18/5/22

Note Title 18-05-2022

$T: V \longrightarrow V$

- $\lambda \in \mathbb{F}$ is an eigenvalue if $\exists\ 0 \neq v \in V$ s.t. $Tv = \lambda v$.
- v is called as the eigenvector.
- λ is an eigenvalue $\iff$ λ is a root of the polynomial $p(x) = \det(xI - A)$.
- The polynomial $\det(xI-A)$ is called as the characteristic polynomial.
- Cayley-Hamilton theorem. Every square matrix satisfies its own ch. polynomial
- $p(x) = \sum_{i=0}^{n} a_i x^i$ is the polynomial,

then $P(A) = \sum_{i=0}^{n} a_i A^i$, where $A^0 = I$ is the identity matrix.

Observation. Consider the characteristic polynomial

$$p(x) = \det(xI - A).$$

Then $p(0) = \det(-A) = (-1)^n \det(A)$. Thus, A is invertible $\iff p(0) \neq 0$.

An application. Let A be an invertible matrix. Let $p(x)$ be the ch. polynomial of A. By Cayley-Hamilton theorem,

$$p(A) = 0$$
$$\Rightarrow A^{-1} p(A) = 0$$

Let $p(x) = \sum_{i=0}^{n} a_i x^i$. Then

$$0 = A^{-1} P(A) = A^{-1} \sum_{i=0}^{n} a_i A^i = A^{-1} a_0 + \sum_{i=1}^{n} a_i A^{i-1}$$

$$\boxed{\therefore\ A^{-1} = \frac{1}{a_0} \sum_{i=1}^{n} -a_i A^{i-1}}$$

Note that $a_0 \neq 0$ as $p(0) = a_0$.

<u>Example</u>

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$p(x) = \det(xI - A) = \begin{vmatrix} x-1 & 0 & 0 \\ 0 & x-1 & -1 \\ -1 & -1 & x \end{vmatrix} =$$

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

- Identity matrix
- Scalar matrix or Diagonal matrix

Question. Given $T: V \to V$, Does $\exists$ a basis $\mathcal{B}$ w.r.t. which $[T]_{\mathcal{B}}$ is a diagonal matrix.

$$[T]_{\mathcal{B}} = [a_{ij}]$$

Let $[T]_{\mathcal{B}} = \begin{pmatrix} a_1 & & 0 \\ & a_2 & \\ 0 & & \ddots \; a_n \end{pmatrix}$

Let $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$

$T(v_1) = a_1 v_1$

Definition. A LT $T: V \to V$ is said to be diagonalizable if V has a basis w.r.t. which the matrix of T is diagonal.

Example, $I: V \to V$
$0: V \to V$
are diagonalizable.

Example $T: \mathbb{R}^2 \to \mathbb{R}^2$
$T(x,y) = (x+y, y)$

Ex. Show that T is not diagonalizable.

Diagonalizability Criteria. An operator $T: V \to V$ is diagonalizable (over $\mathbb{F}$) $\Leftrightarrow$ $\dim V$ is equal to the sum of the dimensions of the eigenspaces of T.

- Suppose that $W_1, W_2, \ldots, W_m$ are sub-spaces of a vector space V. They are called independent if $w_1 + w_2 + \cdots + w_m = 0$ for $w_i \in W_i \Rightarrow w_i = 0 \; \forall i$.

Lemma. Suppose that $W_1, W_2, \ldots, W_m$ are subspaces, then they are independent $\Leftrightarrow$

$$\dim\left(\sum_1^m W_i\right) = \sum_1^m \dim(W_i).$$

Proof. Out of scope.

Lemma. The Eigenspaces corresponding to different eigenvalues of T are independent

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be distinct eigenvalues of T. Let

$$E_{\lambda_i} := \{v \in V : Tv = \lambda_i v\} = \ker(\lambda_i I_V - T), \quad 1 \le i \le m.$$

claim: $E_{\lambda_1}, E_{\lambda_2}, \ldots, E_{\lambda_m}$ are independent.

When $m=1$, then there is nothing to prove.
Let $m=2$. Let $w_1 \in E_{\lambda_1}$ & $w_2 \in E_{\lambda_2}$. Then

$$w_1 + w_2 = 0$$
$$\Rightarrow w_1 = -w_2$$
$$\Rightarrow w_1, w_2 \in E_{\lambda_1} \cap E_{\lambda_2}$$

Thus $\lambda_1 w_1 = T(w_1) = \lambda_2 w_1$
$\Rightarrow (\lambda_1 - \lambda_2) w_1 = 0 \quad \Rightarrow w_1 = 0 \quad \Rightarrow w_2 = 0$

Induction hypothesis. The statement is true for $m-1$.

Let $w_i \in E_{\lambda_i}$, $1 \le i \le m$, such that

$$w_1 + w_2 + \cdots + w_m = 0. \longrightarrow ①$$
$$\Rightarrow T(w_1 + w_2 + \cdots + w_m) = 0$$
$$\Rightarrow \lambda_1 w_1 + \lambda_2 w_2 + \cdots + \lambda_m w_m = 0 \longrightarrow ②$$

From ①, $\lambda_m w_1 + \lambda_m w_2 + \cdots + \lambda_m w_m = 0 \longrightarrow ③$

② − ③ $\Rightarrow (\lambda_1 - \lambda_m) w_1 + (\lambda_2 - \lambda_m) w_2 + \cdots + (\lambda_{m-1} - \lambda_m) w_{m-1} = 0$

By induction hypothesis, $(\lambda_i - \lambda_m) w_i = 0$
$\Rightarrow w_i = 0$ as $\lambda_i \ne \lambda_m \ \forall\, 1 \le i \le m-1$.

$\therefore$ From ①, $w_m = 0$.

Ex. Find the eigenvalues and the eigenvectors of

$$T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$
$$T(x,y,z) = (5x - 6y - 6z,\ -x + 4y + 2z,\ 3x - 6y - 4z).$$

See whether T is diagonalizable or not.