24/5/22

Note Title 24-05-2022

Ordinary Differential Equations

- $F(x, y, y', y'', \ldots, y^{(n)}) = 0$ — ODE
- $F(x, y, y') = 0$ — 1^{st} Order ODE
  — Implicit form
- $y' = f(x, y)$ — Explicit form.

$y' = f(x, y)$ $\quad x \in I \subseteq \mathbb{R}$, where I is an interval

A solution is any fn. h which is diff. on I and satisfies the equation on I.

$y' = f(x, y)$; $y(x_0) = y_0$ — IVP — Initial Value Problem.

Question. Does every 1^{st} order ODE have a solution?

- $(y')^2 + 1 = 0$
  This ODE has no real solutions
- $(y')^2 + y^2 = 0$; $y(0) = y_0$ with $y_0 \neq 0$
  has no solution

Question. Can a 1^{st} order IVP have more the one solution?

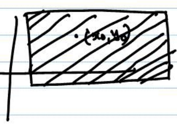
$x y' = y - 1$; $y(0) = 1$.

$y = 1 + cx$

$y' = f(x, y)$; $y(x_0) = y_0$ — $(*)$

Existence theorem. Suppose f is cts. on some closed rectangle
$R = \{(x, y) \in \mathbb{R}^2 : |x - x_0| \leq a, |y - y_0| \leq b\}$.
Then the IVP $(*)$ has atleast one solution $y = h(x)$ defined on some open interval $(x_0 - \alpha, x_0 + \alpha)$. Furthermore, if
$|f(x, y)| \leq K \;\; \forall \; (x, y) \in R$ for some $K > 0$,
then the solution must exist in $(x_0 - \alpha, x_0 + \alpha)$, where $\alpha = \min(a, b/K)$.

Recall. A fn. $f : U \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ is said to be continuous at (x_0, y_0) if for every given $\varepsilon > 0$ $\exists$ $\delta > 0$ s.t.



$|f(x, y) - f(x_0, y_0)| < \varepsilon$ whenever $\left((x - x_0)^2 + (y - y_0)^2\right)^{1/2} < \delta$.
$\left(d((x, y), (x_0, y_0)) = |(x, y) - (x_0, y_0)| < \delta.\right)$

Proof. Beyond the scope of this course.

Remark. The conditions in the above theorem are sufficient to guarentee the existence of a solution. But they are not necessary conditions. Therefore the IVP (*) can have solutions even if the fn. f is discontinuous on every rectangle containing (x_0, y_0).

Example $xy' = y-1$; $y(0) = 1$.

Here $f(x,y) = \frac{y-1}{x}$.

Note that $f(0,1)$ is not defined. Threfore in rectangle R around $(0,1)$ f is not continuous. On the other hand this IVP possesses infinite no. of solutions.

Suppose that $xy' = y-1$; $y(x_0) = 1$, $x_0 \neq 0$

choose $a = \frac{|x_0 - 0|}{2}$. Then in the rectangle

$R = \{(x,y) : |x-x_0| \leq a, \ |y-y_0| \leq b\}$

f is cts.

$\therefore$ By existence theorem IVP has a solution.

Remark. The constant $\alpha = \min(a, b/K)$ need not be the best possible, i.e., the longest interval for a given IVP.

Example $y' = y^2 + 1$; $y(0) = 0$.

$y = \tan x$, $x \in \underline{(-\pi/2, \pi/2)}$

Here $f(x,y) = 1+y^2$

Let $R = \{(x,y) : |x| \leq a, \ |y| \leq b\}$. Then f is cts. on R irrespective of the choice of a & b.

$|f(x,y)| = |1+y^2| \leq 1+b^2$.

$\therefore \min(a, b/K) = \min(a, b/1+b^2) \leq 1/2$

$\Rightarrow$ The maximum possible interval provided by the theorem is $(-1/2, 1/2)$.