30/4/22

Note Title 30-04-2022

$W_1, W_2 \subseteq V$

$W_1 + W_2 = \{ w_1 + w_2 : w_1 \in W_1 \ \& \ w_2 \in W_2 \}$

- $W_1 + W_2$ is a subspace of V.
- If V is finite dimensional, then
$\dim(W_1+W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$.

1) $V = \mathbb{R}^2$ $\quad W_1 = \{(x,x) : x \in \mathbb{R}\}$
$W_2 = \{(x,-x) : x \in \mathbb{R}\}$ $\quad \mathbb{R}^2 = W_1 + W_2$
$W_1 \cap W_2 = \{0\}$.

- The sum $W_1 + W_2$ is called direct if $W_1 \cap W_2 = \{0\}$.
- If V is a direct sum of $W_1 + W_2$ then we write $V = W_1 \oplus W_2$

Theorem. Suppose W_1 & W_2 are subspaces of V so that $V = W_1 + W_2$. Then $V = W_1 \oplus W_2 \iff$ every vector $v \in V$ can be written in a unique way as $w_1 + w_2$ where $w_i \in W_i$, $i = 1,2$.

Proof. $\Rightarrow$ Suppose that $V = W_1 \oplus W_2$. Let $v \in V$. Suppose that
$v = w_1 + w_2 = w_1' + w_2'$
where $w_1, w_1' \in W_1$ & $w_2, w_2' \in W_2$.

Now $w_1 + w_2 = w_1' - w_2'$
$\Rightarrow w_1' - w_1 = w_2 - w_2' \in W_1 \cap W_2$
As $W_1 \cap W_2 = \{0\}$, $w_1' - w_1 = 0$ & $w_2 - w_2' = 0$
i.e., $w_1 = w_1'$ & $w_2 = w_2'$

$\Leftarrow$ Suppose $V = W_1 + W_2$ & every vector $v \in V$ can be written uniquely as $w_1 + w_2$ with $w_i \in W_i$.

Let $w \in W_1 \cap W_2$. If w is non-zero, then

$w = w + 0 = 0 + w$
which is a contradiction.
$\therefore w = 0$.

<u>Examples</u>

1) $V = \mathbb{R}^2$
$W_1 = \{(x, 2x) : x \in \mathbb{R}\}$
$W_2 = \{(x, 3x) : x \in \mathbb{R}\}$.
- $V = W_1 + W_2$ (Verify)
- $W_1 \cap W_2 = \{(0,0)\}$

$\therefore V = W_1 \oplus W_2$

2) $V = M_n(\mathbb{R})$

$W_1 = \{A \in M_n(\mathbb{R}) : A \text{ is upper triangular}\}$
$W_2 = \{A \in M_n(\mathbb{R}) : A \text{ is lower triangular}\}$

$V = W_1 + W_2$ (Verify)

$W_1 \cap W_2 = \{A \in M_n(\mathbb{R}) : A \text{ is a diagonal matrix}\}$.
$\therefore$ V is not a direct sum of W_1 & W_2.

3) $V = M_n(\mathbb{R})$

$W_1 = \{A \in M_n(\mathbb{R}) : A \text{ is symmetric}\}$
$W_2 = \{A \in M_n(\mathbb{R}) : A \text{ is skew-symmetric}\}$

$V = W_1 + W_2$ (Verify)
$W_1 \cap W_2 = \{0\}$

$\therefore V = W_1 \oplus W_2$.

Linear transformation

V, W – Vector spaces over $\mathbb{F}$.

- A map $T : V \longrightarrow W$ is called a linear transformation if
$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v) \quad \forall\, u, v \in V \ \&\ \alpha, \beta \in \mathbb{F}$.

Examples

1) Fix $\alpha \in \mathbb{R}$. Define $T_\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ as $T_\alpha(x) = \alpha x$.

Let $T : \mathbb{R} \longrightarrow \mathbb{R}$ be a linear transformation.
$T(x) = T(x \cdot 1) = T(1)x = T_{T(1)}(x)$

2) Fix $\alpha, \beta \in \mathbb{R}$. Define $T_{\alpha,\beta} : \mathbb{R}^2 \longrightarrow \mathbb{R}$ as $T_{\alpha,\beta}(x, y) = \alpha x + \beta y$.

Given $T : \mathbb{R}^2 \longrightarrow \mathbb{R}$ let $\alpha = T(1,0)$ & $\beta = T(0,1)$. Then
$T(x, y) = \alpha x + \beta y$ (Verify).

3) Define $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ as
$T(x, y) = (ax + by, cx + dy)$

- Verify that T is linear.
- Any LT $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is of the above form.

4) $P_i : \mathbb{F}^n \longrightarrow \mathbb{F}$
$P_i(x_1, x_2, \ldots, x_n) = x_i$

- If $T: V \to W$ is a linear transformation, then $T(0) = 0$

Proof.

$$T(0) = T(0 \cdot 0) = 0 \cdot T(0) = 0$$

Example $T: \mathbb{R}^2 \to \mathbb{R}^2$

$T(x,y) = (x+y+1, x-y)$

$T(0,0) = (1,0) \neq (0,0)$.

$\therefore$ T is not linear.

Let $T: V \to W$ be a linear transformation.

$\ker(T) = \text{null}(T) = \{v \in V : T(v) = 0\}$ Kernel of T or null space of T.

- $\ker(T)$ is a subspace of V.

Let $u, v \in \ker(T)$ & $\alpha \in \mathbb{F}$.

$u \in \ker(T) \Rightarrow T(u) = 0$

III^{ly} $v \in \ker(T) \Rightarrow T(v) = 0$

$\therefore T(\alpha u + v) = \alpha T(u) + T(v) = \alpha \cdot 0 + 0 = 0$

$\Rightarrow \alpha u + v \in \ker(T)$.

- Range space

$R(T) = \{w \in W : w = T(v) \text{ for some } v \in V\}$.

Ex. Show that $R(T)$ is a subspace of W.