Lambda Calculus: Notational Simplification and Examples

We will write λ for the word *lambda* in an expression, and use a vertical bar to separate the parameter of a lambda function from its body. For example,

$$((\lambda x \mid (x\ y))\ a)$$

It is of the form $(Exp1\ Exp2)$ where $Exp1 = (\lambda x \mid (x\ y))$ and $Exp2 = a$.

Since we will use single letters to denote identifiers, we need not leave space between two identifiers. For example, the above expression could be written as

$$((\lambda x \mid (xy))\ a)$$

We may omit some parentheses, for the above example for instance, we may write

$$(\lambda x \mid xy)\ a$$

We omit parentheses by left association for expressions of more than two components, i.e.

$$N\ M\ Q = ((N\ M)\ Q)$$

Note that under this convention, $N\ M\ Q \neq N\ (M\ Q)$.

In what follows, a reduction step indicated by $\rightarrow$ means beta-reduction if not said otherwise.

We extend the syntax of lambda calculus by allowing multiple parameters such as

$$(\lambda xy|...)$$

In this case, when it comes to substitution, all occurrences of x and y are simultaneously replaced by their arguments. That is, they do not interfere with each other even in the case of name conflicts. For example,

$$\begin{aligned}
&(\lambda xy|xy)\ (\lambda y|y)\ (\lambda x|x) \\
&\rightarrow (\lambda y|y)\ (\lambda x|x) \\
&\rightarrow (\lambda x|x)
\end{aligned}$$

In the first reduction step above, the occurrence of x in xy is replaced by $(\lambda y|y)$ and the occurrence y in xy by $(\lambda x|x)$. The two replacements happen simultaneously. Better yet, simultaneous replacement doesn't cause name conflict

As we discussed earlier, an n-ary function can be viewed equivalently as a unary function, each parameter being dealt with at a time. So, a function with multiple parameters such as $(\lambda xy|...)$ can also be viewed equivalently as $(\lambda x|(\lambda y|...))$. Thus,

$$
\begin{aligned}
&(\lambda xy|xy)\ (\lambda y|y)\ (\lambda x|x) = (\lambda x|(\lambda y|xy))\ (\lambda y|y)\ (\lambda x|x) \\
&\to (\lambda y|(\lambda y|y)\ y)\ (\lambda x|x) \\
&\to (\lambda y|y)\ (\lambda x|x) \\
&\to (\lambda x|x)
\end{aligned}
$$

Reducing one parameter at a time is convenient when not enough actual parameters are available.

$$
\begin{aligned}
&(\lambda xz|xz)(\lambda y|x) \\
&\to (\lambda z|(\lambda y|x)\ z) \\
&\to (\lambda z|x)
\end{aligned}
$$

Note that $(\lambda y|x)z$ reduces to x.

Additional Exercises

Reduce the following λ-expressions to their normal forms. Show all the reduction steps.

(a)

$$
\begin{aligned}
&(\lambda x \mid (\lambda xy \mid xy)xy)s \\
&\to (\lambda xy \mid xy)sy \\
&\to sy
\end{aligned}
$$

(b)

$$
\begin{aligned}
&(\lambda xy \mid xxy)(\lambda x \mid xy) \\
&\to (\lambda w \mid (\lambda x \mid xy)(\lambda x \mid xy)w) \\
&\to (\lambda w \mid (\lambda x \mid xy)yw) \\
&\to (\lambda w \mid yyw)
\end{aligned}
$$

(c)

$$
\begin{aligned}
&(\lambda xy \mid x(xy))\ (\lambda z \mid sz) \\
&\to (\lambda y \mid (\lambda z \mid sz)((\lambda z \mid sz)y)) \\
&\to (\lambda y \mid (\lambda z \mid sz)(sy)) \\
&\to (\lambda y \mid s(sy))
\end{aligned}
$$

Note that $(\lambda y \mid s(sy))$ doesn't equal $(\lambda y \mid ssy)$. By the simplified notation, $ssy = ((ss)y)$

(c) This example is more involved notationally.

$$
\begin{aligned}
&((\lambda xy \mid x\,(\lambda y \mid (+\,y\,2)))\;(\lambda fx \mid f\,(f\,x)))\;5 \\
&\to (\lambda y \mid (\lambda fx \mid f\,(f\,x))\,(\lambda y \mid (+\,y\,2)))\;5 \\
&\to (\lambda fx \mid f\,(f\,x))\,(\lambda y \mid (+\,y\,2)) \\
&\to (\lambda x \mid (\lambda y \mid (+\,y\,2))((\lambda y \mid (+\,y\,2))x) \\
&\to (\lambda x \mid (\lambda y \mid (+\,y\,2))(+\,x\,2)) \\
&\to (\lambda x \mid (+\,(+x\,2)\,2)))
\end{aligned}
$$