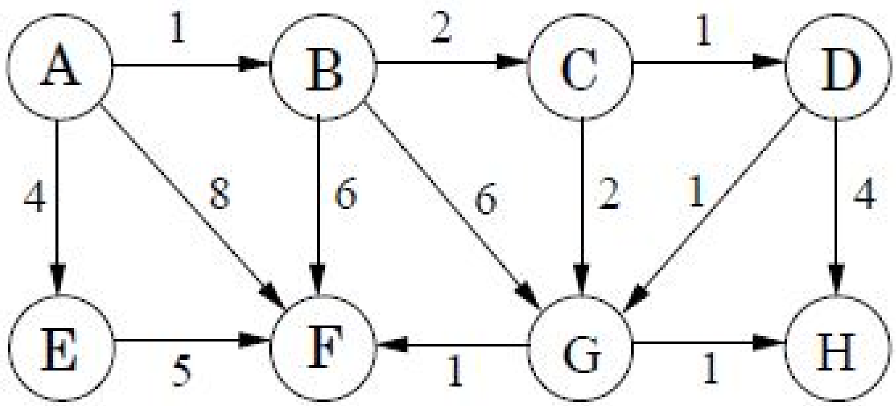
Dijkstra's Algorithm

Consider the following directed graph[1] with edge weights. Trace an execution of Dijkstra's algorithm and highlight the search tree.



[1] `https://sites.google.com/site/markdolanprogramming/cis-3223/assignment-6`

1. Give a simple example of a graph with some negative-cost edges but no negative-cost cycles where Dijkstra's algorithm fails to find a minimum-cost path.

2. Now find values $\phi(v)$ for each vertex v in your graph above so that $c(u, v) + \phi(v) - \phi(u) \geq 0$ for each edge $uv \in E$. Run Dijkstra's, except using edge costs $c(u, v) + \phi(v) - \phi(u)$. Check that the paths it finds are indeed minimum-cost paths under the original costs.

3. Can you see why this would work in any graph? That is, if we are given $\phi(v)$ values for vertices so $c(u, v) + \phi(v) - \phi(u) \geq 0$ for each edge $uv \in E$ then running Dijkstra's with these modified costs will find paths that are minimum-cost paths under the original costs?
   Hint: Try writing out the new cost of a length-2 path. Of a length-3 path. See the pattern?

4. Argue that if G has a negative-cost cycle then no such values $\phi(v)$ exist.
   Hint: Verify that the original cost and modified cost of any cycle is the same.
   Note: It is possible to show these ϕ values always exist (and to find them in $O(|V| \cdot |E|)$ time) if there are no negative-cost cycles.