Orbital Dynamics

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ABSTRACT

Since Kepler's observations, it has long been known that celestial bodies travel in elliptical orbits. Additionally, Newton's law of universal gravitation determined a method to find the gravitational force between objects and has been used in numerous studies since its discovery, thereby solidifying in the study of astronomy. With a growing presence in space, we must be able to predict the location of celestial objects such as planets, asteroids, and comets. In this paper, we will explore how the European Space Agency accurately determined the location of comet 67P, also known as Churyumov–Gerasimenko. through orbital dynamics and Euler's method.

Keywords: Comet Orbits, Differential Equations, Numerical Approximations

INTRODUCTION

Space research and exploration requires the knowledge of immensely precise motions and positions of stars, planets, and other celestial bodies. In order for a spacecraft to land on a planet or comet, engineers and scientists must be able to calculate the exact position of the celestial body when the spacecraft enters the body's orbit. Slight changes in modeling can cause significant differences in accuracy. We will aim to approximate comet 67P's orbit.

The motion of of 2 celestial objects is determined by Newton's law of Universal Gravitation. We will be using Euler's method, taking small steps in time and using the tangent line to estimate the change in position.

$$\mathbf{F}_{CS} = -G\frac{m_c m_s}{|\mathbf{r}_{CS}|^3}\mathbf{r}_{CS}$$

Figure 1. Newton's law of Universal Gravitation

$\mathbf{r}_{CS}$ is a vector describes the x-component and y-component of the celestial body's direction from the sun. $\mathbf{F}_{CS}$ is a vector of the x-component and y-component of force on the celestial body due to the sun. m_c is the mass of the comet, and m_s is the mass of the sun. G is the universal gravitational constant.

We also know from Newton's second law of motion that $\mathbf{F}_{CS} = m_C\mathbf{a}_C$. Furthermore, acceleration is the second derivative of position which gives our differential equation.

$$\mathbf{a}_C(t) = -G\frac{m_s}{|\mathbf{r}_{CS}|^3}\mathbf{r}_{CS} = -G\frac{m_s}{(x^2+y^2)^{3/2}}\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{x}''(t) = \begin{bmatrix} x'' \\ y'' \end{bmatrix}$$

Figure 2. System of Nonlinear Differential Equations for the position of celestial bodies

x represents the x-position of the comet, and y represents the y-position of the comet. $\mathbf{a}_C$ is a vector for the acceleration of the comet.

Since our frame of reference is set at (0,0), the direction vector and position vector are equivalent.

Due to the difficulties of obtaining an exact solution in nonlinear differential equations, we will approximate a celestial bodies orbit using the Euler-Cromer approximation method.

ASSUMPTIONS IN THE MODEL

We will be assuming that there are only the sun and the comet of interest in our system and that orbits occur in 2D rather than 3D.

We will also be assuming that the acceleration that the comet applies on the sun is negligible, and therefore the sun will be stationary at (0,0).

The comet is perfectly spherical where the center of mass is the center of the sphere

The comet does not rotate or spin on its axis.

For comet 67P/Churyumov-Gerasimenko:

- The initial position of the comet will be the point (193,929,400 km, 0 km).
- It is also known that after 67.69 seconds, the comet will be 193,929,000 km with a speed of 33.35 km/s and a positive y-position.
- Thus, it can be calculated that the initial x-velocity will be -5.91 km/s and the initial y-velocity will be 32.82 km/s

For comet 4P/Faye:

- The initial position of the comet will be the point (380,324,265 km, 0 km).
- It can be calculated that the initial x-velocity will be 10.76 km/s and the initial y-velocity will be 10.85 km/s[2]

ANALYSIS

The method to find the estimated position is the Euler-Cromer method[4], which slightly differs from the original Euler's method. The difference in the mathematical sense is that we do not use the velocity of the previous iteration, but instead use the velocity of the current iteration that we are trying to calculate. To be able to use the current iteration of velocity, the Euler-Cromer method has an Euler method meant to be applied to velocity utilizing acceleration. The reason that this happens is that it prevents the trajectory from spiraling out of control. Additionally, this method is meant to conserve energy for oscillatory motions unlike the original method which would increase energy with oscillatory motions.

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \Delta t \times \mathbf{v}_n$$

Figure 3. Traditional Euler Method

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \Delta t \times \mathbf{v}_{n+1}$$

Figure 4. Euler-Cromer Method

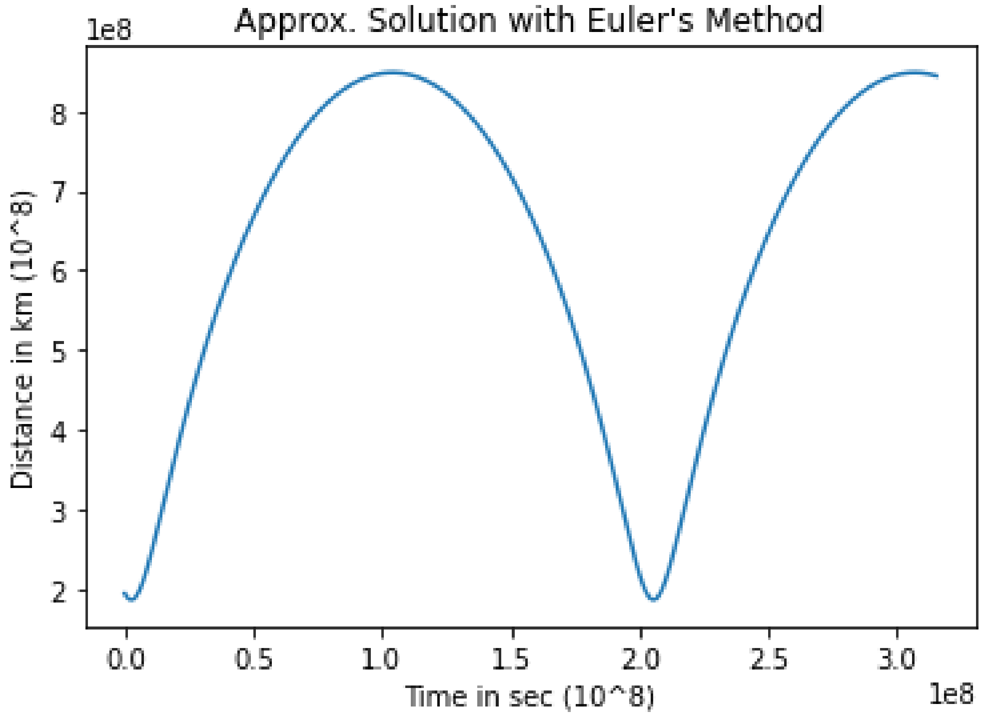

Figure 5. A numerical approximation of 67P's distance from the sun
Results obtained from self-developed code in Figure 11 in Appendix
(step size= 100 seconds)

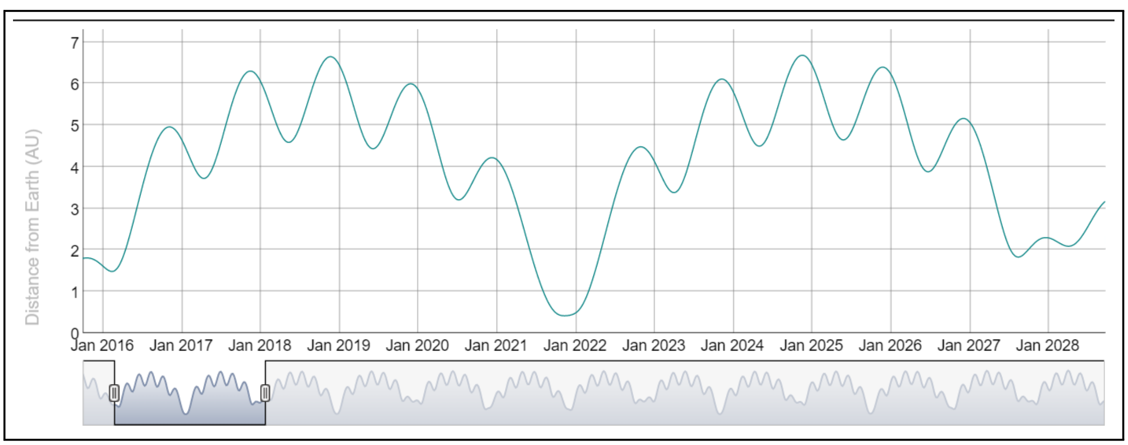

Figure 6. Accurate data of 67P's distance from Earth
Results in Figure 4 above were obtained from The Sky Live[3]

Looking at the two graphs above that measure position as a function of time, there are some obvious discrepancies as the actual position is relative to earth and the position we calculated is relative to the Sun. This is shown by the six small oscillations for every large oscillation representing the earth's orbit around the sun. The minimum distance that the comet is from the sun according to the calculations is 186,105,151 kilometers while in reality it is 185,781,011 kilometers which has an error of 0.00174 percent. The period that was calculated for comet 67P is about 6.516 years and the actual period is about 6.43 years which had an error about 0.0134 percent. This small error for both the period and minimum distance indicates the immense accuracy that the Euler-Cromer method presents even if the method uses simple mathematical calculations.

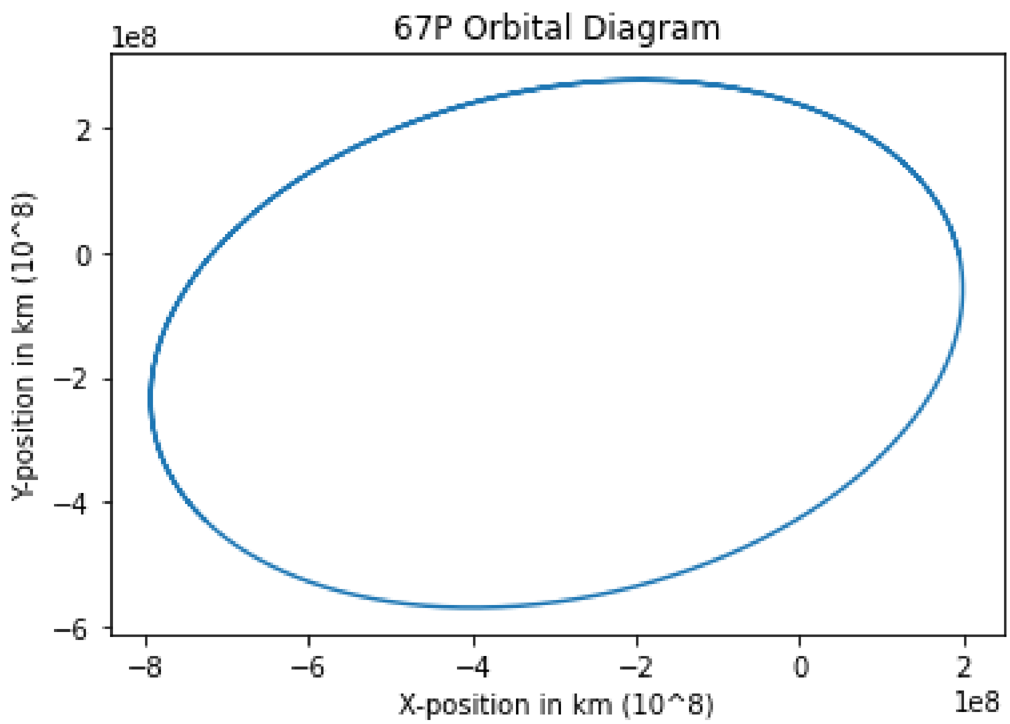

Figure 7. A diagram of 67P's orbit around the sun centered at (0,0) Results obtained from the self-made code in Figure 11 in Appendix (step size= 100 seconds)

The orbital diagram of 67P in Figure 7 graphs an estimation of the orbit around the sun. This graph captures the eccentricity of the real life orbit while also being extremely accurate to the real orbit. Though it is accurate, it is not perfect which is result of using the Euler-Cromer method. The end beahvior predicted by Euler-Cromer method seems to be correct as the ending position is around the same as the starting position which is similar to how this comet is in the real world.

4P/Faye

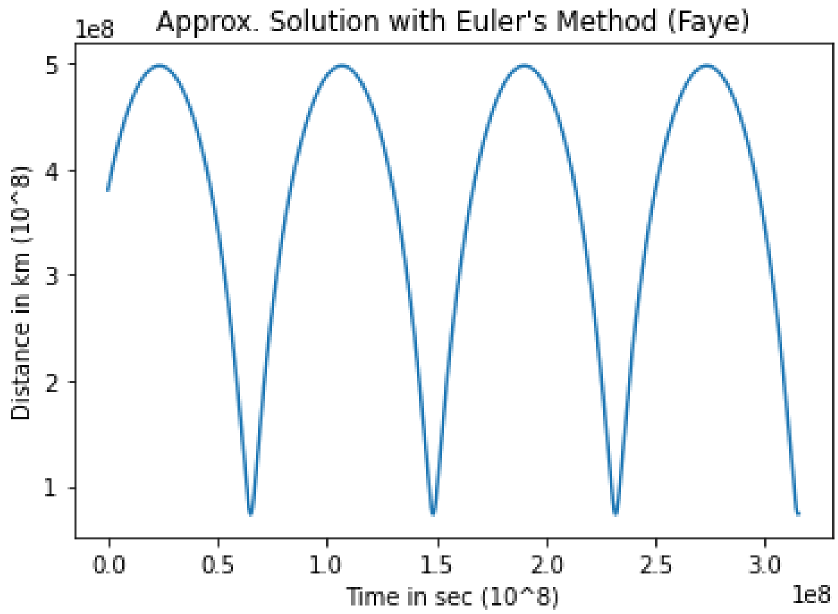

Figure 8. An approximation of Faye Comet's distance from the Sun from code seen in Figure 12 in Appendix
(step size= 100 seconds)

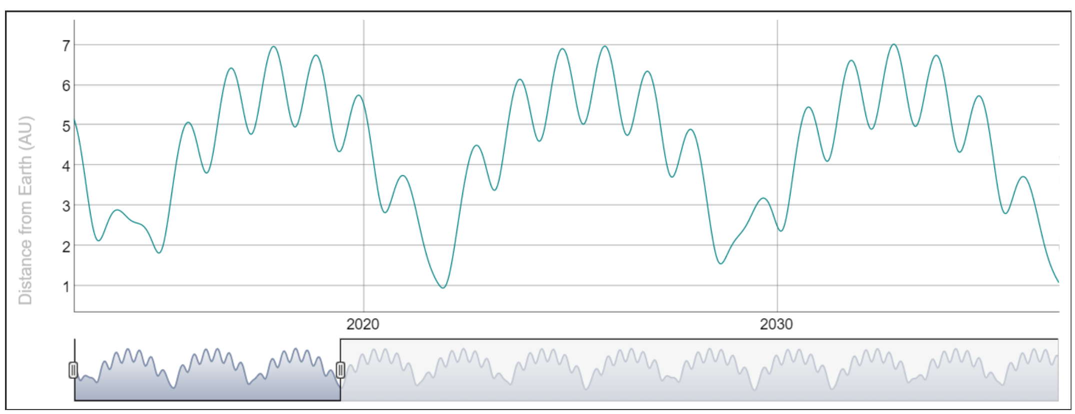

Figure 9. An accurate graph of Faye's distance from Earth found from The Sky Live[3]

The Faye comet is slightly different because we had to determine an initial velocity in the x and y direction. The source used for 67P's initial values,SIMIODE[4], did not provide any initial values for 4P, so we utilized Space Reference[2] in order to determine these values. Due to all the values being self determined rather than given, there will be a greater error when compared to the error obtained from the comet 67P. The minimum distance obtained for this comet was 73,644,934.11 km where in reality the minimum distance of the 234,870,000 which has about a 68.6 percent error. The period of this comet comes out to be 9.99 years and the real period 7.48 years which presents an error of 25.13 percent. The error presented here is far greater than the error presented when analyzing the 67p comet and this may occur due to not knowing the initial position and the initial velocities of the comet. This also highlights the uncertainty and variability of this method as there are instances where the error is almost insignificant and also instances where the error is blatantly obvious.

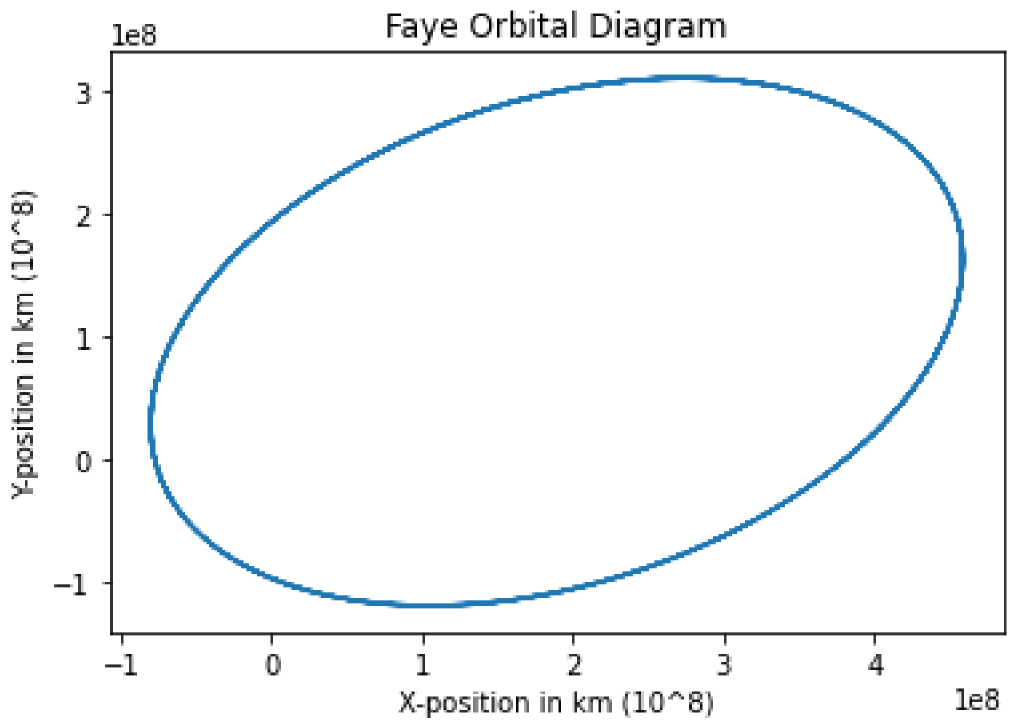

Figure 10. A diagram of the Faye Comet's orbit with the sun centered at (0,0) which was obtained from the code presented in figure 12 in the appendix (step size= 100 seconds)

Similarly with the comet of 67P, Faye Comet also had an estimation of the orbit produced and it shown in the orbital diagram, figure 10. It once again successfully captures the elliptical orbit around the sun and since the estimation uses Euler-Cromer method, the graph is significantly more accurate than a graph just using Euler's method would be. Once again the comet returned to the starting point to make a full ellipse indicating that the end behavior predicted by Euler-Cromer method was correct. Despite the fact that the Faye Orbital Diagram is relatively accurate , it will be far less accurate than the 67P Orbital Diagram as the initial values and the velocity had to be self determined.

CONCLUSION

Euler's method is simple approximation used when ODE cannot be solved or would require too much time to solve. When comparing to Runge-Kutta method that was also learned in class, there would be less drift from the analytical value. However, the method would take significantly longer, and the decreased drift might not be helpful because as seen in each comet's orbital diagram, the comet does not really change its orbit over time.

One limitation of this method is the identification of the initial conditions. While it is relatively simple to identify the initial position, it is very difficult to calculate the initial x-velocity and y-velocity because comet's speed changes over time, but information of comets only provide the average velocity. Also, a second position must be A large cause of inaccuracy from 4P compared to 67p is due to this uncertainty.

Another limitation in our model was that we did not account for the gravitational influence of the planets which would have a non-negligible impact on the comet's acceleration when the planet and comet are close. Regardless of how accurate our approximations were, this is an inherent flaw in our simplified model. In order to expand the model, we would we need to keep track of the planets' motion up to Saturn since the two comets in question do not approach Uranus or Neptune. Each planet would add their own acceleration term to the ODE that describes the comet's motion. Furthermore, each direction vector would be different because each planet is at a different location, so the position vector and direction vector would no longer be equivalent, significantly increasing the complexity of the model.

Another limitation of the model was the assumption that the comet was spherical and did not rotate. In reality, comets are rarely close to a sphere since their gravitational force is not enough to compact itself into a sphere. Since their shapes are not sphere-like, their angular velocity can change from outside influences like gravity, so this would also affect comet's orbits.

With our project, we accomplished a simplified model on how to calculate the position of objects such as comets in the solar system. This model would also work for comets rotating single stars besides our sun by adjusting the star's mass. As long as there are the correct initial conditions, the results are accurate within one-hundredth of a percent.

APPENDIX

```
from matplotlib import pyplot as plt
from math import *

G=6.67384*(10**(-20))                   #gravitational constant in km^3 not m^3
m_s=1988500*(10**(24))                  #mass of sun in kg
t=0                                     #init time
x=193929400                             #init x position relative to sun
y=0                                     #init y position relative to sun
h=100                                   #step size (100 seconds)
ttime=315360000                         #10 years in seconds
nos=ttime//h                            #number of steps
vx=-5.91                                #init x velocity
vy=32.82                                #init y velocity
n=(x*x+y*y)**(1/2)                      #magnitude of distance (r_cs vector on the info sheet)
time=[]                                 #list of times at each step
dis=[]                                  #list of distances at each step
lx=[]                                   #list of x positions at each step
ly=[]                                   #list of y positions at each step

'''euler's method calculations done in a loop
    calculates acceleration, velocity, and position by components
    then calculates the distance for each step'''
for i in range(0,nos):
    a_x=-((G*m_s)/(n*n*n))*x
    a_y=-((G*m_s)/(n*n*n))*y

    vx=vx+h*a_x
    vy=vy+h*a_y

    x=x+h*vx
    y=y+h*vy
    lx.append(x)
    ly.append(y)

    d=(x*x+y*y)**(1/2)                  #calculates the distance for the current step
    n=d                                 #sets r_cs to the current distance
    dis.append(d)
    time.append(t)
    t=t+h

item=min(dis)
print("Minimum Distance:", item)

matchlist=[]                            #creates a list used to match the times for each min distance to calculate period
for i in range(0,len(dis)):             #for loop that finds and adds index of each minimum distance
    if dis[i]==item:
        matchlist.append(i)

lop=[]                                  #list of periods (abbreviated)
for i in matchlist:
    lop.append(time[i])
print(lop)                              #prints period in seconds

#code for plotting graphs
plt.plot(time, dis)
plt.xlabel("Time in sec (10^8)")
plt.ylabel("Distance in km (10^8)")
plt.title("Approx. Solution with Euler's Method")
plt.show()

plt.plot(lx,ly)
plt.xlabel("X-position in km (10^8)")
plt.ylabel("Y-position in km (10^8)")
plt.title("67P Orbital Diagram")
plt.show()
```

Figure 11. Code used to approximate orbit for 67P/Churyumov-Gerasimenko

```
G=6.67384*(10**(-20))          #gravitational constant in km^3 not m^3
m_s=1988500*(10**(24))         #mass of sun in kg
t=0                            #init time
x=380324365                    #init x position relative to sun
y=0                            #init y position relative to sun
vx=10.76                       #init x velocity
vy=10.85                       #init y velocity
h=100                          #step size (100 seconds)
ttime=315360000                #10 years in seconds
```

Figure 12. For 4P/Faye, the following values were changed.
The algorithm for 4P/Faye is same 67P

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