14/6/22

Note Title 14-06-2022

System of linear ODE

$\bar{X}' = A\bar{X} \longrightarrow ①$ − homogeneous eqn

$\bar{x}' = A\bar{x} + \bar{r}(t) \longrightarrow ②$ − non-homogeneous eqn.

$$\left.\begin{aligned} \bar{x}' &= A\bar{x} + \bar{r}(t) \\ \bar{x}(t_0) &= X_0 \end{aligned}\right\} - \text{IVP}$$

- Existence-uniqueness theorem.

- The set $\{\bar{x} : A\bar{x} = \bar{x}'\}$ forms a real vector space
- If $A \in M_{n\times n}$, then the dimension of the solution space is 'n' (Verify!)

Let $\bar{X}_1, \bar{X}_2, \dots, \bar{X}_n$ be any n-functions with values in $\mathbb{R}^n$. Suppose that they are linearly dependent. Then $\exists$ scalars $\alpha_1, \alpha_2, \dots, \alpha_n$ (not all zero) such that

$$\alpha_1 \bar{X}_1 + \alpha_2 \bar{X}_2 + \dots + \alpha_n \bar{X}_n = \bar{0}$$

$$\Leftrightarrow \left(\bar{X}_1 \ \bar{X}_2 \ \cdots \ \bar{X}_n\right)\begin{pmatrix}\alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n\end{pmatrix} = \bar{0}$$

$\Leftrightarrow \left(\bar{X}_1 \ \bar{X}_2 \cdots \bar{X}_n\right)$ is not invertible

$\Leftrightarrow \det\left(\bar{x}_1 \ \bar{x}_2 \cdots \bar{x}_n\right) = 0$

- If $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n$ are linearly dependent, then $\det(\bar{x}_1 \ \bar{x}_2 \cdots \bar{x}_n)(t) = 0 \ \forall t \in I$.
- If $\exists \ t_0 \in I$ such that $\det(\bar{x}_1 \ \bar{x}_2 \cdots \bar{x}_n)(t_0) \neq 0$, then $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n$ are lin. ind.

- $W(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n) = \det(\bar{x}_1 \ \bar{x}_2 \cdots \bar{x}_n)$

- If $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n$ are linearly dependent, then $W(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n) = 0 \ \forall t \in I$.
- If $\exists \ t_0 \in I$ s.t. $W(\bar{x}_1, \dots, \bar{x}_n)(t_0) \neq 0$, then $\bar{x}_1, \dots, \bar{x}_n$ are lin. ind.

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$\bar{x}' = A\bar{x}$, where A is just a constant matrix.

$\bar{x} = e^{\lambda t}\bar{v}$

$(e^{\lambda t}\bar{v})' = A(e^{\lambda t}\bar{v})$

$\lambda e^{\lambda t}\bar{v} = e^{\lambda t}A\bar{v}$

$\Rightarrow A\bar{v} = \lambda\bar{v}$

Cor. If A has n distinct eigenvalues, then the system $\bar{x}' = A\bar{x}$ will have 'n' linearly ind. solutions.

Example

$$\begin{aligned} x' &= -x + 2y + 3z \\ y' &= -2y + z \\ z' &= 3z \end{aligned}$$

$$\begin{array}{ccc} -1 & -2 & 3 \\ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 17 \\ 4 \\ 20 \end{pmatrix} \end{array}$$

$$e^{-t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad e^{-2t}\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \quad e^{3t}\begin{pmatrix} 17 \\ 4 \\ 20 \end{pmatrix}$$

The general solution is given by

$$\bar{X}(t) = c_1 e^{-t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{-2t}\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{3t}\begin{pmatrix} 17 \\ 4 \\ 20 \end{pmatrix}.$$

Cor. If $\bar{v}_1, \bar{v}_2, \dots, \bar{v}_n$ are n linearly ind. eigen vectors with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, then $\{ e^{\lambda_i t}\bar{v}_i : 1 \le i \le n \}$ are 'n'- linearly ind. solns.

Example

$$\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 3 & 3 & -1 \end{bmatrix}$$

$$\begin{array}{ccc} 1 & 2 & \\ \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} & \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \end{array}$$