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Note Title 02-04-2022

Example

$$(A|B) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 3 \end{array}\right)$$

$$\cong \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

x_3 - independent unknown

Let $x_3 = \lambda$. Then $x_1 = \lambda$ & $x_2 = 3-2\lambda$.
Thus the solution set is $\{(\lambda, 3-2\lambda, \lambda) : \lambda \in \mathbb{R}\}$.

1) rank (A) = 2 rank (A|B) = 3
2) rank (A) = 3 rank (A|B) = 3
3) rank (A) = 2 rank (A|B) = 2

Theorem. Let $Ax = B$ be a system of linear equations. Then
- the system has a solution $\iff$ rank (A) = rank (A|B)
- the system has a unique solution $\iff$ rank(A) = no. of unknowns.

Motivation

$$Ax = 0 \longrightarrow (*)$$

- 0 is always a solution
- $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_n)$ - are solutions to $(*)$

$(x_1+y_1, x_2+y_2, \ldots, x_n+y_n)$ is again a solution
- $\alpha(x_1, x_2, \ldots, x_n)$ is again a solution

Definition.
Let V be a non empty set with a binary operation

$$+ : V \times V \longrightarrow V$$

& a scalar multiplication

$$\cdot : \mathbb{F} \times V \longrightarrow V$$

satisfying the following conditions:

- $u + v = v + u \quad \forall\ u, v \in V$ (commutativity)
- $u + (v + w) = (u + v) + w \quad \forall\ u, v, w \in V$ (associativity)

- $\exists$ a vector denoted '0' such that $v+0 = v = 0+v \ \forall\ v \in V$
- for every $v \in V\ \exists\ \underline{w \in V}$ such that $v+w = 0 = w+v$
- $\alpha\cdot(u+v) = \alpha\cdot u + \alpha\cdot v \quad \forall\ \alpha \in \mathbb{F}\ \&\ \forall\ u, v \in V$
- $\alpha\cdot(\beta\cdot v) = (\alpha\beta)\cdot v \quad \forall\ \alpha, \beta \in \mathbb{F}\ \&\ \forall\ v \in V$
- $(\alpha+\beta)\cdot v = \alpha\cdot v + \beta\cdot v \quad \forall\ \alpha, \beta \in \mathbb{F}\ \&\ v \in V$
- $1\cdot v = v \quad \forall\ v \in V$

Then V is called a vector space over $\mathbb{F}$. $\quad$ ($V(\mathbb{F})$ or $(V, \mathbb{F})$ (or) $(V_{\mathbb{F}})$)

<u>Example</u> $\mathbb{R}$ over $\mathbb{R}$

<u>Example</u> $\mathbb{C}$ over $\mathbb{R}$

<u>Example</u> $\mathbb{C}$ over $\mathbb{C}$. More generally, $\mathbb{F}$ over $\mathbb{F}$.

$\mathbb{C}$ over $\mathbb{C}$
$\mathbb{C}$ over $\mathbb{R}$ $\quad$ – all these are different vector spaces.
$\mathbb{C}$ over $\mathbb{Q}$

<u>Example</u>

$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}\ \forall\ 1 \le i \le n\}$
– vector space over $\mathbb{R}$.

<u>Example</u>

$\mathbb{F}^n = \{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{F}\ \forall\ 1 \le i \le n\}$ – vector space over $\mathbb{F}$.

<u>Example</u> $M_{n,m}(\mathbb{F})$ – set of all $n \times m$ matrices over $\mathbb{F}$.
– vector space over $\mathbb{F}$.

Remark. In the 4^{th} axiom of the defn. vector space, it is stated that for each $v \in V\ \exists\ w \in V$ s.t. $v+w = w+v = 0$. We shall denote this w as $-v$.

- Suppose that $\exists\ 0_1\ \&\ 0_2$ such that
  $0_1 + v = v + 0_1 = v$
  $\&\ 0_2 + v = v + 0_2 = v \quad \forall\ v \in V$.

  $0_1 = 0_1 + \underline{0_2} = \underline{0_2}$

- IIIrly one can show that the additive inverse is unique.
  Suppose that for a given $v \in V\ \exists\ w_1\ \&\ w_2$ s.t.

  $v + w_1 = 0 = w_1 + v$
  $\&\ v + w_2 = 0 = w_2 + v$.
  $w_1 = w_1 + 0 = w_1 + (v + w_2) = (w_1 + v) + w_2 = 0 + w_2 = w_2$.

Exercise. $0 \cdot v = 0 \quad \forall\, v \in V$