4/5/22

Note Title 04-05-2022

- V & W – Vector spaces over $\mathbb{F}$.
- $T: V \longrightarrow W$
  $T(\alpha u + \beta v) = \alpha T(u) + \beta T(v) \quad \forall\ u, v \in V\ \&\ \alpha, \beta \in \mathbb{F}$.

- $\ker(T) := \{v \in V : T(v) = 0\}$ – kernel of T or null space of T
- $R(T) := \{w \in W : w = T(v)$ for some $v \in V\}$ – Image space or range space.

Proof. Let $w_1, w_2 \in R(T)$ & $\alpha \in \mathbb{F}$.
Since $w_1 \in R(T)$ $\exists\ v_1 \in V$ s.t. $w_1 = T(v_1)$
$|||^{ly}$, as $w_2 \in R(T)$ $\exists\ v_2 \in V$ s.t. $w_2 = T(v_2)$.

$\therefore\ \alpha w_1 + w_2 = \alpha T(v_1) + T(v_2) = T(\alpha v_1 + v_2) \in R(T)$.

- V, W – FDVS
  Nullity of T = $\dim(\ker(T))$

Rank of T = $\dim(R(T))$.

Lemma. Let $T: V \longrightarrow W$ be a linear transformation. Then T is injective (1-1) $\Leftrightarrow$ $\ker(T) = \{0\}$.

Proof. $\Rightarrow$ Suppose that T is injective. As T is a linear transformation $T(0) = 0$.
$\therefore\ \ker(T) = \{0\}$.

$\Leftarrow$ Suppose that $\ker(T) = \{0\}$. Suppose that $u, v \in V$ s.t. $T(u) = T(v)$.

$$\begin{aligned} & T(u) = T(v) \\ \Rightarrow\ & T(u) - T(v) = 0 \\ \Rightarrow\ & T(u - v) = 0 \\ \Rightarrow\ & u - v \in \ker(T) \\ \Rightarrow\ & u - v = 0 \\ \Rightarrow\ & u = v. \end{aligned}$$

i.e., T is injective

Lemma. Let $T: V \to W$ be a LT. If $\mathcal{B}$ is a basis for V, then $\operatorname{span}(T(\mathcal{B})) = R(T)$.

Proof. Note that $v \in \mathcal{B} \Rightarrow T(v) \in R(T)$
$\Rightarrow T(\mathcal{B}) \subseteq R(T)$
$\Rightarrow \operatorname{span}(T(\mathcal{B})) \subseteq R(T)$.

Let $w \in R(T)$. Then $\exists\ v \in V$ s.t. $w = T(v)$.
$v \in V$ & $\mathcal{B}$ is a basis for V
$\Rightarrow \exists\ v_1, v_2, \ldots, v_n \in \mathcal{B}$ & scalars $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{F}$ s.t.
$v = \alpha_1 v_1 + \cdots + \alpha_n v_n$

$$\begin{aligned} \therefore\ w = T(v) &= T(\alpha_1 v_1 + \cdots + \alpha_n v_n) \\ &= \alpha_1 T(v_1) + \cdots + \alpha_n T(v_n) \in \operatorname{span}(T(\mathcal{B})). \end{aligned}$$

i.e., $R(T) \subseteq \operatorname{span}(T(\mathcal{B}))$.

Example

$T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$T(x,y,z) = (x+y+z,\ x-y+z,\ y-z)$

We want to find the rank of T.

$\boxed{\begin{array}{c} T(x,y,z) = (0,0,0) \\ x+y+z = 0 \\ x-y+z = 0 \\ y-z = 0 \\ \text{This is to find a basis for kernel} \end{array}}$

$T(e_1) = (1,1,0)$
$T(e_2) = (1,-1,1)$
$T(e_3) = (1,1,-1)$

By above lemma span $\{(1,1,0), (1,-1,1), (1,1,-1)\} = R(T)$.

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \qquad RRE = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\boxed{\text{Ex. S.T. Nullity}(T) = 0}$

Rank $(T) = 3$.

Example

$T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$T(x,y,z) = (x+y-z,\ x-y+z,\ y-z)$

Rank $(T) = 2$.

$\boxed{\text{Ex. Nullity}(T) = 1}$

Rank - Nullity Theorem. Suppose $T: V \longrightarrow W$ is a linear transformation. Then
$$\text{Rank}(T) + \text{Nullity}(T) = \dim(V).$$

Proof. Let nullity $(T) = m$.
$\dim(V) = m+r$

$\mathcal{B} = \{v_1, v_2, \dots, v_m\}$ – basis for $\ker(T)$
$\mathcal{B}_1 = \{v_1, v_2, \dots, v_m, u_1, u_2, \dots, u_r\}$ – basis for V.

We know that Range$(T) = \text{span}(T(\mathcal{B}_1)) = \text{span}\{T(\mathcal{B}_1 \setminus \mathcal{B})\}$

$= \text{span}\{T(u_1), T(u_2), \dots, T(u_r)\}$.

claim: Rank $(T) = r$.

In order to prove this claim, it is enough to show that the set $\{T(u_1), T(u_2), \dots, T(u_r)\}$ is linearly independent.

Suppose that
$$\sum_1^r \alpha_i T(u_i) = 0$$
$$\Rightarrow T\left(\sum_1^r \alpha_i u_i\right) = 0$$
$$\Rightarrow \sum_1^r \alpha_i u_i \in \ker(T)$$

$\Rightarrow \exists$ scalars $\beta_1, \beta_2, \dots, \beta_m$ s.t.

$$\sum_1^r \alpha_i u_i = \sum_1^m \beta_j v_j$$

$$\Rightarrow \sum_1^r \alpha_i u_i + \sum_1^m (-1)\beta_j v_j = 0$$

$$\Rightarrow \alpha_i = 0 \ \forall 1 \le i \le r \ \& \ \beta_j = 0 \ \forall 1 \le j \le m \quad \text{(why?)}$$

$\therefore \{T(u_1), \dots, T(u_r)\}$ is a lin. ind. set.