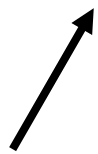
Lecture 6

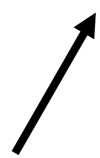
Abstract vector spaces

What are Vectors?

- 2D vector is an arrow on a flat plane that we can describe with coordinates for convenience.

 $= \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

- Or is it fundamentally a pair of real numbers which is nicely visualized as an arrow.

$\begin{bmatrix} 0 \\ 2 \end{bmatrix} =$ 

- Or are both just manifestations of something deeper?

Many Possible Coordinate Systems

- You are dealing with a space that exists independently from the coordinates that you specify.
- Determinant and Eigenvectors do not care about the coordinate system.
- The determinant tells you how much a transformation scales areas.
- Eigenvectors are the ones that stay on their own span during a transformation.

What exactly do you mean by "Space"?

- If vectors are not only lists of real numbers,
- And if their underlying essence is something more special,
- What do we mean by "space" or "spacial"?
- Let us first talk about something else that has some *vectorish* qualities.

Functions

- **Functions are just another type of vectors.**

1. Like vectors, you can add two functions f and g to get a new function (f+g).
2. The output of (f+g) for any input is the sum of outputs of f and g with the same input, $(f+g)(x) = f(x)+g(x)$.
3. You can scale a function by a real number.
4. We can apply the same constructs of linear algebra to functions as well.

Linear Transformations for Functions

- Something that transforms one function into another function? The derivative.

$$\frac{d}{dx}\left(\frac{1}{9}x^3 - x\right) = \frac{1}{3}x^2 - 1$$

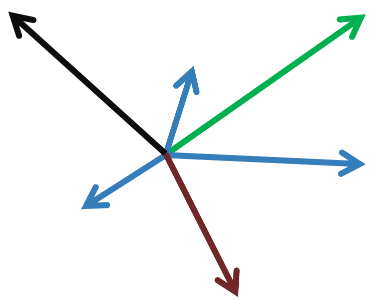
- What does it mean for a transformation of a function to be linear?
- A transformation is linear if it satisfies two properties: **Additivity** and **Scaling**.
- **Additivity**: $L(\vec{v} + \vec{w}) = L(\vec{v}) + L(\vec{w})$
- **Scaling**: $L(c\vec{v}) = \mathrm{c} * \mathrm{L}(\vec{v})$

Linear Transformations for Functions

- $\frac{d}{dx}$ is linear.
- Let's describe the derivative with a matrix.
- Let us limit our selves to polynomials:
$$x^2 + 3x + 5$$
- Each polynomials in our space will have finite terms.
- First, we need to choose a basis (give coordinates to the space).

So back to the first question

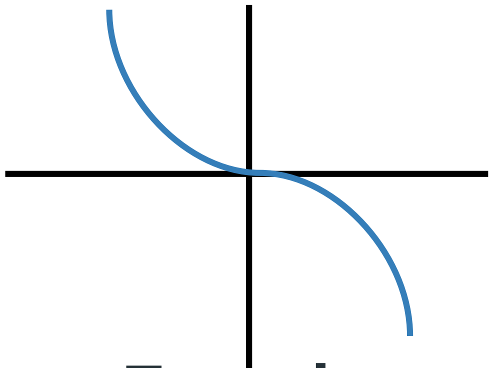
- What vectors really are?
- There are a lot of *vectorish* objects in math.
- as long as we are dealing with a set of objects with reasonable notion of scaling and adding.



Set of Arrows

$$\begin{bmatrix} 4 \\ 7 \\ 1 \end{bmatrix} \quad \begin{bmatrix} -2 \\ 5 \end{bmatrix} \quad \begin{bmatrix} 5 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

Lists of Numbers



Functions

Vector Spaces

- Consider a **field** F, and any **set** X. Consider the set of all **functions** from X to F.
- You can add any two such functions, and you can multiply any such function by an element of F.
- This set of functions is a vector space "over F", and any one of the functions in it is a vector.

What on earth is a field?

What on earth is a field?

- Four million hours of abstract algebra theory or give up.

Now let's distill out the essential ideas involved.

- What is a **monoid**? It's a set M with an operation taking two arguments, denoted by +, with the property that (x+y)+z = x + (y+z), and also having an operation taking no arguments, denoted 0, with the property that 0 + x = x + 0 = x, for all x.
- What's a **group**? It's a monoid with x + y = 0 (for all x, there is a y, or for all y, there is an x.) If x + y = 0 = y + z, then x = x + 0 = x + (y +z) = (x + y) + z = 0 + z = z, so x = z. We denote it by -y; it is clearly unique.
- What's an **Abelian group**? It's a group for which x + y = y + x for all x and y.
- What's an **endomorphism**? It's a map f satisfying f(0) = 0 and f(x + y) = f(x) + f(y). These are also the functorial properties.
- What's a **ring**? It's a set R with two monoid structures. denoted by +, 0 and 1, satisfying a functorial property, that x(y+z) = xy + xz and (x +y)z = xz + yz, with x0 = 0x = 0 for all x.
- We assume also a **nullary operation** denoted -1, satisfying -1 + 1 = 0, or equivalently 1 + -1 = 0. It follows that the addition in fact is an Abelian group. The other operation is called multiplication.
- What is a **division ring**? A ring for which the elements apart from 0 form a group, under the multiplication operation. If that group is Abelian the ring is called a field.
- The endomorphisms of an Abelian group form a ring. The multiplication is given by composition of functions. The addition is given by adding values. 1 is the identity map, and 0 is the constant map taking everything to 0.
- What is a **functor**? a functor is a map between categories. We're going to ignore categories ☺.
- What is a **functor of rings**? It is a functor of each of the two monoid structures a ring possesses: f(0) = 0, f(a+b) = f(a) + f(b), f(1) = 1, f(ab) = f(a)f(b).
- What is a **module**? It is a functor from a ring to the endomorphism ring of an Abelian group.
- When the domain ring is a division ring, the elements of that Abelian group are called vectors, and the Abelian group is called a vector space.

Vector Spaces

- A transformation is linear if it satisfies two properties: **Additivity** and **Scaling**.
- **Additivity**: $L(\vec{v} + \vec{w}) = L(\vec{v}) + L(\vec{w})$
- **Scaling**: $L(c\vec{v}) = \mathrm{c} * \mathrm{L}(\vec{v})$

Rules for vectors addition and scaling

- These rules are called **axioms**.
- There are 8 axioms in the modern theory of linear algebra.
- Any vector space mush satisfy these rules so linear algebra tools can be applied.
- These axioms are an **interface**.
- In this way, the form that vectors take does not matter.
- It can be anything so long there is some notion of adding and scaling vectors that follows these 8 rules.

Rules for vectors addition and scaling

1. $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$
2. $\vec{v} + \vec{w} = \vec{w} + \vec{v}$
3. There is a vector 0 such that $0 + \vec{v} = \vec{v}$ for all $\vec{v}$
4. For every vector $\vec{v}$ there is a vector - $\vec{v}$ so that $\vec{v} + (-\vec{v}) = 0$
5. $a(b\vec{v}) = (ab)\vec{v}$
6. $1\vec{v} = \vec{v}$
7. $a(\vec{v} + \vec{w}) = a\vec{v} + a\vec{w}$
8. $(a + b)\vec{v} = a\vec{v} + b\vec{v}$

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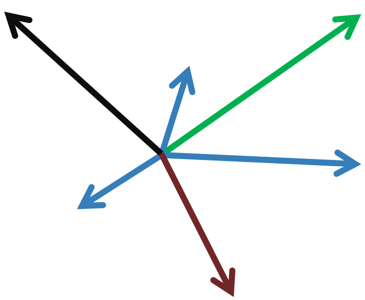

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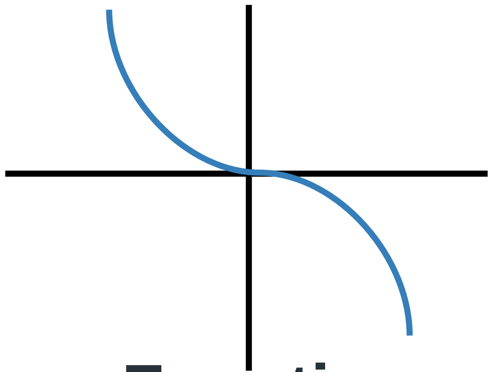

Functions