27/4/22

Note Title 27-04-2022

Subspace basis dimension

Easy observations

- $\text{span}(\text{span}(S)) = \text{span}(S)$
- $S_1 \subseteq \text{span}(S_2) \Rightarrow \text{span}(S_1) \subseteq \text{span}(S_2)$
- $S_1 \subseteq \text{span}(S_2)$ & $S_2 \subseteq \text{span}(S_1) \Rightarrow \text{span}(S_1) = \text{span}(S_2)$.

Fix $A \in M_{m\times n}(\mathbb{F})$.

For $1 \le i \le m$, let R_i denote the i^{th} row of the matrix A.

- $R_i \in \mathbb{F}^n$.

Row space of $A := \text{span}\{R_i : 1 \le i \le m\} \le \mathbb{F}^n$

- Row space of A is a finite dim. vector space.

Theorem. If $A, B \in M_{m\times n}(\mathbb{F})$ and if A & B are row equivalent, then row space (A) = row space (B).

Proof: Exercise.

Defn. The dimension of the row space of A is called as the row rank (A).

Corollary.

i) If $A \in M_{m\times n}(\mathbb{F})$ is a RRE matrix, then the row rank (A) is equal to the no. of non-zero rows.

ii) If $A \in M_{m\times n}(\mathbb{F})$, then row rank$(A)$ = rank(A).

Problem. Find the dimension of $\text{span}\{(1,2,3,4), (2,3,4,5), (3,4,5,6), (4,5,6,7)\}$.

Solution.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ Dimension $= 2$.

C_j, $1 \le j \le n$, the j^{th} column of A.

- $C_j \in \mathbb{F}^m$.

- column space of $A = \text{span}\{C_j : 1 \le j \le n\} \le \mathbb{F}^m$.

column rank of A = dimension of the column space.

$Ax = 0$

$\{ x \in \mathbb{F}^n : Ax = 0 \}$ – solution space.

<u>Example</u>

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 1 & 2 & 1 & 4 \\ 2 & 4 & 1 & 5 \end{pmatrix}$$

$$RRE(A) = \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

x_2 & x_4 are independent / free unknowns

$x_2 = \lambda$

$x_4 = \mu$

$x_1 = -2\lambda - \mu \qquad x_3 = -3\mu.$

solution space $= \{ (-2\lambda - \mu, \lambda, -3\mu, \mu) : \lambda, \mu \in \mathbb{R} \}$

$\lambda = 1 \quad \mu = 0 \; : \; (-2, 1, 0, 0)$

$\lambda = 0 \quad \mu = 1 \; : \; (-1, 0, -3, 1)$

dim of the solution space is 2.

Theorem. The dimension of the solution space of $Ax = 0$ is $n - r$ where n is the no. of unknowns and r is the rank of A.

<u>Example</u>

$$A = \begin{pmatrix} 1 & 2 & 3 & -2 & 4 \\ 2 & 4 & 8 & 1 & 9 \\ 3 & 6 & 13 & 4 & 14 \end{pmatrix}$$

$$RRE(A) = \begin{pmatrix} 1 & 2 & 0 & -19/2 & 5/2 \\ 0 & 0 & 1 & 5/2 & 1/2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$x_2 = \lambda \quad x_4 = \mu \quad x_5 = \delta$

$x_1 = 19/2\,\mu - 5/2\,\delta - 2\lambda$

$x_3 = -5/2\,\mu - \frac{1}{2}\delta$

$\{ (\frac{19}{2}\mu - \frac{5}{2}\delta - 2\lambda, \lambda, -\frac{5}{2}\mu - \frac{1}{2}\delta, \mu, \delta) : \lambda, \mu, \delta \in \mathbb{R} \}$.

$\lambda = 1 \quad \mu = 0 \quad \delta = 0 \; : \; (-2, 1, 0, 0, 0)$

$\lambda = 0 \quad \mu = 1 \quad \delta = 0 \; : \; (\frac{19}{2}, 0, -\frac{5}{2}, 1, 0)$

$\lambda = 0 \quad \mu = 0 \quad \delta = 1 \; : \; (-\frac{5}{2}, 0, -\frac{1}{2}, 0, 1)$

Basis for the solution space is $\{ (-2, 1, 0, 0, 0), (19/2, 0, -5/2, 1, 0), (-5/2, 0, -1/2, 0, 0) \}$

Dimension $= 3$.