20/5/22

Note Title 20-05-2022

- Subspaces $W_1, W_2, \dots, W_m$ of a vs V are ind. $\Leftrightarrow d(\sum_1^m W_i) = \sum \dim(W_i)$.
- $T: V \to V$ a LT.
  Let $\lambda_1, \lambda_2, \dots, \lambda_m$ be distinct eigenvalues of T.
  For $1 \le i \le m$, let $E_{\lambda_i} = \{v \in V : Tv = \lambda_i v\}$

  $\{E_{\lambda_i}\}_1^m$ are independent subspaces of V.
- $T: V \to V$ is diagonalizable if V has a basis w.r.t. which the matrix of T is diagonal.

Diagonalizability criterion. $T: V \to V$ is diagonalizable $\Leftrightarrow \dim(V) =$ sum of the dimensions of the eigenspaces.

Proof. $\Rightarrow$ Suppose that T is diagonalizable. Then $\exists$ a basis $\mathcal{B}$ of V s.t. $[T]_{\mathcal{B}}$ is a diagonal matrix. Let

$$[T]_{\mathcal{B}} = \operatorname{diag}(\mu_1, \mu_2, \dots, \mu_n),$$

where $n = \dim(V)$. Here $\mu_i \in \mathbb{F}$. Also, μ_i's need not be distinct.

Observe that $\lambda \in \mathbb{F}$ is an eigenvalue of $T \Leftrightarrow$ it is one of the μ_i's. Let $\lambda_1, \lambda_2, \dots, \lambda_m$ be the distinct eigenvalues of T. Let r_i denote the no. of times λ_i is repeated. Then $r_1 + r_2 + \dots + r_m = n$. Also, as T is diagonalizable, $\dim(E_{\lambda_i}) = r_i$ i.e.,

$$\sum_1^m \dim(E_{\lambda_i}) = n = \dim(V).$$

$\Leftarrow$ Suppose that $\dim(V) =$ sum of the dimensions of the eigenspaces of T.

Let $\lambda_1, \lambda_2, \dots, \lambda_m$ be the distinct eigenvalues of T and let E_{λ_i} denote the corresponding eigenspace. Let $r_i = \dim(E_{\lambda_i})$. Let $\mathcal{B}_i$ be the basis for E_{λ_i}, $1 \le i \le m$. Let $\mathcal{B} = \bigcup_1^m \mathcal{B}_i$. Observe that $|\mathcal{B}| = \dim V$, because E_{λ_i}'s are independent. Thus,

$$\dim V = \dim(\sum E_{\lambda_i}) = \sum_1^m |\mathcal{B}_i| = |\mathcal{B}|.$$

In particular, $\mathcal{B}$ forms a basis for V. As $\mathcal{B}$ consists only of eigenvectors $[T]_{\mathcal{B}}$ is diagonal.

<u>Example</u> $T: \mathbb{R}^3 \to \mathbb{R}^3$

$T(x, y, z) = (5x - 6y - 6z, -x + 4y + 2z, 3x - 6y - 4z)$

$\lambda = 1$ $(3, -1, 3)$ $\lambda = 2$ $(2, 1, 0), (2, 0, 1)$

$\dim(E_1) = 1$ $\qquad\qquad$ $\dim(E_2) = 2$

$\therefore$ T is diagonalizable.

<u>Example</u>

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix}$$

T is not diagonalizable.

$\lambda = 1$ $\qquad$ $(1, -1, 0)$ $\qquad$ (Verify)

$\lambda = 2$ $\qquad$ $(-2, 0, 1)$

<u>Exercise</u> Let

$$A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$$

Check whether A is diagonalizable or not.