Fuzzy Systems

Fuzzy Arithmetic

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Fuzzy Numbers

Fuzzy sets defined over the domain of real numbers $\mathbb{R}$. They allow us to work with uncertain/fuzzy quantities, such as

- about 5
- smaller than 200
- close to 0.1

Example: Resistors are produced with certain tolerance around their nominal value of resistance, e.g. a 220k resistor has in fact resistance "about 220kΩ"

In turn, we may need to incorporate this uncertainty into our calculations

Fuzzy numbers

Fuzzy number is a fuzzy set

$$A : \mathbb{R} \to [0, 1]$$

that is

- normal,
- unimodal,
- continuous, and has
- bounded support

Interval analysis

Roots of computing with fuzzy numbers originate from interval analysis (a.k.a. calculus of tolerances). The basic objects of interval analysis are intervals I, such as $[1, 2]$, $[a, b]$, etc.

Basic arithmetic operations on intervals are:

$$\begin{aligned}
[a, b] \quad &+ \quad [c, d] = [a + c, b + d] \\
[a, b] \quad &- \quad [c, d] = [a - d, b - c] \\
[a, b] \quad &* \quad [c, d] = [\min(ac, ad, bc, bd), \max(ac, ad, bc, bd)] \\
[a, b] \quad &\div \quad [c, d] = [\min(\tfrac{a}{c}, \tfrac{a}{d}, \tfrac{b}{c}, \tfrac{b}{d}), \max(\tfrac{a}{c}, \tfrac{a}{d}, \tfrac{b}{c}, \tfrac{b}{d})]
\end{aligned}$$

Fuzzy arithmetic

Computing with fuzzy numbers leads to fuzzy arithmetic, which is based on extension principle. Consider a function F taking as arguments two fuzzy numbers A and B and producing a third fuzzy number C

$$C = F(A, B)$$

The extension principle determines m.f. of C as

$$C(z) = \sup_{x,y \in \mathbb{R} : z = f(x,y)} \left[A(x) \wedge B(y) \right]$$

Fuzzy arithmetic

where $z \in \mathbb{R}$, and $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a real function that is point-wise consistent with F, i.e.

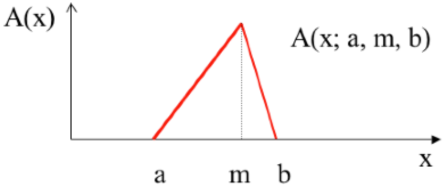
$$F\big(\{x\}, \{y\}\big) = f(x, y)$$

where $\{x\}$ and $\{y\}$ are fuzzy singletons corresponding to values x and y, respectively. The expression for extension principle is also called *sup-min convolution*.

Triangular fuzzy numbers

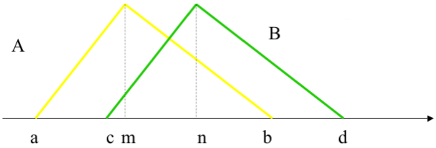
A triangular fuzzy number (TFN) is defined in terms of variable $x \in \mathbb{R}$ with parameters a, m, and b, describing lower boundary, modal value, and upper boundary, respectively

$$A(x; a, m, b) = \begin{cases} \frac{x-a}{m-a} & \text{if } x \in [a, m], \\ \frac{b-x}{b-m} & \text{if } x \in [m, b]. \end{cases}$$



Addition of TFNs

Consider two TFNs $A(x; a, m, b)$ and $B(x; c, n, d)$

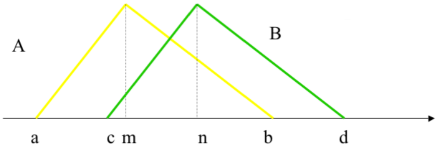


To derive the formula for fuzzy addition, the general form of the extension principle has to be parameterized by $f(x, y) = x + y$, i.e

$$C(z) = \sup_{x,y \in \mathbb{R}: z = x+y} \left[A(x) \wedge B(y) \right]$$

Multiplication of TFNs

Again, consider two TFNs $A(x; a, m, b)$ and $B(x; c, n, d)$



The general form of the extension principle has to be parameterized by $f(x, y) = xy$, i.e

$$C(z) = \sup_{x,y \in \mathbb{R}: z=xy} \left[A(x) \wedge B(y) \right]$$

Multiplication of TFNs (continued)

As in case of addition (done in class), consider two cases:
a) $z < mn$, b) $z > mn$

a) Similarly to addition, we would consider such values of x and y for which memberships $A(x)$ and $B(y)$ are equal and to a constant ω, i.e.

$$A(x) = B(y) = \omega$$

ω can be expressed in terms of parameters/variables of fuzzy numbers A and B as follows

$$\frac{x-a}{m-a} = \omega, \; \frac{y-c}{n-c} = \omega$$

Multiplication of TFNs (continued)

The above expressions can be rewritten as $x = a + (m - a)\omega$ and $y = c + (n - c)\omega$, respectively. So far, this results is the same as for addition. However, since z is now xy, it can be expressed as follows

$$z = ac + \omega c(m - a) + \omega a(n - c) + \omega^2(m - a)(n - c),$$

i.e. z is a quadratic function of ω, and thus the resulting FN is no longer triangular.

[Note: Derivation of expression for case b) is similar]

Measures of fuzziness

Which set is fuzzier?

Degree of fuzziness is a measure of difficulty of ascertaining membership of elements of a fuzzy set:

- if the membership of an element is less than 0.5, the possibility of the element being outside of the fuzzy set is greater and hence it is easier to exclude it from the set
- if the membership of an element is greater than 0.5, the possibility of the element being inside the fuzzy set is greater and hence it is easier to include it in the set

Fuzziness: Simple energy index

$$F_0 = \int_{x \in \mathrm{Supp}(A)} A(x)dx$$

Any problems here? Compare fuzziness of fuzzy set $A(x) = \mathrm{trimf}(x, [0, 2, 4])$ and that of $B(x) = 1; x \in [1, 3]$?

$B(x)$ is a crisp interval nearest to $A(x)$ and

$$F_0(A) = F_0(B)$$

Fuzziness: Closeness to grade 0.5

The closeness to the most fuzzy grade

$$F_1 = \int_{x \in \mathrm{Supp}(A)} f(x),$$

where the function $f(x)$ is defined in the support of the set A, such that $f(x): \mathrm{Supp}(A) \leftarrow [0, 0.5]$, i.e.

$$f(x) = \begin{cases} A(x) & \text{for } A(x) \leq 0.5, \\ 1 - A(x) & \text{otherwise.} \end{cases}$$

Fuzziness: Distance from 0.5-cut

$$F_2 = \int_{x \in \mathrm{Supp}(A)} |A(x) - A_{0.5}(x)| dx,$$

The value of $A_{0.5}$ is 0 for x where $A(x) < 0.5$, and $A_{0.5}$ is 1 for x where $A(x) \geq 0.5$. In effect, $F_1 = F_2$. Other measures have been proposed, e.g. quadratic energy index or inverse of distance from the complement (A_3 in the textbook, p.94)