(\lambda) = \lambda(\lambda-1)^{n-1}.$$

Theorem. Let a and b be two nonadjacent vertices in a graph G. Let G' be a graph obtained by adding an edge between a and b. Let G'' be a simple graph obtained from G by fusing the vertices a and b together and replacing sets of parallel edges

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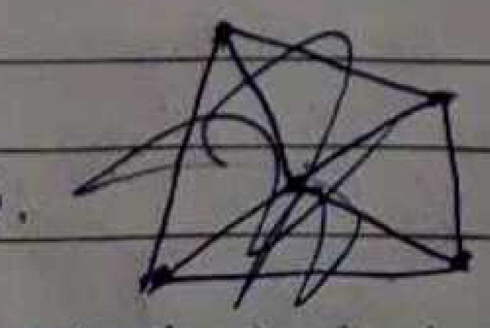
with single edges. Then
$P_n(\lambda)$ of G = $P_n(\lambda)$ of G' + $P_{n-1}(\lambda)$ of G''.

<u>Proof</u>. The number of ways of properly colouring G can be grouped into two cases, one such that ~~a~~ vertices a and b are of the same colour and the other such that a and b are of different colours. Since the number of ways of properly colouring G such that a and b have different colours = number of ways of properly colouring G' ~~and~~ and number of ways of properly colouring G such that a and b have the same colour = number of ways of properly colouring G'',

$$P_n(\lambda) \text{ of } G = P_n(\lambda) \text{ of } G' + P_{n-1}(\lambda) \text{ of } G''.$$

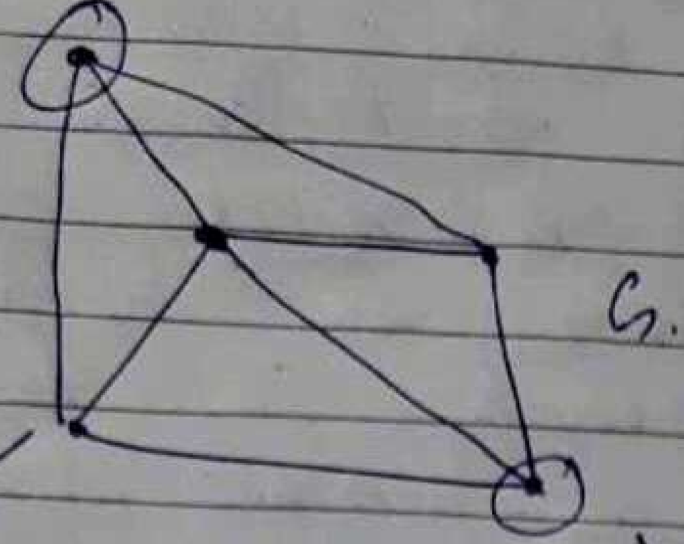
<u>~~Result~~. Note</u>. The above theorem is often used in evaluating the chromatic polynomial of a graph.

<u>Eg</u>:

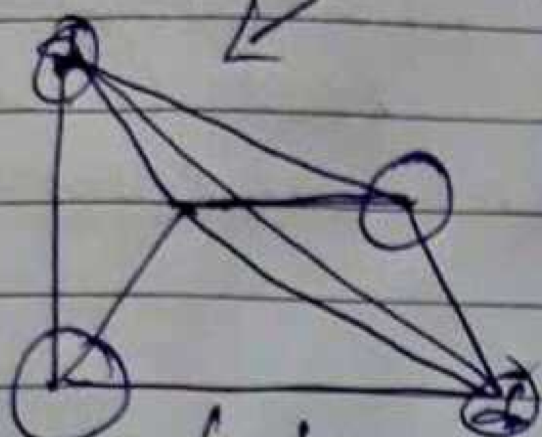
G. 

Eg.

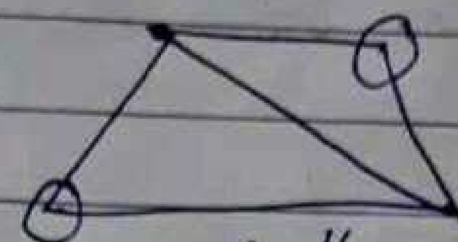
G_1'



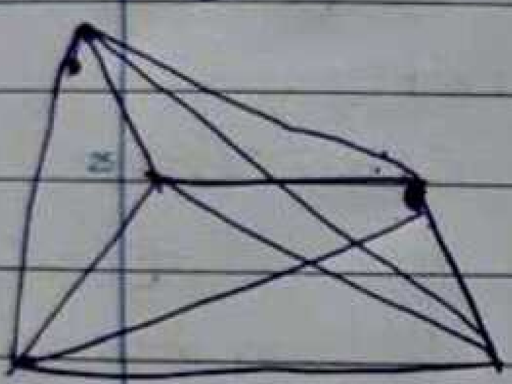
$G.$



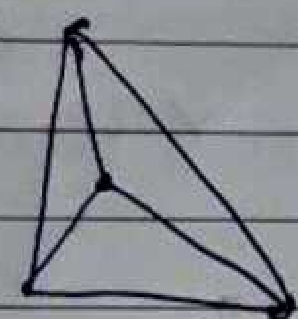
G_1'

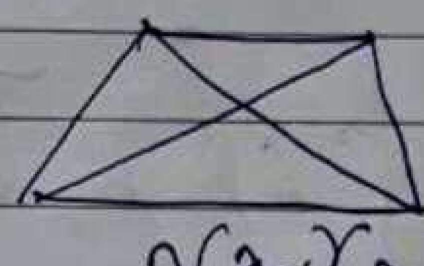


G_1''

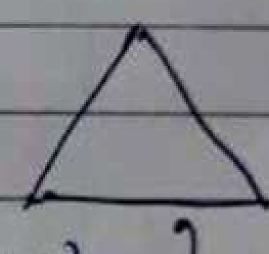


G_2''





$\lambda(\lambda-1)(\lambda-2)(\lambda-3)$



$\downarrow$

$\lambda(\lambda-1)(\lambda-2)$

$\lambda(\lambda-1)(\lambda-2)(\lambda-3)$

$\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)$

$$P_5(\lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4) + 2\lambda(\lambda-1)(\lambda-2)(\lambda-3) + \lambda(\lambda-1)(\lambda-2)$$

$$= \lambda(\lambda-1)(\lambda-2)(\lambda^2-5\lambda+7)$$

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The presence of factors ~~λ~~ $\lambda - 1$ and $\lambda - 2$ indicates that G is at least 3-chromatic.

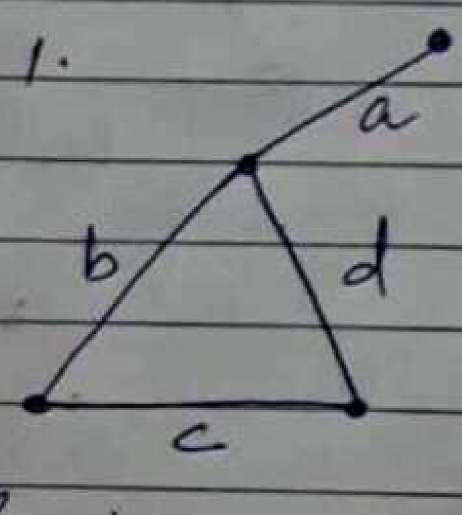
Matching.

A matching in a graph is a subset of edges in which no two edges are adjacent. A single edge in a graph is clearly a matching.

A maximal matching is a matching to which no edge in the graph can be added.

Or a maximal matching is a matching that is not a subset of any other matching.

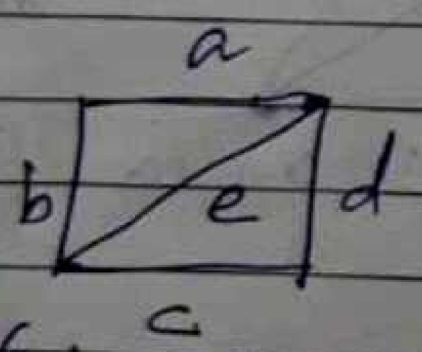
Eg. 1.



~~a and c are~~ $\{a, c\}$ is a matching. also it is a maximal matching.

$\{b\}$ is a maximal matching.

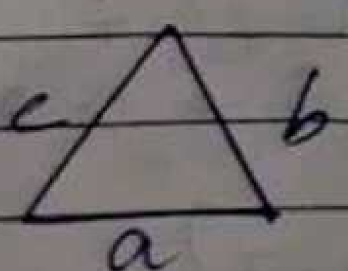
2.



$\{e\}$ is a maximal matching.

$\{a, c\}$ is a maximal matching.

$\{b, d\}$ is a maximal matching



$\{a\}$ is a maximal matching

$\{b\}$ "

$\{c\}$ "

The maximal matching with the largest number of edges are called the largest maximal matching. The number of edges in a largest maximal matching is called the matching number of the graph.

In eg. (2) $\{a, c\}$ is a largest maximal matching.

Matching is defined for any graph, it is mostly studied is the content of bipartite graphs.

In a bipartite graph having a vertex partition V_1 and V_2, a complete matching of vertices in set V_1 into those in V_2 is a matching in which there is one edge incident with every vertex in V_1.

~~Res~~ **Theorem.** A complete matching of V_1 into V_2 is a bipartite graph exists if and only if every subset of r vertices in V_1 is collectively adjacent to r or more vertices in V_2 for all values of r.

Theorem. In a bipartite graph a complete matching of V_1 into V_2 exists if there is a positive integer m for which the following condition is satisfied:

degree of every vertex in $V_1 \geq m \geq$ degree of every vertex in V_2.

<u>Proof.</u> Consider a subset of r vertices in V_1. These r vertices have at least $m \cdot r$ edges incident on them. Each $m \cdot r$ edge is incident to some vertex in V_2. Since the degree of every vertex in set V_2 is no greater than m, these $m \cdot r$ edges are incident on at least $\frac{m \cdot r}{m} = r$ vertices in V_2.

Thus any subset of r vertices in V_1 is collectively adjacent to r or more vertices in V_2. Therefore, by above theorem, there exists a complete matching of V_1 into V_2.

A set of r vertices in V_1 is collectively incident on, say, q vertices of V_2. Then the maximum value of the number $r-q$ taken over all values of $r = 1, 2, 3 \ldots$ and all subsets of V_1 is called the deficiency $\delta(G)$ of the bipartite graph G.

<u>Note</u>. A complete matching in a bipartite graph G exists if and only if

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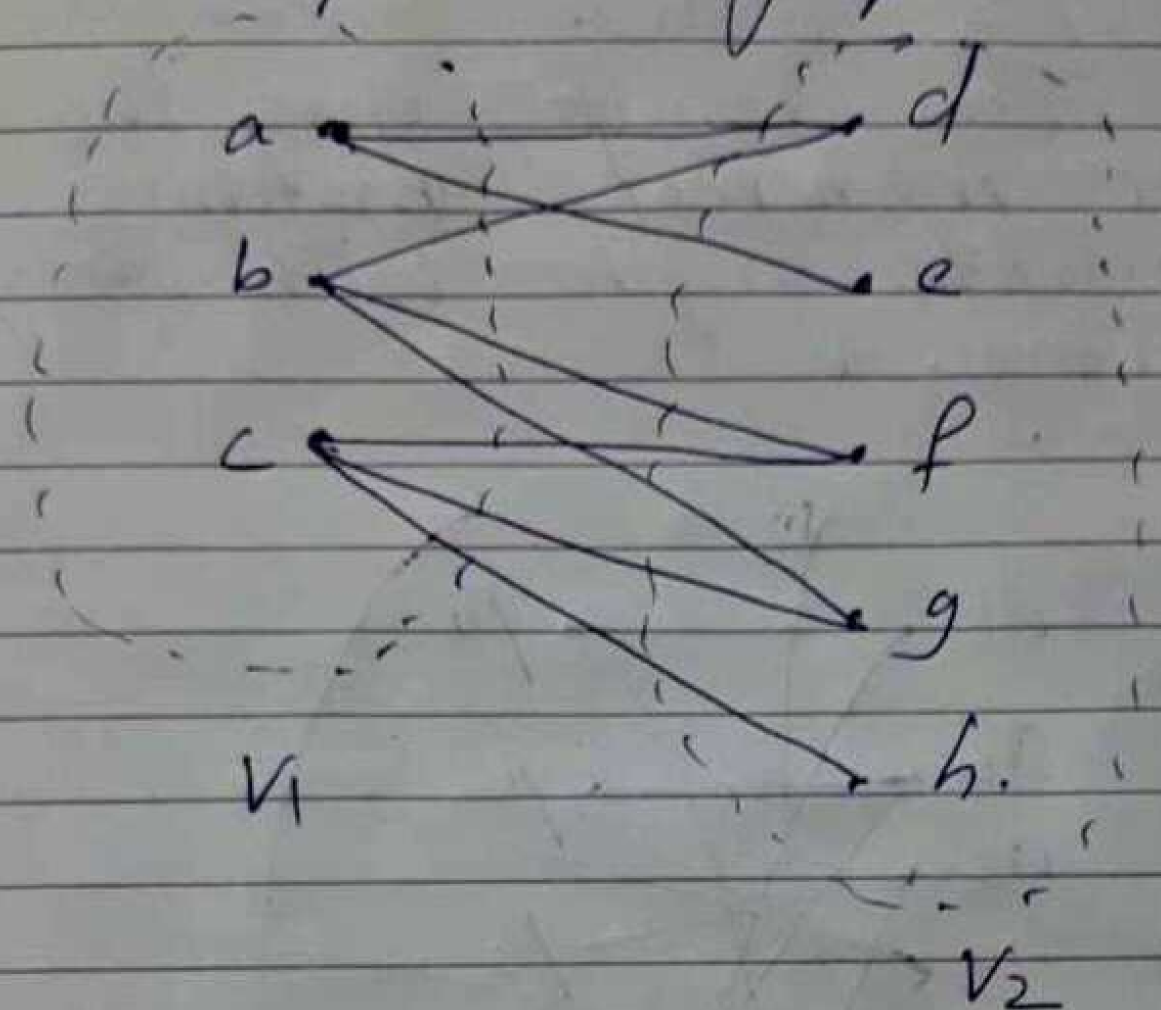
$\delta(G) \leq 0.$

Eg: Consider the following bipartite graph.



	V_1	V_2	$r-q$
$r=1$	$\{a\}$	$\{d, e\}$	-1
	$\{b\}$	$\{d, f, g\}$	-2
	$\{c\}$	$\{f, g, h\}$	-2
$r=2$	$\{a, b\}$	$\{d, e, f, g\}$	-2
	$\{b, c\}$	$\{d, f, g, h\}$	-2
	$\{a, c\}$	$\{d, e, f, g, h\}$	-3
$r=3$	$\{a, b, c\}$	$\{d, e, f, g, h\}$	-2

$\delta(G) =$ maximum value of $r-q = -1$

Also $\delta(G) < 0$, its complete matching exists.

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<u>Result</u>: The maximal ~~num~~ number of vertices in set V_1 that can be matched into V_2 is equal to number of vertices in $V_1 - \delta(G)$.

This result gives the size of the maximal matching for a bipartite graph with a positive deficiency.

Matching and Adjacency Matrix

Consider a bipartite graph G with non adjacent sets of vertices V_1 and V_2 having number of vertices n_1 and n_2, respectively, and let $n_1 \leq n_2$, $n_1 + n_2 = n$, the number of vertices in G. The adjacency matrix $X(G)$ of G can be written in the form.

$$X(G) \begin{bmatrix} 0 & X_{12} \\ X_{12}^T & 0 \end{bmatrix}$$

Where the submatrix X_{12} is the n_1 by n_2 $(0,1)$ - matrix containing the information as to which of the n_1 vertices of V_1 are connected to which of the n_2 vertices of V_2. Matrix X_{12}^T is the transpose of X_{12}

Clearly, all the information about the bipartite graph G is contained in X_{12} matrix.

A matching V_1 into V_2 corresponds to a selection of the k 1's in the matrix X_{12} such that no line (ie a row or a column) has more than one 1.

The matching is complete if the n_1 by n_2 matrix made of selected 1's has exactly one 1.

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in ~~every~~ every row.

Eg: Consider the above bipartite graph.

$$X_{1,2} = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccccc} d & e & f & g & h \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

$n_1 = 3 \quad n_2 = 5 \quad n = 8 \quad \& \quad n_1 \leq n_2$

$V_1 = \{a, b, c\} \quad V_2 = \{d, e, f, g, h\}$.

A complete matching of V_1 onto V_2 is given by

$$M_1 = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccccc} d & e & f & g & h \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$M_2 = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccccc} d & e & f & g & h \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array} \text{ etc}$$