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Note Title 04-06-2022

$$y'' + py' + qy = 0 \longrightarrow (1)$$

- p, q are continuous.

Abel's theorem. If y_1 & y_2 are solutions of (1), then

$$W(y_1, y_2)(t) = c \exp\left(-\int_{t_0}^{t} p(s)\, ds\right)$$

for some $t_0 \in I$.

Theorem. If y_1 & y_2 are solutions of (1), then $W(y_1,y_2)(t_0) = 0 \Rightarrow W(y_1,y_2)(t) = 0$ $\forall t$. Consequently, y_1 & y_2 are linearly dependent on I.

Proof. The first statement follows from Abel's theorem.

$$W(y_1, y_2)(t_0) = 0$$

i.e.,
$$\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = 0$$

$\Rightarrow$ The homogeneous system

$$\begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

will have atleast one non-zero solution.

Let c_1 & c_2 denote that non-zero solution.

claim. $c_1 y_1 + c_2 y_2 = 0$ on I.

Let $y(t) = c_1 y_1(t) + c_2 y_2(t)$.

Then y is a solution of (1).

- $y(t_0) = 0$ & $y'(t_0) = 0$

$\therefore$ y satisfies the IVP $y'' + py' + qy = 0$, $y(t_0) = 0$; $y'(t_0) = 0$.

$\Rightarrow$ By existence & uniqueness $y(t) = 0$ $\forall t \in I$

i.e., $c_1 y_1(t) + c_2 y_2(t) = 0$ $\forall t \in I$.

$$y'' + py' + qy = 0 \quad \text{where } p \text{ \& } q \text{ are constants.} \longrightarrow (2)$$

- Suppose we know one solution of (2), say y_1.

$$y_2(t) = u(t)\, y_1(t).$$

$$y_2'(t) = u'(t)\, y_1(t) + u(t)\, y_1'(t)$$

$$y_2''(t) = u''(t)\, y_1(t) + u'(t)\, y_1'(t) + u'(t)\, y_1'(t) + u(t)\, y_1''(t)$$

$$u''(t)\, y_1(t) + 2u'(t)\, y_1'(t) + u(t)\, y_1''(t) + p\, u'(t)\, y_1(t) + p\, u(t)\, y_1'(t) + q\, u(t)\, y_1(t) = 0$$

$$\Rightarrow u''(t)\, y_1(t) + 2u'(t)\, y_1'(t) - \cancel{u(t)\, p\, y_1'} - \cancel{u(t)\, q\, y_1} + p\, u'(t)\, y_1(t) + \cancel{p\, u(t)\, y_1'(t)} + \cancel{q\, u(t)\, y_1(t)} = 0$$

$$\Rightarrow u''(t)\, y_1(t) + 2u'(t)\, y_1'(t) + p\, u'(t)\, y_1(t) = 0$$

Let $v = u'(t)$. Then

$$v'(t)\, y_1(t) + v(t)\left[2y_1'(t) + p\, y_1(t)\right] = 0$$

$$\Rightarrow \frac{v'(t)}{v(t)} = \frac{-2y_1'(t) + p\, y_1(t)}{y_1(t)} \quad \text{where } y_1(t) \neq 0 \ \forall t \in I.$$

$$\Rightarrow v(t) = \frac{1}{y_1(t)^2} \exp\left(-\int p(t)\, dt\right)$$

$$= \frac{1}{y_1(t)^2} \exp(-pt)$$

- $y'' + ay' + by = 0$, where a & b are constants.

$$y(t) = e^{mt}$$

$$m^2 e^{mt} + am\, e^{mt} + b e^{mt} = 0$$

$$\Rightarrow m^2 + am + b = 0 \quad \text{– characteristic equation}$$

Auxillary equation

Let m_1 & m_2 two roots of $m^2 + am + b = 0$.

Case i) m_1 & m_2 are distinct.

$$y_1(t) = e^{m_1 t} \qquad y_2(t) = e^{m_2 t}$$

$$y(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t}$$

Case ii) $m_1 = m_2 := m = -a/2$

$$y_1(t) = e^{mt}.$$

$$y_2(t) = u(t)\, y_1(t)$$

$$u(t) = t \Rightarrow y_2(t) = t e^{mt}$$

$$u''(t) + u'(t) + [\cancel{-a + a}] + u(t)\, [\underbrace{\cancel{m^2 + am + b}}_{0}] = 0$$

$$u''(t) = 0$$

$$\Rightarrow u'(t) = c$$

$$\Rightarrow u(t) = ct$$

$$\therefore\ y(t) = c_1 e^{mt} + c_2 t e^{mt}.$$

Case iii) m_1 & m_2 are complex

If $m_1 = \alpha + i\beta$ then $m_2 = \alpha - i\beta$

$e^{(\alpha+i\beta)t}$ $\qquad$ $e^{(\alpha-i\beta)t}$

$= e^{\alpha t}(\cos\beta t + i\sin\beta t)$ $\quad = e^{\alpha t}(\cos\beta t - i\sin\beta t)$

As these two are solutions, their linear combinations are also solutions.

$\therefore$ Let $y_1(t) = e^{\alpha t}\cos\beta t$ & $y_2(t) = e^{\alpha t}\sin\beta t$.

$\therefore y(t) = c_1 e^{\alpha t}\cos\beta t + c_2 e^{\alpha t}\sin\beta t$.

Examples 1) $y'' + y' - 6y = 0$

$y(t) = c_1 e^{2t} + c_2 e^{-3t}$