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Note Title 06-05-2022

Rank - Nullity theorem. If $T: V \longrightarrow W$, where V is finite dimensional, then

$$\dim(V) = \text{Rank}(T) + \text{Nullity}(T).$$

Corollary. Suppose that $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a LT. If T is

i) injective then $m \geq n$
ii) surjective then $m \leq n$
iii) bijective then $m = n$.

Proof. i) Suppose that T is injective. Then $\ker(T) = \{0\}$ i.e., nullity$(T) = 0$. Now by Rank-Nullity theorem

$$\text{Rank}(T) + \text{nullity}(T) = \dim(\mathbb{R}^n) = n$$

i.e., $\text{Rank}(T) = n$

As $R(T)$ is a subspace of $\mathbb{R}^m$, $\text{Rank}(T) \leq m$ i.e., $m \geq n$.

ii) Suppose that T is surjective, i.e., $\text{Range}(T) = \mathbb{R}^m$ or equivalently $\text{Rank}(T) = m$.

$\therefore$ By Rank-nullity theorem,

$$\text{Rank}(T) + \text{nullity}(T) = n$$

$$\therefore n = \text{nullity}(T) + m \geq m$$

iii) Follows from i) & ii).

Corollary.

i) There is no injective linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ if $n > m$.
ii) There is no surjective linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ if $m > n$.

iii) There is a $\underbrace{\text{bijective linear transformation}}_{\text{isomorphism}}$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ $\Leftrightarrow$ $m = n$.

Corollary. Suppose V & W are V.S.'s over $\mathbb{F}$ such that $\dim(V) = \dim(W)$. Let $T: V \to W$ be a LT. Then TFAE:

i) T is injective
ii) T is bijective
iii) $\ker(T) = \{0\}$
iv) T is surjective.

Proof. Exercise

Corollary. For any matrix A, row rank and column rank are identical.

Note. Let $A \in M_{m\times n}(\mathbb{F})$. Define $T_A : \mathbb{F}^n \longrightarrow \mathbb{F}^m$ as $T_A(x_1, x_2, \cdots, x_n) = A\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$.

Verify that T_A is a linear transformation.

Proof. Let $A \in M_{m\times n}(\mathbb{F})$. Define $T_A : \mathbb{F}^n \longrightarrow \mathbb{F}^m$ as $T_A(x_1, x_2, \cdots, x_n) = A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.

Let $C_1, C_2, \cdots, C_n$ denote the n-columns of the matrix A, so that
$A = (C_1\ C_2\ C_3\ \cdots\ C_n)$.

Then $A\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \sum_1^n x_i C_i \in$ column space of A. On the other hand $A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ is also in the range of T_A. Thus range $(T_A) \subseteq$ Column space of A.

If $(y_1, y_2, \cdots, y_m) \in$ Column space of A, then $\exists\ x_1, x_2, \cdots, x_n \in \mathbb{F}$ s.t.

$$(y_1, y_2, \cdots, y_m)^t = \sum_1^n x_i C_i = A\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \text{Range}(T).$$

Thus column space of $A \subseteq$ range (T_A).

$\therefore$ column space (A) = range (T_A)

$\Rightarrow$ column rank (A) = rank (T_A).

Observe that $\ker(T_A)$ = solution space of the system $AX=0$.

$\therefore$ Nullity (T_A) = dim (solution space).

$\Rightarrow$ Nullity $(T_A) = n -$ row rank (A)

By Rank - Nullity theorem

rank (T_A) + Nullity (T_A) = dim $(\mathbb{F}^n)$

Column rank $(A) + n -$ row rank $(A) = n$

$\Rightarrow$ Column rank (A) = row rank (A).

Matrix representation.

V, W are FDVS's over $\mathbb{F}$.

$T : V \longrightarrow W$ LT.

$\left.\begin{array}{l} \mathcal{B} = \{v_1, v_2, \cdots, v_n\} \text{ - basis for } V. \\ \mathcal{B}' = \{w_1, w_2, \cdots, w_m\} \text{ - basis for } W. \end{array}\right\}$ ordered.

$$T(v_j) = \sum_1^m a_{ij} w_i \qquad 1 \le j \le n$$

$$[T]_{\mathcal{B}}^{\mathcal{B}'} = [a_{ij}] \in M_{m\times n}(\mathbb{F})$$

Notation. If $V=W$ & $\mathcal{B}=\mathcal{B}'$, then $[T]_{\mathcal{B}} := [T]_{\mathcal{B}}^{\mathcal{B}'}$.

Example

$$T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$
$$T(x,y,z) = (2x+z, y+3z)$$

$\mathcal{B} = \{(1,1,0), (1,0,1), (1,1,1)\}$
$\mathcal{B}' = \{(2,3), (3,2)\}$.

$T(1,1,0) = (2,1) = -\frac{1}{5}(2,3) + \frac{4}{5}(3,2)$

$T(1,0,1) = (3,3) = \frac{3}{5}(2,3) + \frac{3}{5}(3,2)$

$T(1,1,1) = (3,4) = \frac{6}{5}(2,3) + \frac{1}{5}(3,2)$

$$[T]_{\mathcal{B}}^{\mathcal{B}'} = \begin{bmatrix} -1/5 & 3/5 & 6/5 \\ 4/5 & 3/5 & 1/5 \end{bmatrix}$$

$\mathcal{B}_1$, $\mathcal{B}_2$ – bases for V
$\mathcal{B}_1'$, $\mathcal{B}_2'$ – bases for W.

$[T]_{\mathcal{B}_1}^{\mathcal{B}_1'}$ $\quad$ $[T]_{\mathcal{B}_2}^{\mathcal{B}_2'}$.

1) Lemma. Suppose that $\mathcal{B}$ & $\mathcal{B}'$ are bases for V & W respectively & $T: V \to W$ a LT. If $v \in V$, then
$$[T(v)]_{\mathcal{B}'} = [T]_{\mathcal{B}}^{\mathcal{B}'} [v]_{\mathcal{B}}$$
Proof. Exercise.

2) Lemma. Suppose $A, B \in M_{m\times n}(\mathbb{F})$. If $Ax = Bx$ for every $x \in M_{n\times 1}(\mathbb{F})$, then $A=B$.
3) Lemma. The map $v \longmapsto [v]_{\mathcal{B}}$ is an isomorphism between V & $\mathbb{F}^n$.

$T: V \longrightarrow W$

V	$\mathcal{B}_1$	$\mathcal{B}_2$
W	$\mathcal{B}_1'$	$\mathcal{B}_2'$

Let $v \in V$.

$[v]_{\mathcal{B}_2} = P\, [v]_{\mathcal{B}_1}$

$[Tv]_{\mathcal{B}_2'} = Q\, [Tv]_{\mathcal{B}_1'}$

$[T]_{\mathcal{B}_2}^{\mathcal{B}_2'} [v]_{\mathcal{B}_2} = Q\, [T]_{\mathcal{B}_1}^{\mathcal{B}_1'} [v]_{\mathcal{B}_1}$

$[T]_{\mathcal{B}_2}^{\mathcal{B}_2'} P [v]_{\mathcal{B}_1} = Q\, [T]_{\mathcal{B}_1}^{\mathcal{B}_1'} [v]_{\mathcal{B}_1}$

$\Rightarrow [T]_{\mathcal{B}_2}^{\mathcal{B}_2'} P = Q\, [T]_{\mathcal{B}_1}^{\mathcal{B}_1'}$

$\Rightarrow [T]_{\mathcal{B}_2}^{\mathcal{B}_2'} = Q\, [T]_{\mathcal{B}_1}^{\mathcal{B}_1'} P^{-1}$

Corollary. If $V = W$, $\mathcal{B}_1 = \mathcal{B}_1'$ & $\mathcal{B}_2 = \mathcal{B}_2'$ then $P = Q$ i.e.,

$$[T]_{\mathcal{B}_2} = Q\, [T]_{\mathcal{B}_1}\, Q^{-1}.$$