9/4/22

Note Title 09-04-2022

Let $S \subseteq V$ be a nonempty set.

$$\text{Span}(S) = \left\{ \sum_{1}^{n} \alpha_i v_i : v_i \in S \ \& \ \alpha_i \in \mathbb{F} \ \forall \ 1 \le i \le n \ \& \ n \in \mathbb{N} \right\}$$

- span (S) is a subspace of V.

Proof.

Let $u, v \in$ span (S) & $\alpha \in \mathbb{F}$.

$u \in$ span (S) $\Rightarrow \exists \ u_1, u_2, \dots, u_n \in S$ & $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{F}$ s.t.

$$u = \sum_{1}^{n} \alpha_i u_i$$

lllly $v \in$ span $(S) \Rightarrow \exists \ v_1, v_2, \dots, v_m \in S$ & $\beta_1, \beta_2, \dots, \beta_m \in \mathbb{F}$ s.t.

$$v = \sum_{1}^{m} \beta_j v_j$$

$$\therefore \quad \alpha u + v = \alpha \left(\sum_{1}^{n} \alpha_i u_i \right) + \sum_{1}^{m} \beta_j v_j$$

$$= \sum_{1}^{n} \alpha(\alpha_i u_i) + \sum_{1}^{m} \beta_j v_j \quad (\because \text{Distributive property})$$

$$= \sum_{1}^{n} (\alpha \alpha_i) v_i + \sum_{1}^{m} \beta_j v_j \in \text{span}(S).$$

Thus, span (S) is a subspace of V.

Remark. $S \subseteq$ span (S) because $v = 1 \cdot v$.

Corollary. For any nonempty set S, $W(S) \subseteq$ span (S).

Theorem. For any nonempty set S, span $(S) = W(S)$.

Proof.

In order to prove this, it is enough to show that span $(S) \subseteq W(S)$.

Let $v \in$ span (S). Then $\exists \ v_1, v_2, \dots, v_n \in S$ & $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{F}$ s.t.

$$v = \sum_{1}^{n} \alpha_i v_i$$

As $S \subseteq W(S) \Rightarrow v_i \in W(S) \ \forall \ 1 \le i \le n$. Since $W(S)$ is a subspace, any linear combination of v_i's also belong to $W(S)$. In particular, $v \in W(S)$. Thus span $(S) \subseteq W(S)$.

Examples

i) $V = \mathbb{R}^2$. $S = \{(1,0)\}$.

span $(S) = \{\alpha(1,0) : \alpha \in \mathbb{R}\} = \{(\alpha, 0) : \alpha \in \mathbb{R}\}$ — X-axis

ii) $V = \mathbb{R}^2$ $S = \{(1,0), (0,0)\}$

$(x,y) \in$ span (S)

$\Rightarrow (x,y) = \alpha(1,0) + \beta(0,0) = (\alpha, 0) + (0,0) = (\alpha, 0)$

$\Rightarrow x = \alpha$ & $y = 0$

$\therefore$ span $(S) =$ X-axis

iii) $V = \mathbb{R}^3$

$S = \{ (1,0,0) , (1,1,0) \}$

span (S) = xy- plane.

iv) $V = \mathbb{R}^3$

$S = \{ (1,1,1), (1,2,3), (1,0,-1) \}$

$(2)(1,1,1) + (-1)(1,2,3) = (1,0,-1)$

$(2)(1,1,1) + (-1)(1,2,3) + (-1)(1,0,-1) = (0,0,0)$

$(x, y, z) \in$ span (S)

$\Rightarrow (x,y,z) = \alpha (1,1,1) + \beta(1,2,3) + \gamma (1,0,-1)$

$= (\alpha,\alpha,\alpha) + (\beta+2\beta+3\beta) + (\gamma,0,-\gamma)$

$= (\alpha+\beta+\gamma, \alpha+2\beta, \alpha+3\beta-\gamma)$

$x = \alpha+\beta+\gamma$

$y = \alpha+2\beta$ $\quad \boxed{x-2y+z=0}$

$y = \alpha+3\beta-\gamma$

Notation. We shall denote by $\langle S \rangle$ the smallest subspace of V containing S. Hence $\langle S \rangle$ = span (S). We will also call this as the smallest subspace generated by S.

Definition. A subset S of a vector space V is said to be linearly dependent if $\exists$ a finite subset $\{ v_1, v_2, \ldots, v_m \}$ of S & scalars $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{F}$ such that atleast one of these scalars is non-zero &

$$\sum_{1}^{m} \alpha_i v_i = 0$$

<u>Example</u> $V = \mathbb{R}^3$, $S = \{(1,1,1), (1,2,3), (1,0,-1)\}$

S is linearly dependent.

Ex. Verify whether the set $S = \{ (1,2,3), (2,3,4), (1,1,2) \}$ is linearly dependent or not.

Defn. A subset S of a vector space V is linearly independent if it is not linearly dependent.