CS648 : Randomized Algorithms
Semester II, 2016-17, CSE, IIT Kanpur

Practice sheet 3

The topics are:

- Random sampling
- Hashing
- Expected duration of a randomized experiment.

1. **Dominating set** Given an undirected graph $G = (V, E)$, a subset $S \subseteq V$ is said to be a dominating set if for each vertex $v \in V$ the following condition is satisfied:

   either v is present in S or some neighbor of v is present in S

   Suppose degree of each vertex in G is at least r. Design an $O(n)$ time Monte Carlo algorithm which computes a dominating set of size $O(n/r \log n)$ with high probability. Convert this algorthm into Las Vegas algorithm. What is its expected running time ?

2. **Hashing with larger space for hash table** Let H be a universal hash family. Suppose we wish to build a single level hash table only. Fortunately, we are allowed to have a hash table of size larger than s. Suppose we wish to have a worst case search time $O(s^{1/4})$. What should be the minimum value of n (size of hash table) that would serve this purpose. Design a Las Vegas algorithm to build such a hash table.

3. **Universal Hashing** Let p be a prime number larger than m (the size of universe U). Let $h_{a,b}$ be a hash function which maps any element $x \in U$ to $(ax + b)$mod p mod n. Let $H = \{h_{a,b} | 1 \le a < p, 0 \le b < p\}$. Prove that for any two elements $x_1, x_2 \in U$,

$$\mathbf{P}_{h \in_r H}(h(x_1) = h(x_2)) \le 1/n$$

4. **Coupon Collector Problem** Recall the coupon collector problem with n different coupons. What is the expected number of coupons to be collected to have $0.99n$ different coupons ?

5. **A biased random walk** Recall the random walk on a line discussed in the class. Suppose, probability of taking a step in the right direction is 3/4 and the probability of taking a step in the left direction is 1/4. What is the expected number of steps till the particle reaches nth milestone ?

6. **Random walk on a complete graph** We can extend the notion of random walk on a line to a graph in a very natural manner. Suppose we are at a vertex v. If $u_1, ..., u_t$ are the neighbors of v. The particle selects a vertex randomly uniformly from $\{u_1, \ldots, u_t\}$ and moves to that vertex in the next step.

   We are given a complete graph G on n vertices (there is an edge between each pair of vertices). Consider a random walk in G starting from a vertex u.

   (a) Let v be a vertex other than u. What is the expected number of steps of the random walk to visit v ?

   (b) What is the expected number of steps to visit each vertex of the graph at least once ?

   (c) What is the expected number of steps to visit half of the vertices ?

7. **A variant of client server problem**

Consider a parallel computer consisting of n processors and n memory modules. In the first step, each processor sends a memory request to a memory module selected randomly uniformly. If more than one processor sends request to the same memory module, the memory module discards all of them. If a memory module receives only one request, that request gets satisfied. After the first step, all those processors whose memory request was satisfied, leave the system. The remaining processors follow the same protocol in the following round. What is the expected number of rounds when all the memory request have been satisfied ?