5/4/22

Note Title 05-04-2022

Problem. Consider the system

$$x + \lambda y = 1$$
$$2x + y = \mu.$$

Find $\lambda, \mu \in \mathbb{R}$ such that the system has

i) no solutions
ii) a unique solution
iii) infinite no of solutions.

Solution. Consider the augmented matrix

$$\left(\begin{array}{cc|c} 1 & \lambda & 1 \\ 2 & 1 & \mu \end{array}\right)$$

$R_2 \longrightarrow R_2 - 2R_1$ $$\left(\begin{array}{cc|c} 1 & \lambda & 1 \\ 0 & 1-2\lambda & \mu-2 \end{array}\right)$$

i) If $\lambda = 1/2$ & $\mu \neq 2$

Then the rank $(A) = 1$ but rank $(A|B) = 2$

$\therefore$ The system has no solutions

ii) If $\lambda = 1/2$ & $\mu = 2$

Then rank (A) = rank $(A|B) = 1 <$ no. of unknowns

$\Rightarrow$ The system has infinite no. of solutions

$y = \theta \Rightarrow x = 1 - \theta/2$

$\therefore$ The solution set is

$$\{ (1-\theta/2, \theta) : \theta \in \mathbb{R} \}$$

iii) If $\lambda \neq 1/2$

rank $(A) = 2$ = rank $(A|B)$ = no. of unknowns

$\Rightarrow$ The system has a unique solution.

$$\left(\begin{array}{cc|c} 1 & \lambda & 1 \\ 0 & 1-2\lambda & \mu-2 \end{array}\right)$$

$R_2 \longrightarrow \frac{1}{1-2\lambda} R_2$ $$\left(\begin{array}{cc|c} 1 & \lambda & 1 \\ 0 & 1 & \frac{\mu-2}{1-2\lambda} \end{array}\right)$$

$$y = \frac{\mu - 2}{1 - 2\lambda}$$

$$x = \frac{1 - \lambda\mu}{1 - 2\lambda}$$

— x —

1) $\mathbb{R}$ over $\mathbb{R}$. More generally $\mathbb{F}$ over $\mathbb{F}$
2) $\mathbb{C}$ over $\mathbb{R}$, $\mathbb{C}$ over $\mathbb{Q}$
3) $\mathbb{F}^n$ over $\mathbb{F}$
4) $M_n(\mathbb{F})$ over $\mathbb{F}$

Let $A, B \in M_n(\mathbb{F})$.

$A = [a_{ij}]$ & $B = [b_{ij}]$. Then $A + B = [a_{ij} + b_{ij}]$

$\alpha \in \mathbb{F}$ & $A \in M_n(\mathbb{F})$, $\alpha \cdot A = \alpha \cdot [a_{ij}] = [\alpha a_{ij}]$

i) $A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}] = [b_{ij} + a_{ij}] = [b_{ij}] + [a_{ij}] = B + A$

ii) $(A + B) + C = A + (B + C)$ (Verify)

iii) Let $\bar{O} = [0]$. Then $\bar{O} + A = A + \bar{O} = A \ \forall A \in M_n(\mathbb{F})$ (Verify)

iv) Let $A = [a_{ij}] \in M_n(\mathbb{F})$. Let $B = [-a_{ij}]$. Then
$A + B = B + A = \bar{O}$ (Verify)

v) $\alpha \cdot (A + B) = \alpha \cdot ([a_{ij}] + [b_{ij}])$

$= \alpha \cdot ([a_{ij} + b_{ij}])$

$= [\alpha(a_{ij} + b_{ij})]$

$= [\alpha a_{ij} + \alpha b_{ij}]$

$= [\alpha a_{ij}] + [\alpha b_{ij}]$

$= \alpha \cdot [a_{ij}] + \alpha \cdot [b_{ij}] = \alpha A + \alpha B$

vi) $(\alpha + \beta) A = \alpha A + \beta A$ (Verify)

vii) $\alpha \cdot (\beta A) = (\alpha\beta) \cdot A$ (Verify)

viii) $1 \cdot A = A$ where $1 \in \mathbb{F}$.

5) $C([0,1]) = \{f : [0,1] \longrightarrow \mathbb{R} : f$ is continuous$\}$

Ex. Show that $C([0,1])$ is a vector space over $\mathbb{R}$.