S4

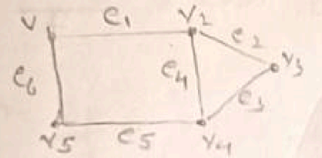
Graph theory

M1 simple graph : a graph without loop and parallel edges.

complete graph : a simple graph which there is an edge between each vertices. i.e $\frac{n(n-1)}{2}$.

Finite graph : a graph with finit number of edges and vertices

Incidence : when a vertices is on end vertex of some edge then the vertex and edge is called incident



v_1 & e_1 are incident

e_1 & e_5 are adjacent

v_4 & v_5 are adjacent

Degree of vertex : the number of edges connected to the vertex.

even vertex : a vertex with degree of even

odd vertex : " " " odd.

Regular graph : the degree of every vertex in a graph is same.

Note : Complete graph is always regular

Theorem 1

$\sum d(v_i) = 2e$.

Proof: explain with example

Theorem 2

The number of vertices of odd degree in a graph is even

proof: using $\sum d(v) = 2e$, i.e the odd degree 'e' is odd, $\therefore$ 2e will be a even number. or it can't be devided by 2.

Theorem 3:

A complete graph with n vertices has $\frac{n(n-1)}{2}$ edges.

Proof:

$$\sum d(v) = 2e$$

total number of edges $\sum d(v) = n(n-1)$

$$2e = n(n-1)$$
$$e = \frac{n(n-1)}{2}$$

Problems:

1. is it possible to have a group of 9 people, each knowing exactly 5 other.

A: 9 vertices 2 vertices are adjacent if they knew each other.

for any vertex v, d(v) is the number of friends given that $d(v) = 5 \ \forall v \in V$

it is not possible, because in any graph G, $\exists$ even number of odd values.

2. Is it possible to construct a simple graph with 12 vertices with 2 of them with $d(v)=1$, 3 of them $d(v)=3$ & 7 of them with $d(v)=10$.

Ans.

$$\sum d(v) = 1\times 2 + 3\times 3 + 7\times 10 = 81$$

$$2e = 81$$

$$e = \frac{81}{2}$$

$$= 40.5$$

$\therefore$ not possible

3. what is the largest number of vertices in a graph with 35 edges if all vertices are of degree at least 3

Ans:

$$\sum d(v) = 2\times 35$$

$$= 70$$

$$n = \frac{70}{3}$$

$$= 23$$

$$= \underbrace{3+3+\cdots+3}_{23\ \text{vertex}} + \underset{1\ \text{vertex}}{4}$$

$$= \underline{\underline{24}}$$

4. Is it possible to make a graph with 12 vertices which 2 of them $d(v)=3$ & other is 4.

$$d(v) = 2\times 3 + 10\times 4$$

$$= 46$$

$$2e = 46$$

$$e = 23$$

$\therefore$ possible

? Find the smallest 'n' such that K_n has at least 500 edges.

Ans: K_n - complete graph with n vertices

$\sum d(v) = e2$

$= 2 \times 500$

$= 1000$

$\underbrace{d + d + d \cdots + d}_{n \text{ times}} = 1000$

$d \times n = 1000$

$= \frac{1000}{2}$

$d = 500$

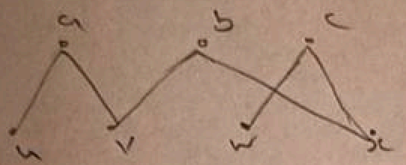
$n = 2$

? Let G be a n vertices and m edges. Assume that each vertex of G has degree either K or K+1. that is no. of vertices of degree K in G is $(K+1)n - 2m$

Bipartite Graph

The graph $G = (V, E)$ is bipartite is the vertex set V can be partitioned into X & Y ($X \cup Y = V$ & $X \cap Y = \phi$), such that each edge in E has one end in X and other in Y.

eg:

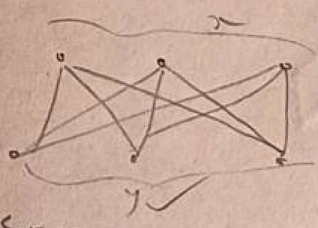


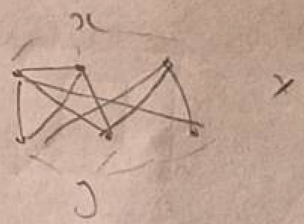
$V = \{a, b, c, u, v, w, x\}$

$X = \{a, b, c\}$, $Y = \{u, v, w, x\}$

complete bipartite graph

Vertices in x and y have edge between each other don't have edge to $x \to x$ & $y \to y$.





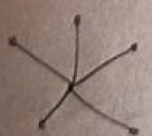
denoted by = K_{nm}, n & m is the no: of vertex

order = no. of vertices

size = no. of edges.

star graph

bipartite graph that K_{1m}



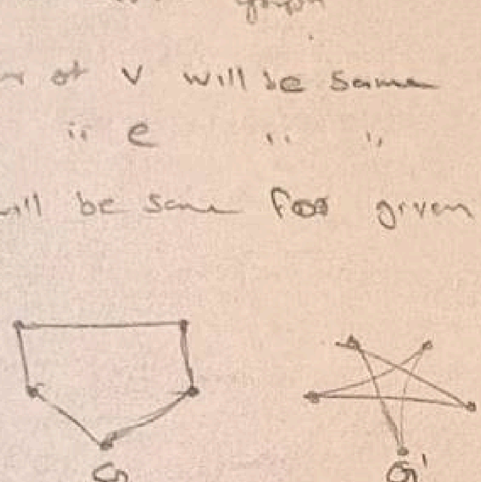
K_{15}

Graph Isomorphism

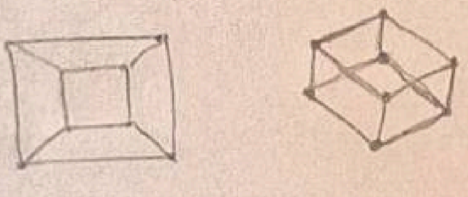
An edge 'e' is incident on vertices v_1 & v_2 in 'G', then the corresponding edge e' in G' must be incident on the vertices v_1' & v_2' that correspond to v_1 & v_2 respectively.

- for two isometric graph
- number of V will be same
- " " e " "
- d(v) will be same for given v.

eg:



eg:



not isometric

Sub-graphs

if a graph is said to be subgraph Gs of graph G if all the vertex and edges of Gs is present in G.

edge disjoint subgraph: a sub graph without common edge

vertex disjoint subgraph: a subgraph without common vertex walk : an alternative sequence of edges & vertices begin and end in a vertex , which each edge is incident to a vertex , no edges are appeared more than one .

terminal vertex : the ends of walk

path : a walk which no vertex is repeated

length : no: of edges .

circuit : a walk which start and end in same vertex .
or cycle
no vertex are repeated .

Connected and disconnected graph

A graph is said to be connected if there is atleast one path between every pair of vertices in G . other wise it is called disconnected .

A disconnected graph consist of two or more connected graph called components .

Euler line: is a closed walk in a graph which contains all the edges in the graph, exactly once.

Euler graph: a graph with euler line.

Theorem:

a given connected graph is a euler graph if and only if all the vertices of G are of even degree.

Fleury's Algorithm

to construct eluer tour in a Euler graph.

1. choos a vertex to start and end

2. select edge which incident to vi

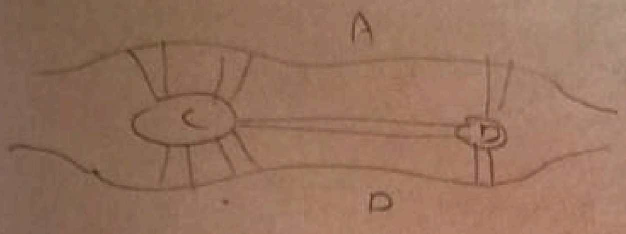
3. the next edge should be adjacent

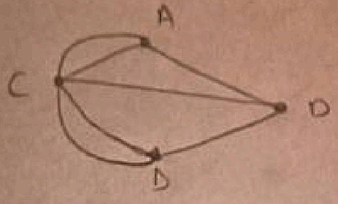
4. repeat until every edges are covered.

Theorem

A connected graph G is euler if and only if graph G can be decomposed into edge-disjoint cycle (G is the union of edge-disjoint cycles)

Konigsberg bridge problem





The degree of vertices in this odd, so can't find a

Operations on graph

1. Union
2. Intersection
3. Ringsum $(\oplus)$

$G = G_1 \oplus G_2$ $\quad V_1 \oplus V_2$, $E_1 \oplus E_2$ = edges which not in both graph

4. Decomposition

a graph G said to have be decomposed into two subg

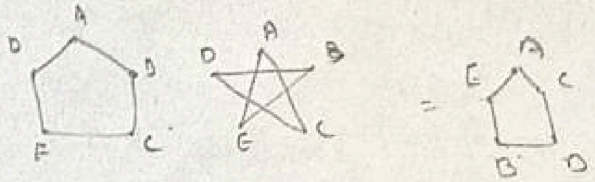
if $G = G_1 \cup G_2$ & $G_1 \cap G_2 = \phi$

Complement of graph

a complement of graph which is made by making a complete graph and removing the previous edges

Self complementary graph

A graph said to be self complementary if the complement of the graph is isomorphic.



Deletion

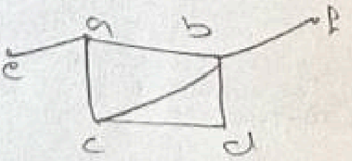
edge: if e is a edge which removed to create a subgraph.

vertex: if v is a vertex which removed to create a subgraph

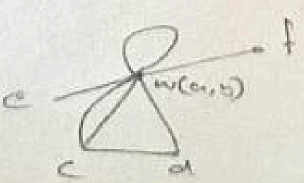
Fusion of vertices

A pair of vertex and edge are said to be fused if it is replaced by a vertex.

Eg:



fusion of a & b



Cut edge

An edge e of a graph is cut edge if G - e is disconnected

Cut vertex

an vertex v of a graph G is cut vertex if G-v is disconnected

Theorem

An edge e of a graph G is cut edge if and only if it is not in any cycle.

proof:

Hamiltonian graph

Hamiltonian path: path that visits each and every vertex of a graph once. A graph that contain Hamiltonian path is called hamiltonian graph.

Hamiltonian circuit

hamiltonian path which start and end in a same vertex.

Note

every complete graph is hamiltonian if $n \geq 3$

all cycle are hamiltonian

Theorem

If G is a simple graph with $n \geq 3$ and degree of every vertex in G be atleast $n/2$ Then G is Hamiltonian.

Proof

Let u & v are two non-adjacent vertices of G

Then $d(v) \geq n/2$ $d(u) \geq n/2$

$d(v) + d(u) = n/2 + n/2 = \underline{\underline{n}}$

$\therefore$ the graph is hamiltonian

Theorem

In a complete graph K_n, where $n \geq 3$ is odd, there are $(n-1)/2$ edge disjoint hamiltonian cycles.

weighted graph

the edges are assigned a real no: $w(e)$ called the weight

Directed graph:

graph with edge with direction

Degree in a Digraph

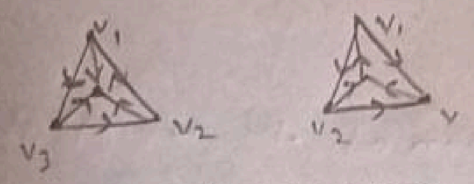
Indegree & outdegree

source : a vertex with zero in-degree

sink : a vertex with zero out degree

Isometric digraph.

Two digraphs are said to be isometric if and only if the direction of the corresponding edges must be same.



not isometric

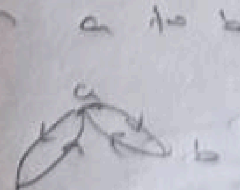
Asymmetric Digraph

Digraph that have atmost one directed edge b/w a pair of vertices but are allowed to have loops are called

$(v_1 v_2)$ ✓ $(v_2 v_1)$ ✗ ✓

Symmetric digraph

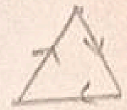
If a and b are vertices in a digraph then there will be edge from a to b & b to a



Complete symmetric digraph

$n(n-1)$ edges

Complete asymmetric digraph



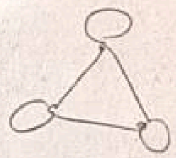
$\frac{n(n-1)}{2}$ edges

Balanced digraph

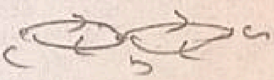
If each vertex, the sum of the weight of in-degree is equal to the sum of weight of out-degree

Digraph and binary relation

1. reflexive : each vertex with loop

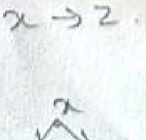


2. Symmetric : there is a edge between a to b & b to a



3. Transitive : A digraph G is said to be transitive if any 3 vertex (x, y, z) & edge from $x \to y$ & $y \to z$ then $x \to z$.

Eg:



or

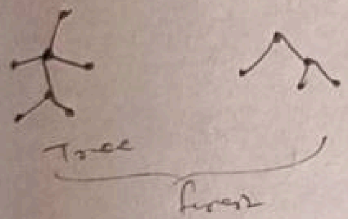
equivalence diagraph

a graph with thes three properties.

M-III

Tree: A graph G is called a tree, if it is connected and has no cycles,

Forest: A tree which disconnected graph with componets are tree called forest



Minimal connected graph: A graph is said to be minimally connected if of any edge disconnect the graph

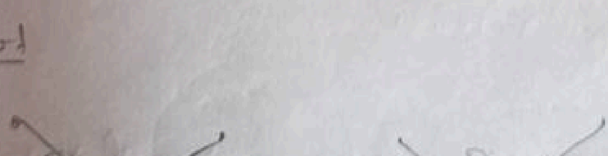
Distance in a tree: The number of in the shortest path

eccentricity of a vertex: $E(v) = \max d(v, v_i)$

Center: a vertex with minimum eccentricity

Theorem:
a tree has either one or two centers.

Proof



Radius: the eccentricity of a center in a tree

diameter: the longest path in a Tree

root of a tree: A tree in which one vertex is distinguished from all other vertices.

Path length of a tree

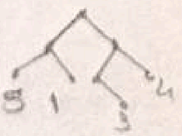
is the sum of length of pendant vertices

$P.L = 2+2+3+2 = 9$ //

weighted path length

If the pendant vertex have a value of +ve number

then $W.P.L = .5\times2 + 1\times2 + 3\times3 + 2\times4$



Binary tree

only one vertex with degree 2 and other with 3 or 1

1-level   2-level   3-level

a binary tree can't be made any even number of vertices.

Theorem

The no. of vertices in a binary tree is odd.

Proof

Let T be a binary tree, then the only vertex of T which has even degree is the root. we also have the result that the number of odd degree vertices in any graph is even.

Hence the number of vertices in T is odd.

Theorem

a binary tree on n vertices has $\frac{n+1}{2}$ pendant vertices.

Proof

p is the no. of pendant vertex in T, the number of vertices of degree 3 is $n-p-1$

$$\text{sum of degree} = 2e$$

$$2+p+(n-p-1)3 = 2(n-1)$$

$$2+p+3n-3p-3 = 2(n-1)$$

$$3n-2p-1 = 2(n-1)$$

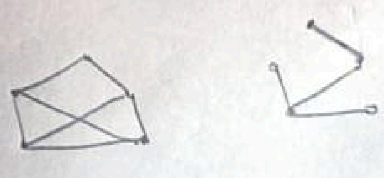
$$3n-2p-1 = 2n-2$$

$$3n-2n+2-1 = 2p$$

$$\frac{n+1}{2} = p$$

Spanning tree

a spanning tree of a connected graph G is a tree with all the vertices in the graph



* Branchs : an edge that present in the spanning tree
* chords : an edge which is not present in the spanning tree

Theorem

there is $n-1$ branchs and $e-n+1$ chords

Proof :

a tree is the minimal edge graph

$\therefore e = n-1$

$\therefore$ chord $= e-(n-1)$

$= e-n+1$

no. of branch = rank

no. of chord = Nullity

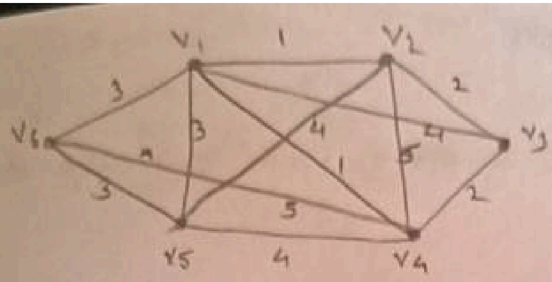
Fundemental circuit

A cycle formed in a graph G by adding a chord of a spanning tree T of G

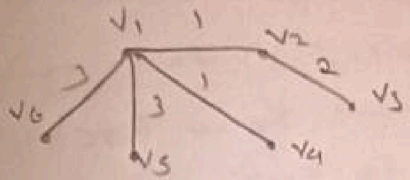
minimal spanning tree algorithm

a spanning tree with minimum weight in a weighted graph.

7.

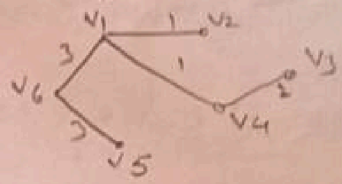


Kruskal's algorithm



min weight = 10

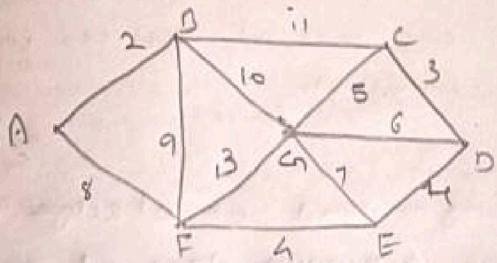
prim



=10

Dijkstra's Algorithm

To find the shortest path. A to D



Steps	A	B	C	D	E	F	G
7	0	α	α	α	α	α	α
	—	$\boxed{2}$	α	α	α	8	α
	—	—	13	α	α	$\boxed{8}$	12
	—	—	13	α	12	—	$\boxed{12}$
	—	—	13	18	$\boxed{12}$	—	—
	—	—	$\boxed{13}$	16	—	—	—
	—	—	—	$\boxed{16}$	—	—	—

Shortest path length = 16//

ABCD / AFED

M-4

Connectivity and planer graph

Cut edge: An edge 'e' of a graph G is a cut edge of G if G-e is disconnect.

Cut set: In a connected graph G, a cut-set is a set of edge whose removal from G leaves G disconnected, provided removal of no proper subset of these edge disconnect G.

Properties of cut-set

- every edge of a tree is cut-set
- every cut-set in a graph G must contain atleast one branch of every spanning tree of G

Proof: Let F be a cut-set in G and T be any spanning tree of G. If F does not contain any edge of T, then there will be a unique path between any pair of vertices in F and hence G-F, i.e. G-F is connected.

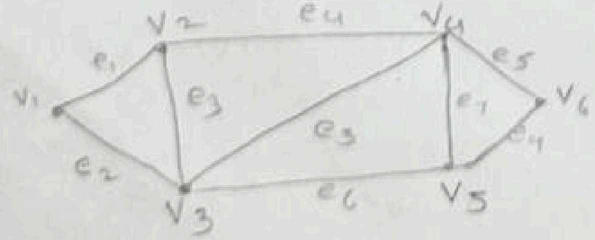
- In any connected graph G, any minimal set of edge consisting of atleast one, branch of every spanning tree of G is a cut set.

- every circuit has even number of edge in common with any cut-set

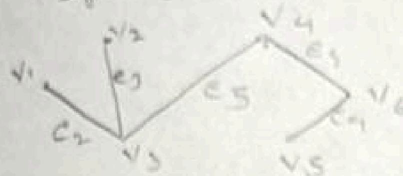
Fundamental cut-set

Let T be a spanning tree of a connected graph G. A cut set F of G containing exactly one branch of T is called a fundamental cut-set of G w.r.t T.

Eg:



Ans: Spanning tree of G



fundamental cut set

$\{e_1, e_3, e_4\}$

$\{e_1, e_2\}$ $\{e_4, e_5, e_6\}$

$\{e_6, e_7, e_4\}$ $\{e_7, e_1, e_6\}$

Result:

every connected graph of order n has $(n-1)$ distinct fundamental cut-set corresponding to any spanning tree of G

Vertex-cut (cut-vertices)

A subset W of the vertex set V of a graph G is said to be a vertex cut of G if $G-W$ is disconnected.

Theorem

every internal vertex of a tree is a cut vertex.

Proof

An internal vertex of a tree is a vertex with degree greater than or equal to 2.

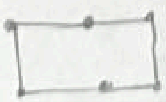
Let 'v' be an internal vertex, there will be a unique path b/w its neighbors $\therefore$ $T - v$ = disconnected graph

Theorem

every connected graph on three or more vertices has at least two vertices which are not cut-vertices. (pendent vertices)

Connectivity in Graphs

Separable graph:- A connected graph if it has a cut-vertex. A graph which is not separable is called non-separable graph



non-separable

separable

Block

A non-separable subgraph of a separable graph is called block

separable

block

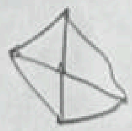
edge connectivity of a graph

Let G be a graph having K component. The minimum no of edge whose deletion from G increase the no: of components of G (λ)

edge connectivity of tree = $\lambda(T)=1$

Vertex connectivity of a graph

Let G be a graph the minimum no: of vertices whose deletion from G ↑ the no: of components of G is called the vertex connectivity of G (K)



$K(G)=2$

Theorem

The maximum vertex connectivity of a connected graph G with n vertices and 'e' edges is $[\frac{2e}{n}]$

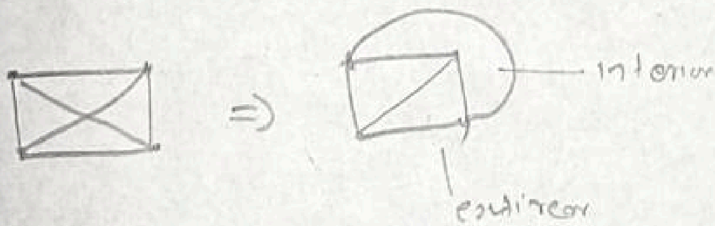
Proof: Using $d(v)=2e$

planar graphs

A face of a graph G is the region formed by a cycle or parallel edges on loops in G. A face of a graph G is also called a region or mesh. it is denoted by 'f'.

Jordan curve: non-self intersecting

planar graph: it can be re-drawn on a plane without any crossovers (without edge intersecting) such a drawing of G is called planar graph.



embedding of graph: making a graph planar by rearranging the edge.

intersection number of graph: minimum no. of edge crossing we draw G on a plane.

spherical embedding: A G is said to be embeddable on a sphere if it can be drawn on surface of sphere without crossing of edge.

Theorem:

A GBA connected planar simple graph, with V vertices, E edges & F faces

$$V - E + F = 2$$

Proof:

consider a graph with 2 vertex

$V = 2 \quad E = 1 \quad F = 1$

$V - E + F = 2$

assume graph G_k with k edges

then

$V = V_k$

$E = E_k$

$F = F_k$

added an edge to it

$V_k = V_k$

$E_k = E_k + 1$

$F_k = F_k + 1$

$$V_{k+1} - E_{k+1} + F_{k+1} = V_k - (F_k + 1 +) + (E_k + 1)$$

add an vertex to it

$V_k = V_k + 1$

$E_k = E_k + 1$

$F_k = F_k$

$$V_{k+1} - E_{k+1} + \underline{F_{k+1}} = V_{k+1} - E_{k+1} + F_k$$

Theorem

If any simple connected planar graph with F region E edges V vertices then

$E \geq 3/2 F$

$E \leq 3V - 6$

Proof

let B be the total no. of boundary around all the planes in graph $3F \leq B$

Since each edge is used as bounding exactly twice $B = 2E$

$3F \leq 2E$

$E\,2 \geq 3F$

$E \geq \frac{3F}{2}$

$V - E + F = 2$

$F = 2 - V + E$

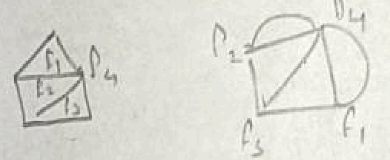
$2E \geq 3(2 - V + E)$

$2E \geq 6 - 3V + 3E$

$3V - 6 \geq 3E - 2E$

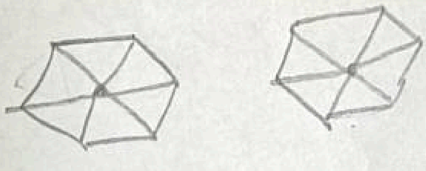
$3V - 6 \geq E$

dual graph



self dual

If the dual of a graph is same as the graph



M-5.

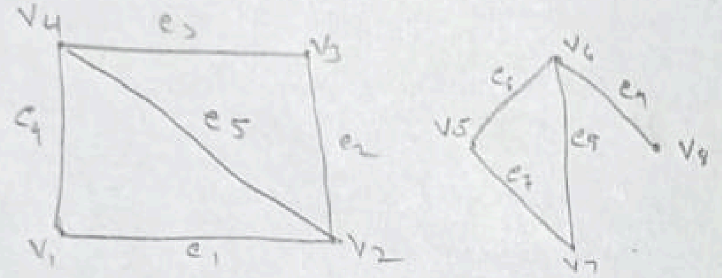
Incidence matrix of a graph

(minimal vertex coloring
greedy theorem algo

let G be a graph with 'n' vertices, m edges without any loop. The incidence matrix A of G is an $n \times m$ matrix defined by $A(G) = [a_{ij}]$ $1 \le i \le n$, $1 \le j \le m$ where

$a_{ij} = \{$ 1 if jth edge incident on ith vertex, 0 otherwise $\}$.

?. consider a disconnected graph



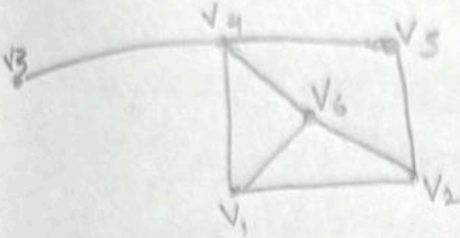
A(G)

$$A(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_1 \end{array} \begin{array}{c} \begin{array}{ccccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 \end{array} \\ \left(\begin{array}{ccccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right) \end{array} = A(G).$$

If a graph can't match from incident matrix, just traspose it and find the graph.

parallel edges in a graph produce by identical columns.

Adjacency Matrix

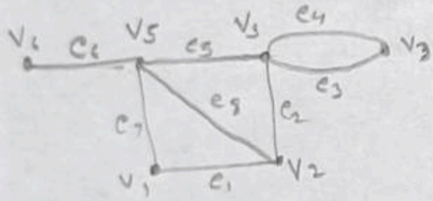


Adjacency matrix $\quad X^T = X$

$$X = \begin{array}{c|cccccc} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\ \hline v_1 & 0 & 1 & 0 & 1 & 0 & 1 \\ v_2 & 1 & 0 & 0 & 0 & 1 & 1 \\ v_3 & 0 & 0 & 0 & 1 & 0 & 0 \\ v_4 & 1 & 0 & 1 & 0 & 1 & 1 \\ v_5 & 0 & 1 & 0 & 1 & 0 & 0 \\ v_6 & 1 & 1 & 0 & 1 & 0 & 0 \end{array} \quad 6 \times 6$$

Principal diagonal

Circuit or cycle Matrix



$c_1 = \{e_1, e_7, e_8\}$

$c_2 = \{e_2, e_4, e_5\}$

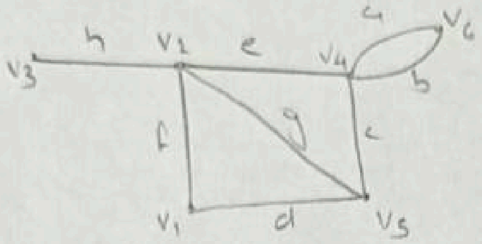
$c_3 = \{e_3, e_4\}$

$c_4 = \{e_1, e_2, e_5, e_7\}$

$$\begin{array}{c|cccccccc} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \\ \hline c_1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ c_2 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ c_3 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ c_4 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}$$

Path matrix

Consider a graph G



Path b/w v_3 & v_4

$P_1 = \{h, e\}$

$P_2 = \{h, g, c\}$

$P_3 = \{h, f, d, c\}$

$$\begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \end{array} \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

GRAPH coloring

no adjacent vertices share the same color

If the graph G has a loop at the vertex V, then V is adjacent to it self and hence no coloring of G is possible.

∴ we will assume that in any vertex coloring context, graph has no loop.

CHROMATIC NUMBER $\chi(G)$

minimum no: of color to colour a graph.

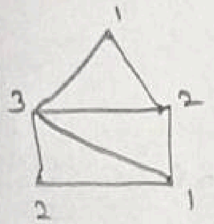
subset of vertices of $V(G)$ having same color is called colour class

Greedy Algorithms for coloring vertices of a graph

1. Color a vertex with colour 1

2. pick an uncoloured vertex v. colour it with the lowest numbered colour that has not been used on any previously colored vertices adjacent to v. If all previously used colour appear on vertices adjacent to v, we must introduce a new color and number it.

3. repeat it until all vertices are colored.



$\chi(G) = 3$

CHROMATIC POLYNOMIAL $P_n(\lambda)$

If a graph G can colour using sufficient no. of colour in different ways.

$$P_n(\lambda) = \sum_{i=1}^{n} c_i \binom{\lambda}{i}$$

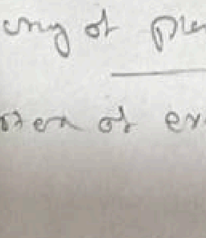
Given 4 colors

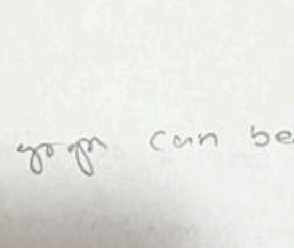
$P_n(\lambda) = 4!$

$$= c_1 \frac{\lambda}{1!} + c_2 \frac{\lambda(\lambda-1)}{2!} + c_3 \frac{\lambda(\lambda-1)(\lambda-2)}{3!}$$

colouring of planar graphs

Vertex of every planar graph can be colored with five colour





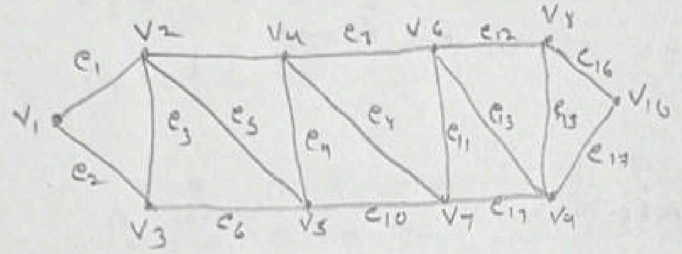
Matching of Graphs

A matching M in G is a set of edges of G such that no two edges share a common vertex.

Saturated vertex

Let m be a matching on a graph G. A vertex v of G is said to be m-saturated if some edge $e \in M$ incidents with v.

Eg:



$M = \{e_1; e_6, e_7, e_{14}\}$ is a matching in G

Ans. Saturated vertices are $\{v_1, v_2, v_3, V_4, v_5, v_6, v_7, v_9\}$

Maximal matching

in G is a matching in G which contains the largest possible no: of edges.

Perfect matching

A matching M of G is a perfect matching if and only if all vertices of G are m-saturated

Covering of Graph

edge covering (F)

set of edge F if a graph G is said to be a cover of G if every vertex of G is incident with atleast one edge in F.

Minimal edge covering

set of edge F of a graph G is said to be a minimal edge-covering of G if no proper subset of F is an edge covering of G.

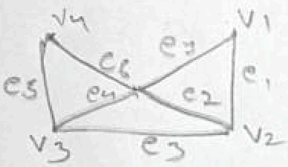
edge covering number

The edge covering number of a graph G is the minimal cardinality of a minimum edge covering of G.

Vertex covering Number

is the minimal cardinality of a minimal vertex covering of G

Eg:



edge covering - $\{e_1, e_2, e_5\}$ (minimal edge covering)

edge covering number - 3

vertex covering - $\{v_1\ v_5\ v_3\}$ (minimal vertex covering)

vertex covering number - 3