Problem 1.

Prove that, if $P(X, Y) = f(X)g(Y)$ for some function f on Xonly, and $g$on Yonly, then X and Y are independent.

Hint: Use $P(X, Y) = P(X)P(Y)$ to prove independence, which is simply the definition. To obtain $P(X)$ and $P(Y)$, use the definition of marginal probability. The rule of the thumb is that, when we don't know how to start a proof, it's a good indicator that the proof is simple. Just follow the definitions.

Problem 2.

Prove that the expectation is a linear system.

$$\mathbb{E}_{X \sim P(X)}[a\, f(X)\, +\, b\, g(X)] = a\, \mathbb{E}_{X \sim P(X)}[f(X)]\, +\, b\, \mathbb{E}_{X \sim P(X)}[g(X)]$$

You may treat X as a discrete variable.

Hint: Again, use the definition $\mathbb{E}_{X \sim P(X)}[f(X)] = \sum_{X} P(X) f(X)$

Problems 3 and 4 concern the following setting.

Let $X \sim U[a, b]$ be a continuous random variable uniformly distributed in the interval $[a, b]$, where a and b are unknown parameters.

We have a dataset $\{x^{(m)}\}_{m=1}^{M}$, where each data sample is iid drawn from the above distribution, and we would like to estimate the parameters a and b.

Problem 3.

(a) Give the likelihood of parameters.

(b) Give the maximum likelihood estimation of parameters.

Problem 4.

(c) Prove that MLE is biased in this case.

(d) Prove that MLE is asymptotically unbiased if $M \to +\infty$.

Hint: A parameter estimation being biased is said in terms of the current dataset $\{x^{(m)}\}_{m=1}^{M}$, although we imagine that $\{x^{(m)}\}_{m=1}^{M}$ could be repeatedly drawn from a repeatable trial. In other words, we do not assume M goes to infinity in (c). Being *asymptotically unbiased* means that, if M goes to infinity, the bias would become smaller and converge to 0.

END OF W4