31/5/22

Note Title 31-05-2022

Exact equations

$$M + N y' = 0 \longrightarrow (1)$$

(1) is said to be exact if $\exists\ u = u(x,y)$ s.t. $M = u_x$ & $N = u_y$.

Theorem. Suppose that M & N have cts. 1st order partial derivatives, then the eqn. (1) is exact $\Leftrightarrow M_y = N_x$.

Remark. The proof of the above theorem gives us a way to construct a solution.

Example. $(2x+y) + (x+2y)y' = 0$.
This is exact.

$$u = \int M\,dx = \int (2x+y)\,dx = x^2 + xy + h(y)$$

Now, $u_y = x + h'(y)$
$\Rightarrow x + 2y = x + h'(y)$

$\Rightarrow h'(y) = 2y$
$\Rightarrow h(y) = y^2 + c$

$\therefore$ The general solution is given by

$$x^2 + xy + y^2 = c.$$

Example $(y - y^2) + x y' = 0$.
This eqn. is not exact.

Consider $M + N y' = 0 \longrightarrow (1)$

Suppose (1) is not exact.
$\Leftrightarrow M_y \neq N_x$.

Suppose $\exists\ \mu\ (= \mu(x,y))$ s.t.

$$\mu M + \mu N y' = 0 \longrightarrow (2)$$

is exact.

$$\Rightarrow (\mu M)_y = (\mu N)_x$$

$$\Rightarrow \mu_y M + \mu M_y = \mu_x N + \mu N_x \longrightarrow (3)$$

a) Suppose that μ is a fn. of x only.

$\therefore$ ③ gives

$$\mu M_y = \mu_x N + \mu N_x$$

$$\Rightarrow \mu (M_y - N_x) = \mu_x N$$

$$\Rightarrow \frac{M_y - N_x}{N} = \frac{\mu_x}{\mu}$$

$\Rightarrow \dfrac{M_y - N_x}{N}$ is a fn. of x only.

$$\therefore \int \frac{\mu_x}{\mu}\, dx = \int \frac{M_y - N_x}{N}\, dx$$

$$\ln(\mu(x)) = \int \frac{M_y - N_x}{N}\, dx$$

$$\Rightarrow \mu(x) = \exp\left(\int \frac{M_y - N_x}{N}\, dx\right)$$

b) Suppose that μ is a fn. of y only.

$$\mu(y) = \exp\left(\int \frac{N_x - M_y}{M}\, dy\right) \qquad \text{(Verify!)}$$

Example. $(x^2 y + y + 1) + x(1+x^2)\, y' = 0$

$$\mu(x) = \frac{1}{1+x^2}$$

$$xy + \tan^{-1} x = c$$

Example $(y - y^2) + xy' = 0$

$$\mu(y) = \frac{1}{(1-y)^2}$$

c) Suppose that $\mu = x^p y^q$

$x^p y^q M + x^p y^q N\, y' = 0$ is exact

$$\Leftrightarrow (x^p y^q M)_y = (x^p y^q N)_x$$

$$\Leftrightarrow xy(M_y - N_x) = pyN - qxM$$

$$\Leftrightarrow M_y - N_x = p\frac{N}{x} - q\frac{M}{y} \longrightarrow ④$$

Thus, if we can find p & q satisfying (4), then we are done.

<u>Example</u> $y - y^2 + xy' = 0$

Verify that $x^{-2} y^{-2}$ is an integrating factor.