CMPUT 466 Assignment 2

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Problem 1

Part 1

$$\{(x_1, x_2) : x_1^2 + x_2^2 = 1\}$$

This set is not convex.

Proof. Assume the set is convex.

Let's draw a line from $(x_1, x_2) = (-1, 0)$ to $(1, 0)$

These two points are part of the set because $(-1)^2 + 0^2 = 1$,

and $1^2 + 0^2 = 1$

That is, $x_1 \in [-1, 1], x_2 = 0$

If this set is convex, then any point on this line must also be in the set.

Let's select the point $(0, 0)$. i.e., $\lambda = 0.5$

This point is clearly part of the line defined above, however, it is definitely not in the set because $0^2 + 0^2 = 0 \neq 1$.

Therefore, we have found a counterexample that proves that this set is not convex. □

Part 2

$$\{(x_1, x_2) : |x_1| + |x_2| \leq 1\}$$

This set is convex.

We begin by looking at the definition:

let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ be any two points in the set.

Using the definition, if the set is convex, then

$$\lambda x + (1 - \lambda)y, \lambda \in (0, 1)$$

is also in the set.

Applying this definition, we get the point:

$$\begin{pmatrix} \lambda x_1 + (1 - \lambda)y_1 \\ \lambda x_2 + (1 - \lambda)y_2 \end{pmatrix}$$

If this point is in the set, we need to prove that:

$$|\lambda x_1 + (1-\lambda)y_1| + |\lambda x_2 + (1-\lambda)y_2| \leq 1$$

Proof. Using the triangle inequality: $|x+y| \leq |x| + |y|$:

$$|\lambda x_1 + (1-\lambda)y_1| + |\lambda x_2 + (1-\lambda)y_2| \leq |\lambda x_1| + |(1-\lambda)y_1| + |\lambda x_2| + |(1-\lambda)y_2|$$

we can factor out λ and $(1-\lambda)$ from the right side, because by definition, $\lambda \in (0,1)$:

$$\lambda(|x_1| + |x_2|) + (1-\lambda)(|y_1| + |y_2|)$$

Because $\mathbf{x}$ and $\mathbf{y}$ are in the set, we know that:

$$|x_1| + |x_2| \leq 1$$

$$|y_1| + |y_2| \leq 1$$

Because of the above and $\lambda \in (0,1)$, we also know that:

$$\lambda(\dots) + (1-\lambda)(\dots) \leq 1$$

(The values where the dots are were determined to be ≤ 1)

Thus,

$$\lambda(|x_1| + |x_2|) + (1-\lambda)(|y_1| + |y_2|) \leq 1$$

And because:

$$|\lambda x_1 + (1-\lambda)y_1| + |\lambda x_2 + (1-\lambda)y_2| \leq \lambda(|x_1| + |x_2|) + (1-\lambda)(|y_1| + |y_2|) \leq 1$$

We have finally proven that:

$$|\lambda x_1 + (1-\lambda)y_1| + |\lambda x_2 + (1-\lambda)y_2| \leq 1$$

, which means that the set is indeed convex.

□

Problem 2

$$f(x_1, x_2) = x_1^2 + x_2^2 - 4x_1x_2$$

a) x_1 variable, x_2 constant:

$$\begin{aligned} \nabla^2 f(x_1) &= \nabla 2x_1 - 4x_2 \\ &= 2 \geq 0 \end{aligned}$$

Because the second order gradient of $f(x_1, x_2)$ is positive when x_2 is constant, we have determined that f is convex in x_1

b) x_2 variable, x_1 constant

$$\nabla^2 f(x_2) = \nabla 2x_2 - 4x_1$$
$$= 2 \geq 0$$

Because the second order gradient of $f(x_1, x_2)$ is positive when x_1 is constant, we have determined that f is convex in x_2

c) $f : \mathbb{R}^2 \to \mathbb{R}$

```
import numpy as np
from numpy import linalg as LA
# in a) and b) we found the values 2 and 2 in the Hessian matrix
# -4 is obtained because del/ (del x_1 del_x2) f(x_1, x_2) = -4
# same for del/ (del x_2 del_x1) because the matrix is symmetric
H = np.array([
    [2, -4],
    [-4, 2]
            ])
eigenval, eigenvec = LA.eig(H)
print(f"eigenval: {eigenval} | eigenvector: {eigenvec} ")

if any(map(lambda x: x < 0, eigenval)):
    print("Because one of the eigenvalues are negative, we conclude that the function is not
else:
    print("Because all of the eigenvalues are positive, we conclude that the function is con
```

```
eigenval: [ 6. -2.] | eigenvector: [[ 0.70710678  0.70710678]
 [-0.70710678  0.70710678]]
Because one of the eigenvalues are negative, we conclude that the function is not convex
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Problem 3

If f is a differentiable convex function, then f satisfies the first-order condition

Proof. Given f is a differentiable convex function, the definition holds:

$$\forall x, y \in dom f, \forall \lambda \in (0, 1), f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

Let's do some re-arranging:

$$f(\lambda x + y - \lambda y)) \leq \lambda f(x) + f(y) - \lambda f(y)$$

$$f(\lambda(x - y) + y)] \leq \lambda(f(x) - f(y)) + f(y)$$

$$\Rightarrow \frac{f(\lambda(x-y)+y)-f(y)}{\lambda} \leq f(x)-f(y)$$

The left side of this equation looks exactly like the definition of the derivative using limits.

As $\lambda \to 0$,

$$\Rightarrow \nabla[f(y)]^T(x-y) \leq f(x)-f(y)$$

$$\nabla[f(y)]^T(x-y)+f(y) \leq f(x)$$

This is equivalent to the first-order condition:

$$f(y) \geq f(x) + \nabla[f(x)]^\top(y-x), \forall x, y \in dom f$$

□

Problem 4

Proof. Because

$$z = (1-\lambda) + \lambda y$$

,we can find $||y-x||$ with some rearranging:

$$z = x - \lambda x + \lambda y = x + \lambda(y-x)$$

$$z - x = \lambda(y-x)$$

$$||z-x|| = ||\lambda(y-x)|| = \lambda||y-x||$$

Using $\lambda = \frac{\epsilon}{2||y-x||}$:

$$\lambda||y-x|| = ||z-x|| = \frac{\epsilon}{2}$$

So,

$$\lambda||y-x|| < \epsilon$$

Because $\lambda \in (0,1)$, λ cannot be greater than 1, so:

$$||y-x|| < \epsilon$$

□

Problem 5

Intuitively, we know that if the gradient of f is $\mathbf{0}$ at a point, it is either a local minimum or local maximum of the function. We know that it must be a local minimum because no part of the function curves down We know this, because if any neighboring point around $\mathbf{x}$ does not have a gradient equal to $\mathbf{0}$, we can simply do gradient descent to get to the minimum, where the gradient is $\mathbf{0}$. In class, we went over a theorem with a proof that if $\mathbf{x}$ is a local minimum, and f is convex, then $\mathbf{x}$ is a global minimum. Because $\mathbf{x}$ is a local minimum, by this theorem, $\mathbf{x}$ is also a global minimum.

Now, for a more formal proof:

Proof. From the first-order condition for convex functions we have:

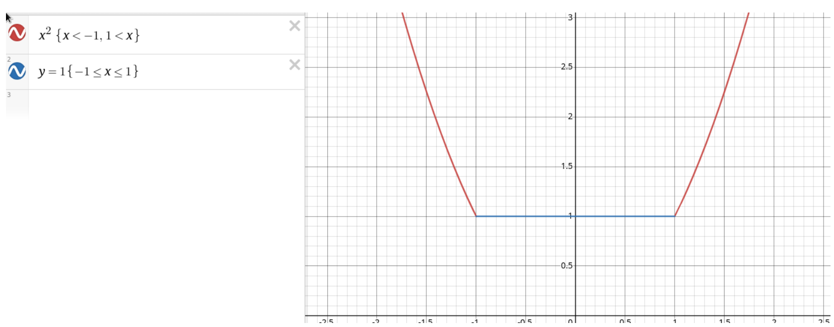
$$\forall x, y \in dom f, f(y) \geq f(x) + [\nabla_x f(x)]^\top (y - x)$$

Because $\nabla f(x) = \mathbf{0}$,

$$\forall x, y \in dom f, f(y) \geq f(x)$$

This is precisely the definition of a global optimum. $\square$

The global optimum is not necessarily unique. Picture a convex function where the minimum is a straight line. Any point along this line would be the minimum of the function, while still satisfying the condition $\forall x, y \in dom f, f(y) \geq f(x)$. For example, the plot below:



Problem 6

$$J = \sum_{m=1}^{M}\Big(\underbrace{\sum_{i=0}^{d} w_i x_i^{(m)}}_{Xw} - \underset{\sim}{t^{(m)}}_{M}\Big)^2 + \underbrace{\sum_{i=0}^{d} w_i^2}_{w^T w}$$

$M \times (d+1)\ (d+1)$ $\qquad\qquad$ $1 \times (d+1)\ (d+1) \times 1$

M $\qquad\qquad$ $\in \mathbb{R}$

$$\sum_{m=1}^{M} (Xw - t)^2 + w^T w$$

$$(Xw - t)^T (Xw - t) + w^T w$$

$$(w^T X^T - t^T)(Xw - t) + w^T w$$

$$\underbrace{w^T X^T X w}_{1 \times (d+1)\ (d+1) \times M\ M \times (d+1)\ (d+1) \times 1} - \underbrace{t^T X w}_{1 \times M\ M \times (d+1)\ (d+1) \times 1} - \underbrace{w^T X^T t}_{1 \times (d+1)\ (d+1) \times M\ M \times M} + \underbrace{t^T t}_{1 \times M\ M \times 1} + \underbrace{w^T w}_{\in \mathbb{R}}$$

$\in \mathbb{R}$ $\qquad$ $\in \mathbb{R}$ $\qquad$ $\in \mathbb{R}$ $\qquad$ $\in \mathbb{R}$

$$(w^T X^T t)^T = t^T X w$$

scalar transpose is itself.

$$J = w^T X^T X w - 2 t^T X w + t^T t + w^T w$$

$$\nabla J(w) = 2X^TXw - 2(t^TX)^T + 2w$$

$$= 2X^TXw - 2X^Tt + 2w$$

$$\nabla\nabla J(w) = 2(X^TX)^T + 2$$

$$= \underbrace{2\underbrace{X^TX}_{\geq 0} + 2}_{\text{definitely } > 0}$$

because this value is greater than 0, J is convex in w

Problem 7

Prove $J(w - \alpha u) \leq J(w)$ for small enough α

Taylor approximation:

$$J(w - \alpha u) \approx J(w) + [\nabla J(w)]^T[-\alpha u] + \underbrace{o(\alpha^2)}_{\text{some other higher order terms}}$$

as $\alpha \to 0$

$$= J(w) - \alpha [\nabla J(w)]^T u + \underbrace{\cancel{o(\alpha^2)}}_{\text{small, can ignore as } \alpha \to 0}$$

and $\alpha \geq 0$

$$J(w - \alpha u) + \underbrace{\alpha \underbrace{[\nabla J(w)]^T u}_{\geq 0}}_{\geq 0} = Jw + \cancel{o(\alpha^2)}$$

so,

$$J(w - \alpha u) \leq Jw$$

Problem 8

In #6 we computed the gradient:

$$\nabla J(w) = 2X^TXw - 2X^Tt + 2w$$

Gradient descent:

initialize $w^{(0)}$ randomly

for iteration $t = 1, 2, 3, \ldots$ until satisfied

Compute gradient $\nabla J(w)|_{w=w^{(t-1)}}$

$$g(t) = 2X^TXw^{(t-1)} - 2X^Tt + 2w^{(t-1)}$$

update parameters $w^{(t)} = w^{(t-1)} - \alpha J(w)|_{w=w^{(t-1)}}$

$$w(t) = w^{(t-1)} - \underset{\uparrow \\ \text{step size}}{\alpha}\left[2X^TXw^{(t-1)} - 2X^Tt + 2w^{(t-1)}\right]$$

return last $w^{(t)}$