8/4/22

Note Title 08-04-2022

Q. Does $\exists$ a smallest subspace containing a given set S?

- Intersection of two subspaces of a vector space is again a subspace.

Proof. Let W_1 & W_2 be any two subspaces of V.

claim. $W_1 \cap W_2$ is a subspace.

Let $u, v \in W_1 \cap W_2$ & $\alpha \in \mathbb{F}$.

$u \in W_1 \cap W_2 \Rightarrow u \in W_1$ & $u \in W_2$

$|||^{ly}$ $v \in W_1 \cap W_2 \Rightarrow v \in W_1$ & $v \in W_2$.

Now, $u, v \in W_1$ & $\alpha \in \mathbb{F} \Rightarrow \alpha u + v \in W_1$ ($\because W_1$ is a subspace)

Also, $u, v \in W_2$ & $\alpha \in \mathbb{F} \Rightarrow \alpha u + v \in W_2$ ($\because W_2$ is a subspace)

Therefore, $\alpha u + v \in W_1 \cap W_2$

i.e., $W_1 \cap W_2$ is a subspace.

- Arbitrary intersection of subspaces is again a subspace.

Proof. Let $\{W_\alpha : \alpha \in \Lambda\}$ be a collection of subspaces of a vector space V.

claim. $\bigcap_{\alpha \in \Lambda} W_\alpha$ is a subspace.

Let $u, v \in \bigcap_{\alpha \in \Lambda} W_\alpha$ & $\beta \in \mathbb{F}$.

Λ – Wedge
– Lambda

$u \in \bigcap_{\alpha \in \Lambda} W_\alpha \Rightarrow u \in W_\alpha \ \forall \alpha \in \Lambda$

$|||^{ly}$ $v \in \bigcap_{\alpha \in \Lambda} W_\alpha \Rightarrow v \in W_\alpha \ \forall \alpha \in \Lambda$

For any $\alpha \in \Lambda$,

$u, v \in W_\alpha$ & $\beta \in \mathbb{F} \Rightarrow \beta u + v \in W_\alpha$ ($\because W_\alpha$ is a subspace)

Since α is arbitrary, $\beta u + v \in W_\alpha \ \forall \alpha \in \Lambda$

i.e., $\beta u + v \in \bigcap_{\alpha \in \Lambda} W_\alpha$

$\Rightarrow \bigcap_{\alpha \in \Lambda} W_\alpha$ is a subspace.

- Let S be any nonempty subset of a vector space V. Let

$W(S) = \bigcap \{W \subseteq V : W \text{ is a subspace of } V \text{ containing } S\}$

- $W(S)$ is a sub-space containing S.
- $W(S)$ is the smallest among all the sub-spaces containing S.

Let W' be any sub-space of V containing S.

$\Rightarrow W'$ is one among the collection of all subspaces containing S.

$\Rightarrow \bigcap_{\substack{W \subseteq V \\ W \text{ is a subspace} \\ \text{containing } S}} W \subseteq W'$

i.e, $W(S) \subseteq W'$

$\Rightarrow W(S)$ is the smallest among all subspaces containing S.

Linear combination.

v_1, v_2 $\qquad \alpha v_1 + \beta v_2$

$v_1, v_2, \dots, v_n$ $\qquad \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$

Let S be a nonempty subset of V.

$span(S) = \{\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n : \alpha_i \in \mathbb{F}, v_i \in S \ \forall 1 \le i \le n\}$

Example

Let $V = \mathbb{F}[x]$

$S = \{1, x, x^2, x^3, \dots\} \subseteq \mathbb{F}[x]$

$span(S) = \mathbb{F}[x]$.

Ex. Show that $Span(S)$ is a subspace of V containing S.