20/4/22

Note Title 20-04-2022

Let V be a finite-dimensional vector space over $\mathbb{F}$. Let $\mathcal{B}$ be a basis for V.
Let $\mathcal{B} = \{v_1, v_2, \dots, v_n\}$ – Ordered basis.

Let $v \in V$. Then
$v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$.

$$[v]_{\mathcal{B}} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \in M_{n,1}(\mathbb{F}) \cong \mathbb{F}^n$$

↳ coordinate vector of v w.r.t. the Ordered basis $\mathcal{B}$.

Remark. $[v]_{\mathcal{B}}$ depends on the order in which the vectors in $\mathcal{B}$ are placed.

Example

1) $V = \mathbb{R}^2$

$\mathcal{B} = \{(1,0), (0,1)\}$

$(x,y) = x(1,0) + y(0,1)$

$\therefore [(x,y)]_{\mathcal{B}} = \begin{bmatrix} x \\ y \end{bmatrix}$.

$\mathcal{B}' = \{(0,1), (1,0)\}$

$(x,y) = y(0,1) + x(1,0)$

$[(x,y)]_{\mathcal{B}'} = \begin{bmatrix} y \\ x \end{bmatrix}$.

$\mathcal{B}'' = \{(1,1), (1,-1)\}$ – Verify that $\mathcal{B}$ is a basis!

$(x,y) = \alpha(1,1) + \beta(1,-1)$

$\alpha = \dfrac{x+y}{2}$ $\quad \beta = \dfrac{x-y}{2}$.

$$\therefore [(x,y)]_{\mathcal{B}''} = \begin{bmatrix} \frac{x+y}{2} \\ \frac{x-y}{2} \end{bmatrix}$$

$\mathcal{B}''$ – Old basis
$\mathcal{B}'''$ – New basis

$\mathcal{B}''' = \{(1,2), (2,1)\}$

Ex. Find the change of basis matrix.

$(x,y) = \alpha(1,2) + \beta(2,1)$

$\alpha = \dfrac{2y-x}{3}$ $\quad \beta = \dfrac{2x-y}{3}$

Remark. The coordinate vector $[v]_{\mathcal{B}}$ depends on the choice of the basis.

Let $\underbrace{\mathcal{B} = \{v_1, v_2, \dots, v_n\}}_{\text{old basis}}$ $\quad \underbrace{\mathcal{B}' = \{w_1, w_2, \dots, w_n\}}_{\text{new basis}}$.

For any $v \in V$,

$[v]_{\mathcal{B}} = [a_1 a_2 \cdots a_n]^t$ $\qquad$ $[v]_{\mathcal{B}'} = [b_1\, b_2 \cdots b_n]^t$

For $1 \le j \le n$, $\qquad$ $v = \sum_{i=1}^{n} b_i w_i \longrightarrow ②$

$$v_j = \sum_{i=1}^{n} p_{ij}\, w_i \longrightarrow ①$$

Let $[v]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$

$$\Rightarrow v = \sum_{j=1}^{n} a_j v_j = \sum_{j=1}^{n} a_j \sum_{i=1}^{n} p_{ij}\, w_i = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} p_{ij}\, a_j \right) w_i \longrightarrow ③$$

From ② & ③, $\quad b_i = \sum_{j=1}^{n} p_{ij}\, a_j \quad \forall\ 1 \le i \le n$

In matrix notation

$$[v]_{\mathcal{B}'} = \underbrace{[p_{ij}]}_{\to P \text{ matrix}} [v]_{\mathcal{B}}$$

- The matrix P is invertible.

Proof. Suppose $w_k = \sum_{j=1}^{n} q_{jk}\, v_j \longrightarrow ④ \qquad 1 \le k \le n.$

$Q = [q_{ij}]$ – $n \times n$ matrix.

$$w_k = \sum_{j=1}^{n} q_{jk}\, v_j \qquad \text{(From ④)}$$

$\mathcal{B} = \{v_1, v_2, \dots, v_n\}$

$[v_i]_{\mathcal{B}} = e_i$

$$= \sum_{j=1}^{n} q_{jk} \sum_{i=1}^{n} p_{ij}\, w_i$$

$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} p_{ij}\, q_{jk} \right) w_i$$

$$\Rightarrow \sum_{j=1}^{n} p_{ij}\, q_{jk} = \delta_{ik}$$

i.e., $PQ = I$

III^{ly} verify that $QP = I$.

Ex.

1) Let W_1 & W_2 be any two subspaces of a vector space V. Let

$W_1 + W_2 = \{ w_1 + w_2 : w_1 \in W_1 \ \& \ w_2 \in W_2 \}$.

a) Show that $W_1 + W_2$ is a subspace of V.

b) Show that $W_1 + W_2 = \text{span}\{W_1 \cup W_2\}$.

c) Give an example to show that $W_1 \cup W_2$ need not be a subspace.