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- Row Reduced Echelon (RRE) Matrix
- Elementary Row operations

$P_1 : R_i \longrightarrow \lambda R_i \quad \lambda \neq 0 \qquad P_1^{-1} : R_i \longrightarrow (1/\lambda) R_i$

$P_2 : R_i \longleftrightarrow R_j \quad i \neq j \qquad P_2^{-1} : R_i \longleftrightarrow R_j$

$P_3 : R_i \longrightarrow R_i + \lambda R_j \quad i \neq j \qquad P_3^{-1} : R_i \longrightarrow R_i - \lambda R_j$

- Every elementary row operation is invertible
- Inverse of an elementary row operation is also an elementary row operation.

Observation. Let $A \in M_{m \times n}(\mathbb{F})$ & let P be an elementary row operation. Then $P(A) = P(I) \cdot A$, where I denotes the $m \times m$ identity matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$B = P(A) = [b_{ij}]$

$C = P(I) = [c_{ij}]$

$D = P(I) \cdot A = [d_{ij}]$

$$(P_1 \circ P_2 \circ \cdots \circ P_n)(A) = P_1(I) P_2(I) \cdots P_n(I) A$$
$$= (P_1 \circ P_2 \circ \cdots P_n)(I) A$$

Row Equivalence of Matrices

Two matrices A and B are said to be row equivalent if $\exists$ (there exists) finitely many elementary operations $(P_1, P_2, \cdots, P_n)$ s.t.

$$B = (P_1 \circ P_2 \circ \cdots \circ P_n)(A).$$

Relation. A relation on a set X is any subset of $X \times X$.

$R \subseteq X \times X$.

- We say that x is related to y, denoted xRy if $(x, y) \in R$.

xRy (or) $x \lesssim y$ (or) $x \cong y$ (or) $x \prec y$

A relation is said to be

i) reflexive if $x R x \ \forall x \in X$
ii) symmetric if $xRy \Rightarrow yRx$
iii) transitive if xRy & $yRz \Rightarrow xRz$.

Any relation which is reflexive, symmetric and transitive is called an equivalence relation.

<u>Example</u> $X = \mathbb{Z}$ – the set of integers.
Fix $m \in \mathbb{N}$.
Define R as follows:

xRy if $x \equiv y \pmod{m}$
i.e., $x-y$ is divisible by m

i.e., $x-y$ is a multiple of m.

Ex. Show that R is an equivalence relation.

<u>Example</u> $X = M_{m,n}(\mathbb{F})$.

ARB if A is row equivalent to B.

R is an equivalence relation

Reflexive: $A = \rho(A)$ where $\rho : R_i \to 1 \cdot R_i$

Symmetric: Suppose that ARB. Then $\exists$ elementary row operations $\rho_1, \rho_2, \dots, \rho_n \ni$
$$B = (\rho_1 \circ \rho_2 \circ \cdots \circ \rho_n)(A)$$

$$\Rightarrow A = (\rho_n^{-1} \circ \rho_{n-1}^{-1} \circ \cdots \circ \rho_2^{-1} \circ \rho_1^{-1})(B)$$
$$\Rightarrow B R A.$$

Transitive. Suppose that ARB and BRC. Then $\exists$ elementary row operations $\rho_1, \rho_2, \dots, \rho_n, \rho_{n+1}, \dots, \rho_{n+m}$ s.t.
$$B = (\rho_1 \circ \rho_2 \circ \cdots \circ \rho_n)(A)$$
$$\& \ C = (\rho_{n+1} \circ \rho_{n+2} \circ \cdots \circ \rho_{n+m})(B)$$
$$\therefore C = (\rho_{n+1} \circ \rho_{n+2} \circ \cdots \circ \rho_{n+m} \circ \rho_1 \circ \rho_2 \circ \cdots \circ \rho_n)(A)$$

As C is obtained from A just by performing $n+m$ elementary row operations, ARC.

Let X be any set and let R be an equivalence relation on X.

Fix $x \in X$. Let

$[x] = \{y \in X : x R y\}$ – Equivalence class containing 'x'

- $[x] \neq \phi$ for each $x \in X$.
- For $x, y \in X$, either $[x] \cap [y] = \phi$ or $[x] = [y]$.

Proof. If $[x] \cap [y] = \phi$, then we are done.

Suppose that $[x] \cap [y] \neq \phi$.

Let $z \in [x] \cap [y]$

$\Rightarrow z \in [x]$ & $z \in [y]$

$z \in [x] \Rightarrow x R z \longrightarrow ①$

III^{ly} $z \in [y] \Rightarrow y R z \Rightarrow z R y \longrightarrow ②$

① & ② $\Rightarrow x R y$

$\Rightarrow x \in [y]$ & $y \in [x]$.

Thus if $u \in [x]$ then $u \in [y]$ & vice-versa.

$\Rightarrow [x] = [y]$.

- Equivalent classes are disjoint

Theorem. Given $A \in M_{n,m}(F)$ $\exists$ a unique RRE matrix $\tilde{A}$ such that $A R \tilde{A}$.

Gauss Elimination method

<u>Algorithm</u>

1) Apply interchange of rows to push all the rows to the bottom of the matrix

2) If the leading coefficient is not equal to 1, use the 1st elementary row operation to convert it into 1.

3) Use the 3rd elementary row operation to convert the other coefficients of that column into zero.

<u>Example</u>

$$\begin{pmatrix} 0 & 0 & 4 & 1 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & 2 & 0 \end{pmatrix}$$

$R_3 \longleftrightarrow R_4$ $\begin{pmatrix} 0 & 0 & 4 & 1 \\ 0 & 3 & 0 & 1 \\ 0 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$R_1 \longleftrightarrow R_2$ $\begin{pmatrix} 0 & 3 & 0 & 1 \\ 0 & 0 & 4 & 1 \\ 0 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$R_1 \longrightarrow \frac{1}{3} R_1$ $\begin{pmatrix} 0 & 1 & 0 & 1/3 \\ 0 & 0 & 4 & 1 \\ 0 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$R_3 \longrightarrow R_3 - 4R_1$ $\begin{pmatrix} 0 & 1 & 0 & 1/3 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 2 & -4/3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$R_2 \longrightarrow \frac{1}{4} R_2$ $\begin{pmatrix} 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & 1/4 \\ 0 & 0 & 2 & -4/3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$R_3 \longrightarrow R_3 - 2R_2$ $\begin{pmatrix} 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & 1/4 \\ 0 & 0 & 0 & -11/6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$R_3 \longrightarrow -6/11\ R_3$ $\begin{pmatrix} 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & 1/4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$R_1 \longrightarrow R_1 - 1/3 R_3$
$R_2 \longrightarrow R_2 - 1/4 R_3$ $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$