29/4/22

Note Title 29-04-2022

V - vector space over $\mathbb{F}$
W_1, W_2 are subspaces of V.

- $W_1 + W_2 = \{ \omega_1 + \omega_2 : \omega_1 \in W_1 \ \& \ \omega_2 \in W_2 \}$.
- $W_1 + W_2 = \text{span}(W_1 \cup W_2)$

Proof.
$\Rightarrow$ Let $v \in W_1 + W_2$
$\Rightarrow v = \omega_1 + \omega_2$ for some $\omega_1 \in W_1$ & $\omega_2 \in W_2$.
Then $\omega_1, \omega_2 \in W_1 \cup W_2$
$\Rightarrow \omega_1, \omega_2 \in \text{span}(W_1 \cup W_2)$
$\Rightarrow \omega_1 + \omega_2 \in \text{span}(W_1 \cup W_2)$
i.e., $v \in \text{span}(W_1 \cup W_2)$
Thus $W_1 + W_2 \subseteq \text{span}(W_1 \cup W_2)$.

$\Leftarrow$ $W_1 \subseteq W_1 + W_2$ & $W_2 \subseteq W_1 + W_2$
$\Rightarrow W_1 \cup W_2 \subseteq W_1 + W_2$
$\Rightarrow \text{span}(W_1 \cup W_2) \subseteq W_1 + W_2$ (as $\text{span}(W_1 \cup W_2)$ is the smallest subspace containing $W_1 \cup W_2$).

Ex. If $W_1 = \text{span}(S_1)$ & $W_2 = \text{span}(S_2)$, then show that $W_1 + W_2 = \text{span}(S_1 \cup S_2)$.

Examples

1) $V = \mathbb{R}^2$
$W_1 = \{(x, x) : x \in \mathbb{R}\}$
$W_2 = \{(x, -x) : x \in \mathbb{R}\}$

$W_1 + W_2 = \{(x, x) + (y, -y) : x, y \in \mathbb{R}\}$
$= \{(x+y, x-y) : x, y \in \mathbb{R}\}$

Let $(x, y) \in \mathbb{R}^2$. Then

$$(x, y) = \left(\frac{x+y}{2} + \frac{x-y}{2}, \frac{x+y}{2} - \frac{x-y}{2}\right)$$
$$= \left(\frac{x+y}{2}, \frac{x+y}{2}\right) + \left(\frac{x-y}{2}, -\frac{x-y}{2}\right) \in W_1 + W_2$$

$\therefore W_1 + W_2 = \mathbb{R}^2$.

2) $V = \mathbb{R}^4$.
$W_1 = \{(x, y, z, w) : x + y + z = 0, \ x + 2y - z = 0\}$
$W_2 = \{(s - 3t, 2s + 2t, 3s + t, t) : s, t \in \mathbb{R}\}$

$W_1 + W_2$
What is the dimension of $W_1 + W_2$?

$\mathcal{B}_1 = \{(-3,2,1,0), (0,0,0,1)\}$
$\mathcal{B}_2 = \{(1,2,3,0), (-3,2,1,1)\}$

$W_1 + W_2 = \operatorname{span}(\mathcal{B}_1 \cup \mathcal{B}_2)$.

$$\begin{pmatrix} -3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 2 & 3 & 0 \\ -3 & 2 & 1 & 1 \end{pmatrix} \qquad \text{RRE} = \begin{pmatrix} 1 & 0 & 1/2 & 0 \\ 0 & 1 & 5/4 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Basis for $W_1 + W_2 = \{(1,0,1/2,0), (0,1,5/4,0), (0,0,0,1)\}$.
$\dim(W_1 + W_2) = 3$.

Theorem (Dimension formula)

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).$$

Proof.
Let
$\dim(W_1 \cap W_2) = r \qquad \mathcal{B} = \{u_1, u_2, \dots, u_r\}$
$\dim(W_1) = r + s \qquad \mathcal{B}_1 = \{u_1, u_2, \dots, u_r, v_1, v_2, \dots, v_s\}$
$\dim(W_2) = r + t \qquad \mathcal{B}_2 = \{u_1, u_2, \dots, u_r, w_1, w_2, \dots, w_t\}$

Let $\mathcal{B}' = \{u_1, u_2, \dots, u_r, v_1, v_2, \dots, v_s, w_1, w_2, \dots, w_t\}$

claim. $\mathcal{B}'$ forms a basis for $W_1 + W_2$.

Suppose that
$$\sum_{i=1}^{r} \alpha_i u_i + \sum_{j=1}^{s} \beta_j v_j + \sum_{k=1}^{t} \gamma_k w_k = 0 \longrightarrow (1)$$

$$\Rightarrow \underbrace{\sum_{i=1}^{r} \alpha_i u_i + \sum_{j=1}^{s} \beta_j v_j}_{\in W_1} = \underbrace{-\sum_{k=1}^{t} \gamma_k w_k}_{\in W_2} \in W_1 \cap W_2. \longrightarrow (2)$$

$$\therefore \; -\sum_{k=1}^{t} \gamma_k w_k = \sum_{i=1}^{r} \delta_i u_i$$

$$\Rightarrow \sum_{i=1}^{r} \delta_i u_i + \sum_{k=1}^{t} \gamma_k w_k = 0$$

Since $\mathcal{B}_2$ is a basis, it follows that $\delta_i = 0 \ \forall\, 1 \le i \le r$ & $\gamma_k = 0 \ \forall\, 1 \le k \le t$.

Plugging this in (2), we get

$$\sum_{i=1}^{r} \alpha_i u_i + \sum_{j=1}^{s} \beta_j v_j = 0$$

As $\mathcal{B}_1$ is a basis, it follows that $\alpha_i = 0 \ \forall\ 1 \le i \le r$ & $\beta_j = 0 \ \forall\ 1 \le j \le s$.

Thus $\mathcal{B}'$ is linearly independent.

Let $w \in W_1 + W_2$. Then $\exists\ w_1 \in W_1$ & $w_2 \in W_2$ such that

$$w = w_1 + w_2 \longrightarrow (3)$$

$\mathcal{B}_1$ forms a basis for W_1.

$$\therefore\ w_1 = \sum_{i=1}^{r} \alpha_i u_i + \sum_{j=1}^{s} \beta_j v_j \longrightarrow (4)$$

where $\alpha_i \in \mathbb{F}$ & $\beta_j \in \mathbb{F}$ $\forall\ 1 \le i \le r$ & $1 \le j \le s$.

IIIrly, $\mathcal{B}_2$ forms a basis for W_2

$$\therefore\ w_2 = \sum_{i=1}^{r} \gamma_i u_i + \sum_{k=1}^{t} \delta_k w_k \longrightarrow (5)$$

where $\gamma_i \in \mathbb{F}$ & $\delta_k \in \mathbb{F}$ $\forall\ 1 \le i \le r$ & $1 \le k \le t$.

Inserting (4) & (5) into (3), we get

$$w = \sum_{i=1}^{r} \alpha_i u_i + \sum_{j=1}^{s} \beta_j v_j + \sum_{i=1}^{r} \gamma_i u_i + \sum_{k=1}^{t} \delta_k w_k$$

$$= \sum_{i=1}^{r} (\alpha_i + \gamma_i) u_i + \sum_{j=1}^{s} \beta_j v_j + \sum_{k=1}^{t} \delta_k w_k \in \operatorname{span}(\mathcal{B}').$$