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Note Title 06-04-2022

$C([0,1]) = \{f : [0,1] \longrightarrow \mathbb{R} \ : \ f \text{ is continuous}\}$

$f, g \in C([0,1])$
$(f+g)(t) := f(t) + g(t) \quad \forall\, t \in \mathbb{R}$

$f \in C([0,1])$ & $\alpha \in \mathbb{R}$
$(\alpha \cdot f)(t) := \alpha f(t) \quad \forall\, t \in \mathbb{R}$.

Examples

1) $\mathbb{F}[X] := \{p : p \text{ is a polynomial over } \mathbb{F}\}$

$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ for some $n \in \mathbb{N}$
$a_i \in \mathbb{F} \quad \forall\, 0 \le i \le n$

n - degree of the polynomial.

- $\mathbb{F}[X]$ is a vector space over $\mathbb{F}$.

2) $P_n(\mathbb{F}) := \{p \in \mathbb{F}[X] : \deg p < n\}$
- Vector space over $\mathbb{F}$

3) $\{(a_n)_{n\in\mathbb{N}} : a_n \in \mathbb{F} \ \forall\, n \in \mathbb{N}\}$

$f : \mathbb{N} \longrightarrow \mathbb{R}$
$\text{Maps}(\mathbb{N}, \mathbb{F})$
$= \{f : \mathbb{N} \longrightarrow \mathbb{F}\} \cong \{(a_n)_{n\in\mathbb{N}} : a_n \in \mathbb{F} \ \forall\, n \in \mathbb{N}\}$

4) $\mathbb{F}^\infty = \{(a_n) \in \text{Maps}(\mathbb{N}, \mathbb{F}) \ : \ a_n = 0 \ \forall n$ but finitely many$\}$

- $P_n(\mathbb{F}) \subseteq \mathbb{F}[X]$
- $\mathbb{F}^\infty \subseteq \text{Maps}(\mathbb{N}, \mathbb{F})$.
- $\{x \in \mathbb{R}^m : AX = 0\} \subseteq \mathbb{R}^m$

$V = \mathbb{R}^2 \qquad W = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$

(x_1, y_1) & (x_2, y_2) $\qquad (1,0) \quad (0,1)$

$(1,1)$

$(x_1 + x_2, \ y_1 + y_2) \qquad 1^2 + 1^2 = 2$

$(x_1 + x_2)^2 + (y_1 + y_2)^2 = x_1^2 + x_2^2 + 2x_1x_2 + y_1^2 + y_2^2 + 2y_1y_2$

W is not a vector space.

Definition. A non empty subset W of V is called a subspace of V if W is a vector space w.r.t. the operations obtained by restricting the operations on V to W.

$\mathbb{R}$ over $\mathbb{R}$

$\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$

- $\mathbb{R}^+$ is not a subspace of $\mathbb{R}$
- $x, y \in \mathbb{R}^+$

$$x + y := e^{\log x + \log y}$$
$$\alpha \cdot x := e^{\alpha \log x}$$

- $V \supseteq W$

$+ : V \times V \longrightarrow V$ $\qquad$ $+ : W \times W \longrightarrow V$

$\cdot : \mathbb{F} \times V \longrightarrow V$ $\qquad$ $\cdot : \mathbb{F} \times W \longrightarrow V$

- $P_n(\mathbb{F})$ is a subspace of $\mathbb{F}[x]$
- $\mathbb{F}^\infty$ is a subspace of Maps $(\mathbb{N}, \mathbb{F})$.

Theorem. A necessary & sufficient condition for a non empty set W of a V.S. V over $\mathbb{F}$ to be a subspace is that $u, v \in W$ & $\alpha \in \mathbb{F} \Rightarrow \alpha u + v \in W$.

Proof. $\Rightarrow$ Suppose that W is a subspace of V

i.e., W is a vector space w.r.t. the $+$ & $\cdot$ on V.

$\therefore$ If $u, v \in W$ then $u + v \in W$ $\longrightarrow$ ①

lly if $\alpha \in \mathbb{F}$ & $v \in W$ then $\alpha \cdot v \in W$ $\longrightarrow$ ②

Thus if $u, v \in W$ & $\alpha \in \mathbb{F}$, then $\alpha u + v \in W$ (from ① & ②)

$\Leftarrow$ Conversely, suppose that $u, v \in W$ & $\alpha \in \mathbb{F} \Rightarrow \alpha u + v \in W$.

As W is non-empty, choose $u \in W$. Then $(-1) \cdot u + u \in W$ i.e., $0 \in W$.

Let $u, v \in W$. Then $u + v = 1 \cdot u + v \in W$

Let $u \in W$ & $\alpha \in \mathbb{F}$. Then $\alpha \cdot u = \alpha \cdot u + 0 \in W$

Hence the proof.

Q. Given a set $S \subseteq V$ does $\exists$ a smallest subspace containing S?

Remark. Given any vector space V it always possesses two subspaces namely V & $\{0\}$.

$$\{0\} \subsetneq W \subsetneq V$$