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Note Title 25-05-2022

Uniqueness theorem.

Example $xy' = y-1$; $y(0) = 1$.
$y = 1 + Cx$, $c \in \mathbb{R}$

Definition. A fn. $f : R \subseteq \mathbb{R}^2 \to \mathbb{R}$ is said to satisfy the Lipschitz condition on R if $\exists$ a constant $L > 0$ such that

$$|f(x, y_1) - f(x, y_2)| \le L |y_1 - y_2| \quad \forall (x, y_1), (x, y_2) \in R.$$

[$g : S \subseteq \mathbb{R} \to \mathbb{R}$ is Lipschitz if $\exists$ $L > 0$ s.t.

$$|g(x_1) - g(x_2)| \le L |x_1 - x_2| \quad \forall x_1, x_2 \in S.$$]

Lemma. If f_y is bounded on R, then f satisfies the Lipschitz condition.
Proof. Follows from MVT. (How?)

Remark. Note that f might be Lipschitz in the y-variable even if $\partial f/\partial y$ does not exist at all pts. in R.

Example
$f(x, y) = |\sin y|$ $(x, y) \in \mathbb{R}^2$.
If $y = n\pi$, $n \in \mathbb{Z}$, then $\partial f/\partial y$ does not exist. But f satisfies the Lipschitz condition (Exercise).

$$y' = f(x, y) \; ; \; y(x_0) = y_0 \longrightarrow (*) \quad \text{IVP}.$$
$$x \in I$$

$$R = \{(x, y) \in \mathbb{R}^2 : |x - x_0| \le a, \; |y - y_0| \le b\}.$$

Uniqueness theorem. Suppose f is cts. on a closed rectangle R. Also, suppose f is Lipschitz in the y-variable. Then, the IVP (*) has a unique solution defined on some small enough interval $(x_0 - \alpha, x_0 + \alpha)$.

Remark. If the conditions in the uniqueness theorem fails, then we cannot conclude that the IVP does not have a unique solution.

Example $y' = 2\sqrt{|y|}$, $y(0) = 0$.

- $y_1 = 0$
- $y_2(x) = \begin{cases} x^2 & \text{if } x \ge 0 \\ -x^2 & \text{if } x < 0 \end{cases}$

Then y_2 is also a solution.

- f is not Lipschitz.

- If the initial condition is $y(x_0) = y_0$ with $y_0 \neq 0$, then f is Lipschitz & hence will have a unique solution.

Problem (Tut 4, Q_{29}). Consider the IVP

$$y y' = x, \quad y(0) = \beta.$$

Find all possible $\beta \in \mathbb{R}$ for which the IVP has
a) a unique solution
b) more than one solution
c) no solutions.

Solution. $y y' = x$

$$\frac{y^2 - x^2}{2} = c$$

Putting $y(0) = \beta$, we get $c = \beta^2/2$.

$$\therefore \; y^2 = x^2 + \beta^2$$

$$\Rightarrow y = \sqrt{x^2+\beta^2} \quad \text{or} \quad y = -\sqrt{x^2+\beta^2}.$$

Case i) $\beta > 0$.
As $\beta > 0$, $y = -\sqrt{x^2+\beta^2}$ is not a solution.
But $y = \sqrt{x^2+\beta^2}$ is a solution.

Case ii) $\beta < 0$.
As $\beta < 0$, $y = \sqrt{x^2+\beta^2}$ is not a solution.
But $y = -\sqrt{x^2+\beta^2}$ is a solution.

Case iii) $\beta = 0$. Here both $y = x$ & $y = -x$ are solutions

Conclusion.

a) $\exists$ no $\beta \in \mathbb{R}$ for which the above IVP has no solutions.
b) If $\beta = 0$, then the above IVP has more the one solution.
c) Suppose $\beta \neq 0$.
Here $f(x,y) = x/y$. $y_0 = \beta \neq 0$. $x_0 = 0$.

$R = \{|x-0| \leq a, \; |y-\beta| \leq b\}$

$-b \leq y - \beta \leq b$
$\Rightarrow -b+\beta \leq y \leq b+\beta$
choose $b = \frac{|\beta|}{2}$

$$\partial f/\partial y = -x/y^2$$

$$|\partial f/\partial y| = |x/y^2|$$

$$\leq \frac{a}{|\beta|^2/4}$$

$$= 4a/|\beta|^2.$$

choose $a=1$. Then $|\partial f/\partial y| \leq 4/|\beta|^2$.

i.e., f is Lipschitz in $R = \{(x,y)\in\mathbb{R}^2 : |x|\leq 1,\ |y-\beta| \leq |\beta|/2\}$.

$\therefore$ The IVP has a unique solution for $\beta \neq 0$.