19/4/22

Note Title 19-04-2022

$\mathcal{B}$ - is a basis if
- $\mathcal{B}$ should be linearly ind
- $\text{span}(\mathcal{B}) = V$.

$\Leftrightarrow$ $\mathcal{B}$ is a maximal linearly ind. set.
$\Leftrightarrow$ $\mathcal{B}$ is a minimal spanning set.

- Every vector can be written as a unique linear combination of vectors from $\mathcal{B}$.

Q. If $\mathcal{B}$ is a nonempty set and if every vector can be written as a unique linear combination of vectors from $\mathcal{B}$, then is $\mathcal{B}$ a basis?
Answer: Yes (Exercise).

- Every vector space has a basis.

Q. We already have a linearly ind. set. Can one extend it to a basis?

Lemma. Any linearly independent finite set of vectors is a part of a basis.
Proof.
Let $S = \{v_1, v_2, \dots, v_n\}$ be a linearly independent set. Let $\mathcal{B}$ be a basis.

$v_1 \in S \subseteq V$ & $\mathcal{B}$ is a basis.
$\therefore \exists$ $w_1, w_2, \dots, w_m \in \mathcal{B}$ & $\alpha_1, \alpha_2, \dots, \alpha_m \in \mathbb{F}$ such that

$$v_1 = \alpha_1 w_1 + \alpha_2 w_2 + \dots + \alpha_m w_m$$

As S is linearly ind., $v_1 \neq 0$ and hence atleast one $\alpha_i \neq 0$. WLOG let us assume that $\alpha_1 \neq 0$. Then

$$w_1 = \alpha_1^{-1} v_1 - \sum_{i=2}^{m} \alpha_1^{-1} \alpha_i w_i.$$

Let $\mathcal{B}_1 = (\mathcal{B} \setminus \{w_1\}) \cup \{v_1\}$. It is easy to check that $\mathcal{B}_1$ is a basis.

Choose $v_2 \in S \setminus \{v_1\}$. By following the above procedure, form a basis $\mathcal{B}_2 = (\mathcal{B} \setminus \{w_1, w_2\}) \cup \{v_1, v_2\}$.
Keep doing the same until all the vectors in S are exhausted. As S is a finite set, this process stops after a finite no. of steps.

Theorem. If V has a finite basis consisting of n vectors, then any other basis also has n vectors.
Proof. Let $\mathcal{B}$ be a basis consisting of n vectors. Let $\mathcal{B}'$ be any other basis for V. Let $\mathcal{B}'$ contain m vectors.

Claim. $m = n$.

Suppose to the contradiction that $m \neq n$. Then either $m < n$ or $m > n$.
WLOG let us assume that $m > n$. By following the procedure used in the previous lemma, we can find a basis $\mathcal{B}_1$ such that

$|S|$ - Cardinality of S.

$\mathcal{B} \subsetneq \mathcal{B}_1$ & $|\mathcal{B}_1| = |\mathcal{B}'|$.

But this cannot happen as $\mathcal{B}$ is maximally linearly independent.

Definition. The no. of vectors in a basis is called as the dimension of the vector space.

- If no. of vectors in a basis is finite then the vector is said to be a finite dimensional vector space.
- Otherwise, the vector space is said to be infinite dimensional.

<u>Examples</u> $\dim_{\mathbb{F}}(V) = \dim(V(\mathbb{F}))$ = dimension of V over $\mathbb{F}$.

1) $\mathbb{R}$ $\dim_{\mathbb{R}}(\mathbb{R}) = 1$ $\dim_{\mathbb{Q}}(\mathbb{R}) = \infty$.

(Produce a set which is infinite & linearly ind.)

$\{\sqrt{p} : p \text{ is a prime}\}$

2) $\mathbb{C}$ $\dim_{\mathbb{C}}(\mathbb{C}) = 1$ $\{1+0i\}$

$\dim_{\mathbb{R}}(\mathbb{C}) = 2$ $\{1, i\}$

3) $\mathbb{C}^2$ $\dim_{\mathbb{C}}(\mathbb{C}^2) = 2$ $\{(1,0), (0,1)\}$

$\dim_{\mathbb{R}}(\mathbb{C}^2) = 4$ $\{(1,0), (i,0), (0,1), (0,i)\}$

4) $\mathbb{R}^3$ $\dim_{\mathbb{R}}(\mathbb{R}^3) = 3$ $\{\overset{e_1}{(1,0,0)}, \overset{e_2}{(0,1,0)}, \overset{e_3}{(0,0,1)}\}$

5) $P_n(\mathbb{F})$ $\dim_{\mathbb{F}}(P_n(\mathbb{F})) = n$ $\{1, x, x^2, \dots, x^{n-1}\}$

6) $\mathbb{F}[x]$ $\dim_{\mathbb{F}}(\mathbb{F}[x]) = \infty$

$\{1, x, x^2, x^3, \dots\}$

$\mathcal{B} = \{v_1, v_2, \dots, v_n\}$ - Ordered basis $\quad \overset{v}{=(x,y,z)} = x e_1 + y e_2 + z e_3$

$v = \alpha_1 v_1 + \alpha_2 + \dots + \alpha_n v_n$

$$[v]_{\mathcal{B}} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \in M_{n \times 1}(\mathbb{F})$$

- Coordinate vector of v w.r.t. the basis $\mathcal{B}$.