
KTU
NOTES

15|9|2021

MODULE - 5

GRAPH REPRESENTATION AND VERTEX COLOURING.

GRAPH MATRICES

I. Incidence matrix of a graph.

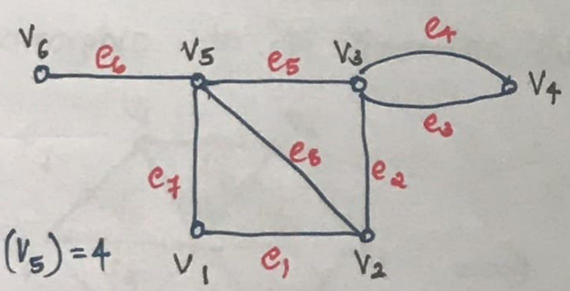
Let G be a graph with n vertices, m edges without any loops. The incidence matrix A if G is an $n \times m$ matrix defined by $A(G) = [a_{ij}]$ $1 \le i \le n$, $1 \le j \le m$ where

$$a_{ij} = \begin{cases} 1 & \text{if } j\text{th edge incident on } i\text{th vertex} \\ 0 & \text{otherwise.} \end{cases}$$

'n' rows of A of G correspond to the n-vertices and 'm' columns of A correspond to m-edges.

Incidence matrix contains only two types of elements 0 and 1.

eg: 1]



$d(v_5) = 4$

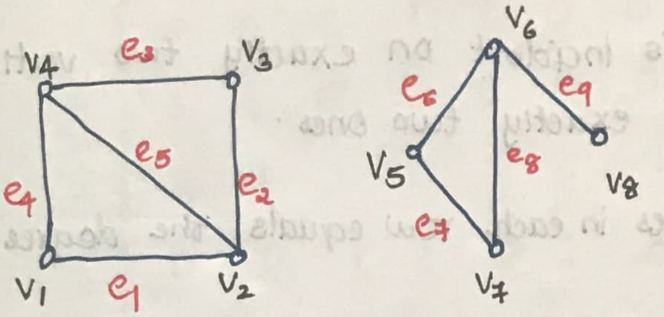
$$A(G) = \begin{array}{c|cccccccc} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \\ \hline v_1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ v_2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ v_3 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ v_4 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ v_5 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ v_6 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}$$

The incidence matrix of G is given by

$$A(G) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \; d=4$$

6×8

eg: 2) Consider a disconnected graph



Find the incidence matrix.

A(G1)

$$A(G) = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \\ V_8 \end{array} \begin{array}{c} \begin{array}{ccccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 \end{array} \\ \left[\begin{array}{ccccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

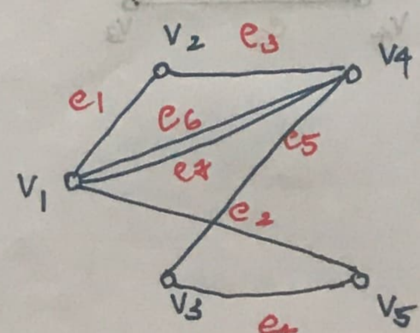
0   0   A(G2) 8×9

eg: 3) Draw the graph with the following matrix as its incidence matrix.

No. of 1's in each column is diff So it is not an incidence matrix but its transpose $[A(G)^T]$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \Longrightarrow A(G) = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{array} \begin{array}{c} \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

5×7



Graph G corresponding A(G).

Any 5

Properties of Incidence matrix

1) Since every edge is incident on exactly two vertices, each column of A has exactly two ones.

2) The number of ones in each row equals the degree of the corresponding vertex.

3) A row with all zeroes respresents an isolated vertex.

4) Parallel edges in a graph produce identical columns.

5) If a graph G is disconnected and consists of two components G_1 and G_2, then its incidence matrix A(G) can be written in a block diagonal form as.

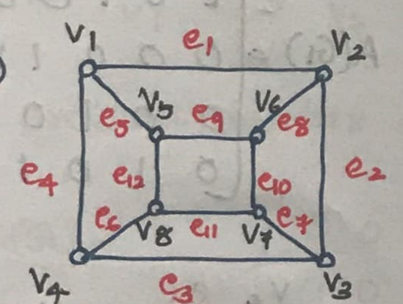
$$A(G) = \begin{bmatrix} A(G) & 0 \\ 0 & A(G) \end{bmatrix}$$

6) If A(G) is an incidence matrix of a connected graph G with n vertices, Then rank $(A(G)) = n-1$.

Problems:

1) Find the incidence matrix for the following graphs.

(i)

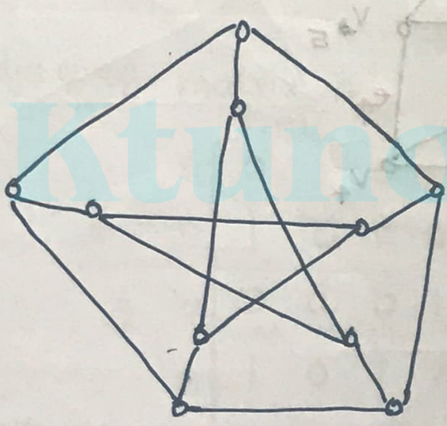


e_1 e_2 e_3 e_4 e_5 e_6 e_7 e_8 e_9 e_{10}

$A(G) =$

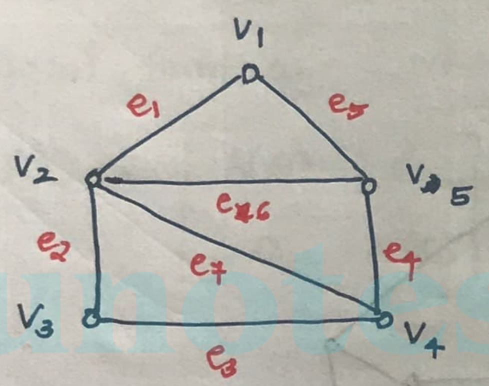
$$
A(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \end{array}
\begin{array}{c}
\begin{array}{cccccccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} & e_{11} & e_{12} \end{array} \\
\begin{bmatrix}
1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1
\end{bmatrix}
\end{array}
$$

(ii)



2) Find the graph whose incidence matrix is

$$A(G)^T = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \Longrightarrow A(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

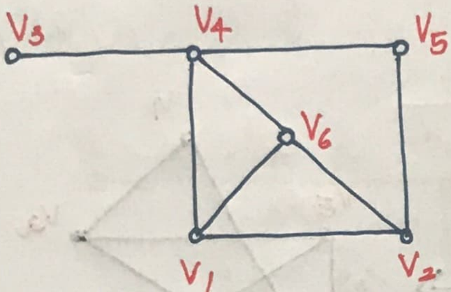


ADJACENCY MATRIX

The adjacency matrix of a graph G with n vertices and no parallel edges is an $n \times n$ symmetric matrix $X = [x_{ij}]$ where

$$x_{ij} = \begin{cases} 1 & \text{if there is an edge b/w } i^{th} \text{ \& } j^{th} \text{ vertices} \\ 0 & \text{if there is no edge b/w them} \end{cases}$$

eg: 1)



Adjacency matrix is

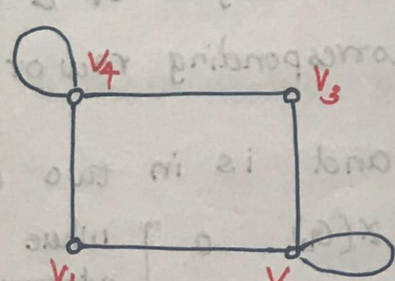
$$X = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{array} \begin{array}{c} \begin{array}{cccccc} V_1 & V_2 & V_3 & V_4 & V_5 & V_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}_{6\times 6} \end{array}$$

first row = first column

$X^T = X$.

∴ Symmetric matrix

2)



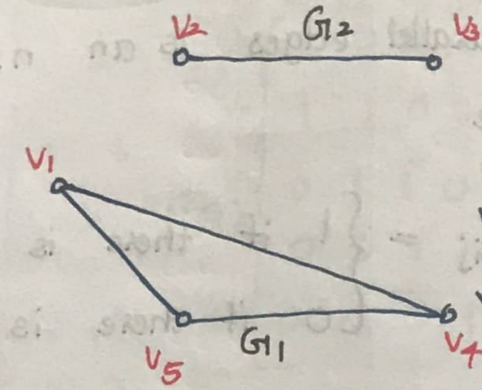
$$X = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \end{array} \begin{array}{c} \begin{array}{cccc} V_1 & V_2 & V_3 & V_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}_{4\times 4} \end{array}$$

Draw the graph with the following as its adjacency matrix.

1)

Symmetric So an adjacency matrix

$$\begin{array}{c|ccccc} & V_1 & V_2 & V_3 & V_4 & V_5 \\ \hline V_1 & 0 & 0 & 0 & 1 & 1 \\ V_2 & 0 & 0 & 1 & 0 & 0 \\ V_3 & 0 & 1 & 0 & 0 & 0 \\ V_4 & 1 & 0 & 0 & 0 & 1 \\ V_5 & 1 & 0 & 0 & 1 & 0 \end{array}_{5\times 5} \Longrightarrow$$



$$\begin{array}{c|ccccc} & V_1 & V_4 & V_5 & V_2 & V_3 \\ \hline V_1 & 0 & 1 & 1 & 0 & 0 \\ V_4 & 1 & 0 & 1 & 0 & 0 \\ V_5 & 1 & 1 & 0 & 0 & 0 \\ V_2 & 0 & 0 & 0 & 0 & 1 \\ V_3 & 0 & 0 & 0 & 1 & 0 \end{array}$$

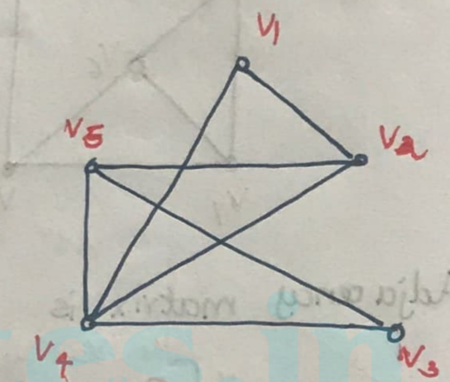
clock diagonal form

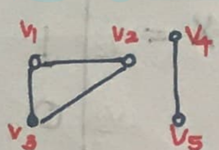
2)

Symmetric So adjacency matrix

$$\begin{array}{c|ccccc} & V_1 & V_2 & V_3 & V_4 & V_5 \\ \hline V_1 & 0 & 1 & 0 & 1 & 0 \\ V_2 & 1 & 0 & 0 & 1 & 1 \\ V_3 & 0 & 0 & 0 & 1 & 1 \\ V_4 & 1 & 1 & 1 & 0 & 1 \\ V_5 & 0 & 1 & 1 & 1 & 0 \end{array} \Longrightarrow$$

Principle diagonal





$$X = \begin{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{bmatrix}$$

X(G1), 0, 0, X(G2)

clock diagonal form

Properties of adjacency matrix.

1) Principal diagonal of X are all zero's if and only if graph has no loops.

2) If the graph has no loops, the degree of a vertex equals the number of 1's in the corresponding row or column.

3) A graph G is disconnected and is in two components G_1 & G_2 if and only if $X = \begin{bmatrix} X(G_1) & 0 \\ 0 & X(G_2) \end{bmatrix}$ where $X(G_1)$ is the adjacency matrix of G_1 and $X(G_2)$ is the adjacency matrix of G_2.

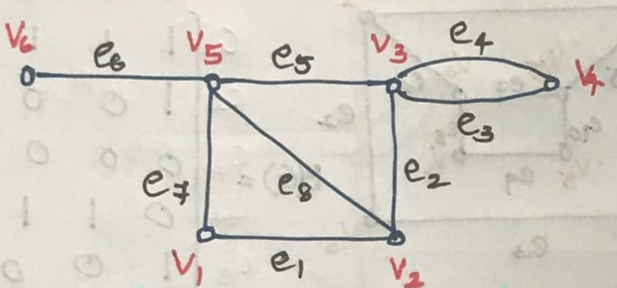
4) Given any square, symmetric, binary matrix Q of order n, we can construct a graph G of n vertices s.t Q is the adjacency matrix of G.

CIRCUIT MATRIX (CYCLE MATRIX)

Let G be a graph with e edges and q different cycles. The current matrix of G, denoted by B(G), is defined as a (0,1) matrix $B(G) = [b_{ij}]$ of order $q \times e$, s.t

$$b_{ij} = \begin{cases} 1 & \text{if the } i\text{th cycle includes } j\text{th edge} \\ 0 & \text{one} \end{cases}$$

eg:1) Consider a graph G.



In G, we have following circuits.

$C_1 = \{e_1, e_7, e_8\}$

$C_2 = \{e_2, e_5, e_8\}$

$C_3 = \{e_3, e_4\}$

$C_4 = \{e_1, e_2, e_5, e_7\}$

Circuit matrix

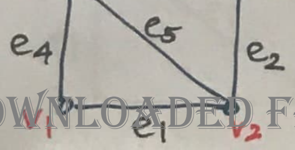
$$B(G) = \begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \\ C_4 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

2) Find the circuit matrix of the disconnected graph

$C_1 = \{e_3, e_5, e_2\}$

$C_2 = \{e_1, e_5, e_4\}$

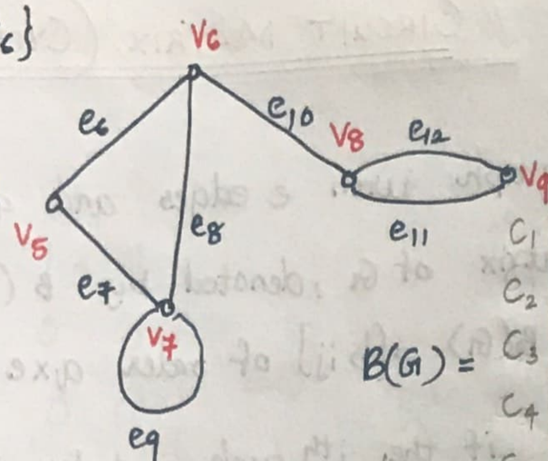
$C_3 = \{e_1, e_2, e_3, e_4\}$



$C_4 = \{e_7, e_8, e_6\}$

$C_5 = \{e_9\}$

$C_6 = \{e_{11}, e_{12}\}$



$B(G) =$

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}
C_1	0	1	1	0	1	0	0	0	0	0	0	0
C_2	1	0	0	1	1	0	0	0	0	0	0	0
C_3	1	1	1	1	0	0	0	0	0	0	0	0
C_4	0	0	0	0	0	1	1	1	0	0	0	0
C_5 (a loop)	0	0	0	0	0	0	0	0	1	0	0	0
C_6	0	0	0	0	0	0	0	0	0	0	1	1

6×12

3) find the cycle matrix of the given graph

$C_1 = \{e_1, e_2, e_3, e_4\}$

$C_2 = \{e_1, e_6, e_7, e_5\}$

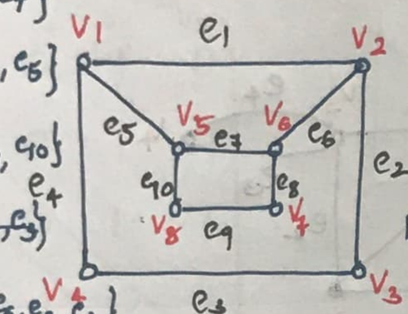
$C_3 = \{e_7, e_8, e_9, e_{10}\}$

~~$C_4 = \{e_4, e_5, e_7, e_6, e_2, e_3\}$~~

$C_4 = \{e_4, e_5, e_{10}, e_9, e_8, e_6, e_2, e_3\}$

$C_5 = \{e_5, e_{10}, e_9, e_8, e_6, e_1\}$

$C_6 = \{e_4, e_5, e_7, e_6, e_2, e_3\}$



$B(G) =$

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}
C_1	1	1	1	1	0	0	0	0	0	0
C_2	1	0	0	0	1	1	1	0	0	0
C_3	0	0	0	0	0	0	1	1	1	1
C_4	0	1	1	1	1	1	0	1	1	1
C_5	1	0	0	0	1	1	0	1	1	1
C_6	0	1	1	1	1	1	1	0	0	0

Properties of Circuit Matrix.

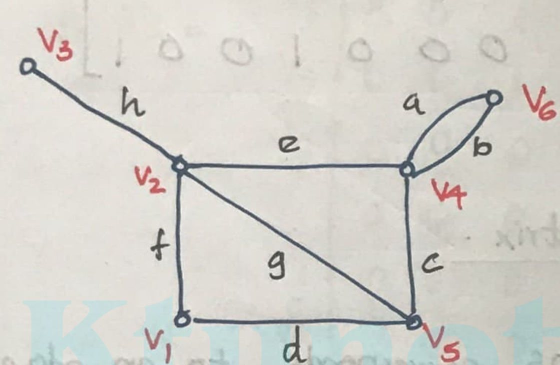
1) A column of all zeroes corresponds to a cut-edge.
2) Number of 1's in a row is equal to the number of edges in the corresponding cycle
3) Row corresponding to a loop to contain a Single 1.
4) If the graph is disconnected and consists of two compounds H_1 & H_2, Then cycle matrix $B(G) = \begin{bmatrix} B(H_1) & 0 \\ 0 & B(H_2) \end{bmatrix}$

PATH MATRIX

Let G be a graph with e edges, and u and v be any two vertices in G. Also let 'q' be the number of different paths between u & v in G. The path matrix for vertices u & v denoted by $P(u,v) = [P_{ij}]$ $1 \le i \le q$, $1 \le j < e$

Such that $P_{ij} = \begin{cases} 1 & \text{if } i^{th} \text{ path contain the } j^{th} \text{edge} \\ 0 & \text{one} \end{cases}$

eg: Consider a graph G,



Let us take two vertices V_3 & V_4.

There are three different paths b/w V_3 & V_4.

1) $\{h, e\}$ - P_1
2) $\{h, g, c\}$ - P_2
3) $\{h, f, d, c\}$ - P_3

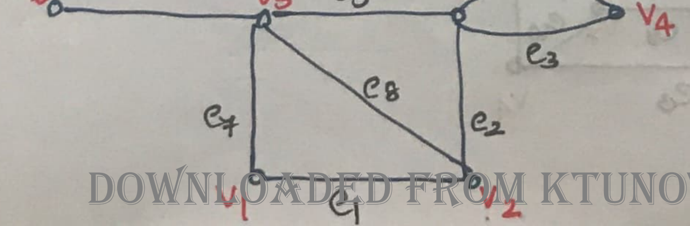
q = 3 → no: of diff path from V_3 & V_4.

Corresponding path matrix is a 3×8 matrix given by

$$P(V_3, V_4) = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \end{array} \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

— edge in all path.

eg:



Find $P(V_1, V_3)$

1) $P_1 = \{e_1, e_2\}$
2) $P_2 = \{e_7, e_5\}$
3) $P_3 = \{e_7, e_8, e_2\}$
4) $P_4 = \{e_1, e_8, e_5\}$

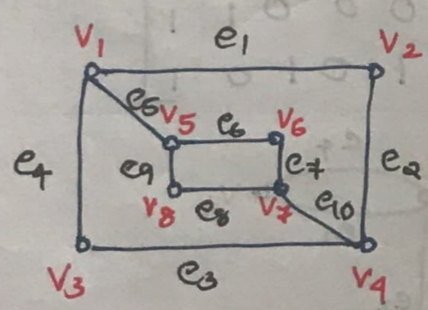
$$P[V_1, V_3] = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \\ P_4 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Properties of Path matrix.

1) A column of all zeros corresponds to an edge that does not be in any path b/w u & v.
2) A column of all ones corresponds to an edge that in every path b/w u & v.
3) There is no row with all zeros.

Problem

Consider a graph G.



Find 1) Incidence matrix.

2) Adjacency matrix

3) Circuit matrix

4) Path matrix $P(V_1, V_4)$

Sol: 1) Incidence matrix $A(G)$

$$A(G) = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \\ V_8 \end{array} \begin{array}{c} \begin{array}{cccccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

~~10×8~~

8×10

2) Adjacency matrix $X(G)$

$$X(G) = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \\ V_8 \end{array} \begin{array}{c} \begin{array}{cccccccc} V_1 & V_2 & V_3 & V_4 & V_5 & V_6 & V_7 & V_8 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

8×8

3) Circuit Matrix $B(G)$

$C_1 = \{e_1, e_2, e_3, e_4\}$

$C_2 = \{e_6, e_7, e_8, e_9\}$

$C_3 = \{e_1, e_2, e_{10}, e_7, e_6, e_5\}$

$C_4 = \{e_4, e_3, e_{10}, e_8, e_9, e_5\}$

$C_5 = \{e_1, e_2, e_{10}, e_8, e_9, e_5\}$

$C_6 = \{e_4, e_3, e_{10}, e_7, e_6, e_5\}$

$$B(G) = \begin{array}{c|cccccccccc} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} \\ C_1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ C_2 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ C_3 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ C_4 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ C_5 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ C_6 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \end{array}$$

~~10×6~~

6×10

4) Path Matrix $P(V_1, V_4)$

$P_1 = \{e_1, e_2\}$

$P_2 = \{e_3, e_4\}$

$P_3 = \{e_5, e_6, e_7, e_{10}\}$

$P_4 = \{e_5, e_9, e_8, e_{10}\}$

$$\begin{array}{c|cccccccccc} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} \\ P_1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ P_2 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ P_3 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ P_4 & 0 & 0 & 0 & 0 & 1 & 0 & & 1 & 1 & 1 \end{array}$$

~~10×4~~

4×10

24/9/2021

GRAPH COLORING.

A Graph colouring is an assignment of colours (labels or weight) to the elements (vertices, edges or faces) of a graph subject to certain constraints.

Note: The graph coloring, we mean the vertex coloring of graph under cosideration.

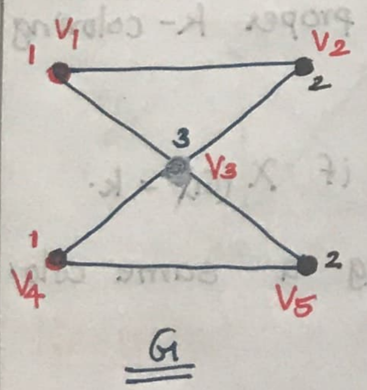
PROPER COLORING:

Given a color set $C = \{c_1, c_2, \dots c_l\}$, a proper coloring in a mapping $c: V(G) \rightarrow C$ defined by $c(v_i) = c_r$, $1 \leq r \leq l$. s.t $c(v_i) \neq c(v_j)$ where v_i & v_j are adjacent vertices.

A coloring of graph G using K colors is called a proper k-coloring of G.

In otherwords, a proper coloring is a way of coloring the vertices of a graph s.t no to adjacent vertices share the same color.

eg:



$\longrightarrow$ 3-colorable graph.

$\chi(G) = 3$

Note:

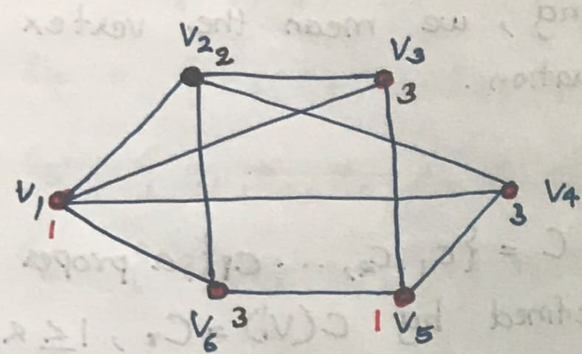
1) Proper edge/face coloring assigns a color to each edge/face so that no two adjacent edge/face share the same color.

2) imp If the graph G has a loop at the vertex v, then v is adjacent to itself and hence no coloring of G is possible
∴ We will assume that in any vertex coloring contex, graph has no loops

CHROMATIC NUMBER $\chi(G)$

The minimum number of colors needed to color (proper color) a graph G is called its chromatic number and is denoted by $\chi(G)$. (chi)

eg:



Color classes

$S_1 = \{v_1, v_5\}$

$S_2 = \{v_2\}$

$S_3 = \{v_3, v_4, v_6\}$

$$\chi(G) = 3$$

Note:

Any empty graph has $\chi(G) = 1$ and $\chi(G) \geq 2$ for a graph with atleast one edge.

⟹ A graph that can be assigned a proper k-coloring is said to be a k-colorable graph.

⟹ A graph is said to be k-chromatic if $\chi(G) = k$.

⟹ A subset of vertices of $V(G)$ having a same color is called a color class of G.

eg: $\{v_3, v_4, v_6\}$ is a color class in the above graph.

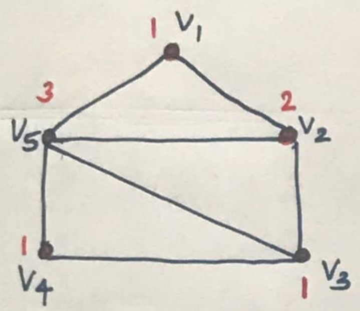
GREEDY ALGORITHM for coloring vertices of a graph

Step 1: Color a vertex with color 1.

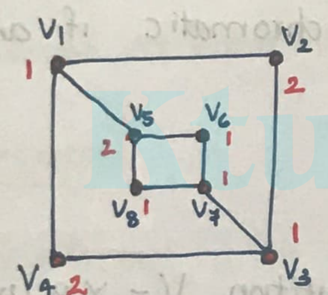
Step 2: Pick an uncolored vertex v. Color it with the lowest numbered color that has not been used on any previosly colored vertices adjacent to v. If all previosly used colors appear on vertices ajacent to v, we must introduce a new color and number it.

step 3: Repeat the previous step until all the vertices are colored.

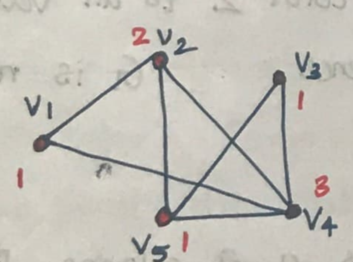
Find the chromatic Number $X(G)$ of the following graphs.

1) 

$X(G) = \underline{\underline{3}}$

2) 

$X(G) = \underline{\underline{2}}$

3) 

$X(G) = \underline{\underline{3}}$

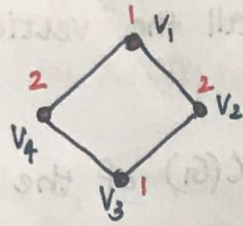
RESULTS:

1) If G is a graph with 'n' vertices then $X(G) \leq n$.
2) $X(K_n) = n$ for all $n > 1$.
3) If H is a sub-graph of G, then $X(H) \leq X(G)$.
4) If the graph G has $G_1, G_2, G_3 \cdots G_n$ as its connected components then we have $X(G) = \max(X(G_i))$, $1 \leq i \leq n$.
5) $X(G) = 1 \iff$ G is an empty graph

6) $\chi(C_n) = \begin{cases} 2 & \text{if } n\text{-even} \\ 3 & \text{if } n\text{-odd.} \end{cases}$

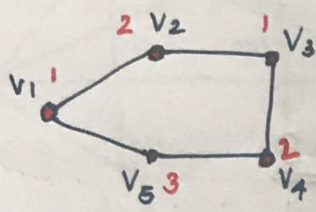
Where C_n denote cycle of length n.

eg:



$n = 4$ (even)

$\therefore \chi(G) = 2$



$n = 5$ (odd)

$\chi(G) = 3$

THEOREM

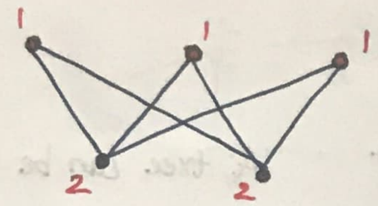
A Non-empty graph G is 2-chromatic if and only if it is bipartite.

Proof:

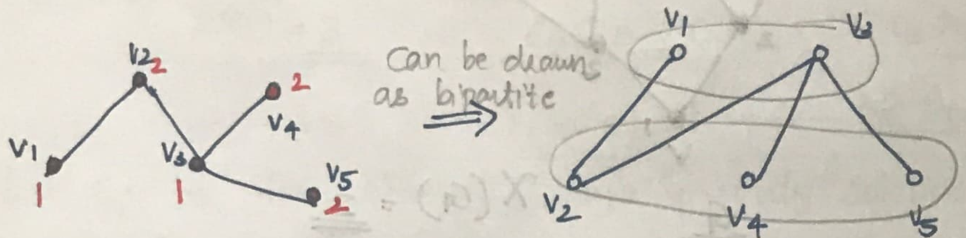
Let G be bipartite with bipartition $V = X \cup Y$. Assigning color 1 to all vertices in X and color 2 to all vertices in Y gives a 2-coloring for G and hence G is non-empty, $\chi(G) = 2$.

Conversely $\chi(G) = 2$. Then G has a 2-coloring. Denote X by the set all vertices colored 1 & Y by set of all vertices colored 2. Then no two vertices in X are adjacent & similarly for Y. Hence any edge in G must join a vertex in X and a vertex in Y.

$\therefore$ G is bipartite.



$\chi(G) = 2$



RESULT:

If $\chi(G) = 2 \iff$ it has no odd cycle. – cycle with odd length

Proof:

We have, A graph is bipartite $\iff$ it has no odd cycle (Use the above theorem to prove the result).

THEOREM.

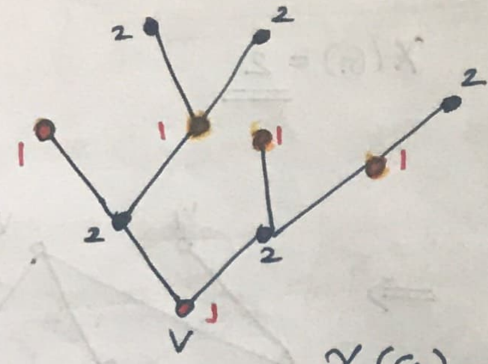
Every tree with two or more vertices is 2-chromatic

Proof:

Select any vertex v in a given tree T. Color vertex v with color 1. Paint all the vertices adjacent to v with color 2. Now paint the vertices adjacent to these (those that just have been colored with color 2) using color 1. Continue this process till every vertex in T has been panted. Now in T we ~~have~~ find that all vertices at odd distances from v have color 2, while v and vertices at even distances from v have color 1.

Now along any path in T the vertices are of alternating colors. There is one and only one path b/w any two vertices in a tree, no two adjacent vertices have the same color. Thus T has been properly colored with two colors.

eg:1)

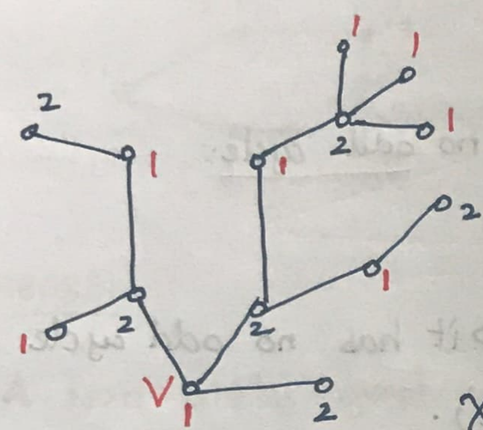


A tree can be completed with 2 colors.

$\chi(G) = \underline{\underline{2}}$

$\underline{G}$

2)



$\chi(T) = \underline{\underline{2}}$

$\underline{T}$

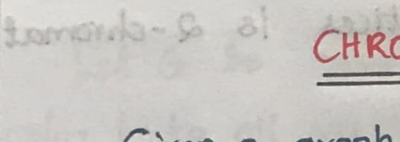
CHROMATIC POLYNOMIAL

Given a graph G if n vertices can be properly colored in many different ways using a sufficiently large number of colors. The chromatic polynomial of a graph G counts the number of its proper vertex coloring and is denoted by $P_n(\lambda)$.

ie; The value of the chromatic polynomial $P_n(\lambda)$ of a graph with n vertices given the number of ways of properly coloring the graph, using λ colors.

~~Let C_1 be the different ways~~

eg:



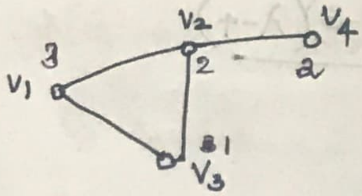
$n = 4$

It can be colored in $4!$ ways $= C_0$

Given 4 colors

1 – Red
2 – Blue
3 – Yellow
4 – Grey.

$\underline{G}$



Proper coloring using 3 colors :

$C_3 = 3! \times 2$

$P_n(\lambda) = \sum_{i=1}^{4} C_i \binom{4}{i}$

$= C_1 \cancel{\frac{\lambda}{1!}} + C_2 \cancel{\frac{\lambda(\lambda-1)}{2!}} +$

$\chi(G) = \underline{\underline{3}}$

$C_2 = 0 \; ; \; C_1 = 0 .$

$C_3 \frac{\lambda(\lambda-1)(\lambda-2)}{2!} + C_4 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{1!}$

Ans : $\lambda(\lambda-1)(\lambda-2)$ ←

Let C_i be the different ways propely coloring G using exactly i different colors. Since i can be chosen out of λ colors in

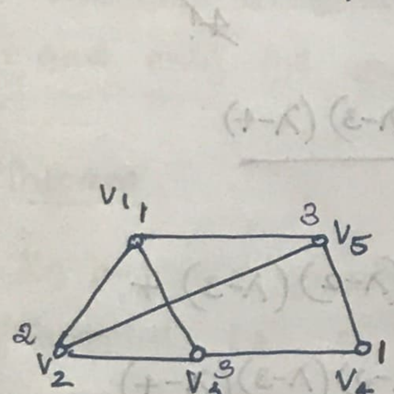
$\lambda_{C_i} \rightarrow \binom{\lambda}{i}$ different ways

$$\boxed{\therefore P_n(\lambda) = \sum_{i=1}^{n} C_i \binom{\lambda}{i}}$$

$$= C_1 \frac{\lambda}{1!} + C_2 \frac{\lambda(\lambda-1)}{2!} + C_3 \frac{\lambda(\lambda-1)(\lambda-2)}{3!} + \cdots$$

$$+ C_n \frac{\lambda(\lambda-1)(\lambda-2)\cdots(\lambda-(n-1))}{n!}$$

eg:



Find chromatic number and chromatic polynomial of G.

$\chi(G) = \underline{\underline{3}}$

Chromatic polynomial is given by :

$$P_n(\lambda) = \sum_{i=1}^{n} C_i \binom{\lambda}{i}$$

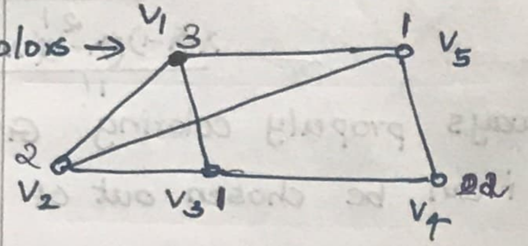
$$\Rightarrow P_5(\lambda) = \sum_{i=1}^{5} C_i \binom{\lambda}{i}$$

$$= C_1 \frac{\lambda}{1!} + C_2 \frac{\lambda(\lambda-1)}{2!} + C_3 \frac{\lambda(\lambda-1)(\lambda-2)}{3!} + C_4 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!}$$

$$+ C_5 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{5!}$$

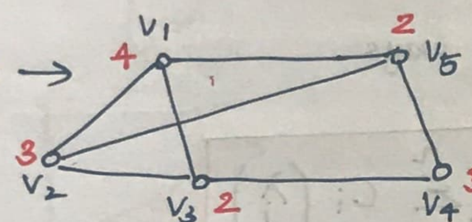
To find $C_1 = 0; C_2 = 0; C_3 = 3! \times 2; C_4 = 4! \times 6; C_5 = 5!$

3 colors →



$= 3! \times 2 \times 1$

4 colors →



$= 4 \times 3 \times 2 \times 2 \times 3$

$= 4! \times 6$

5 colors & 5 vertices $= 5!$

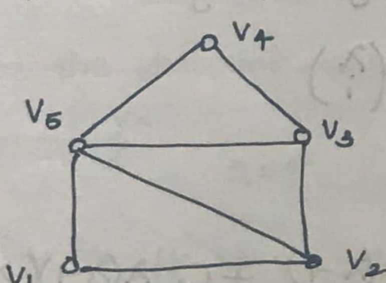
$$P_n(\lambda) = \cancel{3!} \times 2 \frac{\lambda(\lambda-1)(\lambda-2)}{\cancel{3!}} + \cancel{4!} \times 6 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{\cancel{4!}}$$

$$+ \cancel{5!} \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{\cancel{5!}}$$

$$= 2\lambda(\lambda-1)(\lambda-2) + 6\lambda(\lambda-1)(\lambda-2)(\lambda-3) + \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)$$

eg:2) Find the chromatic polynomial if

G →



$\chi(G) = 3$

Since $\lambda(G) = 3$, $C_1 = 0$ & $C_2 = 0$

$C_3 = 3! \times 1 = 6$

$C_4 = 4! \times 2 = 48$

$C_5 = 5!$

Theorem:

A graph on n vertices in a complete graph if and only if its chromatic polynomial is.

$$P_n(\lambda) = \lambda(\lambda-1)(\lambda-2) \dots (\lambda-(n-1))$$

Proof:

With λ colors, there are λ different ways if coloring any selected vertex of a graph. A second vertex can be colored propely in exactly $(\lambda-1)$ ways, the third in $(\lambda-2)$ ways the forth in $(\lambda-3)$ ways and the n^{th} in $(\lambda-(n-1))$ ways if and only if every vertex is adjacent to every other. That is, if and only the graph is complete.

Theorem

An n-vertex graph in a tree if and only if its chromatic Polynomial is $P_n(\lambda) = \lambda(\lambda-1)^{n-1}$

Proof:

Fix an initial vertex V_1. There are λ Possible choice for color V_1. Then consider a vertex V_2 adjacent to V_1. Now there $(\lambda-1)$ ways to choose color V_2. Now consider a vertex V_3 adjacent V_1 or V_2. Note that if it cannot be adjacent to be the V_1 & V_2, or there would be a cycle. Thus there are $(\lambda-1)$ possible choices for color V_3. We repeat this procedure, we will always have a vertex adjacent to exactly one if the previosly.

colored vertices, so it can be colored in k-1 ways. After repeating (n-1) times we get $P_n(\lambda) = \lambda(\lambda-1)^{n-1}$.

COLORING OF PLANAR GRAPH

FIVE - COLOR THEOREM.

The vertex of every planar graph can be properly colored with five colors.

Proof:

The theorem will be proved by mathematical induction on n. Since vertices of all graphs with 1, 2, 3, 4 or 5 vertices can be properly colored with five colors, let us assume that vertices of every planar graph with (n-1) vertices can be properly colored with five colors. Then, if we prove that any planar graph with n vertices will require no more than 5 colors, we shall proved the theorem.

Consider the planar graph G with n vertices. Since G is planar, it must have atleast one vertex with degree five or less. (otherwise it contain K_5).

Let this vertex be v. Let G' be the graph obtained from G by deleting the vertex v. According to induction ~~hey~~ hypothesis, G' require no more than 5 colors (G' having (n-1) vertices). Suppose that the vertices in G' have been properly colored, and now we add to it v and all edges incident on v. If the degree of v is 1, 2, 3 or 4, we have no difficulty in assigning a proper color to v.

This leaves only the case in which the degree of v is five, and all the five colors have been used in coloring the vertices adjacent to v. (figure below).

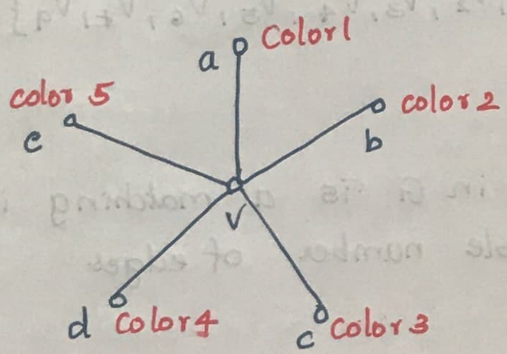
Suppose that there is a path in G' between vertices a and c (figure) colored alternatively with colors 1 & 3 as shown in figure.

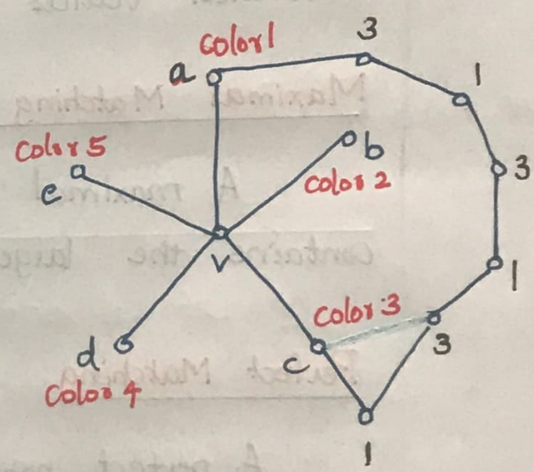
Then a similar path between b and d, colored alternately with colors 2 and 4, cannot exist, otherwise these two paths will intersects and cause G to be nonplanar.

If there is no path between b and d colored alternately with colors 2 and 4, starting from vertex. We can interchange colors 2 and 4, ~~starting from~~ if all vertices connected to b through vertices if alternately color 2 and 4. This interchange will paint vertex b with color 4 and ~~y~~ yet keep G' properly colored. Since vertex d is still with color 4, we have color 2 left over with which to paint vertex v.

∴ G is colored with 5 colors.

∴ Every planar graph is 5 colorable.





FOUR - COLOR THEOREM

Every planar graph has a chromatic number of four or less

Proof:

No proof available.

Matching of Graphs

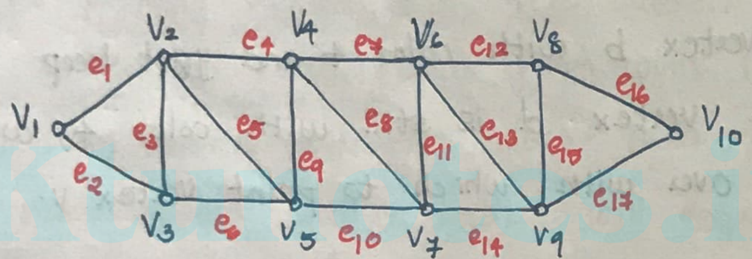
A matching M in G is a set of edges of G such that no two edges share a common vertex.

ie; In a matching M of a graph G, the edges must be pairwise independent.

Saturated vertex.

Let m be a matching on a graph G. A vertex v of G is said to be m-saturated if some edge $e \in M$ incidents with v.

eg:



$M = \{e_1, e_6, e_7, e_{14}\}$ is a matching in G.

Saturated vertices are $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_9\}$

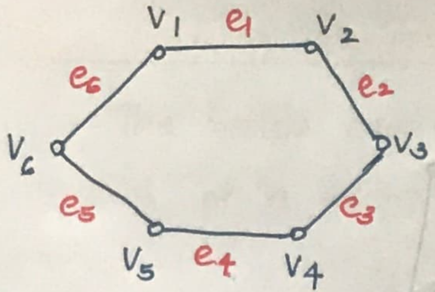
Maximal Matching

A maximal matching in G is a matching in G which contains the largest possible number of edges.

Perfect Matching

A perfect matching is a matching where the edges in M are incident on each vertex in V(G). ie; A matching M of G is a perfect matching if and only if all vertices of G are M-saturated.

eg:

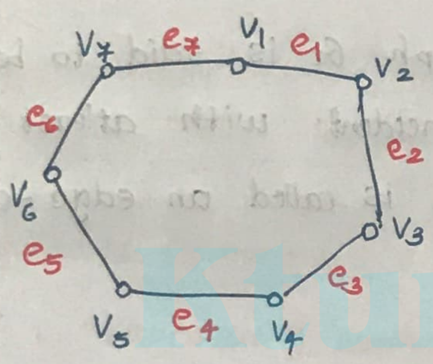


$M = \{e_1, e_3, e_5\}$ which is a maximal matching and is a perfect matching also. (Saturated vertices are $\{v_1, v_2, v_3, v_4, v_5, v_6\}$)

<u>Note:</u>

Every Perfect Matching is a Maximal matching, but a maximal matching need not be a perfect matching.

eg:

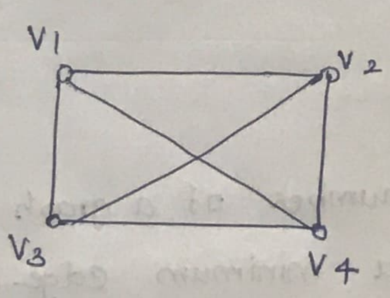


$M = \{e_1, e_3, e_5\}$ is a maximal matching but not a perfect matching.

The matching number of a graph is the number of edges in a maximal matching denoted by $\mu(G)$

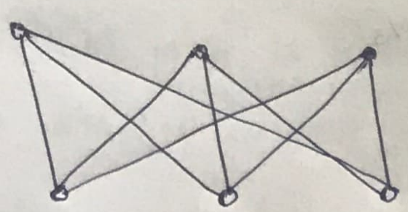
Find the matching number of K_4 & $K_{3,3}$.

1) <u>K_4</u>



$\mu(K_4) = 2$

2) $K_{3,3}$



$\psi(K_{3,3}) = 3$

COVERING OF GRAPHS

Edge Covering

A set of edges F if a graph G is said to be a cover of G if every vertex of G is incident with atleast one edge in F. The set F of edges is called an edge-covering of G.

Note:

1) Edge set of a graph G intrivially an edge covering of G.
2) For any connected graph, a spanning tree, forms an edge-covering of G.

Minimal edge Covering

A set of edges F of a graph G is said to be a minimal edge-covering of G if no proper subset of F is an edge-covering of G.

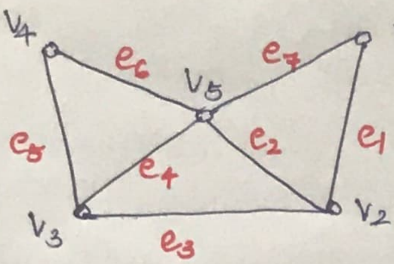
Edge Covering Number

The edge covering number of a graph G is the minimum cardinality of a minimum edge-covering of G.

Vertex Covering Number.

The vertex covering number of a graph G is the minimum cardinality of a minimal vertex covering of G.

eg:



1) Find an edge covering of G and hence find edge covering number

2) Find a vertex covering of G and hence find vertex covering number.

$\Rightarrow$ Edge Covering - $\{e_1, e_2, e_5\}$ (Minimal edge covering)

Edge covering number - 3

Vertex Covering - $\{v_1, v_5, v_3\}$ (Minimal vertex covering)

Vertex covering number = 3.