27/5/22

Note Title 27-05-2022

- $A \subseteq \mathbb{R}$
  $f : A \longrightarrow A$ then $x \in A$ is a <u>fixed point</u> for f if $f(x) = x$.
- $x_{n+1} = f(x_n)$

$$y' = f(x, y) \;;\; y(x_0) = y_0 \quad — \text{ IVP.} \quad \longrightarrow ①$$

Note that ① is equivalent to

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt \quad (\text{Thanks to FTC}).$$

$$y_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0)\, dt$$

$$y_2(x) = y_0 + \int_{x_0}^{x} f(t, y_1(t))\, dt$$

$$y_3(x) = y_0 + \int_{x_0}^{x} f(t, y_2(t))\, dt$$

$$\vdots$$

$$y_n(x) = y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))\, dt \quad — \text{ Picard's approximation.}$$

Remark. Under the assumptions of existence & uniqueness theorems, the seq. $\{y_n\}$ converges to the solution y.

Given $\varepsilon > 0$ $\exists$ $n_0 \in \mathbb{N}$ s.t.

$$|y_n(x) - y(x)| < \varepsilon \quad \forall n \geq n_0$$

Here n_0 depends only on ε and not on the pt. 'x'.

Example

$$y' = y \;;\; y(0) = 1.$$

$x_0 = 0$, $y_0 = 1$, $f(x, y) = y$.

$$y_1 = 1 + \int_0^x f(t, 1)\, dt = 1 + \int_0^x 1\, dt = 1 + x$$

$$y_2 = 1 + \int_0^x f(t, 1+t)\, dt = 1 + \int_0^x (1+t)\, dt = 1 + x + \frac{x^2}{2!}$$

$$y_3 = 1 + \int_0^x f\left(t, 1 + x + \frac{x^2}{2}\right) dt = 1 + \int_0^x 1 + t + \frac{t^2}{2}\, dt = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

$$\vdots$$

$$y_n = \sum_{k=0}^{n} \frac{x^k}{k!} \longrightarrow e^x.$$

$$x + y\frac{dy}{dx} = 0$$

A general solution to this ODE is
$$x^2 + y^2 = c.$$

$$x + y\frac{dy}{dx} = 0 \iff d(x^2+y^2) = 0 \qquad u(x,y) = \frac{x^2+y^2}{2}$$
$$M = \partial u/\partial x \ \& \ N = \partial u/\partial y$$

$$\underbrace{M + Ny' = 0}_{①}, \text{ where } M = M(x,y) \ \& \ N = N(x,y).$$

Equation ① is called an exact equation if $\exists\, u$ s.t.
$$M = u_x \quad \& \quad N = \partial u/\partial y$$

In this case,

$$u_x + u_y\frac{dy}{dx} = 0$$
$$\iff \frac{d}{dx}\left(u(x, y(x))\right) = 0$$

The general solution to ① is given by $u(x,y) = c$.

<u>Example</u>

$$y + x\frac{dy}{dx} = 0$$
$$M = y, \quad N = x$$
$$u(x,y) = xy.$$

$xy = c$ is the general solution.

$$M + Ny' = 0 \quad — ①$$
is an exact equation

$\Rightarrow \exists$ a fn. u s.t. $M = u_x$ & $N = u_y$.

Suppose that $\boxed{\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}}$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

i.e., $N_x = M_y$

Suppose that $N_x = M_y$.

Aim. To find u such that $u_x = M$ & $u_y = N$.

If $u_x = M$

$\Leftrightarrow u(x,y) = \int M(x,y)\,dx + h(y). \longrightarrow$ ②

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\int M(x,y)\,dx + h(y)\right)$$

$$= \frac{\partial}{\partial y}\left(\int M(x,y)\,dx\right) + h'(y) = \int M_y\,dx + h'(y)$$

$\Rightarrow h'(y) = u_y - \int M_y\,dx. \longrightarrow$ ③

This expression is meaningful if RHS is independent of x.

i.e., $\frac{\partial}{\partial x}\left(u_y - \int M_y\,dx\right) = 0$

i.e., $N_x - M_y = 0$

But this is our assumption.

Thus ③ is always meaningful and hence we can use it to find h.

Now the required u can be found using ②.

Theorem. Suppose that M & N have cts. 1^{st} order partial derivatives, then the equation

$$M + N y' = 0$$

is exact $\Leftrightarrow M_y = N_x$.