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Note Title 01-04-2022

- RRE Matrix
- Row equivalence of matrices

(X, R)

$[x]$ - Equivalence class containing x

Theorem. Every matrix is row equivalent to a unique RRE matrix.

Gauss Elimination method.

Applications

Inverse of a matrix

- If ρ is an elementary row operation & $A \in M_n(\mathbb{F})$, then A is inv $\Leftrightarrow \rho(A)$ is invertible.
- $A \in M_n(\mathbb{F})$ is invertible $\Leftrightarrow$ its corresponding RRE is invertible.
- $A \in M_n(\mathbb{F})$ is invertible $\Leftrightarrow$ its corresponding RRE is invertible.

$[I_n]_R = \{A \in M_n(\mathbb{F}) : A \text{ is invertible}\}$

A is invertible

- $\det(A)$
- $\frac{1}{\det(A)} \operatorname{adj}(A)$.

Consider the augmented matrix $(A|I)$ ($n \times 2n$ matrix). Suppose that $\rho_1, \rho_2, \dots, \rho_m$ are the elementary row operations applied on $(A|I)$ such that

$$(\rho_m \circ \rho_{m-1} \circ \dots \circ \rho_1)(A|I) = (R|B),$$

where R is the RRE matrix corresponding to A.

Thus A is invertible
$\Leftrightarrow R$ is invertible
$\Leftrightarrow R = I$

$$\therefore (\rho_m \circ \rho_{m-1} \circ \dots \circ \rho_1)(A) = R$$
$$\Leftrightarrow (\rho_m \circ \rho_{m-1} \circ \dots \circ \rho_1)(I)\, A = I$$
$$\Leftrightarrow A^{-1} = (\rho_m \circ \rho_{m-1} \circ \dots \circ \rho_1)(I) = B$$

Example
$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right)$$

$R_2 \longrightarrow R_2 - R_1$
$R_3 \longrightarrow R_3 - R_1$
$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{array}\right)$$

$R_1 \longrightarrow R_1 - R_2$
$R_3 \longrightarrow R_3 - R_2$
$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & 0 & -1 & 1 \end{array}\right)$$

$R_3 \longrightarrow \frac{1}{2} R_3$
$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1/2 & 1/2 \end{array}\right)$$

$R_1 \longrightarrow R_1 - R_3$
$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1/2 & -1/2 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1/2 & 1/2 \end{array}\right)$$

Thus, $A^{-1} = \begin{pmatrix} 2 & -1/2 & -1/2 \\ -1 & 1 & 0 \\ 0 & -1/2 & 1/2 \end{pmatrix}$

Rank of a matrix.

Rank of a matrix A is the integer $n > 0$ such that $\exists$ an invertible submatrix B of A such that B is a $n \times n$ matrix and there does not exist any other invertible submatrix of A of size larger than n.

- Rank of $A = 0 \iff A = 0$.
- If A & B are row equivalent then they have the same rank.
- If R is a RRE matrix then rank (R) is equal to its no. of non-zero rows.

Theorem. Rank of matrix is equal to the no. of non-zero rows in its corresponding RRE.

Example

$$A = \begin{pmatrix} 0 & 0 & 4 & 1 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & 2 & 0 \end{pmatrix}$$

$$R = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ Rank $(A) = 3$.

Solving a system of linear equations

- Let $Ax = B$ be a system

$$(A|B) \overset{R}{\cong} (A'|B')$$

$$Ax = B \qquad A'x = B'$$

Thus the systems $Ax = B$ & $A'x = B'$ have the same set of solutions

Example

$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ x_1 + 2x_2 + 3x_3 &= 6 \\ x_2 + 2x_3 &= 1 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 1 \end{array}\right)$$

$$R_2 \longrightarrow R_2 - R_1 \quad \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 1 \end{array}\right)$$

$$R_3 \longrightarrow R_3 - R_2 \quad \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & -2 \end{array}\right)$$

The system has no solutions

Example

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 6 \\ 1 & 1 & 2 & 4 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$x_1 = 1$, $x_2 = 1$, $x_3 = 1$