17/5/22

Note Title 17-05-2022

$T: V \longrightarrow V$ where V is a finite dimensional vector space over $\mathbb{F}$ & T is a LT.

- A scalar $\lambda \in \mathbb{F}$ is said to be an eigenvalue if $\exists$ a non-zero vector $v \in V$ such that

$$Tv = \lambda v.$$

The vector v is called as the eigenvector corresponding to the eigenvalue λ.

Remark. Note that $v \neq 0$ is important. Otherwise every $\lambda \in \mathbb{F}$ will become an eigenvalue.

Example $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$
$T(x,y) = (2x+3y, 3x+2y)$.

$$T(x,y) = \lambda(x,y)$$

$\iff (2x+3y, 3x+2y) = \lambda(x,y)$

$\iff 2x+3y = \lambda x$
& $3x+2y = \lambda y$

$\iff (2-\lambda)x + 3y = 0$
$3x + (2-\lambda)y = 0$

$$\iff \begin{pmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This system has a non-zero solution if and only if $\lambda = -1$ or 5.

$\lambda = -1 \quad (x,y) = (-1, 1)$
$\lambda = 5 \quad (x,y) = (1,1)$.

λ - eigenvalue of T

$E_\lambda = \{v \in V : Tv = \lambda v\}$ - Eigenspace corresponding to the eigenvalue λ.

- E_λ is a subspace of V.
- Fix a basis for V. Then consider the matrix $[T]_{\mathcal{B}}$.

Observation.

λ is an eigenvalue for T $\qquad Tv = \lambda v \qquad T: V \longrightarrow V$
$\iff \lambda$ is an eigenvalue for $[T]_{\mathcal{B}}$. $\quad [T]_{\mathcal{B}}[v]_{\mathcal{B}} = \lambda [v]_{\mathcal{B}} \quad [T]_{\mathcal{B}}: \mathbb{F}^n \longrightarrow \mathbb{F}^n$

λ is an eigenvalue
$\iff \exists$ a non-zero $v \in V$ s.t. $Tv = \lambda v$
$\iff (T - \lambda I)v = 0$ where I denotes the identity transformation.

$\Leftrightarrow 0 \neq v \in \ker(T-\lambda I)$

$\Leftrightarrow$ $T-\lambda I$ is not bijective

$\Leftrightarrow$ $[T-\lambda I]_{\mathcal{B}}$ is not invertible

$\Leftrightarrow$ $\det([T-\lambda I]_{\mathcal{B}}) = 0$

$\Leftrightarrow$ $\det([\lambda I - T]_{\mathcal{B}}) = 0$ $\det([T]_{\mathcal{B}_1}) = \det([T]_{\mathcal{B}_2})$

- The polynomial $\det(\lambda I - T)$ is called as the characteristic polynomial
- The polynomial equation $\det(\lambda I - T) = 0$ is called as the characteristic equation.

<u>Facts.</u>

i) If V is of n-dimension then the characteristic polynomial has degree n.

ii) The coefficient of λ^n is 1, i.e., $\det(\lambda I - T)$ is a monic polynomial.

<u>Example</u> $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$T(x,y,z) = (x+y, y+z, z+x)$.

$\mathcal{B} = \{e_1, e_2, e_3\}$.

$T(e_1) = (1,0,1)$

$T(e_2) = (1,1,0)$

$T(e_3) = (0,1,1)$

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Ex. Find the corresponding eigenvector(s).

- 2 is the only eigenvalue.

<u>Example</u> $T: \mathbb{C}^2 \longrightarrow \mathbb{C}^2$

$T(z_1, z_2) = (z_1 - z_2, z_1 + z_2)$

$1+i$ & $1-i$ are the eigenvalues of T. Find the corresponding eigenvector.

Cayley - Hamilton theorem. Suppose $A \in M_{n\times n}(\mathbb{F})$ and $p(\lambda) = \det(\lambda I - A)$ is the characteristic polynomial of A, then $p(A) = 0$.