7/5/22

Note Title 07-05-2022

$T: V \longrightarrow W$ $\qquad V \overset{\to \dim n}{-} \mathcal{B}$, $W - \mathcal{B}'$ (→ dim m) $\qquad V \longrightarrow F^n$

$v \in V \quad Tv$ $\qquad [T]_{\mathcal{B}}^{\mathcal{B}'}$ $\qquad v \longmapsto [v]_{\mathcal{B}}$

$v \longmapsto Tv$ $\qquad [T]_{\mathcal{B}}^{\mathcal{B}'} : F^n \longrightarrow F^m$

$$(x_1, x_2, \cdots, x_n) \longrightarrow [T]_{\mathcal{B}}^{\mathcal{B}'} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

- $[T]_{\mathcal{B}_2}^{\mathcal{B}_2'} = Q\,[T]_{\mathcal{B}_1}^{\mathcal{B}_1'}\,P^{-1}$

<u>Example</u>

$T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$T(x, y, z) = (2x + z,\ y + 3z)$

O.B $\mathcal{B}_1 = \{(1,1,0), (1,0,1), (1,1,1)\}$ $\qquad \mathcal{B}_2 = \{(2,3), (3,2)\}$ $\qquad [T]_{\mathcal{B}_1}^{\mathcal{B}_2} = \begin{bmatrix} -1/5 & 3/5 & 6/5 \\ 4/5 & 3/5 & 1/5 \end{bmatrix}$

N.B $\mathcal{B}_1' = \{(1,0,0), (0,1,0), (0,0,1)\}$ $\qquad \mathcal{B}_2' = \{(1,0), (0,1)\}$ $\qquad [T]_{\mathcal{B}_1'}^{\mathcal{B}_2'} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix}$

$$[T]_{\mathcal{B}_1'}^{\mathcal{B}_2'} = Q\,[T]_{\mathcal{B}_1}^{\mathcal{B}_2}\,P^{-1}$$

Express the old basis in terms of new.

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad Q = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$$

$(1,1,0) = 1\,(1,0,0) + 1\cdot(0,1,0) + 0\cdot(0,0,1)$

$(1,0,1) = 1\,(1,0,0) + 0\cdot(0,1,0) + 1\cdot(0,0,1)$

$(1,1,1) = 1\,(1,0,0) + 1\cdot(0,1,0) + 1\cdot(0,0,1)$

Exc.

1) $T: U \longrightarrow V$ & $S: V \longrightarrow W$ are LT's.

a) S.T. $S \circ T$ is also a LT.

b) $\mathcal{B}_U$, $\mathcal{B}_V$ & $\mathcal{B}_W$ be bases for U, V & W respectively. Show that

$$[S \circ T]_{\mathcal{B}_U}^{\mathcal{B}_W} = [S]_{\mathcal{B}_V}^{\mathcal{B}_W}\,[T]_{\mathcal{B}_U}^{\mathcal{B}_V}$$

2) $T: U \longrightarrow V$ a LT & $\lambda \in F$. Define $\lambda T: U \longrightarrow V$ as

$(\lambda T)(u) = \lambda\, T(u)$.

a) Show that λT is a LT

b) $\mathcal{B}_U$ & $\mathcal{B}_V$ are bases for U & V respectively. Show that

$$[\lambda T]_{\mathcal{B}_U}^{\mathcal{B}_V} = \lambda [T]_{\mathcal{B}_U}^{\mathcal{B}_V}$$

3) $S, T: U \to V$ be LT's. Define $S+T: U \to V$ as
$(S+T)(u) = S(u) + T(u)$.

a) Show that $S+T$ is a LT.

b) $\mathcal{B}_U$ & $\mathcal{B}_V$ are bases for U & V respectively. Show that

$$[S+T]_{\mathcal{B}_U}^{\mathcal{B}_V} = [S]_{\mathcal{B}_U}^{\mathcal{B}_V} + [T]_{\mathcal{B}_U}^{\mathcal{B}_V}.$$

Definition. Let $A, B \in M_n(\mathbb{F})$. We say that A & B are similar if $\exists$ an invertible matrix P such that

$$A = PBP^{-1}.$$

$T: V \to V$ $\quad \mathcal{B}_1 \quad \mathcal{B}_1'$

$[T]_{\mathcal{B}_1} \qquad [T]_{\mathcal{B}_1'}$

Ex. i) Show that similarity of matrices forms an equivalence relation.
ii) Show that similar matrices have same determinant.

<u>Example</u>

$T: \mathbb{R}^3 \to \mathbb{R}^2$

$T(x,y,z) = (x-y, \; y-z)$

Old Basis $\quad \mathcal{B}_1 = \{(1,1,0), (1,0,1), (0,1,1)\} \qquad \mathcal{B}_2 = \{(1,2), (2,1)\}$

$$[T]_{\mathcal{B}_1}^{\mathcal{B}_2} = \begin{bmatrix} 2/3 & -1 & 1/3 \\ -1/3 & 1 & -2/3 \end{bmatrix}$$

$T(1,1,0) = (0,1) = \frac{2}{3}(1,2) - \frac{1}{3}(2,1)$

$T(1,0,1) = (1,-1) = (-1)(1,2) + 1(2,1)$

$T(0,1,1) = (-1,0) = (1/3)(1,2) - 2/3\,(2,1)$

New Basis $\quad \mathcal{B}_1' = \{(1,0,0), (0,1,0), (0,0,1)\} \qquad \mathcal{B}_2' = \{(1,0), (0,1)\}$

$$[T]_{\mathcal{B}_1'}^{\mathcal{B}_2'} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

$(1,1,0) = 1\cdot e_1 + 1\cdot e_2 + 0\cdot e_3 \qquad (1,2) = 1\cdot e_1 + 2\cdot e_2$

$(1,0,1) = 1\cdot e_1 + 0\cdot e_2 + 1\cdot e_3 \qquad (2,1) = 2\cdot e_1 + 1\cdot e_2$

$(\cancel{1}0,1,1) = 0\cdot e_1 + 1\cdot e_2 + 1\cdot e_3$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$