Module - 1

Introduction to Graphs

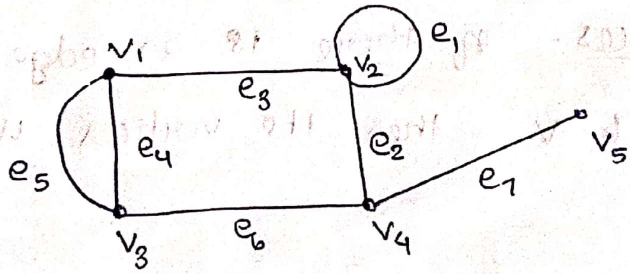
Introduction

What is a graph ?

Defenition-: A graph $G = (V, E)$ consists of a set of objects $V = \{v_1, v_2, v_3, \dots v_n\}$ called vertices and another set $E = \{e_1, e_2, \dots e_n\}$ whose elements are called edges such that each edge e_k is identified with an unordered pairs (v_i, v_j) of vertices. The vertices v_i, v_j associated with edge e_k are called the end vertices of e_k.

The most common representation of a graph in by means of a diagram, in which the vertices are represented as points and each edge as a line segment joining its end vertices.

Fig.



$G = (V, E)$. $V = \{v_1, v_2, v_3, v_4, v_5\}$

$E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$

Note-:

For a graph $G=(V,E)$ V → called vertex set it is non empty

E → called egde set, it can be empty

Definitions -:

Loop -: An edge having the same vertex as both its end vertices is called a loop

In the above figure e_1 is a loop.

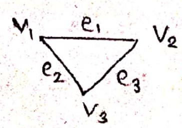
Parallel edges -: If two or more edges have same end vertices, such edges are called Parallel edges

e_4 & e_5 are parallel in above figure

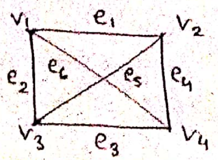
Simple graph -: A graph that has neither loops nor Parallel ~~or~~ edges is called a simple graph, otherwise it is a multigraph or general graph.

Adjacent vertices -: If there is an edge joining two vertices u & v, then its vertices u & v are adjacent vertices.

Eg.

① 

v_1 & v_2 are adjacent
v_2 & v_3 are adjacent
v_1 & v_3 are adjacent

② 

v_1 & v_2 are adjacent
v_1 & v_3 are adjacent
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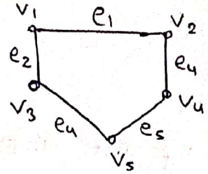
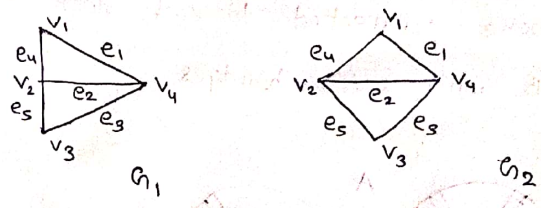
③ 

v_1 & v_2 are adjacent
But v_1 & v_4 are not adjacent

④ 

Here G_1 & G_2 are same.

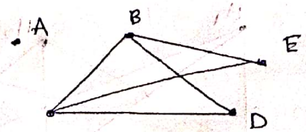
Applications of Graphs

Graph theory has a very wide range of applications in numerous areas. A graph can be used to represent almost any physical situations involving discrete objects and a relationship among them.

Some examples

1) Representing Relation

Let A, B, C, D, E are five students, we can represent this friendship by means of a graph



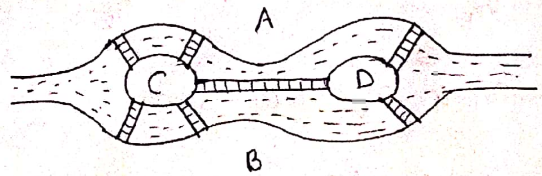
Edge joining two vertices means they are friends.

A has no friends

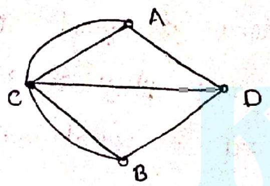
E and D are not directly friends etc.

2) Konigsberg Bridge Problem

Two islands C & D formed by the river in Konigsberg were connected to each other & to the lands A & B with seven bridges.

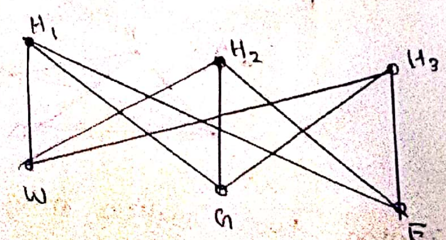


Euler represented this situation by means of a graph



3] Utility Problems

Let H_1, H_2 & H_3 are three houses to be connected with three utilities water (W), Gas (G), Electricity (E).

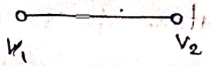


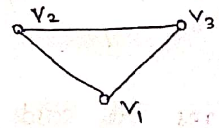
Complete Bipartite graph

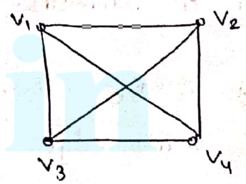
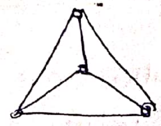
Complete Graph

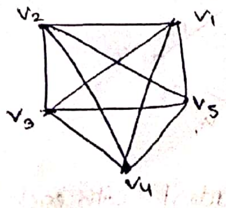
A simple graph in which any two distinct vertices are adjacent is called a complete graph. The complete graph with n vertices is denoted by K_n.

$K_1 \rightarrow$ • v_1 0 edge 1 vertex

$K_2 \rightarrow$  1 e, 2 v.

$K_3 \rightarrow$  3 e, 3 v

$K_4 \rightarrow$   6 e, 4 v

$K_5 \rightarrow$  10 e, 5 v.

* In a complete graph, all vertices will have edge to every other vertices

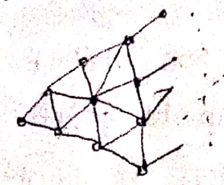
* K_n has $\frac{n(n-1)}{2}$ edges always.

* Complete graph will have maximum edges possible, which is $\frac{n(n-1)}{2}$.

* Finite Graph

A graph with finite number of vertices as well as a finite number of edges is called a finite graph. Otherwise it is called an infinite graph.

Infinite graph



Portions of an infinite graph.

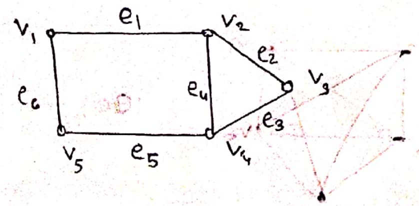
Incidence and Degree

Defenition-: When a vertex V_i is an end vertex of some edge e_j then V_i and e_j are said to be incident with each other.

Two non parallel edges are said to be adjacent if they are incident on a common vertex.

Two vertices are said to be adjacent if they are the end values of the same edge.

eg.



V_1 & e_1 are incident with each other

e_1 & e_5 are adjacent edges

V_4 & V_5 are adjacent vertices.

* Degree of a vertex-:

The number of edges incident on a vertex V_i with loops counted as twice is called the degree, $d(V_i)$ of the vertex.

$d(V_4) = 3$ In above example.

Even and odd vertices

A vertex V of a graph G is called even if d(V) is even and V is called odd if d(V) is odd.

Regular graph

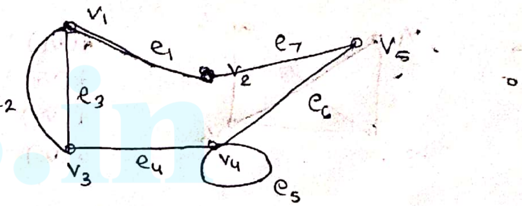
If d(v) is ~~the~~ same for every vertices in V, then G is a regular graph.

V ↓ vertex set

If G is k-regular. Then $d(v) = k$ $\boxed{\forall\ v \in V}$ Refer Eg. 3,4

⇓ degree of vertices is k

* Example 1:



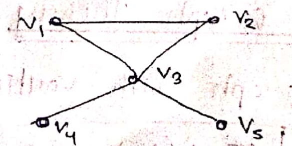
$d(V_1) = 3$, $d(V_2) = 2$, $d(V_3) = 3$,

$d(V_4) = 4$ (loop counted as twice) $d(V_5) = 2$ $d(V_6) = 0$

→ V_1, V_3 are odd vertices

→ V_2, V_4, V_5, V_6 are even vertices.

* Example 2-:



$d(V_1) = 2$

$d(V_2) = 2$

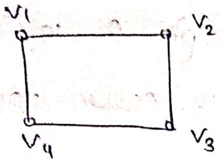
$d(V_3) = 4$

$d(V_4) = 1$

$d(V_5) = 1$

For simple graph → maximum value of degree of vertex $= n-1$ // n = no. of vertices

Example 3 - :



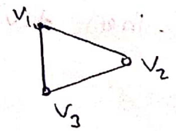
2-Regular graph

$d(v_1) = 2$
$d(v_2) = 2$
$d(v_3) = 2$
$d(v_4) = 2$

$\therefore$ This is an example of regular graph.

* complete graph is always regular graph.

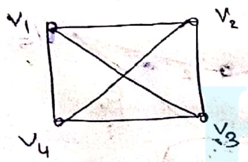
$K_3 \rightarrow$



$d(v_1) = 2$
$d(v_2) = 2$
$d(v_3) = 2$

2-Regular graph

$K_4 \rightarrow$

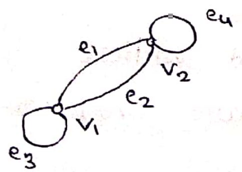


$d(v_1) = 3$
$d(v_2) = 3$
$d(v_3) = 3$
$d(v_4) = 3$

3- Regular graph

$K_n \longrightarrow (n-1)$ - Regular Graph

Example - 4



$\Longrightarrow$ 4- Regular graph

* 1^{st} theorem In Graph Theory

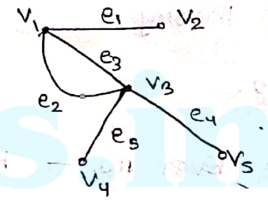
Let G be a graph with vertices $v_1, v_2 \dots v_n$ and 'e' edges, Then $\boxed{\sum_{i=1}^{n} d(v_i) = 2e}$

In any Graph with n vertices, sum of degree of all the vertices is twice the number of edges.

Proof -: Consider any edges joining two vertices v_i & v_j. This edge is counted both in $d(v_i)$ and $d(v_j)$.
Here each edge is counted twice in $\sum_{i=1}^{n} d(v_i)$

$$\therefore \boxed{\sum_{i=1}^{n} d(v_i) = 2e}$$

Example - :



$d(v_1) = 3$ $\quad d(v_2) = 1$
$d(v_3) = 4$ $\quad d(v_4) = 3$
$d(v_5) = 1$

$d(v_1) + d(v_2) + d(v_3) + d(v_4) + d(v_5) = \sum_{i=1}^{5} d(v_i) = 3+1+4+3+$
$= 12$

$e = 6$; $2e = 12$

$\therefore \sum_{i=1}^{n} d(v_i) = 2e$

* 2^{nd} Theorem in Graph Theory

The number of vertices of odd degree in a graph is always even.

Proof -: Let U be the set of even vertices and W be the set of odd vertices of G

ie. $d(u)$ is even $\forall u \in U$ & $d(w)$ is odd $\forall w \in W$

& $V = U \cup W$.

$\therefore \sum d(v) = \sum_{u \in U} d(u) + \sum_{w \in W} d(w)$ ——— (1)

$\because$ Sum of Even numbers are always even

$\therefore \sum_{u \in U} d(u) =$ Even no.

Also By 1st theorem of graph theory,

$\sum d(v) = 2e =$ Even no.

$\therefore$ From (1) $\sum_{w \in W} d(w) =$ Even no. $-$ Even no.

$=$ Even no.

i.e.

Each $d(w)$ is odd & $\sum d(w)$ is Even.

This is possible only if there exist even no. of odd vertices.

$\therefore$ In any graph G

There is even no. of odd degree vertices.

* Isolated vertex, Pendant vertex & Null graph

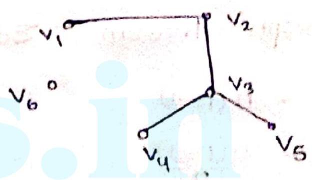
→ Isolated vertex -: A vertex having no incident edge is called an isolated vertex. It is vertex with degree 0.

→ Pendant vertex -: A vertex of degree 1 is called pendant vertex

→ Null graph or Empty Graph -: A graph $G = (V, E)$, it is possible for the edge set E to be empty. Such a graph without any edge is called null graph.

A graph G with every vertex is an isolated vertex is called Null graph.

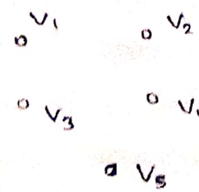
Example 1:



V_6 is isolated vertex

V_1, V_4, V_5 are pendant vertic

Example 2 :



This is an example of null graph.

Theorem 3

Prove that a complete graph is with n vertices has $\frac{n(n-1)}{2}$ edges.

Proof :

Let G be a complete graph with n vertices and e edges.

By the definition of complete graph, every vertex is adjacent to all other $(n-1)$ vertices.

$\therefore d(v) = n-1 \quad \forall v \in V$

Since there are total 'n' vertices,

$\therefore \sum d(v) = n(n-1)$ —— ①

By 1st theorem in Graph theorem ; $\sum d(v) = 2e$ — ②

From ① & ② ; $2e = n(n-1)$

$\Rightarrow e = \frac{n(n-1)}{2}$

$\therefore$ The number of edges in complete graph K_n is $\frac{n(n-1)}{2}$

Problems

1) Is it possible to have a group of 9 people, each knowing exactly 5 others.

Ans) Let 9 people represented by 9 vertices of a graph. Two vertices are adjacent if the corresponding person know each other.

For any vertex v, $d(v)$ is the number of friends

Given that $d(v) = 5 \quad \forall v \in V$

i.e. Each vertex of a graph is odd and there are 9 (odd no) such values.

It is not possible, because in any graph G, $\exists$ even number of odd values.

2) Is it possible to construct a simple graph of 12 vertices with 2 of them having degree 1, 3 of them having degree 3 and the remaining 7 having degree 10.

Soln) $n = 12$

Total degree of vertices $\sum d(v_i) = 1+1+(3\times3) + (7\times10)$

$= \underline{81}$

By 1'st theorem $\sum_{i=1}^{n} d(v_i) = 2\times e$, where e denote number of edges.

$\therefore 2e$ is always an even no.

$\therefore \sum d(v_i) = 81$ is not possible

So it is not possible to construct such a simple graph.

3) What is the largest number of vertices in a graph with 35 edges, if all vertices are of degree atleast 3.

Soln) Given that G has 35 edges & all vertices having atleast degree 3.

∴ Total degree = 2 × 35

e = 35 & $d(v_i) \geq 3$

By the 1st theorem, $\sum d(v_i) = 2e$
$= 2 \times 35$
$= 70$

∴ Sum of degree of vertices = 70

∴ Largest no. of vertices $n = \left\lceil \frac{70}{3} \right\rceil$ (least value)

$= \underline{\underline{23}}$

$= (\underbrace{3+3+3+\ldots 3}_{23 \text{ vertices}} + \underbrace{4}_{1 \text{ vertices}})$

4 Is it possible to construct a graph with 12 vertices such that 2 of the vertices have degree 3 and the remaining vertices have degree 4. Justify.

Ans) Number of vertices, n = 12

$\sum d(v_i) = (3 \times 2) + (4 \times 10) = \underline{\underline{46}}$

∴ Total degree of vertices in G = 46

By 1st theorem of graph theory, $\sum d(v_i) = 2 \times e$

∴ $46 = 2e \Rightarrow e = 23$//

Yes it is possible to construct a graph with 12 vertices and 23 edges having 2 vertices of degree 3 and 10 vertices of degree 4.

5 Find the smallest 'n' such that K_n has atleast 500 edges.

Ans $K_n \rightarrow$ complete graph with n ~~edg~~ vertices

In a complete graph, degree of vertices is same for all vertices.

From 1st theorem of graph, $\sum d(v_i) = 2e$
$= 2 \times 500 = \underline{1000}$

Let d be the degree of vertices

∵ degree of all vertices are same.

$\underbrace{d+d+d+d+\ldots d}_{n \text{ times}} = 1000$

$nd = 1000$

$d = \frac{1000}{n}$

Smallest value of n for which d gets an integer value is when $n = 2$, $d = \frac{1000}{2} = 500$

∴ Smallest n such that K_n has atleast 500 edges is when n = 2//

Q Prove that for any simple graph with atleast two vertices has of the same degree

Ans) Let G be a simple graph with n vertices.

If possible assume all the n-vertices has different degree say $d_1, d_2, \ldots d_n$ (n- different degrees)

We know that max degree of a vertex in a simple graph is $n-1$.

$\therefore$ These 'n' different degrees are $\{0, 1, 2, \ldots n-1\}$.

ie. There exist two vertices having degree 0 & $n-1$.

This is a contradiction.

It is not possible to have different degrees for these n vertices.

$\therefore$ For any simple graph G with atleast two vertices has of same degree.

H.W

1 In any group of n person, where $n \geq 2$, there are atleast two having same no. of friends.

Soln. Answer same as above

2 Let G be a n vertices and m edges. Assume that each vertex of G has ~~either~~ degree either k or $k+1$. show that the no. of vertices of degree k in G is $(k+1)n - 2m$.

Soln. Given that G has n vertices and m edges

Let there are 't' vertices having degree k and remaining $n-t$ vertices having degree $k+1$.

$\therefore$ Total degree of vertices in G is $kt + (k+1)(n-t)$

By First theorem, $\sum_{i=1}^{n} d(v_i) = 2e$

$\therefore kt + (k+1)(n-t) = 2m$

$\Rightarrow kt + kn - kt + n - t = 2m$

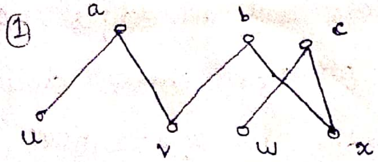
$\Rightarrow (k+1)n - t = 2m$

$\Rightarrow (k+1)n - 2m = t$

$\therefore$ No. of vertices having degree $k = n(k+1) - 2m$

* INP

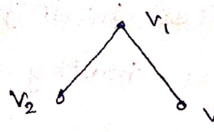
Bipartite Graph

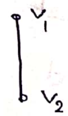
The graph $G = (V, E)$ is bipartite if the vertex set V can be partitioned into X & Y. ($X \cup Y = V$ & $X \cap Y = \phi$), such that each edge in E has one end in X and other end in Y. (No two vertices in X or Y are adjacent) e.g. (1) 

Bipartite graph

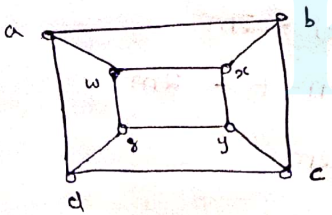
$V = \{a, b, c, u, v, w, x\}$

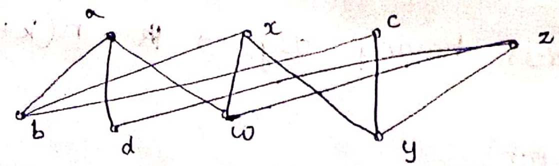
$X = \{a, b, c\}$ $Y = \{u, v, w, x\}$

(2) 

(3) $K_2 \rightarrow$ 

Q Check whether the following graph is Bipartite or not.



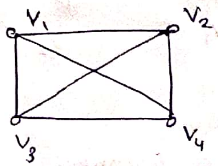
Soln 

$X = \{a, x, c, z\}$ $Y = \{b, d, w, y\}$

$X \cup Y = V$; $X \cap Y = \phi$

Each edge in G has one end in X and other end in Y. $\therefore$ G is bipartite.

Q Is the complete graph K_4 is bipartite or not. Justify

Soln. $K_4 \Rightarrow$ 

This is not bipartite since its vertex set cannot be partitioned into two sets.

* ~~All~~ complete graph K_n is not bartite except $n = 2$.

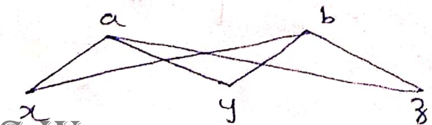
Complete Bipartite Graph

Let $G = (V, E)$ be a bipartite graph with partition $V = X \cup Y$ is called a complete bipartite graph if each vertex in X is adjacent to every vertex in Y.

If X contains 'm' vertices and Y contains 'n' vertices ; thus a complete bipartite graph is denoted by $K_{m,n}$.

Note-:

$K_{m,n}$ has $(m+n)$ vertices has mn edges.

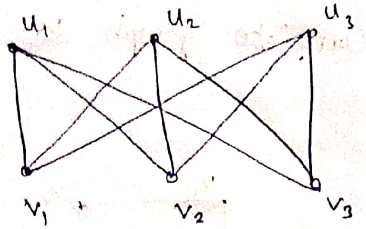
eg.  $\rightarrow K_{2,3}$

$2+3 \Rightarrow$ 5 vertices

$2 \times 3 \Rightarrow$ 6 edges

$V = \{a, b, x, y, z\}$ $X = \{a, b\}$ $Y = \{x, y, z\}$

$K_{3,3}$ →



3+3 → 6 vertices
3×3 → 9 edges

Order and size of a Graph

Order of a graph is the number of vertices in the graph

Size of a graph is the number of edges in the graph.

$K_{m,n}$ has order → $m+n$, size ⇒ mn.

K_n has order → n & size → $\frac{n(n-1)}{2}$

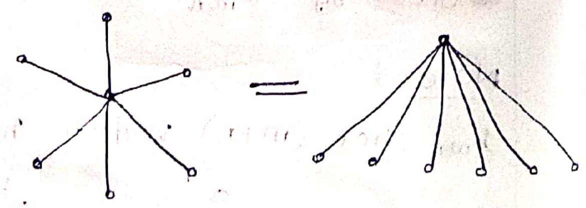
Note –:

Every complete Bipartite graph $K_{m,n}$ is bipartite.

* **Star graph**

A complete bipartite graph $K_{1,n}$ is called a star graph.

eg. 1) $K_{1,6}$ →



* IMP

Graph Isomorphism

Two graphs G and G' are said to be isomorphic to each other if there is one-to-one correspondence between their vertices and between their edges such that the incidence relationship is preserved.

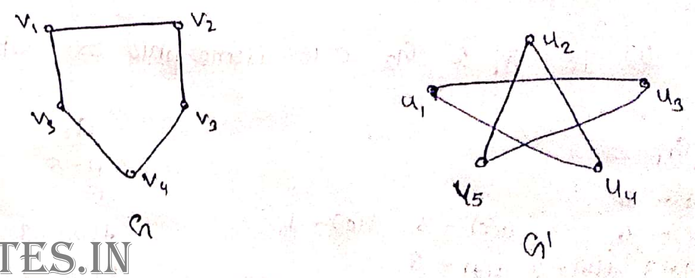
ie. suppose that an edge 'e' is incident on vertices v_1 & v_2 in G, then the corresponding edge e' in G' must be incident on the vertices v_1' & v_2' that correspond to v_1 & v_2 respectively.

Note

1) For two isomorphic graph G & G'
   a) same number of vertices
   b) same number of edges
   c) ~~An~~ equal no. of vertices with a given degree

2) Except for the labels of their vertices and edges, isomorphic graph are the same graphs perhaps drawn differently.

eg.



To check if G and G' are isomorphic :

in G

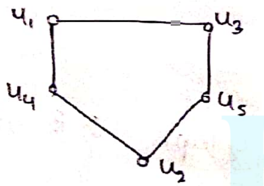
n = 5
e = 5

d(v₁) = d(v₂) = d(v₃) = d(v₄) = d(v₅) = 2

in G'

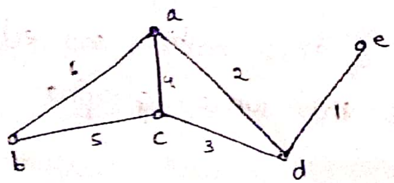
n = 5
e = 5
d(u₁) = d(u₂) = d(u₃) = d(u₄) = d(u₅) = 2

G' can be drawn as :-

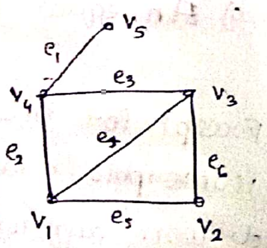


∴ G & G' are isomorphic, $G \cong G'$

eg. 2]



G_1



G_2

Q) Is G_1 & G_2 are isomorphic or not

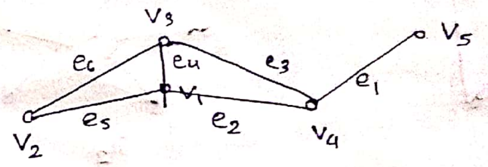
G_1

n = 5
e = 6
d(a) = 3 d(b) = 2 d(c) = 3 d(d) = 3 d(e) = 1

G_2

n = 5
e = 6
$d(v_1) = 3$ $d(v_3) = 3$ $d(v_5) = 1$
$d(v_2) = 2$ $d(v_4) = 3$

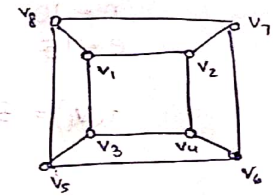
G_2 can be drawn as :-



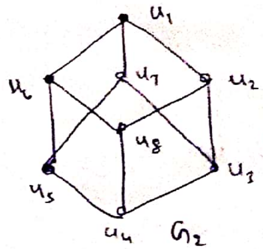
∴ G_1 & G_2 are isomorphic, $G_1 \cong G_2$

H.W

Q)



G_1



G_2

Is G_1 & G_2 isomorphic ?

Ans

in G_1

n = 8
e = 12

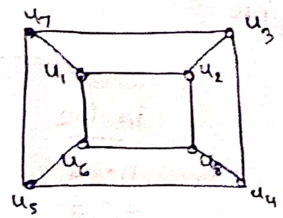
$d(v_1) = 3$ $d(v_2) = 3$
$d(v_3) = 3$ $d(v_4) = 3$
$d(v_5) = 3$ $d(v_6) = 3$
$d(v_7) = 3$ $d(v_8) = 3$

in G_2

n = 8
e = 12

$d(u_1) = 3$ $d(u_2) = 3$
$d(u_3) = 3$ $d(u_4) = 3$
$d(u_5) = 3$ $d(u_6) = 3$
$d(u_7) = 3$ $d(u_8) = 3$

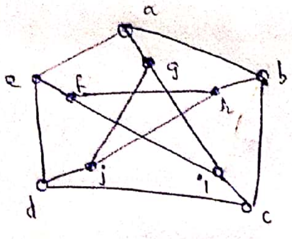
$G_2 \Rightarrow$



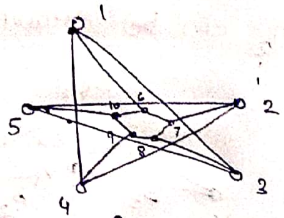
G_2

∴ G_1 & G_2 are isomorphic, $G_1 \cong G_2$

Q)



G_1



G_2

Is G_1 & G_2 is isomorphic ?

Ans) in G_1

$n = 10$

$e = 15$

$d(v_i) = 3$ for $\forall i$

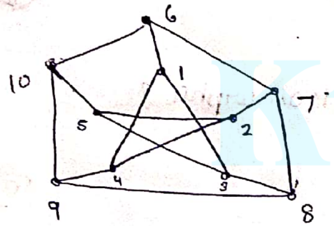
In G_2

$n = 10$

$e = 15$

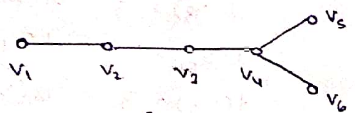
$d(u_i) = 3$ for $\forall i$

$G_2 \Longrightarrow$

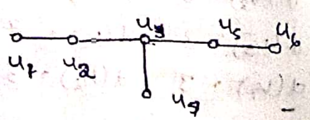


$\therefore$ G_1 and G_2 are isomorphic

Q



G_1



G_2

Is G_1 & G_2 isomorphic ?

Ans in G_1

$n = 6$

$e = 5$

$d(v_1) = 1$ $d(v_2) = 2$ $d(v_3) = 2$

$d(v_4) = 3$ $d(v_5) = 1$ $d(v_6) = 1$

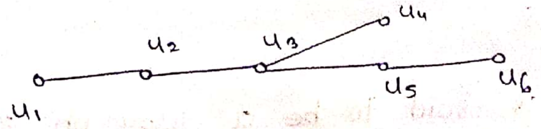
in G_2

$n = 6$

$e = 5$

$d(u_1) = 1$ $d(u_2) = 2$ $d(u_3) = 3$

$d(u_4)$ ... $d(u_5) = 2$ $d(u_6) = 1$



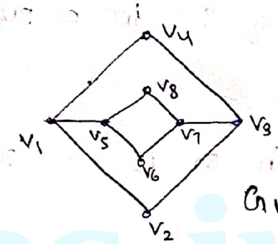
G_2

$\therefore$ Here G_1 & G_2 are not isomorphic

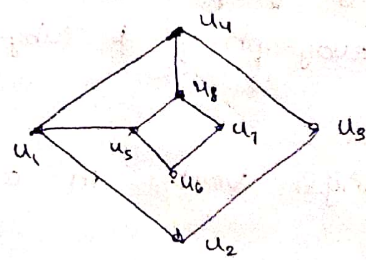
Reason - : In graph G_1 vertex v_3 is incident with 2 vertices whereas in graph G_2 the vertex u_3 is incident with 3 vertices.

$\therefore$ G_1 & G_2 are not isomorphic

Q



G_1



G_2

Soln In G_1

$n = 8$

$e = 10$

$d(v_1) = 3$ $d(v_2) = 2$ $d(v_3) = 3$

$d(v_4) = 2$ $d(v_5) = 3$ $d(v_6) = 2$

$d(v_7) = 3$ $d(v_8) = 2$

In G_2

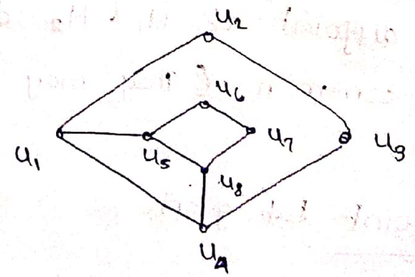
$n = 8$

$e = 10$

$d(u_1) = 3$ $d(u_2) = 2$ $d(u_3) = 2$

$d(u_4) = 3$ $d(u_5) = 3$ $d(u_6) = 2$

$d(u_7) = 2$ $d(u_8) = 3$



G_2

$\therefore$ G_1 & G_2 are not isomorphic

Reason - In graph G_1 vertex v_3 is incident with 3 vertex whereas in graph G_2 u_3 vertex is incident with 2 vertices

Sub - graphs

A graph H is said to be a subgraph of a graph G if all the vertices and all the edges of H are in G, and each edge of H has same end vertices in H as in G.

Note -:

1) Every graph is its own subgraph
2) A subgraph of a subgraph of G is a subgraph of G.
3) A single vertex in a graph G is a subgraph of G.
4) A single edge in G, together with its end vertices, is also a sub-graph of G.

Edge Disjoint sub graph

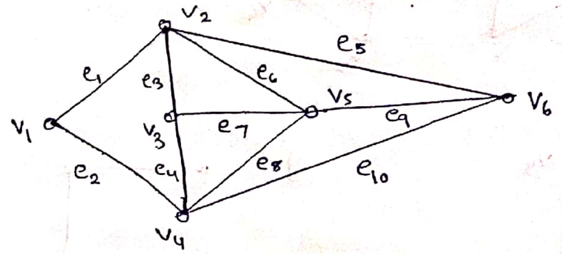
Two subgraphs H_1 & H_2 of a graph G are said to be edge disjoint if H_1 & H_2 do not have any edges in common. (They may have vertex in common)

Vertex Disjoint sub graph

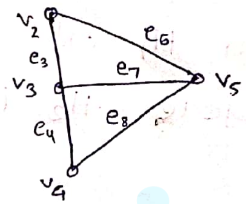
Sub graph that do not have vertices in common is said to be vertex disjoint sub graph (It is always edge disjoint)

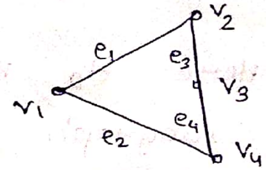
Example -:

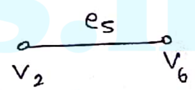
consider the graph G,

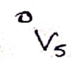


Subgraphs are -:

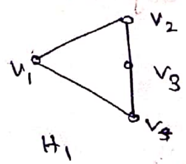
1) 

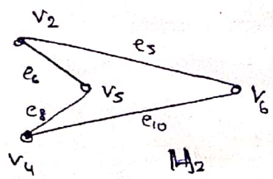
2) 

3) 

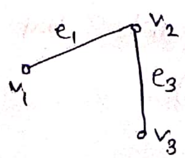
4) 

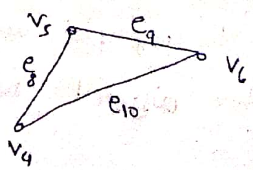
* Edge disjoint subgraphs -: H_1 & H_2 are edge disjoint.



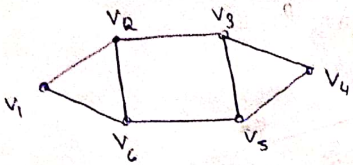


* Vertex disjoint subgraphs -:



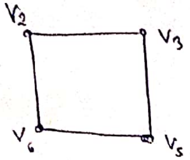


* Consider a graph G



subgraph of G,

1)



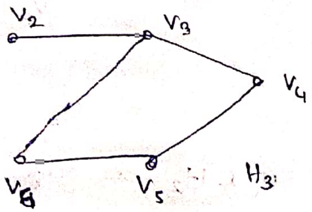
H_1

2)

H_2

Here H_2 is a subgraph H_1 and H_1 is a subgraph of G. Therefore H_2 is a subgraph of G.

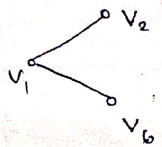
*



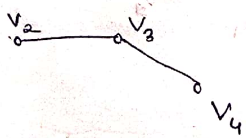
H_3

H_3 is not subgraph of G ∵ v_3 and v_6 are not adjacent in G.

*



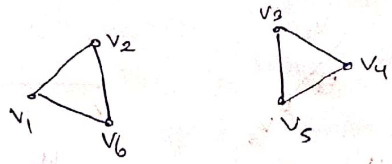
H_1



H_2

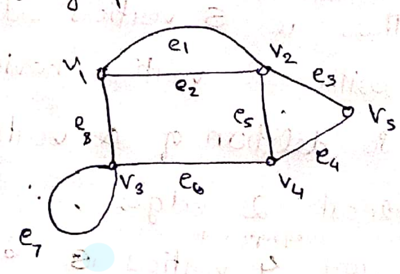
This two subgraphs are edge disjoint but not a vertex disjoint subgraph (v_2)

*



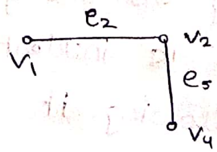
These subgraphs are vertex disjoint. ∴ It is edge disjoint too.

Q. Consider a graph G,

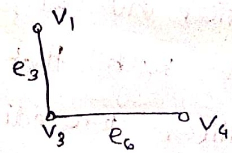


1) Draw 3 - subgraphs of G having 3 vertices
2) Draw any 3 edge disjoint sub graphs of G
3) Is it possible to draw a sub graph with 4 vertices and 7 edges.

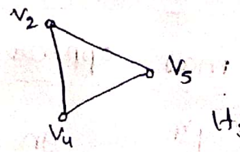
Soln) 1)



H_1



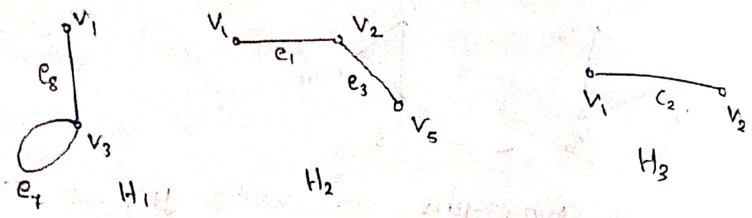
H_2



H_3

H_1, H_2, H_3 are 3 subgraphs with 3 vertices.

2)



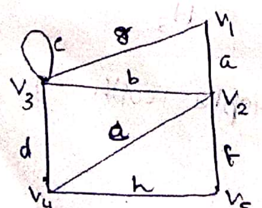
H_1, H_2, H_3 are edge disjoint

3) No, it is not possible to draw a subgraph of G with 4 vertices and 7 edges, ∵ There is 5 vertices and 8 edges in the graph, with all vertices having degree of atleast 2. So, deletion of a vertex lead to deletion of atleast 2 edges, ∴ Maximum edges possible for subgraph with 4 vertices is 6 edges.

* Walks, Paths and Circuit.

A walk is defined as a finite alternating sequence of vertices and edges, beginning and ending with vertices, such that each edge is incident with the vertices preceding and following it. No edge appears more than once in a walk. A vertex, however may appear more than once.

eg.



In graph G,

$v_1 \, a \, v_2 \, b \, v_3 \, c \, v_3 \, d \, v_4 \, e \, v_2 \, f \, v_5$ is walk

Set of vertices and edges constituting a given walk in a graph G is clearly a subgraph of G.

Vertices with which a walk begins and ends are called its terminal vertices. ∴ v_1 & v_5 are terminal vertices of above example.

It is possible for a walk to begin and end at the same vertex; such a walk is called a closed walk. A walk that is not closed (terminal vertices are distinct) is called an open walk.

eg. $v_1 \, g \, v_3 \, c \, v_3 \, b \, v_2 \, a \, v_1 \longrightarrow$ closed walk

$v_2 \, e \, v_4 \, d \, v_3 \, g \, v_1 \longrightarrow$ open walk.

$v_4 \, e \, v_2 \, f \, v_5 \, h \, v_4 \longrightarrow$ closed walk

$v_3 \, c \, v_3 \longrightarrow$ closed walk.

Path -:

An open walk in which no vertex appears more than once is called a path.

eg. $v_2 \, b \, v_3 \, d \, v_4 \, h \, v_5$

Number of edges in a path is called Length of a ~~graph~~ path. In above example Length = 3

Note -:

* A path does not intersect itself.
* Loop can be included in walk but not in Path.

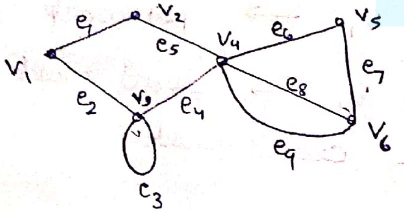
Circuit or cycle

A closed walk in which no vertex (except the initial and the final vertex) appears more than one is called a cycle or circuit.

ie.. A cycle or circuit is a closed, non intersecting walk.

eg) D-, V-g

1) Consider a graph G,,



1) walk of length 5 -:

$V_1\ e_1\ V_2\ e_5\ V_4\ e_6\ V_5\ e_7\ V_6\ e_9\ V_4$

2) A path of length 4

$V_1\ e_2\ V_3\ e_4\ V_4\ e_6\ V_5\ e_7\ V_6$

3) A Circuit of length 3

$V_4\ e_8\ V_6\ e_7\ V_5\ e_6\ V_4$

4) A circuit of length 2

$V_4\ e_8\ V_6\ e_9\ V_4$

5) A circuit of length 1 (loop)

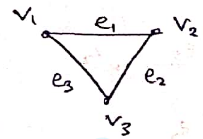
$V_3\ e_3\ V_3$

6) $V_1\ e_1\ V_2\ e_5\ V_4\ e_4\ V_3\ e_2\ V_1 \longrightarrow$ circuit

7) $V_1\ e_1\ V_2\ e_5\ V_4\ e_6\ V_5\ e_7\ V_6\ e_9\ V_4\ e_4\ V_3\ e_2\ V_1$

$\longrightarrow$ Not circuit (V_4 two times)

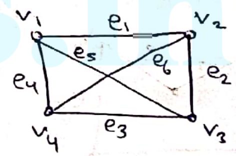
$K_3 \rightarrow$



$\longrightarrow$ cycle of length 3

$v_1\ e_1\ v_2\ e_2\ v_3\ e_3\ v_1$

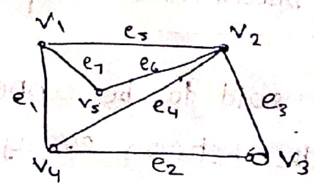
$K_4 \rightarrow$



$[v_1\ e_1\ v_2\ e_2\ v_3\ e_3\ v_4\ e_4\ v_1] \rightarrow 4$

$v_1\ e_5\ v_3\ e_3\ v_4\ e_2\ v_1 \longrightarrow 3$

Q. Consider a graph G



1) List all the paths from the vertex v_1 to vertex v_3

2) Is it possible to construct a path of length 5 in G.

3) Draw all the circuits in G.

Ans)

1) a) $v_1\ e_5\ v_2\ e_3\ v_3$

b) $v_1\ e_7\ v_5\ e_6\ v_2\ e_3\ v_3$

c) $v_1\ e_1\ v_4\ e_2\ v_3$

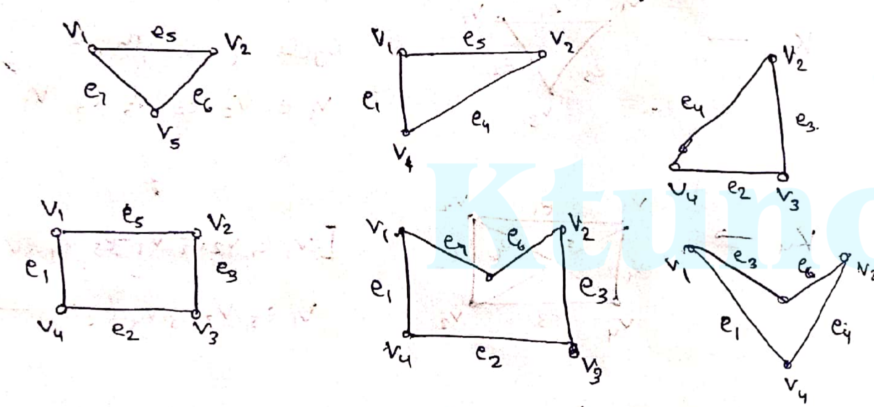
d) $v_1\ e_1\ v_4\ e_4\ v_2\ e_3\ v_3$

e) $v_1\ e_5\ v_2\ e_4\ v_4\ e_2\ v_3$

f) $v_1\ e_7\ v_5\ e_6\ v_2\ e_4\ v_3$

2) Not possible (Maximum length 4)

3)

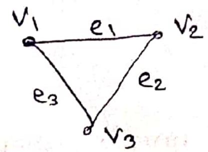


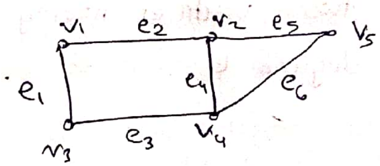
Connected graph, Disconnected graph & components

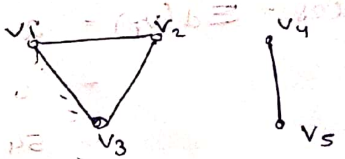
A graph G is said to be connected if there is atleast one path between every pair of vertices in G, otherwise it is called a Disconnected Graph.

A disconnected graph consist of two or more connected graphs. Each of this connected sub graph is called a component.

eg.

1)  → Connected graph

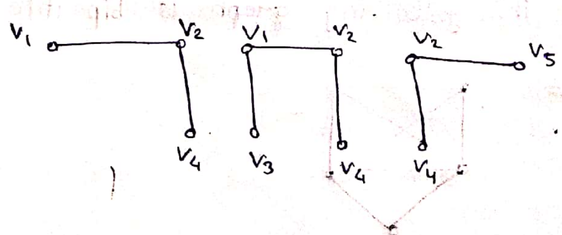
2)  → connected graph

3)  → Disconnected graph

Definition :-

A component of a graph is a maximal connected subgraph of G. The number of components of G is denoted by $\omega(G)$

If G is connected, then $\omega(G) = 1$.



Disconnected graph with 3 component $\omega(G) = 3$

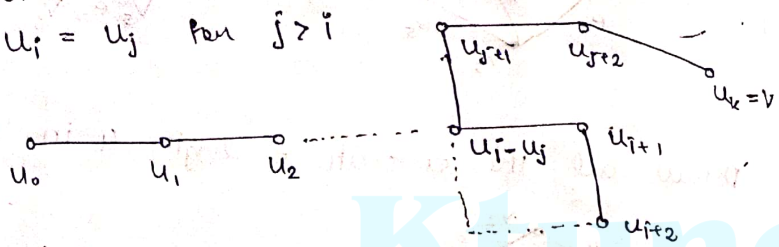
* **Theorem -:** In every graph a u-v walk contains a u-v path where $u, v \in V(G)$.

Proof -:

Let $W = u_0 e_1 u_1 e_2 u_2 e_3 \ldots u_{n-1} e_k u_k$ be a walk in a graph, where $u = u_0$ & $v = u_k$

If all the vertices in this walk are distinct then W is itself a path.

If not, let u_i be the first vertex repeated. ie.. $u_i = u_j$ for $j > i$



From 'w', we delete the vertices $u_{i+1}, u_{i+2} \ldots u_{j-1}$.

Then we have a shorter u-v walk.

$$W = u_0 e_1 u_1 e_2 \ldots u_i e_j u_{j+1} \cdots u_k$$

If all the vertices of our shorter walk W are distinct, Then W is the required path.

If not, repeat the same process, until u get a u-v walk ~~cont~~ with all the vertices distinct.

∴ Every u-v walk in G contains a u-v path.

* **Theorem -:**

If a graph (connected or disconnected) has exactly two vertices of odd degree, There must be a path joining these two vertices.

Proof -:

Let G be a graph, with all the vertices except V_1 & V_2 are even.

ie. V_1 & V_2 are two vertices having odd degree.

We know that every graph is having even no. of odd vertices (by a theorem).

∴ V_1 & V_2 must belong to the same component of a graph (Every component of a graph is itself a graph).

∴ V_1 & V_2 are two vertices in a connected components.

∴ There exist path joining V_1 & V_2 in G (By definition of connectedness)

* **Theorem -:**

A graph G is disconnected if and only if it's vertex set V can be partitioned into two non-empty disjoint sets, V_1 & V_2 such that there is no edge in G whose one end vertex is in subset V_1 and others in subset V_2.

Proof :-

Suppose that such a partition exists. Consider two arbitrary vertices a & b of G, such that $a \in V_1$ & $b \in V_2$.

Since there is no edge in G joining vertices between V_1 & V_2, No path can exist between the vertices a & b

$\therefore$ G is disconnected (By defn. of connectedness)

Conversely suppose G is disconnected,

$\therefore$ there exist min two components in G.

Let V_1 be the set of all vertices that are joined by path A. Since G is disconnected, V_1 does not include all vertices of G. The remaining vertices will form a non-empty, V_2. No vertex in V_1 in joined to any in V_2 by an edge

$\therefore$ Therefore such a partition exist.

* Theorem

A simple graph with n vertices and k components can have atmost $\frac{(n-k)(n-k+1)}{2}$ edges

Proof -:

Let G be a graph with n vertices and k components. Let $n_1, n_2 \dots n_k$ be the number of vertices in each of the k components respectively ... ie..

$$n_1 + n_2 + n_3 + \dots n_k = n$$

$$\Rightarrow \sum_{i=1}^{k} n_i = n \quad \text{--- (1)}$$

The maximum number of edges in the i^{th} component is given by $\frac{n_i(n_i-1)}{2}$.

The maximum number of edges in the graph G is

$$\sum_{i=1}^{k} \frac{n_i(n_i-1)}{2} = \frac{1}{2} \sum_{i=1}^{k} (n_i^2 - n_i)$$

$$= \frac{1}{2} \left[\sum_{i=1}^{k} n_i^2 - \sum_{i=1}^{k} n_i \right]$$

$$= \frac{1}{2} \left[\sum_{i=1}^{k} n_i^2 - n \right] \quad \text{from (1)}$$

$$\text{--- (2)}$$

Now consider $\sum_{i=1}^{k} (n_i - 1) = \sum_{i=1}^{k} n_i - \sum_{i=1}^{k} 1$

$$= n_i - k$$

$$\therefore \sum_{i=1}^{k} (n_i - 1) = n - k$$

Squaring both sides

$$\left[\sum_{i=1}^{k} (n_i - 1) \right]^2 = (n-k)^2$$

$$\Rightarrow \sum_{i=1}^{k} (n_i^2 - 2n_i + 1) = n^2 - 2nk + k^2$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 - 2 \sum_{i=1}^{k} n_i + \sum_{i=1}^{k} 1 = n^2 - 2nk + k^2$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 - 2n + k = n^2 - 2nk + k^2$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 = n^2 - 2nk + k^2 + 2n - k$$

Sub in ②,

Maximum number of edges in a graph G with n vertices and k components is

$$= \frac{1}{2}\left[\sum_{i=1}^{k} n_i^2 - n\right]$$

$$= \frac{1}{2}\left[n^2 - 2nk + k^2 + 2n - k - n\right]$$

$$= \frac{1}{2}\left[n^2 - 2nk + k^2 + n - k\right]$$

$$= \frac{1}{2}\left[(n-k)(n-k+1)\right]$$

$\therefore$ It can have $\frac{(n-k)(n-k+1)}{2}$ edges.

* **Theorem** :-

A simple graph G with n vertices must be connected if it has more than $\frac{(n-1)(n-2)}{2}$ edges.

Proof

Let G be a simple graph with 'n' vertices and 'E' edges.

We want to prove if $e > \frac{(n-1)(n-2)}{2}$ then graph G is connected.

So it is enough to prove if $e \leq \frac{(n-1)(n-2)}{2}$ then G is disconnected.

By a theorem, a graph with n vertices and k components can have atmost $\frac{(n-k)(n-k+1)}{2}$ edges.

i.e., $e \leq \frac{(n-k)(n-k+1)}{2}$

Here $e \leq \frac{(n-1)(n-2)}{2} \Rightarrow k = 2$

ie. the graph G can have more than 2 components.

$\therefore$ Graph G is disconnected.