
KTU NOTES

Module - 3

Relations and Functions

Cartesian Product

For sets A, B the cartesian product or cross product of A and B is denoted by $A \times B$

$$A \times B = \{ (a, b) \mid a \in A, b \in B \}$$

Example:

Let $A = \{2, 3, 4\}$

$B = \{4, 5\}$ then

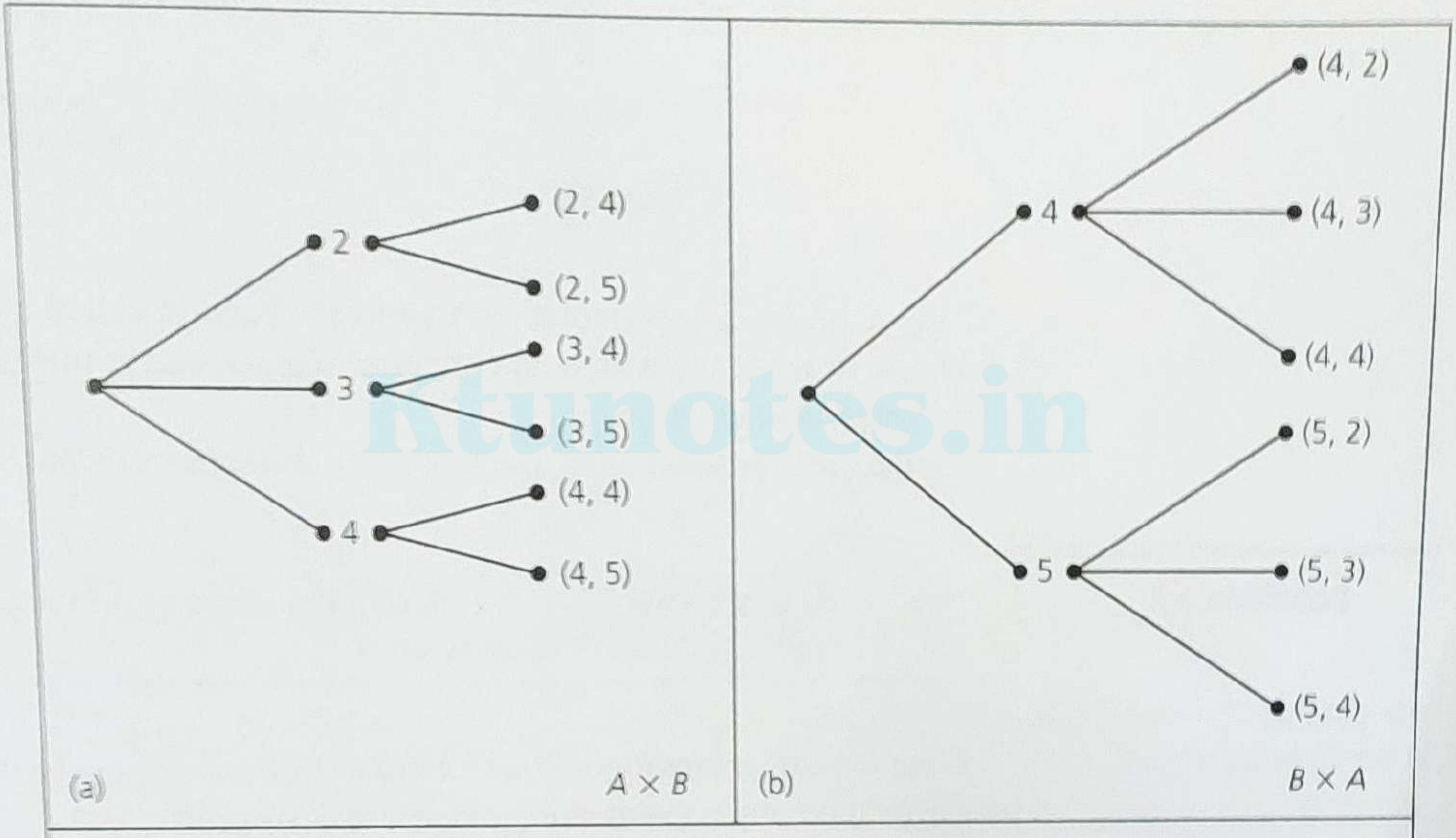
$$A \times B = \{ (2,4), (2,5), (3,4), (3,5), (4,4), (4,5) \}$$

$$B \times A = \{ (4,2), (4,3), (4,4), (5,2), (5,3), (5,4) \}$$

$$B^2 = B \times B = \{ (4,4), (4,5), (5,4), (5,5) \}$$

$$B^3 = B \times B \times B = \{(a,b,c) \mid a,b,c \in B\}$$

$$= \{(4,4,4), (4,5,4), (4,4,5), (5,4,4), (5,4,5), (5,5,4), (5,5,5), (4,5,5)\}$$


2
(2, 4)
(2, 5)
3
(3, 4)
(3, 5)
4
(4, 4)
(4, 5)
(a)
A × B
4
(4, 2)
(4, 3)
(4, 4)
5
(5, 2)
(5, 3)
(5, 4)
(b)
B × A

Binary Relation.

For sets A, B any subset of $A \times B$ is called a (binary) relation from A to B.

Any subset of $A \times A$ is called a (binary) relation on A

NOTE

We use the word "relation" for binary relation, unless something otherwise is specified.

Example

Let $A = \{2, 3, 4\}$

$B = \{4, 5\}$

$A \times B = \{(2,4), (3,4), (4,4), (2,5), (3,5), (4,5)\}$

Subsets of $A \times B$ will be a relation from A to B

* $\{(2,4)\}$
* $\{(2,4), (2,5)\}$
* $\{(2,4), (3,4), (4,5)\}$
* ϕ
* $A \times B$ are some of the relations from A to B

NOTE:

For finite sets A, B with $|A| = m$ and $|B| = n$, there are

2^{mn} relations from A to B (including the empty relation and the relation $A \times B$ itself)

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2. Let $B = \{1, 2\}$

$A = P(B)$, power set of B ie; set of all subsets of B

$= \{\{1,2\}, \{1\}, \{2\}, \phi\}$

$B \times A = \{(1, \{1,2\}), (1, \{1\}), (1, \{2\}), (1, \phi), (2, \{1,2\}), (2, \{1\}), (2, \{2\}), (2, \phi)\}$

$A \times A = \{(\{1,2\}, \{1,2\}), (\{1,2\}, \{1\}), (\{1,2\}, \{2\}), (\{1,2\}, \phi),$

$(\{1\}, \{1,2\}), (\{1\}, \{1\}), (\{1\}, \{2\}), (\{1\}, \phi),$

$(\{2\}, \{1,2\}), (\{2\}, \{2\})$

$(\{2\}, \{1\}), (\{2\}, \phi),$

$(\phi, \{1,2\}), (\phi, \{1\}), (\phi, \{2\}),$

$(\phi, \phi)\}$

Thus any subset of $B \times A$ is a relation from B to A

Example $R_1 = \{(1, \{2\}), (2, \phi)\}$

$R_2 = \{(1, \{1\}), (1, \{2\})\}$

$R_3 = B \times A$

$R_4 = \phi$.

Any subset of $A \times A$ is a relation on A

Example $R_5 = \{(\{1,2\}, \{1\})\}$

$R_6 = A \times A$

$R_7 = \phi$

NOTE

Let $A_1, A_2, \ldots, A_n$ be n sets then

$$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) / a_i \in A_i\}$$

called n-tuples collection

$|A \times B| = |A| \times |B| = |B \times A|$ where $|A|$ denotes the number of elements in A

n tuples $(a_1, a_2, \ldots a_n) = (b_1, b_2, \ldots, b_n)$ only when $a_i = b_i$ for each $1 \leq i \leq n$

For any set A

$$A \times \phi = \phi$$
$$\phi \times A = \phi$$

Since Let $(a, b) \in A \times \phi$.

$\Longrightarrow a \in A$ and $b \in \phi$

But $b \in \phi$ is Impossible since it is empty set.

Thus $(a, b) \notin A \times \phi$.

$\Longrightarrow A \times \phi$ is an empty set

$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$

$$(A \cap B) \times C = (A \times C) \cap (B \times C)$$

$$(A \cup B) \times C = (A \times C) \cup (B \times C)$$

Q: Let $A = \{a, b, c, d\}$

$B = \{w, x, y\}$

a) Give examples of 3 non empty relations from A to B

b) Give 3 examples of non empty relations on A

Soln a) $R_1 = \{(a, w), (a, y)\}$

$R_2 = \{(b, x)\}$

$R_3 = \{(c, w), (c, x), (c, y)\}$

b) $R_1 = \{(a,a), (a,b)\}$

$R_2 = \{(a,c), (a,a), (a,d)\}$

$R_3 = \{(a,a), (b,b), (c,c), (d,d)\}$

Q: Let $A = \{a, b, c, d, e, f, g, h\}$

$B = \{1, 2, 3, 4, 5\}$

a) How many elements are there in $P(A \times B)$?

b) Generalize the result in (a).

Soln $|A| = 8$

$|B| = 5$

$|A \times B| = |A| \times |B| = 8 \times 5 = 40.$

Consequently number of subsets of $A \times B$.

$$= 2^{5 \times 8} = 2^{40}$$

ie; no: of elements in $P(A \times B) = 2^{40}$

b) If $|A| = m$

$|B| = n$

$|A \times B| = mn$

Thus no: of elements in $A \times B = mn$

$\therefore$ There are 2^{mn} elements in $P(A \times B)$

Q: Prove that $(A \cup B) \times C = (A \times C) \cup (B \times C)$

Let $(a, b) \in (A \cup B) \times C$

$\iff a \in A \cup B$ and $b \in C$

$\iff a \in A$ or $a \in B$ and $b \in C$

$\iff [a \in A$ and $b \in C]$ or $[a \in B$ and $b \in C]$

$\iff (a, b) \in A \times C$ or $(a, b) \in B \times C$

$\iff (a, b) \in (A \times C) \cup (B \times C)$

Hence the proof.

Q: Prove that $A \times (B \cup C) = (A \times B) \cup (A \times C)$

Let $(a, b) \in A \times (B \cup C)$

$\iff a \in A$ and $b \in B \cup C$

$\iff a \in A$ and $(b \in B$ or $b \in C)$

$\iff [a \in A$ and $b \in B]$ or $[a \in A$ and $b \in C]$

$\iff (a, b) \in A \times B$ or $(a, b) \in A \times C$

$\iff (a, b) \in (A \times B) \cup (A \times C)$

Hence the proof

Q: Let A, B, C, D be non empty sets

Prove that $A \times B \subseteq C \times D$ iff

$A \subseteq C$ and $B \subseteq D$.

* If $A \times B \subseteq C \times D$ then $A \subseteq C$ and $B \subseteq D$

Let $A \times B \subseteq C \times D$, we need to

prove $A \subset D$ and $B \subseteq D$

Let $(a, b) \in A \times B$

$\Rightarrow a \in A$ and $b \in B$

Now since $A \times B \subseteq C \times D$ and

$(a, b) \in A \times B$

$\Rightarrow (a, b) \in C \times D$

$\Rightarrow a \in C$ and $b \in D$

Thus $A \subseteq C$ and $B \subseteq D$

Conversely let $A \subseteq C$ and $B \subseteq D$

we need to prove $A \times B \subseteq C \times D$

Let $x \in A$ and $y \in B$

$\Rightarrow (x, y) \in A \times B$ —— (1)

Now since $x \in A$ and $A \subseteq C$

$\Rightarrow x \in C$

$y \in B$ and $B \subseteq D$

$\Rightarrow y \in D$

Thus $(x, y) \in C \times D$ —— (2)

Thus from (1) & (2)

$A \times B \subseteq C \times D$.

Hence the proof.

Q: Let $A = \{1, 2, 3\}$ $B = \{2, 4, 5\}$ determine

a) $|A \times B|$

b) the no: of relations from A to B

c) no: of relations from A to B containing (1,2) and (1,5)

d) no: of relations from A to B contain exactly five ordered pairs.

e) no: of relations on A that contain atleast 7 elements.

Soln a) $|A| = 3$

$|B| = 3$

$|A \times B| = 3 \times 3 = 9$

b) no: of relations from A to B $= 2^{mn}$

$= 2^9$

c) $A \times B = \{(1,2), (1,4), (1,5), (2,2), (2,4)$
$(2,5), (3,2), (3,4), (3,5)\}$

Total no: of subsets of $A \times B$

$=$ no: of relations from A to B

$= 2^9$

no: of relations from A to B, that do not contain (1,2) and (1,5)

$=$ no: of subsets of $\{(1,4), (2,2), (2,4), (2,5), (3,2), (3,4), (3,5)\}$

$= 2^7$

Now include (1,2) and (1,5) with all those 2^7 subsets (relations, we will get the required type of relation

Thus required no: of relations $= \underline{\underline{2^7}}$

d) Relations containing 5 elements only

$=$ Choosing 5 elements from $A \times B$

$= \binom{9}{5}$ ways $= \frac{9!}{5!\,4!} = \frac{6\times7\times8\times9}{1\times2\times3\times4}$

$= 126$

e) Relations on A that contain atleast seven elements.

$|A\times A| = 3\times3 = 9$

no: of elements in $A\times A = 9$

Now no: of ways of selecting atleast seven elements from $A\times A$

= no: of relations on A contain atleast seven elements

Selecting atleast 7 $\Rightarrow$ Selecting 7 elements Or

Selecting 8 elements Or

Selecting 9 elements

$\therefore$ Required number $= \binom{9}{7} + \binom{9}{8} + \binom{9}{9}$

$$= \frac{9!}{7!\,2!} + \frac{9!}{8!\,1!} + 1$$

$$= 36 + 9 + 1$$

$$= \underline{\underline{46}}$$

Properties of Relations.

* A relation may be represented algebraically either by set-builder method or Roaster method.

Example. Let $A = \{1, 2, 3, 4, 5, 6\}$

Let $R = \{(1,2), (2,3), (3,4), (4,5), (5,6)\}$

be a relation on A and which is the roaster form of R.

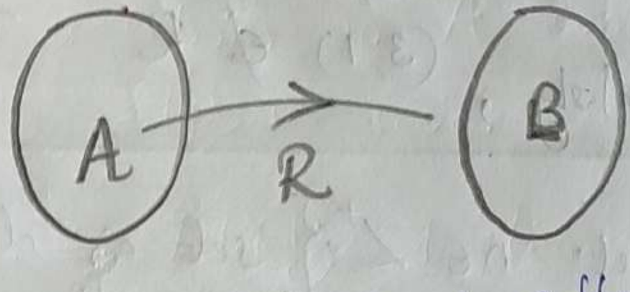
Now, the set-builder form of R is as follows

$$R = \{(x, y) \,/\, y = x + 1\}$$

* Consider two nonempty sets A & B, then any subset of $A \times B$ is a relation from A to B.

* The set of all first elements of the ordered pairs in a relation R from a set A to a set B is called the domain of R

* The set of all second elements in a relation R from a set A to a set B is called the range of the relation R



and the whole set B is called the codomain of R

* Range is a subset of Codomain.

Examples.

1. Define a relation R on the set Z {set of all integers} by $a R b$ or $(a,b) \in R$, $\boxed{\text{if } a \le b}$

Any subset of $Z \times Z$ with ordinary "less than or equal to" is a relation

$(1,2) \in R$ or $1 R 2$

but $(2,1) \notin R$ or $2 \not R 1$

Similarly, $(3,1) \notin R$

2. Let $n \in Z^+ =$ {set of all positive integers}

for $x, y \in Z$, the modulo n relation R defined by $x R y$ if

$\boxed{x - y \text{ is a multiple of } n}$

Eg: Let $n = 7$.

then $9 \, R \, 2$ since $9-2 = 7$ is a multiple of 7

But $5 \not{R} 1$ since $5-1 = 4$ not a multiple of 7.

$-3 \, R \, 11$ since $-3-11 = -14$ multiple of 7

$14 \, R \, 0$ since $14-0 = 14$ multiple of 7.

$3 \not{R} 7$ since $3-7 = -4$ not a multiple of 7

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3 Let $\mathcal{U} = \{1, 2, 3, 4, 5, 6, 7\}$.

Let $C \subseteq \mathcal{U}$ be $C = \{1, 2, 3, 6\}$

Define a relation R on $P(\mathcal{U})$ by

$$\boxed{A \, R \, B \text{ if } A \cap C = B \cap C}$$

where $A, B \in P(\mathcal{U})$

Eg: let $A = \{1, 2, 4, 5\}$

$B = \{1, 2, 5, 7\}$

we have $C = \{1, 2, 3, 6\}$

$A \cap C = \{1, 2\}$

$B \cap C = \{1, 2\}$

Thus $A \cap C = B \cap C$

$\therefore A R B$

Eg: Let $X = \{4, 5\}$

$Y = \{7\}$

$X \cap C = \{\} = \phi$

$\cancel{X} \cap C = \{\} = \phi$

$\therefore X R Y$

Ex: $P = \{1, 2, 3, 4, 5\}$

$Q = \{1, 2, 3, 6, 7\}$

$P \cap C = \{1, 2, 3\}$

$Q \cap C = \{1, 2, 3, 6\}$

$P \cap C \neq Q \cap C$ $\therefore$ $P \not\leq Q$ here

Types of Relations

I Reflexive relation.

A relation R on a set A is called reflexive $\boxed{\text{if } \forall x \in A,\ (x,x) \in R}$

II Symmetric Relation.

A relation R on a set A is called symmetric

$\boxed{\text{if } (x,y) \in R \Longrightarrow (y,x) \in R,\ \forall x,y \in A}$

III Anti symmetric Relation

A relation R on a set A is called antisymmetric

$\boxed{\text{if } \forall a,b \in A\ (aRb \text{ and } bRa) \Longrightarrow a=b}$

ie, If "a related to b" and
"b related to a" is if a
and b are one and the same element
from A.

IV Transitive Relation

For a set A, a relation R on
A is called transitive

$$\boxed{\begin{array}{l} \text{if } \forall\ x, y, z \in A \\ (x,y),\ (y,z) \in R \Longrightarrow (x,z) \in R \end{array}}$$

Examples.

1. For $A = \{1, 2, 3, 4\}$ define relations

$R_1 = \{(1,1), (2,3)\}$

$R_2 = \{(1,1), (2,2), (3,3), (4,4), (1,2)\}$

$R_3 = \{(1,1), (2,1), (1,2)\}$

$R_4 = \{(1,2),(2,1),(1,1),(2,2),(2,1)\}$

$R_5 = \{(x,y) / x \le y,\ x,y \in A\}$

Here R_1 is not reflexive,

R_1 is symmetric.

R_1 is Antisymmetric.

R_1 is transitive.

R_2 is reflexive

R_2 is not symmetric since $(2,1) \notin R_2$

R_2 is Antisymmetric

R_2 is transitive.

$(1,2),(2,2) \in R_2$ also

$(1,2) \in R_2$

R_3 is not reflexive

R_3 is symmetric

R_3 is not Antisymmetric

Since $(1,2),(2,1) \in R_3$

But $1 \neq 2$

R_3 is not transitive

since $(2,1), (1,2) \in R_3$
but $(2,2) \notin R_3$

R_4 is not reflexive

R_4 is symmetric

R_4 is not Antisymmetric

since $(3,2), (2,3) \in R_4$ But
$2 \neq 3$

R_4 is not transitive

$(3,2), (2,3) \in R_4$ But
$(3,3) \notin R_4$

$$R_5 = \{(x,y) / x \leq y ; x,y \in A\}$$

$$= \{(1,1), (1,2), (1,3), (1,4)$$

$$(2,2), (2,3), (2,4)$$

$$(3,3), (3,4)$$

$$(4,4)\}$$

R_5 is reflexive

R_5 is not symmetric

(since $(2,1) \notin R_5$)

R_5 is antisymmetric

R_5 is transitive

7.6

With $A = \{1, 2, 3\}$, we have:

a) $\mathcal{R}_1 = \{(1, 2), (2, 1), (1, 3), (3, 1)\}$, a symmetric, but not reflexive, relation on A;

b) $\mathcal{R}_2 = \{(1, 1), (2, 2), (3, 3), (2, 3)\}$, a reflexive, but not symmetric, relation on A;

c) $\mathcal{R}_3 = \{(1, 1), (2, 2), (3, 3)\}$ and $\mathcal{R}_4 = \{(1, 1), (2, 2), (3, 3), (2, 3), (3, 2)\}$, two relations on A that are both reflexive and symmetric; and

d) $\mathcal{R}_5 = \{(1, 1), (2, 3), (3, 3)\}$, a relation on A that is neither reflexive nor symmetric.

3. Consider Z^+ with a relation R defined by $R = \{ (x,y) / x \text{ divides } y ; \}$

Here R is reflexive

since $\forall x \in Z^+$

x divides x

$\Longrightarrow (x,x) \in R$

R is not symmetric

since if $(x,y) \in R$

ie; x divides y

y ~~need~~ not divide x

ie; (y,x) need not belongs to R

R is Antisymmetric

(Since for $x \neq y$.

If $(x, y) \in R$

$(y, x) \notin R$

R is transitive

Since if $x \neq y \neq z$ and $(x, y) \in R$

$(y, z) \in R$

ie, x divides y and

y divides z

$x\overline{)y}$

0

ie, $y = k_1 x$ and

$z = k_2 y$

Thus. $z = k_2 [k_1 x]$

$= k_3 x$

$\Longrightarrow$ x divides z

$\Longrightarrow$ $(x, z) \in R$

4) Let $\mathcal{U}$ be any non empty set, define a relation R on $P(\mathcal{U})$ by $(A, B) \in R$ if $A \subseteq B$ for any $A, B \in P(\mathcal{U})$.

Here R is a relation defined on $P(\mathcal{U})$

ie; R is a subset of $P(\mathcal{U}) \times P(\mathcal{U})$

and two sets $A, B \in P(\mathcal{U})$ are related under R if $A \subseteq B$

* Now, For any set A, it is clear that

$A \subseteq A$

$\Longrightarrow$ R is reflexive

* R is not symmetric

since if $A \subset B$ then

$B \not\subset A$ for $A \neq B$

ie, if $(A, B) \in R$ then (B, A) need not belongs to R

* R is antisymmetric

since if $A \subseteq B$ and $B \subseteq A$

then clearly $A = B$

* R is transitive

Since if $A \subseteq B$ and

$B \subseteq D$

$\Longrightarrow A \subseteq D$

ie, if $(A, B), (B, D) \in R$

then $(A, D) \in R$

Partial order relations

A relation R on a set A is called a partial order, or a partial ordering relation, if R is

- reflexive
- antisymmetric and
- Transitive

Equivalence Relations

An equivalence relation R on a set A is a relation that is

- reflexive
- symmetric and
- transitive

Example

Let $n \in Z^+$

For $x, y \in Z$, the modulo n relation R is defined by $x R y$ if $x-y$ is a multiple of n is an equivalence relation.

for, $\forall x \in Z$ and $n \in Z^+$

$x - x = 0$, a multiple of n

$\Longrightarrow (x, x) \in R \quad \forall x \in Z$

$\therefore R$ is symmetric.

Now for $x, y \in Z$

if $(x, y) \in R$

$\Longrightarrow$ $x-y$ is a multiple of n

$\Longrightarrow$ $y-x$ is also a multiple of n

$\Longrightarrow$ $(y, x) \in R$

$\therefore$ R is symmetric

Now, for $(x,y) \in R$ and $(y,z) \in R$

ie; $x-y$ is a multiple of n and

$y-z$ is a multiple of n

ie; $x-y = k_1 n$ and

$y-z = k_2 n$

$\Longrightarrow (x-y)+(y-z) = (k_1+k_2) n$

$\Longrightarrow x-z = k_3 n$

$\Longrightarrow x-z$ is a multiple of 7

$\Longrightarrow (x,z) \in R$ 7.16

$\therefore$ R is transitive 7.17

7.18

$\therefore$ R is an equivalence relation

Irreflexive relations.

A relation R on a set A is called irreflexive if for all $a \in A$,

$(a,a) \notin R$

Example.

1. Let $A = \{1, 2, 3\}$

$R = \{(1,2), (2,1)\}$ is irreflexive

2. Let R be a relation defined on $\mathbb{Z}$ as $\{(x,y) / x < y, x, y \in \mathbb{Z}\}$

is clearly irreflexive relation

b) If $A = \{1, 2, 3\}$, then

$\mathcal{R}_1 = \{(1, 1), (2, 2), (3, 3)\}$,

$\mathcal{R}_2 = \{(1, 1), (2, 2), (2, 3), (3, 2), (3, 3)\}$,

$\mathcal{R}_3 = \{(1, 1), (1, 3), (2, 2), (3, 1), (3, 3)\}$, and

$\mathcal{R}_4 = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\} = A \times A$

are all equivalence relations on A.

NOTE

Let A be any non empty set with $|A| = n$ then,

a) The no: of reflexive relations on A $= 2^{(n^2-n)}$

b) The no: of symmetric relations on A $= 2^{\left(\frac{n^2+n}{2}\right)}$

c) The no: of reflexive and symmetric relations on A $= 2^{\left(\frac{n^2-n}{2}\right)}$

d) The no: of antisymmetric relations on A $= 2^n 3^{\left(\frac{n^2-n}{2}\right)}$

e) The no: of asymmetric relations on A $= 3^{\left(\frac{n^2-n}{2}\right)}$

f) The no: of irreflexive relations on A $= 2^{n(n-1)}$

9) The no: of relations which are neither reflexive nor irreflexive are

$$= 2^{n^2} - 2 \cdot 2^{n(n-1)}$$

PLE 7.20

Determine the number of relations on $A = \{a, b, c, d, e\}$ that are **(a)** reflexive **(b)** symmetric **(c)** reflexive and symmetric **(d)** antisymmetric **(e)** asymmetric **(f)** irreflexive **(g)** neither reflexive nor irreflexive.

Here $|A| = n = 5$.

a) The number of reflexive relations on A are $2^{(n^2-n)} = 2^{(5^2-5)} = 2^{20}$.

b) The number of symmetric relations on A are $2^{(n^2+n)/2} = 2^{(5^2+5)/2} = 2^{15}$.

c) The number reflexive and symmetric relations on A are $2^{(n^2-n)/2} = 2^{(5^2-5)/2} = 2^{10}$.

d) The number of antisymmetric relations on A are $2^n 3^{(n^2-n)/2} = 2^5 3^{(5^2-5)/2} = 2^5 \cdot 3^{10}$.

e) The number of asymmetric relations on A are $3^{(n^2-n)/2} = 3^{(5^2-5)/2} = 3^{10}$.

f) The number of irreflexive relations are $2^{n(n-1)} = 2^{5(5-1)} = 2^{20}$.

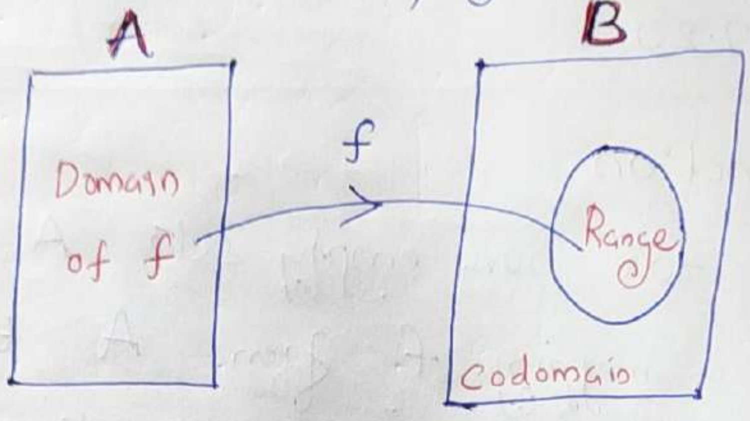
g) The number of relations which are neither reflexive nor irreflexive are $2^{n^2} - 2 \cdot 2^{n(n-1)} = 2^{5^2} - 2 \cdot 2^{5(5-1)} = 2^{25} - 2^{21}$.

Function

For non empty sets A, B a function or a mapping f from A to B, denoted by $f: A \rightarrow B$ is a relation from A to B in which every element of A appears exactly once as the first component of an ordered pair in the relation.

Domain of a function

For the function $f: A \rightarrow B$, A is called the domain of f and B is the Codomain of f. The subset of B consisting of those elements that appear as second components in the ordered pairs of f is called the range of f and it is denoted by f(A) because it is the set of images (of the elements of A) under the mapping f.



Example

1. Greatest integer function or floor function

$f: \mathbb{R} \to \mathbb{Z}$ defined by

$f(x) = \lfloor x \rfloor$ = the greatest integer less than or equal to x

$f(3.8) = \lfloor 3.8 \rfloor = 3$

$f(-3.8) = \lfloor -3.8 \rfloor = -3$

$f(15.3) = \lfloor 15.3 \rfloor = 15$

Here $\mathbb{R}$ is the domain $\mathbb{Z}$ is the codomain also $\mathbb{Z}$ itself is the range of f

2. Ceiling function

$g : R \to Z$ defined by

$g(x) = \lceil x \rceil$ = the least integer greater than or equal to x.

$f(3.01) = \lceil 3.01 \rceil = 4$

$f(3.8) = \lceil 3.8 \rceil = 4$

$f(-0.1) = \lceil -0.1 \rceil = 0$

$f(-3.7) = \lceil -3.7 \rceil = -3$

3. Truncation function.

It is another integer valued function defined on R, This function deletes the fractional part of a real number.

trunc $(3.78) = 3$

trunc $(5) = 5$

trunc $(-7.22) = -7$

trunc $(-9.1) = -9$

AMPLE 5.24

Determine each of the following.

a) $\lfloor 3.2 - 2.5 \rfloor$

b) $\lfloor 3.2 \rfloor - \lfloor 2.5 \rfloor$

c) $\lceil 5.3 \rceil \lfloor 8.1 \rfloor$

d) $\lfloor 5.3 \rfloor \lceil 8.1 \rceil$

e) $\lfloor 5\pi \rfloor$

f) $5\lceil \pi \rceil$

g) $\lceil -\frac{3}{4} \rceil$

h) $\lfloor -\frac{7}{8} \rfloor$

i) $\lfloor -1 \rfloor$

j) $\lceil 6 \rceil$

k) $\lfloor 8 \rfloor$

Note that floor of x, $\lfloor\ \rfloor$, rounds x down while ceiling of x $\lceil\ \rceil$ rounds x up.

a) $\lfloor 3.2 - 2.5 \rfloor = \lfloor 0.7 \rfloor = 0$

b) $\lfloor 3.2 \rfloor - \lfloor 2.5 \rfloor = 3 - 2 = 1$

c) $\lceil 5.3 \rceil \lfloor 8.1 \rfloor = 6 \cdot 8 = 48$

d) $\lfloor 5.3 \rfloor \lceil 8.1 \rceil = 5 \cdot 9 = 45$

e) $\lfloor 5\pi \rfloor = \lfloor 15.70 \rfloor = 15$

f) $5\lceil \pi \rceil = 5\lceil 3.14159 \rceil = 5 \cdot 4 = 20$

g) $\lceil -\frac{3}{4} \rceil = \lceil -0.75 \rceil = 0$

h) $\lfloor -\frac{7}{8} \rfloor = \lfloor -0.875 \rfloor = -1$

i) $\lfloor -1 \rfloor = -1$

j) 6

k) 8

Composition of two functions.

Function composition is an operation that takes two functions f and g and produces a function 'h' such that $h(x) = g(f(x))$. Here the function g is applied to the result of applying the function f to x. [denoted by $g\circ f$]

Example.

1. Let $f(x) = 2x+3$ and $g(x) = x^2$ find $g\circ f$ and $f\circ g$

$$g\circ f = g(f(x)) = g(2x+3)$$
$$= (2x+3)^2 = 4x^2+12x+9$$

$$f\circ g = f(g(x)) = f(x^2)$$
$$= 2(x^2)+3 = 2x^2+3$$

2. Suppose $f(x) = x+2$, $g(x) = x-2$, $h(x) = 3x$ for $x \in \mathbb{R}$, where R is the set of real numbers. find $f\circ g$, $g\circ f$, $f\circ f$, $g\circ g$, $f\circ h$

$$f\circ g = f(g(x)) = f(x-2) = (x-2)+2$$
$$= x$$

$g \circ f = g(f(x)) = g(x+2) = (x+2) - 2$
$= \underline{\underline{x}}$

$f \circ f = f(f(x)) = f(x+2) = (x+2) + 2$
$= \underline{\underline{x+4}}$

$g \circ g = g(g(x)) = g(x-2) = (x-2) - 2$
$= \underline{\underline{x-4}}$

$f \circ h = f(h(x)) = f(3x) = \underline{\underline{3x+2}}$

I Determine whether or not each of the following relations are functions, find its range.

a) $\{(x,y) \mid x, y \in \mathbb{Z}, y = x^2+7\}$, a relation from Z to Z

Here for every $x \in \mathbb{Z}$ we can define unique y. $\therefore$ It is a function

When $x=0 \Rightarrow y = 7$

$x=-1 \Rightarrow y = (-1)^2+7 = 8$

$x=1 \Rightarrow y = 1+7 = 8$

$x=-2 \Rightarrow y = (-2)^2+7 = 11$

$x=2 \Rightarrow y = 2^2+7 = 11$

$x=-3, x=3 \Rightarrow y = 9+7 = 16$

$\therefore$ Range of this function $= \{7, 8, 11, 16, \dots\}$

b) $\{(x,y) \mid x, y \in \mathbb{R}, y^2 = x\}$, a relation from R to R

when $x = 1 \Rightarrow y^2 = 1$

$\Rightarrow y = \pm 1$

ie; $(1,1), (1,-1) \in R$

Thus 1 appears twice as the 1st component in this relation

$\Longrightarrow$ It is not a function

c. $\{(x,y) \mid x, y \in R, \; y = 3x+1\}$, a relation from R to R

for every $x \in R$, x appears as the 1st component only once

$\Longrightarrow$ It is a function

When $x \in R$ $\quad y = 3x+1 \in R$

$\therefore$ Range of this function $= R$

d) $\{(x,y) \mid x, y \in Q, \; x^2+y^2=1\}$ a relation from Q to Q when

$x = 2 \quad y = \pm\sqrt{1-4} = \pm\sqrt{-3} \notin Q$

$\therefore x = 2$ not included in this relation as 1st component

i.e., $2 \in Q \notin$ the set of 1st component in this relation

$\Longrightarrow$ R is not a function

e) R is a relation from A to B where

$|A| = 5, \ |B| = 6$ and $|R| = 6$.

R is a subset of $A \times B$

we know $|A \times B| = 5 \times 6$

$= 30$

$|R| = 6$

i.e., the relation R contains 6 elements

this is possible only if atleast one element from A to repeat. (since A contains 5 elements only)

$\Longrightarrow$ R is not a function.

Q: For $A = \{1,2,3,4,5\}$ and $B = \{w, x, y, z\}$

① Let $f : A \rightarrow B$ be given by

$f = \{(1, w), (2, x), (3, x), (4, y), (5, y)\}$

Then for $A_1 = \{1\}$

$A_2 = \{1, 2\}$

$A_3 = \{1, 2, 3\}$

$A_4 = \{2, 3\}$ and $A_5 = \{2, 3, 4, 5\}$

find the corresponding images under f

$f(A_1) = \{f(a) / a \in A_1\}$

$= \{f(a) / a \in \{1\}\}$

$= \{f(a) / a = 1\}$

$= \{f(1)\} = \{w\}$

$f(A_2) = \{f(a) / a \in A_2\}$

$= \{f(a) / a \in \{1, 2\}\}$

$= \{f(1), f(2)\}$

$= \{w, x\}$

$f(A_3) = \{f(a) / a \in A_3\}$

$= \{f(1), f(2), f(3)\}$

$= \{w, x, x\}$

$= \{w, x\}$

$f(A_4) = \{f(a) / a \in A_4\}$

$= \{f(2), f(3)\}$

$= \{x\}$

$f(A_5) = \{f(a) / a \in A_5\}$

$= \{f(2), f(3), f(4), f(5)\}$

$= \{x, y\}$

Q) Let $g : R \to R$ by $g(x) = x^2$ find the

② range of g next

a) $g(R)$

b) $g(Z)$

c) $g(A_1)$ where $A_1 = [-2, 1]$

(2) soln.

$$g(x) = x^2$$

$$g(R) = \{ x^2 / x \in R \}$$

$$= R$$

$$g(Z) = \{ g(x) / x \in Z \}$$

$$= \{ x^2 / x \in Z \}$$

$$= \{ x^2 / x \in \{ \dots, -2, -1, 0, 1, 2, \dots \} \}$$

$$= \{ 0, 1, 4, 9, 16, \dots \}$$

c) $g(A_1) = \{ g(x) / x \in A_1 \}$

$= \{ g(x) / x \in [-2, 1] \}$

$= \{ x^2 \mid -2 \leq x \leq 1 \}$

$= [0, 4]$

Q: Let $h: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$ where

3 $h(x,y) = 2x+3y$ find

a) range of h

b) $h(A_1)$ where $A_1 = \{(0,n) / n \in \mathbb{Z}^+\}$

soln

$$h(x,y) = 2x+3y$$

a) where $x, y \in \mathbb{Z}$, integers

Ex: $h(0,0) = 0$

$$h(-3,7) = 2(-3)+3(7)$$
$$= -6+21$$
$$= 15$$

for any $n \in \mathbb{Z}$

$$h(2n,-n) = 2[2n]+3[-n]$$
$$= 4n-3n$$
$$= n, \quad n \in \mathbb{Z}$$

Thus we can say all $n \in \mathbb{Z}$ will be in the range of h

Thus range of $h = \mathbb{Z}$

b) $A_1 = \{(0,n) / n \in Z^+\}$

$h(0,n) = 2(0) + 3(n)$

$= 3n, \; n \in Z^+$

Thus range of h of A_1

$= \{3n / n \in Z^+\}$

RESULT

Let $f: A \to B$, with $A_1, A_2 \subseteq A$

then a) $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$

b) $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$

c) $f(A_1 \cap A_2) = f(A) \cap f(A_2)$ when f is one-to-one

Restriction and Extension of a function

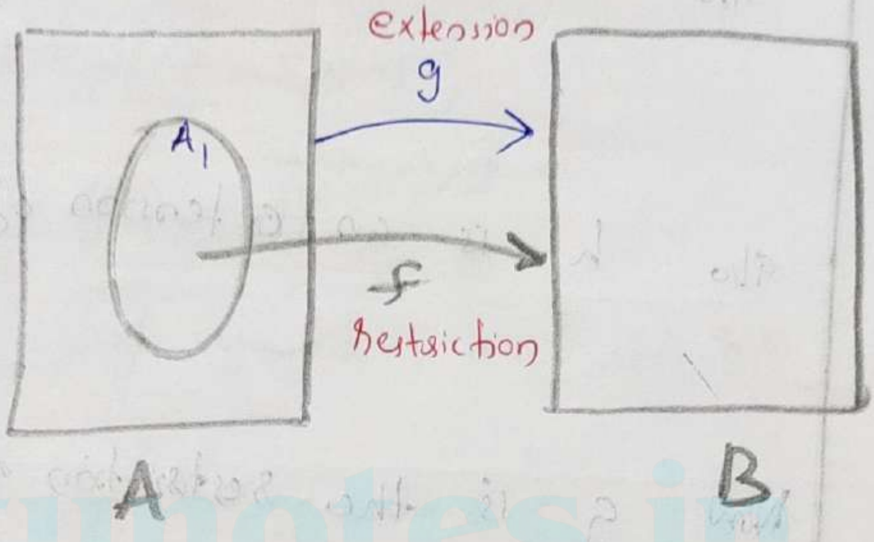
If $f: A \to B$ and $A_1 \subseteq A$ then $f|_{A_1} : A_1 \to B$ is called the restriction of f to A_1 if

$$\boxed{f|_{A_1}(a) = f(a) \text{ for all } a \in A_1}$$

If $A_1 \subseteq A$ and $f : A_1 \to B$

If $g : A \to B$ and $g(a) = f(a)\ \forall a \in A_1$,

then we call g an extension of f to A



Q: For $A = \{1, 2, 3, 4, 5\}$ let $f : A \to \mathbb{R}$ be defined by $f = \{(1,10), (2,13), (3,16), (4,19), (5, 22)\}$

let $g : \mathbb{Q} \to \mathbb{R}$ where $g(q) = 3q + 7\ \forall q \in \mathbb{Q}$

and $h : \mathbb{R} \to \mathbb{R}$ with $h(r) = 3r + 7\ \forall r \in \mathbb{R}$

We know here $A \subseteq \mathbb{Q} \subseteq \mathbb{R}$

$$\begin{array}{ccc} A & \mathbb{Q} & \mathbb{R} \\ \downarrow & \downarrow & \downarrow \\ f & g & h \end{array}$$

Thus

g is an extension of f (domain from A) to Q

also h is an extension of f (from A) to R

also h is an extension of g (from Q) to R

Now g is the restriction of h (from R) to Q

f is the restriction of h (from R) to A

f is the restriction of g (from Q) to A

Q. Suppose $A, B, C \subseteq Z^2$

$A = \{(x,y) / y = 5x-1\}$

$B = \{(x,y) / y = 6x$

$C = \{(x,y) / 3x - y = -7\}$

b)

find a) $A \cap B$

b) $B \cap C$

c) $\overline{\bar{A} \cup \bar{C}}$

d) $\bar{B} \cup \bar{C}$

$A = \{(x,y) / y = 5x-1\}$

$B = \{(x,y) / y = 6x\}$

$A \cap B = \{(x,y) / y = 5x-1 \text{ and } y = 6x\}$

$= \{(x,y) / 5x-1 = 6x\}$

$= \{(x,y) / x = -1\}$

$\therefore y = 6x = 6(-1) = -6$

$y = 5x-1 = 5(-1)-1 = -6$

Thus $A \cap B = \{(-1,-6)\}$

b) $B \cap C = \{(x,y) / y = 6x \text{ and } 3x-y = -7\}$

$= \{(x,y) / y = 6x \text{ and } y = 3x+7\}$

$= \{(x,y) / 6x = 3x+7\}$

$$= \{(x,y) / 3x = 7\}$$

$$= \{(x,y) / x = \frac{7}{3}\}$$

but $\frac{7}{3} \notin \mathbb{Z}$

So $B \cap C = \phi$

c) $\overline{\bar{A} \cup \bar{C}}$

Now $\bar{A} = \{(x,y) / y \neq 5x - 1\}$

$\overline{A \cup B} = \bar{A} \cap \bar{B}$ and $\bar{\bar{A}} = A$

$$\Rightarrow \overline{\bar{A} \cup \bar{C}} = \bar{\bar{A}} \cap \bar{\bar{C}}$$

$$= A \cap C$$

Thus

$$A \cap C = \{(x,y) / y = 5x - 1 \text{ and } 3x - y = -7\}$$

$$= \{(x,y) / y = 5x - 1 \text{ and } y = 3x + 7\}$$

$$= \{(x,y) / 5x - 1 = 3x + 7\}$$

$= \{(x,y) / 2x = 8\}$

$= \{(x,y) / x = 4\}$ $\therefore y = 5x - 1$

$= 5(4) - 1$

$= 19$

$= \{(4,19)\}$

d) $\overline{B} \cup \overline{C}$

Now $\overline{B \cap C} = \overline{B} \cup \overline{C}$

$\therefore \overline{B} \cup \overline{C} = \overline{B \cap C} = \overline{\phi} =$ Universal set

$= \mathbb{Z}^2$

$= \mathbb{Z} \times \mathbb{Z}$

One-to-One function [Injetive]

A function $f: A \to B$ is called One-to-one or injective if each element of B appears at most once as the image of an element of A.

If $f: A \to B$ is one-to-one with A, B finite, we must have $|A| \leq |B|$

For arbitrary sets A and B

$f: A \to B$ is one-to-one iff

$\forall\, a_1, a_2 \in A$

$$f(a_1) = f(a_2)$$

$$\Rightarrow a_1 = a_2$$

Example

Q: Consider the function $f: \mathbb{R} \to \mathbb{R}$ where

$f(x) = 3x + 7 \;\forall\, x \in \mathbb{R}$

Let $a_1, a_2 \in \mathbb{R}$

Let $f(a_1) = f(a_2)$

$\Rightarrow 3a_1 + 7 = 3a_2 + 7$

$\Rightarrow 3a_1 = 3a_2$

$\Rightarrow a_1 = a_2$

So the given function is one-to-one

2) Let $g: \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^4 - x$

Let $a_1, a_2 \in \mathbb{R}$

$g(a_1) = g(a_2)$

$\Longrightarrow a_1^4 - a_1 = a_2^4 - a_2 \not\Rightarrow a_1 = a_2$

$g(0) = 0$

$g(1) = 1 - 1 = 0$

ie, $g(0) = g(1)$ but $0 \neq 1$

$\therefore$ g is not one-one function

3) Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4, 5\}$

The function $f = \{(1,1), (2,3), (3,4)\}$

and $g = \{(1,1), (2,3), (3,3)\}$

Here f is one one, since every element in B appears atmost once in f.

But g is not one-one since $g(2) = 3$ and $g(3) = 3$

But $2 \neq 3$.

4. Determine which of the following functions are one-to-one and find its range

a) $f: Z \to Z \quad f(x) = 2x$.

b) $f: Q \to Q \quad f(x) = 2x$

c) $f: R \to R \quad f(x) = e^{x^2}$

d) $f: \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \to R \quad f(x) = \cos x$

Soln Let $a_1, a_2 \in \mathbb{Z}$

Let $f(a_1) = f(a_2)$

$\Longrightarrow 2a_1 = 2a_2$

$\Longrightarrow a_1 = a_2$

$\therefore f: Z \to Z$ is one-to-one

Range of $f = \{f(x) / x \in Z\}$

$= \{2x / x \in Z\}$

= set of all even integers

b) Let $a_1, a_2 \in Q$

Let $f(a_1) = f(a_2)$

$\Longrightarrow 2a_1 = 2a_2$

$\Longrightarrow a_1 = a_2$

$\therefore f: Q \to Q$ is also one-to-one function.

Range of $f = \{ f(x) / x \in Q \}$

$= \{ 2x / x \in Q \}$

$= Q$, rational numbers

c) Let $f(x) = e^{x^2}$

Clearly $f(1) = e^1$

$f(-1) = e^1$

ie, $f(1) = f(-1)$

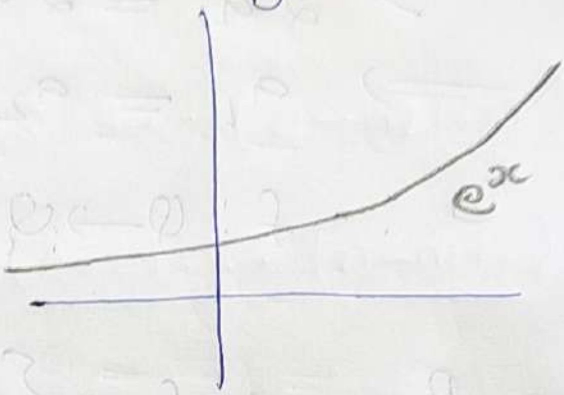
but $1 \neq -1$

$\therefore f$ is not one-to-one function

Range of $f = \{ f(x) / x \in R \}$

$= \{ e^{x^2} / x \in \mathbb{R} \}$

exponential value always lies in $(0, \infty)$



Range of $f = (0, \infty)$

d) $f(x) = \cos x$

Clearly

$f(\frac{\pi}{2}) = \cos \frac{\pi}{2} = 0$

$f(-\frac{\pi}{2}) = \cos(-\frac{\pi}{2}) = \cos \frac{\pi}{2} = 0$

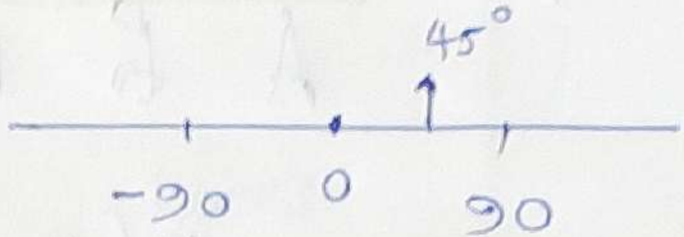
But $\frac{\pi}{2} \neq -\frac{\pi}{2}$

$\therefore$ f is not one to one function

Range of $f = \{ f(x) / x \in [-\frac{\pi}{2}, \frac{\pi}{2}] \}$

$$= \left\{ \cos x \;/\; x \in \left[\frac{-\pi}{2}, \frac{\pi}{2} \right] \right\}$$

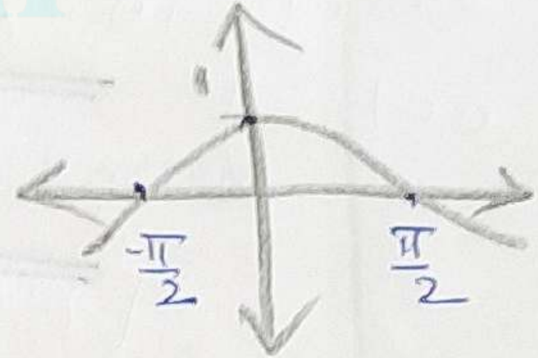
$$\cos\left(\frac{-\pi}{2}\right) = 0$$



$$\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$\cos 0 = 1$$

also $(\cos \theta = \cos -\theta)$



Range = $[0, 1]$

NOTE

*

(1) Let $|A| = m$ and $|B| = n$

Any function $f: A \to B$ defined as

$$f = \{(a_1, x_1), (a_2, x_2) \ldots, (a_m, x_m)\}$$

where x_1 can have n choices

x_2 " " n choices

⋮

x_m can have n choices

Thus Total no of functions which can be defined from $A \to B$ is

$$n \times n \times \ldots \times n \quad m \text{ times} = n^m$$

$$= |B|^{|A|}$$

NOTE

(2) Let $A = \{a_1, a_2, \ldots, a_m\}$

$B = \{b_1, b_2, \ldots, b_n\}$, $m \leq n$

a one-to-one function $f : A \to B$ has the form $\{(a_1, x_1), (a_2, x_2), \ldots, (a_m, x_m)\}$

when x_1 has n choices from B

x_2 has $n-1$ choices from B

$\vdots$

x_m has $n-(m-1)$ choices

Thus by rule of product

* no: of one-to-one functions from A to B is

$$n(n-1)(n-2)\cdots(n-m(-1))$$

$$= n(n-1)(n-2)\cdots(n-m+1)$$

$$= \frac{n!}{(n-m)!}$$

$$= P(n, m) = P(|B|, |A|)$$

$$= {}^{n}P_{m}$$

On-to function
(Surjective)

A function $f: A \to B$ is called onto, or surjective, if $f(A) = B$

ie; for all $b \in B$ there is

at least one $a \in A$ with $f(a) = b$

Example:

(1) If $A = \{1, 2, 3, 4\}$ and $B = \{x, y, z\}$

then $f_1 = \{(1, z)\ (2, y)\ (3, x)\ , (4, y)\}$

and $f_2 = \{(1, x), (2, x)\ (3, y), (4, z)\}$?

$g = \{(1, x), (2, x)\ (3, y), (4, y)\}$

Here f_1 is onto function

Since range of $f_1 = \{x, y, z\}$
$= B$

But f_1 is not one-to-one function

Since $f_1(2) = f_1(4) = y$

But $2 \neq 4$.

Also, f_2 is onto function.

Since $f_2(A) = B$

f_2 is not one-to-one function

$f_2(1) = x = f_2(2)$

But $1 \neq 2$

g is not onto since

$g(A) = \{x, y\} \neq B$

$g(1) = g(2) = x$ But $1 \neq 2$

$\therefore g$ is not one-to-one

---

(2) Check whether the following functions are On-to.

a) $f: R \to R$ defined by

$$f(x) = x^3.$$

$$\text{Range of } f = \{ f(x) / x \in \text{domain of } f \}$$

$$= \{ f(x) / x \in R \} = R$$

$\therefore$ for any $x \in$ codomain of f

there exist an element in R

ie, every element has got a pre-image

$\Rightarrow$ Range$(f) = R$ itself

$\therefore f$ is On-to function

b) $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$

$$\text{Range of } f = \{ f(x) / x \in \mathbb{R} \}$$
$$= \{ x^2 / x \in \mathbb{R} \}$$
$$= \{ \text{positive real numbers} \}$$
$$\neq \mathbb{R}$$

$\therefore f$ is not onto function

c) $f: \mathbb{Z} \to \mathbb{Z}$ where $f(x) = 3x+1$ for each $x \in \mathbb{Z}$.

$$\text{Range of } f = \{ f(x) / x \in \mathbb{Z} \}$$
$$= \{ 3x+1 / x \in \mathbb{Z} \}$$
$$= \{ 3x+1 / \{ \ldots, -2, -1, 0, 1, 2, \ldots \} \}$$
$$= \{ \ldots, -5, -2, 1, 4, \ldots \}$$
$$\neq \mathbb{Z}$$

$\therefore f$ is not on-to function

d) $g : Q \to Q$ where $g(x) = 3x+1$, $x \in Q$

Range $g = \{ g(x) / x \in Q \}$

$= \{ g(x) / x \in Q \}$

$= \{ 3x+1 / x \in Q \}$

$= Q$

$\therefore g$ is onto function

e) $h : \mathbb{R} \to \mathbb{R}$ where $h(x) = 3x+1 / x \in \mathbb{R}$

Range of $h = \{ h(x) / 3x+1, x \in \mathbb{R} \}$

$= \mathbb{R}$

$\therefore h$ is onto function

(8) If $A = \{x, y, z\}$ and $B = \{1, 2\}$
Show that constant function $f_1 : A \to B$
defined by $f_1 = \{(x,1), (y,1), (z,1)\}$ or
$f_2 : A \to B$ defined by $f_2 = \{(x,2), (y,2), (z,2)\}$ are not onto functions?

$$\text{Range of } f_1 = \{f_1(x) / x \in A\}$$
$$= \{1\} \neq B$$

$$\text{Range of } f_2 = \{f_2(x) / x \in A\}$$
$$= \{2\} \neq B$$

Since Range of $f_1 \neq B \neq$ Range of f_2

f_1 and f_2 are not onto functions

NOTE

For finite sets A, B with $|A| = m$ and $|B| = n$ there are

$$\sum_{k=0}^{n} (-1)^k \binom{n}{n-k} (n-k)^m$$ onto functions from A to B.

Maximal - Minimal element

If (A,R) is a poset then an element $x \in A$ is called a maximal element of A if for all $a \in A$, $a \neq x \Rightarrow x \not R a$

An element $y \in A$ is called a minimal element of A. If whenever $b \in A$ and $b \neq y \Rightarrow b \not R y$

Example :

① Let $U = \{1, 2, 3\}$ and $A = P(U)$

Let R be the subset relation on A. Clearly (A, R) is a poset

Here $\{1, 2, 3\}$ is the maximal element and

ϕ is the minimal element

② Consider the poset $(Z, \leq)$

here there is no minimal element as well as no maximal element exist

③ Consider the poset $(N, \leq)$

Minimal element is 1

and maximal element does not exist

④ Let $U = \{1, 2, 4, 8\}$ and R be the relation "exactly divides" defined in U

(U, R) is a poset (verify ?)

Hasse diagram

Minimal element = 1

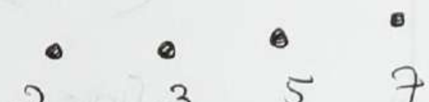
Maximal element = 8

(5) Let $A = \{2, 3, 5, 7\}$ and R be the relation exactly divides.

Then (A, R) will be a poset (Verify?)

Since $R = \{(2,2), (3,3), (5,5), (7,7)\}$

Hasse diagram

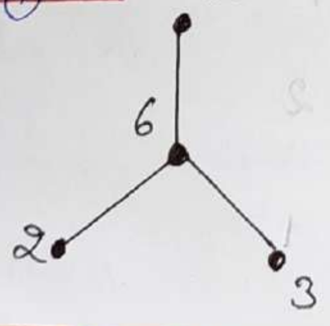


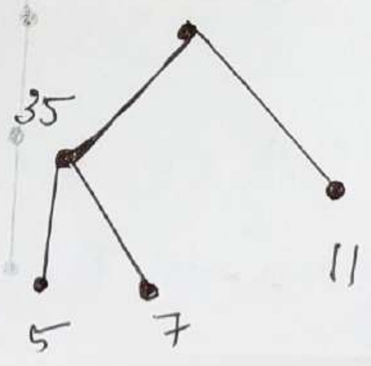
Here maximal elements $\rightarrow$ 2, 3, 5, 7

Minimal elements $\rightarrow$ 2, 3, 5, 7

(6) Let $B = \{2, 3, 5, 6, 7, 11, 12, 35, 385\}$ with relation R as "exactly divides" then (B, R) is a poset (Verify?)

Hasse diagram





Maximal elements = 12, 385

Minimal elements = 2, 3, 5, 7, 11

Least element and greatest element

If (A, R) is a poset, then an element $x \in A$ is called a least element if $x R a \ \forall a \in A$.

Element $y \in A$ is called a greatest element if $a R y \ \forall a \in A$

Example

1. Let $X = \{1, 2, 3\}$, then $(P(X), \subseteq)$ is a poset.

Here greatest element $= \{1, 2, 3\}$

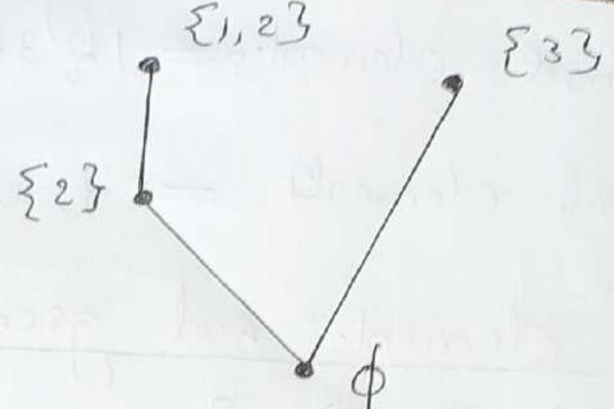
least element $= \phi$.

2. Let $A = \{\phi, \{1, 2\}, \{2\}, \{3\}\}$

then $(A, \subseteq)$ is a poset

Here least element $= \phi$

But no greatest element exist

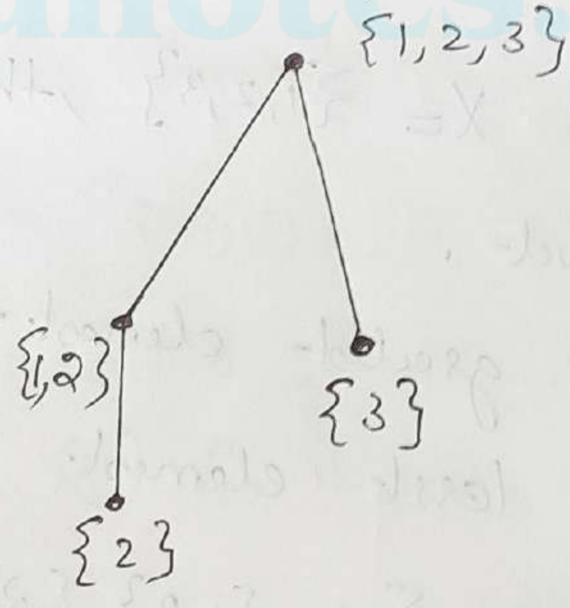


Maximal elements = {1,2}, {3}

Minimal element = ϕ.

3 Let $B = \{ \{1,2\}, \{2\}, \{3\}, \{1,2,3\} \}$

$(B, \subseteq)$ is a poset



Maximal element = {1,2,3} = greatest element

Minimal element = {2}, {3}

least element does not exist

greatest element $\{1,2,3\}$

4. Let $A = \{2, 3, 5, 7\}$ and

$R = \{(2,2), (3,3), (5,5), (7,7)\}$

No least element exist

No greatest element exist

Lower bound and upper bound

Let (A, R) be a poset with $B \subseteq A$

An element $x \in A$ is called a lower bound of B if $x R b \ \forall b \in B$

$x \leq b \ \forall b$

An element $y \in A$ is called upper bound of B if $b R y \ \forall b \in B$

$b \leq y \ \forall b$

Greatest lowerbound and least upper bound

An element $x' \in A$ is called a greatest lower bound (glb) of B if

It is a lower bound of B and if for all other lower bounds x'' of B we have $x'' R x'$.

$y' \in A$ is a least upper bound (lub) of B if it is an upper bound of B and if $y' R y''$ for all other upper bounds y'' of B

Example

Let R be the "less than or equal to" relation for the poset (A, R).

(1) * If $A = \mathbb{R}$, set of all real numbers

$$B = [0, 1]$$

then B has glb 0 and least upper bound 1 $\boxed{0, 1 \in \mathbb{R}}$

(2) * If $A = \mathbb{R}$ and $C = (0, 1]$

Here C has greatest lower bound 0

but $0 \notin C$

and least upper bound 1, $1 \in C$

$0, 1 \in A = \mathbb{R}$

(3)* Let $A = \mathbb{R}$ and

$B = \{q \in Q \mid q^2 < 2\}$

Find glb and lub?

Here $B = \{q \in Q \mid q^2 < 2\}$

$= \{q \in Q \mid -\sqrt{2} < q < \sqrt{2}\}$

$\therefore$ glb of $B = -\sqrt{2}$

lub of $B = \sqrt{2}$ $\Big\} \in A = \mathbb{R}$

But $\sqrt{2}, -\sqrt{2} \notin B$

(4)* Let $A = Q$

$B = \{q \in Q \mid q^2 < 2\}$

$= \{q \in Q \mid -\sqrt{2} < q < \sqrt{2}\}$

Here B has no glb or lub since $A = Q$ here and $\boxed{\sqrt{2}, -\sqrt{2} \notin Q}$

5. Let $U = \{1, 2, 3, 4\}$ with $A = P(U)$ and let R be the subset relation on A. If $B = \{\{1\}, \{2\}, \{1,2\}\}$ then

Upper bounds of $B = \{1,2\}, \{1,2,3\}, \{1,2,4\}, \{1,2,3,4\}$

ie. those subsets (in $P(U)$) which elements in B as subset

lower bounds of $B = \phi$

Now least upper bound $= \{1,2\}$

Greatest lower bound $= \phi$

Here l.u.b $\in B$ but g.l.b $\notin B$

HW Q Let $A = \{1, 2, 3, 4, 6, 8, 12\}$ and R be the partial ordering on A defined by aRb if "a divides b". Determine the Hasse diagram of the poset (A, R)

6. Give an example of a poset with four maximal elements but no greatest element?

Let $U = \{1, 2, 3, 4\}$

A = Collection of all proper subsets of U

R be the relation '$\subseteq$'

then (A, R) will be a poset (verify)

Thus $A = \{ \{1,2,3\}, \{1,3,4\} \{1,2,4\}, \{2,3,4\}$

$\{1,2\}, \{1,3\}, \{1,4\}$

$\{2,3\}, \{2,4\}, \{3,4\}$

$\{1\}, \{2\}, \{3\}, \{4\}, \phi \}$

Maximal elements are $\{1,2,3\}, \{1,3,4\}$,

$\{1,2,4\}$ and $\{2,3,4\}$

Here no greatest element exist

Equivalence relations and partitions

Partition

Given a set A and index set I, let $\phi \neq A_i \subseteq A \ \forall i \in I$, Then $\{A_i\}_{i \in I}$ is a partition of A if

(a) $A = \bigcup_{i \in I} A_i$ and

(b) $A_i \cap A_j = \phi \ \forall i, j \in I$ where $i \neq j$

Each subset A_i is called a cell or block of the partition.

Example:

Let $A = \{1, 2, 3, \ldots, 10\}$ then

(1) $A_1 = \{1, 2, 3, 4, 5\}$

$A_2 = \{6, 7, 8, 9, 10\}$

here $A_1 \cup A_2 = A$ and

$A_1 \cap A_2 = \phi$ also $A_i \neq \phi$

$\therefore$ A_1 and A_2 are partitions of A

(2) $A_1 = \{1, 2, 3\}$ $A_2 = \{4, 5, 9\}$, $A_3 = \{6, 7, 8, 10\}$

$A_1 \cup A_2 \cup A_3 = A$

$A_1 \cap A_2 = A_2 \cap A_3 = A_1 \cap A_3 = \phi$

$A_1, A_2, A_3 \neq \phi$

$\therefore$ A_1, A_2, A_3 is another partition of the set A

(3) $A_1 = \{1, 2, 3, 4, 5\}$ $A_2 = \{5, 6, 7, 8, 9, 10\}$

Here $A_1 \cup A_2 = A$ but

$A_1 \cap A_2 = \{5\} \neq \phi$ $\therefore$ this is not a partition of the set A.

(4) Let $A_i = \{i, i+5\}$ $1 \leq i \leq 5$

is also a partition of A

Since here $A_1 = \{1, 6\}$

$A_2 = \{2, 7\}$

$A_3 = \{3, 8\}$

$A_4 = \{4, 9\}$

$A_5 = \{5, 10\}$

(5) Let $A = \mathbb{R}$ and for each $i \in \mathbb{Z}$

let $A_i = [i, i+1)$ then

$\{A_i\}_{i \in \mathbb{Z}}$ will be a partition

of the set $\mathbb{R}$

NOTE

A relation R is called equivalence relation if R is reflexive, symmetric and transitive.

Equivalence Class

Let R be an equivalence relation on a set A for each $x \in A$, the equivalence class of x denoted by $[x]$ is defined as $[x] = \{ y \in A \mid y R x \}$

RESULT

If R is an equivalence relation on a set A, and $x, y \in A$ then

a) $x \in [x]$

b) $x R y \iff [x] = [y]$

c) $[x] = [y]$ or $[x] \cap [y] = \phi$

---

* If A is a set, then

a) Any equivalence relation R on A induces a partition of A and

b) Any partition of A gives rise to an equivalence relation R on A.

* For any set A, there is a one-to-one correspondence between the set of equivalence relations on A and the set of partitions of A.

Example

1. Let $A = \{1, 2, 3, 4, 5\}$ and

$$R = \{(1,1), (2,2), (2,3), (3,2), (3,3), (4,4), (4,5), (5,4), (5,5)\}$$

Here R is reflexive, symmetric and antisymmetric (Verify)

$\Longrightarrow$ R is an equivalence relation on A

Here $[1] = \{1\}$

$[2] = \{2, 3\}$ (2,2), (3,2)

$[3] = \{3, 2\}$ (3,3), (2,3)

$[4] = \{4, 5\}$

$[5] = \{5, 4\}$

Thus $[1] \cup [2] \cup [4] = A$

also $[1] \cap [2] = [2] \cap [4]$

$= [1] \cap [4] = \phi$

$\therefore$ $[1], [2], [4]$ is a partition of A

6. Let $A = \{1, 2, 3, 4, 5, \ldots, 11, 12\}$ and let R be the equivalence relation on $A \times A$ defined by

$(a, b) \, R \, (c, d) \iff a + d = b + c$

Prove that R is an equivalence relation and find the equivalence class of $(2, 5)$

R is reflexive

Clearly $a+b = b+a$

$\Longrightarrow (a,b) \, R \, (a,b) \quad \forall (a,b) \in R$

$\therefore$ R is reflexive

R is symmetric

Let $(a,b) \, R \, (c,d)$ in R.

$\Longrightarrow a+d = b+c$

$\Longrightarrow b+c = a+d$. (Commutativity of addition)

$\Longrightarrow c+b = d+a$

$\Longrightarrow (c,d) \, R \, (a,b)$ in R

$\therefore$ R is symmetric.

R is transitive

Let $(a,b) \, R \, (c,d)$ and $(c,d) \, R \, (e,f)$

$\Longrightarrow a+d = b+c$ and $c+f = d+e$

Thus $a+d = b+c$ —— ①

$c+f = d+e$ —— ②

① + ②

$\Rightarrow a + \cancel{d} + \cancel{c} + f = b + \cancel{c} + \cancel{d} + e$

$\Rightarrow a + f = b + e$

$\Rightarrow (a, b) \, R \, (e, f)$

$\Rightarrow R$ is transitive

Thus R is an equivalence relation

Now $[(2,5)] = \{ (x,y) \in A \times A \, / \, (x,y) \, R \, (2,5) \}$

$= \{ (x,y) \in A \times A \, / \, x + 5 = y + 2 \}$

where $x, y \in \{1, 2, 3, \dots, 12\}$

$[(2,5)] = \{ (x,y) \in A \times A \, / \, x - y = -3 \}$

$= \{(x,y) \in A \times A \,/\, x = y-3\}$

$= \{(1,4), (2,5), (3,6), (4,7), (5,8)$

$(6,9), (7,10), (8,11), (9,12)\}$

7. Let Z be the set of all integers R is a relation called "congruence modulo 3" defined by $R = \{(x,y) / x, y \in Z$ and $x-y$ is divisible by $3\}$. Show that R is an equivalence relation..? Determine the equivalence classes and partition of A induced by R?

Here $R = \{(x,y) / x-y = 3k,\ x,y \in Z,\ k \in Z\}$

clearly $(x,x) \in R$

since $x-x=0$ is divisible by 3

$\forall x \in Z$ $\therefore R$ is reflexive relation

Let $(x, y) \in R$

$\Longrightarrow x - y = 3k_1 \quad k_1 \in Z$

$\Longrightarrow y - x = -3k_1$ ie; $(y-x)$ is also divisible by 3

$\Longrightarrow (y, x) \in R$

$\Longrightarrow$ <u>R is symmetric</u>

Let $(x, y), (y, z) \in R$

$\Longrightarrow (x - y) = 3k_1$

$(y - z) = 3k_2 \quad k_1, k_2 \in Z$

Thus adding both equations

$x - y + y - z = 3k_1 + 3k_2$

$\Longrightarrow x - z = 3[k_3]$ let $k_1 + k_2 = k_3$

$\Longrightarrow (x, z) \in R$

$\therefore R$ is transitive

Thus R is an equivalence relation

Now $[0] = \{y \in Z \mid y R 0\}$

$= \{y \in Z \mid (y, 0) \in R\}$

$= \{y \in Z \mid y - 0 = 3k,\ k \in Z\}$

$= \{y \in Z \mid y = 3k,\ k \in Z\}$

$= \{3k \mid k \in Z\}$

$[1] = \{y \in Z \mid (y, 1) \in R\}$

$= \{y \in Z \mid y - 1 = 3k,\ k \in Z\}$

$= \{y \in Z \mid y = 3k + 1,\ k \in Z\}$

$= \{y = 3k + 1,\ k \in Z\}$

$[2] = \{y \in Z \mid (y, 2) \in R\}$

$= \{y \in Z \mid y - 2 = 3k,\ k \in Z\}$

$= \{y \in Z \mid y = 3k + 2,\ k \in Z\}$

$= \{y = 3k + 2 \mid k \in Z\}$

$[3] = \{y = 3k + 3,\ k \in Z\}$

$= \{ y = 3(k+1), k \in Z \}$

$= \{ y = 3k_1, k_1 \in Z \}$

$= [0]$ and so on

Thus $Z = [0] \cup [1] \cup [2]$

Thus $\{[0], [1], [2]\}$ provides a partition of Z under the relation "Congruence modulo 3"

HW 8. Define the relation R on Z by $x R y$ if $4 / (x-y)$. Show that R is equivalence relation and obtain a partition for Z under this relation.

9. Define a relation R on the set $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

by $R = \{(x,y) / x-y$ is a multiple of $5\}$
Show that R is an equivalence relation on A? Determine the equivalence classes and partition of A induced by R?

10 Define the relation R on the set Z by $R = \{(a,b) / a^2 = b^2 ; a,b \in Z\}$
ST R is equivalence relation. Obtain the equivalence classes? Determine a partition for Z under this relation?

Here, $a^2 = a^2 \ \forall a \in Z$

$\Longrightarrow (a,a) \in R \ \forall a \in Z$

$\Longrightarrow$ R is reflexive.

Now $(a,b) \in R$

$\Longrightarrow a^2 = b^2$

$\Longrightarrow b^2 = a^2$

$\Longrightarrow (b,a) \in R \quad \forall (a,b) \in R$

$\therefore$ R is Symmetric

Let $(a,b), (b,d) \in R$

$\Longrightarrow a^2 = b^2$ and $b^2 = d^2$

$\Longrightarrow a^2 = d^2$

$\Longrightarrow (a,d) \in R$

$\therefore R$ is transitive

Thus R is an equivalence relation on Z

Now $[0] = \{0\}$

$[1] = \{y \in Z / yR1\}$

$= \{y \in Z / (y,1) \in R\}$

$= \{y \in Z / y^2 = 1\}$

$= \{1, -1\}$

$[2] = \{y \in Z / (y,2) \in R\}$

$= \{y \in Z / y^2 = 2^2\}$

$= \{y \in Z / y^2 = 4\}$

$= \{2, -2\}$

$[3] = \{3, -3\}$

$\vdots$

$[n] = \{n, -n\} \quad n \in \mathbb{Z}^+$

Clearly $Z = \bigcup_{n=0}^{\infty} [n] = \bigcup_{n \in N} [n]$

also intersection any two such sets will be ϕ.

Thus $\{[n], n \in Z^+\}$ is the partition of Z induced by the given relation R

Lattice

A lattice is a partially ordered set (A, R) in which for every pair of element $a, b \in A$, the least upper bound and greatest lower bound ie, lub $\{a, b\}$ and glb $\{a, b\}$ both exists in A

NOTE:

* lub $\{a, b\}$ is denoted by $a \vee b$ or $a \oplus b$ or $a + b$ called the join or sum of 'a' and 'b'
* glb $\{a, b\}$ is denoted by $a \wedge b$ or $a * b$, or $a \cdot b$ or ab called the meet or product of 'a' and 'b'
* All totally ordered set is trivially a lattice.
* All posets are not lattices.
* l.u.b = least element of the set of upper bound
  g.lb = greatest element of the set of lower bound

Example.

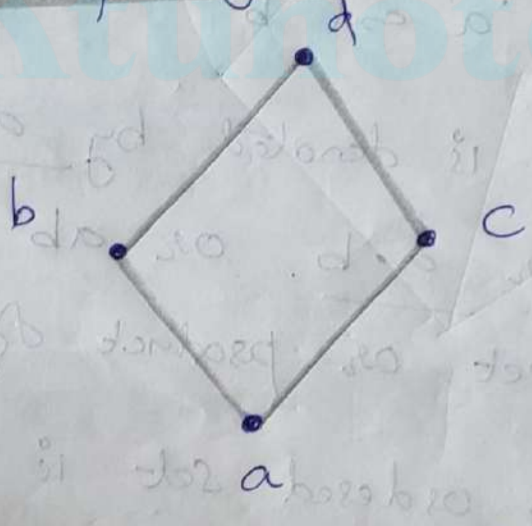
1. Let $A = \{a, b, c, d\}$

$R = \{(a,b), (a,d), (a,c), (b,d), (c,d), (a,a), (b,b), (c,c), (d,d), (a,d)\}$

Here (A, R) is a poset. (Verify)

Now to check whether (A, R) is a lattice or not!

Corresponding Hasse diagram



$gl \cdot b\{b, c\} = a \in A$

$l \cdot u \cdot b\{b, c\} = d \in A$

$glb\{b, a\} = a$

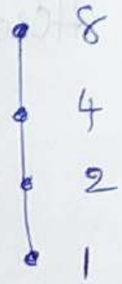
$l \cdot u \cdot b\{b, a\} = b$

similarly we can define glb and l.u.b for any pair of elements

∴ Here (A, R) is a lattice.

2. Let $B = \{1, 2, 4, 8\}$ with relation

$R = \{(x, y) / x \text{ divides } y\}$



g.l.b $\{1, 2\} = 1$

l.u.b $\{1, 2\} = 2$

glb $\{2, 8\} = 2$

l.u.b $\{2, 8\} = 8$

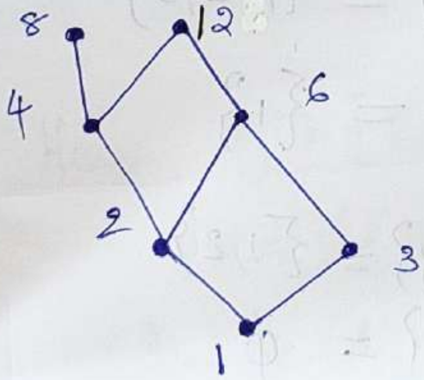
Similarly g.l.b and l.u.b exist for all pairs of elements in B

∴ (B, R) is a lattice.

3. $S = \{1, 2, 3, 4, 6, 8, 12\}$

Let the partial order be the division

ie, $R = \{(x, y) / x \text{ divides } y\}$



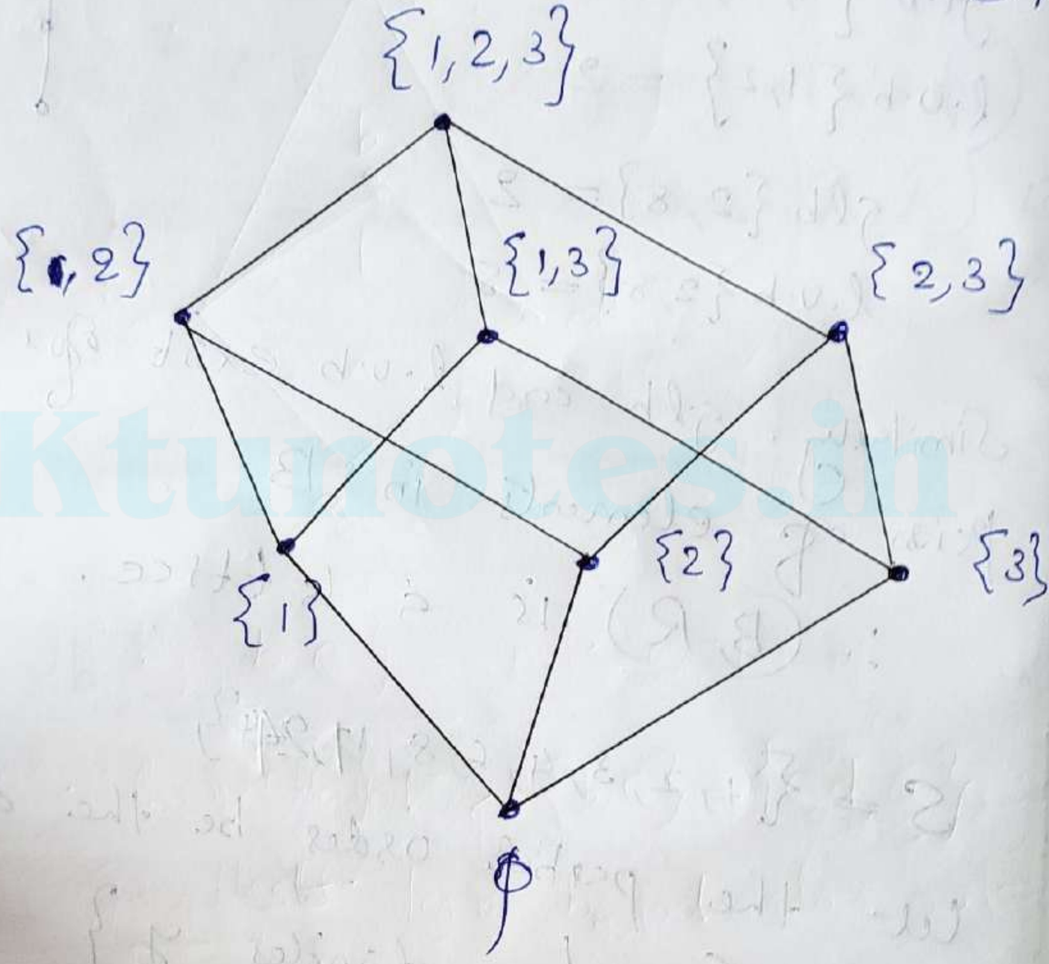
Here glb $\{8, 12\} = 4$

l.u.b $\{8, 12\}$ = does not exist

∴ (S, R) is not a lattice

4. Let $S = \{a, b, c\}$ with the poset $(P(S), \subseteq)$.

Soln: Here $R = \{(A, B) / A \subseteq B\}$ where $A, B \in P(S)$



$l.u.b \{\{1,2\}, \{1,3\}\} = \{1,2,3\}$

$g.l.b \{\{1,2\}, \{1,3\}\} = \{1\}$

$l.u.b \{\{1\}, \{2\}\} = \{1,2\}$

$g.l.b \{\{1\}, \{2\}\} = \phi$

glb $\{\{1,3\}, \{2\}\} = \phi$

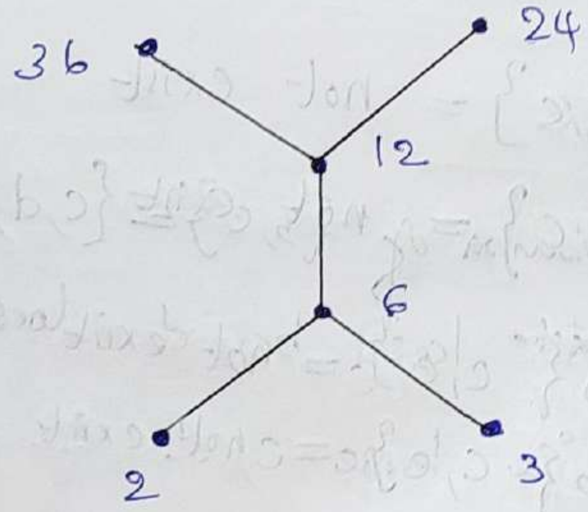
l.u.b $\{\{1,3\}, \{2\}\} = \{1,2,3\}$

Similarly for any pair of elements, we can define l.u.b and glb in $P(S)$

$\Longrightarrow (P(S), \subseteq)$ is a lattice

5. Let $A = \{2, 3, 6, 12, 24, 36\}$ let

$R = \{(x,y) / x \text{ divides } y\}$



Here glb $\{2,3\}$ does not exist [also

l.u.b $\{36,24\}$ does not exist]

$\therefore (A, R)$ is not a lattice

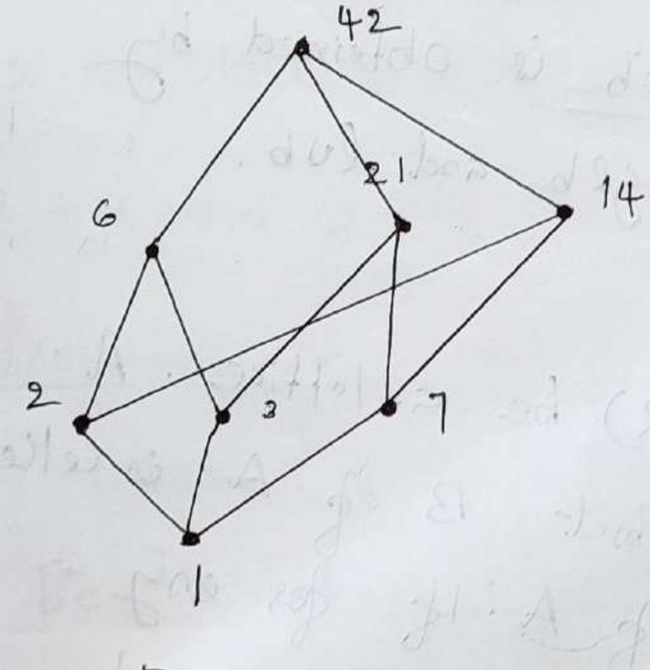
7. Let n be a positive integer and D_n be the set of all positive divisors of n. The poset $(D_n, /)$ is a lattice with respect to the relation of divisibility "/". Check this for n = 42 and n = 35?

when n = 42

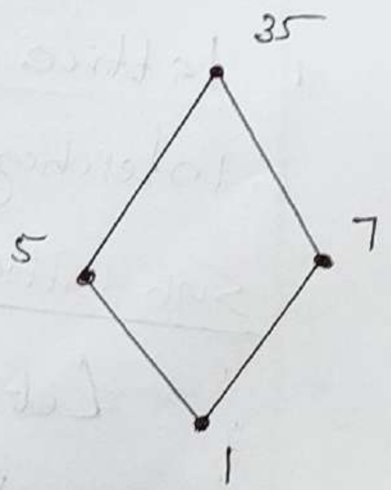
$D_n = \{1, 2, 3, 6, 7, 14, 21, 42\}$

when n = 35

$D_n = \{1, 5, 7, 35\}$



D_{42}



D_{35}

For any two pair of elements

$a, b \in D_n$

glb $\{a, b\}$ and l.u.b $\{a, b\}$ exist

in D_n itself $\therefore$ $(D_{35}, /)$ and

$(D_{42}, /)$ are lattices

Dual or reversed lattice

If $(A, \leq) = [A, \vee, \wedge]$ is

a lattice then $(A, \geq) = [A, \wedge, \vee]$

is known as dual or reversed

lattice, which is obtained by

interchanging glb and lub.

Sub lattice

Let (A, R) be a lattice. A

non empty subset B of A is called

a sublattice of A if for any

$a, b \in B$, $a \vee b$ and $a \wedge b$

exist in B.

Example

1. Find all sublattices of D_{24} that contains atleast 5 elements. Give examples of posets which are not sublattices with respect to D_{24}. (Pg 7.82) 392

$D_{24} = \{1, 2, 3, 6, 4, 8, 12, 24\}$

Properties of glb and lub

$\underbrace{\quad}_{glb} \; x, y \; \underbrace{\quad}_{l.ub}$

1. $glb(x, y) \leq x$ and $glb(x, y) \leq y$

2. $m \leq x$ and $m \leq y \implies m \leq glb(x, y)$

3. $x \leq lub(x, y)$ and $y \leq lub(x, y)$

4. $x \leq u$ and $y \leq u \implies lub(x, y) \leq u$

Properties of Lattices

Let $(A, R) = (A, \leq) = [A, \underset{glb}{\cdot}, \underset{lub}{+}]$ be a lattice for any $x, y \in A$ then

I $\left.\begin{aligned} x + x &= x \\ x \cdot x &= x \end{aligned}\right\}$ Idempotent

II

$$x + y = y + x$$

$$x \cdot y = y \cdot x$$

Commutative

III

$$x + (y + z) = (x + y) + z$$

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z$$

Associative

IV

$$x + (x \cdot y) = x$$

$$x \cdot (x + y) = x$$

Absorption

Some special lattices

* Complete lattice

A lattice is called complete if each of its non empty subsets have a lub and glb.

The greatest element of a lattice, if it exists, is denoted by I and is known as unit element.

The least element, if it exists

is denoted by **0** and is known as zero element.

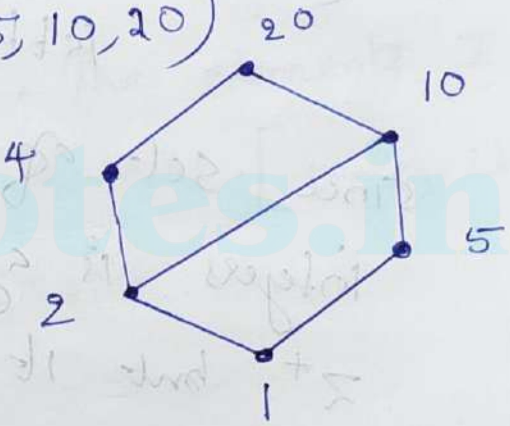
Every complete lattice must have I and **0**.

Example.

1. Every finite lattice is complete

Consider, $(D_{20}, /)$

$D_{20} = \{1, 2, 4, 5, 10, 20\}$

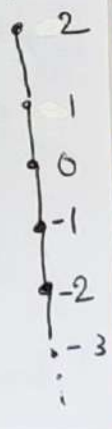


$(D_{20}, /)$ is a lattice, is a complete lattice.

Since we can find glb and l.u.b for any subset of D_{20} in D_{20}.

2. Consider $(Z, \leq)$ which is a lattice.

But the subset of Z



Say the set of even positive integers $\{2, 4, 6, \ldots\}$

Here l.u.b does not exist

So $(Z, \leq)$ is not a complete lattice

3. $(Z^+, \leq)$ is not a complete lattice

Since set of all even positive integers is a subset of Z^+ but it does not posses l.u.b

4. Consider the lattice $(Q, \leq)$ is a lattice

$B = \{q \in Q / q^2 < 2\}$ be a subset of Q

$B = \{q \in Q / -\sqrt{2} < q < \sqrt{2}\}$

$glb(B) = -\sqrt{2}$ $\quad$ $l.u.b(B) = \sqrt{2}$

But $\sqrt{2}, -\sqrt{2} \notin B$

$\therefore$ glb as well as lub does not belongs to the subset B

$\Longrightarrow$ $(Q, \leq)$ is not a complete lattice.

Bounded lattice

A lattice is said to be bounded if it has a greatest element I and least element 0. The elements I and 0 are known as bounds [units, universal bounds] of the lattice.

Example:

1. The lattice $(P(A), \subseteq)$ is bounded lattice for any set A.

   Since here $I = P(A)$

   $0 = \phi$ both belongs to $P(A)$.

2. The lattice $(Z^+, /)$ is not bounded

   Since here $0 = 1$ (ie, least element)

but I, greatest element does not exist

3. The lattice $(\mathbb{Z}, \leq)$ is not bounded since here I & 0 does not exist

4. For any finite set A, (A, R) is bounded. Let $A = \{a_1, a_2, \ldots, a_n\}$

Here $I = a_1 + a_2 + \cdots + a_n = \text{l.u.b}$

$0 = a_1 a_2 \cdots a_n = \text{g.l.b}$

Always $0 \leq A \leq I$ for any $a \in A$.

NOTE:

$*$

$a + 0 = a$

$a \cdot 0 = 0$ $\quad + \rightarrow \text{l.u.b.}$

$a + I = I$ $\quad \cdot \rightarrow \text{glb}$

$a \cdot I = a$

Complemented lattice

A lattice is said to be complemented lattice if every element has at least one complement.

For a bounded lattice an element b is said to be a complement of 'a' if

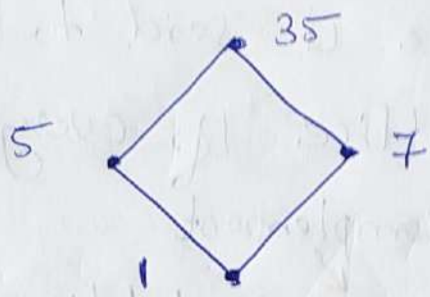
$a \cdot b = 0$ and $a + b = I$

* Complement is symmetric
  ie, a is complement of b if b is complement of a.
* Complement is not unique, need not exist
* Complement of "a" is denoted by a'.

Example.

Let $(P(A), \subseteq)$ will be lattice (with a finite set A) then for every element $B \in P(A)$ has a complement $A-B$ and Here the unit $I = A$ and $0 = \phi$

2. Let $B = \{1, 5, 7, 35\}$



Here $I = 35$ and $O = 1$
greatest elmt   least element

Let Complement of $5 = b$

$\therefore 5 \cdot b = 1$ and

$5 + b = 35$

ie, $glb(5, b) = 1$

$lub(5, b) = 35$

then $b = 7$

Thus 7 is complement of 5
$\Rightarrow$ 5 is complement of 7

Complement of $35 = x$

$\Rightarrow glb(35, x) = 1$

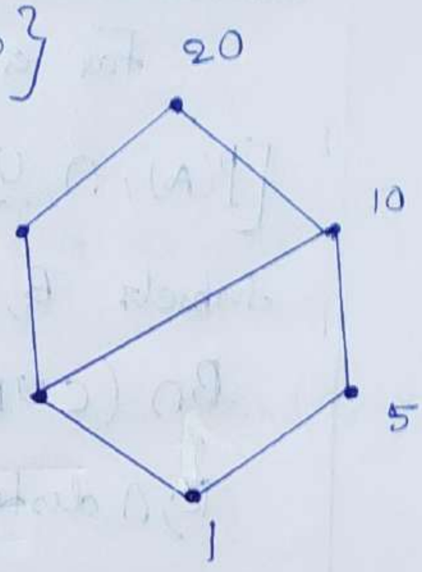
$lub(35, x) = 35$

$\Rightarrow x = 1$

Thus every element possess a complement
$\Rightarrow (B, /)$ is complemented lattice

3 Consider the lattice $(D_{20}, /)$

$D_{20} = \{1, 2, 4, 5, 10, 20\}$



Here $I = 20$ greatest elmt &

$0 = 1$ least element

To find

Complement of 2

Let x be the complement of 2

$\Longrightarrow$ $2 \cdot x = 1$

$2 + x = 20$

$\Longrightarrow$ $\left.\begin{array}{l} glb(2, x) = 1 \\ lub(2, x) = 20 \end{array}\right\}$ no such x exist

So complement of 2 does not exist

Illy for 10

$\left.\begin{array}{l} glb(10, y) = 1 \\ lub(10, y) = 20 \end{array}\right\}$ no such y exist

10 have no complement

$\therefore$ It is not a complemented lattice

Distributive lattice

For a non empty set A, the lattice $[P(A), \cap, \cup]$ is distributive. For any subsets B, C, D of A we have

$$B \cap (C \cup D) = (B \cap C) \cup (B \cap D)$$

ie, $\cap$ distributive over $\cup$ also

$$B \cup (C \cap D) = (B \cup C) \cap (B \cup D)$$

ie, $\cup$ is distributive over $\cap$.

Q: Show that lattice $(Z^+, \leq)$ is distributive

Under the partial order '$\leq$'

$$a+b = lub(a,b) = max\{a,b\}$$

$$a \cdot b = glb(a,b) = min(a,b)$$

Let $a, b \in Z^+$

then $a \cdot (b+c) = min\{a, (b+c)\}$

$= min\{a, max(b,c)\}$

$= max(min(a,b), min(a,c))$

$= max(a \cdot b, a \cdot c)$

$= a \cdot b + a \cdot c$

Similarly $a + (b \cdot c) = max\{a, min(b, c)\}$

$= min(max(a,b), max(a,c))$

$= min(a+b, a+c)$

$= (a+b) \cdot (a+c)$

Q. Let $A = \{1, 2, 3, 5, 30\}$ consider $(A, /)$ Show that

$\cdot$ is not distributive over $+$

(ie. Prove that there exist a, b, c in A such that

$a \cdot (b + c) \neq a \cdot b + a \cdot c$)

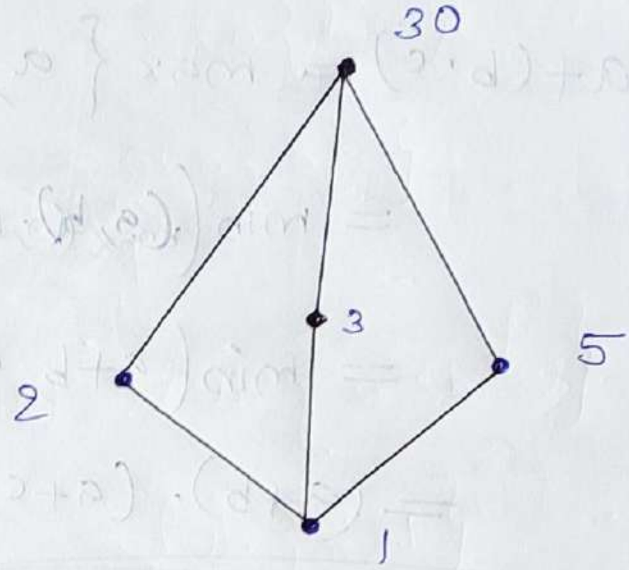
also prove that $+$ is not distributive over $\cdot$

(ie. prove that there exist a, b, c in A such that

$a + (b \cdot c) \neq (a+b) \cdot (a+c)$

Here (A, /) is a lattice.

$A = \{1, 2, 3, 5, 30\}$



Clearly glb(a, b) and lub(a, b) exist for every a, b ∈ A.

∴ (A, /) is a lattice.

Now to prove

· is not distributive over + (lub)

(g.l.b)

Let $a = 2$, $b = 3$, $c = 5$

need to prove

$a \cdot (b + c) \neq (a \cdot b) + a \cdot c$

$b+c = lub\{b, c\} = lub\{3, 5\} = 30$

$$a \cdot (b+c) = glb\{a, lub(b,c)\}$$

$$= glb\{2, 30\}$$

$$= 2$$

Now $a \cdot b = glb\{a, b\}$

$$= glb\{2, 3\}$$

$$= 1$$

$$a \cdot c = glb\{a, c\}$$

$$= glb\{2, 5\}$$

$$= 1$$

$(a \cdot b) + (a \cdot c) = lub\{1, 1\} = 1$

Thus $a \cdot (b+c) = 2 \neq 1 = (a \cdot b) + a \cdot c$

Thus lattice $(A, /)$ is not distributive

* To prove $+$ is not distributive over $\cdot$

Let $a = 2$ $b = 3$ $c = 5$

Need to prove

$$a+(b\cdot c) \neq (a+b)\cdot(a+c)$$

$$b\cdot c = glb\{b,c\}$$
$$= glb\{3,5\}$$
$$= 1$$

$$a+(b\cdot c) = lub\{a,(b\cdot c)\}$$
$$= lub\{2,1\}$$
$$= 2$$

$$a+b = lub(a,b)$$
$$= lub(2,3)$$
$$= 30$$

$$a+c = lub(a,c)$$
$$= lub(2,5)$$
$$= 30$$

Thus $(a+b)\cdot(a+c) = glb\{(a+b),(a+c)\}$

$$= glb\{30,30\} = 1$$

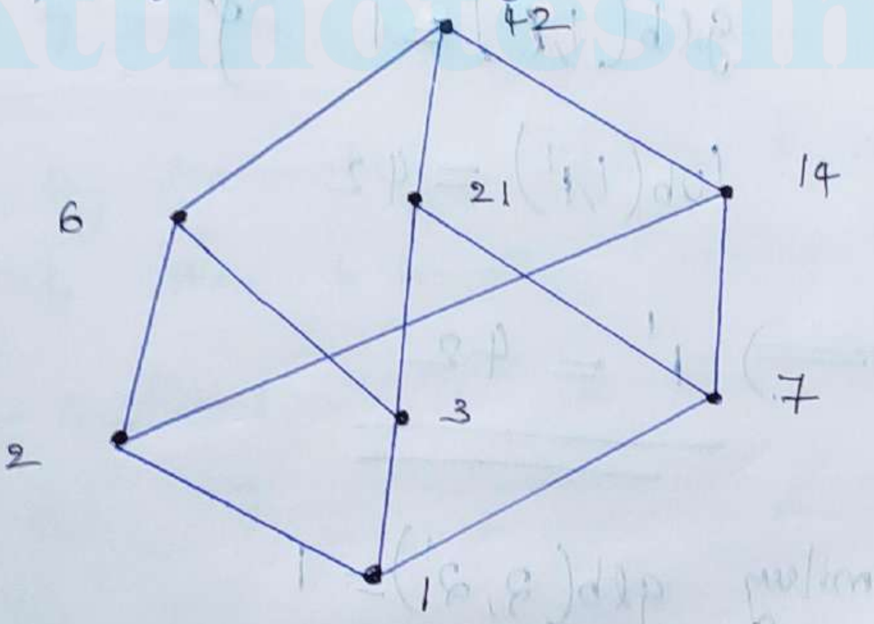
Thus $a+(b\cdot c) = a \neq 1 = (a+b)\cdot(a+c)$

Thus $(A, \wedge)$ is not distributive

Q: Find the complement of each element in D_{42}?

$D_{42} = \{1, 2, 3, 6, 7, 14, 21, 42\}$

Corresponding Hasse diagram



Clearly $0 = 1$ (least element)

$I = 42$ (greatest element)

Let a' denote the complement of a

ie, $a \cdot a' = 0$ (glb)

$a + a' = I$ (lub)

Let $a = 1$ and $1'$ be it complement

$\Rightarrow 1 \cdot 1' = glb(1, 1') = 0 = 1$

and $1 + 1' = lub(1, 1') = I = 42$

$glb(1, 1') = 1$ &

$lub(1, 1') = 42$

$\Rightarrow 1' = 42$

Similarly $glb(2, 2') = 1$

$lub(2, 2') = 42$

$\Rightarrow 2' = 21$

Now to find 3', we have

$$glb(3, 3') = 1$$

$$lub(3, 3') = 42$$

$$\Longrightarrow 3' = 14$$

To find 7'

$$glb(7, 7') = 1$$

$$lub(7, 7') = 42$$

$$\Longrightarrow 7' = 6$$

Now By symmetry, if a is complement of b then b is the complement of a

$$\Longrightarrow 42' = 1$$

$$21' = 2$$

$$14' = 3$$

$$6' = 7$$