CMPUT 466 Assignment 4

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Problem 1

If $P(X,Y) = f(X)g(Y)$ for some function f on X only, then X and Y are independent.

Proof. Using marginal probability,

$$P(X)P(Y) = \int_X P(X,Y)dX \int_Y P(X,Y)dY$$

$$= \int_X f(X)g(Y)dX \int_Y f(X)g(Y)dY$$

$$= f(X)g(Y) \int_X f(X)dX \int_Y g(Y)dY$$

$$= P(X,Y) \int_X f(X)dX \int_Y g(Y)dY$$

$$= P(X,Y)$$

Thus, we have proven that if $P(X,Y) = f(X)g(Y)$, $P(X,Y) = P(X)P(Y)$.

This is the definition for independent random variables, so X and Y must be independent.

The proof is similar in the discrete case, just replace $\int$ with $\sum$

□

Problem 2

$\mathbb{E}_{X\sim P(X)}[af(X) + bg(X)]$ is a linear system.

Proof. Using the definition for $\mathbb{E}$,

$$\mathbb{E}_{X\sim P(X)}[af(X) + bg(X)] = \sum_X P(X)(af(X) + bg(X))$$

$$= \sum_X aP(X)f(X) + \sum_X bP(X)g(X)$$

$$= a \sum_X P(X) f(X) + b \sum_X P(X) g(X)$$

applying the definition again,

$$\mathbb{E}_{X \sim P(X)}[af(X) + bg(X)] = a\mathbb{E}_{X \sim P(X)}[f(X)] + b\mathbb{E}_{X \sim P(X)}[g(X)]$$

This precisely fits the definition of a linear system. The proof is similar in the continuous case, just replace $\sum$ with $\int$

$\square$

Problem 3

- $X \sim U[a, b]$ continuous random variable
- uniformly distributed
- a, b unknown parameters
- dataset $\{x^{(m)}\}_{m=1}^M$

a) Likelihood of parameters

because of the above information,

$$\mathcal{L}(a, b; \mathcal{D}) = \Pi_{m=1}^m \frac{1}{b-a} = \frac{1}{(b-a)^m}$$

- note: if $x^{(m)} = a = b$, then the likelihood is infinite
- if $x^{(m)} \notin [a, b]$, then the likelihood is zero

The log likelihood is:

$$\log \frac{1}{(b-a)^m} = -m \log(b-a)$$

b) MLE of parameters

- the derivative of the log likelihood with respect to a is $\frac{m}{b-a}$
  - we notice that this is monotonically increasing, so MLE for a is the largest a possible, i.e.

$$\hat{a} = \min_m \{x^{(m)}\}$$

- the derivative of the log likelihood with respect to b is $-\frac{m}{b-a}$
  - we notice that this is monotonically decreasing, so MLE for b is the smallest b possible, i.e.

$$\hat{b} = \max_m \{x^{(m)}\}$$

Problem 4

c) Prove MLE is biased in this case

- let $B \in [a, b]$

  Then,

$$
\begin{aligned}
Pr[\hat{b} \leq B] &= \Pi_{m=1}^{M} Pr[x^{(m)} \leq B] \\
&= \left(\frac{B - a_*}{b_* - a_*}\right)^M
\end{aligned}
$$

This is the cumulative probability density function $F_{\hat{b}}(B)$

$$
f_{\hat{b}}(B) = \frac{d}{dB} F_{\hat{b}}(B) = M \frac{(B - a_*)^{(M-1)}}{(b_* - a_*)^M}
$$

$$
\mathbb{E}_{x^{(m)} \sim^{iid} U[a_*, b_*]}[\hat{b}] = \int_{a_*}^{b_*} \frac{\hat{b} M (\hat{b} - a_*)^{(M-1)}}{(b_* - a_*)^M} d\hat{b}
$$

$$
\begin{aligned}
&= \frac{M}{(b_* - a_*)^M} \int_{a_*}^{b_*} \hat{b}(\hat{b} - a_*)^{(M-1)} d\hat{b} \\
&= \frac{M}{(b_* - a_*)^M} \int_{a_*}^{b_*} a_*(\hat{b} - a_*)^{(M-1)} + (\hat{b} - a_*)^M d\hat{b} \\
&= \frac{M}{(b_* - a_*)^M} \left(a_* \int_{a_*}^{b_*} (\hat{b} - a_*)^{(M-1)} d\hat{b} + \int_{a_*}^{b_*} (\hat{b} - a_*)^M d\hat{b} \right) \\
&= \frac{M}{(b_* - a_*)^M} \left(\left. \frac{a_*(\hat{b} - a_*)^M}{M} \right|_{a_*}^{b_*} + \left. \frac{(\hat{b} - a_*)^{(M+1)}}{M+1} \right|_{a_*}^{b_*} \right) \\
&= \frac{M}{(b_* - a_*)^M} \left(\frac{a_*(b_* - a_*)^M}{M} + \frac{(b_* - a_*)^{(M+1)}}{M+1} \right) \\
&= a_* + \frac{M(b_* - a_*)}{M+1} \\
&\neq b_*
\end{aligned}
$$

This proves that the MLE is biased.

Similarly, we can see that:

$$
\mathbb{E}[\hat{a}] = a_* + \frac{b_* - a_*}{M+1}
$$

, which is $\neq a_*$

d) Prove MLS is asymptotically unbiased if $M \to +\infty$

$$\lim_{M\to\infty} \mathbb{E}[\hat{a}] = \lim_{M\to\infty} a_* + \frac{b_* - a_*}{M+1}$$

,

$$= a_*$$

$$\lim_{M\to\infty} \mathbb{E}[\hat{b}] = \lim_{M\to\infty} a_* + \frac{M(b_* - a_*)}{M+1}$$
$$= \lim_{M\to\infty} \frac{(a_*M + a_*) + b_*M - a_*M}{M+1}$$
$$= \lim_{M\to\infty} \frac{a_*}{M+1} + \frac{b_*M}{M+1}$$
$$= 0 + b_*$$
$$= b_*$$

This proves that as $M \to +\infty$, $\mathbb{E}[\hat{a}] = a$ and $\mathbb{E}[\hat{b}] = b$, which fits the definition for unbiased.

- note: $x^{(m)} \sim^{iid} U[a_*, b_*]$ should be under the $\mathbb{E}$ symbols but was dropped for brevity