4/6/22

Note Title 04-06-2022

Examples

2) $y'' - 6y' + 25y = 0$

$$y(t) = c_1 e^{3t}\cos 4t + c_2 e^{3t}\sin 4t$$

3) $y'' + 8y = 0$

$$y(t) = c_1 \cos(2\sqrt{2}t) + c_2 \sin(2\sqrt{2}t)$$

4) $16y'' - 8y' + y = 0$

$$y(t) = c_1 e^{t/4} + c_2 t e^{t/4}$$

5) $y''' - 3y'' + 4y' - 2y = 0$

$$y(t) = c_1 e^{t} + c_2 e^{t}\cos t + c_3 e^{t}\sin t$$

6) $y''' - 3y'' + 2y' = 0$

$$y(t) = c_1 + c_2 e^{t} + c_3 e^{2t}$$

7) $y^{(4)} - 8y'' + 16y = 0$

$$y(t) = (c_1 + c_2 t)e^{2t} + (c_3 + c_4 t)e^{-2t}$$

Euler-Cauchy equation.

$$at^2 y'' + bty' + cy = 0, \quad t > 0, \; a \neq 0.$$

$$y(t) = t^m$$

$$at^2 \cdot m(m-1)t^{m-2} + btm t^{m-1} + c t^m = 0$$

$$\Rightarrow (am(m-1) + bm + c)\, t^m = 0$$

$$\Rightarrow am(m-1) + bm + c = 0$$

Let $m_1$ & $m_2$ be the roots

Case i) $m_1$ & $m_2$ are real & distinct.

$$y(t) = c_1 t^{m_1} + c_2 t^{m_2}$$

Case ii) $m_1$ & $m_2$ are real & equal to say $m$.

$y_1(t) = t^m$

$y_2(t) = u(t)\, t^m$

Show that $u(t) = \ln t$.

$\therefore\ y_2(t) = (\ln t)\, t^m$.

$\therefore\ y(t) = (c_1 + c_2 \ln t)\, t^m$.

Case iii) $m_1$ & $m_2$ are complex.

$t^{\alpha+i\beta} = t^{\alpha} e^{i\beta \ln t}$ $\quad t^{\alpha-i\beta} = t^{\alpha} e^{-i\beta \ln t} = t^{\alpha}[\cos(\beta \ln t) - i \sin(\beta \ln t)]$

$= t^{\alpha}[\cos(\beta \ln t) + i \sin(\beta \ln t)]$

$y_1(t) = t^{\alpha} \cos(\beta \ln t)$ $\qquad y_2(t) = t^{\alpha} \sin(\beta \ln t)$

Examples

1) $x^2 y'' + 2x y' - 6y = 0$

$y(t) = c_1 t^{-3} + c_2 t^{2}$

2) $x^2 y'' + x y' - 2y = 0$

$y(t) = c_1 t^{\sqrt{2}} + c_2 t^{-\sqrt{2}}$

Non-homogeneous equation.

$$y'' + p(t) y' + q(t) y = r(t) \longrightarrow ①$$

$p, q$ & $r$ are cts. fns.

$$y'' + p(t) y' + q(t) y = 0 \longrightarrow ②$$

- If $y_1$ & $y_2$ are solutions of ①, then $y_1 - y_2$ is a solution for ②.

The general solution for ① is of the form

$$y(t) = \underbrace{y_h(t)}_{\text{solution for ②}} + \underbrace{y_p(t)}_{\text{Particular solution}}$$

1) Method of undetermined coefficients
2) Variation of parameters

$$y'' + p(t)y' + q(t)y = r(t) \longrightarrow (*)$$
$$y'' + p(t)y' + q(t)y = 0 \longrightarrow (**)$$

$$y_h(t) = c_1 y_1(t) + c_2 y_2(t).$$

Assume that $y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$

$$y_p'(t) = u_1'(t)y_1(t) + u_1(t)y_1'(t) + u_2'(t)y_2(t) + u_2(t)y_2'(t)$$
$$= (u_1'(t)y_1(t) + u_2'(t)y_2(t)) + (u_1(t)y_1'(t) + u_2(t)y_2'(t))$$

Extra condition: $u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0 \longrightarrow (\text{III})$

$$\therefore y_p'(t) = u_1(t)y_1'(t) + u_2(t)y_2'(t)$$
$$y_p''(t) = u_1(t)y_1''(t) + u_2(t)y_2''(t) + u_1'(t)y_1'(t) + u_2'(t)y_2'(t)$$

Substituting $y_p$, $y_p'$ & $y_p''$ into $(*)$,

$$u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = r(t) \longrightarrow (\text{IV})$$

Use (III) & (IV) to solve for $u_1'$ & $u_2'$.

Example $\quad y'' + y = \sec t$

$$y'' + y = 0$$
$$y_h(t) = c_1\cos t + c_2 \sin t.$$

$$y_p(t) = u_1(t)\cos t + u_2(t)\sin t.$$

Then $u_1'$ & $u_2'$ satisfy

$$\left.\begin{aligned} u_1'\cos t + u_2'\sin t &= 0 \\ -u_1'\sin t + u_2'\cos t &= \sec t \end{aligned}\right\} \Rightarrow \begin{aligned} u_1'(t) &= -\tan t \Rightarrow u_1(t) = \ln(\cos t) \\ u_2'(t) &= 1 \Rightarrow u_2(t) = t. \end{aligned}$$

$$y_p(t) = \ln|\cos t|\cos t + t\sin t.$$

$$y(t) = \underbrace{c_1\cos t + c_2\sin t}_{} + \underbrace{\ln|\cos t|\cos t + t\sin t}_{}.$$