26/4/22

Note Title 26-04-2022

1) $v_1 = (1,2)$ & $v_2 = (0,1)$.

a) $W_1 = \{ t v_1 : t \in \mathbb{R} \}$ — straight line joining (1,2) & (0,0)
$W_2 = \{ t v_2 : t \in \mathbb{R} \}$ —
$W_3 = \{ t v_1 + s v_2 : t, s \in \mathbb{R} \}$ — plane
$W_4 = \{ t v_1 + s v_2 : 0 \le t, s \le 1 \}$ — parallelogram (1,2), (1,3), (0,0), (0,1).

b)

c) $\alpha(1,2) + \beta(0,1) = (0,0)$
$$\left.\begin{aligned} \alpha &= 0 \\ 2\alpha + \beta &= 0 \end{aligned}\right\} \Rightarrow \alpha = \beta = 0 \Rightarrow v_1 \,\&\, v_2 \text{ are linearly independent.}$$

d) $v_3 = (2,3)$. Is $\{v_1, v_2, v_3\}$ lin. ind ?

$2v_1 - v_2 - v_3 = 0$ — Verify

2) $V = \mathbb{C}^2$ over $\mathbb{C}$

a) $\{(1+i, 2), (2,1)\}$ — linearly ind.

b) $\{(1,2), (0,i), (i, 1-i)\}$

$\alpha(1,2) + \beta(0,i) + \gamma(i, 1-i) = (0,0)$
$$\left.\begin{aligned} \alpha + i\gamma &= 0 \\ 2\alpha + i\beta + (1-i)\gamma &= 0 \end{aligned}\right\} \Rightarrow \alpha = -i \quad \beta = 3+i, \ \gamma = 1$$

c) $v_1 = (1+i, 2)$ & $v_2 = (2,1)$.
Show that any ordered pair (x,y) can be written as a linear comb. of v_1 & v_2.

$(x,y) = \alpha(1+i, 2) + \beta(2,1)$
$$\alpha = \frac{x-2y}{i-3} \qquad \beta = \frac{-2x + (1+i)y}{i-3}$$

3) $V = \mathbb{C}^2(\mathbb{R})$.
$X = \{(1+i, 1-i), (1-i, 1+i), (2,i), (3,2i)\}$

- Show that X is linearly ind.

$\alpha(1+i, 1-i) + \beta(1-i, 1+i) + \gamma(2,i) + \delta(3,2i) = (0,0)$
$$\begin{aligned} \alpha + \beta + 2\gamma + 3\delta &= 0 \\ \alpha - \beta + 0\gamma + 0\delta &= 0 \\ \alpha + \beta + 0\cdot\gamma + 0\delta &= 0 \\ -\alpha + \beta + \gamma + 2\delta &= 0 \end{aligned}$$

Ex. $V = \mathbb{C}^2(\mathbb{C})$. Show that X is linearly dependent

4) Let u, v & $w \in V$. S.T. $\{u, v, w\}$ is lin. ind. $\Leftrightarrow$ $\{u+v, v+w, w+u\}$ is lin. ind.
Solution.

$\Rightarrow$ Suppose that $\{u, v, w\}$ is linearly ind.

$$\alpha(u+v) + \beta(v+w) + \gamma(w+u) = 0$$
$$\Rightarrow (\alpha+\gamma)u + (\alpha+\beta)v + (\beta+\gamma)w = 0$$
$$\Rightarrow \alpha+\gamma=0, \ \alpha+\beta=0 \ \& \ \beta+\gamma=0$$
$$\Rightarrow \alpha=\beta=\gamma=0$$
i.e., $\{u+v, v+w, w+u\}$ is linearly ind.

$\Leftarrow$ Suppose that $\{u+v, v+w, w+u\}$ is linearly ind.

$\alpha u + \beta v + \gamma w = 0$

Q. Can we write u, v & w as linear combinations of $u+v$, $v+w$ & $w+u$?

$$u = \alpha_{11}(u+v) + \alpha_{12}(v+w) + \alpha_{13}(w+u)$$
$$v = \alpha_{21}(u+v) + \alpha_{22}(v+w) + \alpha_{23}(w+u)$$
$$w = \alpha_{31}(u+v) + \alpha_{32}(v+w) + \alpha_{33}(w+u)$$

$$\boxed{\begin{array}{lll} \alpha_{11} = 1/2 & \alpha_{12} = -1/2 & \alpha_{13} = 1/2 \\ \alpha_{21} = 1/2 & \alpha_{22} = 1/2 & \alpha_{23} = -1/2 \\ \alpha_{31} = -1/2 & \alpha_{32} = 1/2 & \alpha_{33} = 1/2 \end{array}}$$

$\therefore$ $\alpha u + \beta v + \gamma w = 0$
$$\Rightarrow \alpha\left(\tfrac{1}{2}(u+v) - \tfrac{1}{2}(v+w) + \tfrac{1}{2}(w+u)\right) + \beta\left(\tfrac{1}{2}(u+v) + \tfrac{1}{2}(v+w) - \tfrac{1}{2}(w+u)\right)$$
$$+ \gamma\left(-\tfrac{1}{2}(u+v) + \tfrac{1}{2}(v+w) + \tfrac{1}{2}(w+u)\right) = 0$$
$$\Rightarrow \left(\frac{\alpha}{2} + \frac{\beta}{2} - \frac{\gamma}{2}\right)(u+v) + \left(-\frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2}\right)(v+w) + \left(\frac{\alpha}{2} - \frac{\beta}{2} + \frac{\gamma}{2}\right)(w+u) = 0$$

$$\Rightarrow \left.\begin{array}{r} \alpha + \beta - \gamma = 0 \\ -\alpha + \beta + \gamma = 0 \\ \alpha - \beta + \gamma = 0 \end{array}\right\} \Rightarrow \alpha = \beta = \gamma = 0.$$

6) $V = \mathbb{R}^2$ over $\mathbb{R}$
$W_1 = \{(x,0) : x \in \mathbb{R}\}$
$W_2 = \{(0,y) : y \in \mathbb{R}\}$

$W_1 \cup W_2$ $\quad (0,1) + (1,0) = (1,1) \notin W_1 \cup W_2$.

S.T. $W_1 \cup W_2$ is a subspace $\Leftrightarrow$ either $W_1 \subset W_2$ or $W_2 \subset W_1$.
Solution. $\Leftarrow$ Easy.

$\Rightarrow$ Suppose that $W_1 \cup W_2$ is a subspace.

Suppose to the contrary that neither $W_1 \subset W_2$ nor $W_2 \subset W_1$.
Choose $w_2 \in W_2 \setminus W_1$ & $w_1 \in W_1 \setminus W_2$. Then

$$w_1 + w_2 \in W_1 \cup W_2$$
$$\Rightarrow w_1 + w_2 \in W_1 \quad \text{or} \quad w_1 + w_2 \in W_2$$

$w_1 + w_2 \in W_1$
$\Rightarrow w_2 = (w_1 + w_2) - w_1 \in W_1$

llly $w_1 + w_2 \in W_2$
$\Rightarrow w_1 = (w_1 + w_2) - w_2 \in W_2$

Both cannot happen, which is a contradiction.