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Note Title 03-06-2022

$$y'' + p(t)\,y' + q(t)\,y = 0 \longrightarrow (1)$$ – Linear & homogeneous

Here p & q are continuous.

- The solution space forms a 2-dimensional real vector space.
- The general solution for (1) is given by
  $$y(t) = c_1 y_1(t) + c_2 y_2(t)$$
  where y_1 & y_2 are any two L.I. solutions

Q. How to check whether two fns. are LI or not?

Suppose $f_1, f_2, \dots, f_n$ are fns. defined on I. Also, suppose that they are lin. dep.

Then $\exists$ scalars $c_1, c_2, \dots, c_n$, not all equal to zero, s.t.

$$c_1 f_1 + c_2 f_2 + \dots + c_n f_n = 0$$
i.e., $$c_1 f_1(t) + c_2 f_2(t) + \dots + c_n f_n(t) = 0 \quad \forall\, t \in I.$$

Suppose that f_i's are $(n-1)$-times diff. on I.

$$c_1 f_1 + c_2 f_2 + \dots + c_n f_n = 0$$
$$c_1 f_1' + c_2 f_2' + \dots + c_n f_n' = 0$$
$$c_1 f_1'' + c_2 f_2'' + \dots + c_n f_n'' = 0$$
$$\vdots$$
$$c_1 f_1^{(n-1)} + c_2 f_2^{(n-1)} + \dots + c_n f_n^{(n-1)} = 0$$

$$\begin{bmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & & & \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

As f_i's are lin. depnd. this system will have atleast one one non-zero solution, say $(c_1, c_2, \dots, c_n)$ and hence the determinant of the matrix is zero.

Wronskian. For any n real valued fns. $f_1, f_2, \dots, f_n$ which are $(n-1)$-times diff. on an interval I, the Wronskian, denoted $W(f_1, f_2, \dots, f_n)$, is defined as

$$W(f_1, f_2, \dots, f_n) := \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & & & \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

Theorem. If $f_1, f_2, \dots, f_n$ are $(n-1)$-times diff. fns. which are also linearly dependent, then $W(f_1, f_2, \dots, f_n) = 0$, i.e., $W(f_1, f_2, \dots, f_n)(t) = 0 \ \forall\ t$

Corollary. If $f_1, f_2, \dots, f_n$ are $(n-1)$-times diff-fns. and if $\exists\ t_0 \in I$ s.t. $W(f_1, f_2, \dots, f_n)(t_0) \neq 0$
then $f_1, \dots, f_n$ are linearly independent on I.

Remark. The converse of the above theorem need not be true.
Choose $f_1 = t^2$ & $f_2 = t|t|$ $\quad t \in (-1, 1)$.

- $W(f_1, f_2) = 0$
- f_1 & f_2 are linearly independent.

Remark. It is actually possible that the Wronskian is zero at some pts & non-zero at other points.
Choose $f_1(x) = x$ & $f_2(x) = x^2$ $\quad x \in \mathbb{R}$.

Abel's theorem. If y_1 & y_2 are solutions of (1), then the

$$W(y_1, y_2)(t) = c \exp\left(-\int_{t_0}^{t} p(t)\, dt\right)$$

for some constant c.

Proof. Let $W(t) = W(y_1, y_2)(t)$. Then

$$W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)\, y_2'(t) - y_2(t)\, y_1'(t).$$

$$\begin{aligned}
\therefore\ W'(t) &= y_1'(t)\, y_2'(t) + y_1(t)\, y_2''(t) - y_2'(t)\, y_1'(t) - y_2(t)\, y_1''(t) \\
&= y_1(t)\, y_2''(t) - y_1''(t)\, y_2(t). \\
&= y_1\left[-p(t)\, y_2'(t) - q(t)\, y_2(t)\right] - y_2(t)\left[-p(t)\, y_1'(t) - q(t)\, y_1(t)\right] \\
&= -p(t)\left[y_1 y_2'(t) - y_1'(t)\, y_2(t)\right] \\
&= -p(t)\, W(t)
\end{aligned}$$

Thus W satisfies the 1^{st} order ODE $y' = py$.

$$\therefore\ W(t) = c \exp\left(-\int_{t_0}^{t} p(s)\, ds\right).$$

Theorem. If y_1 & y_2 are solutions of (1), then $W(y_1, y_2)(t_0) = 0$ for some $t_0 \in I$
$\Rightarrow W(y_1, y_2)(t) = 0 \quad \forall\ t \in I$ i.e., y_1 & y_2 are linearly dependent.
Proof. Follows from Abel's theorem.