CS648 : Randomized Algorithms
Semester II, 2016-17, CSE, IIT Kanpur

Practice sheet 5

The topics are:

- Random walks and Electric Networks
- Delay Sequence
- Probabilistic methods (Alteration)
- Miscellaneous Problems

1. **Commute time of a dense graph**
   Let G be an undirected graph on n vertices where degree of each vertex is at least $2n/3$. What is the maximum commute time in this graph ?

2. **Commute time of a lollipop graph**
   Let G be a complete graph on n vertices and H be a line graph on n vertices. We join one end of the graph H with any arbitrary vertex of G. This graph is called lollipop graph. What is the maximum commute time of this graph. Calculate it using

   (a) the relation between random walks and electric networks.

   (b) appropriate equations from scratch.

   Compare the two expressions you get.

3. **Cover time of a graph**
   Cover time of graph Let $G = (V, E)$ be an undirected graph on n vertices and m edges. Let $C(v)$ be the expected number of steps of a uniform random walk originating from v to visit every vertex at least once. Let $C(G) = \max_{v \in V} C(v)$. Show that with very high probability that $C(G) = O(mR \log n)$ where R is the maximum effective resistance between any two nodes in the electric circuit associated with G.
   Hint: Use the formula for commute time along with the tool of partitioning of an experiment into stages suitably.

4. **Generalizing the delay sequences to d-regular graphs**
   Recall the problem we used to demonstrate the concept of delay sequence. The counters were arranged along a line, the coin used by each counter was fair, and the maximum permissible difference in the count-value between any two neighboring counters was 1. The aim of the current exercise is to ensure that you fully internalize the elegant analysis we carried out.

   We would like to generalize this problem and analyse it using delay sequence. Suppose the counters are located on the vertices of a $d$regular graph, the probability of getting heads is p and the maximum permissible difference in the count-value of two neighboring counters is b. Try to derive the best possible bound on the number of rounds till every counter reaches a

value m. You may assume that $m > c \log n$ for any c that you will like to choose.
Hint: Proceed from scratch.

5. **Edge-coloring**
   Prove that, for every integer n, there exists a coloring of the edges of the complete graph K_n by two colors so that the total number of monochromatic copies of K_4 is at most $\binom{n}{4}2^{-5}$.

6. **Independent Set**
   Let G be a graph on n vertices, and $nd/2$ edges. Consider the following probabilistic experiment for finding an independent set in G.

   Delete each vertex of G (together with its incident edges) independently with probability $1-1/d$.

   (a) Compute the expected number of vertices and edges that remain after the deletion process.

   (b) Infer that there is an independent set with at least $n/2d$ vertices in any graph with $nd/2$ edges.

7. **Optional problem only for fans of backward analysis**
   If we select k points independently at random from interval $[0,1]$, we get a partition of the interval into $k+1$ sub-intervals. We shall try to calculate the expected value of the smallest of the $k+1$ intervals. This is a problem in continuous probability and since most of you are not so comfortable with continuous probability, we shall solve its discrete version :

   Given a set of n integers $1, 2, ..., n$, we select k integers (k is much smaller than n) from this set one after another, independently and without replacement. In this way, the set gets partitioned into $k+1$ intervals defined by the sampled points. For example, if $n = 10$ and we select two numbers : 4 and 8, then there are three intervals $[1,4], [4,8], [8,10]$. The smallest interval would be $[8,10]$ and its length is 2.

   Let random variable X_i denote the length of the smallest interval when i in tegers are selected according to the way described above. It can be observed that X_{i+1} is either X_i or smaller than X_i. We shall show that the expected length of the smallest interval is $\Theta(n/k^2)$ by using two different ways to calculate the probability of event $\mathcal{E}_{i+1}$ - "$X_{i+1} < X_i$". Here is a sketch.

   (a) Using Backward analysis, compute the probability of event $\mathcal{E}_{i+1}$.

   (b) Conditioned on $X_i = r$, show that the probability of event $\mathcal{E}_{i+1}$ is at least $\frac{(i+1)(r-1)}{n}$ and at most $\frac{2(i+1)(r-1)}{n}$. In other words,

   $$\frac{(i+1)(r-1)}{n} \leq \mathbf{P}[X_{i+1} < X_i | X_i = r] \leq \frac{2(i+1)(r-1)}{n} \tag{1}$$

   (c) Use (a) and (b) carefully to conclude that

   $$\mathbf{E}[X_i] = \Theta\left(\frac{n}{i^2}\right)$$