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Note Title 01-06-2022

Second Order ODE's.

$$f(x, y, y', y'') = 0$$

Linear - linear in 'y'.

$$a(x)y + b(x)y' + c(x)y'' = r(x).$$

Assumption: $c(x) \neq 0 \quad \forall x$

$$y'' + a(x)y' + b(x)y = r(x).$$

IVP: $\left.\begin{array}{l} y'' + a(x)y' + b(x)y = r(x) \\ y(t_0) = y_0 \quad y'(t_0) = y_1. \end{array}\right\} \longrightarrow (1)$

Existence and Uniqueness theorem. Let a, b & r be cts. fns. on some open interval I. Let $t_0 \in I$. Then the IVP (1) has a unique solution defined on the interval I.

Remark. The above theorem can be extended to higher order ODE's as well.

Exercise. Write the corresponding IVP and the existence-uniqueness theorem.

Consider the general 2^{nd} order ODE

$$y'' + py' + qy = 0 \longrightarrow (2)$$

where p & q are cts. fns.

$Y = \{y : y \text{ is a solution of } (2)\} \subseteq \{f : I \to \mathbb{R} : f \text{ is twice diff.}\}$

$\hookrightarrow$ Vector space over $\mathbb{R}$.

claim. $\dim(Y) = 2$.

Fix $t_0 \in I$. Consider the IVP's

$$\begin{array}{ll} y'' + py' + qy = 0 & \qquad y'' + py' + qy = 0 \\ y(t_0) = 1 \,;\, y'(t_0) = 0 & \qquad y(t_0) = 0 \,;\, y'(t_0) = 1. \end{array}$$

By assumption p & q are continuous functions and hence the above IVP's will have unique solution. Let y_1 & y_2 denote the corresponding solution.

- $\{y_1, y_2\}$ is linearly independent.

$$\alpha y_1 + \beta y_2 = 0 \longrightarrow (3)$$

i.e., $\alpha y_1(t) + \beta y_2(t) = 0 \quad \forall t$

$\Rightarrow \alpha y_1(t_0) + \beta y_2(t_0) = 0 \qquad$ i.e $\alpha = 0$

Diff. ⑤ $\therefore$ $\alpha y_1' + \beta y_2' = 0$

i.e., $\alpha y_1'(t) + \beta y_2' = 0$

$\Rightarrow \alpha y_1'(t_0) + \beta y_2'(t_0) = 0$

i.e., $\beta = 0$

- Let y be a solution of ②. Let

$$\tilde{y} = y(t_0)\, y_1 + y'(t_0)\, y_2 .$$

Then $\tilde{y}$ is also a solution of ②.

$\tilde{y}(t_0) = y(t_0)$

$\tilde{y}'(t_0) = y'(t_0)$

$\therefore$ Both y & $\tilde{y}$ satisfy the IVP $y'' + py' + qy = 0$, $y(t_0) = y_0$, $y'(t_0) = y_1$.

$\Rightarrow$ By existence uniqueness theorem $y = \tilde{y}$, i.e., y is a linear combination of y_1 & y_2.

Ex. Consider the n^{th} order ODE

$$y^{(n)} + p_1 y^{(n-1)} + p_2 y^{(n-2)} + \cdots + p_n y = 0 .$$

Show that the solution set for this ODE forms a n-dimensional vector space over $\mathbb{R}$.