Problem 1

Since 90% of sampless are positive and we always predict positive, we have the following:

$$\begin{cases} TP = 0.9 \\ FP = 0.1 \\ TN = 0.0 \\ FN = 0.0 \end{cases}$$

$$\begin{aligned} Precision &= \frac{TP}{TP + FP} = \frac{0.9}{0.9 + 0.1} = 0.9 \\ Recall &= \frac{TP}{TP + FN} = \frac{0.9}{0.9 + 0.0} = 1.0 \\ F_1 &= \frac{2 \times Precision \times Recall}{Precision + Recall} = \frac{2 \times 0.9 \times 1.0}{0.9 + 1.0} \approx 0.947 \end{aligned}$$

F_1 score is superficially high in this case because the positive case is the majority case. It indicates that our predictor is doing a decent job at identifying the positive cases.

However, we sometimes care more about the minority cases. We can resurrect the metrics in this case by flipping our definition of positives and negatives.

Suppose 90% of the samples are negative, and we still take the majority guess (always predicting negative). If we apply the same formulas for calculating precision, recall and F_1, then we will encounter division by zero.

Therefore, let x be the probability for predicting negative, and we will calculate the metrics via taking the limit:

$$\begin{cases} TP = 0.1(1 - x) = 0.1 - 0.1x \\ FP = 0.9(1 - x) = 0.9 - 0.9x \\ TN = 0.9x \\ FN = 0.1x \end{cases}$$

$$\begin{aligned} Precision &= \frac{TP}{TP + FP} \\ &= \lim_{x \to 1} \frac{0.1 - 0.1x}{0.1 - 0.1x + 0.9 - 0.9x} \\ &= \lim_{x \to 1} \frac{0.1 - 0.1x}{1 - x} \\ &= \lim_{x \to 1} \frac{-0.1}{-1} && \text{[L'Hopital's rule]} \\ &= 0.1 \end{aligned}$$

$$
\begin{aligned}
Recall &= \frac{TP}{TP + FN} \\
&= \lim_{x \to 1} \frac{0.1 - 0.1x}{0.1 - 0.1x + 0.1x} \\
&= \lim_{x \to 1} \frac{0.1 - 0.1x}{0.1} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
F_1 &= \frac{2PR}{P + R} \\
&= \frac{2 \times 0.1 \times 0}{0.1 + 0} \\
&= 0
\end{aligned}
$$

Problem 2

To show $\sigma(z)$ is symmetric about $(0, 0.5)$, we need to show $\sigma(-z) = 1 - \sigma(z)$. Equivalently, we can show $\sigma(-z) + \sigma(z) = 1$.

$$
\begin{aligned}
\sigma(-z) + \sigma(z) &= \frac{1}{1 + e^{z}} + \frac{1}{1 + e^{-z}} \\
&= \frac{1 + e^{-z} + 1 + e^{z}}{(1 + e^{z})(1 + e^{-z})} \\
&= \frac{1 + e^{-z} + 1 + e^{z}}{1 + e^{-z} + e^{z} + 1} \\
&= 1
\end{aligned}
$$

Problem 3

$$
\begin{aligned}
D_{KL}(\mathbf{t}||\mathbf{y}) &= t\ log(\frac{t}{y}) + (1-t)log(\frac{1-t}{1-y}) \\
&= t\ log(t) - t\ log(y) + (1-t)log(1-t) - (1-t)log(1-y) \\
\operatorname*{argmin}_y D_{KL}(\mathbf{t}||\mathbf{y}) &= \operatorname*{argmin}_y[-t\ log(y) - (1-t)log(1-y)] \qquad \text{[Drop constant terms w.r.t y]} \\
&= \operatorname*{argmin}_y J
\end{aligned}
$$