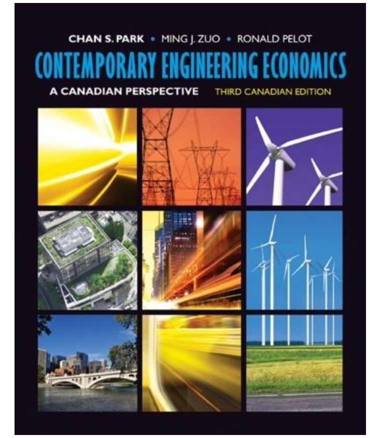
Understanding Money and Its Management



Lecture No. 8
Chapter 4
Contemporary Engineering Economics
Third Canadian Edition
Copyright © 2012

Chapter Opening Story: Mortgages

- Canadians have the choice of using a variable-rate mortgage or a fixed-rate mortgage to finance their homes.
- The interest rate charged on a variable-rate mortgage is tied to the prime rate, which is adjusted from time to time.
- A fixed-rate mortgage charges a fixed interest rate over a specified period of time.

Changing Rates

	Five-Year Fixed Mortgage Rate	Prime Rate
February 2008	5.89%	5.75%
August 2008	5.65%	4.75%
February 2009	4.29%	3.00%
June 2009	4.00%	2.25%

January 2016 (TD 5-yr Special): 2.94%   2.75%

Chapter 4 Objectives

- What is the difference between the nominal interest rate and the effective interest rate?
- What is the procedure for computing the effective interest rate per payment period?
- How do you perform equivalence analysis with effective interest rates?
- How are commercial loans and mortgages structured in terms of interest and principal payments?
- What are some basics of investing in bonds?

Lecture 8 Objectives

- What is the difference between the **nominal interest rate** and the **effective interest rate**?

Nominal and Effective Interest Rates

Nominal Interest Rate:
is a **stated** rate of interest for a given period (usually a year)

APR

Effective Interest Rate:
is the **actual** rate of interest, which accounts for the interest amount accumulated over a given period

Nominal Interest Rates

- the rate of interest that banks state (annual percentage rate or APR) for interest arrangements. It is the yearly cost of a loan expressed as a percentage.
- ***18% APR compounded monthly*** means

1. that each month the bank will charge 1.5% interest on an unpaid balance.
2. you will earn 1.5% interest each month on your remaining balance, if you deposited money.

Effective Annual Interest Rates

Transaction Frequency

- The effective annual interest rate is the rate that truly represents the interest earned or paid in one year — that is, compounding within the year is considered.
- The 18% APR compounded monthly means the bank will charge 1.5% interest on any unpaid balance at the end of each month. Therefore, the 1.5% rate represents the effective interest rate per month. On a yearly basis, you are looking for a cumulative rate — 1.5% each month for 12 months.

Savings Account With Interest Rate of 9% Compounded Quarterly

- You deposit $10,000 in a savings account with an **APR of 9% compounded quarterly**. The interest rate per quarter is 2.25% (9%/4). The following is an example of how interest is compounded when it is paid quarterly:

End of Period	Base Amount	Interest Earned 2.25% × (Base Amount)		New Base
First quarter	$10,000.00	2.25% × $10,000.00 =	$225.00	$10,225.00
Second quarter	$10,225.00	2.25% × $10,225.00 =	$230.06	$10,455.06
Third quarter	$10,455.06	2.25% × $10,455.06 =	$225.24	$10,690.30
Fourth quarter	$10,690.30	2.25% × $10,690.30 =	$240.53	$10,930.83

1st

Comp. times 4/yr

Effective Annual Interest Rate Formula

$$i_a = \left(1 + \frac{r}{M}\right)^M - 1$$

$$i_a = e^r - 1, \qquad when\ M \to \infty$$

r = nominal interest rate per year APR

M = number of compounding periods per year

i_a = effective annual interest rate

Example: Find i_a given different M values

Given an APR value of 15% compounded annually, semi-annually, monthly, daily, or continuously, find the annual effective interest rate in each case.

Example: Find i_a given different M values

Given an APR value of 15% compounded annually, semi-annually, monthly, daily, or continuously, find the annual effective interest rate in each case.

1. $i_a = APR = 15\%$
2. $m = 2$ $i_a = \left(1 + \frac{15\%}{2}\right)^2 - 1 = 15.563\%$
3. $m = 12$ $i_a = \left(1 + \frac{15\%}{12}\right)^{12} - 1 = 16.0755\%$
4. $m = 365$
5. Continuous $i_a = e^r - 1 =$

Example 4.1: Determining the Compounding Frequency

- The following table summarizes interest rates on several term deposits (TDs) and guaranteed investment certificates (GICs) offered by TD Canada Trust during December 2008:

APR

Product	Minimum	Rate	APY*	
3-Month TD	$5000	0.95%	0.95%	
1-Year TD	$1000	0.95%	0.95%	
1-Year Money Market GIC	$1000	1.50%	1.51%	m = ?
1+1 GIC	$1000	1st year: 1.80% 2nd year: 5.00%	3.388%	
2-Year Premium Rate Redeemable GIC	$1000	3.50%	3.53%	m = ?

*Annual percentage yield = effective annual interest rate (i_a).

a) Find the compounding frequency assumed.

b) Find the total balance two years later for a deposit amount of $100,000.

Example 4.1: Solution

- Given: r = 3.50% per year, i_a *(APY)* = 3.53%, P = \$100,000, and N = 2 years.
- Find: M and the balance at the end of two years

a) The nominal interest rate is 3.50% per year, and the effective annual interest rate (yield) is 3.53%.

$$0.0353 = \left(1 + \frac{0.0350}{M}\right)^M - 1$$

By trial and error $M = 2$

$F = P(1 + 3.53\%)^2$

b) Total balance two years later

$$F = \$100{,}000(F/P, 3.53\%, 2) = \$107{,}185$$

Effective Interest Rates per Payment Period

Not per year

- We can generalize the effective annual interest rate formula to compute the effective interest rate for periods of any duration.

$$i_{\text{per payment period}} = \left(1+\frac{r}{M}\right)^{C} - 1 = \left(1+\frac{r}{CK}\right)^{C} - 1 = \left(1+\frac{r}{M}\right)^{\frac{M}{K}} - 1$$

r, M, C   r, C, K   r, M, K

- M = number of compounding periods per year
- C = number of compounding periods per payment period
- K = number of payment periods per year
- $M = CK$

Example 4.2: Effective Rate per Payment Period

$K=4$ $m = cK$

- Suppose that you make quarterly deposits in a savings account that earns 9% interest compounded monthly. Compute the effective interest rate per quarter.

$r = 9\%$

$m = 12$

$$i_{e,\ \text{quarterly}} = \left[1 + \frac{r}{m}\right]^{\frac{m}{K}} - 1$$

$$= \left[1 + \frac{9\%}{12}\right]^{\frac{12}{4}} - 1 = 2.267\%$$

Example 4.2: Solution

- Given: r = 9%, C = three compounding periods per quarter, K = four quarterly payments per year, and M = 12 compounding periods per year.
- Find: i

$$i = \left(1 + \frac{r}{M}\right)^{C} - 1 = \left(1 + \frac{0.09}{12}\right)^{3} - 1$$

$$= 2.27\%$$

Continuous Compounding

- the process of calculating interest and adding it to existing principal and interest at infinitely short time intervals
- To calculate the effective annual interest rate for continuous compounding, we set *K* equal to unity and allow *M* to go to infinity, resulting in

$$i_a = e^r - 1$$

$$i_e = e^{\frac{r}{K}} - 1$$ – per payment period

Example 4.3: Calculating an Effective Interest Rate With Quarterly Payment

$K=4$

- Find the effective interest rate per quarter at a nominal rate of 8% compounded

APR

a) quarterly, $m=4$

b) monthly, $m=12$

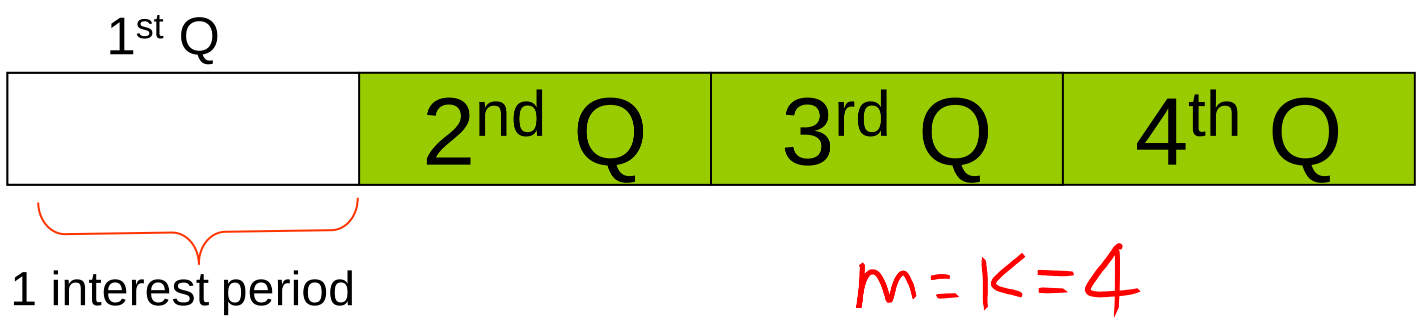
c) weekly, $m=52$

d) daily, and $m=365$

e) continuously. $m \to \infty$

Example 4.3: Quarterly Compounding

(a) r = 8%, M = 4, C = 1 compounding period per quarter, and K = 4 payments per year



m = K = 4
C = 1

$$i_Q = \left(1+\frac{0.08}{4}\right)^1 - 1 = \underline{2.00\%}$$

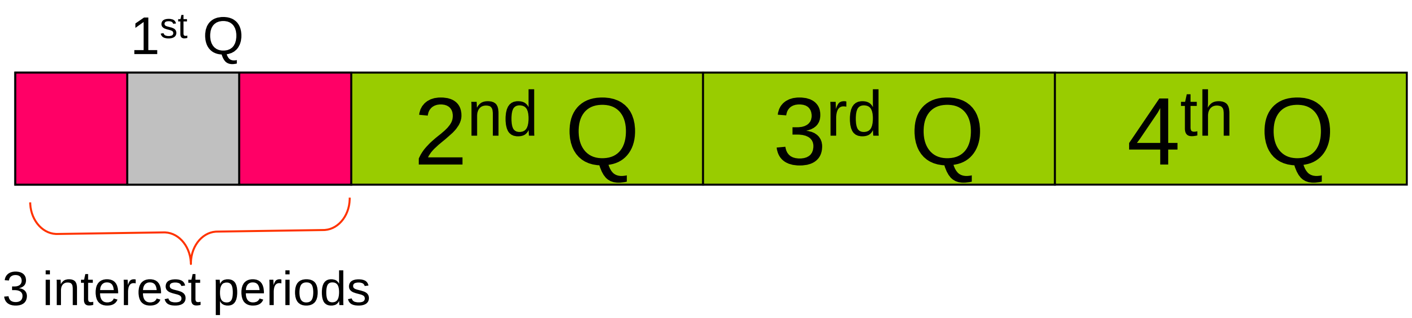
quarterly

$$i_e = \frac{r}{K}$$

Example 4.3: Monthly Compounding

$C = \frac{M}{K} = \frac{12}{4} = 3$

(b) r = 8%, M = 12, C = 3 compounding periods per quarter, and K = 4 payments per year



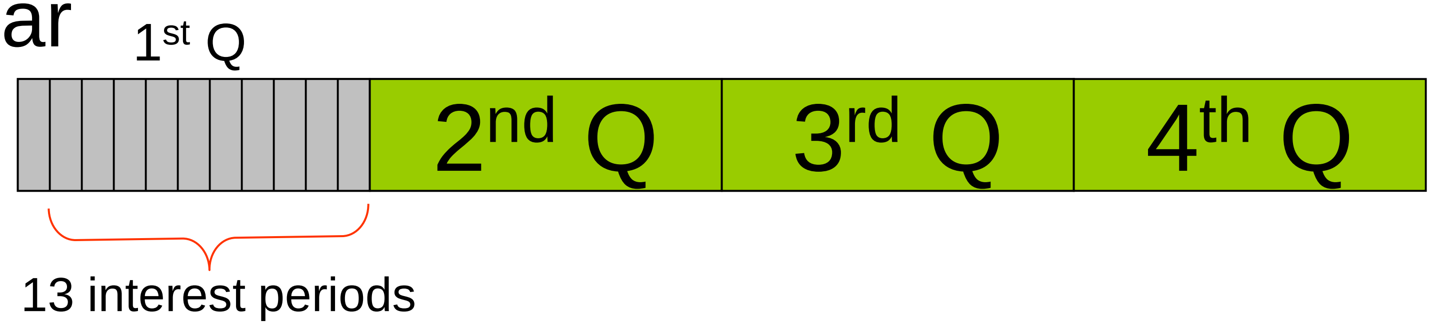
$$i_e = \left(1 + \frac{0.08}{12}\right)^3 - 1 = \underline{2.013\%}$$

Quarterly

Example 4.3: Weekly Compounding

$c = \frac{M}{K} = \frac{52}{4} = 13$

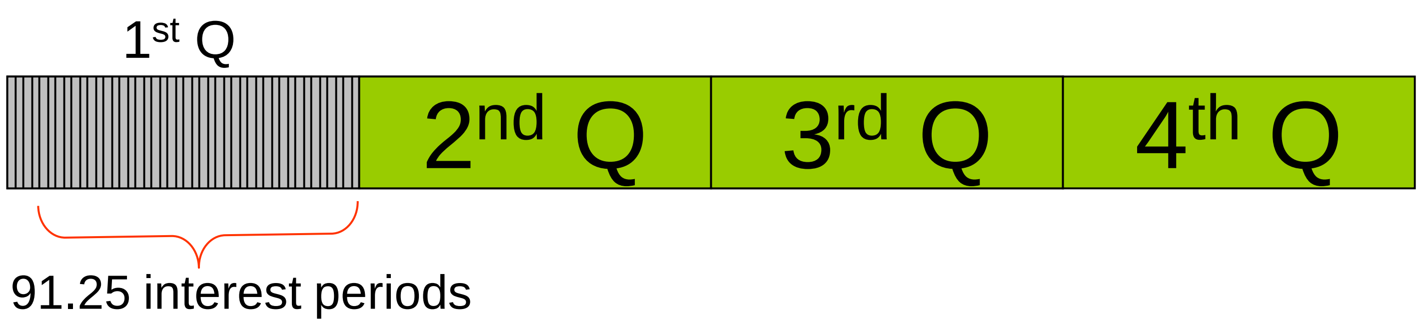
(c) r = 8%, M = 52, C = 13 compounding periods per quarter, and K = 4 payments per year



$$i = \left(1 + \frac{0.08}{52}\right)^{13} - 1 = \underline{2.0186\%}$$

Example 4.3: Daily Compounding

(d) r = 8%, M = 365, C = 91.25 days per quarter, and K = 4 payments per year

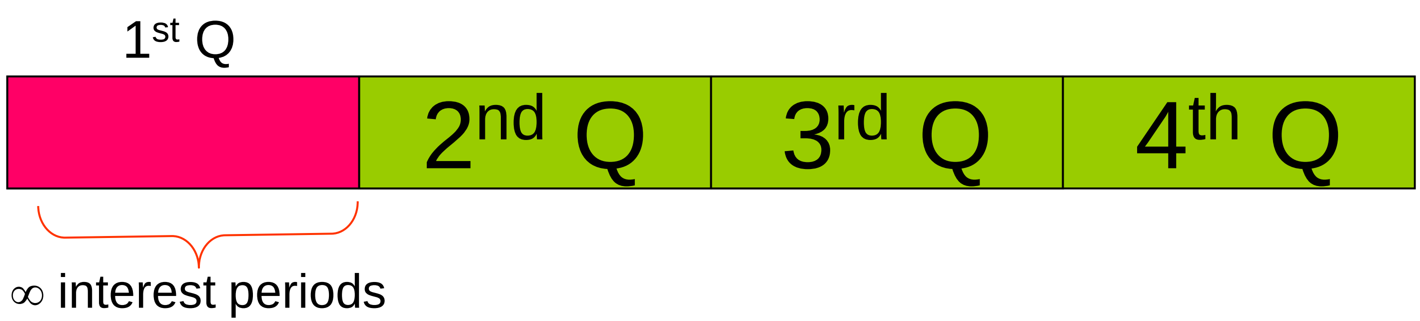


$$i_e = \left(1 + \frac{0.08}{365}\right)^{91.25} - 1 = \underline{2.0199\%}$$

quarterly

Example 4.3: Continuous Compounding

(e) r = 8%, M → ∞, C = ∞, and K = 4 payments per year



$$i_e = e^{0.08/4} - 1 = 2.0201\%$$

quarterly

Example Extra 1

- Suppose you make equal quarterly deposits of \$1500 each into a fund that pays interest at a rate of 6% compounded monthly. Find balance at the end of year 2.

$K=4$

$M=12$

$r=6\%$

$$i_{e\,\text{quarterly}} = \left[1+\frac{r}{m}\right]^{\frac{M}{K}} - 1 = \left[1+\frac{6\%}{12}\right]^{3} - 1$$

$$= 1.5075\%$$

$$N = 4 \times 2 = 8$$

$$F = A \cdot (F/A,\ 1.5075\%,\ 8) = A \cdot \left[\frac{(1+i)^N - 1}{i}\right] = 1500 \times 8.4378$$

$$= \$12,657$$

Example Extra 1

- Suppose you make equal quarterly deposits of $1500 each into a fund that pays interest at a rate of 6% compounded monthly. Find balance at the end of year 2.

Example Extra 2

- A loan company offers money at 1.8% per month, compounded monthly.

$m = 12$
$k = 12$
$c = 1$

  a) What is the nominal interest rate?

  b) What is the effective annual interest rate?

  c) How many years will it take for a borrowed amount to **triple** if no payments are made?

$APR = 1.8\% \times 12 = 21.6\%$ (Nominal)

$i_a = \left[1 + \frac{r}{m}\right]^m - 1 = \left[1 + \frac{21.6\%}{12}\right]^{12} - 1 = 23.87\%$

$\$(1+i_a)^N = 3\$ \rightarrow N\ln(1+i_a) = \ln 3 \quad N = 6$

Example Extra 2

- A loan company offers money at 1.8% per month, compounded monthly.
  a) What is the nominal interest rate?
  b) What is the effective annual interest rate?
  c) How many years will it take for a borrowed amount to **triple** if no payments are made?

Example Extra 3

r

- If the interest rate is 8.5% compounded continuously, what is the required quarterly payment to repay a loan of $12,000 in five years?

Example Extra 3

- If the interest rate is 8.5% compounded continuously, what is the required quarterly payment to repay a loan of $12,000 in five years?

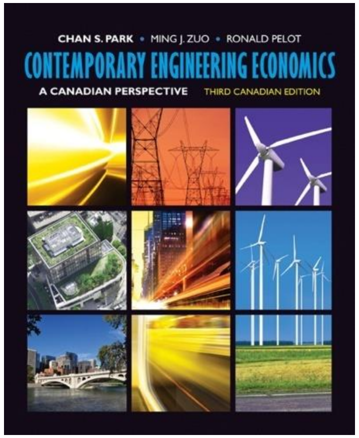
$$i_{e\ \text{quarterly}} = e^{\frac{8.5\%}{4}} - 1 = e^{\frac{0.085}{4}} - 1 = 2.148\%$$

$$A = P \cdot (A/P, \underbrace{2.148\%}, 20)$$

$$= P \cdot \left[\frac{2.148\% \cdot (1+2.148\%)^{20}}{\underbrace{(1+2.148\%)^{20} - 1}_{0.06203}} \right]$$

$$= \$744$$

Summary



Interest is most frequently quoted by financial institutions as an APR. However, compounding frequently occurs more often than once annually. This situation leads to the distinction between nominal and effective interest rates.