Fuzzy Systems

Membership Functions

Dr. Petr Musilek

Department of Electrical and Computer Engineering
University of Alberta

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Membership Functions

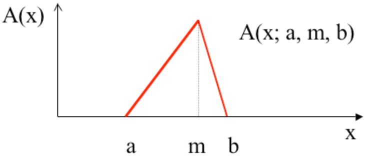
Membership functions are used to represent (capture) notions expressed in problem description:

- elements with complete membership
- elements excluded from fuzzy set
- the form of transition between complete membership and complete exclusion

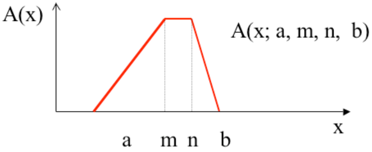
Triangular membership functions

Triangular fuzzy sets:

- modal value *m*
- lower and upper bounds *a* and *b*



Trapezoidal membership functions



Trapezoidal fuzzy sets:

- modal values *m* and *n*
- lower and upper bounds *a* and *b*

Other types of membership functions

Gaussian m.f.	$A(x) = e^{(-{\frac{(x-m)^2}{\sigma}}^2)}$
Exponential m.f.	$A(x) = \frac{k(x-m)^2}{1+k(x-m)^2}$
Sigmoidal m.f.	$A(x) = \frac{1}{1+e^{x-m}}$

Examples of fuzzy sets

Fuzzy sets represent concepts in real wolrd:

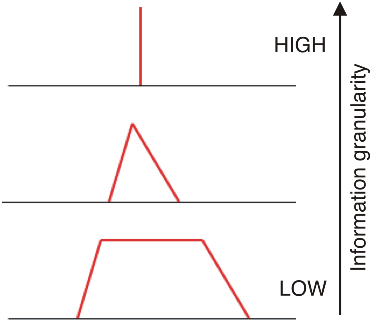
- Size
- Age
- Complexity
- Error
- Reliability
- Power consumption
- Microprocessor speed

Characteristics of Fuzzy Sets

Normality	$A(x)$: $\sup_x A(x) = 1$ (otherwise "subnormal")
Height	$\mathrm{hgt}(A) = \sup_x A(x)$
Support	$\mathrm{Supp}(A) = \{x \in X \mid A(x) > 0\}$
Core	$\mathrm{Core}(A) = \{x \in X \mid A(x) = 1\}$
Cardinality	$\mathrm{Card}(A) = \sum_{x \in X} A(x)$ (or $\int$ for continuous f.s.)

Hierarchy of fuzzy sets

Fuzzy sets representing the same concept but with different resolution lead to different levels of information granularity



Higher granularity: need more constructs (f.s.) to describe the universe of discourse

Unary operations on fuzzy sets

Operations with single argument – a fuzzy set. They serve for manipulation of fuzzy sets to change their meaning or strength of the concept they represent (i.e. change membership function)

Normalization – converting subnormal fuzzy set into its normal counterpart

$$\mathrm{Norm}(A) = \frac{A(x)}{\mathrm{hgt}(A)}$$

Unary operations on fuzzy sets

Concentration – concentrating the membership function around points with higher membership values

$$\mathrm{Con}(A) = A^2(x)$$

Dilatation – effect opposite to concentration

$$\mathrm{Dil}(A) = A^{1/2}(x)$$

Of course, concentration and dilatation can be generally expressed in form

$$A^p(x)$$

with concentration corresponding to values $p > 1$ and dilatation to values $p < 1$.

Equality and Inclusion Relations

Two fuzzy sets A and B can be related with the following two relations

Equality:

$$A = B \text{ iff } \forall x : \; A(x) = B(x)$$

Inclusion:

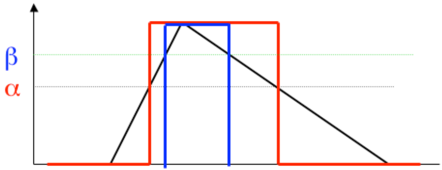
$$A \subset B \text{ iff } \forall x : \; A(x) \leq B(x)$$

The Representation Theorem

Definition

α - cut of fuzzy set A is a set, A_α defined as

$$A_\alpha = \{x|A(x) \geq \alpha\};\ \text{if}\alpha < \beta \text{ then } A_\alpha \supseteq A_\beta$$



Theorem

Any fuzzy set can be represented as a family of sets

Membership Function Determination

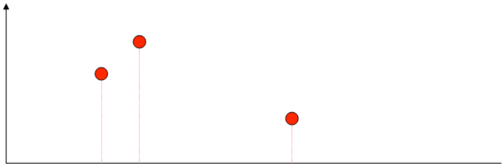
How to determine shape/functional description of membership from empirical data?

- Horizontal approach
- Vertical approach
- Pairwise comparison
- Clustering (grouping) method

Horizontal approach

Gather information about membership values at selected elements of the universe of discourse (space) X - polling mechanism (likelihood measure):

Can value x_i be accepted as compatible with given concept (fuzzy set A)?



Vertical approach

Identify α-cuts and "reconstruct" a fuzzy set using these sets:

What interval corresponds fully/partially/not at all to given concept?

Horizontal and Vertical Approaches

- Pro: easy to use
- Con: “local” character of experiments (isolated experiments dealing with single elements of the universe of discourse)

Pairwise Comparison Method

Rationale:

- assume that membership values $A_{x1}, A_{x2}, \ldots, A_{xn}$ are given
- arrange them as a reciprocal matrix $\boldsymbol{A}$
  - reciprocity $a_{ij} = 1/a_{ji}$
  - transitivity $a_{ik} = a_{ij}a_{jk}$
- multiply $\boldsymbol{A}$ by the vector of membership values $\boldsymbol{a}$

$$\begin{aligned} \boldsymbol{A}\boldsymbol{a} &= n\boldsymbol{a} \\ (\boldsymbol{A} - n\,\boldsymbol{I})\,\boldsymbol{a} &= 0 \end{aligned}$$

where I is a unit matrix, and $\boldsymbol{a}$ and n are eigenvector and eigenvalue of the matrix $\boldsymbol{A}$, respectively.

Pairwise Comparison Method

Realization:

- compare objects pair–wise in the context of $\boldsymbol{A}$ (e.g. ratio scale 1, 2, . . . , 7) and construct the matrix based on these ratios
- assess consistency of the matrix (and, in turn, consistency of the gathered data) by looking at the value of n (should be comparable to the dimension of the matrix)
- normalize the eigenvector $\boldsymbol{a}$ to get an estimate of the membership function