13 | 4 | 22

Theorem. Let $S \subseteq V$ be a non empty. Then TFAE:

a) S is a basis.
b) S is a maximal linearly independent set.
c) S is a minimal spanning set.

Proof.

a) $\Rightarrow$ b) Assume that S is a basis. As S is already a basis (by assumption) it is enough to show that S is maximal w.r.t. the proplity of being linearly independent.

Let $x \in V \setminus S$. Consider the set $S' = S \cup \{x\}$. By assumption S is a basis and hence spans V. Therefore $\exists$ vectors $v_1, v_2, \dots, v_n \in S$ & $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{F}$ such that

$$x = \sum_1^n \alpha_i v_i$$

$\Rightarrow$ S' is linearly dependent.

Thus S is maximally linearly independent.

b) $\Rightarrow$ c) Suppose that S is a maximal linearly independent set.

claim. span $(S) = V$.

Let $x \in V \setminus S$. Consider the set $S' = S \cup \{x\}$. As S is a maximal linearly independent set, S' is linearly dependent.

$\therefore$ $\exists$ vectors $v_1, v_2, \dots, v_n \in S$ & $\alpha_1, \alpha_2, \dots, \alpha_{n+1} \in \mathbb{F}$ s.t.

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n + \alpha_{n+1} x = 0.$$

Note that $\alpha_{n+1} \neq 0$.

$$\therefore \quad x = \sum_1^n (-1)\, \alpha_{n+1}^{-1}\, \alpha_i v_i \in \text{span}(S),$$

thereby proving our claim.

claim. S is a minimal spanning set.

Let $x \in S$ and let $S' = S \setminus \{x\}$. Suppose that S' spans V. Then every element of V can be written as a linear combination of vectors from S'. In particular, x can be written as a linear combination of vectors from S', thereby contradicting the linear independence of S.

Thus S is a minimal spanning set.

c) $\Rightarrow$ a) Exercise.

Examples

$V(\mathbb{F})$ - vector space V over the field $\mathbb{F}$.

1) $\mathbb{R}(\mathbb{R})$ $\quad \mathcal{B} = \{1\}$

$\mathcal{B} = \{x\}, \; x \neq 0$

$$y = \left(\frac{y}{x}\right) x$$

2) $\mathbb{R}^n$ over $\mathbb{R}$

$$e_i^n(j) = \delta_{ij} = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}$$

$$\mathcal{B}_{\mathbb{R}} = \{e_i^n : 1 \leq i \leq n\}$$

$$e_1^n = (1, 0, 0, \dots, 0)$$

3) $\mathbb{C}^n$ over $\mathbb{R}$ $\qquad$ $\mathbb{C}^n$ over $\mathbb{C}$

Write down a basis for both the vector spaces.

4) $P_n(\mathbb{F})$

$$\mathcal{B} = \{1, x, x^2, \dots, x^{n-1}\}$$

5) $M_n(\mathbb{F}) \cong \mathbb{F}^{n^2}$

Write down a basis

6) $C([0,1])$

Theorem. Every vector space has a basis.

Theorem. If $\mathcal{B}$ is a basis for a vector space V, then every vector in V is a unique linear combination of elements of $\mathcal{B}$.

Proof. Let $x \in V$. Suppose that

$$x = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n, \quad v_i \in \mathcal{B} \quad 1 \leq i \leq n$$

$$\& \; x = \beta_1 w_1 + \beta_2 w_2 + \dots + \beta_m w_m, \quad w_j \in \mathcal{B} \quad 1 \leq j \leq m.$$

W.L.G. let us assume that $n = m$ & $v_i = w_i \quad \forall\, 1 \leq i \leq n$.

$$\therefore \quad x = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = \beta_1 w_1 + \beta_2 w_2 + \dots + \beta_n w_n$$

$$\Rightarrow (\alpha_1 - \beta_1) v_1 + (\alpha_2 - \beta_2) v_2 + \dots + (\alpha_n - \beta_n) v_n = 0$$

As v_i's are linearly independent, $\alpha_i - \beta_i = 0 \quad \forall\, 1 \leq i \leq n$

$$\Rightarrow \alpha_i = \beta_i \quad \forall\, 1 \leq i \leq n.$$