CMPUT 275 — Tangible Computing II **Winter 2018**

Big-Oh!

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1. Let $f(n) = 5n^2 + 2n + 3$, $g(n) = n^2$, $n \in \mathbb{N}$. Show that $f = O(g)$. What are a suitable values for c and N that we can use in the big-Oh definition?

   Pick $c = 10$ (the sum of the coefficients) and $N = 1$. Then for $n \geq N$:

   $$5n^2 + 2n + 3 \leq 5n^2 + 2n^2 + 3n^2 = 10n^2 = c \cdot n^2.$$

2. For the same f, g, show that $g = O(f)$. Again, specify suitable values of c and N (and show your work).

   Pick $c = 1$ and $N \geq 0$. Then for $n \geq N$:

   $$5n^2 + 2n + 3 \geq n^2 = c \cdot n^2.$$

3. Show $2^n = o(3^n)$.

   Let $c > 0$. We must show $2^n \leq c \cdot 3^n$ for sufficiently large n. This is equivalent to

   $$\frac{1}{c} \leq (3/2)^n$$

   or

   $$\log_{3/2} c^{-1} \leq n.$$

   So pick $N = \log_{2/3} c^{-1}$: for $n \geq N$ we then have $2^n \leq c \cdot 3^n$.

4. Use the definition of $O()$ to show that if $f = O(g)$ and $h = o(g)$ then $f + h = O(g)$.

   In fact, it holds even if $h = O(g)$, but let's prove this one.

   By definition, there are constants c_1, N_1 such that $f(n) \leq c_1 \cdot g(n)$ for $n \geq N_1$.
   Also, for $c_2 = 1$ there is some N_2 such that $h(n) \leq c_2 \cdot g(n)$ for $n \geq N_2$.
   Thus, for $c = c_1 + 1$ and $N = \max(N_1, N_2)$ we have $(f + h)(n) \leq (c_1 + 1) \cdot g(n)$ for all $n \geq N$.

5. Show that if $f(n) = a_0 + a_1 n + \cdots + a_d n^d$ is a d-degree polynomial wth non-negative coefficients (and $a_d > 0$), then $f = O(n^d)$. Also show $f = \Omega(n^d)$ (so you are really showing $f = \Theta(n^d)$).

   For one side, pick $c = \sum_i a_i$ and $N = 1$. Then for $n \geq N$:

$$f(n) \leq n^d \sum_{i=0}^{d} a_i = c \cdot n^d.$$

   Conversely, let $c' = a_d$ and $N' = 0$. Then for $n \geq N$:

$$f(n) \geq a_d \cdot n^d = c' \cdot n^d.$$

6. **Optional**: Let $f, g : \mathbb{N} \to \mathbb{R}$ be arbitrary functions that are eventually positive and increasing. That is, $f(x) \geq f(y)$ if $x \geq y$ when y is large enough (and the same for g).
   - Show that if $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ is some constant (perhaps 0) then $f = O(g)$.
   - Show that $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$ if and only if $f = o(g)$.
   - Show that if the limit is some strictly positive value then $f = \Theta(g)$.