Fuzzy Systems

Operations on Fuzzy Sets

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Fall 2019

Standard (Fuzzy) Set Operations

Operation	Notation	Standard model
Union	$A \cup B$	$\max[A(x), B(x)]$
Intersection	$A \cap B$	$\min[A(x), B(x)]$
Complement	$\overline{A}$	$1 - A(x)$

Problems of standard models

Although the entire range [0,1] is available for the values of membership, these operations provide no interaction among the variables, i.e.

- union: no matter how small the other variable, the result of max operation is given by the larger value alone
- intersection: no matter how large the other variable is, the result of min operation is given by the smaller value alone

Alternative definitions of FS operations

Triangular norms and co-norms

- Triangular norms are operations that satisfy reasonable axioms for the definition of intersection and union; they were introduced in probabilistic metric spaces
- models of fuzzy set operations
  - Intersection: **t**-norms (triangular norm)
  - Union: **s**-norms (triangular co-norm)

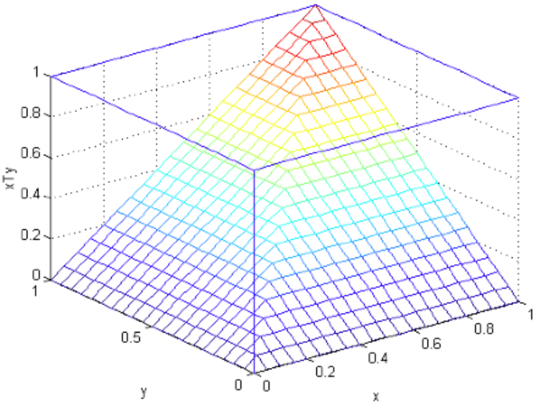
Axioms (intersection)

Commutativity:	$x \mathbf{t} y = y \mathbf{t} x$
Associativity:	$x \mathbf{t} (y \mathbf{t} z) = (x \mathbf{t} y) \mathbf{t} z$
Monotonicity:	if $x \leq y$ and $w \leq z$ then $x \mathbf{t} w \leq y \mathbf{t} z$
Boundary Conditions:	$0 \mathbf{t} x = 0$, $1 \mathbf{t} x = x$

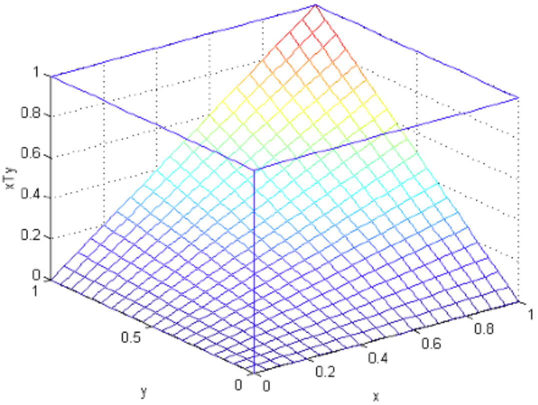
Triangular Norms - models of intersection

Minimum **t**–norm	$x\,\mathbf{t}\,y = \min(x, y)$
Product **t**–norm	$x\,\mathbf{t}\,y = xy$
Lukasiewicz **t**–norm	$x\,\mathbf{t}\,y = \max(x + y - 1, 0)$
Drastic product **t**–norm	$x\,\mathbf{t}\,y = \begin{cases} 0 & \text{if } \max(x, y) < 1, \\ \min(x, y) & \text{otherwise.} \end{cases}$

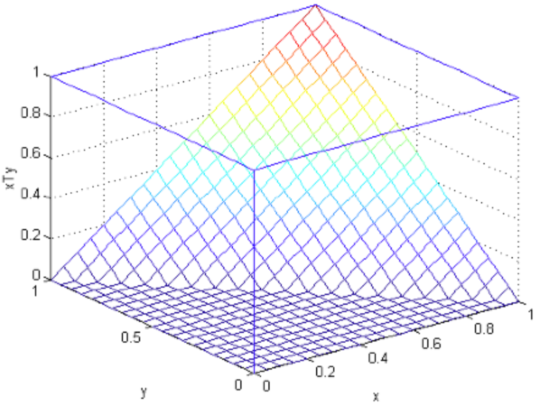
Standard **t**–norm $x \, \mathbf{t} \, y = \min(x, y)$



Product t–norm $x \mathbf{t} y = xy$



Lukasiewicz t–norm $x \mathbf{t} y = \max(x + y - 1, 0)$



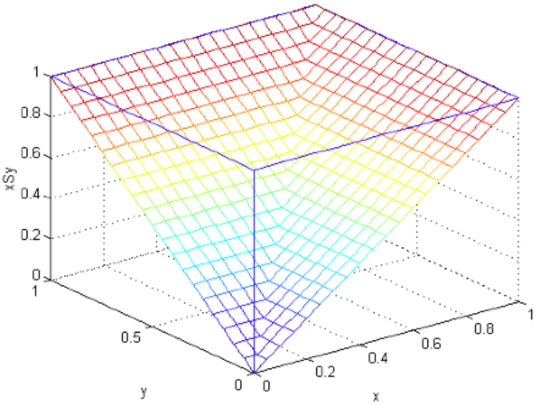
Axioms (union)

Commutativity:	$x \, \mathbf{s} \, y = y \, \mathbf{s} \, x$
Associativity:	$x \, \mathbf{s} \, (y \, \mathbf{s} \, z) = (x \, \mathbf{s} \, y) \, \mathbf{s} \, z$
Monotonicity:	if $x \leq y$ and $w \leq z$ then $x \, \mathbf{s} \, w \leq y \, \mathbf{s} \, z$
Boundary Conditions:	$0 \, \mathbf{s} \, x = x, \; 1 \, \mathbf{s} \, x = 1$

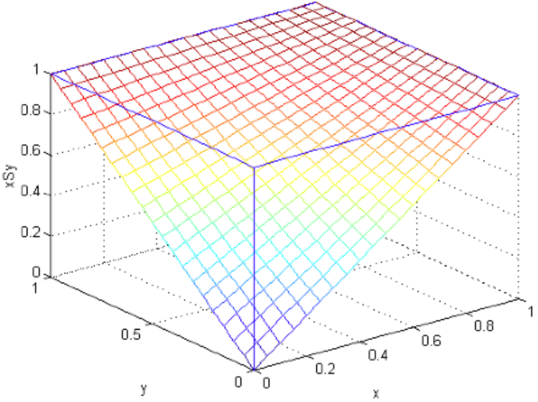
Triangular Co-norms - models of union

Maximum **s**–norm	$x \mathbf{s} y = \max(x, y)$
Bounded sum **s**–norm	$x \mathbf{s} y = x + y - xy$
Lukasiewicz **s**–norm	$x \mathbf{s} y = \min(x + y, 1)$
Drastic sum **s**–norm	$x \mathbf{s} y = \begin{cases} 1 & \text{if } \min(x, y) > 0, \\ \max(x, y) & \text{otherwise.} \end{cases}$

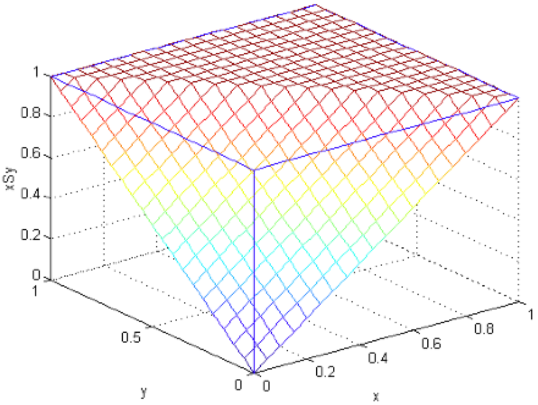
Standard **s**–norm $x \mathbf{s} y = \max(x, y)$



Bounded sum s–norm $x \, \mathbf{s} \, y = x + y - xy$



Lukasiewicz **s**–norm $x \, \mathbf{s} \, y = \min(x + y, 1)$



Triangular Norms and Co-norms

Cannot be linearly ordered.

However, there are bounds on their values:

$$\text{drasticproduct} \leq \mathbf{t} \leq \mathsf{min}$$

and

$$\mathsf{max} \leq \mathbf{s} \leq \text{drasticsum}$$

Triangular Norms and Co-norms: Duality

For each t–norm, there exists a dual s–norm.

Corresponding to DeMorgan's laws:

$$x \,\mathbf{s}\, y = 1 - (1 - x) \,\mathbf{t}\, (1 - y)$$

and

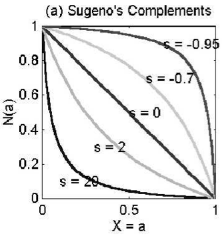
$$x \,\mathbf{t}\, y = 1 - (1 - x) \,\mathbf{s}\, (1 - y)$$

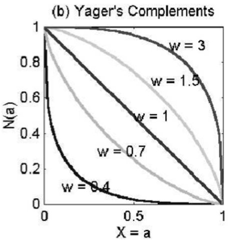
Axioms (Fuzzy Complement)

Monotonicity:	if $a < b$ then $\mathrm{N}(a) > \mathrm{N}(b)$
Involution:	$\mathrm{N}(\mathrm{N}(a)) = a$
Boundary Conditions:	$\mathrm{N}(0) = 1$ and $\mathrm{N}(1) = 0$

Fuzzy Complements

Standard fuzzy complement	$\mathrm{N}(a) = 1 - a$
Sugeno's fuzzy complement	$\mathrm{N}(a) = (1 - a)/(1 + sa)$
Yager's fuzzy complement	$\mathrm{N}(a) = (1 - a^w)^{1/w}$





Comparison Operations on Fuzzy Sets

So far: set operations on fuzzy sets; i.e. how to combine two fuzzy sets and how to find a fuzzy set which complements another fuzzy set.
Now: comparison operations, to find how similar two fuzzy sets are; there are several possibilities to measure this:

- distance measures
- possibility measure
- necessity measure

Distance Measures

In general, distance between two fuzzy sets can be measured using their membership functions

$$d(A, B) = \sqrt[p]{\int_X |A(x) - B(x)|^p dx}; \; p \geq 1$$

For different values of p, we obtain different measures, such as Hamming distance ($p = 1$), Euclidean distance ($p = 2$), Tchebyschev distance ($p = \infty$), etc.

Set based comparison operations

Calculation of distance involves two functions (membership functions *A* and *B*) – this measure therefore emphasizes functional aspect of fuzzy sets and not their set–based characteristic.

Set based comparison operations

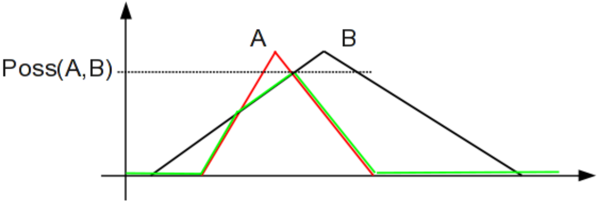
- possibility
- necessity

alleviate this problem by taking into account set–based operations instead of functions.

Possibility Measure

Possibility measure of fuzzy set *A* with respect to fuzzy set *B* is defined as

$$\mathrm{Poss}(A, B) = \sup_{x \in X} \left[\min \left(A(x), B(x) \right) \right]$$



This measure quantifies the extent to which fuzzy sets *A* and *B* overlap.

Possibility measure: main properties

Possibility

$$\mathrm{Poss}(A, B) = \sup_{x \in X} \left[\min \left(A(x), B(x) \right) \right]$$

$$\mathrm{Poss}(A, B) = \mathrm{Poss}(B, A)$$

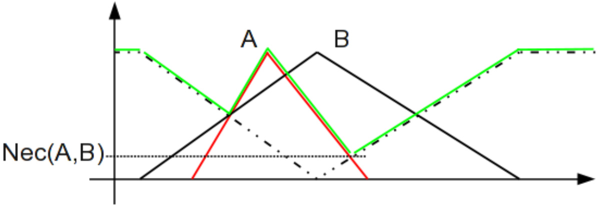
$$\mathrm{Poss}(A \cup B, C) = \max \left[\mathrm{Poss}(A, C), \mathrm{Poss}(B, C) \right]$$

$$\mathrm{Poss}(\{x\}, B) = B(x)$$

Necessity Measure

Necessity measure of fuzzy set *A* with respect to fuzzy set *B* is defined as

$$\mathrm{Nec}(A, B) = \inf_{x \in X} \left[\max \left(A(x), 1 - B(x) \right) \right]$$



This measure quantifies the extent to which fuzzy sets *A* is included in fuzzy set *B*.

Necessity measure: main properties

Necessity

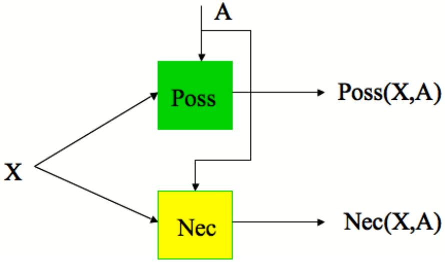
$$\mathrm{Nec}(A, B) = \inf_{x \in X} \left[\max \left(A(x), 1 - B(x) \right) \right]$$

$$\mathrm{Nec}(A, B) \neq \mathrm{Nec}(B, A)$$

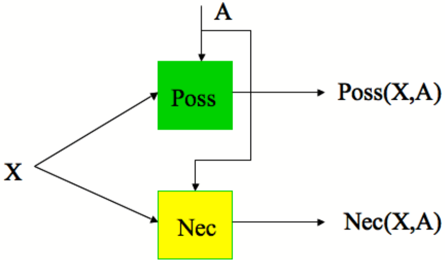
$$\mathrm{Nec}(A \cap B, C) = \min \left[\mathrm{Nec}(A, C), \mathrm{Nec}(B, C) \right]$$

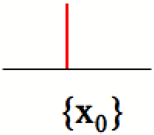
$$\mathrm{Nec}(\{x\}, B) = B(x)$$

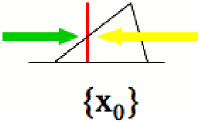
Possibility and Necessity: a matching interface



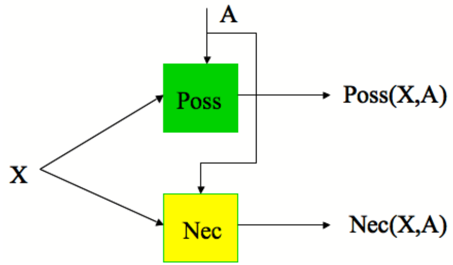
Possibility and Necessity: a matching interface

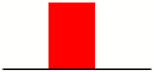


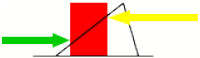




Possibility and Necessity: a matching interface







Equality index

Consider two fuzzy sets A and B defined in the same finite space $X = \{x_1, x_2, \ldots, x_n\}$. Their equality ($\equiv$) can be assessed as follows:

$$A \equiv B : \ (A \subset B) \wedge (B \subset A)$$

Inclusion ($\subset$) can be modelled with implication ($\rightarrow$)

$$A \equiv B = \frac{1}{n} \sum_{i=1}^{n} \min \left[\big(A(x_i) \rightarrow B(x_i)\big), \big(B(x_i) \rightarrow A(x_i)\big) \right]$$

Operation of Implication

In case of fuzzy sets $a, b, \in [0, 1]$:

$$a \rightarrow b = \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{if } a > b. \end{cases}$$

$$a \rightarrow b = \begin{cases} 1 & \text{if } a \leq b, \\ b/a & \text{if } a > b. \end{cases}$$