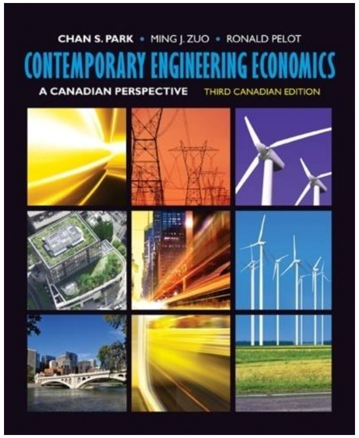
Equivalence Analysis Using Effective Interest Rates



Lecture No. 9
Chapter 4
Contemporary Engineering Economics
Third Canadian Edition
Copyright © 2012

Lecture 9 Objectives

- How do you perform equivalence analysis with effective interest rates?

When Payment Period Is Equal to Compounding Period

$M = K$
$C = 1$

- **Step 1:** Identify the number of compounding periods (M) per year.
- **Step 2:** Compute the effective interest rate per payment period (i).

$$i = r/M$$

- **Step 3:** Determine the total number of payment periods (N).

$$N = M \times (\text{number of years})$$

K = payment#/year

M = Comp pd/yr

Example 4.4: Calculating Auto Loan Payments

- Given:

MSRP	$23,798
Dealer's discount	$1,143
Manufacturer rebate	$800
University graduate cash discount	$500
Sale price	$21,355

$15,000

Down payment = $6,355

Dealer's interest rate = 6.25% APR (monthly compounding)

Length of financing = 72 months

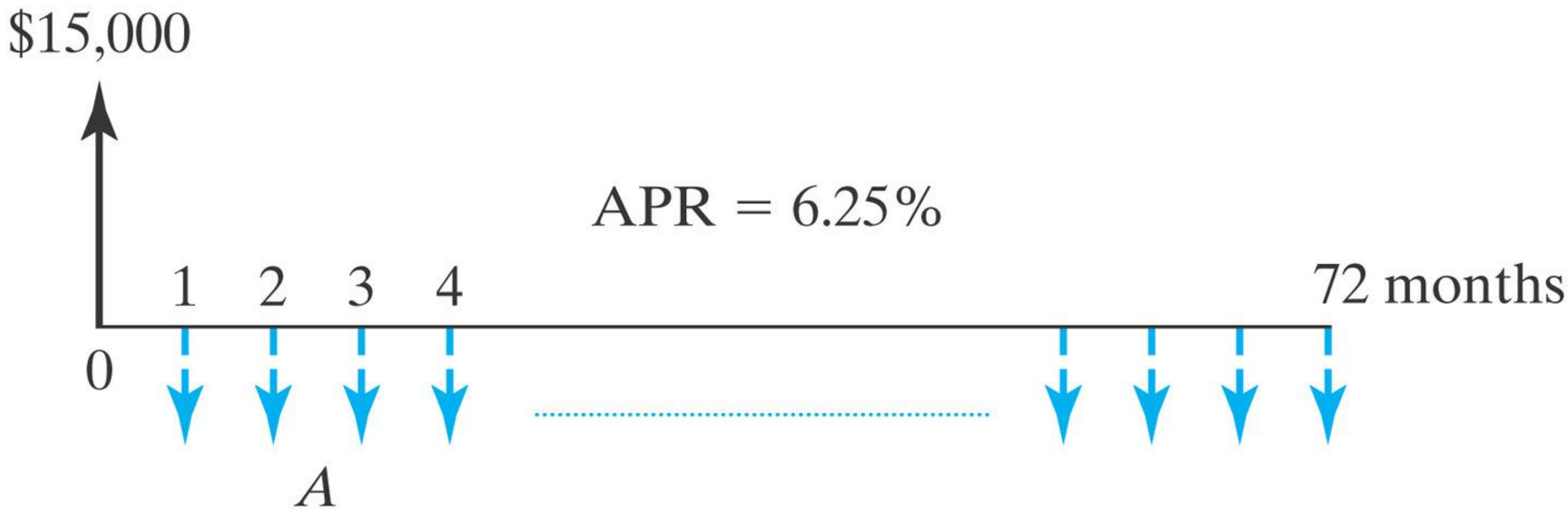
$M = K$ ie monthly $= \frac{6.25\%}{12}$

$A = P \cdot (A/P, 0.5208\%, 72)$

$= 15000 \times 0.0167 = \$250.37$

- Find: the monthly payment (*A*)

Example 4.4: Solution



Step 1: $M = 12$
Step 2: $i = r/M = 6.25\%/12 = 0.5208\%$ per month
Step 3: $N = (12)(6) = 72$ months
Step 4: $A = \$15{,}000(A/P, 0.5208\%, 72) =$ **$250.37**

Compounding Occurs at a Different Frequency from the Payment Frequency

- We will consider two situations:

1) compounding is more frequent than payments
2) compounding is less frequent than payments (less ~~popular~~)

Compounding Occurs at a Different Rate Than That at Which Payments Are Made

- Step 1: Identify the following parameters.
  - M = number of compounding periods
  - K = number of payment periods
  - C = number of interest periods per payment period
- Step 2: Compute the effective interest rate per payment period.
  - For discrete compounding

$$i_e = [1 + r/M]^C - 1 \leftarrow \text{either}$$

  - For continuous compounding

$$i_e = e^{r/K} - 1 \leftarrow \text{or}$$

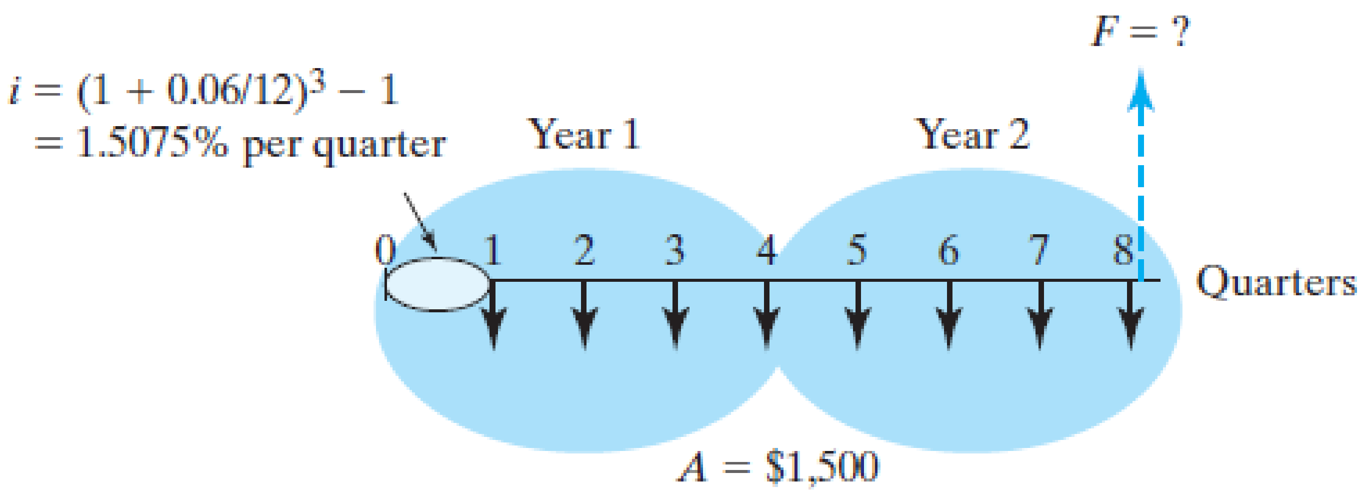
- Step 3: Find the total number of payment periods

$$N = \underline{\underline{K}} \times (\text{number of years})$$

- Step 4: Use i and N in the appropriate compounding formula.

Example 4.5: Compounding Occurs More Frequently Than Payments Are Made (Discrete-Compounding Case)

- Suppose you make equal quarterly deposits of $1,500 into a fund that pays interest at a rate of 6% compounded monthly. Find the balance at the end of year 2.



Example 4.5: Solution

- Given: A = \$1,500 per quarter, r = 6% per year, M = 12 compounding periods per year, and N = 8 quarters.
- Find: F
- Step 1: M = 12 compounding periods/year

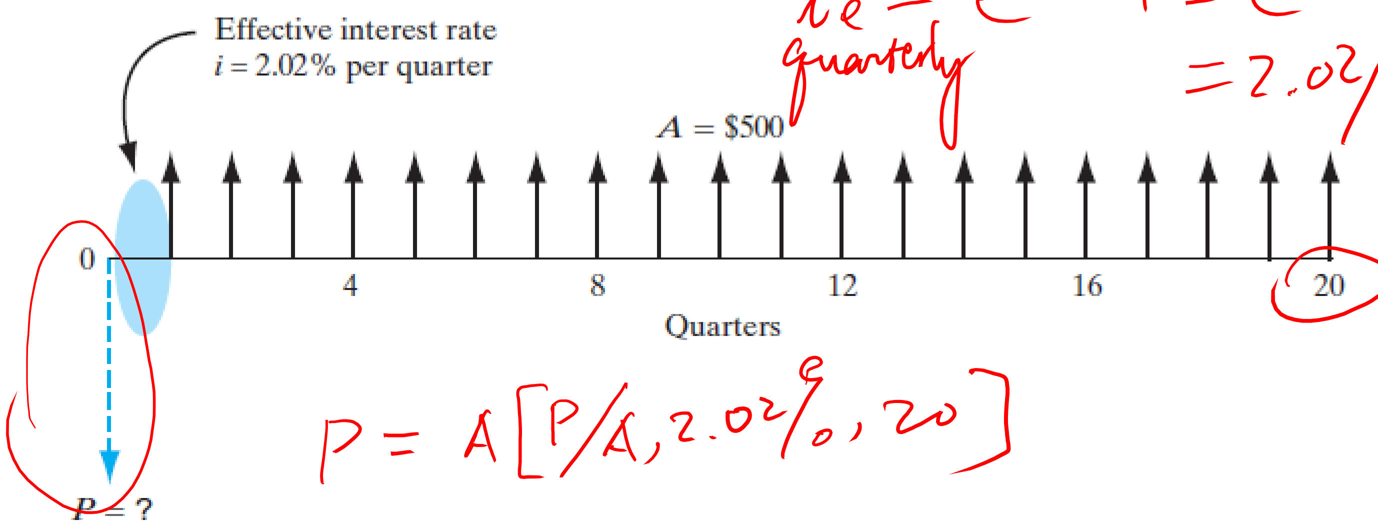
  K = 4 payment periods/year

  C = 3 interest periods per quarter
- Step 2: $i = \left(1 + \dfrac{0.06}{12}\right)^3 - 1 = 1.5075\%$ per quarter
- Step 3: $N = (4)(2) = 8$
- Step 4: F = \$1500 (F/A,1.5075%,8) = \$12,652.60

Example 4.6: Compounding Occurs More Frequently Than Payments Are Made (Continuous-Compounding Case)

- A series of equal quarterly receipts of $500 extends over a period of five years. What is the present worth of this quarterly payment series at 8% interest compounded continuously?

$i_e = e^{\frac{r}{K}} - 1 = e^{\frac{8\%}{4}} - 1$
quarterly
$= 2.02\%$



$P = A[P/A, 2.02\%, 20]$

Example 4.6: Solution

- Given: A = \$500 per quarter, r = 8% per year, and N = 20 quarters.
- Find: P
- Step 1: K = 4 payment periods/year

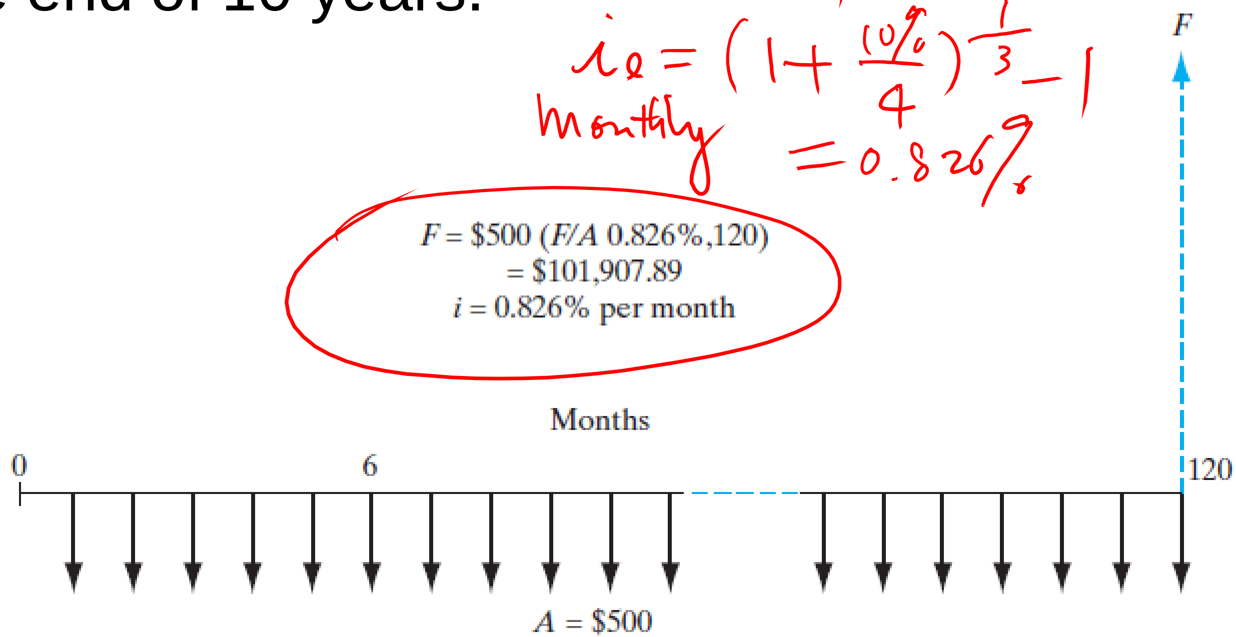
  $C = \infty$ interest periods per quarter
- Step 2: $i = e^{r/K} - 1 = e^{0.08/4} - 1 = \underline{2.02\%}$ per quarter
- Step 3: $N = (4)(5) = 20$
- Step 4: $P = \$500\ \underbrace{(P/A, 2.02\%, 20)}_{\left[\frac{(1+i)^N - 1}{i(1+i)^N}\right] = 16.32} = \underline{\$8{,}159.96}$

Example 4.7: Compounding Is Less Frequent Than Payments: Effective Interest Rate per Payment Period

- Suppose you make $500 monthly deposits to a registered retirement savings plan (RRSP) that pays interest at a rate of 10% compounded quarterly. Compute the balance at the end of 10 years.

$m = 4 \quad K = 12 \quad C = \frac{m}{K} = \frac{4}{12} = \frac{1}{3}$

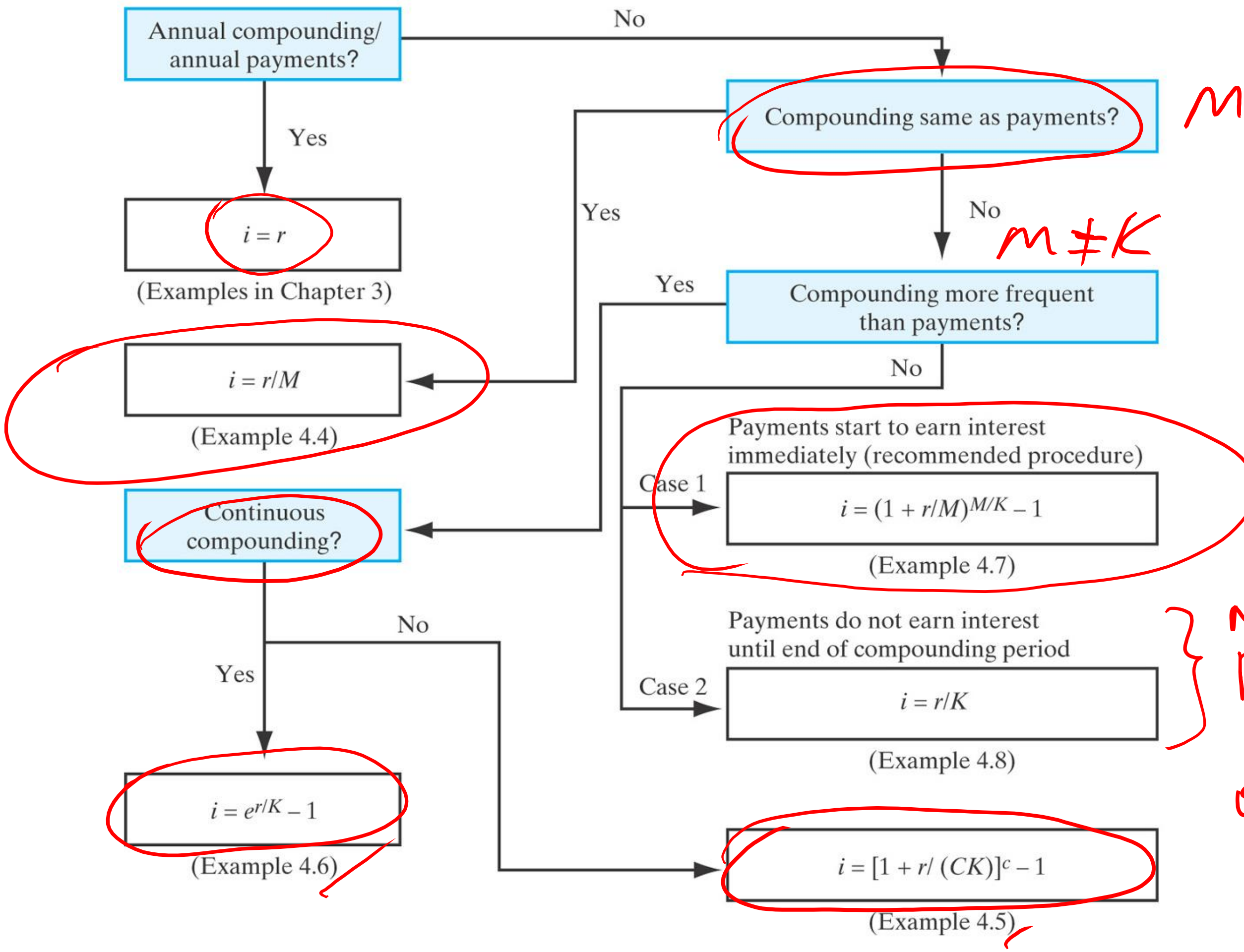
$i_e = \left(1 + \frac{10\%}{4}\right)^{\frac{1}{3}} - 1 = 0.826\%$ monthly



Example 4.7: Solution

- Given: A = \$500 per month, r = 10% per year, M = 4 compounding periods per year, K = 12 payment periods per year, N = 8 quarters, and interest is accrued during the compounding period.
- Find: F
- Step 1: M = 4 compounding periods/year
  K = 12 payment periods/year
  C = 1/3 interest periods per quarter
- Step 2: $i = \left(1 + \frac{0.10}{4}\right)^{1/3} - 1 = 0.826\%$ per month
- Step 3: $N = (12)(10) = 120$
- Step 4: F = \$500 (F/A,0.826%,120) = \$101,907.89

A Decision Flow Chart on How to Compute the Effective Interest Rate per Payment Period



$M = K$?

$M \neq K$

Not popular, ignore for our course.

Changing Interest Rates

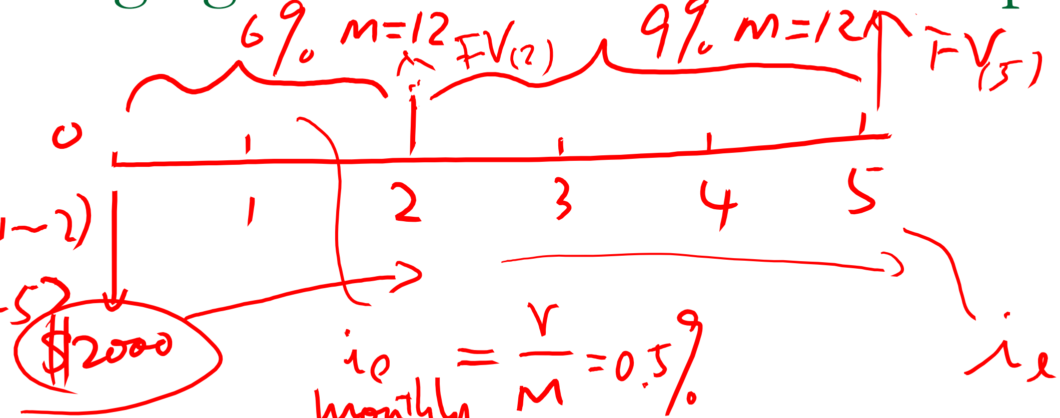
- When an equivalence calculation extends over several years, more than one interest rate may be applicable to properly account for the time value of money.
- We will consider variable interest rates in both

a) single payments

b) single payments and a series of cash flows.

Example 4.10: Changing Interest Rates with a Lump-Sum Amount

- RRSP
- APR 6% (1~2)
  9% (3-5)



$i_p = \frac{r}{M} = 0.5\%$ (monthly)

$i_e = \frac{9\%}{12} = 0.75\%$

$FV_{(2^{nd})} = P \cdot (F/P, 0.5\%, 24) = 2000 \times 1.12716$
$= \$2254$

$FV_{(5^{th})} = FV_{(2^{nd})} \cdot (F/P, 0.75\%, 36)$
$= \$2254 \times 1.3086$
$= \$2950$

Example 4.10: Solution

Example 4.11: Changing Interest Rates with Uneven Cash Flow Series

- Consider the cash flows in Figure 4.13 with the interest rates indicated. Determine the uniform series equivalent of the given cash flow series.

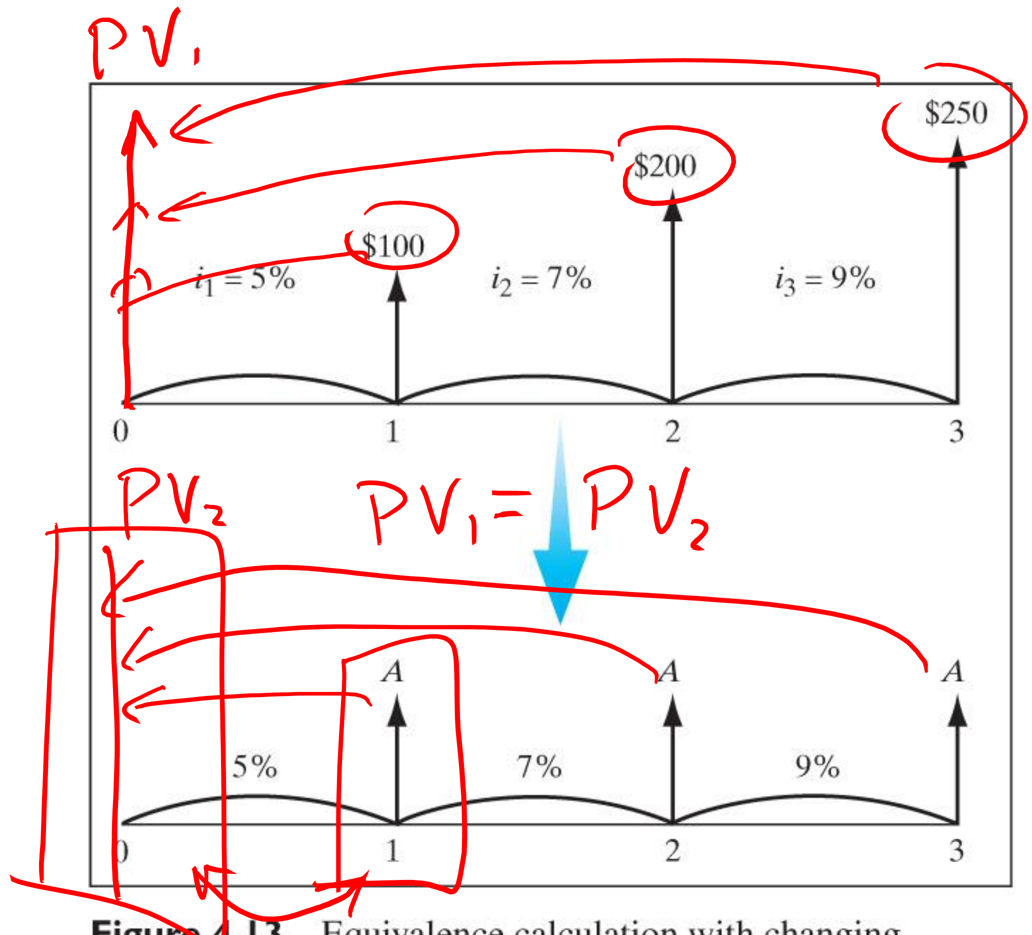


Figure 4.13 Equivalence calculation with changing interest rates (Example 4.11).

Example 4.11: Solution

check on interest table p. 113 in text book

- Given: *The cash flows and the interest rates*.
- Find: *A*
- Set the present worth of the first cash flow series equal to the present worth of the second cash flow series.

2nd 1st – Time shifting

Using Eq. (4.9), we find the present worth:

$$
\begin{aligned}
P &= \$100(P/F, 5\%, 1) + \$200(P/F, 5\%, 1)\boxed{(P/F, 7\%, 1)} \\
&\quad + \$250(P/F, 5\%, 1)(P/F, 7\%, 1)(P/F, 9\%, 1) \\
&= \$477.41.
\end{aligned}
$$

Then we obtain the uniform series equivalent as follows:

2nd 1st

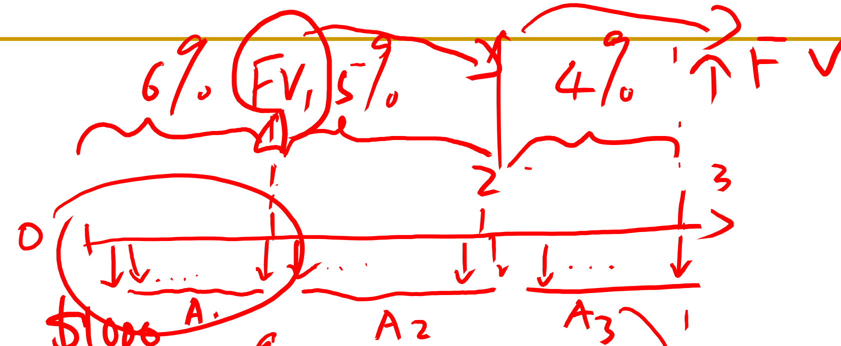
Time-shifting

$$
\begin{aligned}
\$477.41 &= A(P/F, 5\%, 1) + A(P/F, 5\%, 1)(P/F, 7\%, 1) \\
&\quad + A(P/F, 5\%, 1)(P/F, 7\%, 1)(P/F, 9\%, 1) \\
&= 2.6591A \\
A &= \$179.54.
\end{aligned}
$$

3rd 2nd 1st

Extra Example: Changing interest rate

- Over the past three years, you have been making monthly deposits of $1000 each. The interest rates you have earned from this account have been 6% in year 1, 5% in year 2, and 4% in year 3 (all based on monthly compounding). What is the balance today?
- Answer: $38,496.53

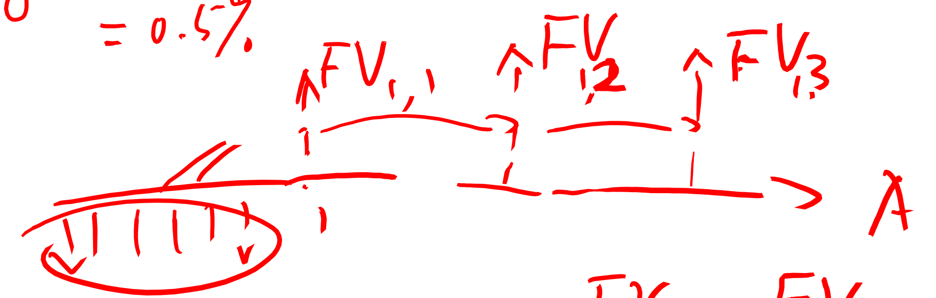


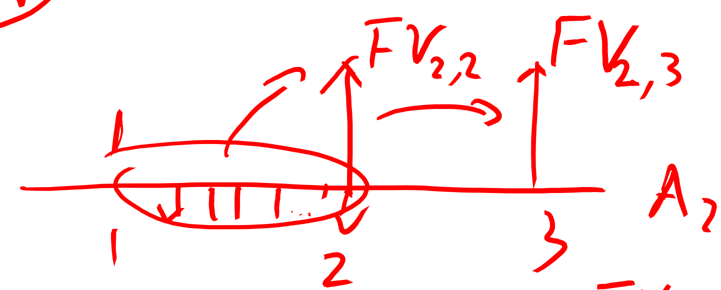
$$FV = A_1 (F/A, 0.5\%, 12) \cdot (F/P, 0.4167\%, 12) \cdot (F/P, 0.333\%, 12)$$
$$+ A_2 (F/A, 0.4167\%, 12) \cdot (F/P, 0.333\%, 12)$$
$$+ A_3 (F/A, 0.333\%, 12)$$
$$= \$38,496.52$$

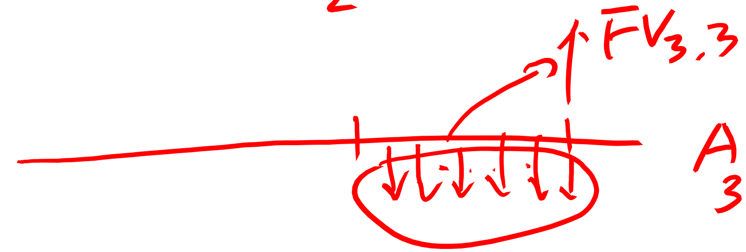
Monthly $i_{e1} = \frac{6\%}{12} = 0.5\%$

$i_{e2} = 0.4167\%$

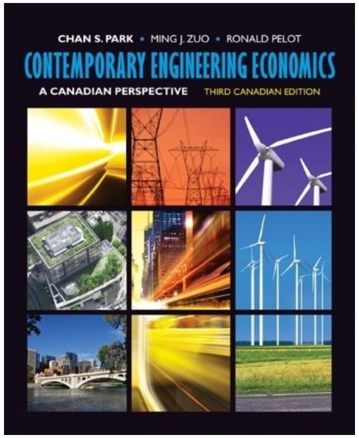
$i_{e3} = 0.333\%$







Summary



In any equivalence problem, the interest rate to use is the effective interest rate per payment period.